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The Riemann-Roch theorem for elliptic operators and solvability of elliptic equations with additional conditions on compact subsets

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1 Introduction

The classical Riemann-Roch theorem (see e.g. [G-H]) can be considered as a connection between two dimensions. The first one is the dimension of a linear space of meromorphic functions on a compact Riemann surface (or a non-singular algebraic curve over **C**) which are allowed to have poles up to an assigned order at any point from a given finite set, and are required to have zeros of at least assigned order at any point from another finite set. The second one is the dimension of a space of meromorphic $(1, 0)$-forms with similar restrictions but with poles and zeros changing places. The information about poles and zeros is conveniently encoded into a notion of divisor which is just a finite subset in the given Riemann surface with integers (multiplicities) assigned to every point in this subset. The result includes the degree of the divisor which is just the sum of all multiplicities.

In our previous paper [G-S] we proved a version of the classical Riemann-Roch theorem for solutions of general elliptic equations with point singularities. Here we extend the results to much more general singularities supported on arbitrary compact nowhere dense sets. The only restriction is that the allowed singularities should be taken from a finite-dimensional space. Dually a finite set of conditions may be imposed on another nowhere dense compact set (which should be disjoint with the set where singularities are allowed). This leads to a notion of rigged divisor which includes two disjoint nowhere dense compact sets with finite-dimensional distribution spaces supported on them. Then the allowed singularities on the first given set are described as singularities of solutions which may be extended as distributions to the whole given manifold so that after applying the given elliptic operator we get into the first given space of distributions. The conditions imposed on the second compact set are just orthogonality conditions to the second space of distributions. The main theorem then connects the dimension of the space of solutions having the allowed singularities and satisfying the imposed conditions, with another dimension defined in the same way from the dual (or

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inverse) divisor which is obtained by changing places of two given compact subsets and distribution spaces (and replacing the given operator by the adjoint operator). As in the classical Riemann-Roch theorem the corresponding formula includes a degree of the divisor. The degree is defined in terms of the dimensions of the given distribution spaces and two other naturally arising "secondary" distribution spaces. This clarifies the appearance of combinations of the binomial coefficients in the case of point divisors in [G-S].

The Riemann-Roch type formula described above is proved at the same time with a duality theorem which gives necessary and sufficient solvability conditions of elliptic equations (with a right-hand side) if the solution is allowed to have singularities of the described type and is required to satisfy finite number of orthogonality conditions. This result seems important from the analytical point of view. It implies for instance local solvability results in smooth sections near a compact set with a finite number of conditions imposed on the solution. Basically the result here is that the local solution always exists provided obvious necessary conditions are satisfied.

The simplest example is that the Poisson equation $\Delta u = f$ can be always solved in a neighbourhood of a compact set $D \subset \mathbf{R}^n$, $\operatorname{mes} D = 0$, with any finite number of additional conditions on u which are orthogonality conditions to measures supported on D or to first order derivatives of such measures. This can be interpreted as an approximate solution of the Cauchy problem for the Poisson equation:

$$\Delta u = f \text{ near } D; \quad u \approx 0 \text{ and } \nabla u \approx 0 \text{ on } D.$$

Here the approximate equality might mean e.g. equality of any number of Fourier coefficients with respect to any orthonormal system in $L^2(D, d\nu)$ where $d\nu$ is a finite Borel measure on D. Note that D might be a complicated set (e.g. a Sierpiński carpet) and not just a hypersurface as for the classical Cauchy problem.

In Sect. 2 we shall give necessary definitions and precise formulations of the main results for the case of compact manifolds. Section 3 contains the proofs. A special situation of a non-compact manifold with boundary (with boundary conditions and conditions at infinity) is discussed in Sect. 4.

This paper is written completely independently of [G-S]; in fact the proofs are even simpler in this generality. But the reader should keep in mind that applications and examples given in [G-S] are not repeated here being specific for the case of point divisors.

2 Preliminaries and main results

A. Let X be a compact closed C^∞-manifold, E a C^∞ complex vector bundle over X. For any open subset $U \subset X$ denote by $C^\infty(U, E)$ the linear space of all C^∞-sections of E over U. We shall also need the space of all distributional sections of E over U which will be denoted $\mathscr{D}'(U, E)$. If D is a compact (closed) subset in X then $\mathscr{E}'_D(X, E)$ denotes the linear space of all $f \in \mathscr{D}'(X, E)$ such that $\operatorname{supp} f \subset D$.

For any C^∞ vector bundle E over X denote by E^* any vector bundle which is supplied with a C^∞ bilinear or sesquilinear nondegenerate duality of bundles $E \times E^* \to \Omega(X)$ where $\Omega(X)$ is the density bundle over X. Then we obviously have bilinear or sesquilinear duality on sections:

$$(\cdot,\cdot): C^\infty(X, E) \times C^\infty(X, E^*) \to \mathbf{C},$$

$$(u, v) = \int_X (u(x), v(x))_x,$$

where $(\cdot\,,\cdot)_x$ denotes the given duality in the fibers over the point $x \in X$.

Let E, F be C^∞ vector bundles over X,

$$A: C^\infty(X, E) \to C^\infty(X, F)$$

an elliptic linear differential operator of order d. Then the adjoint operator is again an elliptic linear differential operator of order d

$$A^*: C^\infty(X, F^*) \to C^\infty(X, E^*),$$

such that

$$(Au, v) = (u, A^*v);\ u \in C^\infty(X, E),\ v \in C^\infty(X, F^*).$$

Definition 2.1 ***Rigged divisor*** *(associated with A) is a tuple*

$$\mu = (D^+, L^+; D^-, L^-),$$

where $D^\pm$ are compact (closed) nowhere dense disjoint subsets in X, $L^\pm$ are finite-dimensional linear spaces of distributional sections,

$$L^+ \subset \mathscr{E}'_{D^+}(X, F),\ L^- \subset \mathscr{E}'_{D^-}(X, E^*).$$

So the sections in L^+ (L^-) are supported on D^+ (resp. D^-). Denote also $l^\pm = \dim L^\pm$. Hereafter $\dim L$ for a complex linear space L will always mean $\dim_{\mathbf{C}} L$.

We shall also need "secondary" spaces of distributional sections which are defined as follows:

$$\tilde{L}^+ = \{u \mid u \in \mathscr{E}'_{D^+}(X, E),\ Au \in L^+\},\ \tilde{L}^- = \{v \mid v \in \mathscr{E}'_{D^-}(X, F^*),\ A^*v \in L^-\}.$$

Denote also $\tilde{l}^\pm = \dim \tilde{L}^\pm$. Note that $\tilde{l}^\pm \leqq l^\pm$ because A, A^* are injective on $\mathscr{E}'_{D^\pm}$ due to the standard elliptic regularity result.

Definition 2.2 ***Degree of the rigged divisor μ*** *is the following integer:*

$$\deg_A(\mu) = (l^+ - \tilde{l}^+) - (l^- - \tilde{l}^-).$$

Definition 2.3 ***Inverse divisor*** *to a divisor $\mu = (D^+, L^+; D^-, L^-)$ associated with the elliptic operator A is the rigged divisor*

$$\mu^{-1} = (D^-, L^-; D^+, L^+),$$

associated with the adjoint operator A^*.

Note that

$$\deg_{A^*}(\mu^{-1}) = -\deg_A(\mu).$$

Now we shall introduce the main space of solutions with allowed singularities on D^+ and vanishing conditions on D^-.

Definition 2.4 *Denote*

$$L(\mu, A) = \{u \mid u \in C^\infty(X - D^+, E), \exists \tilde{u} \in \mathscr{D}'(X, E), \tilde{u} = u \text{ on } X - D^+, A\tilde{u} \in L^+,$$
$$(u, L^-) = 0\}; \; r(\mu, A) = \dim L(\mu, A).$$

Here (u, L^-) means the set $\{(u, g) \mid g \in L^-\} \subset \mathbf{C}$ and we write 0 instead of $\{0\}$, so the equality $(u, L^-) = 0$ means the u is orthogonal to L^- with respect to the given duality. This makes sense (in spite of the fact that u is defined on $X - D^+$ only) because all distribution sections from L^- are supported on D^- and $D^- \cap D^+ = \emptyset$ according to Definition 2.1.

We shall use the notation $\operatorname{ind} A$ for the standard index of A

$$\operatorname{ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A$$

in spaces of C^∞-sections. This index is given by the Atiyah-Singer index formula.

Now we can formulate our first main result.

Theorem 2.5 (The Riemann-Roch theorem for the rigged divisor μ)

$$r(\mu, A) = \operatorname{ind} A + \deg_A(\mu) + r(\mu^{-1}, A^*). \tag{2.1}$$

Corollary 2.6 $r(\mu, A) \geqq \operatorname{ind} A + \deg_A(\mu)$.
In particular $\operatorname{ind} A + \deg_A(\mu) > 0$ *implies that* $L(\mu, A) \neq \{0\}$.

This is actually an existence result for solutions with allowed singularities and prescribed orthogonality conditions.

Example 2.7 Let us consider a particular case of "point divisors". Namely, let $D^\pm$ be finite sets. Suppose that $D^+ = (x_1, \ldots, x_k\}$, $D^- = \{x_{k+1}, \ldots, x_m\}$ and let also integers $p_1, \ldots, p_k > 0$ and $p_{k+1}, \ldots, p_m < 0$ be given. This corresponds to the point divisor $\mu = x_1^{p_1} x_2^{p_2} \ldots x_m^{p_m}$ in notations of [G-S]. Then we can introduce the distribution section spaces $L^\pm$ which are locally represented as

$$L^\pm = \{f \mid f(x) = \sum_{\pm p_i > 0} \sum_{|\alpha| \leqq |p_i| - 1} c_{i\alpha} \delta^{(\alpha)}(x - x_i), c_{i\alpha} \in \mathbf{C}^q\},$$

where δ means the Dirac δ-function, $\delta^{(\alpha)}$ denotes its derivative corresponding to the multiindex α, and $c_{i\alpha}$ are vector coefficients from $\mathbf{C}^q$ where q is the dimension of the fibers of the bundles E and F (they are equal due to the ellipticity of A).

Since A is elliptic (of order d) it is easy to check that the "secondary" spaces have a similar form

$$\tilde{L}^\pm = \{v \mid v(x) = \sum_{\pm p_i > 0} \sum_{|\alpha| \leqq |p_i| - 1 - d} c_{i\alpha} \delta^{(\alpha)}(x - x_i), c_{i\alpha} \in \mathbf{C}^q\}.$$

Now an easy combinatorial exercise shows that

$$\dim L^\pm = q \sum_{\pm p_i > 0} \binom{n + |p_i| - 1}{n}, \quad \dim \tilde{L}^\pm = q \sum_{\pm p_i > 0} \binom{n + |p_i| - 1 - d}{n},$$

where

$$\binom{N}{n} = \frac{N!}{n!(N-n)!} \quad \text{if } N \geqq n \text{ and } 0 \text{ otherwise.}$$

It follows that

$$\deg_A(\mu) = \sum_{1 \leqq i \leqq m} \operatorname{sign} p_i \left[\binom{|p_i| + n - 1}{n} - \binom{|p_i| + n - 1 - d}{n} \right].$$

It is explained in [G-S] that the space $L(\mu, A)$ in this particular case can be described in terms of the behaviour of solutions near the points $x_1, \ldots, x_m$ without referring to their distribution extensions on X. In this way we arrive to the main Riemann-Roch type theorem in [G-S] as a particular case of Theorem 2.5 when μ is a point divisor.

Example 2.8 Let X be a compact Riemann surface, $g = \text{genus}(X)$. Let us consider the operator

$$A = \bar{\partial}: C^\infty(X) \to \Lambda^{0,1}(X).$$

Suppose that $k + l$ distinct points $x_1, \ldots, x_k, y_1, \ldots, y_l$ are given in X and define

$$D^+ = \{x_1, \ldots, x_k\}, D^- = \{y_1, \ldots, y_l\}.$$

Define also in local real coordinates near the given points

$$L^+ = \left\{ \sum_{i=1}^{k} c_i \delta(x - x_i),\ c_i \in \mathbf{C} \right\},$$

$$L^- = \left\{ \sum_{j=1}^{l} \sum_{|\alpha|=1} c_{j\alpha} \partial^\alpha \delta(x - y_j),\ c_{j\alpha} \in \mathbf{C} \right\}.$$

Let us consider the rigged divisor $\mu = (D^+, L^+; D^-, L^-)$. Then the space $L(\mu, A)$ is the space of all meromorphic functions f on X which are allowed to have at most simple poles at $x_1, \ldots, x_k$ and are required to have critical points at $y_1, \ldots, y_l$ i.e. $f'(y_j) = 0$, $j = 1, \ldots, l$. Note now that

$$A^* = \bar{\partial}: \Lambda^{1,0} \to \Lambda^{1,1}(X) = \Lambda^2(X).$$

Therefore $L(\mu^{-1}, A^*)$ is the space of all meromorphic $(1,0)$-forms which are allowed to have poles of second order with vanishing residues at $y_1, \ldots, y_l$ and are required to vanish at all the points $x_1, \ldots, x_k$.

Obviously $l^+ = k$ and $l^- = 2l$. The secondary spaces are as follows:

$$\tilde{L}^+ = \{0\}, \tilde{L}^- = \left\{ \sum_{j=1}^{l} \tilde{c}_j \delta(x - y_j),\ \tilde{c}_j \in \mathbf{C} \right\}.$$

It follows that $\tilde{l}^+ = 0$ and $\tilde{l}^- = l$, hence $\deg_A(\mu) = k - l$. Since $\operatorname{ind} A = 1 - g$, Theorem 2.5 gives in this case

$$r(\mu, A) = 1 - g + k - l + r(\mu^{-1}, A^*).$$

It follows that

$$r(\mu, A) \geqq 1 - g + k - l, r(\mu^{-1}, A^*) \geqq g - 1 + l - k,$$

which implies the corresponding existence results.

This example is of course well known and may be easily deduced from the classical Riemann-Roch theorem. Note however that it is a natural example to

Theorem 2.5. Similar example for harmonic functions in $\mathbf{R}^n$ will be considered in Sect. 4 after we discuss a non-compact situation.

B. First idea of the proof of Theorem 2.5 is a localization which begins with the introduction of the following space:

$$\Gamma(X, \mu, A) = \{u \mid u \in C^\infty(X - D^+, E), \exists \tilde{u} \in \mathscr{D}'(X, E), \tilde{u} = u \text{ on } X - D^+,$$
$$A\tilde{u} \in L^+ + C^\infty(X, F), (u, L^-) = 0\}.$$

Here the difference with the definition of $L(\mu, A)$ is that $A\tilde{u} = f$ is allowed to be modified by adding any C^∞ section. In particular $\Gamma(X, \mu, A)$ contains the space $C_c^\infty(X - (D^+ \cup D^-), E)$ of all C^∞-sections of E having a compact support on $X - (D^+ \cup D^-)$.

The next space that we need is the space of all possible regularizations of sections from $\Gamma(X, \mu, A)$:

$$\tilde{\Gamma}(X, \mu, A) = \{\tilde{u} \mid \tilde{u} \in \mathscr{D}'(X, E), A\tilde{u} \in L^+ + C^\infty(X, F), (\tilde{u}, L^-) = 0\}.$$

Lemma 2.9 *The following sequence is exact:*

$$0 \to \tilde{L}^+ \xrightarrow{i} \tilde{\Gamma}(X, \mu, A) \xrightarrow{r} \Gamma(X, \mu, A) \to 0, \tag{2.2}$$

where i and r are natural inclusion and restriction maps.

Proof. The statement is obvious from the definitions of all the spaces involved. □

Now let us find out what happens if we apply A to a section $u \in \Gamma(X, \mu, A)$. Obviously Au can be extended to a C^∞-section of F. Let us denote this extension by $\tilde{A}u$. Besides we have

$$(Au, \tilde{L}^-) = (u, A^*\tilde{L}^-) \subset (u, L^-) = 0.$$

This motivates the introduction of the following space:

$$\tilde{\Gamma}_\mu(X, A) = \{f \mid f \in C^\infty(X, F), (f, \tilde{L}^-) = 0\}; \tag{2.3}$$

then $\tilde{A}$ defines a linear map

$$\tilde{A}: \Gamma(X, \mu, A) \to \tilde{\Gamma}_\mu(X, A).$$

Note that $\operatorname{Ker} \tilde{A} = L(\mu, A)$.

Definition 2.10 *The duality*

$$\Gamma(X, \mu, A) \times \tilde{\Gamma}_{\mu^{-1}}(X, A^*) \to \mathbf{C} \tag{2.4}$$

is defined by

$$(u, f) = (r^{-1}u, f), \ u \in \Gamma(X, \mu, A), \ f \in \tilde{\Gamma}_{\mu^{-1}}(X, A^*),$$

where r is the restriction map from (2.2).

Note that this duality is well defined due to the orthogonality condition in (2.3) (with μ and A replaced by μ^{-1} and A^*).

The duality (2.4) is obviously non-degenerate since both spaces involved can be considered as spaces of C^∞-sections on $X - (D^+ \cup D^-)$ and both include all sections with compact support in $X - (D^+ \cup D^-)$.

Replacing A and μ by A^* and μ^{-1} we get a similar duality

$$\tilde{\Gamma}_\mu(X, A) \times \Gamma(X, \mu^{-1}, A^*) \to \mathbf{C}. \tag{2.5}$$

Lemma 2.11 *We have*

$$(\tilde{A}u, v) = (u, \tilde{A}^*v),\ u \in \Gamma(X, \mu, A),\ v \in \Gamma(X, \mu^{-1}, A^*),$$

where the dualities on the left and right hand sides are the dualities (2.5) *and* (2.4) *respectively.*

Proof. The statement becomes obvious if we pass to the extensions of u and v to the distribution sections in $\tilde{\Gamma}(X, \mu, A)$ and $\tilde{\Gamma}(X, \mu^{-1}, A^*)$ respectively. □

Now let $(\cdot, \cdot): \mathscr{H} \times \mathscr{H}' \to \mathbf{C}$ be a bilinear or sesquilinear duality (or pairing) of complex linear spaces. For any linear subspace $L \subset \mathscr{H}'$ define its **annihilator** or **orthogonal complement** with respect to the duality $(\cdot, \cdot)$ as follows:

$$L^\circ = \{f \mid f \in \mathscr{H}, (f, L) = 0.\}.$$

Hence L° is a linear subspace in $\mathscr{H}$. Similarly if L is a linear subspace in $\mathscr{H}$ then L° is defined as a linear subspace in $\mathscr{H}'$.

In the following theorem which is our second main result L° will mean the annulator of L with respect to the dualities (2.4) or (2.5). So if $L \subset \Gamma(X, \mu^{-1}, A^*)$ then $L^\circ \subset \tilde{\Gamma}_\mu(X, A)$ etc.

Theorem 2.12 (Duality theorem) (i) $\operatorname{Im} \tilde{A} = (\operatorname{Ker} \tilde{A}^*)^\circ$ *i.e.* $f \in \operatorname{Im} \tilde{A}$ *if and only if* $f \in \tilde{\Gamma}_\mu(X, A)$ *and* $(f, \operatorname{Ker} \tilde{A}^*) = 0$. (ii) $\dim \operatorname{Coker} \tilde{A} = \dim \operatorname{Ker} \tilde{A}^*$.

This theorem gives solvability conditions of the equation $Au = f$ in the class $\Gamma(X, \mu, A)$ which consists of sections which may have some singularities on D^+ and should satisfy some orthogonality conditions on D^-.

C. Now we turn to a local solvability result of the equation $Au = f$ in smooth sections near a closed nowhere dense set $D \subset X$ with a finite number of additional orthogonality conditions imposed on u. First introduce the space of germs of C^∞-sections of E on D:

$$C^\infty(D, E) = \lim_{\mathscr{U} \supset D} C^\infty(\mathscr{U}, E),$$

$\mathscr{U}$ runs through the set of all open neighbourhoods of D. Suppose a finite-dimensional linear subspace $L \subset \mathscr{E}'_D(X, E^*)$ is given. Given $f \in C^\infty(D, F)$ we want to find a solution $u \in C^\infty(D, E)$ of the equation $Au = f$ such that $(u, L) = 0$.

Let us introduce the "secondary" space

$$\tilde{L} = \{v \mid v \in \mathscr{E}'_D(X, F^*),\ A^*v \in L\}.$$

Then the obvious necessary condition on f is $(f, \tilde{L}) = 0$ since $(Au, \tilde{L}) = (u, A^*\tilde{L}) \subset (u, L)$.

Theorem 2.13 (Local solvability theorem) *If* $f \in C^\infty(D, F)$ *and* $(f, \tilde{L}) = 0$ *then there exists* $u \in C^\infty(D, E)$ *such that* $Au = f$ *and* $(u, L) = 0$.

A particular case of Theorem 2.13 when D is a point, say $0 \in \mathbf{R}^n$, and L is obtained as L^+ in Example 2.7 (with $k = 1$), was discussed in [G-S]. It says that if the right-hand side of an elliptic equation $Au = f$ of order d has a zero of order m at the origin, then locally near the origin there always exists a solution u having there a zero of order $m + d$.

In some particular cases the necessary conditions might become void, so the solvability holds without any conditions on f. We shall give an example of this situation now. In this example $C^\infty(D)$ will denote the set of germs of all C^∞-functions near D (or $C^\infty(D, E)$ with $E = X \times \mathbf{C}$).

Corollary 2.14 *Let Δ be the standard Laplacian on $\mathbf{R}^n$, D a compact subset in $\mathbf{R}^n$ with the Lebesgue measure 0. Suppose that for any multiindex α with $|\alpha| \leqq 1$ and any $j = 1, \ldots, k$ a complex-valued Borel measure $\mu_{\alpha j}$ supported on D is given. Then for any $f \in C^\infty(D)$ there exists $u \in C^\infty(D)$, such that $\Delta u = f$ and*

(2.6) $$\sum_{|\alpha| \leqq 1} \int_D u^{(\alpha)} d\mu_{\alpha j} = 0, \; j = 1, \ldots, k.$$

Note that derivatives of order 2 cannot be allowed here. For example the condition $(\Delta u)(0) = 0$ (which can be obviously written in a form similar to (2.6) but with second order derivatives) implies $f(0) = 0$ for the right-hand side of the equation $\Delta u = f$, so f cannot be an arbitrary C^∞ function then.

Corollary 2.14 can be interpreted as an approximate solution of the Cauchy problem for the Poisson equation $\Delta u = f$ near D (see also Introduction for additional comments).

3 Proofs

We shall use the notations from Sect. 2. Let us start with the proof of the following important lemma:

Lemma 3.1

(3.1) $$\operatorname{ind} \tilde{A} = \operatorname{ind} A + \deg_A(\mu).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{L}^+ & \xrightarrow{i} & \tilde{\Gamma}(X, \mu, A) & \xrightarrow{r} & \Gamma(X, \mu, A) & \longrightarrow & 0 \\
 & & \downarrow A_S & & \downarrow \tilde{A} & & \downarrow \hat{A} & & \\
0 & \longrightarrow & L^+ & \xrightarrow{i_1} & \tilde{\Gamma}_\mu(X, F) \oplus L^+ & \xrightarrow{\pi_1} & \tilde{\Gamma}_\mu(X, F) & \longrightarrow & 0
\end{array}$$

where the first row is as in (2.2), i_1 and π_1 are natural inclusion and projection respectively and A_S, $\hat{A}$ are restrictions of A to the corresponding spaces of distributions. Both rows in the diagram are exact. Due to the well known algebraic property of the Euler characteristic we have

$$\operatorname{ind} \tilde{A} = \operatorname{ind} \hat{A} - \operatorname{ind} A_S.$$

But

$$\operatorname{ind} A_S = \dim \tilde{L}^+ - \dim L^+ = \tilde{l}^+ - l^+.$$

Hence

$$\operatorname{ind} \tilde{A} = \operatorname{ind} \hat{A} + (l^+ - \tilde{l}^+). \tag{3.2}$$

Now consider the following space of smooth sections:

$$\Gamma_\mu(X, A) = \{u \mid u \in \Gamma(X, E), (u, L^-) = 0\}.$$

Then the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_\mu(X, E) & \longrightarrow & \tilde{\Gamma}(X, \mu, A) & \xrightarrow{\pi_2 \circ \tilde{A}} & L^+ & \longrightarrow & 0 \\
 & & \downarrow A_\mu & & \downarrow \hat{A} & & \downarrow \mathrm{Id} & & \\
0 & \longrightarrow & \tilde{\Gamma}_\mu(X, F) & \longrightarrow & \tilde{\Gamma}_\mu(X, F) \oplus L^+ & \xrightarrow{\pi_2} & L^+ & \longrightarrow & 0
\end{array}$$

where π_2 is a natural progection, A_μ is the restriction of A.

The rows of this diagram are again exact. This is not obvious in the term L^+ of the first row only. In this term exactness means that the equation $Au = f \in \mathscr{E}'_{D^+}(X, F)$ can be always solved modulo C^∞ sections (with the solution $u \in \tilde{\Gamma}(X, \mu, A)$). But this follows e.g. from the existence of a pseudodifferential parametrix (see e.g. [H, vol. 3] or [S] for necessary facts). Namely, let $B: C^\infty(X, F) \to C^\infty(X, E)$ be a (classical or polyhomogeneous) pseudodifferential operator such that $BA = I - T$ with an infinitely smoothing operator T (an operator with a C^∞ Schwartz kernel). Using the standard extension of B to distribution sections we can now take $u = Bf$ to obtain $Au = f + g$ with $g \in C^\infty(X, F)$. Replacing u by $u + v$ with $v \in C^\infty(X, E)$ supported near D^- we can make $u = 0$ near D^-; then we shall obviously have $u \in \tilde{\Gamma}(X, \mu, A)$.

From the last diagram we find

$$\operatorname{ind} \hat{A} = \operatorname{ind} A_\mu. \tag{3.3}$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_\mu(X, A) & \xrightarrow{i_\mu} & \Gamma(X, E) & \xrightarrow{p_\mu} & (L^-)' & \longrightarrow & 0 \\
 & & \downarrow A_\mu & & \downarrow A & & \downarrow (A^*)' & & \\
0 & \longrightarrow & \tilde{\Gamma}_\mu(X, F) & \xrightarrow{\tilde{i}_\mu} & \Gamma(X, F) & \xrightarrow{\tilde{p}_\mu} & (\tilde{L}^-)' & \longrightarrow & 0
\end{array}$$

where for a finite-dimensional complex linear space L we denote its dual (or antidual) space by L', $i_\mu, \tilde{i}_\mu$ are natural inclusion maps and $p_\mu, \tilde{p}_\mu$ are defined as follows:

$$(p_\mu u)(s) = (u, s), u \in \Gamma(X, E), s \in L^-; (\tilde{p}_\mu f)(t) = (f, t), f \in \Gamma(X, F), t \in \tilde{L}^-.$$

The maps $p_\mu, \tilde{p}_\mu$ are surjective since the dual maps $L^- \to \mathscr{D}'(X, E^*)$, $\tilde{L}^- \to \mathscr{D}'(X, F^*)$ are just canonical injections. Hence the rows are exact and we find

$$\operatorname{ind} A_\mu = \operatorname{ind} A - \operatorname{ind}(A^*)' = \operatorname{ind} A - (l^- - \tilde{l}^-).$$

Now using (3.2) and (3.3) we obtain

$$\operatorname{ind} \tilde{A} = \operatorname{ind} A + (l^+ - \tilde{l}^+) - (l^- - \tilde{l}^-) = \operatorname{ind} A + \deg_A(\mu). \quad \square$$

Remark 3.2 Lemma 3.1 means that

$$\dim \operatorname{Ker} \tilde{A} = \operatorname{ind} A + \deg_A(\mu) + \dim \operatorname{Coker} \tilde{A}, \tag{3.4}$$

so to prove Theorem 2.5 it suffices to prove the equality (ii) in Theorem 2.12. But it is not easy to do it directly since no Hilbert space duality technique is available for the spaces involved. So actually Theorems 2.5 and 2.12 will be proved simultaneously.

Now we need the following abstract lemma from [G-S] which we reproduce with the proof for the sake of completeness:

Lemma 3.3 *Let* $(\cdot,\cdot): \mathscr{H} \times \mathscr{H}' \to \mathbf{C}$ *be a non-degenerated bilinear (or sesquilinear) pairing between two complex linear spaces* $\mathscr{H}$, $\mathscr{H}'$. *Let* L *be a linear subspace in* $\mathscr{H}'$, L° *its annihilator in* $\mathscr{H}$ *and* $(L^{\circ})^{\circ}$ *the annihilator of* L° *in* $\mathscr{H}'$. *Then*

$$L \subset (L^{\circ})^{\circ} \tag{3.5}$$

and

$$\operatorname{codim} L \geqq \dim L^{\circ}. \tag{3.6}$$

Furthermore if F *is a linear subspace in* $\mathscr{H}'$ *then*

$$\operatorname{codim} F^{\circ} = \dim F. \tag{3.7}$$

Proof. The inclusion (3.5) is obvious. It follows that $\operatorname{codim} L \geqq \operatorname{codim} (L^{\circ})^{\circ}$. Hence (3.7) implies (3.6) and we have only to prove (3.7). Clearly $\operatorname{codim} F^{\circ} \leqq \dim F$, so it remains to prove that

$$\operatorname{codim} F^{\circ} \geqq \dim F.$$

It is sufficient to do it in the case when $\dim F < \infty$. Consider then the natural map $j: F \to (\mathscr{H}/F^{\circ})'$, where L' means the space of all complex linear (or antilinear) maps of L to $\mathbf{C}$,

$$j(f)(x + F^{\circ}) = (f, x),\ x \in \mathscr{H}.$$

Then j is injective due to the non-degeneracy of the pairing. Hence

$$\operatorname{codim} F^{\circ} = \dim \mathscr{H}/F^{\circ} = \dim (\mathscr{H}/F^{\circ})' \geqq \dim F$$

as required. □

Lemma 3.4 *In the pairing* (2.4)

$$(\operatorname{Im} \tilde{A})^{\circ} = \operatorname{Ker} \tilde{A}^{*}.$$

Proof. Clearly

$$\operatorname{Ker} \tilde{A}^{*} = \{v \mid v \in \Gamma(X, \mu^{-1}, A^{*}),\ A^{*}v = 0 \text{ on } X - (D^{+} \cup D^{-})\}.$$

Lemma 3.4 follows because $\operatorname{Im} \tilde{A}$ contains all sections Au with $u \in \Gamma(X, E)$ and $\operatorname{supp} u \subset X - (D^{+} \cup D^{-})$. □

Proof of Theorems 2.5 and 2.12 Due to Lemmas 3.3 and 3.4 we have

$$\operatorname{Im} \tilde{A} \subset (\operatorname{Ker} \tilde{A}^{*})^{\circ}, \tag{3.8}$$

$$\operatorname{codim} \operatorname{Im} \tilde{A} \geqq \dim \operatorname{Ker} \tilde{A}^{*} \tag{3.9}$$

and we have to prove that both these inclusion and inequality are actually equalities. Furthermore (3.7) gives that

$$\operatorname{codim}(\operatorname{Ker}\tilde{A}^*)^o = \dim \operatorname{Ker}\tilde{A}^*,$$

hence equality in (3.9) implies equality in (3.8). Since $\operatorname{codim}\operatorname{Im}\tilde{A} = \dim\operatorname{Coker}\tilde{A}$ we have only to prove that

$$\dim\operatorname{Coker}\tilde{A} = \dim\operatorname{Ker}\tilde{A}^* \tag{3.10}$$

which will immediately give us the proof of Theorems 2.5 and 2.12 due to (3.4). Clearly (3.9) and (3.4) imply

$$\begin{aligned}\dim\operatorname{Ker}\tilde{A} &= \operatorname{ind} A + \deg_A(\mu) + \dim\operatorname{Coker}\tilde{A}\\ &\geqq \operatorname{ind} A + \deg_A(\mu) + \dim\operatorname{Ker}\tilde{A}^*.\end{aligned} \tag{3.11}$$

But now we can apply the same results to the divisor μ^{-1} (instead of μ) and the operator $\tilde{A}^*$ (instead of $\tilde{A}$). Then we obtain

$$\begin{aligned}\dim\operatorname{Ker}\tilde{A}^* &\geqq \operatorname{ind} A^* + \deg_{A^*}(\mu^{-1}) + \dim\operatorname{Ker}\tilde{A}\\ &= -\operatorname{ind} A - \deg_A(\mu) + \dim\operatorname{Ker}\tilde{A}.\end{aligned}$$

But this is the opposite inequality to (3.11), hence we actually have equalities in (3.11) and (3.10). Thus the proofs of Theorems 2.5 and 2.12 are completed. □

Proof of Theorem 2.13 The idea is to use Theorem 2.12(i). Suppose that we are given $f\in C^\infty(\mathscr{U}, F)$, where $\mathscr{U}$ is a neighbourhood of D, and $(f, \tilde{L}) = 0$. We want to find $u\in C^\infty(\mathscr{U}_1, E)$ in another (possibly smaller) neighbourhood $\mathscr{U}_1$ of D, so that $Au = f$ in $\mathscr{U}_1$. Using a cut-off C^∞-function supported in $\mathscr{U}$ and equal 1 in another smaller neighbourhood of D we can suppose that $f\in C^\infty(X, F)$.

Now let us consider a rigged divisor $\mu = (\emptyset, 0; D, L)$ i.e. we take

$$D^+ = \emptyset, L^+ = 0, D^- = D, L^- = L,$$

so no singularities are allowed for the sections in $\Gamma(X, \mu, A)$ but the orthogonality conditions are imposed on D. Let us consider the operator $\tilde{A}$ and try to solve the equation $\tilde{A}u = \hat{f}$ where $\hat{f}$ is an extension of f from a neighbourhood of D to a section in $\Gamma(X, F)$ (which will be authomatically in $\tilde{\Gamma}_\mu(X, A)$ because $(f, \tilde{L}) = 0$). We want this extension to satisfy the orthogonality condition $(\hat{f}, \operatorname{Ker}\tilde{A}^*) = 0$ to apply Theorem 2.12.

Denote

$$N_D(X, F) = \{g \mid g\in C^\infty(X, F), g = 0 \text{ in a neighbourhood of } D\}.$$

Then we have to find $g\in N_D(X, F)$ such that $f - g\in(\operatorname{Ker}\tilde{A}^*)^o$ (hence $f - g$ will be the desired modified section). Consider the natural map

$$j:\tilde{\Gamma}_\mu(X, A)\to(\operatorname{Ker}\tilde{A}^*)', \quad j(f)(v) = (f, v).$$

We want to prove that $j(f)\in j(N_D(X, F))$. But actually $j: N_D(X, F)\to(\operatorname{Ker}\tilde{A}^*)'$ is surjective because of the obvious injectivity of the dual map $j':\operatorname{Ker}\tilde{A}^*\to(N_D(X, F))'$ which is defined similarly to j, namely: $j'(v)(f) = (v, f)$. The map j' is injective because $\operatorname{Ker}\tilde{A}^*\subset C^\infty(X - D, F^*)$ and $N_D(X, F)$ includes all sections $g\in C^\infty(X, F)$ with $\operatorname{supp} g\subset X - D$. □

Proof of Corollary 2.14 First note that instead of considering the Laplacian on $\mathbf{R}^n$ we can consider the Laplacian on the flat torus $T^n = \mathbf{R}^n/N\mathbf{Z}^n$ (with large N) since a neighbourhood of D can be considered as an open subset in this torus as well. Let us introduce then a space $L \subset \mathscr{D}'(T^n)$ spanned by the distributions

$$\sum_{|\alpha| \leqq 1} (-1)^{|\alpha|} \partial^\alpha \mu_{\alpha j},\ j = 1, \ldots, k.$$

Then the conditions (2.6) acquire the form $(u, L) = 0$ allowing an application of the Theorem 2.13. So we have to check only that the "secondary" space $\tilde{L}$ is trivial. By definition

$$\tilde{L} = \{g \mid g \in \mathscr{D}'(T^n),\ \Delta g \in L\}.$$

Obviously L belongs to the dual space to $C^1(T^n)$. By the standard Sobolev embedding theorems $C^1(T^n) \supset H^{1+n/p+\varepsilon,p}(T^n)$ for any $p > 1$ and any $\varepsilon > 0$. (See e.g. [St] or [Tr]). But L obviously belongs to the dual space to $C^1(T^n)$. Hence L is in the dual space to $H^{1+n/p+\varepsilon,p}(T^n)$ that is in $H^{-1-n/p-\varepsilon,p'}(T^n)$ where $1/p + 1/p' = 1$ (let us suppose that $n/p + \varepsilon$ is not an integer). But then the standard regularity results for the equation $\Delta g = f$ imply that $\tilde{L} \subset H^{1-n/p-\varepsilon,p'}(T^n)$. Taking p sufficiently large (so that $1 - n/p > 0$) we see that $\tilde{L} \subset L^{p'}(T^n)$ then. Therefore $\tilde{L} = 0$ because all distributions from $\tilde{L}$ are supported on a set of the Lebesgue measure 0. □

4 Non-compact case

In this section we shall briefly describe generalizations of all results formulated in Sect. 2 to non-compact manifolds (possibly with boundary). This generalization supposes that boundary conditions and conditions at infinity are given, so that the operator A on an appropriate domain will still be a Fredholm operator in the usual sense. We shall describe the corresponding generalization axiomatically so as to avoid technicalities and achieve bigger generality. The corresponding context was introduced in [G-S] for the same purpose and we shall follow closely the exposition given there.

In a subsequent paper we shall consider generalizations to the case of elliptic operators on covering manifolds and L^2-solutions with singularities. But this case requires completely different technique since it leads to operators which are Fredholm in the sense of Breuer in appropriate von Neumann algebras.

So let X be a non-compact manifold with boundary ∂X (which need not be compact either).

Let E, F be complex vector bundles over the open manifold of all interior points of X, which we denote $\operatorname{Int} X$. Let

(4.1) $$A : C^\infty(\operatorname{Int} X, E) \to C^\infty(\operatorname{Int} X, F)$$

be an elliptic differential operator. Let E^*, F^* be another pair of complex vector bundles over $\operatorname{Int} X$, such that non-degenerate bilinear or sesquilinear C^∞-pairings of bundles

$$E \times E^* \to \Omega(\operatorname{Int} X),\ F \times F^* \to \Omega(\operatorname{Int} X)$$

are given. For any vector bundle E over $\operatorname{Int} X$ denote $C_c^\infty(\operatorname{Int} X, E)$ the set of all C^∞-sections of E over $\operatorname{Int} X$, having compact support inside $\operatorname{Int} X$. So if $u \in C_c^\infty(\operatorname{Int} X, E)$ then $\operatorname{supp} u$ is a compact set in $\operatorname{Int} X$; in particular $\operatorname{supp} u$ does not intersect ∂X.

In this case the adjoint elliptic differential operator

$$A^*: C^\infty(\operatorname{Int} X, F^*) \to C^\infty(\operatorname{Int} X, E^*)$$

is defined again by the identity

$$(Au, v)_F = (u, A^*v)_E, \quad u \in C_c^\infty(\operatorname{Int} X, E), \quad v \in C_c^\infty(\operatorname{Int} X, F^*),$$

where the dualities $(\cdot, \cdot)_E$, $(\cdot, \cdot)_F$ are defined exactly as for the compact case.

Now suppose that the domains of A and A^* are distinguished as linear subspaces $\operatorname{Dom} A$ and $\operatorname{Dom} A^*$ such that

$$C_c^\infty(\operatorname{Int} X, E) \subset \operatorname{Dom} A \subset C^\infty(\operatorname{Int} X, E),$$

$$C_c^\infty(\operatorname{Int} X, F^*) \subset \operatorname{Dom} A^* \subset C^\infty(\operatorname{Int} X, F^*).$$

They may be defined e.g. by a choice of boundary conditions and appropriate conditions at infinity. Then let us define images of A, A^* as

$$\operatorname{Im} A = A(\operatorname{Dom} A), \quad \operatorname{Im} A^* = A^*(\operatorname{Dom} A^*).$$

Suppose also that linear subspaces $\operatorname{Dom}' A$ and $\operatorname{Dom}' A^*$ are given such that

$$C_c^\infty(\operatorname{Int} X, E^*) \subset \operatorname{Dom}' A \subset C^\infty(\operatorname{Int} X, E^*), \operatorname{Im} A^* \subset \operatorname{Dom}' A,$$

$$C_c^\infty(\operatorname{Int} X, F) \subset \operatorname{Dom}' A^* \subset C^\infty(\operatorname{Int} X, F), \operatorname{Im} A \subset \operatorname{Dom}' A^*.$$

We shall suppose that the following integrability condition is satisfied:

(4.2) $$x \mapsto (v, \tilde{v})_x \quad \text{and} \quad x \mapsto (u, \tilde{u})_x$$

are L^1-densities on $\operatorname{Int} X$ for any $v \in \operatorname{Dom}' A^*$, $\tilde{v} \in \operatorname{Dom} A^*$, $u \in \operatorname{Dom} A$, $\tilde{u} \in \operatorname{Dom}' A$ (i.e. these densities are absolutely integrable in Lebesgue sense over $\operatorname{Int} X$).

Integrating these densities over $\operatorname{Int} X$ we obtain bilinear or sesquilinear pairings

$$\operatorname{Dom}' A^* \times \operatorname{Dom} A^* \to \mathbf{C}, \quad \operatorname{Dom} A \times \operatorname{Dom}' A \to \mathbf{C}$$

which will be denoted $(\cdot, \cdot)_F$ and $(\cdot, \cdot)_E$, the same way as for sections with compact support.

Our next requirement is

(4.3) $$(Au, v)_F = (u, A^*v)_E, \quad u \in \operatorname{Dom} A, v \in \operatorname{Dom} A^*.$$

Now if we are given a linear subspace $L \subset \operatorname{Dom} A$, then its annihilator L° is naturally defined as a linear subspace in $\operatorname{Dom}' A$:

$$L^\circ = \{\tilde{u} \mid \tilde{u} \in \operatorname{Dom}' A, (u, \tilde{u})_E = 0 \text{ for every } u \in L\}.$$

Similarly for a linear subspace $M \subset \operatorname{Dom} A^*$ its annihilator $M^\circ \subset \operatorname{Dom}' A^*$ is naturally defined.

Define also $\operatorname{Ker} A$ and $\operatorname{Ker} A^*$ as linear subspaces in $\operatorname{Dom} A$ and $\operatorname{Dom} A^*$ respectively, e.g.

$$\operatorname{Ker} A = \{u \mid u \in \operatorname{Dom} A, Au = 0\}.$$

Now our next requirement is

(4.4) *A and A^* are Fredholm in the following sense:*
(i) $\dim \operatorname{Ker} A < \infty$, $\dim \operatorname{Ker} A^* < \infty$;
(ii) $\operatorname{Im} A = (\operatorname{Ker} A^*)^{\circ}$, $\operatorname{Im} A^* = (\operatorname{Ker} A)^{\circ}$.

Hence

$$\operatorname{ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A = \dim \operatorname{Ker} A - \dim \operatorname{Ker} A^*$$

is well defined.

Example 4.1 Let X be a compact Riemannian manifold with a smooth boundary, $A = A^* = \Delta$ is the Laplacian of the given Riemannian metric. Then taking $E = F = E^* = F^* = \mathbf{C}_X$ (the trivial vector bundle with the fiber $\mathbf{C}$ over X) and defining the duality by the use of the Riemannian volume, we can take

$$\operatorname{Dom} A = \operatorname{Dom} A^* = \{u \mid u \in C^\infty(X),\ u|_{\partial X} = 0\}$$

and

$$\operatorname{Dom}' A = \operatorname{Dom}' A^* = C^\infty(X),$$

i.e. define A as the Laplacian with the Dirichlet boundary condition. Then conditions (4.2)–(4.4) are satisfied and $\operatorname{ind} A = 0$.

Similarly the Neuman condition can be also considered.

Example 4.2 Let $X = \mathbf{R}^n$, $n \geqq 3$, $A = A^* = \Delta$ (the standard Laplacian or the Laplacian of the flat metric), $E = F = E^* = F^* = \mathbf{C}_X$ and

$$\operatorname{Dom} A = \operatorname{Dom} A^* = \{u \mid u \in C^\infty(\mathbf{R}^n),\ \Delta u \in C_c^\infty(\mathbf{R}^n) \text{ and } u(x) \to 0 \text{ as } |x| \to \infty\}$$

$$\operatorname{Dom}' A = \operatorname{Dom}' A^* = C_c^\infty(\mathbf{R}^n).$$

Here $C_c^\infty(\mathbf{R}^n) = C^\infty(\mathbf{R}^n) \cap \mathscr{E}'(\mathbf{R}^n)$ is the set of all C^∞-functions with a compact support.

Note that the condition $u \to \infty$ as $|x| \to \infty$ can be replaced by a formally stronger but in fact equivalent condition

$$u(x) = O(|x|^{2-n}) \text{ as } |x| \to \infty$$

because $u \in \operatorname{Dom} A$ is a harmonic function near infinity. It easily follows that all conditions (4.2)–(4.4) are satisfied because $\operatorname{Ker} A = \operatorname{Ker} A^* = \{0\}$ by the Liouville theorem and $\operatorname{Im} A = \operatorname{Im} A^* = C_c^\infty(\mathbf{R}^n)$ because the equation $\Delta u = f$ with any $f \in C_c^\infty(\mathbf{R}^n)$ can be solved by taking the convolution of f with the standard fundamental solution $c_n|x|^{2-n}$.

Definition 4.3 ***Rigged divisor*** *(associated with A) is a tuple*

$$\mu = (D^+, L^+; D^-, L^-),$$

where $D^\pm$ are compact nowhere dense disjoint subsets in $\operatorname{Int} X$, *$L^\pm$ are finite-dimensional linear spaces of distributional sections,*

$$L^+ \subset \mathscr{E}'_{D^+}(\operatorname{Int} X, F),\ L^- \subset \mathscr{E}'_{D^-}(\operatorname{Int} X, E^*).$$

Denote $l^\pm = \dim L^\pm$.

The "secondary" spaces $\tilde{L}^{\pm}$ are defined exactly as in Sect. 2, X should be replaced by $\operatorname{Int} X$. We do not change the notation for their dimensions. No change is necessary in Definitions 2.2 and 2.3.

We shall write that $u \in \operatorname{Dom} A$ outside a compact set $K \subset \operatorname{Int} X$ if $u \in C^{\infty}(\operatorname{Int} X - K, E)$ and there exists $\tilde{u} \in \operatorname{Dom} A$ such that $\tilde{u} = u$ on $X - K$.

Definition 4.4 *Denote*

$$L(\mu, A) = \{u \mid u \in \operatorname{Dom} A \text{ outside a compact neighbourhood of } D^+;$$
$$\text{there exists } \tilde{u} \in \mathscr{D}'(X, E), \text{ such that } \tilde{u} = u \text{ on } X - D^+ \text{ and } A\tilde{u} \in L^+;$$
$$(u, L^-) = 0\};$$

$$r(\mu, A) = \dim L(\mu, A).$$

Theorem 4.5 *Let A be an elliptic operator* (4.1) *such that the conditions* (4.2)–(4.4) *are satisfied, and μ is a rigged divisor associated with A. Then*

$$r(\mu, A) = \operatorname{ind} A + \deg_A(\mu) + r(\mu^{-1}, A^*). \tag{4.5}$$

Let us introduce necessary spaces to extend the localization and the duality used in the compact case. Denote

$$\Gamma(X, \mu, A) = \{u \mid u \in \operatorname{Dom} A \text{ outside a compact neighbourhood of } D^+,$$
$$\exists \tilde{u} \in \mathscr{D}'(\operatorname{Int} X, E), \tilde{u} = u \text{ on } \operatorname{Int} X - D^+, A\tilde{u} \in L^+ + C^{\infty}(\operatorname{Int} X, F),$$
$$(u, L^-) = 0\}.$$

The space of all possible regularizations $\tilde{u}$ is naturally introduced as follows:

$$\tilde{\Gamma}(X, \mu, A) = \{\tilde{u} \mid \tilde{u} \in \mathscr{D}'(\operatorname{Int} X, E), \tilde{u} \in \operatorname{Dom} A \text{ outside a compact neighbourhood of } D^+,$$
$$A\tilde{u} \in L^+ + C^{\infty}(\operatorname{Int} X, F), (\tilde{u}, L^-) = 0\}.$$

Then the exact sequence (2.2) still holds.

Denote also

$$\Gamma_{\mu}(X, A) = \{u \mid u \in \operatorname{Dom} A, (u, L^-) = 0\}$$

and

$$\tilde{\Gamma}_{\mu}(X, A) = \{f \mid f \in \operatorname{Dom}' A^*, (f, \tilde{L}^-) = 0\}.$$

Then as in the compact case for any $u \in \Gamma(X, \mu, A)$ we can consider Au on $\operatorname{Int} X - D^+$ and denote its extension by continuity to $\operatorname{Int} X$ by $\tilde{A}u$. In this way we again obtain a linear map

$$\tilde{A}: \Gamma(X, \mu, A) \to \tilde{\Gamma}_{\mu}(X, A).$$

Now the extensions of Definition 2.8 (duality), Lemma 2.9, Theorem 2.10 and the proofs of Theorem 4.4 and the extended duality theorem do not require any changes.

Example 4.6 Let us consider the situation of the Example 4.2 and take

$$D^+ = \{x_1, \ldots, x_k\}, D^- = \{y_1, \ldots, y_l\},$$

where all the points $x_1, \ldots, x_k, y_1, \ldots, y_l$ are distinct. Define the distribution spaces $L^\pm \subset \mathcal{D}'(\mathbf{R}^n)$ as follows:

$$L^+ = \left\{ \sum_{i=1}^{k} c_i \delta(x - x_i),\ c_i \in \mathbf{C} \right\},$$

$$L^- = \left\{ \sum_{j=1}^{l} \sum_{\alpha=1}^{n} c_{j\alpha} \frac{\partial}{\partial x_\alpha} \delta(x - y_j),\ c_{j\alpha} \in \mathbf{C} \right\}.$$

Consider the rigged divisor $\mu = (D^+, L^+; D^-, L^-)$. Then $L(\mu, \Delta)$ consists of functions of the form

$$u(x) = \sum_{i=1}^{k} \frac{q_i}{|x - x_i|^{n-2}}$$

(which are generalized Coulomb potentials of point charges $q_1, \ldots, q_k \in \mathbf{C}$, situated at the points $x_1, \ldots, x_k \in \mathbf{R}^n$), such that $\nabla u(y_j) = 0, j = 1, \ldots, l$, i.e. the points y_j are equilibrium positions in the electrostatic field of the given system of charges.

Note also that on the other hand $L(\mu^{-1}, \Delta)$ consists of the functions of the form

$$v(x) = \sum_{j=1}^{l} \sum_{\alpha=1}^{n} c_{j\alpha} \frac{\partial}{\partial x_\alpha} \frac{1}{|x - y_j|^{n-2}}, \quad c_{j\alpha} \in \mathbf{C}$$

(which are dipole potentials of dipoles situated at the points y_j) such that $v(x_i) = 0, i = 1, \ldots, k$.

It is easy to check that the secondary spaces $\tilde{L}^\pm$ vanish in this case. Therefore $\deg_\Delta(\mu) = k - 3l$. Since the index vanishes too, the Theorem 4.5 gives in this example

$$r(\mu, \Delta) = k - 3l + r(\mu^{-1}, \Delta).$$

It follows that

$$r(\mu, \Delta) \geqq k - 3l,\ r(\mu^{-1}, \Delta) \geqq 3l - k,$$

though these inequalities have an easy elementary proof.

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