Geometric and Functional Analysis
Vol. 1, No. 4 (1991)
1016-443X/91/0400375-30$1.50+0.20/0

VON NEUMANN SPECTRA NEAR ZERO

M. GROMOV AND M.A. SHUBIN

Introduction

We study in this paper the spectral distribution of the Laplace operator acting on k-forms on a complete Riemannian manifold M. If M is compact this distribution is expressed by the (spectral density) function $N_k(\lambda)$ that represents the number of eigenvalues $\leq \lambda$. In the general case where M is non-compact this number may very well be infinite. Yet, if M comes along with a discrete isometric action of a group Γ, then one can "renormalize" the naive definition of $N_k(\lambda)$ in order to have $N_k(\lambda) < \infty$ whenever the quotient space M/Γ is compact. The renormalization is achieved with the notion of the *von Neumann dimension* (or Γ-*trace*) as is explained in §1.

The most important characteristic of $N_k(\lambda)$ is

$$\bar{b}_k = \lim_{\lambda \to +0} N_k(\lambda) ,$$

which equals the von Neumann dimension of the space of harmonic ℓ-forms on M of degree k. This was introduced by M. Atiyah [A] and called the k-th L^2-*Betti number* of M. Atiyah also asked in [A] whether the L-Betti numbers are *homotopy invariants*. A positive answer was given by J. Dodziuk [D].

Next, it was observed in [NS1,NS2] that the asymptotic behavior of $N_k(\lambda)$ for $\lambda \to +0$ also has a topological meaning which has led to new differential-topological invariants of compact manifolds with infinite fundamental groups.

The purpose of the present paper is to improve the results in [NS1,NS2] by showing that the asymptotics of $N_k(\lambda)$ at $\lambda = 0$ is in fact a *homotopy* invariant. This is done by expressing this asymptotics as a *chain homotopy invariant* of the Rham L^2-complex of M in the category of Hilbert spaces and *bounded* linear operators.

The authors wish to thank V. Ivrii, to whom we owe the first version of the proof of Corollary 3.1. (See the remark following this Corollary.)

1. Preliminaries

Let M be a Riemannian Γ-manifold with a discrete infinite group Γ of isometries of M. We shall always suppose that Γ acts freely (i.e. without fixed points) and that $X = M/\Gamma$ is a compact manifold. Let us consider the Laplacians $\Delta_k = d\delta + \delta d$ on exterior k-forms on M (here δ is the formal adjoint to d).

Let $L^2\Lambda^k(M)$ be the Hilbert space of all square-integrable exterior k-forms on M and $C_0^\infty\Lambda^k(M)$ be the space of all smooth (i.e. C^∞) k-forms with compact support on M. Since M is complete as a Riemannian manifold, the Laplacian Δ_k is essentially self-adjoint in $L^2\Lambda^k(M)$ ([Ch]), i.e. its closure $\overline{\Delta}_k$ (with the initial domain $C_0^\infty\Lambda^k(M)$) is a self-adjoint operator in $L^2\Lambda^k(M)$. So we can take the spectral decomposition

$$\overline{\Delta}_k = \int \lambda dE_\lambda^{(k)} .$$

Let $e_k(\lambda, x, y)$ be the Schwartz kernel of the projection $E_\lambda^{(k)}$ so that $e_k(\lambda, x, y)$ defines a linear map $\Lambda^k T_y^* M \to \Lambda^k T_x^* M$ and

$$(E_\lambda^{(k)}\omega)(x) = \int_M e_k(\lambda, x, y)\omega(y)d\mu(y) ,$$

where $\omega \in C_0^\infty\Lambda^k(M)$ and $d\mu$ is the Riemannian density on M. It is well known that $e_k \in C^\infty$ with respect to x, y. So we can define the spectrum distribution function

$$N_k(\lambda) = \operatorname{Tr}_\Gamma E_\lambda^{(k)} = \int_F \operatorname{tr} e_k(\lambda, x, x)d\mu(x) , \tag{1.1}$$

where F is a fundamental domain of the action of Γ on M, tr is the usual matrix trace and Tr_Γ defined by the integration over F is the Γ-*trace* on the von Neumann algebra $\mathcal{A}_\Gamma^{(k)} = \mathcal{A}_\Gamma^{(k)}(M)$ of the Γ-invariant operators in $L^2\Lambda^k(M)$.

The trace Tr_Γ was introduced by M. Atiyah [A] who used it to define and calculate the Γ-index of elliptic Γ-invariant operators on vector Γ-bundles over M. Atiyah also defined the L^2-Betti numbers,

$$\bar{b}_k = N_k(+0) = \lim_{\lambda\to+0} N_k(\lambda) = \operatorname{Tr}_\Gamma B_k ,$$

where B_k is the orthogonal projection in $L^2\Lambda^k(M)$ onto the space of the harmonic L^2-forms. It was proved in [NS1] that these Betti numbers satisfy the same Morse inequalities as the ordinary Betti numbers. It was conjectures in [NS1] that the asymptotic behaviour of $N_k(\lambda)$ as $\lambda \to +0$ should contain interesting topological information. It was announced in [NS2] that if N_k, N'_k are spectrum distribution functions corresponding to Γ-invariant Riemannian metrics g, g' on M then there exists $C > 0$ such that

$$N_k(C^{-1}\lambda) \leq N'_k(\lambda) \leq N_k(C\lambda) \tag{1.2}$$

for every $\lambda \in \mathbf{R}$. The proof (see [ES]) uses the variational principle

$$N_k(\lambda) = \sup_{P \in \mathcal{P}_\lambda^{(k)}} \operatorname{Tr}_\Gamma P \ , \tag{1.3}$$

where $\mathcal{P}_\Gamma^{(k)}$ is the set of all bounded operators P in $L^2\Lambda^k(M)$ such that the following conditions are satisfied:

(i) $P^2 = P = P^*$

i.e. P is an orthogonal projection in $L^2\Lambda^k(M)$;

(ii) $P \in \mathcal{A}_\Gamma^{(k)}$

i.e. P is a Γ-invariant operator;

(iii) $\operatorname{Im} P \subset D(\overline{\Delta}_k)$

where $D(\overline{\Delta}_k)$ is the domain of $\overline{\Delta}_k$;

(iv) $P(\overline{\Delta}_k - \lambda I)P \leq 0$

i.e. the quadratic form of $\overline{\Delta}_k - \lambda I$ is negative on $\operatorname{Im} P$ (here I is the identity operator).

We can omit the condition of selfadjointness of P, replacing (i),(iv) by the following conditions

(i)′ $P^2 = P$

(iv)′ $P^*(\overline{\Delta}_k - \lambda I)P \leq 0$.

Also we can forget about the operator $\overline{\Delta}_k$ and consider the corresponding closed quadratic form Q_k (with the domain $D(Q_k)$). If $\omega \in D(\overline{\Delta}_k)$ then by definition

$$Q_k(\omega) = (\overline{\Delta}_k\omega, \omega) \ ,$$

where the inner product is taken in $L^2\Lambda^k(M)$. Now we can replace (iii), (iv)′ by the conditions

(iii)′ $\operatorname{Im} P \subset D(Q_k)$;

(iv)″ $Q_k(\omega) \leq \lambda(\omega, \omega)$ for every $\omega \in \operatorname{Im} P$.

If we introduce the Γ-*dimension* $\dim_\Gamma$ on the Γ-invariant linear subspaces $L \subset L^2\Lambda^k(M)$ by $\dim_\Gamma L = \operatorname{Tr}_\Gamma P_L$ for the orthogonal projection $P_L : L^2\Lambda^k(M) \to L$, then (1.3) can be written as

$$N_k(\lambda) = \sup_{L \in \mathcal{L}_\lambda^{(k)}} \dim_\Gamma L \tag{1.3′}$$

where $\mathcal{L}_\lambda^{(k)}$ is the set of all Γ-invariant closed linear subspaces $L \subset L^2\Lambda^k(M)$ such that $L \subset D(Q_k)$ and $Q_k(\omega) \le \lambda(\omega,\omega)$ for every $\omega \in L$.

Let us give a more explicit description of $D(Q_k)$ in our case. Let us define

$$W^1\Lambda^k(M) = \{\omega \mid \omega \in L^2\Lambda^k(M)\ ,\ d\omega \in L^2\Lambda^{k+1}(M)\ ,\ \delta\omega \in L^2\Lambda^{k-1}(M)\}\ ,$$

where d, δ should be understood in the distributions sense. Then

$$D(Q_k) = W^1\Lambda^k(M) \tag{1.4}$$

and

$$Q_k(\omega) = \|d\omega\|^2 + \|\delta\omega\|^2\ , \qquad \omega \in D(Q_k)\ . \tag{1.5}$$

Indeed it is well known that any essential domain of a selfadjoint operator is an essential domain for the corresponding quadratic form (see, e.g. [RS]). So it is sufficient to prove $C_0^\infty\Lambda^k(M)$ is dense in the Hilbert space $W^1\Lambda^k(M)$ equipped with the natural norm

$$\|\omega\|_{W^1}^2 = \|\omega\|^2 + \|d\omega\|^2 + \|\delta\omega\|^2\ , \tag{1.6}$$

where $\|\ \ \|$ denotes the usual L^2-norm. But this is trivial because we can apply Γ-invariant mollifiers to the forms from $W^1\Lambda^k(M)$ and then multiply by appropriate cut-off functions. This will give us small errors because d, δ are first order operators.

Let us also define the Laplace transform of the Stieltjes measure defined by the spectrum distribution function $N_k(\lambda)$:

$$\theta_k(t) = \mathrm{Tr}_\Gamma \exp(-t\Delta_k) = \int \exp(-\lambda t)dN_k(\lambda)\ , \qquad t > 0\ .$$

Then θ_k is a decreasing positive real-valued function on the open half-axis $\mathbf{R}_+ = \{t \mid t > 0\}$.

It is well known that the asymptotic behaviour of $N_k(\lambda)$ as $\lambda \to +0$ or $\lambda \to +\infty$ is closely connected with the asymptotic behaviour of $\theta_k(t)$ as $t \to +\infty$ or $t \to +0$ respectively (see Appendix). The main subject of this paper concerns the asymptotic behaviour of $N_k(\lambda)$ as $\lambda \to +0$ or of $\theta_k(t)$ as $t \to +\infty$. Therefore now we shall introduce an appropriate terminology.

Denote by $\mathcal{N}$ the set of all (non-strictly) increasing functions on $\mathbf{R}$ which vanish on the open negative half-line, i.e. $\mathcal{N}$ contains all increasing functions $N(\cdot)$ such that $N(\lambda) = 0$ when $\lambda < 0$. Choosing $N_1, N_2 \in \mathcal{N}$ we shall write $N_1 \overset{d}{\sim} N^2$ (and say that N_1 and N_2 are dilatationally equivalent) if there exists a constant $C > 0$ such that

$$N_1(C^{-1}\lambda) \le N_2(\lambda) \le N_1(C\lambda)$$

for every $\lambda \in \mathbf{R}$. We shall write that $N_1 \overset{d}{\sim} N_2$ *near* 0 if these inequalities are satisfied on $(-\infty, \varepsilon]$ for some $\varepsilon > 0$.

Similar terminology can be applied to the Laplace transforms. Namely, let Θ be the set of all (non-strictly) decreasing positive real-valued functions on $\mathbf{R}_+$. Let $\theta_1, \theta_2 \in \Theta$. We shall write $\theta_1 \overset{d}{\sim} \theta_2$ if there exists $C > 0$ such that

$$\theta_1(Ct) \le \theta_2(t) \le \theta_1(C^{-1}t)$$

for every $t > 0$. We shall write that $\theta_1 \overset{d}{\sim} \theta_2$ near infinity if this holds true on $[t_0, +\infty)$ for some $t_0 > 0$.

The Laplace transform of a distribution function $N \in \mathcal{N}$ is defined as

$$\theta(t) = \int \exp(-\lambda t) dN(\lambda)$$

and $\theta \in \Theta$ provided $\theta(t)$ is finite for all $t > 0$ (which is the case for all functions N_k which were introduced above). It is easy to prove that $N_1 \overset{d}{\sim} N_2$ (everywhere or near 0) implies that $\theta_1 \overset{d}{\sim} \theta_2$ (everywhere or near infinity respectively) for the corresponding Laplace transforms.

There is not much known about the asymptotic behaviour of $N_k(\lambda)$ as $\lambda \to +0$ or of $\theta_k(t)$ as $t \to +\infty$. The limits

$$\bar{b}_k = \lim_{\lambda \to +0} N_k(\lambda) = \lim_{t \to +\infty} \theta_k(t)$$

are called *von Neumann Betti numbers* of the base manifold $X = M/\Gamma$ in the case when M is simply connected (and $\Gamma = \pi_1(X)$). They are homotopy invariants of X which were introduced in [A] and used in [NS1] to improve Morse inequalities for non-simply connected manifolds.

It was noticed in [NS2] that if N_k, N'_k are the spectrum distribution functions corresponding to the Laplacians Δ_k, Δ'_k constructed by means of two Γ-invariant metrics g, g' on M then $N_k \overset{d}{\sim} N'_k$ (the proof is published in [ES]). It follows that the dilatational equivalence class of N_k in $\mathcal{N}$ does not depend on the choice of a Γ-invariant Riemannian metric on M. Hence the same is true for the corresponding functions θ_k. So the dilatational equivalence classes of N_k and θ_k depend only on the Γ-invariant smooth structure on M. Supposing that $X = M/\Gamma$ is fixed we can say that the classes of N_k and θ_k depend only on the smooth structure on X (and on the choice of a normal subgroup $K \subset \pi_1(X)$ which gives us a covering M of X with an action of $\Gamma = \pi_1(X)/K$). The main result of this paper will imply that these classes near 0 for N_k and near infinity for θ_k are in fact homotopy invariant.

Let us write that $f(\lambda) \asymp \lambda^\alpha$ as $\lambda \to +0$ if there exists $C > 0$ such that $C^{-1}\lambda^\alpha \le f(\lambda) \le C\lambda^\alpha$ in $(0, \lambda_0)$ for some $\lambda_0 > 0$. The notation $g(t) \asymp t^{-\alpha}$ as $t \to +\infty$ has a similar meaning. Now let us suppose that there exists $\alpha_k \in \mathbf{R}^+$ such that

$$N_k(\lambda) - \bar{b}_k \asymp \lambda^{\alpha_k} , \qquad \lambda \to +0 , \tag{1.7}$$

or

$$\theta_k(t) - \bar{b}_k \asymp t^{-\alpha_k} , \qquad t \to +\infty , \tag{1.8}$$

(it was proved in [ES] that these estimates are equivalent). Clearly the number α_k depends only on the dilatation equivalence class of N_k near 0 or θ_k near infinity. Hence the homotopy invariance of the classes of N_k and θ_k implies that the numbers α_k are also homotopy invariants.

Let us mention the results of calculations which are known to the authors. All of them concern the case when M is simply connected and so $\Gamma = \pi_1(X)$.

If $M = \mathbf{R}^n$ with the usual flat metric and Γ is a lattice in $\mathbf{R}^n$ then clearly

$$\alpha_0 = \alpha_1 = \ldots = \alpha_n = n/2$$

due to the standard Poisson formula for the heat kernel.

It was mentioned in [ES] that $\alpha_1 = \alpha_2 = 1/2$ for the case $M = \mathbf{H}^3$ where $\mathbf{H}^n$ is the n-dimensional hyperbolic space (it is a corollary of calculations in [Vi]). Using the results of [M] and [F], J. Lott [L] calculated the numbers α_k in the case $M = \mathbf{H}^{2n+1}$ and has shown that $\alpha_n = \alpha_{n+1} = 1/2$. Note that the spectrum of Δ_k in $\mathbf{H}^{2n}$ for all k and in $\mathbf{H}^{2n+1}$ for all $k \neq n$, $n+1$ has a gap near 0 which is equivalent to the fact that $N_k(\lambda) - \bar{b}_k \equiv 0$ in a neighbourhood of 0 (or to the fact that $\theta_k(t) - \bar{b}_k = O(\exp -\varepsilon t)$ for some $\varepsilon > 0$ as $t \to +\infty$).

In the case when the estimates (1.7) or (1.8) are not valid we can introduce the numbers (from $[0, +\infty]$)

$$\underline{\alpha}_k = \liminf_{\lambda \to +0} \frac{\log\left(N_k(\lambda) - \bar{b}_k\right)}{\log \lambda} = \liminf_{t \to +\infty} \frac{-\log\left(\theta_k(t) - \bar{b}_k\right)}{\log t} \tag{1.9}$$

(the last equality is proved in the Appendix, Proposition A.1) and the numbers

$$\overline{\alpha}_k = \limsup_{\lambda \to +0} \frac{\log\left(N_k(\lambda) - \overline{b}_k\right)}{\log \lambda} , \qquad \overline{\alpha}'_k = \limsup_{t \to +\infty} \frac{-\log\left(\theta_k(t) - \overline{b}_k\right)}{\log t} , \tag{1.10}$$

which satisfy the inequality

$$\overline{\alpha}_k \geq \overline{\alpha}'_k . \tag{1.11}$$

It becomes an equality provided one of the two following equivalent conditions is fulfilled:

$$N_k(\lambda) - \overline{b}_k = \mathcal{O}(\lambda^\delta) \quad \text{as} \quad \lambda \to +0 \tag{1.12}$$

or

$$\theta_k(t) - \overline{b}_k = \mathcal{O}(t^{-\delta}) \quad \text{as} \quad t \to +\infty , \tag{1.13}$$

with some $\delta > 0$ (see Proposition A.1). It is clear that if (1.7) or (1.8) are satisfied then

$$\underline{\alpha}_k = \overline{\alpha}_k = \overline{\alpha}'_k = \alpha_k . \tag{1.14}$$

It is not probable that (1.7) and (1.8) are always fulfilled because these estimates are not satisfied for the functions $N_k(\lambda)$ and $\theta_k(t)$ having logarithmic terms in their asymptotics. But these logarithmic terms are not important for the limits in (1.9) and (1.10) (which are the same, e.g. if $N_k(\lambda) \sim \lambda^{\alpha_k}(\log\lambda)^m$ as $\lambda \to +0$ whatever $m > 0$). It is not known whether

$$\underline{\alpha}_k = \overline{\alpha}_k = \overline{\alpha}'_k \tag{1.15}$$

in the general situation we consider.

J. Lott [L] considers the numbers $\underline{\alpha}_k$ (in a different normalization: in fact his notation α_k means our $2\underline{\alpha}_k$) and obtains some estimates and explicit results about them, mostly in the case of a simply-connected M (i.e. when $\Gamma = \pi_1(X)$) which we shall also suppose for the sake of simplicity until the end of this section. Using Fourier analysis he proves that if Γ is an abelian group then $\underline{\alpha}_k$ is rational and $\underline{\alpha}_k > 0$. Moreover if $\Gamma = \mathbb{Z}^\ell$ then $\underline{\alpha}_0 = \underline{\alpha}_1 = \ell/2$ and if $\ell = 1$ then $\underline{\alpha}_k = 1/2n_k$ with some positive integers n_k for all k. He also remarks that it follows from [Va] that $\underline{\alpha}_0 = \infty$ unless Γ has polynomial growth (i.e. is almost nilpotent according to [G]), in which case $2\underline{\alpha}_0$ is the growth rate of Γ. It is also proved in [L] by a use of CR-analysis that if M is the $(2m+1)$-dimensional Heisenberg group then $\underline{\alpha}_k \leq m+1$ if $k \neq m, m+1$; $\underline{\alpha}_k \leq (m+1)/2$ if $k = m, m+1$. Moreover, if $m = 1$ then $\underline{\alpha}_0 = 2$, $\underline{\alpha}_1 = 1$. We also refer the reader to [L] about some information on the finiteness of the numbers $\underline{\alpha}_k$ in the case when M is a locally symmetric space. Let us also mention that the result of R. Brooks [B] implies that if Γ is not amenable then $\underline{\alpha}_0 = +\infty$.

2. Formulation of the Main Results

Let us fix a discrete group Γ and define a category $\mathcal{M}_\Gamma$. Its objects are smooth Γ-manifolds M which are equipped with a free action of Γ with a compact quotient $X = M/\Gamma$. The morphisms in $\mathcal{M}_\Gamma$ are homotopy classes of smooth Γ-maps.

Without loss of generality we may suppose all manifolds to be connected.

An equivalent category is obtained if we define objects to be smooth compact manifolds X equipped with a given epimorphism $j_X : \pi_1(X) \to \Gamma$ and define morphisms as the homotopy classes of smooth maps which agree with the given epimorphism j_X, i.e. smooth maps $f : X_1 \to X_2$ such that the diagram

$$\begin{array}{ccc} \pi_1(X_1) & \xrightarrow{j_{X_1}} & \Gamma \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \pi_1(X_2) & \xrightarrow{j_{X_2}} & \Gamma \end{array}$$

is commutative. We will not distinguish these two categories. Evidently the same category is obtained if we consider continuous maps instead of smooth ones.

Now we can formulate one of the main results.

THEOREM 2.1. *Let M, M' be Γ-manifolds which are Γ-homotopy equivalent, i.e. isomorphic in $\mathcal{M}_\Gamma$. Let N_k, N'_k be the corresponding spectrum distribution functions. Then $N_k \overset{d}{\sim} N'_k$ near 0 for every $k = 0, 1, \ldots, \dim M$.*

COROLLARY 2.2. *Under the same conditions $\theta_k \overset{d}{\sim} \theta'_k$ near infinity for the Laplace transforms θ_k, θ'_k of N_k, N'_k respectively.*

COROLLARY 2.3. *Let us suppose that M, M' are the same as in Theorem 2.1. Then they have the same numbers $\underline{\alpha}_k, \overline{\alpha}_k, \overline{\alpha}'_k$. If the invariant α_k is defined on M for some k, i.e. (1.7) (or (1.8)) is satisfied then the same is true for M' with the same exponent α_k.*

So all the exponents $\underline{\alpha}_k, \overline{\alpha}_k, \overline{\alpha}'_k$ are homotopy invariants (and so is α_k when it is well defined).

A particularly interesting case is where M is the universal covering of a compact manifold X and the group $\Gamma = \pi_1(X)$ acts on X by the deck transformations. Then the dilatational equivalence classes of N_k can be thought of as some invariants of the manifold X. Theorem 2.1 implies then that the classes of N_k near 0 and of θ_k near infinity are homotopy invariants of X. Hence all exponents $\underline{\alpha}_k, \overline{\alpha}_k, \overline{\alpha}'_k$ are also homotopy invariants of X and so is α_k provided it is well defined. Note that in fact these invariants depend only on the universal covering of X. Hence they are not local invariants because they do not change if we pass from X to its finite covering (a covering manifold which corresponds to a cofinite subgroup in $\pi_1(X)$).

Later we shall formulate the main results in a more general and refined form which includes the case of manifolds with a piecewise smooth boundary. But to do this we shall need some additional preparations which are also necessary for the proof of Theorem 2.1.

3. Kodaira Decomposition and Generalization to Manifolds with Boundary

3.1. Let us describe the Kodaira decomposition [K] for the case of a Riemannian manifold M with a piecewise smooth boundary. By a manifold with a piecewise smooth boundary we mean a closed part M of an (open) manifold $\widetilde{M}$ such that in a neighbourhood $\widetilde{U} \subset \widetilde{M}$ of a point $x \in M$ the part $M \cap \widetilde{U}$ can be represented as (a part of) a polyhedron $P \subset \mathbf{R}^n$ in appropriate local coordinates on $\widetilde{M}$. In fact we need only that M is locally Lipschitz equivalent to a manifold with a smooth boundary but we shall write about a piecewise smooth boundary for the sake of simplicity.

First let us define Hilbert spaces

$$E_k(M) = \overline{d\big(C_0^\infty\Lambda^{k-1}(\operatorname{Int} M)\big)} \ , \quad E_k^*(M) = \overline{\delta\big(C_0^\infty\Lambda^{k+1}(\operatorname{Int} M)\big)} \ ,$$

where $\operatorname{Int} M = M \backslash \partial M$ and the bars denote the closures in $L^2\Lambda^k(M)$.

Let us define the space of all harmonic square integrable k-forms

$$\mathcal{H}_k(M) = \big\{\omega \mid \omega \in L^2\Lambda^k(M) \ , \ d\omega = 0 \ , \ \delta\omega = 0\big\} \ ,$$

where d, δ are applied in the sense of distributions. The Kodaira decomposition has the form

$$L^2\Lambda^k(M) = E_k(M) \oplus \mathcal{H}_k(M) \oplus E_k^*(M) \ . \tag{3.1}$$

It is an orthogonal decomposition with respect to the scalar product in $L^2\Lambda^k(M)$. If we suppose that M has no boundary and is complete then (1.5) and usual ellipticity arguments imply that

$$\mathcal{H}_k(M) = \big\{\omega \mid \omega \in \Lambda^k(M) \cap L^2\Lambda^k(M) \ , \ \Delta_k\omega = 0\big\}$$

where $\Lambda^k(M)$ denotes the space of all smooth k-forms on M.

The kernels $\operatorname{Ker} d$ and $\operatorname{Ker}\delta$ can be easily described in terms of the Kodaira decomposition if d and δ are understood in the sense of distributions:

$$\operatorname{Ker} d_k = E_k(M) \oplus \mathcal{H}_k(M)\ , \quad \operatorname{Ker}\delta_{k-1} = \mathcal{H}_k(M) \oplus E_k^*(M)\ , \tag{3.2}$$

where d_k and δ_{k-1} denote the restrictions of d and δ to k-forms. Indeed, $\operatorname{Ker} d_k$ is evidently the orthogonal complement of the closure of $\delta\big(C_0^\infty\Lambda^{k+1}(\operatorname{Int} M)\big)$, so the first equality in (3.2) follows from (3.1) and the proof of the second one is similar.

Now let us suppose that M is a Riemannian Γ-manifold with a piecewise smooth boundary. As before we suppose that Γ acts freely and $X = M/\Gamma$ is compact (so X is a compact Riemannian manifold with a piecewise smooth boundary). Then all the spaces $L^2\Lambda^k(M), E_k(M), E_k^*(M), \mathcal{H}_k(M)$ are Hilbert Γ-modules. There exists a representation

$$L^2\Lambda^k(M) = L^2\Gamma \otimes L^2\Lambda^k(X)\ , \tag{3.3}$$

where the tensor product is understood as the tensor product of Hilbert spaces (with the completion with respect to the natural scalar product), $L^2\Gamma$ is the Hilbert space of all square-integrable complex-valued functions on Γ with respect to the uniform discrete measure, $L^2\Gamma$ considered with the natural left action of Γ and $L^2\Lambda^k(X)$ with the trivial action of Γ. The representation (3.3) is induced by a choice of a fundamental domain of the action of Γ on M. The essential properties of this representation do not depend on this choice. For example, let us define a trace on the von Neumann algebra $\mathcal{A}_\Gamma^{(k)}$ of the Γ-invariant operators in $L^2\Lambda^k(M)$ by the formula

$$\operatorname{Tr}_\Gamma = \operatorname{tr}_\Gamma \otimes \operatorname{Tr}\ .$$

Here tr_Γ is the natural finite trace on the von Neumann algebra of all Γ-invariant operators in $L^2\Gamma$ (it is 1 at the identity operator and 0 on all right translation operators) and Tr is the usual trace on the algebra of all bounded operators $L^2\Lambda^k(X)$. Then Tr_Γ does not depend on the choice of a fundamental domain and is an extension of the trace which was introduced in §1. We shall also denote the corresponding dimension function by $\dim_\Gamma$. This function is defined on the set of all Γ-invariant closed linear subspaces in $L^2\Lambda^k(M)$.

3.2. Now let us return for a while to the case of Riemannian Γ-manifolds M without boundary and with compact M/Γ. In §1 we introduced a quadratic form Q_k in $L^2\Lambda^k(M)$ with the domain $D(Q_k) = W^1\Lambda^k(M)$ (see (1.4),(1.5)). It follows from (3.2) that the Kodaira decomposition (3.1) gives a splitting of the form Q_k:

$$Q_k = Q'_k \oplus 0 \oplus Q''_k \ .$$

For the spectrum distribution function N_k this gives the splitting

$$N_k(\lambda) = G_k(\lambda) + \bar{b}_k + F_k(\lambda) \ , \tag{3.4}$$

where G_k, F_k are defined by the formulas obtained from $(1.3)'$ by changing $L^2\Lambda^k(M)$ by $E_k(M)$ and $E_k^*(M)$ respectively.

LEMMA 3.1. *$F_k(\lambda) = G_{k+1}(\lambda)$, $k = -1, 0, 1, \ldots, n$ where $n = \dim M$ and by definition $F_{-1}(\lambda) \equiv G_{n+1}(\lambda) \equiv 0$.*

Proof: The main point here is that the Laplacian Δ_k is essentially self-adjoint in $L^2\Lambda^{k+1}(M)$ with the initial domain $C_0^\infty\Lambda^k(M)$. Now let us consider the closures $\overline{d}_k$ and $\overline{\delta}_k$ of the operators d_k and δ_k in $L^2\Lambda^k(M)$ and $L^2\Lambda^{k+1}(M)$ with the initial domains $C_0^\infty\Lambda^k(M)$ and $C_0^\infty\Lambda^{k+1}(M)$ respectively. The operators d_k and δ_k are formally adjoint to each other and using the decomposition (3.1) we easily obtain that the operator $\delta_k d_k$ is essentially self-adjoint. It follows, due to the inverse von Neumann theorem (see §4 of [S1]), that $\overline{d}_k$ and $\overline{\delta}_k$ are adjoint in the exact Hilbert sense as unbounded operators in Hilbert spaces:

$$\overline{\delta}_k = d_k^* \ , \quad \overline{d}_k = \delta_k^*$$

(bars on the right hand sides are omitted because the adjoint operators of an operator and of its closure are the same). It follows that $\overline{\Delta}_k = \overline{\delta}_k\overline{d}_k$ on $E_k^*(M)$, $\overline{\Delta}_k = \overline{d}_{k-1}\overline{\delta}_{k-1}$ on $E_k(M)$ and $\overline{\Delta}_k = \overline{d}_{k-1}\overline{\delta}_{k-1} + \overline{\delta}_k\overline{d}_k$ on $L^2\Lambda^k(M)$, i.e. both sides of all these equalities have the same domains and coincide on these domains.

Now if $\omega \in D(\overline{\Delta}_k \mid E_k^*)$ then

$$(\overline{\Delta}_k\omega, \omega) = (\overline{\delta}_k\overline{d}_k\omega, \omega) = (\overline{d}_k\omega, \overline{d}_k\omega) = (d_k\omega, d_k\omega) \tag{3.5}$$

(we can omit the bars keeping in mind that d_k, δ_k should be applied in the sense of distributions). So $F_k(\lambda)$ coincides with the spectrum distribution function of the self-adjoint operator $\overline{\Delta}_k \mid E_k^*(M)$.

Furthermore, $\Delta_{k+1}d_k = d_k\Delta_k$ on $C_0^\infty\Lambda^k(M)$. It follows that

$$\overline{d}_k(\overline{\Delta}_k - \mu)^{-1} = (\overline{\Delta}_{k+1} - \mu)^{-1}\overline{d}_k \quad \text{on} \quad (\Delta_k - \mu)(C_0^\infty\Lambda^k(M)) \ ,$$

if $\mu \in \mathbf{C}\backslash\sigma(\overline{\Delta}_k)$ where $\sigma(\overline{\Delta}_k)$ is the spectrum of $\overline{\Delta}_k$ in $L^2\Lambda^k(M)$. Taking the closures we obtain

$$(\overline{\Delta}_{k+1} - \mu)^{-1}\overline{d}_k \subset \overline{d}_k(\overline{\Delta}_k - \mu)^{-1} ,$$

since the closure of $d|C_0^\infty\Lambda^k(M)$ coincides with the closure of $d|(\Delta_k - \mu)C_0^\infty\Lambda^k(M)$ because of evident essential self-adjointness of Δ_k on $(\Delta_k - \mu)C_0^\infty\Lambda^k(M)$. It follows that

$$E_\lambda^{(k+1)}\overline{d}_k \subset \overline{d}_k E_\lambda^{(k)} .$$

Restricting this equality to $E_k^*(M)$ we see that $\overline{d}_k$ gives a similarity of $E_\lambda^{(k)}|E_k^*(M)$ and $E_\lambda^{(k+1)}|E_{k+1}(M)$ (this is an "unbounded" similarity but it can be changed to a bounded one or to a unitary equivalence if we pass from $\overline{d}_k$ to its polar decomposition – see, e.g. Lemma 5.1 in [S2]). So we obtain

$$F_k(\lambda) = \mathrm{Tr}_\Gamma\, E_\lambda^{(k)} \mid E_k^*(M) = \mathrm{Tr}_\Gamma\, E_\lambda^{(k+1)} \mid E_{k+1}(M) = G_{k+1}(\lambda) . \qquad \square$$

So due to (3.4) and Lemma 3.1 the functions $N_k(\lambda)$ are expressed in terms of all functions $G_\ell(\lambda)$ or in terms of $F_\ell(\lambda)$ (and also the numbers $\overline{b}_k$ are needed):

$$N_k(\lambda) = G_k(\lambda) + \overline{b}_k + G_{k+1}(\lambda) = F_{k-1}(\lambda) + \overline{b}_k + F_k(\lambda) . \tag{3.6}$$

We shall use the last expression and work with the functions $F_k(\lambda)$ because they can be expressed in a more convenient form which does not include δ:

LEMMA 3.2. *For every $\lambda \in \mathbf{R}_+$*

$$F_k(\lambda) = \sup_{L \in \mathcal{S}_\lambda^{(k)}} \dim_\Gamma L , \tag{3.7}$$

where $\mathcal{S}_\lambda^{(k)}$ is the set of all closed Γ-invariant subspaces $L \subset L^2\Lambda^k(M)/\operatorname{Ker} d$ such that $d(L) \subset L^2\Lambda^{k+1}(M)$ and

$$\|d\omega\| \le \sqrt{\lambda}\|\omega\| , \qquad \omega \in L . \tag{3.8}$$

(The norm on the left-hand side is the usual norm in $L^2\Lambda^{k+1}(M)$ but the norm on the right-hand side is the quotient Hilbert norm in $L^2\Lambda^k(M)/\operatorname{Ker} d$.)

Proof: The statement becomes evident if we use the natural isometry

$$L^2\Lambda^k(M)/\operatorname{Ker} d \cong E_k^*(M)$$

and (3.5). □

Now let us note that $F_k(+0) = 0$ and if

$$F_\ell(\lambda) \asymp \lambda^{\beta_\ell} \quad \text{as} \quad \lambda \to +0\ , \tag{3.9}$$

for $\ell = k-1$ and $\ell = k$ with some $\beta_\ell > 0$, then (1.7) is true due to (3.6) with

$$\alpha_k = \min(\beta_{k-1}, \beta_k)\ . \tag{3.10}$$

So the numbers α_k can easily be expressed in terms of similar numbers defined for the functions F_ℓ.

If the numbers β_ℓ are not defined then as in the end of §1 we can define numbers $\underline{\beta}_\ell, \overline{\beta}_\ell, \overline{\beta}'_\ell$. Then

$$\underline{\alpha}_k = \min(\underline{\beta}_{k-1}, \underline{\beta}_k)\ , \quad \overline{\alpha}_k \le \min(\overline{\beta}_{k-1}, \overline{\beta}_k)\ , \quad \overline{\alpha}'_k \le \min(\overline{\beta}'_{k-1}, \overline{\beta}'_k) \tag{3.10'}$$

and if $\underline{\beta}_{k-1} = \overline{\beta}_{k-1}$, $\underline{\beta}_k = \overline{\beta}_k$ then $\underline{\alpha}_k = \overline{\alpha}_k = \min(\underline{\beta}_{k-1}, \underline{\beta}_k)$ (see Proposition A.2 and Corollary A.2 in the Appendix).

3.3. Now we can return to the case of Γ-manifolds M with piecewise smooth boundaries. Lemma 3.2 allows us to give a definition of $F_k(\lambda)$ in this situation.

DEFINITION: 3.1 For every Γ-manifold M with a piecewise smooth boundary the function F_k is defined by the formula (3.7).

In the case where we need an explicit dependence of F_k on M we shall write $F_k(\lambda; M)$ instead of $F_k(\lambda)$.

It is clear that F_k is a (non strictly) increasing function on $\mathbb{R}$ with values in $[0, +\infty]$ and $F_k(\lambda) = 0$ if $\lambda < 0$.

In fact, only a Γ-invariant Lipschitz structure and a Γ-invariant Lipschitz Riemannian metric on M are needed to make this definition meaningful (see, e.g. [T] for the necessary definitions about Lipschitz manifolds). Changing the Lipschitz metric to another Γ-invariant one leads to a changing of all the functions F_k to dilatationally equivalent ones. Also if two manifolds are Lipschitz homeomorphic then the corresponding functions are dilatationally equivalent.

We still do not know whether $F_k(\lambda)$ is finite or not. To prove the finiteness (for the case of compact M/Γ) we shall need the following important result about comparing the functions F_k for two different manifolds.

PROPOSITION 3.1. *Let M_1, M_2 be two Γ-manifolds of the same dimension with piecewise smooth boundaries and suppose there exists an isometric Γ-inclusion $M_1 \subset M_2$ (i.e. M_1 is a part of M_2). Then*

$$F_k(\lambda; M_1) \le F_k(\lambda; M_2)\ , \qquad k = 0, 1, \ldots, n\ , \tag{3.11}$$

where $n = \dim M_1 = \dim M_2$.

Proof: Let us use the formula

$$F_k(\lambda; M) = \sup \dim_\Gamma L$$

where the supremum is taken over all Γ-subspaces $L \subset E_k^*(M)$ such that $d(L) \subset L^2\Lambda^{k+1}(M)$ and (3.8) is satisfied. Now under the conditions of Proposition 3.1 we have a natural inclusion

$$C_0^\infty \Lambda^{k+1}(\operatorname{Int} M_1) \subset C_0^\infty \Lambda^{k+1}(\operatorname{Int} M_2)$$

which induces the inclusion

$$\delta C_0^\infty \Lambda^{k+1}(\operatorname{Int} M_1) \subset \delta C_0^\infty \Lambda^{k+1}(\operatorname{Int} M_2)$$

so after taking closures we obtain an isometric inclusion

$$E_k^*(M_1) \subset E_k^*(M_2)\ ,$$

so (3.11) evidently follows. □

COROLLARY 3.1. *Let M be a Γ-manifold with a piecewise smooth boundary and a compact quotient manifold $X = M/\Gamma$. Then $F_k(\lambda) < \infty$ for every $\lambda \in \mathbf{R}$ and for every $k = 0, 1, \ldots, n$.*

Proof: We refer to [A] for the case where $\partial M = \emptyset$.

First we reduce the problem to the case of smooth (C^∞) boundary. This can be done either by passing to a Lipschitz homeomorphic manifold or by taking an inclusion $M \subset M_1$ where M_1 is a "neighbourhood" of M with a C^∞-boundary such that $\dim M_1 = \dim M$, M_1 is Γ-invariant and M_1/Γ is compact.

So we can suppose that ∂M is C^∞. Then we can construct an isometric inclusion $M \subset \widehat{M}$ where $\widehat{M}$ is a Riemannian Γ-manifold such that $\widehat{M}/\Gamma$ is a compact closed manifold. (E.g. one can take the double $\widehat{M}$ of M). So the statement follows from Proposition 3.1. □

Remark: A direct proof of Corollary 3.1 for the case of a smooth boundary was suggested by V. Ivrii, who used the ellipticity of a boundary value problem on $X = M/\Gamma$.

We shall define later on the functions $F_k(\lambda)$ (up to dilatational invariance) for every CW-complex M with a free action of a discrete group Γ with a compact quotient space $X = M/\Gamma$. We shall also prove that the statement of Corollary 3.1 remains true in this situation.

4. Homotopy (abstract setting)

Let us recall necessary definitions about $L^2\Gamma$-modules in the sense described in [Co]. A free $L^2\Gamma$-module is a Hilbert space of the form $L^2\Gamma \otimes A$ where A is a (complex) Hilbert space. We consider this free $L^2\Gamma$-module with an action of Γ which is defined by the left action of Γ on $L^2\Gamma$ and the trivial action of Γ on A. There is a natural trace Tr_Γ in the von Neumann algebra of all Γ-invariant operators in $L^2\Gamma \otimes A$:

$$\mathrm{Tr}_\Gamma = \mathrm{tr}_\Gamma \otimes \mathrm{Tr} ,$$

where tr_Γ is defines in §3 and Tr is a usual trace in the algebra of all bounded operators in A. The corresponding dimension-function will be denoted by $\dim_\Gamma$. It is defined on the set of all Γ-invariant closed subspaces in $L^2\Gamma \otimes A$ and has values in $[0, +\infty]$.

A $L^2\Gamma$-module M is a closed Γ-invariant subspace in a free $L^2\Gamma$-module. The dimension function $\dim_\Gamma$ is defined also on all closed Γ-invariant subspaces in M and does not depend on the choice of a Γ-invariant inclusion of M into a free $L^2\Gamma$-module.

Let us consider a complex M of $L^2\Gamma$-modules, i.e. a sequence

$$0 \longrightarrow M_0 \xrightarrow{d_0} M_1 \longrightarrow \dots \longrightarrow M_k \xrightarrow{d_k} M_{k+1} \longrightarrow \dots \xrightarrow{d_{n-1}} M_n \longrightarrow 0$$

where M_k are $L^2\Gamma$-modules, $d_k : M_k \to M_{k+1}$ are closed densely defined linear operators such that $d_{k+1}d_k = 0$, $k = 0, 1, \dots, n-1$, $d_k\gamma = \gamma d_k$ for all $\gamma \in \Gamma$ and $k = 0, 1, \dots, n-1$ (where we use the same notation for the elements $\gamma \in \Gamma$ and the corresponding operators in M_k). For the sake of simplicity of notation we shall always consider complexes of the same length and put by definition $M_{-1} = M_{n+1} = 0$, $d_{-1} = d_n = 0$.

Let N be another complex of $L^2\Gamma$-modules. A morphism $f : M \to N$ is a sequence $f_k : M_k \to N_k$ of bounded linear Γ-operators (i.e. operators which commute with the action of Γ in M_k and N_k) such that

$$f_{k+1}d_k \subset d_k f_k$$

(we denote the differentials in N by the same letters d_k), i.e. $f_{k+1}d_k\omega = d_k f_k\omega$ for every $\omega \in D(d_k)$. A homotopy between two morphisms $f,g: M \to N$ is a sequence of bounded Γ-operators $T_k : M_k \to N_{k-1}$, $k = 0,1,\dots,n$, such that

$$f_k - g_k - T_{k+1}d_k \subset d_{k-1}T_k \; . \tag{4.1}$$

Since f_ℓ, g_ℓ, T_ℓ are bounded this implies, in particular, that the domain of the operator on the right-hand side contains $D(d_k)$, so (4.1) is equivalent to the equality

$$f_k - g_k = T_{k+1}d_k + d_{k-1}T_k \quad \text{on} \quad D(d_k) \; . \tag{4.1$'$}$$

It is clear that homotopy in this sense is an equivalence relation.

Let us say that two complexes of $L^2\Gamma$-modules M and N are homotopy equivalent if there exist two morphisms $f : M \to N$ and $g : N \to M$ such that both compositions fg and gf are homotopic to the corresponding identity morphisms of N and M respectively. It is easy to check that this homotopy equivalence is really an equivalence relation.

If M is a complex of $L^2\Gamma$-modules then we can define the functions $F_k = F_k(\lambda) = F_k(\lambda; M)$, $k = 0,1,\dots,n$, in the same way as we did in §3:

$$F_k(\lambda) = \sup_{L \in \mathcal{S}_\lambda^{(k)}(M)} \dim_\Gamma L \tag{4.2}$$

where $\mathcal{S}_\lambda^{(k)}(M)$ is the set of all closed Γ-invariant subspaces $L \subset M_k/\operatorname{Ker} d_k$ such that $L \subset D(d_k)/\operatorname{Ker} d_k$ and

$$\|d_k\omega\| \le \sqrt{\lambda}\|\omega\| \, , \qquad \omega \in L \; . \tag{4.3}$$

So F_k is a (non strictly) increasing function on $\mathbf{R}$ with values in $[0,+\infty]$ and $F_k(\lambda) = 0$ if $\lambda < 0$.

Now we can formulate the main abstract statement.

PROPOSITION 4.1. *Let M, M' be complexes of $L^2\Gamma$-modules which are homotopy equivalent, F_k, F_k' the corresponding functions. Then $F_k \overset{d}{\sim} F_k'$ near 0.*

Proof: Due to the symmetry of the requirements it is sufficient to prove that there exists $C > 0$ such that near 0

$$F_k(\lambda) \le F'_k(C\lambda) . \tag{4.4}$$

Let the homotopy equivalence of the complexes M and M' be realized by morphisms $f : M \to M'$ and $g : M' \to M$ such that fg and gf are homotopic to the identity morphisms of M' and M respectively. Let us denote by T the homotopy of gf and the identity morphism 1_M of M.

To prove (4.4) it is sufficient to check that there exist $\lambda_0 > 0$ and $C > 0$ such that for every $\lambda \in (0, \lambda_0)$ and L from (4.2) (satisfying (4.3)) there exists a closed Γ-invariant subspace $L' \subset M'_k/\operatorname{Ker} d_k$ such that $\dim_\Gamma L' = \dim_\Gamma L$ and

$$\|d_k\alpha'\| \le \sqrt{C\lambda}\|\alpha'\| , \qquad \alpha' \in L' . \tag{4.5}$$

We shall take $L' = f_k(L) \bmod \operatorname{Ker} d_k$ and check that there exist $\lambda_0 > 0$ and $C > 0$ depending only on the norms of f_ℓ, g_ℓ and T_ℓ, such that if $\lambda \in (0, \lambda_0)$ then the following is true:

i) L' is closed and f_k is injective on L (hence $\dim_\Gamma L' = \dim_\Gamma L$);

ii) (4.5) is satisfied.

Let us choose an arbitrary $\omega \in L$ and a representative $\omega_1 \in M_k$ of the class ω such that $\omega_1 \perp \operatorname{Ker} d_k$, hence $\|\omega\| = \|\omega_1\|$. So we have

$$\|d_k\omega_1\| = \|d_k\omega\| \le \sqrt{\lambda}\|\omega\| = \sqrt{\lambda}\|\omega_1\| .$$

Now we should consider the element $f_k\omega_1$ and its class modulo $\operatorname{Ker} d_k$ in $M'_k/\operatorname{Ker} d_k$ which we shall denote by α'. So we have $d_k\alpha' = d_kf_k\omega_1 = f_{k+1}d_k\omega_1$, hence

$$\|d_k\alpha'\| = \|f_{k+1}d_k\omega_1\| \le \|f_{k+1}\|\|d_k\omega_1\| \le \sqrt{\lambda}\|f_{k+1}\|\|\omega_1\| . \tag{4.6}$$

Now using the homotopy T we can write

$$\omega_1 = g_kf_k\omega_1 + d_{k-1}T_k\omega_1 + T_{k+1}d_k\omega_1 .$$

We can split $f_k\omega_1$ into the sum

$$f_k\omega_1 = \omega'_1 + \omega'_2$$

with $\omega'_2 \in \operatorname{Ker} d_k$ and $\omega'_1 \perp \operatorname{Ker} d_k$, hence $\|\alpha'\| = \|\omega'_1\|$. Clearly $d_kg_k\omega'_2 = g_{k+1}d_k\omega'_2 = 0$, hence the classes of $g_kf_k\omega_1$ and $g_k\omega'_1$ modulo $\operatorname{Ker} d_k$ coincide, so we have

$$\omega_1 = g_k\omega'_1 + T_{k+1}d_k\omega_1 \quad \bmod \operatorname{Ker} d_k .$$

Since $\omega_1 \perp \operatorname{Ker} d_k$ we obtain

$$\|\omega_1\| \leq \|g_k\omega_1' + T_{k+1}d_k\omega_1\| \leq \|g_k\|\|\omega_1'\| + \|T_{k+1}\|\|d_k\omega_1\| \leq$$
$$\leq \|g_k\|\|\omega_1'\| + \sqrt{\lambda}\|T_{k+1}\|\|\omega_1\| \ .$$

Choosing $\lambda_0 > 0$ such that $\sqrt{\lambda_0}\|T_{k+1}\| < 1/2$ we obtain for $\lambda \in (0, \lambda_0)$

$$\|\omega_1\| \leq \left(1 - \sqrt{\lambda_0}\|T_{k+1}\|\right)^{-1}\|g_k\|\|\omega_1'\| \leq 2\|g_k\|\|\omega_1'\| = 2\|g_k\|\|\alpha'\| \ .$$

It follows that f_k is injective on L and $L' = f_k(L)$ is closed. Moreover (4.6) now implies

$$\|d_k\alpha'\| \leq 2\sqrt{\lambda}\|f_{k+1}\|\|g_k\|\|\alpha'\| = \sqrt{C\lambda}\|\alpha'\| \ ,$$

for every $\alpha' \in L'$ provided $\lambda \in (0, \lambda_0)$ where $0 < \lambda_0 < \left(2\|T_{k+1}\|\right)^{-2}$. This proves (4.5) and concludes the proof of Proposition 4.1. □

5. Homotopy (geometrical setting)

Now we are going to prove a generalization of Theorem 2.1 to the case of Γ-manifolds with piecewise smooth boundaries.

Let us define the category $\mathcal{B}_\Gamma$, whose objects $\mathrm{Ob}(\mathcal{B}_\Gamma)$ are the smooth Γ-manifolds M (i.e. C^∞-manifolds which are equipped by a free action of Γ) with piecewise smooth boundaries and compact quotients M/Γ and morphisms are the homotopy classes of smooth Γ-maps (we could also take homotopy classes of continuous Γ-maps which lead to an equivalent category). Two Γ-manifolds $M_1, M_2 \in \mathrm{Ob}(\mathcal{B}_\Gamma)$ are called Γ-homotopy equivalent if they are isomorphic in $\mathcal{B}_\Gamma$.

If $M \in \mathrm{Ob}(\mathcal{B}_\Gamma)$ then the functions $F_k = F_k(\cdot; M)$ are defined up to the dilatational equivalence (§3).

THEOREM 5.1. *Let $M_1, M_2 \in \mathrm{Ob}(\mathcal{B}_\Gamma)$ and let M_1, M_2 be Γ-homotopy equivalent. Then*

$$F_k(\lambda; M_1) \stackrel{d}{\sim} F_k(\lambda; M_2) \quad \text{near} \quad 0$$

for every $k = 0, 1, \ldots, n$, where $n = \max(\dim M_1, \dim M_2)$.

Remark: Here M_1, M_2 are not required to have the same dimension.

For the proof of Theorem 5.1 we shall use the abstract setting of §4 by considering complexes of $L^2\Gamma$-modules of the form

$$L^2\Lambda^{\cdot}(M) : 0 \longrightarrow L^2\Lambda^0(M) \xrightarrow{d_0} L^2\Lambda^1(M) \longrightarrow \ldots \longrightarrow L^2\Lambda^k(M) \xrightarrow{d_k}$$

$$L^2\Lambda^{k+1}(M) \longrightarrow \ldots \xrightarrow{d_{n-1}} L^2\Lambda^n(M) \longrightarrow 0 \ ,$$

where $M \in \mathrm{Ob}(\mathcal{B}_\Gamma)$, M is equipped with a Γ-invariant Riemannian metric and d_k are usual exterior differentials which are taken on the maximal domain containing all forms $\omega \in L^2\Lambda^k(M)$ such that $d\omega \in L^2\Lambda^{k+1}(M)$ if d is applied in the sense of distributions.

Theorem 5.1 will evidently follow from Proposition 4.1 and the following

THEOREM 5.2. *Let $M_1, M_2 \in \mathrm{Ob}(\mathcal{B}_\Gamma)$ and let M_1, M_2 be Γ-homotopy equivalent. Then the complexes of $L^2\Gamma$-modules $L^2\Lambda^{\cdot}(M_1)$ and $L^2\Lambda^{\cdot}(M_2)$ are homotopy equivalent.*

We shall begin with some preparations on the passage from geometrical homotopy to analytical homotopy of complexes of $L^2\Gamma$-modules. The main idea of this passage is a use of submersions.

Let us recall that a *submersion* $f : M_1 \to M_2$ between two manifolds (with boundaries) is a C^∞-map which has surjective derivative maps on tangent spaces $d_x f : T_x M_1 \to T_{f(x)} M_2$ for all $x \in M_1$. The importance of submersions is clear from the observation that if we have a C^∞-map $f : M_1 \to M_2$ between two Riemannian manifolds which is *not* a submersion and take the induced map $f^* : \Lambda^k(M_2) \to \Lambda^k(M_1)$ then it usually cannot be extended to a bounded linear operator $f^* : L^2\Lambda^k(M_2) \to L^2\Lambda^k(M_1)$ even if $k = 0$ and M_1, M_2 are compact (the simplest example: take $M_1 = M_2 = [0,1]$, $f(x) = x^2$, then $\varphi : x \mapsto x^{-1/3}$ is in L^2 but $f^*\varphi : x \mapsto x^{-2/3}$ is not in L^2). In the case of compact M_1 and M_2 the induced map can be extended to a bounded linear operator $f^* : L^2(M_2) \to L^2(M_1)$ (the case $k = 0$) if and only if f is a submersion. Note that a submersion is not necessarily surjective.

Now let M_1, M_2 be Γ-manifolds. Then f is called a Γ-*submersion* if it is a submersion and a Γ-map. The important point is that if $M_1, M_2 \in \mathrm{Ob}(\mathcal{B}_\Gamma)$ and $f : M_1 \to M_2$ is a Γ-submersion then the corresponding maps $f^* : L^2\Lambda^k(M_2) \to L^2\Lambda^k(M_1)$ are bounded linear operators, hence they constitute a morphism of complexes of $L^2\Gamma$-modules

$$f^* : L^2\Lambda^{\cdot}(M_2) \longrightarrow L^2\Lambda^{\cdot}(M_1) \ .$$

Let us denote $I = [0,1]$ and suppose that there are two homotopic C^∞-maps $f_0, f_1 : M_1 \to M_2$ of manifolds with boundaries. Let us denote the corresponding smooth homotopy by F, i.e. $F : M_1 \times I \to M_2$ is a C^∞-map such that $F(x,j) = f_j(x)$, $j = 0,1$. Then the chain homotopy formula gives for every $\omega \in \Lambda^k(M_2)$

$$f_1^*\omega - f_0^*\omega = (dT + Td)\omega \tag{5.1}$$

where

$$T\omega = \int_0^1 (\partial/\partial t \rfloor F^*\omega)dt \ . \tag{5.2}$$

Now let us suppose that $M_1, M_2 \in \mathcal{B}_\Gamma$ and f_0, f_1 be Γ-submersions. We shall always suppose that Γ acts trivially on I. Then evidently $M_1 \times I \in \mathcal{B}_\Gamma$. Let us suppose that F is a Γ-submersion. Then T defines bounded linear operators

$$T_k : L^2\Lambda^k(M_2) \to L^2\Lambda^{k-1}(M_1)$$

which consititute a homotopy of morphisms f_0^*, f_1^* as morphisms of complexes of $L^2\Gamma$-modules.

Now we need some analytic and geometric lemmas.

LEMMA 5.1. *Let $M \in \mathrm{Ob}(\mathcal{B}_\Gamma)$. Then the complexes $L^2\Lambda^\cdot(M)$ and $L^2\Lambda^\cdot(M \times I)$ are homotopy equivalent as complexes of $L^2\Gamma$-modules.*

Proof: Let $p : M \times I \to M$ be the natural projection. It induces bounded linear maps

$$p^* : L^2\Lambda^k(M) \to L^2\Lambda^k(M \times I)$$

which define a morphism of complexes of $L^2\Gamma$-modules

$$p^* : L^2\Lambda^\cdot(M) \to L^2\Lambda^\cdot(M \times I) \ .$$

Let us also define a morphism

$$J : L^2\Lambda^\cdot(M \times I) \to L^2\Lambda^\cdot(M)$$

by the formula

$$J\omega = \int_0^1 (i_t^*\omega)dt$$

where $i_t : M \to M \times I$ is the following inclusion: $i_t(x) = (x,t)$, t is the natural parameter on I. It is easy to prove that the morphisms p^* and J give the desired homotopy equivalence. This can be seen, e.g. by using the chain homotopy formula (5.1) with $f_0 = \mathrm{id}_{M\times I}$ and $f_{1,t} = i_t \circ p$ with a natural homotopy between these maps and the integration with respect to t over $[0,1]$ leading to bounded homotopy operators. ◻

COROLLARY 5.1. *Let $M \in \mathrm{Ob}(\mathcal{B}_\Gamma)$ and B^ℓ be a closed ball in $\mathbf{R}^\ell$. Then the complexes $L^2\Lambda^\cdot(M)$ and $L^2\Lambda^\cdot(M \times B^\ell)$ are homotopy equivalent as complexes of $L^2\Gamma$-modules.*

Proof: Using the Lipschitz invariance of the homotopy classes we can change B^ℓ to the cube I^ℓ and repeatedly apply Lemma 5.1. □

Later we shall always suppose for the sake of simplicity that B^ℓ is the unit ball with the center at the origin $0 \in \mathbf{R}^\ell$.

Now we shall need the following trivial geometric statement.

LEMMA 5.2. *For every compact C^∞-manifold X with a C^∞-boundary there exist an integer $\ell > 0$ and a submersion $s : B^\ell \to X$ with a given $s(0) = x_0 \in \mathrm{Int}\, X$. This submersion can be chosen smoothly depending on x_0.*

Proof: We can imbed X into $\mathbf{R}^\ell$ and take the orthogonal projection of a small ball $B(x_0, r(x_0))$ in $\mathbf{R}^\ell$ centered at x_0 with radius $r(x_0)$ to a neighbourhood of x_0 in X, composing it with an affine map of B^ℓ on $B(x_0, r(x_0))$ ($r(x_0)$ and the affine map can be chosen smoothly depending on x_0 for $x_0 \in \mathrm{Int}\, X$). □

LEMMA 5.3. *Let $M_1, M_2 \in \mathrm{Ob}(\mathcal{B}_\Gamma)$ have smooth boundaries and $f : M_1 \to \mathrm{Int}\, M_2$ be a smooth Γ-map. Then there exist an integer $\ell \geq 0$ and a smooth Γ-map $F : M_1 \times B^\ell \to \mathrm{Int}\, M_2$ such that F is a submersion and $f = F \circ i$ where $i : M_1 \to M_1 \times B^\ell$ is the natural inclusion, i.e. $i(x) = (x, 0)$, $x \in M_1$.*

Proof: Let us denote $X_j = M_j/\Gamma$, and introduce the notations $\pi_j : M_j \to X_j$ for the natural projections, $j = 1, 2$. Due to Lemma 5.2 we can choose a submersion $s : X_1 \times B^\ell \to X_2$ such that $s(x, 0) = \pi_2 f(\widetilde{x})$, $x \in X_1$, $\widetilde{x} \in M_1$, $\pi_1(\widetilde{x}) = x$. Then there exists a unique smooth map $F : M_1 \times B^\ell \to M_2$ such that $F \circ i = f$ and $\pi_2 \circ F = s \circ (\pi_1 \times Id)$, i.e. the diagram

$$\begin{tikzcd} M_1 \arrow[r, "i"] \arrow[dr, "f"] & M_1 \times B^\ell \arrow[r, "(\pi_1 \times Id)"] \arrow[d, "F"] & X_1 \times B^\ell \arrow[d, "s"] \\ & M_2 \arrow[r, "\pi_2"] & X_2 \end{tikzcd}$$

is commutative. It is easy to check that F satisfies all the conditions. □

LEMMA 5.4. *Let $M_1, M_2 \in \mathrm{Ob}(\mathcal{B}_\Gamma)$ have smooth boundaries and $f_0, f_1 : M_1 \to M_2$ be smooth Γ-submersions which are homotopic in the class of all (smooth or continuous) Γ-maps. Then there exist an integer $\ell \geq 0$ and a homotopy $F : M_1 \times B^\ell \times I \to M_2$ such that F is a smooth Γ-submersion (the action of Γ on $B^\ell \times I$ is trivial) and $F(x, b, j) = f_j(x)$, $j = 0, 1$, for all $x \in M$, $b \in B^\ell$.*

Proof: Modifying f_0, f_1 near $f_0^{-1}(\partial M_2), f_1^{-1}(\partial M_2)$ respectively, we may suppose that $f_j(M_1) \subset \operatorname{Int} M_2$, $j = 0, 1$. Let $G : M_1 \times I \to M_2$ be a smooth homotopy of f_0 and f_1 in the class of all smooth Γ-maps. Using Lemma 5.3 we can construct a smooth Γ-submersion $F : M_1 \times B^\ell \times I \to M_2$ such that $F(x, 0, t) = G(t, x)$ for all $x \in M_1$, $t \in I$. Since f_0, f_1 are submersions, the maps $F(\cdot, b, t) : M_1 \to M_2$ have surjective derivatives $d_x F(x, b, t) : T_x M_1 \to T_{F(x,b,t)} M_2$ for all $x \in M_1$ provided b is close to 0 and t is close to 0 or 1. Using the homotheties of the ball B^ℓ it is easy to modify F near $t = 0$ and $t = 1$ to satisfy all the conditions of Lemma 5.4. □

Proof of Theorem 5.2: Using Lipschitz homeomorphisms we can suppose without loss of generality that M_1, M_2 have smooth boundaries. Let $f : M_1 \to M_2$ and $g : M_2 \to M_1$ be smooth Γ-maps which define an isomorphism of M_1 and M_2 in $\mathcal{B}_\Gamma$ (a Γ-homotopy equivalence). Due to Lemma 5.3 we can find an integer $\ell \geq 0$ and submersions $\widehat{f} : M_1 \times B^\ell \to M_2$ and $\widehat{g} : M_2 \times B^\ell \to M_1$ such that $\widehat{f}$ is homotopic to $f \circ p_1$ and $\widehat{g}$ is homotopic to $g \circ p_2$ where $p_j : M_j \times B^\ell \to M_j$ ($j = 1, 2$) are the natural projections.

Let us construct morphisms

$$J_j : L^2\Lambda^\cdot(M_j \times B^\ell) \to L^2\Lambda^\cdot(M_j) , \qquad j = 1, 2 ,$$

as in the proof of Lemma 5.1 and Corollary 5.1 so that p_j^* and J_j give a Γ-homotopy equivalence of complexes of $L^2\Gamma$-modules $L^2\Lambda^\cdot(M_j)$ and $L^2\Lambda^\cdot(M_j \times B^\ell)$, $j = 1, 2$. Then we can define the following morphisms of complexes of $L^2\Gamma$-modules:

$$F = J_2 \circ \widehat{g}^* : L^2\Lambda^\cdot(M_1) \to L^2\Lambda^\cdot(M_2) ,$$
$$G = J_1 \circ \widehat{f}^* : L^2\Lambda^\cdot(M_2) \to L^2\Lambda^\cdot(M_1) .$$

Using Lemma 5.4 it is easy to check that F and G constitute a Γ-homotopy equivalence of complexes of $L^2\Gamma$-modules. The required chain homotopy is constructed from the homotopy operators like (5.2) by integrating over parameters involved in the construction of J_1 and J_2. □

Remark 1: Let us describe a definition of the functions $F_k(\lambda)$ (up to dilatational equivalence near 0) for CW-complexes M with a free action of a discrete group Γ and with a compact quotient $X = M/\Gamma$. First we replace X by a homotopy equivalent simplicial complex and embed X into a vector space $\mathbf{R}^N$. Then we can replace X by a homotopy equivalent closed neighbourhood of X in $\mathbf{R}^N$ which is a C^∞-manifold with a C^∞-boundary. After that we have the corresponding covering manifold M with a free action of Γ such that $X = M/\Gamma$. This M is a C^∞-manifold with C^∞-boundary. So we can consider the de Rham L^2-complex on M and take the corresponding functions F_k which have classes of dilatational equivalence near 0 which do not depend on the arbitrary elements of the construction above, due to the homotopy invariance of these classes.

So for any k the dilatational equivalence class of functions F_k is well defined for every CW-complex M with a free action of the discrete group Γ such that $X = M/\Gamma$ is compact. It is clear from this construction that $F_k(\lambda) < \infty$ for all $\lambda > 0$. In particular this construction works for all Lipschitz Γ-manifolds (and not only those having a C^∞-structure). We do not prove now that for Lipschitz manifolds this topological construction gives the same result as the analytical definition but we hope to do it in the next paper.

Remark 2: Let us consider a Γ-invariant simplicial complex which is obtained by taking a triangulation of X and then lifting it to a triangulation of M. Taking then the corresponding dual combinatorial L^2-complex of simplicial L^2-cochains we obtain a discrete approximation of the earlier considered complex of L^2-forms. Now we can define the corresponding combinatorial functions $F_k^c(\lambda)$. Our technique allows us to prove that their dilatational equivalence class near 0 coincides with that of the earlier defined functions $F_k(\lambda)$ by proving that the combinatorial L^2-complex of cochains and the de Rham L^2-complex are homotopy equivalent in the sense of §4; this was done by A. Efremov [E]. Note that in a recent paper J. Lott [L] proves the inequality $\underline{\alpha}_k^c \leq \underline{\alpha}_k$ for the corresponding combinatorial and analytical numbers. In fact these numbers coincide as follows from the coincidence of the dilatational equivalence classes of the functions F_k^c and F^k.

Appendix

Decay exponents and the Laplace transform.

In this Appendix we investigate connections between the behaviour of a (non-strictly) increasing function $F : \mathbb{R} \to \mathbb{R}_+ = [0, +\infty]$ near 0 (we shall always suppose that $F(\lambda) = 0$ if $\lambda < 0$) and the behaviour of its Laplace transform

$$\theta(t) = \int_{\mathbb{R}} e^{-\lambda t} dF(\lambda) = \lim_{N \to +\infty} \int_{-0}^{N} e^{-\lambda t} dF(\lambda) . \tag{A.1}$$

It was proved in [ES] by some elementary considerations that if $\bar{b} = F(+0)$ (i.e. $\bar{b}$ is the jump of F at 0) and $\theta(t) < \infty$ for all $t > 0$ then the estimates $F(\lambda) - \bar{b} \asymp \lambda^\alpha$, i.e.

$$C^{-1}\lambda^\alpha \leq F(\lambda) - \bar{b} \leq C\lambda^\alpha \quad \text{near} \quad 0$$

with some $\alpha > 0$ (and a constant $C > 0$) are fulfilled if and only if $\theta(t) - \bar{b} \asymp t^{-\alpha}$, i.e.

$$C^{-1}t^{-\alpha} \le \theta(t) - \bar{b} \le Ct^{-\alpha} \quad \text{near infinity}$$

(possibly with another constant C). Here we use similar reasoning to establish other connections between asymptotics of $F(\lambda)$ near 0 and $\theta(t)$ near infinity. Note that the reasoning that we use is close to the standard arguments which are used to prove Tauberian theorems of Karamata's type for the Laplace transform.

The Laplace transform (A.1) is finite for every $t > 0$ if and only if F satisfies the subexponential estimate

$$F(\lambda) = \mathcal{O}(e^{\varepsilon\lambda}) \quad \text{for all} \quad \varepsilon > 0\ . \tag{A.2}$$

Indeed, integrating by parts in (A.1) gives

$$\int_{-0}^{N} e^{-\lambda t} dF(\lambda) = e^{-tN}F(N) + t\int_{-0}^{N} e^{-\lambda t}F(\lambda)d\lambda\ ,$$

hence $\left|e^{-tN}F(N)\right| \le C_t$ for all $t > 0$ which is equivalent to (A.2). Furthermore if (A.2) holds then the same calculation shows that

$$\theta(t) = t\int_{\mathbb{R}} e^{-\lambda t}F(\lambda)d\lambda\ , \qquad t > 0\ , \tag{A.3}$$

where the integral converges absolutely. So from now on we shall always suppose that (A.2) is satisfied. Denote $\bar{b} = F(+0)$. Then

$$\lim_{t\to+\infty} \theta(t) = \bar{b}\ . \tag{A.4}$$

Indeed subtracting $\bar{b}H(\lambda)$ from $F(\lambda)$ (here H is the Heaviside function) we reduce the proof to the case when $\bar{b} = 0$. Then (A.4) follows from the dominant convergence theorem. The main result of this Appendix is the following elementary

PROPOSITION A.1. *(i) In the notations above we have*

$$\liminf_{\lambda\to+0} \frac{\log\left(F(\lambda) - \bar{b}\right)}{\log\lambda} = \liminf_{t\to+\infty} \frac{-\log\left(\theta(t) - \bar{b}\right)}{\log t}\ ; \tag{A.5}$$

(ii) the estimate $F(\lambda) - \bar{b} = O(\lambda^\delta)$ holds near 0 with some $\delta > 0$ if and only if $\theta(t) - \bar{b} = O(t^{-\delta})$ near infinity;

(iii) the following inequality is true

$$\limsup_{\lambda\to+0} \frac{\log\left(F(\lambda) - \bar{b}\right)}{\log\lambda} \ge \limsup_{t\to+\infty} \frac{-\log\left(\theta(t) - \bar{b}\right)}{\log t}\ ; \tag{A.6}$$

(iv) if one of the two equivalent conditions in (ii) is satisfied then in fact there is equality in (A.6).

Proof: 1) By subtraction of $\bar{b}H(\lambda)$ from $F(\lambda)$ we reduce the proof to the case $\bar{b} = 0$ which we shall suppose from now on.

Let us denote the left-hand side of (A.5) by $\underline{\alpha}$ and the right-hand side by $\underline{\alpha}'$. We have to prove that $\underline{\alpha} = \underline{\alpha}'$. First we shall prove that $\underline{\alpha}' \geq \underline{\alpha}$. If $\underline{\alpha} = 0$ then this is evidently true. Now let us suppose that $0 < \underline{\alpha} < \infty$. For every $\varepsilon > 0$ we have

$$F(\lambda) \leq \lambda^{\underline{\alpha}-\varepsilon} \quad \text{if} \quad \lambda \in (0, \lambda_0) ,$$

where $\lambda_0 = \lambda_0(\varepsilon)$. Using (A.3) we obtain that

$$\theta(t) \leq t \int_0^{\lambda_0} \lambda^{\underline{\alpha}-\varepsilon} e^{-\lambda t} d\lambda + t \int_{\lambda_0 - 0}^{\infty} e^{-\lambda t} F(\lambda) d\lambda .$$

The second term on the right-hand side is estimated here as $\mathcal{O}(e^{-\mu t})$ for any $\mu < \lambda_0$ due to (A.2). The first one is estimated by

$$\int_0^{\infty} t \lambda^{\underline{\alpha}-\varepsilon} e^{-\lambda t} d\lambda = \Gamma(\underline{\alpha} - \varepsilon + 1) t^{-\underline{\alpha}+\varepsilon} .$$

Therefore $\theta(t) = O(t^{-\underline{\alpha}+\varepsilon})$.

It follows that
$$\frac{-\log \theta(t)}{\log t} \geq -(\underline{\alpha} - \varepsilon)$$

for large t, hence $\underline{\alpha}' \geq \underline{\alpha}$.

The same reasoning shows that $\underline{\alpha} \geq \alpha > 0$ with $\alpha < \infty$ implies that $\underline{\alpha}' \geq \alpha$. It follows that $\underline{\alpha} = \infty$ implies $\underline{\alpha}' = \infty$ and thus the inequality $\underline{\alpha}' \geq \underline{\alpha}$ is proved for all cases.

Let us prove the inverse inequality $\underline{\alpha} \geq \underline{\alpha}'$. Again this is evident if $\underline{\alpha}' = 0$. Now suppose that $0 < \underline{\alpha}' < \infty$. Note first that

$$\theta(t) \leq t^{-(\underline{\alpha}'-\varepsilon)} \quad \text{if} \quad t \geq t_0(\varepsilon) .$$

Now the Chebyshev inequality gives.

$$F(\lambda - 0) \leq e^{\lambda t} \theta(t) , \qquad \lambda \in \mathbf{R} , \ t > 0 ;$$

therefore
$$F(\lambda) \leq e^{\lambda t} t^{-(\underline{\alpha}'-\varepsilon)} \quad \text{if} \quad t \geq t_0(\varepsilon) .$$

Taking $t = \lambda^{-1}$ we obtain for small λ

$$F(\lambda) \leq C \lambda^{(\underline{\alpha}'-\varepsilon)} ,$$

hence

$$\frac{\log F(\lambda)}{\log \lambda} \geq \underline{\alpha}' - \varepsilon \quad \text{if} \quad 0 < \lambda \leq \lambda_0(\varepsilon) \; .$$

It follows that $\underline{\alpha} \geq \underline{\alpha}'$ which proves (A.5) provided $\underline{\alpha}' < \infty$. But the same reasoning shows that if $\underline{\alpha}' \geq \alpha$ with $0 < \alpha < \infty$ then $\underline{\alpha} \geq \alpha$. It follows that $\underline{\alpha} \geq \underline{\alpha}'$ for all cases (including $\underline{\alpha}' = \infty$). So (A.5) is proved in full generality.

2) Part *(ii)* of the statement of Proposition A.1 is proved by the same arguments as in the first part of the proof.

3) Now let us denote the left-hand side of (A.6) by $\overline{\alpha}$ and the right-hand side by $\overline{\alpha}'$. As before we assume $\overline{b} = 0$. The inequality (A.6) is evidently fulfilled if $\overline{\alpha} = \infty$. So we shall suppose without loss of generality that $\overline{\alpha} < \infty$. Then we obtain

$$F(\lambda) \geq \lambda^{(\overline{\alpha}+\varepsilon)} \; , \qquad 0 < \lambda < \lambda_0(\varepsilon) \; ,$$

hence

$$\begin{aligned}\theta(t) &= t\int_0^{\lambda_0(\varepsilon)} e^{-\lambda t} F(\lambda) d\lambda + t\int_{\lambda_0(\varepsilon)}^{\infty} e^{-\lambda t} F(\lambda) d\lambda \geq \\ &\geq t\int_0^{\infty} \lambda^{(\overline{\alpha}+\varepsilon)} e^{-\lambda t} d\lambda - t\int_{\lambda_0(\varepsilon)}^{\infty} \lambda^{(\overline{\alpha}+\varepsilon)} e^{-\lambda t} d\lambda + t\int_{\lambda_0(\varepsilon)}^{\infty} e^{-\lambda t} F(\lambda) d\lambda = \\ &= \Gamma(\overline{\alpha}+\varepsilon+1) t^{-(\overline{\alpha}+\varepsilon)} + O(e^{-\delta t})\end{aligned}$$

if $0 < \delta < \lambda_0(\varepsilon)$. It follows that

$$\frac{-\log \theta(t)}{\log t} \leq \overline{\alpha} + \varepsilon \quad \text{if} \quad t \geq t_0(\varepsilon) \; ,$$

hence $\overline{\alpha}' \leq \overline{\alpha}$, which proves *(iii)*.

4) Now let one of the two equivalent conditions in *(ii)* be satisfied. Let us prove that $\overline{\alpha} \leq \overline{\alpha}'$ in the notations of the previous part of the proof. This is evident if $\overline{\alpha}' = \infty$, so we shall assume that $\overline{\alpha}' < \infty$. Let us suppose first that also $\overline{\alpha}' > 0$. We shall use the inequality

$$F(\lambda) \geq \theta(t) - e^{-\lambda t/2}\theta(t/2) \; , \qquad t > 0 \; , \; \lambda > 0 \; , \tag{A.7}$$

which is proved in [ES] by the following use of the Chebyshev inequality

$$\begin{aligned}\theta(t) &= \int_{-0}^{\infty} e^{-\lambda t} dF(\lambda) \leq \int_{-0}^{\lambda-0} e^{-\lambda t} dF(\lambda) + \int_{\lambda-0}^{\infty} e^{-\lambda t} dF(\lambda) \leq \\ &\leq F(\lambda) + e^{-\lambda t/2}\int_{\lambda-0}^{\infty} e^{-\lambda t/2} dF(\lambda) \leq F(\lambda) + e^{-\lambda t/2}\int_{-0}^{\infty} e^{-\lambda t/2} dF(\lambda) = \\ &= F(\lambda) + e^{-\lambda t/2}\theta(t/2) \; .\end{aligned}$$

(We have repeated this proof for the convenience of readers.) Now using the estimate

$$\theta(t) \geq t^{-(\overline{\alpha}'+\varepsilon)} \quad \text{if} \quad t \geq t_0(\varepsilon) ,$$

we obtain from (A.7) that

$$F(\lambda) \geq t^{-(\overline{\alpha}'+\varepsilon)} - e^{-\lambda t/2} t^{-\delta} \quad \text{if} \quad t \geq t_0(\varepsilon) .$$

Now let us take $t = -M\lambda^{-1}\log\lambda$ where M will be chosen sufficiently large. Then we obtain

$$F(\lambda) \geq M|\log\lambda|^{-(\overline{\alpha}'+\varepsilon)}\lambda^{\overline{\alpha}'+\varepsilon} - M^{-\delta}|\log\lambda|^{-\delta}\lambda^{M/2+\delta} , \qquad \lambda \geq \lambda_0(\varepsilon, M) .$$

We should choose M such that $M/2+\delta > \overline{\alpha}'+\varepsilon$. Then we evidently obtain

$$F(\lambda) \geq c\lambda^{\overline{\alpha}'+\varepsilon'} , \qquad 0 < \lambda < \lambda_0(\varepsilon') ,$$

for any $\varepsilon' > \varepsilon$ with $c = c(\varepsilon') > 0$. It follows that

$$\frac{\log F(\lambda)}{\log\lambda} \leq \overline{\alpha}' + \varepsilon' \quad \text{near} \quad 0 .$$

We can make $\varepsilon' > 0$ arbitrarily small, hence $\overline{\alpha} \leq \overline{\alpha}'$.

The same reasoning shows that if $\alpha \geq \overline{\alpha}'$ and $0 < \alpha < \infty$ then $\overline{\alpha} \leq \alpha$. It follows that if $\overline{\alpha}' = 0$ then $\overline{\alpha} = 0$ too. This completes the proof of Proposition A.1. □

COROLLARY A.1. *For every α with $0 < \alpha < \infty$ the equalities*

$$\lim_{\lambda\to+0} \frac{\log\left(F(\lambda) - \overline{b}\right)}{\log\lambda} = \alpha$$

and

$$\lim_{\lambda\to+\infty} \frac{-\log\left(\theta(t) - \overline{b}\right)}{\log t} = \alpha$$

are equivalent.

Now let $F_1(\lambda), F_2(\lambda)$ be such functions as $F(\lambda)$ before and $F(\lambda) = F_1(\lambda)+F_2(\lambda)$. Then we can consider the numbers, corresponding to F, F_1, F_2. We shall denote them with the corresponding subscripts (1 and 2 for F_1 and F_2 respectively and without subscripts for F). So we have the set of numbers

$$\underline{\alpha}, \underline{\alpha}_1, \underline{\alpha}_2, \overline{\alpha}, \overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}', \overline{\alpha}'_1, \overline{\alpha}'_2$$

and we want to establish connections between them

PROPOSITION A.2. *The following relations are fulfilled:*

$$\underline{\alpha} = \min(\underline{\alpha}_1, \underline{\alpha}_2) , \tag{A.8}$$

$$\overline{\alpha} \leq \min(\overline{\alpha}_1, \overline{\alpha}_2) , \tag{A.9}$$

$$\overline{\alpha}' \leq \min(\overline{\alpha}'_1, \overline{\alpha}'_2) . \tag{A.10}$$

Proof: 1) Without loss of generality we may suppose that $F_1(+0) = F_2(+0) = 0$. We can also suppose that $\underline{\alpha}_1 \le \underline{\alpha}_2$. Let us prove first that

$$\underline{\alpha} \ge \min(\underline{\alpha}_1, \underline{\alpha}_2) = \underline{\alpha}_1 \ . \tag{A.11}$$

This inequality is trivial if $\underline{\alpha}_1 = 0$. Let us suppose first that $0 < \underline{\alpha}_1 < \infty$. Then we have

$$F_1(\lambda) \le \lambda^{\underline{\alpha}_1 - \varepsilon} \ , \quad F_2(\lambda) \le \lambda^{\underline{\alpha}_2 - \varepsilon} \le \lambda^{\underline{\alpha}_1 - \varepsilon} \quad \text{if} \quad \lambda \in \big(0, \lambda_0(\varepsilon)\big) \ .$$

Therefore near 0

$$F(\lambda) \le 2\lambda^{\underline{\alpha}_1 - \varepsilon} \ ,$$

hence

$$\frac{\log F(\lambda)}{\log \lambda} \ge \underline{\alpha}_1 - 2\varepsilon \quad \text{if} \quad \lambda \in \big(0, \lambda_0(\varepsilon)\big)$$

and (A.11) follows.

Now let $\underline{\alpha}_1 = \underline{\alpha}_2 = +\infty$. Then the same reasoning works if we change $\underline{\alpha}_1$ to every $\alpha \in (0, \infty)$. It follows that $\underline{\alpha} = +\infty$ which proves (A.8) in this case.

Let us prove the inverse inequality

$$\underline{\alpha} \le \min(\underline{\alpha}_1, \underline{\alpha}_2) = \underline{\alpha}_1 \ .$$

The inequality $F(\lambda) \ge F_1(\lambda)$ implies that $\underline{\alpha} \le \underline{\alpha}_1$ as required.

2) Since $F(\lambda) \ge F_j(\lambda)$, $j = 1, 2$, we obtain $\underline{\alpha} \le \underline{\alpha}_j$, $j = 1, 2$, hence (A.9). The inequality (A.10) is proved by the same arguments. □

COROLLARY A.2. *If* $\underline{\alpha}_1 = \overline{\alpha}_1$, $\underline{\alpha}_2 = \overline{\alpha}_2$ *then*

$$\underline{\alpha} = \overline{\alpha} = \min(\underline{\alpha}_1, \underline{\alpha}_2) \ . \tag{A.11}$$

Remark: Notice that the relation between the spectral density and the decay of the heat flow has been known to Jeff Cheeger for quite a while. We present our proof as Cheeger has never published his.

References

[A] M.F. ATIYAH, Elliptic operators, discrete groups and von Neumann algebras, Asterisque, 32-33 (1976), 43-72.

[B] R. BROOKS, The fundamental group and the spectrum of the Lapacian, Comment. Math. Helv. 56 (1981), 581-598.

[Ch] P. CHERNOFF, Essential self-adjointness of powers of generators of hyperbolic equations, J. Funct. Anal. 12 (1973), 401-414.

[Co] J.M. COHEN, Von Neumann dimension and the homology of covering spaces, Quart. J. Math. 30 (1979), 133-142.

[D] J. DODZIUK, De Rham-Hodge theory for L^2-cohomology of infinite coverings, Topology 16 (1977), 157-165.

[E] A.V. EFREMOV, Combinatorial and analytic Novikov-Shubin invariants, preprint, 1991.

[ES] D.V. EFREMOV, M.A. SHUBIN, Spectrum distribution function and variational principle for automorphic operators on hyperbolic space, Séminaire Equations aux Dérivées Partielles, Ecole Polytechnique, Centre de Mathématiques, Exposé VII, 1988-1989.

[F] D. FRIED, Analytic torsion and closed geodesics on hyperbolic manifolds, Invent. Math. 84 (1986), 523-540.

[G] M. GROMOV, Groups of polynomial growth and expanding maps, Publ. Math. IHES 53 (1981), 53-73.

[K] K. KODAIRA, Harmonic fields in Riemannian manifolds, Ann. of Math. 50 (1949), 587-665.

[L] J. LOTT, Heat kernels on covering spaces and topological invariants, Preprint IHES/M/90/47, June 1990.

[M] R.J. MIATELLO, On the Plancherel measure for linear Lie groups of rank one, Manuscripta Math. 29 (1979), 249-276.

[NS1] S.P. NOVIKOV, M.A. SHUBIN, Morse theory and von Neumann II_1-factors, Doklady Akad. Nauk SSSR 289 (1986), 289-292.

[NS2] S.P. NOVIKOV, M.A. SHUBIN, Morse theory and von Neumann invariants on non-simply connected manifolds, Uspekhi Nauk 41, 5 (1986), 222-223 (in Russian).

[RS] M. REED, B. SIMON, Methods of Modern Mathematical Physics, I, Acad. Press, New York e.a., 1972.

[S1] M.A. SHUBIN, Theorems on the equality of the spectra of a pseudo-differential almost periodic operator in $L^2(\mathbb{R}^n)$ and $B^2(\mathbb{R}^n)$, Sibirsk. Mat. Zh. 17 (1976), 200-215.

[S2] M.A. SHUBIN, The spectral theory and the index of elliptic operators with almost periodic coefficients, Russian Math. Surveys 34, 2 (1979), 109-157.

[T] N. TELEMAN, The index of signature operators on Lipschitz manifolds, Publications IHES 58 (1983), 39-78.

[Va] N. Varopoulos, Random walks and Brownian motion on manifolds, Symposia Mathematica XXIX (1987), 97-109.
[Vi] S.M. Vishik, Analogs of Selberg's ζ-function, Func. Anal. and Appl. 9, 3 (1975), 256-257.

M. Gromov
IHES
35 route de Chartres
91440 Bures sur Yvette
France

M.A. Shubin
Institute of New Technologies
Kirovogradskaya, 11
Moscow, 113587
USSR

Submitted: July 1990

M.A. Shubin's address as of September 1, 1991:
Department of Mathematics
M.I.T.
Cambridge, MA 02139
USA