Équations aux dérivées partielles/*Partial Differential Equations*

# The Riemann-Roch theorem for general elliptic operators

Mikhaël GROMOV and Mikhail A. SHUBIN

***Abstract*** – The classical Riemann-Roch theorem is generalized to the solution of general elliptic equations with isolated singularities on an arbitrary compact manifold.

## Le théorème de Riemann-Roch pour les opérateurs elliptiques généraux

***Résumé*** – *Le théorème de Riemann-Roch classique est généralisé pour les solutions d'équations elliptiques générales avec singularités isolées sur une variété compacte arbitraire.*

***Version française abrégée*** – Soient X une variété $C^\infty$ fermée, $\dim X = n \geqq 2$, E et F des fibrés vectoriels sur X, $q = \dim_{\mathbb{C}} E_x = \dim_{\mathbb{C}} F_x$ la dimension de leurs fibres, $\Gamma(U, E)$ l'espace des sections $C^\infty$ de E sur un ouvert $U \subset X$. Soit $A : \Gamma(X, E) \to \Gamma(X, F)$ un opérateur elliptique différentiel d'ordre $d$, ind A son indice qui peut être calculé par le théorème de l'indice d'Atiyah-Singer. Soient $\Omega(X)$ le fibré des densités complexes sur X, $E^* = \mathrm{Hom}(E, \Omega(X))$, $\langle .,. \rangle$ la dualité bilinéaire naturelle $\Gamma(X, E) \times \Gamma(X, E^*) \to \mathbb{C}$ et $A^t$ l'opérateur transposé défini par les dualités $\langle .,. \rangle$ dans E et F, ainsi $A^t : \Gamma(X, F^*) \to \Gamma(X, E^*)$ et

$$\langle A u, v \rangle = \langle u, A^t v \rangle, \qquad u \in \Gamma(X, E), \quad v \in \Gamma(X, F^*).$$

Introduisons un diviseur $\mu = x_1^{p_1} x_2^{p_2} \ldots x_m^{p_m}$, $x_i \in X$, $x_i \neq x_j$ si $i \neq j$, $p_i \in \mathbb{Z} \setminus \{0\}$, $\mu^{-1} = x_1^{-p_1} x_2^{-p_2} \ldots x_m^{-p_m}$ est le diviseur dual. Le degré de μ est défini par

$$d(\mu) = \sum_{1 \leqq i \leqq m} \operatorname{sign} p_i \left[ \binom{|p_i| + n - 1}{n} - \binom{|p_i| + n - 1 - d}{n} \right]$$

où $\binom{N}{n} = N!/n!(N-n)!$ si $N \geqq n$ et 0 sinon.

Soient $\operatorname{supp} \mu = \{x_1, \ldots, x_m\}$, $\lambda = \prod_{p-i \geqq 0} x_i^{p_i}$, $\nu = \prod_{p_i \leqq 0} x_i^{p_i}$. Soient

$$\begin{aligned} L(\mu, A) = \{ u \mid u \in \Gamma(X - \operatorname{supp} \nu, E),\ A u = 0 \text{ dans } X - \operatorname{supp} \nu; \\ u(x) = o(|x - x_i|^{d-n-|p_i|}) \text{ lorsque } x \to x_i,\ x_i \in \operatorname{supp} \nu; \\ u(x) = O(|x - x|^{p_i}) \text{ lorsque } x \to x_i,\ x \in \operatorname{supp} \lambda \}, \end{aligned}$$

$$r(\mu, A) = \dim_{\mathbb{C}} L(\mu, A).$$

THÉORÈME 1. – $r(\mu, A) = \operatorname{ind} A - q d(\mu) + r(\mu^{-1}, A^t)$.

Le théorème de Riemann-Roch classique est un cas particulier si X est une courbe algébrique complexe non-singulière, $A = \bar{\partial} : C^\infty(X) \to \Lambda^{0,1}(X)$. Le cas de l'opérateur de Laplace scalaire sur une variété riemannienne a été considéré par N. S. Nadirashvili [1].

Donnons quelques applications simples :

COROLLAIRE 3. – $r(\mu, A) \geqq \operatorname{ind} A - q d(\mu)$.

PROPOSITION 4. – *Supposons que* $A^t$ *a la propriété de prolongement unique, c'est-à-dire si* $A^t U = 0$ *et* $u$ *a un zéro d'ordre infini en* $x_0$, *alors* $u = 0$ *dans un voisinage de* $x_0$. *Fixons* $x_1, \ldots, x_m \in X$ *et* $N_0 > 0$.

---

Note présentée par Mikhaël GROMOV.

0764-4442/92/03140363 $ 2.00 

*Alors il existe* $N>0$ *tel que, si* $\sum_{p_i>0} p_i \leqq N_0$ *et* $\sum_{p_i<0} |p_i| \geqq N$, *alors*

$$r(\mu, A) = \operatorname{ind} A - q d(\mu).$$

PROPOSITION 5. — *Soit* A *une matrice* $(q \times q)$ *opérateur elliptique d'ordre* $d$ *définie dans un voisinage de* 0 *dans* $\mathbb{R}^n$. *Soit* $\mathscr{E}_0^{(k)}$ *l'espace des germes de fonctions* $f: U \to \mathbb{C}^q$ (U *est un voisinage de* 0 *dans* $\mathbb{R}^n$) *telles que* $\partial^d f(0) = 0$ *pour* $|d| \leqq k$.

*Alors l'application* $A = \mathscr{E}_0^{(k+d)} \to \mathscr{E}_0^{(k)}$ *est surjective.*

---

1. INTRODUCTION. — The classical Riemann-Roch theorem for non-singular complex algebraic curves has been generalized in different ways to $n$-dimensional manifolds. The most known generalizations are the Riemann-Roch-Hirzebruch theorem and the Riemann-Roch-Grothendieck theorem in the algebraic geometry. In this Note we suggest a generalization motivated by the theory of solutions of elliptic equations with point singularities. The case of the scalar Laplacian was considered in a beautiful paper by N. S. Nadirashvili [1], which we took as a starting point. But our proof is simpler than that in [1] due to a use of duality arguments. We also give a number of applications including those which are similar to those of the classical Riemann-Roch theorem, and a local solvability result with an additional condition on the order of zero of the solution at the given point. Further applications will be given in a subsequent detailed paper.

2. NOTATIONS AND THE MAIN RESULT. — Let X be a compact $C^\infty$-manifold, $\dim X = n \geqq 2$, E and F complex vector bundles over X, $q = \dim_{\mathbb{C}} E_x = \dim_{\mathbb{C}} F_x$ the dimension of their fibres, $\Gamma(U, E)$ the space of the $C^\infty$-sections of E over an open set $U \subset X$, $A: \Gamma(X, E) \to \Gamma(X, F)$ an elliptic differential operator of order $d$, ind A its index which can be calculated by the Atiyah-Singer index formula, $\Omega(X)$ the bundle of complex densities on X, $E^* = \operatorname{Hom}(E, \Omega(X))$, $\langle ., . \rangle$ the natural bilinear pairing $\Gamma(X, E) \times \Gamma(X, E^*) \to \mathbb{C}$ and $A'$ the transposed operator to A defined by the pairings $\langle ., . \rangle$ in E and F, *i.e.* $A': \Gamma(X, F^*) \to \Gamma(X, E^*)$ and

$$\langle A u, v \rangle = \langle u, A' v \rangle, \qquad u \in \Gamma(X, E), \quad v \in \Gamma(X, F^*).$$

Let us introduce a divisor $\mu = x_1^{p_1} x_2^{p_2}, \ldots, x_m^{p_m}$, $x_i \in X$, $x_i \neq x_j$ if $i \neq j$, $p_i \in \mathbb{Z} \setminus \{0\}$, and $\mu^{-1} = x_1^{-p_1} x_2^{-p_2} \ldots x_m^{-p_m}$ is the dual divisor. The degree of $\mu$ is defined as

$$d(\mu) = \sum_{1 \leqq i \leqq m} \operatorname{sign} p_i \left[ \binom{|p_i| + n - 1}{n} - \binom{|p_i| + n - 1 - d}{n} \right]$$

where $\binom{N}{n} = N!/n!(N-n)!$ if $N \geqq n$ and 0 otherwise. Let $\operatorname{supp} \mu = \{x_1, \ldots, x_m\}$, $\lambda = x_1^{p_1^+} \ldots x_m^{p_m^+}$, $\nu = x_1^{p_1^-} \ldots x_m^{p_m^-}$, where $p_i^+ = \max(p_i, 0)$, $p_i^- = \min(p_i, 0)$, so $\lambda$ and $\nu$ are positive and negative part of $\mu$ respectively, $\operatorname{supp} \mu = \operatorname{supp} \lambda \cup \operatorname{supp} \nu$, $\operatorname{supp} \lambda \cap \operatorname{supp} \nu = \varnothing$.

Now let us introduce the space of solutions of the equation $A u = 0$ with the prescribed point singularities ("poles") on supp $\nu$ and zeros of prescribed orders on supp $\lambda$:

$$L(\mu, A) = \{ u \mid u \in \Gamma(X - \operatorname{supp} \nu, E),\ A u = 0 \text{ on } X - \operatorname{supp} \nu;$$

$$u(x) = o(|x - x_i|^{d-n-|p_i|}) \text{ as } x \to x_i,\ x_i \in \operatorname{supp} \nu;$$

$$u(x) = O(|x - x|^{p_i}) \text{ as } x \to x_i,\ x \in \operatorname{supp} \lambda \}.$$

The latter condition can be also written as $j_{x_i}^{p_i-1} u=0$, $x \in \operatorname{supp} \lambda$, where $j_{x_i}^{p_i-1} u$ is the jet of order $p_i^{-1}$ of the section $u$ at $x_i$. Now let $r(\mu, A)=\dim_{\mathbb{C}} L(\mu, A)$.

THEOREM 1:

$$r(\mu, A)=\operatorname{ind} A - qd(\mu)+r(\mu^{-1}, A'). \tag{1}$$

*Examples.* — (*a*) Let X be a compact connected complex manifold, $\dim_{\mathbb{C}} X=1$, *i.e.* X is a complex non-singular algebraic curve. Take $A=\bar\partial : C^\infty(X) \to \Lambda^{0,1}(X)$ and identify $\Omega(X)=\Lambda^2(X)$ using the canonical orientation. Then $A'=\bar\partial : \Lambda^{1,0}(X) \to \Lambda^2(X)$ and $\operatorname{ind} A=1-g$ where $g$ is the genus of X. Theorem 1 then becomes the classical Riemann-Roch theorem if we note that here $n=2$, $d=1$, hence $d(\mu)=\sum_{1 \leq i \leq m} p_i$.

(*b*) Let X be a compact Riemannian manifold and $A=\Delta$ is the scalar Laplace-Beltrami operator. Then $A'=A$ and denoting $r(\mu)=r(\mu, \Delta)$ we obtain

$$r(\mu)=-d(\mu)+r(\mu^{-1}).$$

This was proved in [1] where also non-compact manifolds with boundary were allowed (with suitable boundary conditions and conditions at infinity).

3. PROOF OF THEOREM 1. — Denote $\mathscr{D}'(X, F)$ the space of distribution sections of F (dual space to $\Gamma(X, F^*)$) and $S(\mu, F)$ the space of $s \in \mathscr{D}'(X, F)$, such that $\operatorname{supp} s \subset \operatorname{supp} \nu$ and locally near the points $x_i \in \operatorname{supp} \nu$ (*i.e.* the points entering in $\mu$ with negative exponents) $s$ can be written as

$$s=\sum_{\{i \mid p_i<0\}} \sum_{|\alpha| \leq |p_i|-1} C_{i\alpha} \partial_x^\alpha \delta(x-x_i)$$

where $\delta$ is the Dirac measure, $C_{i\alpha} \in F_{x_i}$. So actually $S(\mu, F)=S(\nu, F)$.

Introduce the space $\Gamma(X, \mu, A)$ of sections $u \in \Gamma(X-\operatorname{supp} \nu, E)$ having prescribed poles and zeros [as in definition of $L(\mu, A)$] but not necessarily solutions of $Au=0$:

$\Gamma(X, \mu, A)=\{u \mid u \in \Gamma(X-\operatorname{supp} \nu, E), j_{x_i}^{p_i-1} u=0$ if $x_i \in \operatorname{supp} \lambda$;

for every $x_i \in \operatorname{supp} \nu$ there exists a neighbourhood U of $x_i$

and a representation $u=u_s+u_r$ where $u_s \in \Gamma(U \setminus x_i, E)$,

$Au=0$ in $U \setminus x_i$, $u(x)=O(|x-x_i|^{d-n-|p_i|})$ as $x \to x_i$,

and $u_r$ can be extended to a section $\tilde u_r \in \Gamma(U, E)\}$.

For every $u \in \Gamma(X, \mu, A)$ we can find a "regularization" $\tilde u \in \mathscr{D}'(X, E)$ such that $\tilde u=u$ on $X-\operatorname{supp} \nu$ and $A\tilde u=f+s$ with $f \in \Gamma(X, F)$ and $s \in S(\mu, F)$. Using the standard elliptic regularity result and the well-known structure of fundamental solutions of elliptic operators [2] it is easy to check that the space $\tilde\Gamma(X, \mu, A)$ of all such regularizations $\tilde u$ can be described as a set of $\tilde u \in \mathscr{D}'(X, E)$ such that $\tilde u \in C^\infty$ is a neighbourhood of $\operatorname{supp} \lambda$, $j_{x_i}^{p_i-1} u=0$ for every $x_i \in \operatorname{supp} \lambda$ and $A\tilde u=f+s$ with $f \in \Gamma(X, F)$ and $s \in S(\mu, F)$.

Now introduce the "reduced" divisor

$$\tilde u=x_1^{\tilde p_1} x_2^{\tilde p_2} \ldots x_m^{\tilde p_m},$$

where $\tilde p_i=\operatorname{sign} p_i . (|p_i|-d)_+$ (the factors $x_i^{\tilde p_i}$ with $\tilde p_i=0$ have to be omitted) and denote

$$\Gamma_{\tilde\mu}(X, F)=\{f \mid f \in \Gamma(X, F), j_{x_i}^{\tilde p_i} f=0 \text{ if } \tilde p_i>0\}.$$

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & S(\tilde{\mu}, E) & \xrightarrow{i_1} & \tilde{\Gamma}(X, \mu, A) & \xrightarrow{r} & \Gamma(X, \mu, E) & \to 0 \\
 & \downarrow{\scriptstyle A_S} & & \downarrow{\scriptstyle \tilde{A}} & & \downarrow{\scriptstyle \hat{A}} & \\
0 \to & S(\mu, F) & \xrightarrow{i_2} & \Gamma_{\tilde{\mu}}(X, F) \oplus S(\mu, F) & \xrightarrow{\pi_1} & \Gamma_{\tilde{\mu}}(X, F) & \to 0
\end{array}$$

where $i_1$, $i_2$ are inclusion maps, $r$ is the restriction map, $\pi$ is the natural projection and the vertical maps $A_S$, $\hat{A}$ and $\tilde{A}$ are the restrictions of A to the corresponding function or distribution spaces. Both rows in the diagram are exact, hence due to the well-known property of Euler characteristic we have

$$\operatorname{ind} \tilde{A} = \operatorname{ind} \hat{A} - \operatorname{ind} A_S = \operatorname{ind} \hat{A} - \dim S(\tilde{\mu}, F) + \dim S(\mu, F) = \operatorname{ind} \hat{A} - qd(\nu),$$

since

$$\dim S(\mu, F) = q \sum_{p_i < 0} \binom{n + |p_i| - 1}{n},$$

$$\dim S(\tilde{\mu}, E) = q \sum_{p_i < 0} \binom{n + |p_i| - d - 1}{n}.$$

Now use the following two commutative diagrams with exact rows:

$$\begin{array}{ccccccc}
0 \to & \Gamma\mu(X, E) & \longrightarrow & \tilde{\Gamma}(X, \mu, E) & \xrightarrow{\pi_2 \cdot A} & S(\mu, F) & \to 0 \\
 & \downarrow{\scriptstyle A_\mu} & & \downarrow{\scriptstyle \hat{A}} & & \downarrow{\scriptstyle \mathrm{id}} & \\
O \to & \Gamma_{\tilde{\mu}}(X, F) & \to & \Gamma_{\tilde{\mu}}(X, F) \oplus S(\mu, F) & \xrightarrow{\pi_2} & S(\mu, F) & \to 0
\end{array}$$

and

$$\begin{array}{ccccccc}
0 \to & \Gamma\mu(X, E) & \xrightarrow{i_\mu} & \Gamma(X, E) & \xrightarrow{P_\mu} & J_\mu(E) & \to 0 \\
 & \downarrow{\scriptstyle A_\mu} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle J_\mu(A)} & \\
0 \to & \Gamma_{\tilde{\mu}}(X, F) & \to & \Gamma(X, F) & \to & J_{\tilde{\mu}}(F) & \to 0
\end{array}$$

where $J_\mu(E) = \Gamma(X, E)/\Gamma_\mu(X, E) = \bigoplus_{p_i > 0} J_{x_i}^{p_i - 1}(E)$, $i_\mu$ and $p_\mu$ are natural inclusion and restriction maps, $J_\mu(A)$ is the natural quotient map. Then we find

$$\operatorname{ind} \hat{A} = \operatorname{ind} A_\mu = \operatorname{ind} A - \operatorname{ind} J_\mu(A) = \operatorname{ind} A - \dim J_\mu(E) + \dim J_{\tilde{\mu}}(F) = \operatorname{ind} A - qd(\lambda).$$

Hence

$$\operatorname{ind} \tilde{A} = \operatorname{ind} A - q(d(\nu) + d(\nu)) = \operatorname{ind} A - qd(\mu). \tag{2}$$

Now evidently $\operatorname{Ker} \tilde{A} = L(\mu, A)$ so we only have to identify $\operatorname{Coker} \tilde{A}$ with $\operatorname{Ker} \tilde{A}^t$. This is done by using nondegenerated pairings

$$\Gamma(X, \mu, A) \times \Gamma_{\tilde{\mu}^{-1}}(X, E^*) \to \mathbb{C}, \ \Gamma_{\tilde{\mu}}(X, F) \times \Gamma(X, \mu^{-1}, A^t) \to \mathbb{C}$$

which are defined by usual integration over $X - \operatorname{supp} \mu$ (all integrals converge !) and denoted by $\langle \cdot, \cdot \rangle$.

Theorem 2. — (i) $\langle \tilde{A} u, v \rangle = \langle u, \tilde{A}^t v \rangle$, $u \in \Gamma(X, \mu, A)$, $v \in \Gamma(X, \mu^{-1}, A^t)$.

(ii) $\operatorname{Im} \tilde{A} = (\operatorname{Ker} \tilde{A}^t)^0$ *i. e.* $f \in \operatorname{Im} A$ *iff* $f \in \Gamma_{\tilde{\mu}}(X, F)$ *and* $\langle f, v \rangle = 0$ *for all* $v \in \operatorname{Ker} \tilde{A}^t$.

(iii) $\dim \operatorname{Coker} \tilde{A} = \dim \operatorname{Ker} \tilde{A}^t$.