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HOLOMORPHIC L^2 FUNCTIONS ON COVERINGS OF PSEUDOCONVEX MANIFOLDS

M. Gromov, G. Henkin and M. Shubin

0 Preliminaries and Main Results

1. Let M be a complex manifold with a smooth boundary which will be denoted bM, $\dim_{\mathbf{C}} M = n$. Let us denote $\overline{M} = M \cup bM$ and assume for simplicity that $\overline{M} \subset \tilde{M}$ where $\tilde{M}$ is a complex neighbourhood of $\overline{M}$, $\dim_{\mathbf{C}} \tilde{M} = n$, so that every point $z \in bM$ is an interior point of $\tilde{M}$. Let us take a C^∞-function $\rho : \tilde{M} \to \mathbf{R}$ such that

$$M = \{z \mid \rho(z) < 0\}, \quad bM = \{z \mid \rho(z) = 0\}; \quad d\rho(z) \neq 0 \text{ for all } z \in bM . \tag{0.1}$$

For any $z \in bM$ denote by $T_z^c(bM)$ the complex tangent space to bM: the maximal complex subspace in the real tangent space $T_z(bM)$, $\dim_{\mathbf{C}} T_z^c(bM) = n-1$. If $z_1, \dots, z_n$ are complex local coordinates in $\tilde{M}$ near $z \in bM$, then $T_z\tilde{M}$ is identified with $\mathbf{C}^n$ and

$$T_z^c(bM) = \left\{ w = (w_1, \dots, w_n) \,\Big|\, \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z) w_j = 0 \right\}. \tag{0.2}$$

The *Levi form* is an hermitian form on $T_z^c(bM)$ defined in the local coordinates as follows:

$$L_z(w, \bar{w}) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k . \tag{0.3}$$

The manifold M is called *pseudoconvex* if $L_z(w, \bar{w}) \geq 0$ for all $z \in bM$ and $w \in T_z^c(bM)$. It is called *strongly pseudoconvex* if $L_z(w, \bar{w}) > 0$ for all $z \in bM$ and all $w \neq 0$, $w \in T_z^c(bM)$. In this case replacing ρ by $e^{\lambda\rho} - 1$ with sufficiently large $\lambda > 0$ we can assume that $L_z(w, \bar{w}) > 0$ for all $w \neq 0$ (not only for w satisfying the condition in (0.2)).

Equivalently, strongly pseudoconvex manifolds can be described as the ones which locally, in a neighbourhood of any boundary point, can be presented as strongly convex domains in $\mathbf{C}^n$.

Denote by $\mathcal{O}(M)$ the set of all holomorphic functions on M.

A point $z \in bM$ is called a *peak point* for $\mathcal{O}(M)$ if there exists a function $f \in \mathcal{O}(M)$ such that f is unbounded on M but bounded outside $U \cap M$ for any neighbourhood U of z in $\tilde{M}$.

A point $z \in bM$ is called a *local peak point* for $\mathcal{O}(M)$ if there exists a neighbourhood U of z in $\tilde{M}$ such that z is a peak point for $\mathcal{O}(M)|(U \cap M)$ which is the space of all restrictions of functions from $\mathcal{O}(M)$ to $U \cap M$. In other words there exists a function $f \in \mathcal{O}(M)$ such that f is unbounded in $U \cap M$ for any neighbourhood U of z in $\tilde{M}$ and there exists a neighbourhood U of z in $\tilde{M}$ such that for any neighbourhood V of z in $\tilde{M}$ the function f is bounded in $U - V$.

Note that the pseudoconvexity and strong pseudoconvexity at a point $z \in bM$ are local notions, whereas being a peak point or a local peak point for $\mathcal{O}(M)$ depends on the global structure of M.

The Oka-Grauert theorem ([Gr1], see also [FKo], [He]) states that if M is strongly pseudoconvex, bM is not empty and M is compact, then every point $z \in bM$ is a peak point for $\mathcal{O}(M)$. (Moreover for every $z \in bM$ there exist functions $f_1, \dots, f_n \in \mathcal{O}(M)$ which are local complex coordinates in $U \cap M$ for a neighbourhood U of z in $\tilde{M}$.) It follows in particular that the space $\mathcal{O}(M)$ is infinite-dimensional. If M is weakly pseudoconvex then the space $\mathcal{O}(M)$ is not necessarily infinite-dimensional (see [Gr2]). But if M is weakly pseudoconvex, $\overline{M}$ is compact and, in addition, in the neighbourhood of bM there exists a strictly plurisubharmonic function (not necessarily vanishing on bM) then M can be exhausted by strongly pseudoconvex manifolds (see e.g. (i) in the Lemma 1.10 below). In this case $\mathcal{O}(M)$ is again infinite-dimensional by the Oka-Grauert theory. Note that for the case of domains in $\mathbf{C}^n$ we can even construct the required exhaustion using a global plurisubharmonic function in M (see e.g. [Hö2]). This is not true in general case (see e.g. an example in [Gr2]).

One of the goals of this paper is to extend these results to the case when $\overline{M}$ is not necessarily compact but admits a free holomorphic action of a discrete group Γ such that the orbit space $\overline{M}/\Gamma$ is compact (or in other words $\overline{M}$ is a regular covering of a compact complex manifold with a strongly pseudoconvex boundary). In this case we shall use the von Neumann Γ-dimension $\dim_\Gamma$ to measure Hilbert spaces of holomorphic functions (or some exterior forms) which are in L^2 with respect to a Γ-invariant smooth measure on $\overline{M}$. In case when the group Γ is trivial (i.e. has only one element) the Γ-dimension is just the usual dimension $\dim_{\mathbf{C}}$. We shall prove that the space of L^2-holomorphic functions on a strongly pseudoconvex regular covering $\overline{M}$ of a compact manifold with a non-empty boundary has an infinite Γ-dimension and every point $z \in bM$ is a local peak point for this space.

We shall also prove that $\dim_\Gamma L^2\mathcal{O}(M) = \infty$, if covering $\overline{M}$ is only weakly pseudoconvex, but with strictly plurisubharmonic Γ-invariant function existing in a neighbourhood of bM.

The arguments given in the proof can be carried over if instead of the discrete group Γ we have an arbitrary unimodular Lie group G (not necessarily connected) with free action on $\overline{M}$ (holomorphic in M) such that the quotient $\overline{M}/G$ is compact. The unimodularity is used just to introduce a von Neumann trace and the corresponding dimension as in [CoMo].

A natural question arises: is the cocompact group action really relevant for the existence of many holomorphic L^2-functions or is it just an artifact of the chosen methods which require a use of von Neumann algebras? Can we at least get rid of the unimodularity requirement? In section 3 we give an example which shows that the answer to the last question is negative. In this example $\dim_{\mathbf{C}} M = 2$, bM is strongly pseudoconvex, G is a solvable non-unimodular connected Lie group, $\dim_{\mathbf{R}} G = 3$, G has a free action on $\overline{M}$ which is holomorphic on M, $\overline{M}/G = [-1, 1]$, but $L^2\mathcal{O}(M) = \{0\}$.

It follows in particular that if we only impose bounded geometry conditions with uniform strong pseudoconvexity, then the space of holomorphic L^2-functions may be trivial. It is not clear how to formulate conditions assuring that $\dim L^2\mathcal{O}(M) = \infty$ without any group action.

About half of the presented results are contained in [GroHeS] which can be considered as a preliminary version of this paper.

2. Let us choose a boundary point x for a strongly pseudoconvex manifold M and describe the classical E. Levi construction of a locally defined holomorphic function on $U \cap M$ (here U is a neighbourhood of x in $\tilde{M}$) with the peak point x. Let us consider the Taylor expansion of ρ at x:

$$\rho(z) = \rho(x) + 2\,\mathrm{Re}\, f(x,z) + L_x(z-x, \bar{z}-\bar{x}) + O\big(|z-x|^3\big)\,, \qquad (0.4)$$

where L_x is the Levi form at x and $f(x,z)$ is a complex quadratic polynomial with respect to z:

$$f(x,z) = \sum_{1\le\nu\le n} \frac{\partial\rho}{\partial z_\nu}(x)(z_\nu - x_\nu) + \frac{1}{2}\sum_{1\le\mu,\nu\le n} \frac{\partial^2\rho}{\partial z_\mu \partial z_\nu}(x)(z_\mu - x_\mu)(z_\nu - x_\nu)\,.$$

The complex quadric hypersurface $S_x = \{z \mid f(x,z) = 0\}$ has $T_x^c(bM)$ as its tangent plane at x. Therefore the strong pseudoconvexity implies that $\rho(z) > 0$ if $f(x,z) = 0$ and $z \neq x$ is close to x. This means that near x the intersection of the hypersurface S_x with $\overline{M}$ consists of one point x. Hence the function $1/f(x,\cdot)$ is holomorphic in $U\cap M$ (where U is a neighbourhood of x in $\tilde{M}$) and x is its peak point.

The technique which allows to pass from locally defined holomorphic functions to global ones is $\bar\partial$-cohomology on complex manifolds. For any integers p,q with $1\le p,q\le n$ denote by $\Lambda^{p,q}(M)$ the space of all C^∞ forms of type p,q on M, i.e. forms which can be written in local complex coordinates as

$$\omega=\sum\nolimits'_{|I|=p,|J|=q}\omega_{I,J}dz^I\wedge d\bar z^J\,,$$

where $dz^I=dz^{i_1}\wedge\cdots\wedge dz^{i_p}$, $dz^J=d\bar z^{j_1}\wedge\cdots\wedge d\bar z^{j_q}$, $I=(i_1,\dots,i_p)$, $J=(j_1,\dots,j_q)$, $i_1<\cdots<i_p$, $j_1<\cdots<j_q$, and $\omega_{I,J}$ are C^∞ functions in local coordinates. Here and later $\sum'$ stands for summing over increasing multiindices. For such a form ω its $\bar\partial$ differential is written as

$$\bar\partial\omega=\sum\nolimits'_{|I|=p,|J|=q}\sum_{k=1}^{n}\frac{\partial\omega_{I,J}}{\partial\bar z^k}d\bar z^k\wedge dz^I\wedge d\bar z^J\,,$$

so $\bar\partial$ defines a linear map $\bar\partial:\Lambda^{p,q}(M)\to\Lambda^{p,q+1}(M)$. All these maps constitute a complex of vector spaces

$$\Lambda^{p,\bullet}:0\longrightarrow\Lambda^{p,0}\longrightarrow\Lambda^{p,1}\longrightarrow\cdots\longrightarrow\Lambda^{p,n}\longrightarrow 0\,.$$

Its cohomology spaces are denoted $H^{p,q}(M)$.

An important part of the Grauert theorem is the fact that $\dim_{\mathbf{C}}H^{p,q}(M)<\infty$ for all p,q with $q>0$ provided M is strongly pseudoconvex and $\overline{M}$ is compact. A refinement of this fact is used in constructing global holomorphic functions on M with a peak point $x\in bM$ as follows. We start with a locally defined function $g\in\mathcal{O}(U\cap M)$ (here U is a neighbourhood of x in $\tilde{M}$) with a peak point at x, multiply it by a cut-off function $\chi\in C_0^\infty(U)$ which equals 1 in a neighbourhood of x, then solve the equation $\bar\partial f=\bar\partial(\chi g)$ on M in appropriate function spaces consisting of bounded functions on M. If we can do this then the function $\chi g-f$ is holomorphic on M and x is its peak point. The existence of a bounded solution for the equation $\bar\partial f=\alpha\in\Lambda^{0,1}(\overline{M})$ for all forms α with $\bar\partial\alpha=0$ is equivalent to the vanishing of an appropriate refinement of the cohomology space $H^{0,1}(M)$ (we should actually consider cohomology $H^{p,q}(M)$ with estimates). If we only know that the latter space has a finite dimension then we still can solve the equation $\bar\partial f=\alpha$ for all $\bar\partial$-closed forms α in the space of finite codimension in the space of all $\bar\partial$-closed forms. This is sufficient to construct holomorphic functions on M with the peak point x because it is easy to provide an infinite-dimensional space of holomorphic functions in a neighbourhood of x having x as its peak point (e.g. we can take a linear space spanned by all powers of one function with the peak point x).

3. Now we shall give a very brief description of the Γ-dimension. It will be used to measure Γ-invariant spaces (of functions and forms) which are infinite-dimensional in the usual sense. It is also convenient to use the Γ-trace. For more details we refer the reader to [At], [C] and textbooks on von Neumann algebras (e.g. [D], [N], [Ta]).

We shall denote the Γ-dimension by $\dim_\Gamma$. It is defined on the set of all (projective) Hilbert Γ-modules and takes values in $[0,\infty]$. The simplest Hilbert Γ-module is given by a left regular representation of Γ: it is the Hilbert space $L^2\Gamma$ consisting of all complex-valued L^2-functions on Γ. The group Γ acts unitarily on $L^2\Gamma$ by $\gamma \mapsto L_\gamma$ where L_γ is defined as follows:

$$L_\gamma f(x) = f(\gamma^{-1}x)\,, \quad x \in \Gamma\,, \quad f \in L^2\Gamma\,.$$

By definition $\dim_\Gamma L^2\Gamma = 1$.

For any (complex) Hilbert space $\mathcal{H}$ define a free Hilbert Γ-module $L^2\Gamma \otimes \mathcal{H}$. Its Γ-dimension equals $\dim_{\mathbf{C}} \mathcal{H}$. The action of Γ in $L^2\Gamma \otimes \mathcal{H}$ is defined by $\gamma \mapsto L_\gamma \otimes I$.

A general Hilbert Γ-module is a closed Γ-invariant subspace in a free Hilbert Γ-module. It would be natural to call such subspaces *projective* Hilbert modules, but the word "projective" is usually omitted, so only projective Hilbert modules are considered.

For any Hilbert space $\mathcal{H}$ denote by $\mathcal{A}_\Gamma$ a von Neumann algebra which consists of all bounded linear operators in $L^2\Gamma \otimes \mathcal{H}$ which commute with the action of Γ there. This algebra is in fact generated by the operators of the form $R_\gamma \otimes B$, $B \in \mathcal{B}(\mathcal{H})$, $\gamma \in \Gamma$, where $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators in $\mathcal{H}$, R_γ is the operator of the right translation in $L^2\Gamma$, i.e.

$$R_\gamma f(x) = f(x\gamma)\,, \quad x \in \Gamma\,, \quad f \in L^2\Gamma\,.$$

This means that the algebra $\mathcal{A}_\Gamma$ is the weak closure of all finite linear combinations of the operators of the form $R_\gamma \otimes B$. So in fact $\mathcal{A}_\Gamma$ is a tensor product (in the sense of von Neumann algebras) of $\mathcal{R}_\Gamma$ and $\mathcal{B}(\mathcal{H})$ where $\mathcal{R}_\Gamma$ is the von Neumann algebra generated by the operators R_γ in $L^2\Gamma$ (it consists of all operators in $L^2\Gamma$ which commute with all operators $L_\gamma,\ \gamma \in \Gamma$).

There is a natural trace on $\mathcal{R}_\Gamma$. It is denoted by tr_Γ and defined as the diagonal matrix element (all of them are equal) in the δ-functions basis.

For example we can define it by

$$\mathrm{tr}_\Gamma S = (S\delta_e, \delta_e)\,, \qquad S \in \mathcal{R}_\Gamma\,,$$

where e is the neutral element of Γ, $\delta_e \in L^2\Gamma$ is the "Dirac delta-function" at e, i.e. $\delta_e(x) = 1$ if $x = e$ and 0 otherwise. There is also a natural trace on $\mathcal{A}_\Gamma$ too: $\mathrm{Tr}_\Gamma = \mathrm{tr}_\Gamma \otimes \mathrm{Tr}$ where Tr is the usual trace on $\mathcal{B}(\mathcal{H})$.

Now for any Hilbert Γ-module which is a closed Γ-invariant subspace L in $L^2\Gamma \otimes \mathcal{H}$, its Γ-dimension is defined by the natural formula

$$\dim_\Gamma L = \mathrm{Tr}_\Gamma P_L\,,$$

where P_L is the orthogonal projection on L in $L^2\Gamma \otimes \mathcal{H}$.

4. Let us describe the reduced L^2 Dolbeault cohomology spaces on a complex (generally non-compact) manifold M with a given hermitian metric. Denote the Hilbert space of all (measurable) square-integrable (p,q)-forms on M by $L^2\Lambda^{p,q} = L^2\Lambda^{p,q}(M)$. The operator

$$\bar\partial : L^2\Lambda^{p,q}(M) \longrightarrow L^2\Lambda^{p,q+1}(M)$$

is defined as the maximal operator, i.e. its domain $D^{p,q} = D^{p,q}(\bar\partial; M)$ is the set of all $\omega \in L^2\Lambda^{p,q}$ such that $\bar\partial\omega \in L^2\Lambda^{p,q+1}$ where $\bar\partial\omega$ is applied in the sense of distributions. Obviously $\bar\partial^2 = 0$ on $D^{p,q}$ and we can form a complex

$$L^2\Lambda^{p,\bullet} : 0 \longrightarrow D^{p,0} \longrightarrow D^{p,1} \longrightarrow \cdots \longrightarrow D^{p,n} \longrightarrow 0\,.$$

Its cohomology spaces are denoted $L^2H^{p,q}(M)$ and called L^2 Dolbeault cohomology spaces of M:

$$L^2H^{p,q}(M) = \mathrm{Ker}(\bar\partial : D^{p,q} \to D^{p,q+1}) / \,\mathrm{Im}(\bar\partial : D^{p,q-1} \to D^{p,q})\,.$$

We actually need reduced L^2 Dolbeault cohomology spaces

$$L^2\bar H^{p,q}(M) = \mathrm{Ker}(\bar\partial : D^{p,q} \to D^{p,q+1}) / \overline{\mathrm{Im}(\bar\partial : D^{p,q-1} \to D^{p,q})}\,,$$

where the line over $\mathrm{Im}\,\bar\partial$ means its closure in the corresponding L^2 space. Since $\mathrm{Ker}\,\bar\partial$ is a closed subspace in L^2, the reduced cohomology space $L^2\bar H^{p,q}(M)$ is a Hilbert space.

Note that the space $L^2\bar H^{0,0}(M)$ coincides with the space $L^2\mathcal{O}(\mathcal{M})$ of all square-integrable holomorphic functions on M.

5. Let us assume now that M is a complex manifold (with boundary) with a free action of a discrete group Γ on $\overline{M}$ such that $\overline{M}/\Gamma$ is compact and the action is holomorphic on M. (Here $\overline{M} = M \cup bM$.) Let us assume that an hermitian Γ-invariant metric is given on $\overline{M}$. Then the reduced L^2 Dolbeault cohomology spaces become Hilbert Γ-modules. Hence they have a well defined Γ-dimension (possibly infinity).

We will also assume for simplicity that $\overline{M}$ is a closed subset in $\tilde{M}$ where $\tilde{M}$ is a complex neighbourhood of $\overline{M}$ with a free holomorphic action of Γ so that this action and the complex structure on $\tilde{M}$ extend the corresponding structures of M, and every point of $\overline{M}$ is an interior point in $\tilde{M}$.

Now we will formulate our main results.

Theorem 0.1. *If M is strongly pseudoconvex, then $\dim_\Gamma L^2\bar{H}^{p,q}(M) < \infty$ for all p, q provided $q > 0$.*

Theorem 0.2. *If M is strongly pseudoconvex and bM is non-empty, then $\dim_\Gamma L^2\mathcal{O}(M) = \infty$ and each point in bM is a local peak point for $L^2\mathcal{O}(M)$.*

Under the same conditions it is also possible to construct holomorphic functions which have stronger local singularities (not L^2) but are in L^2 in a generalized sense. For any $s \in \mathbf{R}$ denote by $W^s = W^s(M)$ the uniform (Γ-invariant) Sobolev space of distributions on M, based on the space $W^0 = L^2(M)$ constructed with the use of a smooth Γ-invariant measure on $\overline{M}$ (see e.g. [S1] for the details on the Sobolev spaces). The space W^{-s} for large $s > 0$ contains in particular holomorphic functions on M with power singularities at the boundary. For any $s \in \mathbf{R}$ the space W^s is a Hilbert Γ-module with respect to the natural action of Γ. Denote by $W^s\mathcal{O}(M)$ the space of all elements in W^s which are actually holomorphic functions on M. Now we can formulate another version of Theorem 0.2.

Theorem 0.3. *If M is strongly pseudoconvex and $bM \neq \emptyset$, then for any $x \in bM$ and any integer $N > 0$ there exists $s > 0$ and a closed Γ-invariant subspace $L \subset W^{-s}\mathcal{O}(M)$ such that*

(i) $\dim_\Gamma L = N$;
(ii) $L \cap L^2(M) = \{0\}$ *but for any $f \in L$ and any Γ-invariant neighbourhood U of x in $\overline{M}$ we have $f \in L^2(M - U)$.*

It is also possible to construct L^2-holomorphic functions on M which are in $C^\infty(\overline{M})$:

Theorem 0.4. *If M is strongly pseudoconvex and $bM \neq \emptyset$, then for any integer $N > 0$ there exists a Γ-invariant subspace $L \subset L^2\mathcal{O}(M) \cap C^\infty(\overline{M})$ such that $\dim_\Gamma \bar{L} = N$ where $\bar{L}$ is the closure of L in $L^2(M)$.*

Examples. 1) Let X be a compact real-analytic manifold with an infinite fundamental group $\Gamma = \pi_1(X)$. Assume that X is imbedded into its complexification Y and a Riemannian metric is chosen on Y. Let X_ε be a ε-neighbourhood of X in Y where $\varepsilon > 0$ is sufficiently small. It is known ([M1], [Gr1]) that then X_ε is strongly pseudoconvex. Let M be the universal covering of X_ε. Theorems 0.1–0.4 can be applied to M and we conclude in particular that there are sufficiently many L^2 holomorphic functions on M.

A particular case: strip $\{z \mid |\operatorname{Im} z| < 1\}$ in $\mathbf{C}$ with the action of $\Gamma = \mathbf{Z}$ by translations along $\mathbf{R}$. Of course in this case L^2 holomorphic functions can be obtained by the Fourier transform or explicitly (e.g. take $1/(a^2+z^2)$ where $a > 1$, or $\exp(-z^2)\log(z-i)$).

2) Let X be a compact complex manifold with a holomorphic negative vector line bundle E on X. The negativity means that E is supplied with an hermitian metric and ε-neighbourhood X_ε of X in the total space of E is strongly pseudoconvex (for some $\varepsilon > 0$ or, equivalently, for any $\varepsilon > 0$). Denote by M the universal covering of X.

Note that X_ε is not a Stein manifold because it has a non-trivial compact complex submanifold X (the zero section of E). But we are again in the situation of Theorems 0.1–0.4 and these theorems give extensions of some results of Napier [Na] and Gromov [Gro]. Namely let M_ε be the universal covering of X_ε. Theorems 0.2–0.4 guarantee that there are many L^2 holomorphic functions on M_ε. In particular, $\dim_\Gamma L^2\mathcal{O}(M_\varepsilon) = \infty$. Using the Taylor expansion of $f \in L^2\mathcal{O}(M_\varepsilon)$ along the fibers we obtain L^2-spaces with a finite positive Γ-dimension, such that they consist of holomorphic functions which are polynomial along the fibers. This means that E^{-k} has many holomorphic L^2-sections over M ([Na]). Under the Kähler hyperbolicity condition Gromov [Gro] proved that in fact there are sufficiently many holomorphic L^2-forms (of the type $(n,0)$) on M.

6. The following results give rather general conditions when the statements of Theorems 0.1, 0.2 are still valid for weakly pseudoconvex (i.e. pseudoconvex but not necessarily strongly pseudoconvex) manifolds.

Theorem 0.5. *Let M be a pseudoconvex manifold with holomorphic action of a discrete group Γ on $\overline{M}$ such that $\overline{M}/\Gamma$ is compact. As before we assume that $bM \neq \emptyset$ and $\overline{M}$ is a closed subset in $\tilde{M}$ where $\tilde{M}$ is a complex neighbourhood of $\overline{M}$ with a free holomorphic action of Γ so that this action and the complex structure on $\tilde{M}$ extend the corresponding structures of M, and every point of $\overline{M}$ is an interior point in $\tilde{M}$. Let g be a Γ-invariant hermitian metric on $\overline{M}$. Suppose that in a Γ-invariant neighbourhood U of bM (in $\tilde{M}$) there exist a Γ-invariant strictly plurisubharmonic function Φ and a constant $\delta > 0$ such that $i\partial\bar\partial\Phi \geq \delta \cdot g$ in U. Then $\dim_\Gamma L^2\bar{H}^{p,q}(M) < \infty$ for all p, q with $q > 0$.*

Remark. The inequality $i\partial\bar{\partial}\Phi \geq \delta \cdot g$ is automatically true in a possibly smaller Γ-invariant neighbourhood of $\overline{M}$ provided Φ is in C^2, strictly plurisubharmonic and Γ-invariant.

Theorem 0.6. *Under the assumptions of Theorem 0.5*

(i) $\dim_\Gamma L^2\mathcal{O}(M) = \infty$*;*

(ii) *each point of strong pseudoconvexity in* bM *is a local peak point for* $L^2\mathcal{O}(M)$*.*

Theorem 0.7. *If under the assumptions of Theorem 0.5* M/Γ *is a Stein manifold, then* $L^2H^{p,q}(M) = 0$ *for all* p, q *with* $q > 0$ *(i.e.* $\operatorname{Im}\bar{\partial}$ *exactly coincides with* $\operatorname{Ker}\bar{\partial}$ *in the corresponding spaces* $L^2\Lambda^{p,q}$*), and the functions from* $L^2\mathcal{O}(M)$ *separate all points in* M*.*

Examples. (i) Any pseudoconvex domain X in $\mathbf{C}^n$ or $\mathbf{CP}^n$ is a Stein manifold ([Hö2], [Fu]). Therefore Theorem 0.7 can be applied to its regular covering manifolds.

(ii) For any real-analytic manifold X_0, $\dim_{\mathbf{R}} X_0 = n$, we can find its complex neighbourhood X, $\dim_{\mathbf{C}} X = n$, such that X is a Stein manifold [Gr2] (then X is called a *Grauert tube* for X_0). We can also assume that $\overline{X}$ is a manifold with a smooth boundary. If in addition X_0 is compact, Theorem 0.7 can be applied to any regular covering manifold of X.

7. Remarks. 1) If under the assumptions of the theorems above M/Γ is a Stein manifold then Stein [St] proved that M is also a Stein manifold. It follows from this result that there are sufficiently many holomorphic functions on M, but it does not follow that there exist non-trivial L^2 holomorphic functions. On the other hand it can happen that M/Γ is not Stein (see Example 2 above). Then even the existence of any holomorphic function on M which is not constant along orbits of Γ is not obvious.

2) If $bM = \emptyset$ then it follows from the arguments of Atiyah [At] that $\dim_\Gamma L^2\bar{H}^{p,q}(M) < \infty$ for all p, q (including $q = 0$). In this case in fact $L^2\mathcal{O}(M) = \{0\}$. But if E is a positive Γ-invariant holomorphic line bundle, then the Kodaira imbedding theorem [K] and the Atiyah index theorem [At] imply that $\dim_\Gamma L^2\mathcal{O}(M, E^k) > 0$ for large k; in particular, this again gives the result of [Na]: the space of all holomorphic sections of E^k is infinite-dimensional in the usual sense.

3) Theorems 0.2–0.4 remain valid if we replace holomorphic functions by holomorphic $(p, 0)$-forms. More generally all Theorems 0.1–0.4 are true for sections of arbitrary holomorphic vector Γ-bundles over M.

4) Theorems 0.1–0.4 can be extended to the case when M is strongly pseudoconvex but with possibly non-smooth boundary, i.e. we can drop the requirement $d\rho \neq 0$ on bM in (0.1) but require instead that the Levi form (0.3) is positive for all $z \in bM$ and all $w \neq 0$, $w \in \mathbf{C}^n$.

5) Let us assume that the Levi form (0.3) is non-degenerate on $T_z^c(bM)$ for all $z \in bM$ and the boundary bM is connected. Note that bM will be automatically connected if M is connected and the Levi form has at least one plus at every boundary point and there exists at least one non-trivial holomorphic function on M (in particular this is true if bM is strongly pseudoconvex). Indeed, J. Kohn and H. Rossi [KoRo] proved that in this case every CR-function on bM can be extended to a holomorphic function on M. If we assume that bM is not connected, this leads to a contradiction if we consider a locally constant CR-function which equals 1 on one of the connected components of bM and 0 on all others.

Let r be the number of negative eigenvalues of the Levi form in $T_z^c(bM)$. Then

$$\dim_\Gamma L^2\bar{H}^{0,r}(M) = \infty$$

and

$$\dim_\Gamma L^2\bar{H}^{p,q}(M) < \infty\,, \quad q \neq r\,.$$

This is a generalization to the covering case of the classical theorems by Andreotti-Grauert and Andreotti-Norguet (see [AV], [FKo], [Hö1], [AHi]).

6) There are analogues of Theorems 0.1–0.4 for regular coverings of compact strongly pseudoconvex CR-manifolds. Let N be a strongly pseudoconvex CR-manifold with a free action of a discrete group Γ such that $Y = N/\Gamma$ is compact, $\dim_{\mathbf{R}} N = 2k-1$.

Assume that $k \geq 3$. Let us denote by $L^2H^{p,q}_{CR}(N)$ and $L^2\bar{H}^{p,q}_{CR}(N)$ the L^2 Kohn-Rossi cohomology spaces and reduced cohomology spaces respectively (see [KoRo] for the usual version of these cohomology spaces on compact CR-manifolds). They are defined similarly to the Dolbeault cohomology spaces by using the tangent Cauchy-Riemann operator $\bar\partial_b$ instead of $\bar\partial$. Then the following statements are true:

(i) $\dim_\Gamma L^2\bar{H}^{p,q}_{CR}(N) < \infty$ if $1 \leq q \leq k-2$.
(ii) $\dim_\Gamma L^2\bar{H}^{p,0}_{CR}(N) = \infty$ and $\dim_\Gamma L^2\bar{H}^{p,k-1}_{CR}(N) = \infty$ for all p with $0 \leq p \leq k$.

Note that if $k=2$ the statement (i) is empty and (ii) is not true even in the compact case.

7) First applications of von Neumann algebras to constructions of non-trivial spaces of L^2-holomorphic functions or sections of holomorphic vector bundles are due to M. Atiyah [At] and A. Connes [Co]. J. Roe proved ex-

istence of an infinite-dimensional space of L^2 holomorphic sections of a power E^k for a uniformly positive holomorphic line bundle E over a complete Kähler simply connected manifold of non-positive curvature without any action of a discrete group (see [R] for further results and references).

1 $\bar{\partial}$-cohomology Spaces of Pseudoconvex Coverings

1. In this section we will prove Theorems 0.1 and 0.5. We will start by extending the Kohn-Morrey estimates ([FKo], [M2]) to our case. We will always assume that M is pseudoconvex.

First we will consider a general Γ-invariant analytic situation. Namely let M be a C^∞-manifold (possibly with boundary) with a free action of a discrete group Γ such that $\overline{M}/\Gamma$ is compact. Let E be a (complex) vector Γ-bundle on $\overline{M}$ with a Γ-invariant hermitian metric in the fibers of E. We shall use Γ-invariant Sobolev spaces W^s of sections of E over M. The scale of the Hilbert spaces $W^s = W^s(M,E)$ is based on the Hilbert space $L^2(M,E)$ which is taken with respect to a smooth positive Γ-invariant measure on $\overline{M}$ and the given Γ-invariant hermitian metric on E over $\overline{M}$. Let $\tilde{M}$ be a Γ-invarant complex neighbourhood of $\overline{M}$. Assume that E and the measure on M are extended to $\tilde{M}$ in a smooth Γ-invariant way. For any $s \in \mathbf{R}$ the space $W^s = W^s(M,E)$ is a Hilbert space which consists of all restrictions to M of finite linear combinations of all sections Au where $u \in L^2(\tilde{M},E)$ and A is a properly supported Γ-invariant pseudodifferential operator of order $-s$ on $\tilde{M}$ (see e.g. [At] or [S1]). The norm in W^s is denoted $\|\cdot\|_s$.

In particular Γ-invariant Sobolev spaces $W^s\Lambda^{p,q}$ of (p,q)-differential forms on M are well defined.

Let us consider $\bar{\partial}$ as the maximal operator in L^2 and let $\bar{\partial}^*$ be the Hilbert space adjoint operator. We shall also use the corresponding Laplacian

$$\Box = \Box_{p,q} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \quad \text{on} \quad L^2\Lambda^{p,q}(M)\,.$$

We shall denote the domain of any operator A by $D(A)$. Let $\Lambda_c^\bullet(\overline{M})$ denote the set of all C^∞ forms with compact support on $\overline{M}$.

The following lemma gives a description of the operators $\bar{\partial}^*$, $\Box$ (as well as their domains $D(\bar{\partial}^*)$, $D(\Box)$). Let θ be the formal adjoint operator to $\bar{\partial}$, $\sigma = \sigma(\theta,\cdot)$ its principal symbol.

Lemma 1.1. *Let us assume that M is strongly pseudoconvex.*

(i) *The operator $\bar{\partial}^*$ can be obtained as the closure of θ from the initial* *domain*

$$D_0(\bar\partial^*) = \left\{\omega \;\middle|\; \omega \in \Lambda_c^\bullet(\overline{M}),\ \sigma(\theta, d\rho)\omega = 0 \ \text{ on } \ bM\right\}. \tag{1.1}$$

(ii) *The space* $D_0(\bar\partial^*)$ *is dense in* $D(\bar\partial^*) \cap D(\bar\partial)$ *in the norm*

$$\left(\|\omega\|_0^2 + \|\bar\partial^*\omega\|_0^2 + \|\bar\partial\omega\|_0^2\right)^{1/2}, \quad \omega \in D(\bar\partial^*) \cap D(\bar\partial).$$

(iii) *The operator* $\Box = \Box_{p,q}$ *can be obtained as the closure of the operator* $\bar\partial\theta + \theta\bar\partial$ *from the initial domain*

$$\begin{aligned} D_0(\Box) = \{\omega \mid \omega, \bar\partial\omega, \theta\omega \in \Lambda^\bullet(\overline{M}) \cap L^2(M),& \\ \sigma(\theta, d\rho)\omega = 0 \ \text{ and } \ \sigma(\theta, d\rho)\bar\partial\omega = 0 \ \text{ on } \ bM\}&. \end{aligned} \tag{1.2}$$

(iv) *For any* $\omega \in D(\Box) \overset{\text{def}}{=} \{\omega \in D(\bar\partial) \cap D(\bar\partial^*) : \bar\partial\omega \in D(\bar\partial^*),\ \bar\partial^*\omega \in D(\bar\partial)\}$ *the following integral identity holds*

$$(\Box\,\omega, \omega) = \|\bar\partial\omega\|_0^2 + \|\bar\partial^*\omega\|_0^2. \tag{1.3}$$

Remark. The boundary conditions on ω in (1.2) are called the $\bar\partial$-Neumann conditions.

Proof of Lemma 1.1. Assume first that ω is a smooth compactly supported form, i.e. $\omega \in \Lambda_c^\bullet(\overline{M})$. Then integration by parts formulas for $\bar\partial$ (see e.g. [FKo], [GriH], [Hö1], [T]) show that the inclusion $\omega \in D(\bar\partial^*)$ is equivalent to the boundary condition $\sigma(\theta, d\rho)\omega = 0$ on bM. The same is true if instead of inclusion $\omega \in \Lambda_c^\bullet(\overline{M})$ we only require that $\omega, \theta\omega \in \Lambda^\bullet(\overline{M}) \cap L^2\Lambda^\bullet(M)$.

Note that any Γ-invariant Riemannian metric on $\overline{M}$ is complete in the following sense. For any point $x_0 \in \overline{M}$ and for any $r \in \mathbf{R}$ the ball of the corresponding geodesic metric $\{x \in \overline{M} : \operatorname{dist}(x_0, x) < r\}$ is relatively compact in $\overline{M}$. Using this fact we can construct Lipschitz cut-off functions of the form $a_\varepsilon(x) = A_\varepsilon(\operatorname{dist}(x_0, x))$ on $\overline{M}$ with the following properties: a_ε has values in $[0,1]$ and a compact support on $\overline{M}$, the subsets $\{x \in \overline{M} : a_\varepsilon(x) = 1\}$ exhaust $\overline{M}$ as $\varepsilon \to 0$, and $\sup_{x\in X}|da_\varepsilon(x)| = O(\varepsilon)$ as $\varepsilon \to 0$.

For any form $\omega \in D(\bar\partial^*) \cap D(\bar\partial)$ let us consider the form $a_\varepsilon\omega$. This form belongs to $D(\bar\partial^*) \cap D(\bar\partial)$ and satisfies the estimates:

$$\begin{aligned} &\left\|\bar\partial^*(a_\varepsilon\omega) - a_\varepsilon\bar\partial^*\omega\right\|_0 = O(\varepsilon)\|\omega\|_0 \ \text{ and} \\ &\left\|\bar\partial(a_\varepsilon\omega) - a_\varepsilon\bar\partial\omega\right\|_0 = O(\varepsilon)\|\omega\|_0. \end{aligned}$$

Hence $a_\varepsilon\omega \to \omega$, $\bar\partial(a_\varepsilon\omega) \to \bar\partial\omega$, and $\bar\partial^*(a_\varepsilon\omega) \to \bar\partial^*\omega$ in $L^2(M)$ as $\varepsilon \to 0$.

So to prove (i) and (ii) we can start with ω which have compact support in $\overline{M}$. We need to approximate them by smooth forms in appropriate norms. With the help of partition of unity in the neighbourhood of $\operatorname{supp}\omega$ we can reduce the statements (i) and (ii) to the known Friedrichs results [Fr] asserting the identity of weak and strong extensions of differential operators (see also Proposition 1.2.4 in [Hö1]).

The statement (iv) follows directly from definitions of $\Box$ and $D(\Box)$.

In order to prove (iii) let us consider the operator $I+\Box$. It follows from the well known functional analysis result that the operator $(I+\Box)^{-1}$ exists, is everywhere defined and bounded (see also [G] and Proposition 1.3.8 in [FKo]).

Let $\omega \in D(\Box_{p,q})$. Then $\omega + \Box\,\omega = \alpha \in L^2\Lambda^{p,q}(M)$. Let us choose a sequence $\{\alpha_j\} \subset \Lambda_c^{p,q}(\overline{M})$ converging to α in $L^2\Lambda^{p,q}(M)$ and define $\omega_j = (I+\Box)^{-1}\alpha_j$. Then $\omega_j \in D(\Box)$ and $\omega_j \to \omega$ in $L^2\Lambda^{p,q}(M)$. By the Kohn regularity theorem $\omega_j \in \Lambda^{p,q}(\overline{M}) \cap D(\Box)$ (see Propositions 3.1.4 and 3.1.10 in [FKo]).

The equalities $(\Box\,\omega_j, \omega_j) = \|\bar\partial\omega_j\|_0^2 + \|\bar\partial^*\omega_j\|_0^2$ give us the inclusions $\bar\partial\omega_j \in \Lambda^{p,q}(\overline{M}) \cap L^2\Lambda^{p,q}(M)$ and $\bar\partial^*\omega_j \in \Lambda^{p,q}(\overline{M}) \cap L^2\Lambda^{p,q}(M)$. □

We will use the standard splitting of the complexified tangent bundle $TM \otimes \mathbf{C}$ to (1,0)-part $T_{1,0}(M)$ ("holomorphic" part) and (0,1)-part ("antiholomorphic" part) $T_{0,1}(M)$:

$$TM \otimes \mathbf{C} = T_{1,0}(M) \oplus T_{0,1}(M)\,.$$

Here $T_{1,0}(M)$ is generated by the vector fields $\partial/\partial z_j$, $j = 1, \dots, n$, and $T_{0,1}(M) = \overline{T_{1,0}(M)}$ is generated by the vector fields $\partial/\partial \bar z_j$, $j = 1, \dots n$, at every point $x \in M$.

Let us denote by ∇ the covariant differentiation related with the ∂-connection by the fixed Γ-invariant hermitian metric in the complexified tangent bundle [AV]. We will mainly use the complex conjugate operator $\overline\nabla$ which is the covariant differentiation related with the $\bar\partial$-connection:

$$\overline\nabla : \Lambda^{p,q}(M) \longrightarrow \Lambda^{0,1}(M) \otimes \Lambda^{p,q}(M)\,.$$

It has the property

$$\overline\nabla(f\omega) = f\overline\nabla\omega + \bar\partial f \otimes \omega\,, \quad f \in C^\infty(M)\ ,\ \omega \in \Lambda^{p,q}(M)\,.$$

We shall denote by $\bar\partial_{p,q}$ the operator $\bar\partial$ restricted to (p,q)-forms, by $\bar\partial^*_{p,q}$ the corresponding adjoint operator (i.e. $\bar\partial^*$ restricted to $(p,q+1)$-forms).

Proposition 1.2. *Let us assume that M is strongly pseudoconvex.*

(i) *There exists $\gamma > 0$ such that the following Morrey type estimate holds:*

$$\|\bar\partial\omega\|_0^2 + \|\bar\partial^*\omega\|_0^2 + \|\omega\|_0^2 \ge \gamma\big(\|\overline\nabla\omega\|_0^2 + \|\omega\|_0^2 + \|\omega\|_{L^2(bM)}^2\big) \qquad \omega \in D(\bar\partial^*_{p,q-1}) \cap D(\bar\partial_{p,q})\,, \qquad q > 0\,. \tag{1.4}$$

(ii) *The domain $D(\bar\partial^*_{p,q}) \cap D(\bar\partial_{p,q})$, $q > 0$, is included into $W^{1/2}$ and there exists $\gamma_1 > 0$ such that*

$$\|\omega\|_{1/2}^2 \le \gamma_1\big(\|\overline\nabla\omega\|_0^2 + \|\omega\|_0^2 + \|\omega\|_{L^2(bM)}^2\big)\,, \quad \omega \in D(\bar\partial^*_{p,q-1}) \cap D(\bar\partial_{p,q})\,.$$

(iii) *The domain $D(\square_{p,q})$, $q>0$, is included into W^1 and there exists a constant $\gamma_2>0$ such that the Kohn type estimate holds*

$$\|\omega\|_1^2 \le \gamma_2\big(\|\square\,\omega\|_0^2 + \|\omega\|_0^2\big)\,, \quad \omega \in D(\square_{p,q})\,.$$

Proof. Let us fix a Γ-invariant partition of unity $\{a_\nu\}$ subordinate to a Γ-invariant covering of $\overline{M}$ by contractible neighbourhoods $\{U_\nu\}$. Let $\nabla_{1,\nu},\dots,\nabla_{n,\nu}$ be complex vector fields over $U_\nu\cap\overline{M}$ which constitute an orthonormal basis of $T_{1,0}(\overline{M})$ at each point $x\in U_\nu\cap\overline{M}$). Let the forms

$$\varphi_\nu^1,\dots,\varphi_\nu^n \in \Lambda^{1,0}(U_\nu\cap\overline{M})$$

constitute the dual orthonormal basis at each point $x\in U_\nu\cap\overline{M}$. For any $\omega\in\Lambda^{p,q}(\overline{M})$ we have

$$a_\nu\omega = \sum_{\substack{|I|=p\\|J|=q}} a_\nu\omega_{I,J}^{(\nu)}\varphi^I\wedge\overline{\varphi}^J \quad\text{and}$$

$$\|\overline{\nabla}_\nu\omega\|_0^2 = \sum_{I,J}\big\|\bar\partial(a_\nu\omega_{I,J}^{(\nu)})\big\|_0^2 + O\big(\|a_\nu\omega\|_0^2\big)\,.$$

For any form $a_\nu\omega\in\Lambda^{p,q}(\overline{M})$ with support in U_ν we have the following local Morrey type inequalities (see (3.1.20) in [Hö1] and also [M1,2], [Ko], [AV], [FKo])

$$\begin{aligned}&\big\|\overline{\nabla}(a_\nu\omega)\big\|_0^2 + \|a_\nu\omega\|_0^2 + \|a_\nu\omega\|_{L^2(bM)}^2\\ &\qquad\le C_0\big(\|\bar\partial^*(a_\nu\omega)\|_0^2 + \|\bar\partial(a_\nu\omega)\|_0^2 + \|a_\nu\omega\|_0^2\big)\,.\end{aligned}\tag{1.5}$$

It follows from compactness of $\overline{M}/\Gamma$ and from Γ-invariance of the covering $\{U_\nu\}$ that the constant $C_0>0$ in these inequalities can be chosen independent of ν.

Summing up these inequalities we obtain the following global Morrey type inequality

$$\|\overline{\nabla}\omega\|_0^2+\|\omega\|_0^2+\|\omega\|_{L^2(bM)}^2\le C\big(\|\bar\partial^*\omega\|_0^2+\|\bar\partial\omega\|_0^2+\|\omega\|_0^2\big)\,,\ \omega\in\Lambda_c^{p,q}(\overline{M})\cap D(\bar\partial^*)\,.$$

It follows from Lemma 1.1(ii) that this Morrey inequality is valid for $\omega\in D(\bar\partial)\cap D(\bar\partial^*)$.

Using further Lemma 1.1(iv) and the Kohn type inequality

$$\|f\|_{1/2}^2 \le C_1\big(\|f\|_{L^2(bM)}^2 + \|\bar\partial f\|_{L^2(M)}^2\big)\,,\qquad f\in\Lambda_c^{0,0}(\overline{M})\,,$$

(see e.g. Theorem 2.4.4 in [FKo]) we obtain the statement (ii).

We suppose further that the Γ-invariant covering $\{U_\nu\}$ is included in the bigger Γ-invariant covering $\{\tilde U_\nu\}$, $\overline{U}_\nu\subset\tilde U_\nu$.

Let $\{a_\nu\}$ and $\{\tilde a_\nu\}$ be partitions of the unity subordinate respectively to coverings $\{U_\nu\}$ and $\{\tilde U_\nu\}$, such that $\tilde a_\nu\equiv 1$ on $\operatorname{supp} a_\nu$.

To prove (iii) we use another inequality of Kohn [Ko]

$$\|a_\nu\omega\|_1^2 \le C_2\big(\|\tilde a_\nu\square\,\omega\|_0^2 + \|\tilde a_\nu\omega\|_0^2\big)\,,\qquad \omega\in\Lambda^{p,q}(\overline{M})\cap D(\square_{p,q})\,.$$

For the same reasoning as above, the constant C_2 in this inequality can be chosen independent of ν.

Summing up these inequalities and using Lemma 1.1 (iii) we obtain the statement (iii). $\square$

COROLLARY 1.3. *If M is strongly pseudoconvex then there exists a constant $\gamma_2 > 0$ such that $\|\omega\|_1 \le \gamma_2 \|\omega\|_0$, $\forall\, \omega \in \operatorname{Ker} \square_{p,q}$, $q > 0$, where* $\operatorname{Ker} \square = \{\omega \mid \omega \in D(\square), \square\, \omega = 0\}$.

REMARK. Corollary 1.3 follows from Proposition 1.2(iii). Proposition 1.2(ii) gives us a weaker statement which is also sufficient for our applications: there exists a constant $\gamma_1 > 0$ such that $\|\omega\|_{1/2} \le \gamma_1 \|\omega\|_0$, $\forall\, \omega \in \operatorname{Ker} \square_{p,q}$, $q > 0$.

The advantage of this weaker inequality is that the constant γ_1 in it depends only on the first two derivatives of the function ρ defining bM. So the last inequality is valid if we assume that bM has only C^2-smoothness but keep all other assumptions.

Let us formulate the necessary version of the weak Hodge-Kodaira decomposition (see e.g. [G], [FKo]).

PROPOSITION 1.4. *The following orthogonal decomposition holds:*

$$L^2\Lambda^\bullet(M) = \overline{\operatorname{Im} \bar\partial} \oplus \operatorname{Ker} \square \oplus \overline{\operatorname{Im} \bar\partial^*} \quad \textit{and}$$

$$\operatorname{Ker} \bar\partial = \overline{\operatorname{Im} \bar\partial} \oplus \operatorname{Ker} \square\,.$$

In particular, we have an isomorphism of Hilbert Γ-modules

$$L^2 \bar H^{p,q}(M) = \operatorname{Ker} \square_{p,q}.$$

Proof. First note that $\bar\partial^2 = 0$ implies $\overline{\operatorname{Im} \bar\partial} \perp \overline{\operatorname{Im} \bar\partial^*}$. The orthogonal complement of $\overline{\operatorname{Im} \bar\partial} \oplus \overline{\operatorname{Im} \bar\partial^*}$ is $\operatorname{Ker} \bar\partial \cap \operatorname{Ker} \bar\partial^*$. From the definition of $D(\square)$ we have $\operatorname{Ker} \square \supset \operatorname{Ker} \bar\partial \cap \operatorname{Ker}(\bar\partial^*)$. Vice versa $\omega \in \operatorname{Ker} \square$ implies $0 = (\square\, \omega, \omega) = \|\bar\partial \omega\|^2 + \|\bar\partial^* \omega\|^2$, so $\omega \in \operatorname{Ker} \bar\partial \cap \operatorname{Ker} \bar\partial^*$. This gives us the first decomposition. The decomposition for $\operatorname{Ker} \bar\partial$ follows now from the fact that $\operatorname{Ker} \bar\partial$ is the orthogonal complement of $\operatorname{Im} \bar\partial^*$. Finally the decomposition for $\operatorname{Ker} \bar\partial$ implies the last equality.

2. For the proof of Theorem 0.1 we will need the following general statement where we use notation from the beginning of section 1.

PROPOSITION 1.5. *Let L be a closed Γ-invariant subspace in $L^2(M, E)$, $L \subset W^\varepsilon$ for some $\varepsilon > 0$ and there exists $C > 0$ such that*

$$\|u\|_\varepsilon \le C \|u\|_0, \ u \in L\,. \tag{1.6}$$

Then $\dim_\Gamma L < \infty$.

To prove this proposition we need the following simple statement about estimates of Sobolev norms on compact manifolds with boundary.

PROPOSITION 1.6. *Let X be a compact Riemannian manifold, possibly with a boundary. Let E be a (complex) vector bundle with an hermitian metric over $\overline{X}$. Denote by $(\cdot,\cdot)$ the induced hermitian inner product in the Hilbert space $L^2(X,E)$ of square-integrable sections of E over X. Denote by $W^s = W^s(X,E)$ the corresponding Sobolev space of sections of E over X, $\|\cdot\|_s$ the norm in this space. Let us choose a complete orthonormal system $\{\psi_j;\ j = 1,2,\dots\}$ in $L^2(X,E)$. Then for all $\varepsilon > 0$ and $\delta > 0$ there exists an integer $N > 0$ such that*

$$\|u\|_0 \le \delta\|u\|_\varepsilon \text{ provided } u \in W^\varepsilon \text{ and } (u,\psi_j) = 0\,,\ j = 1,\dots,N\,.$$

Proof. Assuming the opposite we conclude that there exist $\varepsilon > 0$ and $\delta > 0$ such that for every $N > 0$ there exists $u_N \in W^\varepsilon$ with $(u_N,\psi_j) = 0$, $j = 1,\dots,N$, satisfying the estimate $\|u_N\|_\varepsilon \le \delta^{-1}\|u_N\|_0$. Normalizing u_N we can assume that $\|u_N\|_0 = 1$, so the previous estimate gives $\|u_N\|_\varepsilon \le \delta^{-1}$ for all N. It follows from the Sobolev compactness theorem that the set $\{u_N \mid N = 1,2,\dots\}$ is compact in $L^2 = L^2(X,E)$. On the other hand obviously $u_N \to 0$ weakly in L^2 as $N \to \infty$. Therefore $\|u_N\|_0 \to 0$ as $N \to \infty$ which contradicts to the chosen normalization. □

Proof of Proposition 1.5. Let us choose a Γ-invariant covering of M by balls γB_k, $k = 1,\dots,m$, $\gamma \in \Gamma$, so that all the balls have smooth boundary (e.g. have sufficiently small radii). Let us choose a complete orthonormal system $\{\psi_j^{(k)}, j = 1,2,\dots\}$ in $L^2(B_k,E)$ for every $k = 1,\dots,m$. Then $\{(\gamma^{-1})^*\psi_j^{(k)}, j = 1,2,\dots\}$ will be an orthonormal system in γB_k (here we identify the element γ with the corresponding transformation of M).

Given the subspace L satisfying the conditions in the lemma let us define a map

$$P_N : L \longrightarrow L^2\Gamma \otimes \mathbf{C}^{mN}$$
$$u \mapsto \left\{(u,(\gamma^{-1})^*\psi_j^{(k)})\,,\ j = 1,2,\dots,N\,;\ k = 1,\dots,m\,;\ \gamma \in \Gamma\right\}.$$

Since $\dim_\Gamma L^2\Gamma \otimes \mathbf{C}^{mN} = mN < \infty$ the desired result will follow if we prove that P_N is injective for large N. Assume that $u \in L$ and $P_N u = 0$. Using Lemma 1.6 we get then

$$\|u\|^2_{0,\gamma B_k} \le \delta_N^2 \|u\|^2_{\varepsilon,\gamma B_k}\,, \qquad k = 1,\dots,m\,;\ \gamma \in \Gamma\,,$$

where $\delta_N \to 0$ as $N \to \infty$ and $\|\cdot\|_{s,\gamma B_k}$ means the norm in the Sobolev space W^s over the ball γB_k. Summing over all k and γ we get

$$\|u\|_0^2 \le C_1^2 \delta_N^2 \|u\|_\varepsilon^2\,,$$

where $C_1 > 0$ does not depend on N. This clearly contradicts (1.6) unless $u = 0$. □

Remark. It is not necessary to require that L is closed in L^2 in Proposition 1.5. For any L satisfying (1.6) we can consider its closure $\bar{L}$ in L^2. Then obviously $\bar{L} \subset W^\varepsilon$ and Proposition 1.5 implies that $\dim_\Gamma \bar{L} < \infty$.

Proof of Theorem 0.1. Propositions 1.2, 1.4 and 1.5 immediately imply Theorem 0.1. □

3. To prove Theorem 0.5 we need a refined version of the Hörmander and Andreotti-Vesentini weighted L_2-estimates ([Hö1], [AV]).

Let M be a pseudoconvex manifold with a holomorphic action of the discrete group Γ such that $\overline{M}/\Gamma$ is compact, and with a Γ-invariant hermitian metric.

In what follows Φ will be a Γ-invariant function which is defined in a complex neighbourhood of $\overline{M}$, $\Phi \in C^1(M)$ and Φ is strictly plurisubharmonic in a Γ-invariant neighbourhood of bM. In this case we will call Φ an *admissible weight function*.

For an admissible weight function Φ we will define weighted Hilbert spaces $L^2_\Phi \Lambda^{p,q}(M)$ of (p,q)-forms with a finite norm given by

$$\|\omega\|^2_\Phi = \int_M |\omega(x)|^2 e^{-\Phi} dv(x)\,,$$

where the norm $|\omega(x)|$ and the volume element $dv(x)$ are induced by the given hermitian metric. The cohomology and reduced cohomology of the corresponding L^2 Dolbeault complexes will be denoted $L^2_\Phi H^{p,q}(M)$ and $L^2_\Phi \bar{H}^{p,q}(M)$ respectively.

We will consider the operator $\bar\partial = \bar\partial_{p,q} = \bar\partial_\Phi = \bar\partial_{p,q;\Phi}$ as the maximal operator in $L^2_\Phi \Lambda^{p,q}(M)$. Though it is given by the standard differential expression $\bar\partial$, its domain may depend on Φ if Φ is unbounded on M. Denote the corresponding adjoint operator by $\bar\partial^*_\Phi = \bar\partial^*_{p,q;\Phi}$. We will also use the corresponding Laplacian

$$\Box_\Phi = \Box_{p,q;\Phi} = \bar\partial^*_\Phi \bar\partial + \bar\partial \bar\partial^*_\Phi\,.$$

Note that the operator $\bar\partial^*_\Phi$ differs from $\bar\partial^*$ by 0-order terms which are expressed in terms of Φ and its first derivatives. It follows that the domain $D(\bar\partial^*_\Phi)$ coincides with $D(\bar\partial^*)$ provided $\Phi \in C^1$ in a neighbourhood of $\overline{M}$.

Proposition 1.7. *Let M be a pseudonconvex manifold with a holomorphic action of the discrete group Γ such that $\overline{M}/\Gamma$ is compact, and with a Γ-invariant hermitian metric. Suppose that an admissible weight function*

Φ is given which is strictly plurisubharmonic in a Γ-invariant neighbourhood U of bM. Then for any Γ-invariant neighbourhood U_0 of bM with $\bar U_0 \subset U$ there exist positive constants γ_0, δ_0 and δ_1 such that the following inequalities hold

$$\begin{aligned} &1/3\,\|\overline{\nabla}\omega\|^2_{t\Phi} + 2/3\,(\delta_0 t - \gamma_0)\,\|\omega\|^2_{t\Phi} \\ &\qquad \le \|\bar\partial\omega\|^2_{t\Phi} + \|\bar\partial^*_{t\Phi}\omega\|^2_{t\Phi} + \delta_1 t \|\omega\|^2_{L^2_{t\Phi}(M\setminus U_0)} \end{aligned} \tag{1.7}$$

for any $t > 0$, $\omega \in D(\bar\partial^*_{p,q-1;t\Phi}) \cap D(\bar\partial_{p,q;t\Phi})$, $q > 0$. Here the constant γ_0 depends only on M, δ_0 depends only on M and the minimal eigenvalue of $i\partial\bar\partial\Phi$ on U_0, δ_1 depends only on M and the maximal eigenvalue of $i\partial\bar\partial\Phi$ on $M\setminus U_0$.

Corollary 1.8. *Under the assumptions of Proposition 1.7 for any $\omega \in \operatorname{Ker}\Box_{p,q;t\Phi}$, $q > 0$, the following inequality holds*

$$1/3\,\|\overline{\nabla}\omega\|^2_{t\Phi} + 2/3\,(\delta_0 t - \gamma_0)\,\|\omega\|^2_{t\Phi} \le \delta_1 t \|\omega\|^2_{L^2_{t\Phi}(M\setminus U_0)}\,.$$

Corollary 1.9. *If under the assumptions of Proposition 1.7 the function Φ is strictly plurisubharmonic on $\overline{M}$, i.e. on a neighbourhood U of $\overline{M}$, then for any $\omega \in D(\bar\partial^*_{p,q-1;t\Phi}) \cap D(\bar\partial_{p,q;t\Phi})$, $q > 0$, the following inequality holds*

$$2/3\,(\delta_0 t - \gamma_0)\|\omega\|^2_{t\Phi} \le \left(\|\bar\partial^*_{t\Phi}\omega\|^2_{t\Phi} + \|\bar\partial\omega\|^2_{t\Phi}\right).$$

In particular this implies vanishing of (non-reduced) L^2 Dolbeault cohomology spaces of M:

$$L^2_{t\Phi}H^{p,q}(M) = 0\,, \qquad q > 0\,,\ t > \gamma_0/\delta_0\,.$$

Proof. We can take U_0 such that $U_0 \supset \overline{M}$ and use (1.7), so the estimate follows. The vanishing of the cohomology spaces follows from the estimate. □

To prove Proposition 1.7 we shall use the following refined version of Lemma 1.1.

Lemma 1.10. *Under the assumptions of Proposition 1.7*

(i) *M can be exhausted by strongly pseudoconvex Γ-invariant manifolds M_j, $j = 1, 2, \ldots$.*

(ii) *Let $D^0_\Phi(\bar\partial^*_\Phi) \cap D^0_\Phi(\bar\partial)$ be the space of all $\omega \in L^2_\Phi\Lambda^\bullet(M)$ such that ω can be obtained as a weak limit in $L^2_\Phi\Lambda^\bullet(M)$ of forms ω_j such that $\omega_j\big|_{M\setminus M_j} = 0$, $\omega_j\big|_{M_j} \in D_0(\bar\partial^*|M_j)$, and*

$$\|\omega_j\|_{L^2_\Phi(M_j)} + \|\Box_\Phi\omega_j\|_{L^2_\Phi(M_j)} \le c\,.$$

with a constant c independent of j. Then the space $D^0_\Phi(\bar\partial^*_\Phi) \cap D^0_\Phi(\bar\partial)$ is contained in $D(\bar\partial^*_\Phi) \cap D(\bar\partial_\Phi)$ and is dense in this space in the graph norm $(\|\omega\|^2_\Phi + \|\bar\partial\omega\|^2_\Phi + \|\bar\partial^*_\Phi\omega\|^2_\Phi)^{1/2}$.

Proof. If $\Phi \in C^2(\overline{M})$, then the statement (ii) of this lemma follows from Lemma 1.1(ii), because in this case the norms $\|\omega\|_\Phi$ and $\|\omega\|_0$ are equivalent.

To prove the statement (i) let us denote $M/\Gamma = X$. It follows from assumptions of Proposition 1.7 that X is a pseudoconvex manifold with a compact boundary $bX = \{x \in \overline{X} :\ \rho(x) = 0\}$ and there exists a continuous function $\tilde{\Phi}$ which is a well defined function on $\overline{X}$, strictly plurisubharmonic in a neighbourhood of bX. If we suppose that $\tilde{\Phi} > 0$ and consider the function $\rho_j(x) = e^{j\rho(x)} - 1 + \tilde{\Phi}(x)$, then for sufficiently large j the domains $X_j = \{x \in X : \rho_j(x) < 0\}$ are strongly pseudoconvex subdomains in X and X_j exhaust X, when $j \to \infty$. Moreover, for j large enough there exists a strongly pseudoconvex covering M_j of X_j such that $M_j/\Gamma = X_j$.

Now to prove the statement (ii) in full generality, we should modify the proof of Lemma 1.1. Namely, we can use the classical Friedrichs result [Fr] only to obtain the density of $D_0(\bar{\partial}^*|M_j)$ in $D(\bar{\partial}^*|M_j)$. After that by elementary agruments like the ones in [HeI] we can obtain the density of $D^0_\Phi(\bar{\partial}^*) \cap D^0_\Phi(\bar{\partial})$ in $D(\bar{\partial}^*_\Phi) \cap D(\bar{\partial}_\Phi)$ in the graph norm. To do this we first check directly from the definitions that

$$D^0_\Phi(\bar{\partial}^*_\Phi) \cap D^0_\Phi(\bar{\partial}) \subset D(\bar{\partial}^*_\Phi) \cap D(\bar{\partial}_\Phi)\,.$$

To show the density in the graph norm in this inclusion let us consider $\psi \in D(\bar{\partial}^*_\Phi) \cap D(\bar{\partial}_\Phi)$ such that ψ is orthogonal to $D^0_\Phi(\bar{\partial}^*_\Phi) \cap D^0_\Phi(\bar{\partial})$ in the graph inner product, i.e.

$$(\psi, \omega)_\Phi + (\bar{\partial}\psi, \bar{\partial}\omega)_\Phi + (\bar{\partial}^*_\Phi\psi, \bar{\partial}^*_\Phi\omega)_\Phi = 0$$

for any $\omega \in D^0_\Phi(\bar{\partial}^*_\Phi) \cap D^0_\Phi(\bar{\partial})$.

We want to show that $\psi = 0$. To do this let us find $g_j \in L^2_\Phi(M_j)$, $j = 1, 2, \dots$, such that

$$\psi|_{M_j} = \bar{\partial}\bar{\partial}^*_\Phi g_j + \bar{\partial}^*_\Phi\bar{\partial} g_j + g_j\,.$$

Clearly

$$\sup_j\big(\|g_j\|^2_{L^2_\Phi(M_j)} + \|\Box_\Phi g_j\|^2_{L^2_\Phi(M_j)}\big) < \infty\,.$$

Replacing $\{g_j\}$ by a subsequence we can assume that $g_j \to g$ in $L^2_\Phi(M)$ where $g \in D^0_\Phi(\bar{\partial}^*_\Phi) \cap D^0_\Phi(\bar{\partial})$. Therefore

$$\begin{aligned}(\psi, \psi)_\Phi &= \lim_{j\to\infty}\big[(\psi, g_j)_{L^2_\Phi(M_j)} + (\psi, \bar{\partial}\bar{\partial}^*_\Phi g_j)_{L^2_\Phi(M_j)} + (\psi, \bar{\partial}^*_\Phi\bar{\partial} g_j)_{L^2_\Phi(M_j)}\big]\\ &= (\psi, g)_\Phi + (\bar{\partial}^*_\Phi\psi, \bar{\partial}^*_\Phi g)_\Phi + (\bar{\partial}\psi, \bar{\partial} g)_\Phi = 0\,.\end{aligned}$$

Hence $\psi = 0$ as required. □

Proof of Proposition 1.7. Let M_j be as in Lemma 1.10. We fix on M a Γ-invariant partition of unity $\{a_\nu\}$ subordinate to a Γ-invariant covering of $\overline{M}$

by coordinate neighbourhoods $\{U_\nu\}$. So, for any $\omega \in \Lambda^{p,q}(\overline{M}_j)\cap D(\bar\partial^*_{p,q}|M_j)$ and any ν we have inclusions $a_\nu\omega\in\Lambda^{p,q}(\overline{M}_j)\cap D(\bar\partial^*_{p,q}|M_j)$ and $\operatorname{supp} a_\nu\omega\subset U_\nu$.

Using in $U_\nu \cap M_j$ the inequality (3.1.20) from [Hö1] we obtain the following estimate

$$\begin{aligned} 1/2\big\|e^{-t\Phi}\overline{\nabla}(a_\nu\omega)\big\|^2_{L^2(M_j)} \\ + {\sum_{I,K}}' \sum_{j,l} t\int_{U_\nu\cap\overline{M}_j} \frac{\partial^2\Phi}{\partial z_j\partial \bar z_l} e^{-2t\Phi}(a_\nu\omega_{I,jK})\cdot(\overline{a_\nu\omega_{I,lK}})dV \\ + 1/2{\sum_{I,K}}' \sum_{j,l} \int_{U_\nu\cap bM_j} \frac{\partial^2\rho}{\partial z_j\partial \bar z_l}(a_\nu\omega_{I,jK})\cdot(\overline{a_\nu\omega_{I,lK}})e^{-2t\Phi}dV \\ \le 3/2\big(\|e^{-t\Phi}\bar\partial^*_{t\Phi}(a_\nu\omega)\|^2_{L^2(M_j)} + \|e^{-t\Phi}\bar\partial(a_\nu\omega)\|^2_{L^2(M_j)}\big) \\ + \gamma_0\|e^{-t\Phi}a_\nu\omega\|^2_{L^2(M_j)}\,, \end{aligned} \tag{1.8}$$

where ρ_j is a defining function for bM_j in $U_\nu\cap\overline{M}_j$; the constant γ_0 does not depend on Φ and j.

In the case $\rho_j = \Phi$ such inequality was obtained also in [AV]. The term with ρ in (1.8) is non-negative due to the pseudoconvexity of bM_j and the boundary condition for ω. Now using strict plurisubharmonicity of Φ in the neighbourhood U of bM, we obtain for any neighbourhood U_0 of bM with $\bar U_0 \subset U$ and any $\omega \in \Lambda^{p,q}(\overline{M}_j)\cap D(\bar\partial_{p,q}|M_j)$

$$\begin{aligned} 1/2\big\|e^{-t\Phi}\overline{\nabla}(a_\nu\omega)\big\|^2_{L^2(M_j)} + \delta_0 t\|e^{-t\Phi}a_\nu\omega\|_{L^2(U_0\cap M_j)} \\ \le 3/2\big(\|e^{-t\Phi}\bar\partial(a_\nu\omega)\|^2_{L^2(M_j)} + \|e^{-t\Phi}\bar\partial^*_{t\Phi}(a_\nu\omega)\|^2_{L^2(M_j)}\big) \\ + \gamma_0\|e^{-t\Phi}a_\nu\omega\|^2_{L^2(M_j)} + t\cdot\delta_1\,\|e^{-t\Phi}a_\nu\omega\|^2_{L^2(M_j\setminus U_0)}\,, \end{aligned}$$

where δ_0 depends only on M and the minimal eigenvalue of $i\partial\bar\partial\Phi$ on U_0 and δ_1 depends only on M and the maximal eigenvalue of $i\partial\bar\partial\Phi$ on $M\setminus U_0$.

It follows from compactness of $\overline{M}_j/\Gamma$ and from Γ-invariance of the neighbourhood U and of the covering $\{U_\nu\}$, that the constant γ_0 in the last inequalities can be chosen independent of ν. We shall use also the obvious estimates

$$\big\|e^{-t\Phi}L(a_\nu\omega) - e^{-t\Phi}a_\nu L\omega\big\|_{L^2(M_j)} \le C\sup|\operatorname{grad} a_\nu|\,\|e^{-t\Phi}\omega\|_{L^2(M_j)}$$

where $L = \bar\partial$, $\bar\partial^*_{t\Phi}$ or $\overline{\nabla}$ on $U_\nu\cap\overline{M}_j$.

Using these estimates and summing the above inequalities we obtain

$$\begin{aligned} 1/2\|e^{-t\Phi}\overline{\nabla}\omega\|^2_{L^2(M_j)} + \tilde\delta_0 t\|e^{-t\Phi}\omega\|^2_{L^2(U_0\cap M_j)} \\ \le 3/2\big(\|e^{-t\Phi}\bar\partial\omega\|^2_{L^2(M_j)} + \|e^{-t\Phi}\bar\partial^*_{t\Phi}\omega\|^2_{L^2(M_j)}\big) + \tilde\gamma_0\|e^{-t\Phi}\omega\|_{L^2(M_j)} \\ + \tilde\delta_1\cdot t\|e^{-t\Phi}\omega\|_{L^2(M_j\setminus U_0)} \end{aligned}$$

with constants $\tilde\gamma_0$, $\tilde\delta_0$ and $\tilde\gamma_1$ depending only on the corresponding constants $\gamma_0, \delta_0, \delta_1$ and the partition of unity $\{a_\nu\}$.

Inequality (1.7) follows if we take $j \to \infty$ and use Lemma 1.10 (ii). □

We shall also use the following statement similar to Proposition 1.5.

PROPOSITION 1.11. *Let Φ be as in Proposition 1.7 and L be a Γ-invariant subspace in $L^2\Lambda^{p,q}_\Phi(M)$, $q > 0$ such that for a Γ-invariant neighbourhood $U_0 \subset U$ of bM and for some constant $\gamma > 0$ the following estimate holds*

$$\|\overline\nabla\omega\|^2_{L^2_\Phi(M)} + \|\omega\|^2_{L^2_\Phi(M)} \le \gamma\|\omega\|^2_{L^2(M\setminus U_0)}\,, \qquad \omega \in L\,. \tag{1.9}$$

Then $\dim_\Gamma L < \infty$.

The proof of Proposition 1.11 is a copy of the proof of Proposition 1.5, where instead of Proposition 1.6 the following Proposition 1.12 must be used.

PROPOSITION 1.12. *Let M and Φ be as in Proposition 1.7. Let B_0 be a geodesic ball in $\overline M$ with sufficiently small radius. Let $\{\psi_j\}$ be a complete orthonormal system in $L^2_\Phi\Lambda^{p,q}(B_0)$. Let L_0 be a subspace in $L^2_\Phi\Lambda^{p,q}(B)$ such that for some neighbourhood $U_0 \subset U$ of bB_0 and for some $\gamma_0 > 0$ we have*

$$\|\overline\nabla\omega\|_{L^2_\Phi(B_0)} + \|\omega\|_{L^2_\Phi(B_0)} \le \gamma_0\|\omega\|_{L^2_\Phi(B_0\setminus U_0)}\,.$$

Then for all $\delta > 0$ there exists $N > 0$ such that

$$\|\omega\|_{L^2_\Phi(B_0\setminus U_0)} \le \delta\big(\|\overline\nabla\omega\|_{L^2_\Phi(B_0)} + \|\omega\|_{L^2_\Phi(B_0)}\big)$$

provided $\omega \in L_0$ and $(\omega, \psi_j) = 0$, $j = 1, 2, \dots, N$.

Proof of Theorem 0.5. It follows from the assumptions of Theorem 0.5 that in a neighbourhood of $\overline M$ there exists a Γ-invariant function $\tilde\Phi \in C^2$ which is strictly plurisubharmonic in a Γ-invariant neighbourhood $U_0 \subset U$ of bM. We note further that for such $\tilde\Phi$ all the norms $\|\omega\|_{t\Phi}$, $t \ge 0$ are equivalent. Hence, Proposition 1.4, Corollary 1.8 and Proposition 1.11 imply Theorem 0.5. □

2 L^2 Holomorphic Functions

1. We shall use some simple linear algebra and Γ-Fredholm operators in Hilbert Γ-modules. Necessary background and similar arguments can be found in [Br] and [S2].

LEMMA 2.1. *Let L be a Hilbert Γ-module, L_1, L_2 its Hilbert Γ-submodules such that $\dim_\Gamma L_1 > \mathrm{codim}_\Gamma L_2$ where $\mathrm{codim}_\Gamma L_2$ means the Γ-dimension* *of the orthogonal complement of L_2 in L. Then $L_1 \cap L_2 \neq \{0\}$. Moreover*

$$\dim_\Gamma L_1 \cap L_2 \geq \dim_\Gamma L_1 - \operatorname{codim}_\Gamma L_2 \,. \tag{2.1}$$

Proof. Denote by $L_1 \ominus L_2$ the orthogonal complement of $L_1 \cap L_2$ in L_1. Clearly $\dim_\Gamma L_1 \ominus L_2 \leq \operatorname{codim}_\Gamma L_2$. Therefore if (2.1) is not true, then we get

$$\dim_\Gamma L_1 = \dim_\Gamma L_1 \cap L_2 + \dim_\Gamma L_1 \ominus L_2 \leq \dim_\Gamma L_1 \cap L_2 + \operatorname{codim}_\Gamma L_2 < \dim_\Gamma L_1$$

which is a contradiction. □

We will use unbounded Γ-Fredholm operators. The corresponding definition slightly extends the corresponding definition for bounded operators given by M. Breuer [Br] (see also [S2]).

DEFINITION 2.2. Let L_1, L_2 be Hilbert Γ-modules, $A : L_1 \to L_2$ a closed densely defined linear operator (with the domain $D(A)$) which commutes with the action of Γ in L_1 and L_2. The operator A is called Γ-*Fredholm* if the following conditions are satisfied:

(i) $\dim_\Gamma \operatorname{Ker} A < \infty$;
(ii) there exists a closed Γ-invariant subspace $Q \subset L_2$ such that $Q \subset \operatorname{Im} A$ and $\operatorname{codim}_\Gamma Q (= \dim_\Gamma (L_2 \ominus Q)) < \infty$.

Let us also recall the following definition from [S2]:

DEFINITION 2.3. Let L be a Hilbert Γ-module, $Q \subset L$ is a Γ-invariant subspace (not necessarily closed). Then

(i) Q is called Γ-*dense* in L if for every $\varepsilon > 0$ there exists a Γ-invariant subspace $Q_\varepsilon \subset Q$ such that Q_ε is closed in L and $\operatorname{codim}_\Gamma Q_\varepsilon < \varepsilon$ in L.
(ii) Q is called *almost closed* if Q is Γ-dense in its closure $\bar{Q}$.

If Q is Γ-dense in L then it is also dense in L in the usual sense, i.e. $\bar{Q} = L$ (see Lemma 1.8 in [S2]). Note also that if Γ is trivial (or finite) then Q is Γ-dense in L if and only if $Q = L$ (in particular in this case Q is almost closed if and only if it is closed).

LEMMA 2.4. *If $A : L_1 \to L_2$ is a Γ-Fredholm operator then* $\operatorname{Im} A$ *is almost closed.*

Proof. This statement can be reduced to the case when A is bounded by replacing L_1 by $D(A)$ with the graph norm. Then the statement is due to M. Breuer [Br] (see also Lemma 1.15 in [S2]). □

LEMMA 2.5 ([S2]). *Let L be a Hilbert Γ-module, $L_1 \subset L$ and $Q \subset L$ its Γ-invariant subspaces in L such that L_1 is closed and Q is Γ-dense in L. Then $Q \cap L_1$ is Γ-dense in L_1. More generally, if Q is almost closed then $Q \cap L_1$ is almost closed and its closure equals $\bar{Q} \cap L_1$.*

Corollary 2.6. *Let $A : L_1 \to L_2$ be a Γ-Fredholm operator, $L_3 \subset L_2$ is a closed Γ-invariant subspace such that $L_3 \subset \overline{\operatorname{Im} A}$. Then $L_3 \cap \operatorname{Im} A$ is Γ-dense in L_3.*

Now let us return to the analytic situation described above. For proving Theorems 0.2–0.4 we will have need of the following

Proposition 2.7. *Let A be a self-adjoint linear operator in $L^2(M,E)$ such that A commutes with the action of Γ, $D(A) \subset W^\varepsilon$ where $\varepsilon > 0$ and*

$$\|u\|_\varepsilon^2 \le C\big(\|Au\|^2 + \|u\|_0^2\big)\,, \qquad u \in D(A)\,. \tag{2.2}$$

Then A is Γ-Fredholm.

Proof. It follows from (2.2) that the estimate (1.6) is satisfied on $L = \operatorname{Ker} A$. Therefore Proposition 1.5 implies that $\dim_\Gamma \operatorname{Ker} A < \infty$.

Let $\tilde E_\delta$ be the spectral projection of A corresponding to the interval $(-\delta,\delta)$. Then again $\dim_\Gamma \operatorname{Im} \tilde E_\delta < \infty$ by Proposition 1.5. On the other hand

$$\operatorname{Im}(I - \tilde E_\delta) = (\operatorname{Im} \tilde E_\delta)^\perp \subset \operatorname{Im} A\,,$$

which immediately implies that A is Γ-Fredholm. □

For the proof of Theorem 0.6 we will need the following

Proposition 2.8. *Under the assumptions of Proposition 1.7 the operator $\Box_{t\Phi} = \bar\partial\bar\partial_{t\Phi}^* + \bar\partial_{t\Phi}^*\bar\partial$ in $L^2_{t\Phi}\Lambda^{p,q}(M)$, $q > 0$, is Γ-Fredholm for $t > \gamma_0/\delta_0$.*

Proof. It follows from (1.7) that the estimate (1.9) is satisfied on $L = \operatorname{Ker} \Box_{t\Phi}$ in $L^2\Lambda^{p,q}_{t\Phi}(M)$, $q > 0$, for $t > \gamma_0/\delta_0$ with constant $\gamma = \delta_1 t/\min(1/3, 2/3(\delta_0 t - \gamma_0))$. Therefore, Proposition 1.9 implies that $\dim_\Gamma \operatorname{Ker} \Box_{t\Phi} < \infty$. Let $\tilde E_\delta$ be the spectral projection of $\Box$ corresponding to the interval $(-\delta,\delta)$. If $\delta < 2/3(\delta_0 t - \gamma_0)$ then from (1.7) we obtain the estimate (1.9) for all $\omega \in \operatorname{Im} \tilde E_\delta$ with the constant $\tilde\gamma = \delta_1 t/\min(1/3, 2/3(\delta_0 t - \gamma_0) - \delta)$. Hence, by Proposition 1.9 we have $\dim_\Gamma \operatorname{Im} \tilde E_\delta < \infty$. The inclusion $\operatorname{Im}(I - \tilde E_\delta) = (\operatorname{Im} \tilde E_\delta)^\perp \subset \operatorname{Im} \Box_{t\Phi}$ implies that $\Box_{t\Phi}$ is Γ-Fredholm in $L^2_{t\Phi}\Lambda^{p,q}(M)$, $q > 0$, for all $t > \gamma_0/\delta_0$. □

2. Now using Propositions 1.2, 2.7 and 2.8 we will be able to provide the complete proofs of Theorems 0.2–0.4, 0.6, 0.7. We shall start with the following elementary

Lemma 2.9. *Let U be an arbitrary set, $g : U \to \mathbf{C}$ an unbounded function. Then for any integer $N > 0$ the functions $g, g^2, \dots, g^N$ are linearly independent modulo bounded functions, i.e. if $B(U)$ is the space of all bounded functions on U and*

$$c_1 g + c_2 g^2 + \cdots + c_N g^N \in B(U)\,, \tag{2.3}$$

then $c_1 = \dots c_N = 0$.

Proof. Assuming that (2.3) is fulfilled consider the polynomial

$$p(t) = c_1 t + c_2 t^2 + \dots c_N t^N, \ t \in \mathbf{C}.$$

Then (2.3) implies that this polynomial is bounded along an unbounded sequence of complex values of t. Clearly this is only possible if the polynomial p is identically 0. □

Proof of Theorem 0.2. We shall use the notation from the introduction to this paper.

Let us choose a defining function ρ of the manifold M (see (0.1)) so that the Levi form (0.3) is positive for all $w \in \mathbf{C}^n - \{0\}$ (and not only for $w \in T_z^c(bM) - \{0\}$) at all points $z \in bM$. Using (0.4) we see that $\operatorname{Re} f(x,z) < 0$ if $x \in bM$ and $z \in M$ is sufficiently close to x. It follows that we can choose a branch of $\log f(x,z)$ so that $g_x(z) = \log f(x,z)$ is a holomorphic function in $z \in M \cap U_x$ where U_x is a sufficiently small neighbourhood of x in $\overline{M}$. Note that we can (and will) choose $U_{\gamma x} = \gamma U_x$.

Let us fix $x \in bM$. Clearly $g_x^m \in L^2(\overline{M} \cap U_x)$ for all $m = 1, 2, \dots$, and all functions g_x^m have a peak point at x. Besides all these functions are linearly independent modulo bounded functions by Lemma 2.9.

Let us choose a cut-off function $\chi \in C_c^\infty(U_x)$, so that $\chi = 1$ in a neighbourhood of x. We shall identify χ with its extension by 0 to $\overline{M}$, so it becomes a function from $C_c^\infty(\overline{M})$. The translation of χ by $\gamma \in \Gamma$ is a function $\gamma^*\chi$ which is supported in a small neighbourhood of γx: $\gamma^*\chi(z) = \chi(\gamma^{-1}z)$.

Denote by L the closed Γ-invariant subspace in $L^2(M)$ generated by all functions χg_x^m; $m = 1, \dots, N$. Clearly

$$L = \left\{ f \;\Big|\; f = \sum_{\gamma \in \Gamma} \sum_{m=1}^{N} c_{m,\gamma} \gamma^*(\chi g_x^m) \, ; \ \sum_{m,\gamma} |c_{m,\gamma}|^2 < \infty \right\}, \qquad (2.4)$$

where $c_{m,\gamma}$ are complex constants. It follows that L has the form $L^2\Gamma \otimes \mathbf{C}^N$, hence $\dim_\Gamma L = N$.

Let us consider the set of (0,1)-forms (which are smooth on $\overline{M}$ and have compact support):

$$\bar\partial(\chi g_x^m) \, ; \qquad m = 1, 2, \dots, N \, . \qquad (2.5)$$

They are linearly independent for any integer $N > 0$. Indeed, assuming that

$$c_1 \bar\partial(\chi g_x) + c_2 \bar\partial(\chi g_x^2) + c_N \bar\partial(\chi g_x^N) = 0$$

with some complex constants $c_1, \dots, c_N$, we see that

$$c_1\chi g_x + c_2\chi g_x^2 + \cdots + c_N\chi g_x^N$$

is holomorphic on M and has a compact support, hence it is identically 0, which implies that $c_1 = \cdots = c_N = 0$ due to Lemma 2.9.

Let L_1 be a closed Γ-invariant subspace in $L^2\Lambda^{0,1}(M)$ generated by the set of forms (2.5). Then again

$$L_1 = \left\{\omega \;\Big|\; \omega = \sum_{\gamma\in\Gamma}\sum_{m=1}^{N} c_{m,\gamma}\bar\partial\big(\gamma^*(\chi g_x^m)\big)\,,\ \sum_{m,\gamma}|c_{m,\gamma}|^2 < \infty\right\},$$

where $c_{m,\gamma}$ are complex constants, and $\dim_\Gamma L_1 = N$. Clearly $L_1 \subset C^\infty\Lambda^{0,1}(\overline{M})$, i.e. all elements of L_1 are C^∞ forms of type (0,1) on $\overline{M}$. Also $L_1 \subset \operatorname{Im}\bar\partial$, hence $L_1 \subset \overline{\operatorname{Im}\Box}$ due to the orthogonal decomposition (1.3).

Now we can apply Proposition 1.2, Proposition 2.7 and Corollary 2.6 to conclude that $\operatorname{Im}\Box \cap L_1$ is Γ-dense in L_1. Hence for any $\delta > 0$ there exists a closed Γ-invariant subspace $Q_1 \subset L_1$ such that $Q_1 \subset \operatorname{Im}\Box$ and $\dim_\Gamma Q_1 > N-\delta$. Solving the equation $\Box\omega = \alpha$ with $\alpha \in Q_1$ we can assume that $\omega \perp \operatorname{Ker}\Box$ and in this case the solution ω will be unique. Denote the space of all such solutions by K. Then $\dim_\Gamma K = \dim_\Gamma Q_1 > N - \delta$.

Applying $\bar\partial$ to both sides of the equation $\Box\omega = \alpha$ we see that $\bar\partial\bar\partial^*\bar\partial\omega = 0$, hence $\bar\partial^*\bar\partial\omega = 0$ and $\bar\partial\omega = 0$. Therefore $\bar\partial\bar\partial^*\omega = \alpha$. Also $\omega \in \Lambda^{0,1}(\overline{M})$ (i.e. $\omega \in C^\infty$ on $\overline{M}$) for any such solution ω due to the local regularity theorem for the $\bar\partial$ Neumann problem (see [FKo]).

Now denote

$$Q = \big\{f \;\big|\; |f \in L\,,\ \bar\partial f = \alpha \in Q_1\big\}\,.$$

As we have seen earlier $\bar\partial$ is injective on L, hence $\dim_\Gamma Q = \dim_\Gamma Q_1 > N-\delta$. If $f \in Q$ then we can find a (unique) solution $\omega \in K$ of the equation $\Box\omega = \alpha = \bar\partial f$ and then $h = f - \bar\partial^*\omega \in L^2\mathcal{O}(M)$. All these functions h form a closed Γ-invariant subspace $H \subset L^2\mathcal{O}(M)$ with $\dim_\Gamma H > N - \delta$. Hence $\dim_\Gamma L^2\mathcal{O}(M) = \infty$. Besides using the Γ-invariance of H we see that we can always find a function $h \in H$ such that one of the coefficients $c_{m,e}$; $m = 1, \dots, N$, in the expansion (2.4) (for the corresponding function f) does not vanish. The point x will be a local peak point for this function. This completes the proof of Theorem 0.2. □

Proof of Theorem 0.3. We should modify the proof of Theorem 0.2 by another choice of locally given holomorphic functions with singularities at a point $x \in bM$. Namely, if f is a holomorphic polynomial from (0.4), then we should use $\{f^{-k}, f^{-2k}, \dots, f^{-kN}\}$ with sufficiently large integer $k > 0$

instead of $\{\log f, \dots, (\log f)^N\}$ as we did in the proof of Theorem 0.2. It is easy to check that all functions χf^{-k} are in appropriate Sobolev spaces. Then we should apply Lemma 2.1 to evaluate the Γ-dimension of the intersection $L_1 \cap L_2$ where L_1 is the Γ-invariant subspace generated by all forms $\bar\partial(\chi f^{-km})$, $m = 1, \dots, N$, and $L_2 = \overline{\operatorname{Im}\bar\partial}$ in $L^2\Lambda^{0,1}(M)$.

All other arguments are similar to the ones used in the proof of Theorem 0.2. □

Remark. An interesting feature of Theorem 0.3 is that its proof does not use the regularity results for the $\bar\partial$-Neumann problem and so this theorem can be extended to a number of less regular situations.

Proof of Theorem 0.4. We should apply the arguments given in the proof of Theorem 0.3 to a strongly pseudoconvex Γ-invariant neighbourhood $\hat M$ of $\overline{M}$, find a sufficiently large space H of holomorphic functions on $\hat M$ with singularities on the boundary of $\hat M$ and then take the space L of restrictions of all functions from H to M. Since the restriction operator is injective the closure of L will have the same Γ-dimension as H. □

Proof of Theorem 0.6. The statement (ii) can be proved exactly as the statement of Theorem 0.2. So, if there exists on bM a point of strong pseudoconvexity then the statement (i) is also valid. Let us prove (i) without this assumption.

We can suppose, further, that in a neighbourhood of $\overline{M}$ we have a Γ-invariant function $\Phi \in C^2$ which is strictly plurisubharmonic in a Γ-invariant neighbourhood U of bM.

Let us fix a sufficiently small ball B in U so that $B \cap bM \neq \emptyset$ and $\gamma B \cap B = \emptyset$ for any $\gamma \in \Gamma$, $\gamma \neq 1$. Let $x_1, \dots, x_N$ be different points in $B \cap bM$ and $a_1(x), \dots, a_N(x)$ be non-negative cut-off functions from $C_c^\infty(B)$ such that $a_m = 1$ in a neighbourhood of x_m, and $a_m = 0$ in a neighbourhood of x_k, $k \neq m$, $m = 1, 2, \dots, N$.

Let us consider the following function

$$\tilde\Phi(x) = A\Phi(x) + \sum_{\gamma, m} \gamma^*\big(a_m(x) \cdot \ln \operatorname{dist}(x, x_m)\big)\,. \qquad (2.6)$$

For any sufficiently large A the function $\tilde\Phi$ is Γ-invariant and strictly plurisubharmonic in a neighbourhood $U_0 \subset U$ of bM. With this choice of $\tilde\Phi$ we will have $\gamma^* a_m \notin L^2_{t\tilde\Phi}(M)$ for any $\gamma \in \Gamma$ $m = 1, 2, \dots, N$ and $t \geq n$, where $n = \dim_{\mathbf{C}} M$.

Denote by L the following closed Γ-invariant subspace in $L^2(M)$

$$L = \left\{ f \;\Big|\; f(x) = \sum_{\gamma,m} c_{m,\gamma}\gamma^* a_m(x)\,;\ \sum_{m,\gamma} |c_{m,\gamma}|^2 < \infty \right\}.$$

The subspace L has the form $L^2\Gamma \otimes \mathbf{C}^N$ and hence $\dim_\Gamma L = N$.

Let L_1 be the following closed Γ-invariant subspace in $L^2_{t\tilde\Phi}\Lambda^{0,1}(M) \cap L^2\Lambda^{0,1}(M)$

$$L_1 = \left\{ \omega \;\Big|\; \omega \in \sum_{\gamma,m} c_{m,\gamma}\bar\partial\gamma^* a_m(x)\,,\quad \sum_{m,\gamma} |c_{m,\gamma}|^2 < \infty \right\}.$$

We have again $\dim_\Gamma L_1 = N$. Applying Proposition 1.7, Proposition 2.8 and Corollary 2.6 we conclude that $\operatorname{Im}\Box_{t\tilde\Phi} \cap L_1$ is Γ-dense in L_1. Hence, for any $\delta > 0$ there exists a closed Γ-invariant subspace $Q_1 \subset L_1$ such that $Q_1 \subset \operatorname{Im}\Box_{t\tilde\Phi}$ and $\dim_\Gamma Q_1 > N - \delta$.

Denote $Q = \{f \mid f \in L, \bar\partial f = \alpha \in Q_1\}$. Since $\bar\partial$ is injective on L we have $\dim_\Gamma Q = \dim_\Gamma Q_1 > N - \delta$. If $f \in Q$ then we can find a solution ω, $\omega \in L^2_{t\tilde\Phi}\Lambda^{0,1}(M)$, of the equation $\alpha = \Box_{t\tilde\Phi}\omega$, hence a solution $\beta = \bar\partial^*_{t\tilde\Phi}\omega \in L^2_{t\tilde\Phi}(M)$ for the equation $\bar\partial f = \alpha = \bar\partial\beta$. Hence the functions $h = f - \bar\partial^*_{t\tilde\Phi}\omega$ form a closed Γ-invariant subspace $H \subset L^2\mathcal{O}(M)$. It follows from the construction of $\tilde\Phi$ in the form (2.6) that $H \cap L^2_{t\tilde\Phi}\mathcal{O}(M) = \{0\}$, if $t \ge n$. Hence $\dim_\Gamma H = \dim_\Gamma Q > N - \delta$ and $\dim_\Gamma L^2\mathcal{O}(M) = \infty$. □

Proof of Theorem 0.7. The proof of this theorem does not use Γ-dimensions and is based only on the Hörmander type estimate from Corollary 1.9. The idea of the construction below goes back to Bombieri [B].

Under the assumptions of Corollary 1.9 it follows that for any $t > \gamma_0/\delta_0$ the equation $\bar\partial\beta = \alpha$, $\bar\partial\alpha = 0$ has a solution with the estimate

$$\|\beta\|_{t\Phi} \le \frac{1}{\sqrt{2/3(\delta_0 t - \gamma_0)}}\|\alpha\|_{t\Phi}\,. \tag{2.7}$$

Since the constant in (2.7) depends only on M and the minimal eigenvalue of $i\partial\bar\partial\Phi$ on M, this result is valid for a singular strictly plurisubharmonic function $\tilde\Phi$ of the form (2.6) (see [B]).

Let us fix points $x_1, \dots, x_N$ on M and complex numbers $c_1, \dots, c_N$ and find $h \in L^2\mathcal{O}(M)$ such that $h(x_m) = c_m$, $m = 1, \dots, N$. To this end let us take cut-off functions $a_1(x), \dots, a_N(x)$ from $C^\infty_c(\overline{M})$ such that $a_m = 1$ in a neighbourhood of x_m, $\operatorname{supp} a_m \cap \operatorname{supp} a_k = \emptyset$ for $m \ne k$ and all $\operatorname{supp} a_m$ are sufficiently small.

We can assume that Φ is continuous on $\overline{M}$. Let us introduce a singular strictly plurisubharmonic function $\tilde\Phi$ of the form (2.6). Let $\alpha =$

$\sum_{m=1}^{N} c_m \bar{\partial}\, a_m(x)$, where $\{c_m\}$ are any complex numbers. We have $\alpha \in L^2_{t\tilde{\Phi}}\Lambda^{0,1}(\overline{M})$ and $\alpha = \bar{\partial}\, f$, where $f = \sum c_m a_m(x) \in L^2(M)$.

Applying (2.7) we can find $\beta \in L^2_{t\tilde{\Phi}}\Lambda^{0,1}(M)$ such that $\bar{\partial}\beta = \alpha$. The function β is holomorphic in a neighbourhood of any $\{x_m\}$ because $\alpha = 0$ in such a neighbourhood.

It follows from the inclusion $\beta \in L^2_{t\tilde{\Phi}}\Lambda^{0,1}(M)$ that $\beta(x_m) = 0$, $m = 1, \dots, N$, if $t \geq n$. Then $h = f - \beta \in L^2\mathcal{O}(M)$ and $h(x_m) = c_m$, $m = 1, \dots, N$. □

3 An Example

1. In this section we will give an example which shows that the action of a discrete group Γ in the previous results is important: just bounded geometry with uniformity of all conditions is not sufficient even for the existence of a single non-trivial holomorphic L^2 function. In fact our manifold M will have a free holomorphic action of a solvable Lie group G such that the quotient $\overline{M}/G$ is the closed interval $[-1, 1]$. Also M will be a Stein manifold. Its non-compact boundary will be strongly pseudoconvex. Therefore the uniformity of all the local conditions will be guaranteed. The only thing which is missing is the free action of a discrete (or in fact any unimodular) group with a compact quotient. In particular the group G itself does not have any discrete cocompact subgroups.

Let B be the unit ball in $\mathbf{C}^2$:

$$B = \left\{ (w_1, w_2) \in \mathbf{C}^2 \;\middle|\; |w_1|^2 + |w_2|^2 < 1 \right\}.$$

We will consider it as a homogeneous complex manifold with the Bergman metric. The holomorphic automorphisms of B preserve the metric. Our manifold M will be a subdomain in B.

It is more convenient to work with a different representation of the classical domain B: we prefer to present it as a Siegel domain of the second kind:

$$\Omega = \left\{ (z_1, z_2) \in \mathbf{C}^2 \;\middle|\; \operatorname{Im} z_2 > |z_1|^2 \right\}.$$

The isomorphism of B and Ω is given by the formulas:

$$w_1 = 2z_1(z_2 + i)^{-1}, \quad w_2 = (z_2 - i)(z_2 + i)^{-1}. \tag{3.1}$$

The manifold M will be a δ-neighbourhood of the hyperplane section $\operatorname{Im} z_1 = 0$ in the Bergman metric on Ω, for some $\delta > 0$.

We shall always use the notation $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then

$$\Omega = \left\{ (x_1 + iy_1, x_2 + iy_2) \;\middle|\; y_2 > x_1^2 + x_2^2 \right\}.$$

The linear automorphisms of Ω are given by the formulas

$$(z_1, z_2) \mapsto \left(\rho z_1 + \xi, |\rho|^2 z_2 + t + 2i\rho z_1 \bar{\xi} + i|\xi|^2\right), \tag{3.2}$$

where $\rho, \xi \in \mathbf{C}$, $t \in \mathbf{R}$.

Now let us consider a subdomain M in Ω given by

$$M = M_\varepsilon = \left\{ (z_1, z_2) \;\Big|\; y_2 > x_1^2 + \frac{y_1^2}{\varepsilon^2} \right\},$$

where $0 < \varepsilon < 1$. As a limit case we will also use

$$M_0 = \Omega \cap \{y_1 = 0\} = \left\{ (z_1, z_2) \mid y_2 > x_1^2 + y_1^2,\ y_1 = 0 \right\}.$$

Let us consider a subgroup G of the group of automorphisms of Ω which is given by the restrictions $\rho = \lambda > 0$ and $\xi \in \mathbf{R}$ in (3.2). Clearly this will be a Lie group with $\dim_{\mathbf{R}} G = 3$. It is easy to see that G consists of all transformations of the form $T_{\xi,t} H_\lambda$ (or, equivalently, of the form $H_\lambda T_{\xi,t}$), where $\lambda > 0$, $\xi, t \in \mathbf{R}$, H_λ is a "similarity", $T_{\xi,t}$ is a "translation" given by the formulas

$$H_\lambda(z_1, z_2) = (\lambda z_1, \lambda^2 z_2), \quad T_{\xi,t}(z_1, z_2) = (z_1 + \xi, z_2 + t + 2iz_1\xi + i\xi^2).$$

The presentation of g in each of the forms $T_{\xi,t} H_\lambda, H_\lambda T_{\xi,t}$ is unique. The transformations $\{T_{\xi,t} \mid \xi, t \in \mathbf{R}\}$ form an abelian subgroup of the Heisenberg group acting on the boundary of Ω.

If follows from (3.2) that the group G can be represented as the group of matrices

$$g = \begin{bmatrix} \lambda^2 & 2i\lambda\xi & t + i\xi^2 \\ 0 & \lambda & \xi \\ 0 & 0 & 1 \end{bmatrix}$$

where $\xi, t \in \mathbf{R}$, $\lambda > 0$.

LEMMA 3.1. *The action of G on Ω has the following properties:*

(i) *This action preserves M and $\overline{M}$.*
(ii) *It is free on $\overline{M}$.*
(iii) *The space of orbits on $\overline{M}$ is the closed interval* $[-1, 1]$.

Proof. The proofs of (i) and (ii) are straightforward. To prove (iii) we can do the following. For any point $(z_1, z_2) \in \overline{M}$ (here and below the closure is always understood in the Bergman metric), adjusting parameters ξ, t, λ we can find a (unique) transformation from G which maps (z_1, z_2) to a point with $x_1 = x_2 = 0$ and $y_2 = 1$, i.e. a point of the form $(i\tau, i)$. A simple calculation shows that in fact $\tau = y_1/\sqrt{y_2 - x_1^2}$. It is easy to see that the formula for τ defines τ as a continuous function on $\overline{M}$. The range of this function is in fact $[-\varepsilon, \varepsilon]$, where the endpoints $-\varepsilon$ and ε correspond to the two connected components of bM ($bM \cap \{y_1 < 0\}$ and $bM \cap \{y_1 > 0\}$) so that G acts transitively on each of these components. Therefore the map $\varepsilon^{-1}\tau : \overline{M} \to [-1,1]$ identifies the space of orbits with $[-1,1]$. □

A straightforward calculation shows that M is strongly pseudoconvex. (This follows basically from the fact that the function $(x_1, y_1) \mapsto (x_1^2 + \varepsilon^{-2}y_1^2)$ is convex.)

Also uniformity of all the metric conditions is obvious because of the free cocompact action of G as a group of isometries of $\overline{M}$.

2. Here we will prove

PROPOSITION 3.2. *There is no nontrivial holomorphic L^2-functions on M, i.e. $L^2\mathcal{O}(M) = \{0\}$.*

Here L^2 is understood with respect to the measure dv which corresponds to the restriction of the Bergman metric to M.

We will start with the following

LEMMA 3.3. *If $f \in L^2\mathcal{O}(M)$ then $f|_{M_0} \in L^2(M_0, dv_0)$, where dv_0 is the volume element on M_0 corresponding to the restriction of the Bergman metric to M_0.*

Proof. Let us use the standard elliptic estimate

$$|f(x)|^2 \le C_\delta \int_{B(x,\delta)} |f(z)|^2 dv(z)\,, \qquad x \in M_0\,,$$

where $B(x,\delta)$ is the ball with the center x and the radius δ with respect to the Bergman metric, and δ is chosen so that these balls belong to M. Integrating this estimate over M_0 we arrive to the desired result.

In more detail we can first choose a small $\delta > 0$ and integrate over a small ball in M_0 which gives the estimate

$$\int_{B(x,\delta)\cap M_0} |f(x)|^2 dv_0(x) \le C_\delta \int_{B(x,2\delta)} |f(z)|^2 dv(z)\,.$$

Then using the bounded geometry property of M we can find a covering of M_0 by the balls $B(x_j,\delta)$, $j = 1, 2, \dots$, such that the balls $B(x_j, 2\delta)$ have bounded multiplicity of intersections. Summing over all j we get the estimate

$$\int_{M_0} |f(x)|^2 dv_0(x) \le C \int_M |f(z)|^2 dv(z)\,.$$

□

Proof of Proposition 3.2. Let us recall that the Bergman metric on the unit ball $B \subset \mathbf{C}^2$ has the form

$$ds^2 = 3\Big[\sum_{j=1}^{2} \frac{dw_j d\bar{w}_j}{(1-|w_1|^2-|w_2|^2)} + \sum_{j,k=1}^{2} \frac{\bar{w}_k w_j dw_k d\bar{w}_j}{(1-|w_1|^2-|w_2|^2)^2}\Big]\,.$$

The change of variables (3.1) transforms it to the Bergman metric on Ω which is given by the formula

$$ds^2 = 3\big(\operatorname{Im} z_2 - |z_1|^2\big)^{-2}\big(\operatorname{Im} z_2\, dz_1 d\bar z_1 + \tfrac{1}{4} dz_2 d\bar z_2 + \tfrac{i}{2} z_1 dz_2 d\bar z_1 - \tfrac{i}{2}\bar z_1 dz_1 d\bar z_2\big)\,. \tag{3.3}$$

The corresponding volume element has the form

$$dv = \tfrac{9}{4}(y_2 - x_1^2 - y_1^2)^{-3} dx_1 dy_1 dx_2 dy_2\,.$$

Taking $y_1 = 0$ in (3.3) we get the induced metric on M_0:

$$ds_0^2 = 3(y_2 - x_1^2)^{-2}\big(y_2 dx_1^2 + \tfrac{1}{4} dx_2^2 + \tfrac{1}{4} dy_2^2 - x_1 dx_1 dy_2\big)\,.$$

The corresponding volume element is

$$dv_0 = \frac{3\sqrt{3}}{4}(y_2 - x_1^2)^{-5/2} dx_1 dx_2 dy_2\,.$$

Consider the foliation of M_0 by half-planes

$$H_a = M_0 \cap \{z_1 = a\} = \big\{(a, z_2)\ \big|\ \operatorname{Im} z_2 > |a|^2\big\}\,,$$

where $a \in \mathbf{R}$. The restriction of the metric (3.3) to H_a is the Poincaré metric

$$\tfrac{3}{4}(y_2 - a^2)^{-2}(dx_2^2 + dy_2^2)\,,$$

with the volume element

$$dv_{0,a} = \tfrac{3}{4}(y_2 - x_1^2)^{-2} dx_2 dy_2\,.$$

Let $f \in L^2\mathcal{O}(M)$. Denote $f_0 = f|M_0 = f_0(x_1, z_2) = f_0(x_1, x_2, y_2)$ and $f_{0,a} = f_0|H_a = f_0(a, z_2) = f_0(a, x_2, y_2)$. Then $f_0 \in L^2(M_0, dv_0)$ by Lemma 3.3. Comparing the expressions of dv_0 and $dv_{0,a}$ and applying the Fubini theorem we see that the condition $f_0 \in L^2(M_0, dv_0)$ implies that

$$f_{0,a}(y_2 - a^2)^{-1/4} \in L^2(H_a, dv_{0,a})$$

for almost all $a \in \mathbf{R}$. Let us consider only values of a which have this property.

Note that for any fixed a we have $(y_2 - a^2)^{-1/4} \to \infty$ as z_2 tends to any finite boundary point $z_2 \in H_a$ (i.e. to a point $z_2 \in \mathbf{C}$ such that $\operatorname{Im} z_2 = a^2$). Therefore $f_{0,a} \in L^2(U, dv_{0,a})$ in a neighbourhood U of such a point. Note that $f_{0,a}$ is harmonic with respect to the Poincaré metric. Again using the standard elliptic estimate for the Laplacian of the Poincaré metric, we see that $f_{0,a}(z_2) \to 0$ as z_2 tends to any finite boundary point of H_a. Since f is also holomorphic it should be identically 0 on H_a. (If we map H_a biholomorphically to the unit disc D, then f will be transformed to a holomorphic function in D such that f vanishes on the boundary of D with a possible exception of one point which is the image of ∞ in H_a). Since this is true for almost all a we see that $f|M_0 = 0$. It follows that f is identically 0 on M. □

REMARK. Under the same conditions it may happen that G is not unimodular but there are plenty of L^2 holomorphic functions on M. For example this is possible for the simplest non-unimodular group: the group G of the matrices

$$g = \begin{bmatrix} 1 & t \\ 0 & 2^n \end{bmatrix}$$

where $t \in \mathbf{R}$, $n \in \mathbf{Z}$. Indeed, consider M which consists of all matrices

$$h = \begin{bmatrix} 1 & z \\ 0 & 2^m \end{bmatrix}$$

where $m \in \mathbf{Z}$, $z \in \mathbf{C}$, $|\operatorname{Im} z| < 1$. Then M is a disjoint countable union of strips $\{z \mid z \in \mathbf{C}, |\operatorname{Im} z| < 1\}$. Consider M as a (non connected) complex manifold with boundary with $\overline{M}$ obtained by taking closure of each strip, so that $\overline{M}$ consists of the matrices of the same form with $|\operatorname{Im} z| \leq 1$. Clearly M is strongly pseudoconvex.

The action of G on M is obtained by left multiplication of the matrices: $g \cdot h = gh$. It amounts to interchanging the strips with extra translations by real numbers (depending on the number of the strip). This action is obviously holomorphic and free. Introducing the standard Euclidean metric on every strip (the metric induced by the standard metric on $\mathbf{C}$), we obtain an invariant metric. There are plenty L^2 holomorphic functions on each strip, and they can be extended to M by 0. On the other hand it is easy to see that $\overline{M}/G = [-1, 1]$.

4 Open Questions

Here we give a list of open questions of various difficulty. It is assumed in all questions that we are in the situation of Theorems 0.1–0.5.

Let M be strongly pseudoconvex.

1. Does there exist a finite number of functions in $L^2\mathcal{O}(M) \cap C(\overline{M})$ which separate all points in bM?

2. Assume that $\dim_{\mathbf{C}} M = 2$. Does there exist $f \in L^2\mathcal{O}(M) \cap C(\overline{M})$ such that $f(x) \neq 0$ for all $x \in bM$?

3. Is it true that for every CR-function $f \in L^2(bM) \cap C(bM)$ ($\bar{\partial}_b f = 0$) there exists $F \in L^2\mathcal{O}(M) \cap C(\overline{M})$ such that $F|_{bM} = f$?

References

[AHi] A. ANDREOTTI, C.D. HILL, E.E. Levi convexity and the Hans Lewy problem, Ann. Scuola Norm. Sup. Pisa (3) 26 (1972), 325–363, 747–806.

[AV] A. Andreotti, E. Vesentini, Carleman estimates for the Laplace-Beltrami equation on complex manifolds, Publications Math. IHES 25 (1965), 81–130.

[At] M. Atiyah, Elliptic operators, discrete groups and von Neumann algebras, Astérisque 32-33 (1976), 43–72.

[B] E. Bombieri, Algebraic values of meromorphic maps, Invent. Math. 10 (1970), 267–287; 11 (1970), 163–166.

[Br] M. Breuer, Fredholm theories in von Neumann algebras, I, II, Math. Annalen 178 (1968), 243–254; 180 (1969), 313–325.

[C] J.M. Cohen, Von Neumann dimension and the homology of covering spaces, Quart. J. Math. Oxford, Ser. (2) 30 (1974), 133–142.

[Co] A. Connes, A survey of foliations and operator algebras, Proc. Symp. Pure Math. 38, Part I (1982), 521–628.

[CoMo] A. Connes, H. Moscovici, The L^2-index theorem for homogeneous spaces of Lie groups, Annals of Math. 115 (1982), 291–330.

[D] J. Dixmier, Von Neumann algebras, North Holland Publishing Co., Amsterdam, 1981.

[FKo] G.B. Folland, J.J. Kohn, The Neumann Problem for the Cauchy-Riemann Complex, Annals of Mathematics Studies 75, Princeton Univ. Press, Princeton, 1972.

[Fr] K. Friedrichs, The identity of weak and strong extensions of differential operators, Trans. Amer. Math. Soc. 55 (1944), 132–151.

[Fu] R. Fujita, Domaines sans point critique interieur sur l'espace projectif complexe, J. Math. Soc. Japan 15 (1963), 443–473.

[G] M. Gaffney, Hilbert space methods in the theory of harmonic integrals, Trans. Amer. Math. Soc. 78 (1955), 426–444.

[Gr1] H. Grauert, On Levi's problem and the imbedding of real-analytic manifolds, Annals of Math. 68 (1958), 460-472.

[Gr2] H. Grauert, Bemerkenswerte pseudokonvexe Manninfaltigkeiten, Math. Z. 81 (1963), 377–391.

[GriH] P. Griffiths, J. Harris, Principles of Algebraic Geometry, A Wiley-Interscience Publication, John Wiley & Sons, New York e.a., 1978.

[Gro] M. Gromov, Kähler hyperbolicity and L_2-Hodge theory, J. Diff. Geom. 33 (1991), 263–292.

[GroHeS] M. Gromov, G. Henkin, M. Shubin, L^2 holomorphic functions on pseudo-convex coverings, IHES/M/95/58 preprint.

[He] G. Henkin, The Lewy equation and analysis on pseudoconvex manifolds, Russian Math. Surveys 32 (1977), 59–130.

[HeI] G. Henkin, A. Iordan, Compactness of the Neumann operator for hyperconvex domains with non-smooth B-regular boundary, Institut de Mathématiques de Jussieu, Universités Paris VI et Paris VII, Prépublication 54, Décembre 1995.

[Hö1] L. Hörmander, L^2-estimates and existence theorem for the $\bar{\partial}$-operator, Acta Math. 113 (1965), 89–152.

[Hö2] L. Hörmander, Complex Analysis in Several Complex Variables, Van Nostrand, Princeton e.a., 1966.

[K] K. Kodaira, On Kähler varieties of restricted type, Ann. of Math. 60 (1954), 28–48.

[Ko] J.J. Kohn, Harmonic integrals on strongly pseudoconvex manifolds, Ann. of Math. 78 (1963), 112–148; 79 (1964), 450–472.

[KoRo] J.J. Kohn, H. Rossi, On the extension of holomorphic functions from the boundary of a complex manifold, Ann. of Math. 81 (1965), 451–472.

[M1] C.B. Morrey, Jr., The analytic embedding of abstract real analytic manifolds, Annals of Math. 68 (1958), 159–201.

[M2] C.B. Morrey, Jr., The $\bar{\partial}$-Neumann problem on strongly pseudoconvex manifolds, Differential Analysis, Bombay Colloq., Oxford University Press, London (1964), 81–133.

[N] M.A. Naimark, Normed Rings, Noordhoff, Groningen, 1964.

[Na] T. Napier, Convexity properties of coverings of smooth projective varieties, Math. Annalen 286 (1990), 433–479.

[R] J. Roe, Coarse Cohomology and Index Theory on Complete Riemannian Manifolds, Mem. Amer. Math. Soc. 104:497 (1993).

[S1] M.A. Shubin, Spectral theory of elliptic operators on non-compact manifolds, Astérisque 207 (1992), 35–108.

[S2] M.A. Shubin, L^2 Riemann-Roch theorem for elliptic operators, Geometric And Functional Analysis 5 (1995), 482–527.

[St] K. Stein, Überlagungen holomorph-vollstandiger komplexe Räume, Arch. Math. 7 (1956), 354–361.

[T] K. Takegoshi, Global regularity and spectra of Laplace-Beltrami operators on pseudoconvex domains, Publ. RIMS, Kyoto Univ. 19 (1983), 275–304.

[Ta] M. Takesaki, Theory of Operator Algebras, I, Springer-Verlag, 1979.

Mikhail Gromov
IHES
35 route de Chartres
91440 Bures sur Yvettte
France

G. Henkin
Dept. of Math.
Université Paris-VI
4 place Jussieu
75252 Paris Cedex 05
France

Mikhail A. Shubin
Dept. of Math.
Northeastern University
Boston, MA 02115
USA

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