ADVANCES IN SOVIET MATHEMATICS
Volume 16, Part 1, 1993

The Riemann-Roch Theorem for Elliptic Operators

MIKHAEL GROMOV AND MIKHAIL A. SHUBIN

§1. Introduction

The classical Riemann-Roch theorem for nonsingular complex algebraic curves has been generalized in different ways to multidimensional situations. The best known generalizations are the Riemann-Roch-Hirzebruch theorem and the Riemann-Roch-Grothendieck theorem, both of them motivated by algebraic geometry. In this paper we suggest a generalization that is motivated by classical analysis of solutions of general elliptic equations with point singularities. Namely, for any real C^∞-manifold we introduce point divisors which are elements of a free abelian group generated by the points of this manifold. Then we define spaces of "meromorphic" solutions of an elliptic equation depending on a given divisor: these solutions are allowed to have some poles (at points that enter to the divisor with positive degrees) and are required to have zeros (at the points that enter to the divisor with negative degrees). The main theorem gives a formula that connects dimensions of two such spaces of "meromorphic" solutions corresponding to a given divisor and its dual (inverse) divisor. This formula contains also the index of the given elliptic operator (that can be calculated by the Atiyah-Singer formula) and a degree of the divisor, which is written in terms of binomial coefficients and depends on the dimension of the manifold and on the order of the operator. It is a direct generalization of the classical Riemann-Roch theorem, but on the other hand it can be considered as an extension of the Atiyah-Singer index formula. We consider first the case of compact closed manifolds and then give a generalization to noncompact manifolds with a compact boundary. In this case we need appropriate boundary conditions and conditions at infinity are imposed in order to ensure that the given elliptic operator defines a Fredholm operator in appropriate spaces.

A special case of the scalar Laplacian on a Riemannian manifold was considered in a beautiful paper by Nadirashvili [1], which we took as a starting point. But our proof is based on different ideas and is even simpler than the

1991 *Mathematics Subject Classification.* Primary 58G03, 58G10.

proof in [1] due to its generality that allows us a more transparent use of duality arguments.

The main theorem is proved simultaneously with a duality theorem that gives a solvability condition for the given elliptic equation if the solution is allowed to have poles and is required to have zeros at prescribed points.

We also give a number of applications including those similar to well-known applications of the classical Riemann-Roch theorem. The first of them is an analogue of the Riemann inequality, and it gives an existence result for nontrivial meromorphic solutions that have zeros at a given finite set with multiplicities bounded from below by given numbers. This can be achieved provided that a sufficient number of poles (or a pole of sufficiently high order) is allowed. We also supply a more elementary proof of the inequality (this proof does not use our main theorem).

Next we prove that if the adjoint operator has a unique continuation property, then sometimes even the equality can be proved; however, this result is not as effective as in the classical case of ample divisors. On the other hand, we present simple arguments showing that no effective general result is possible in the same terms as in the classical case. We also show that a condition of unique continuation type is necessary for the equality to be true.

As an application of the duality theorem we prove a local solvability result for an elliptic equation with a right-hand side and with an additional condition on the order of zero of the solution at a given point.

Then we investigate simplest properties of sheaves of solutions, naturally associated with point divisors, i.e., sheaves of meromorphic solutions in the sense described above. The mentioned local solvability result leads to a construction of a fine resolution for such a sheaf. Applying this resolution we obtain triviality of cohomologies of these sheaves in dimensions 2 and higher and also a Serre type duality between cohomologies in dimensions 0 and 1.

Using the Riemann-Roch theorem for the Laplacian in the Euclidean space, Nadirashvili [1] proved an estimate for the maximal possible multiplicity of zeros for the Coulomb potential of a finite system of point charges in $\mathbb{R}^3$. At the end of the last section of this paper we reproduce his arguments in more detail, extending them to the Coulomb potentials in $\mathbb{R}^n$. We put basic arguments of Nadirashvili in a more general context (Riemannian manifolds) and also give a more elementary proof of his estimate (a proof that does not use a Riemann-Roch-type theorem). After that we study a similar problem for Riemann surfaces and prove estimates for the maximal possible multiplicity of zeros for harmonic functions with simple (logarithmic) singularities at points from a given finite subset of a Riemann surface. For the case of the Riemann sphere we also prove a precise (though elementary) estimate for this maximal multiplicity and give necessary and sufficient conditions for the existence of a harmonic function with a given configuration of several poles and a zero, so that the zero has the maximal possible multiplicity. This condition is naturally formulated in terms of cross ratios in this configuration.

It is also possible to prove an L^2-version of the main result (on covering manifolds) but we will do it in a separate paper.

Another possible generalization of the main result deals with the case of multidimensional divisors. More exactly, one can introduce the so called "rigged divisors" that consist of a pair of disjoint closed subsets in the manifold and a finite-dimensional space of distributions supported on each of them. Then a finite-dimensional set of singularities of solutions on the first subset may be allowed and finitely many vanishing conditions on the second subset imposed; both singularities and vanishing conditions are defined by the given distribution spaces. Then the Riemann-Roch-type theorem holds in this case too and we will also treat it in a separate paper.

The results of this paper were partially announced in [5], but the reader should keep in mind that a different sign convention was used there. Here we put the sign conventions in agreement with the classical Riemann-Roch theorem.

The second author is grateful to Forschungsinstitut für Mathematik ETH Zürich for the support during the time when this paper was being written, and also to Rahel Boller who partially typed the manuscript.

§2. Notations and main results

A. Let X be a compact closed C^∞-manifold, $\dim X = n \geqslant 2$, E and F complex vector bundles over X, $q = \dim_{\mathbb{C}} E_x = \dim_{\mathbb{C}} F_x$ the dimension of their fibres, and $\Gamma(U, E)$ the space of all C^∞-sections of E over an open set $U \subset X$. Let

$$A: \Gamma(X, E) \longrightarrow \Gamma(X, F) \tag{2.1}$$

be an elliptic differential operator of order d and $\operatorname{ind} A$ its index that can be calculated by the Atiyah-Singer index formula. Let $\Omega(X)$ be the bundle of complex densities on X, so for every $\omega \in \Gamma(X, \Omega(X))$ the integral $\int_X \omega \in \mathbb{C}$ is naturally defined. If E is a complex vector bundle over X then the bundle $E^* = \operatorname{Hom}_{\mathbb{C}}(E, \Omega(X))$ is called the dual bundle and there is a natural pairing of bundles $E \otimes E^* \to \Omega(X)$ which gives the pairing in sections

$$\langle \cdot, \cdot \rangle : \Gamma(X, E) \times \Gamma(X, E^*) \longrightarrow \mathbb{C}, \qquad \langle u, v \rangle = \int_X \langle u(x), v(x) \rangle, \tag{2.2}$$

where $\langle u(x), v(x) \rangle \in \Omega_x(X)$ is obtained by use of the pairing between E_x and E_x^*. Now using the pairing (2.2) define the transposed operator $A^t : \Gamma(X, F^*) \to \Gamma(X, E^*)$ by the formula

$$\langle Au, v \rangle = \langle u, A^t v \rangle, \qquad u \in \Gamma(X, E), \ v \in \Gamma(X, F^*). \tag{2.3}$$

Then A^t is again an elliptic differential operator. Note that $\operatorname{ind} A^t = -\operatorname{ind} A$.

Actually it is often convenient to use more general "dual" (adjoint or transpose) operators to a given differential operator (2.1). Namely, suppose that $\widetilde{E}$, $\widetilde{F}$ are vector bundles given together with nondegenerate bilinear or sesquilinear C^∞-pairings of bundles

$$E \times \widetilde{E} \longrightarrow \Omega(X), \qquad F \times \widetilde{F} \longrightarrow \Omega(X).$$

We denote these pairings at a point $x \in X$ by $\langle \cdot , \cdot \rangle_x$.

An example of such a situation arises e.g. if positive definite hermitian C^∞-metrics h_E, h_F are given in E and F and a positive smooth density $d\mu$ on X is fixed. Then we can take $\widetilde{E} = E$, $\widetilde{F} = F$, defining the pairings by

$$\langle u, \widetilde{u} \rangle_x = h_E(u, \widetilde{u})\, d\mu(x), \qquad u, \widetilde{u} \in E_x,$$

and similarly for F.

In the general case we have bilinear or sesquilinear pairings

$$\Gamma(X, E) \times \Gamma(X, \widetilde{E}) \longrightarrow \mathbb{C}, \qquad \Gamma(X, F) \times \Gamma(X, \widetilde{F}) \longrightarrow \mathbb{C}$$

obtained by the fiberwise pairing and integration. Denoting these pairings $\langle \cdot , \cdot \rangle_E$ and $\langle \cdot , \cdot \rangle_F$, we can define the operator $A^* : \Gamma(X, \widetilde{F}) \longrightarrow \Gamma(X, \widetilde{E})$ by the formula

$$\langle Au, v \rangle_F = \langle u, A^* v \rangle_E, \qquad u \in \Gamma(X, E),\ v \in \Gamma(X, \widetilde{F}).$$

Let us describe a connection between the general adjoint operator A^* and the canonical transpose A^t. To do this define a (linear or antilinear) bundle isomorphism $J_E : \widetilde{E} \longrightarrow E^*$ by

$$J_E \widetilde{e} = \langle \cdot , \widetilde{e} \rangle_x, \qquad \widetilde{e} \in \widetilde{E}_x,$$

where $\langle \cdot , \cdot \rangle_x$ denotes the given pairing $E_x \times \widetilde{E}_x \longrightarrow \Omega(X)_x$. Similarly define a bundle isomorphism $J_F : \widetilde{F} \longrightarrow F^*$. We denote the corresponding isomorphisms on sections by the same letters.

LEMMA 2.1. $A^* = J_E^{-1} A^t J_F$.

PROOF. For any $u \in \Gamma(X, E)$, $\widetilde{v} \in \Gamma(X, \widetilde{F})$ we have

$$\begin{aligned} \langle Au, \widetilde{v} \rangle_E &= \langle Au, J_F \widetilde{v} \rangle = \langle u, A^t J_F \widetilde{v} \rangle \\ &= \langle u, J_E (J_E^{-1} A^t J_F) \widetilde{v} \rangle = \langle u, J_E^{-1} A^t J_F \widetilde{v} \rangle_F, \end{aligned}$$

hence $A^* = J_E^{-1} A^t J_F$ as required.

It follows in particular that $\dim \operatorname{Ker} A^* = \dim \operatorname{Ker} A^t$, hence $\dim \operatorname{Ker} A^*$ does not depend on the choice of arbitrary elements in the definition of A^*.

We use point divisors, which are elements of free abelian groups generated by points $x \in X$. For brevity we refer to them simply as divisors. We write the operation in the group of the divisors as multiplication. Hence such a divisor can be written in the form

$$\mu = x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m}, \tag{2.4}$$

where $x_i \neq x_j$ if $i \neq j$ and $p_i \in \mathbb{Z} - \{0\}$. We fix such a divisor for some time. The dual divisor is defined as

$$\mu^{-1} = x_1^{-p_1} x_2^{-p_2} \cdots x_m^{-p_m}$$

and we introduce also the positive and negative parts of μ as divisors

$$\lambda = x_1^{p_1^+} \cdots x_m^{p_m^+}, \qquad \nu = x_1^{p_1^-} \cdots x_m^{p_m^-}$$

with $p^+ = \max(p, 0)$, $p^- = \min(p, 0)$. Here all factors of the form x_i^r with $r = 0$ have to be omitted. We evidently have $\mu = \lambda\nu$ and $\mu^{-1} = \nu^{-1}\lambda^{-1}$, where ν^{-1} and λ^{-1} are the positive and negative part of μ^{-1} respectively.

For the divisor (2.4) (with $p_i \neq 0$) we use the notation

$$\operatorname{supp} \mu = \{x_1, \ldots, x_m\}.$$

Therefore, $\operatorname{supp} \mu$ is a finite subset in X such that $\operatorname{supp} \mu = \operatorname{supp} \lambda \cup \operatorname{supp} \nu$, $\operatorname{supp} \lambda \cap \operatorname{supp} \nu = \varnothing$.

DEFINITION 2.1. The *degree* of the divisor μ is defined as

$$d(\mu) = \sum_{1 \leqslant i \leqslant m} \operatorname{sign} p_i \left[\binom{|p_i| + n - 1}{n} - \binom{|p_i| + n - 1 - d}{n} \right], \tag{2.5}$$

where $\binom{N}{n} = \frac{N!}{n!(N-n)!}$ if $N \geqslant n$ and 0 otherwise.

Note that in this definition $d(\mu)$ depends also on the order d of the operator A. Actually if $|p_i|$ is sufficiently large then the expression in the square brackets can be written as $f_{n,d}(|p_i|)$ where $f_{n,d}$ is a polynomial of degree $n-1$ with the coefficient $d/(n-1)!$ by x^{n-1}:

$$f_{n,d}(x) = \frac{d}{(n-1)!} x^{n-1} + \cdots. \tag{2.6}$$

In particular

$$f_{2,d}(x) = dx - \frac{d(d-1)}{2}.$$

It is easy to check that if $n = 2$, $d = 1$ then

$$d(\mu) = \sum_{1 \leqslant i \leqslant m} p_i, \tag{2.7}$$

which gives the usual degree of the divisor on a (2-dimensional) surface.

Now we introduce solutions with poles and zeros. Let $x_0 \in X$, $p \in \mathbb{Z} - \{0\}$, $u \in \Gamma(U - \{x_0\}, E)$ where U is a neighbourhood of x_0 in X, and $Au = 0$ in $U - \{x_0\}$.

DEFINITION 2.2. (i) Suppose $p > 0$. Then we write $\operatorname{ord}_{x_0} u \geqslant -p$ iff $u = u_s + u_r$, where $u_r \in \Gamma(U, E)$ and

$$u_s(x) = o(|x - x_0|^{d-n-p}) \quad \text{as } x \longrightarrow x_0. \tag{2.8}$$

(ii) Suppose $p < 0$. Then we write $\operatorname{ord}_{x_0} u \geqslant -p$ iff u can be extended to a section $u \in \Gamma(U, E)$ and

$$u(x) = o(|x - x_0|^{|p|-1}) \quad \text{as } x \longrightarrow x_0. \tag{2.9}$$

Equivalently we could also write in this case $u(x) = O(|x-x_0|^{|p|})$ as $x \to x_0$, or $j_{x_0}^{|p|-1} u = 0$ where $j_{x_0}^{|p|-1} u$ is the jet of order $|p|-1$ of the section u at x_0.

We write $\operatorname{ord}_{x_0} u = -p$ if $\operatorname{ord}_{x_0} u \geqslant -p$ but it is not true that $\operatorname{ord}_{x_0} u \geqslant -p+1$.

The given definition is compatible with the standard definition of the order of a pole or a zero for a meromorphic function on a Riemann surface (this corresponds to the case $A = \bar{\partial}$, $n = 2$, $d = 1$). A motivation of (2.8) in the general case becomes clear if we consider the well-known structure of the fundamental solution of the elliptic operator A (see, e.g., [2]). If x_0 is an isolated singularity of a solution of the equation $Au = 0$ (i.e., u is a solution in $U - \{x_0\}$ where U is a neighbourhood of x_0) and $u(x) = o(|x - x_0|^{-N})$ with $N \in \mathbb{R}$ then we can find a "regularization" of u, a distributional section $\widetilde{u}$ of E over U such that $\widetilde{u} = u$ on $U - \{x_0\}$. Then $A\widetilde{u}$ is a distributional section of F supported at x_0, hence locally

$$A\widetilde{u}(x) = \sum_{|\alpha| \leqslant M} c_\alpha \partial_x^\alpha \delta(x - x_0)$$

with $c_\alpha \in F_{x_0}$. Here $\delta(x-x_0)$ is the standard Dirac measure. Now if B is a local parametrix of the operator A (so B is a classical or polyhomogeneous pseudodifferential operator of order $-d$) then we can conclude that near x_0 section $\widetilde{u}$ has the form

$$\widetilde{u}(x) = \sum_{|\alpha| \leqslant M} (-\partial_{x_0})^\alpha K_B(x, x_0) c_\alpha + u_r(x),$$

where $u_r \in \Gamma(U, E)$, K_B is the Schwartz kernel of B. Now K_B is a Fourier transform of the symbol b of $B = b(x, D_x)$, i.e.,

$$K_B(x, x_0) = (2\pi)^{-n} \int e^{i(x-x_0)\cdot\xi} b(x, \xi)\, d\xi.$$

But since b is polyhomogeneous, K_B is the sum of homogeneous functions (with respect to $x - x_0$) up to logarithmic terms and up to a sufficiently smooth remainder. The orders of homogeneity of terms in K_B are actually $d-n, d-n+1, d-n+2, \ldots$, hence in $\partial_{x_0}^\alpha K_B$ we have terms of homogeneity degree $d - n - |\alpha|$, $d - n - |\alpha| + 1, \ldots$. Hence the condition $u(x) = o(|x - x_0|^{d-n})$ eliminates all singularities, the condition $u(x) = o(|x-x_0|^{d-n-1})$ allows the simplest singularity of the form $K_B(x, x_0)c$ with $c \in F_{x_0}$, etc.

Remark. If $d - n - p < 0$ then the condition on u in Definition 2.2(i) can be replaced by a simpler condition (given in terms of u instead of u_s):

$$u(x) = o(|x - x_0|^{d-n-p}) \quad \text{as } x \to x_0.$$

In the general case the condition can be written in terms of derivatives of u:

$$\partial^\alpha u(x) = o(|x - x_0|^{d-n-p-|\alpha|}) \quad \text{if } |\alpha| \geqslant d - n - p + 1,$$

and in fact, equivalently, it is sufficient to consider only α with $|\alpha| = d - n - p - 1$.

Now we introduce the space of solutions with poles and zeros subordinated to the divisor μ (of the form (2.4)) and the dimension of this space.

DEFINITION 2.3. Denote

$$L(\mu, A) = \{u \mid u \in \Gamma(X - \operatorname{supp}\mu, E),\ Au = 0 \text{ on } X - \operatorname{supp}\mu,\ \operatorname{ord}_{x_i} u \geqslant -p_i \text{ for each } i = 1, \dots, m\},$$

$$r(\mu, A) = \dim_{\mathbb{C}} L(\mu, A).$$

Our first main result is

THEOREM 2.1. *We have*

$$r(\mu, A) = \operatorname{ind} A + qd(\mu) + r(\mu^{-1}, A^*). \tag{2.10}$$

Note that $r(\mu^{-1}, A^*) = r(\mu^{-1}, A')$ due to Lemma 2.1, hence $r(\mu^{-1}, A^*)$ does not depend on the choice of the arbitrary elements involved in the definition of A^*.

Now let us mention two particular cases of Theorem 2.1.

EXAMPLE 2.1 (Riemann-Roch). Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = 1$, hence $n = 2$. (In other words we might say that X is a compact Riemannian surface or a nonsingular algebraic curve over $\mathbb{C}$.) Consider the standard splitting of the cotangent bundle

$$T^*X = T^{1,0}(X) \oplus T^{0,1}(X),$$

where $T^{1,0}(X)$ and $T^{0,1}(X)$ consist of $(1, 0)$- and $(0, 1)$-forms respectively, the spaces of their C^∞-sections being denoted $\Lambda^{1,0}(X)$ and $\Lambda^{0,1}(X)$. Now let us take

$$A = \bar\partial : C^\infty(X) \longrightarrow \Lambda^{0,1}(X).$$

Using the standard orientation on X we can identify $\Omega(X)$ with $\Lambda^2(T^*X)$, hence $\Gamma(X, \Omega(X))$ with $\Lambda^2(X)$. We have the canonical pairing (given by the wedge product)

$$T^{0,1}(X) \times T^{1,0}(X) \longrightarrow \Lambda^2(T^*X).$$

Now if $\mathbb{C}_X$ is the trivial vector bundle (with the fiber $\mathbb{C}$) over X then $(\mathbb{C}_X)^* = \Lambda^2(T^*X)$. Using the Stokes formula it is easy to check that

$$A^* = \bar\partial : \Lambda^{1,0}(X) \longrightarrow \Lambda^{1,1}(X) = \Lambda^2(X)$$

for the given dualities.

It is well known [3] that $\operatorname{ind} A = 1 - g$, where g is the genus of X (actually $\dim \operatorname{Ker} A = 1$, since $\operatorname{Ker} A = \{\text{const}\}$ and $\dim \operatorname{Ker} A^* = g$, since

$\operatorname{Ker} A^*$ is the space of all holomorphic forms on X). Now $L(\mu, A)$ and $L(\mu^{-1}, A^*)$ become spaces of meromorphic functions and forms respectively with the restriction on orders of poles and zeros exactly as in the classical Riemann-Roch theorem.

Now note that in this case $q = 1$ and (2.7) is true. Hence (2.10) leads in this example to the classical Riemann-Roch theorem:

$$r(\mu) = 1 - g + d(\mu) + r'(\mu^{-1})$$

where

$$r(\mu) = r(\mu, \bar\partial), \quad \bar\partial : C^\infty(X) \to \Lambda^{0,1}(X),$$
$$r'(\mu^{-1}) = r(\mu^{-1}, \bar\partial^*), \quad d(\mu) = \sum p_i.$$

EXAMPLE 2.2 (Nadirashvili [1]). Let X be a compact Riemannian manifold and Δ_g the scalar Laplace-Beltrami operator, $\Delta_g : C^\infty(X) \to C^\infty(X)$. Using the standard Riemannian density we can identify $\Omega(X)$ with $\mathbb{C}_X$ and Δ_g^* with Δ_g. Evidently, $\operatorname{ind} \Delta_g = 0$, so (2.10) here has the form

$$r(\mu, \Delta_g) = d(\mu) + r(\mu^{-1}, \Delta_g), \tag{2.11}$$

and this was proved in [1]. Actually Nadirashvili considers some noncompact Riemannian manifolds with boundary introducing elliptic boundary conditions and conditions at infinity satisfying appropriate conditions (to make the corresponding closure of Δ_g a Fredholm selfadjoint operator). As a corollary of (2.11) he proves that the Coulomb potential of m point charges cannot have a zero of order $\geqslant m$ (unless it is identically zero). We shall make further comments about the noncompact case later, and will also consider estimates of multiplicities of zeros in Section 4.

B. Now we shall introduce new spaces that play an important role in the proof of Theorem 2.1 but on the other hand allow us to formulate a duality theorem which is important by itself.

First we introduce the space $\Gamma(X, \mu, A)$ of sections $u \in \Gamma(X - \operatorname{supp}\lambda, E)$ having prescribed poles and zeros but not necessarily solutions of $Au = 0$:

$$\begin{aligned}\Gamma(X, \mu, A) = \{u \mid u \in \Gamma(X - \operatorname{supp}\lambda, E);\ & j_{x_i}^{|p_i|-1} u = 0 \text{ if } \\ & x_i \in \operatorname{supp}\nu;\ \text{for every } x_i \in \operatorname{supp}\lambda \text{ there exist a} \\ & \text{neighbourhood } U \text{ of } x_i \text{ and a representation} \\ & u = u_s + u_r, \text{ where } u_s \in \Gamma(U - \{x_i\}, E),\ Au_s = \\ & 0 \text{ in } U - \{x_i\},\ \operatorname{ord}_{x_i} u_s \geqslant -p_i, \text{ and } u_r \\ & \text{can be extended to a section } \bar u_r \in \Gamma(U, E)\}.\end{aligned} \tag{2.12}$$

So actually we allow only singularities that can occur as singularities of solutions of $Au = 0$. The space $\Gamma(X, \mu, A)$ includes sections with the same zeros and singularities as allowed in the definition of $L(\mu, A)$, but on the other hand (2.12) contains no global restrictions, so all possible local singularities and zeros can be present at every point $x_i \in \operatorname{supp}\mu$ independently from what happens at other points.

Now introduce the reduced divisor

$$\widetilde{\mu} = x_1^{\widetilde{p}_1} x_2^{\widetilde{p}_2} \cdots x_m^{\widetilde{p}_m} \tag{2.13}$$

where $\widetilde{p}_i = \operatorname{sign} p_i(|p_i| - d)^+$ and the factors $x_i^{\widetilde{p}_i}$ with $\widetilde{p}_i = 0$ have to be omitted. So compared with μ, the absolute value of every exponent decreases by d (or becomes 0 if it initially was less than d). Note that $\widetilde{\mu^{-1}} = (\widetilde{\mu})^{-1}$.

For every divisor μ of the form (2.4) and every vector bundle E over X we define the space

$$\Gamma_\mu(X, E) = \{u \mid u \in \Gamma(X, E),\ j_{x_i}^{|p_i|-1} u = 0 \text{ if } p_i < 0\}. \tag{2.14}$$

So $\Gamma_\mu(X, E) \subset \Gamma(X, E)$ and $\Gamma_\mu(X, E)$ depends actually on ν only, hence $\Gamma_\mu(X, E) = \Gamma_\nu(X, E)$.

Now define a linear operator

$$\begin{gathered} \widetilde{A}\colon \Gamma(X, \mu, A) \longrightarrow \Gamma_{\widetilde{\mu}}(X, F), \\ \widetilde{A}u = \operatorname{ext}(Au), \end{gathered} \tag{2.15}$$

where $\operatorname{ext}(Au)$ is the extension by continuity of the section Au (defined a priori over $X - \operatorname{supp}\lambda$, but extendable to a C^∞-section on X due to the definition of $\Gamma(X, \mu, A)$ in (2.12)).

Now we introduce an important duality of the spaces introduced before.

DEFINITION 2.4. Bilinear or sesquilinear pairings

$$\begin{gathered} \Gamma_{\widetilde{\mu}}(X, F) \times \Gamma(X, \mu^{-1}, A^*) \longrightarrow \mathbb{C}, \\ \Gamma(X, \mu, A) \times \Gamma_{\widetilde{\mu}^{-1}}(X, \widetilde{E}) \longrightarrow \mathbb{C} \end{gathered} \tag{2.16}$$

are defined by integration over $X - \operatorname{supp}\mu$ and denoted by $\langle \cdot, \cdot \rangle$.

Note that the integrals that appear here actually converge. Indeed let $f \in \Gamma_{\widetilde{\mu}}(X, F)$, $v \in \Gamma(X, \mu^{-1}, A^*)$. Then near a point x_i with $p_i < 0$ we have $f(x) = O(|x - x_i|^{|p_i|-d})$, $v(x) = O(|x - x_i|^{d-n-|p_i|+\varepsilon})$ with $\varepsilon > 0$ (actually the last estimate is satisfied for any $\varepsilon < 1$), hence

$$\langle f(x), v(x) \rangle_x = O(|x - x_i|^{-n+\varepsilon})$$

and the integral

$$\langle f, v \rangle = \int_{X - \operatorname{supp}\mu} \langle f(x), v(x) \rangle$$

converges near all points x_i with $p_i < 0$ which are the only possible singularities.

Now let $\langle \cdot, \cdot \rangle\colon \mathscr{H} \times \mathscr{H}' \to \mathbb{C}$ be a bilinear or sesquilinear pairing between two complex linear spaces $\mathscr{H}$ and $\mathscr{H}'$. For any linear subspace $L \subset \mathscr{H}'$

we define its *annihilator* or *orthogonal complement* L° with respect to the pairing $\langle\cdot\,,\cdot\rangle$ as follows:

$$L^\circ = \{f \mid f \in \mathscr{H}\,,\ \langle f\,, v\rangle = 0 \text{ for every } v \in L\}.$$

Hence L° is a linear subspace in $\mathscr{H}$. Similarly, if L is a linear subspace in $\mathscr{H}$ then L° is defined and is a linear subspace in $\mathscr{H}'$.

In the following theorem L° will mean the annihilator with respect to the pairings (2.16). So if $L \subset \Gamma(X\,, \mu^{-1}\,, A^*)$ then $L^\circ \subset \Gamma_{\tilde{\mu}}(X\,, F)$, etc.

THEOREM 2.2. (i) $\langle \widetilde{A}u\,, v\rangle = \langle u\,, \widetilde{A^*}v\rangle$, *where* $u \in \Gamma(X\,, \mu\,, A)$, $v \in \Gamma(X\,, \mu^{-1}\,, A^*)$.

(ii) $\operatorname{Im}\widetilde{A} = (\operatorname{Ker}\widetilde{A^*})^\circ$, *i.e.*, $f \in \operatorname{Im}\widetilde{A}$ *iff* $f \in \Gamma_{\tilde{\mu}}(X\,, F)$ *and* $\langle f\,, v\rangle = 0$ *for all* $v \in \operatorname{Ker}\widetilde{A^*}$.

(iii) $\dim\operatorname{Coker}\widetilde{A} = \dim\operatorname{Ker}\widetilde{A^*}$.

Note that $\operatorname{Ker}\widetilde{A} = L(\mu\,, A)$, $\operatorname{Ker}\widetilde{A^*} = L(\mu^{-1}\,, A^*)$. Theorem 2.2 (ii) gives conditions of solvability of the equation $Au = f$ in the class $\Gamma(X\,, \mu\,, A)$ that consists of sections with prescribed orders of zeros and poles.

Theorem 2.2 will be proved simultaneously with Theorem 2.1. We shall apply it in Section 4 to prove a local solvability result when we look for a solution with a prescribed order of a zero at a given point.

C. Now let us describe generalizations of Theorems 2.1 and 2.2 to a special case of noncompact manifolds with boundary. This case will cover important situations when the operator A on an appropriate domain will still be a Fredholm operator in the usual sense. (Note that considering the case of elliptic operators on covering manifolds leads to operators which are Fredholm in the sense of Breuer in appropriate von Neumann algebras, but we will treat this case in a subsequent paper.)

So let X be a noncompact manifold with boundary ∂X (which need not be compact either).

Let $E\,, F$ be complex vector bundles over the open manifold of all interior points of X, which we denote $\operatorname{Int} X$. Let

$$A\colon \Gamma(\operatorname{Int} X\,, E) \to \Gamma(\operatorname{Int} X\,, F) \tag{2.17}$$

be an elliptic differential operator. As before, denote by $q = \dim_{\mathbb{C}} E_x = \dim_{\mathbb{C}} F_x$ the dimensions of fibres of E and F. Let $\widetilde{E}$, $\widetilde{F}$ be another pair of vector bundles over $\operatorname{Int} X$ given together with nondegenerate bilinear or sesquilinear C^∞ pairings of bundles

$$E \times \widetilde{E} \to \Omega(\operatorname{Int} X)\,, \quad F \times \widetilde{F} \to \Omega(\operatorname{Int} X).$$

For any vector bundle E over $\operatorname{Int} X$ denote by $\Gamma_c(\operatorname{Int} X)$ the space of all C^∞-sections of E over $\operatorname{Int} X$ with compact supports lying inside $\operatorname{Int} X$. Hence if $u \in \Gamma_c(\operatorname{Int} X)$ then $\operatorname{supp} u$ is a compact subset of $\operatorname{Int} X$; in particular, $\operatorname{supp} u$ does not intersect ∂X. In this case the adjoint elliptic operator

$A^*: \Gamma(\operatorname{Int} X, \widetilde{F}) \to \Gamma(\operatorname{Int} X, \widetilde{E})$ is again defined by the identity

$$\langle Au, \widetilde{v}\rangle_E = \langle u, A^*\widetilde{v}\rangle_F, \qquad u \in \Gamma_c(\operatorname{Int} X, E),\ \widetilde{v} \in \Gamma_c(\operatorname{Int} X, \widetilde{F}),$$

where the dualities $\langle\cdot, \cdot\rangle_E$, $\langle\cdot, \cdot\rangle_F$ are defined exactly as for the compact case.

Now suppose that the domains of A and A^* are distinguished as linear subspaces $\operatorname{Dom} A$ and $\operatorname{Dom} A^*$ such that

$$\Gamma_c(\operatorname{Int} X, E) \subset \operatorname{Dom} A \subset \Gamma(\operatorname{Int} X, E),$$
$$\Gamma_c(\operatorname{Int} X, \widetilde{F}) \subset \operatorname{Dom} A^* \subset \Gamma(\operatorname{Int} X, \widetilde{F}).$$

They may be defined, for example, by a choice of boundary conditions and appropriate conditions at infinity. Then let us define images of A, A^* as

$$\operatorname{Im} A = A(\operatorname{Dom} A), \qquad \operatorname{Im} A^* = A^*(\operatorname{Dom} A^*).$$

Suppose also that linear subspaces $\operatorname{Dom}' A$ and $\operatorname{Dom}' A^*$ are given such that

$$\Gamma_c(\operatorname{Int} X, \widetilde{E}) \subset \operatorname{Dom}' A \subset \Gamma(\operatorname{Int} X, \widetilde{E}), \quad \operatorname{Im} A^* \subset \operatorname{Dom}' A,$$
$$\Gamma_c(\operatorname{Int} X, F) \subset \operatorname{Dom}' A^* \subset \Gamma(\operatorname{Int} X, F), \quad \operatorname{Im} A \subset \operatorname{Dom}' A^*.$$

We shall suppose that the following integrability condition is satisfied:

$$\begin{gathered} x \mapsto \langle v, \widetilde{v}\rangle_x \text{ and } x \mapsto \langle u, \widetilde{u}\rangle_x \text{ are } L^1\text{-densities on } \operatorname{Int} X \\ \text{for any } v \in \operatorname{Dom}' A^*,\ \widetilde{v} \in \operatorname{Dom} A^*,\ u \in \operatorname{Dom} A,\ \widetilde{u} \in \operatorname{Dom}' A^* \end{gathered} \tag{2.18}$$

(i.e., these densities are absolutely integrable in Lebesgue sense over $\operatorname{Int} X$).

Integrating these densities over $\operatorname{Int} X$ we obtain bilinear or sesquilinear pairings

$$\operatorname{Dom}' A^* \times \operatorname{Dom} A^* \longrightarrow \mathbb{C}, \qquad \operatorname{Dom} A \times \operatorname{Dom}' A \longrightarrow \mathbb{C}$$

which will be denoted $\langle\cdot, \cdot\rangle_F$ and $\langle\cdot, \cdot\rangle_E$, the same way as for sections with compact support.

Our next requirement is

$$\langle Au, \widetilde{v}\rangle_F = \langle u, A^*\widetilde{v}\rangle_E, \qquad u \in \operatorname{Dom} A,\ \widetilde{v} \in \operatorname{Dom} A^*. \tag{2.19}$$

Now if we are given a linear subspace $L \subset \operatorname{Dom} A$, then its annihilator L° is naturally defined as a linear subspace in $\operatorname{Dom}' A$:

$$L^\circ = \{\widetilde{u} \mid \widetilde{u} \in \operatorname{Dom}' A,\ \langle u, \widetilde{u}\rangle_E = 0 \text{ for every } u \in L\}.$$

Similarly for a linear subspace $M \subset \operatorname{Dom} A^*$ its annihilator $M^\circ \subset \operatorname{Dom}' A^*$ is naturally defined.

Define also $\operatorname{Ker} A$ and $\operatorname{Ker} A^*$ as linear subspaces in $\operatorname{Dom} A$ and $\operatorname{Dom} A^*$ respectively, e.g.,

$$\operatorname{Ker} A = \{u \mid u \in \operatorname{Dom} A,\ Au = 0\}.$$

Now our next requirement is

$$\begin{gathered}\textit{A and } A^* \textit{ are Fredholm in the following sense:}\\ \text{(i) } \dim \operatorname{Ker} A < \infty, \ \dim \operatorname{Ker} A^* < \infty; \\ \text{(ii) } \operatorname{Im} A = (\operatorname{Ker} A^*)^\circ, \ \operatorname{Im} A^* = (\operatorname{Ker} A)^\circ. \end{gathered} \tag{2.20}$$

Hence $\operatorname{ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Ker} A^*$ is well defined.

Let us introduce the (point) divisors on X by (2.4) with $x_i \in \operatorname{Int} X$, $i = 1, \dots, m$. Definition 2.1 for the degree is obviously applicable in this case as well as local Definition 2.2.

We write that $u \in \operatorname{Dom} A$ outside a compact subset $K \subset \operatorname{Int} X$ if $u \in \Gamma(\operatorname{Int} X - K, E)$ and there exists $\bar{u} \in \operatorname{Dom} A$ such that $\bar{u} = u$ on $X - K$. Now we can define

$$\begin{aligned} L(\mu, A) = \{u \mid\, & u \in \operatorname{Dom} A \text{ outside a neighbourhood of } \operatorname{supp} \mu, \\ & Au = 0 \text{ on } X - \operatorname{supp} \mu, \ \operatorname{ord}_{x_i} u \geqslant -p_i \text{ for each } i = 1, \dots, m\}, \\ r(\mu, A) = \ & \dim_{\mathbb{C}} L(\mu, A). \end{aligned}$$

$L(\mu, A^*)$ is defined similarly and $r(\mu, A^*) = \dim_{\mathbb{C}} L(\mu, A^*)$.

Now we can formulate a generalization of Theorem 2.1 to the noncompact case.

THEOREM 2.3. *Let A be an elliptic differential operator* (2.17) *and let* (2.18)–(2.20) *be satisfied. Then for any divisor μ (with $\operatorname{supp} \mu \subset \operatorname{Int} X$)*

$$r(\mu, A) = \operatorname{ind} A + q d(\mu) + r(\mu^{-1}, A^*).$$

Let us give two examples when this theorem may be applied.

EXAMPLE 2.3. Let X be a compact Riemannian manifold with a smooth boundary, $A = A^* = \Delta$ is the Laplacian of the given Riemannian metric. Then taking $E = F = \widetilde{E} = \widetilde{F} = \mathbb{C}_X$ (the trivial vector bundle with the fibre $\mathbb{C}$ over X) and defining the duality by the use of the Riemannian volume, we can take

$$\operatorname{Dom} A = \operatorname{Dom}' A = \operatorname{Dom} A^* = \operatorname{Dom}' A^* = \{u \mid u \in C^\infty(X), \, u|_{\partial X} = 0\},$$

i.e., define A as the Laplacian with the Dirichlet boundary condition. Then conditions (2.18)–(2.20) are satisfied and so Theorem 2.3 is applicable in this case with $\operatorname{ind} A = 0$, $q = 1$.

Similarly we can consider the Neuman condition.

EXAMPLE 2.4. Let $X = \mathbb{R}^n$, $n \geqslant 3$, $A = A^* = \Delta$ (the standard Laplacian or the Laplacian of the flat metric), $E = F = \widetilde{E} = \widetilde{F} = \mathbb{C}_X$, and

$$\begin{aligned} \operatorname{Dom} A &= \operatorname{Dom} A^* = \operatorname{Dom}' A = \operatorname{Dom}' A^* \\ &= \{u \mid u \in C^\infty(\mathbb{R}^n), \, \Delta u \in C_c^\infty(\mathbb{R}^n), \text{ and } u(x) \to 0 \text{ as } |x| \to \infty\}. \end{aligned}$$

Here $C_c^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \cap \mathscr{E}'(\mathbb{R}^n)$ is the set of all C^∞-functions with compact support.

Note that the condition $u(x) \to \infty$ as $|x| \to \infty$ can be replaced by a formally stronger but in fact equivalent condition

$$u(x) = O(|x|^{2-n}) \quad \text{as } |x| \to \infty \tag{2.21}$$

because $u \in \operatorname{Dom} A$ is a harmonic function near infinity. It easily follows that all conditions (2.18)–(2.20) are satisfied because $\operatorname{Ker} A = \operatorname{Ker} A^* = \{0\}$ by the Liouville theorem and $\operatorname{Im} A = \operatorname{Im} A^* = C_c^\infty(\mathbb{R}^n)$ since the equation $\Delta u = f$ with any $f \in C_c^\infty(\mathbb{R}^n)$ can be solved by taking the convolution of f with the standard fundamental solution $c_n|x|^{2-n}$. Hence Theorem 2.3 is applicable to this situation.

We can also replace $\mathbb{R}^n$ by a closed subset $X \subset \mathbb{R}^n$ such that $\mathbb{R}^n - X$ is a bounded open subset with a C^∞-boundary, imposing the Dirichlet or Neuman boundary condition on ∂X.

D. Now let us formulate a generalization of the duality Theorem 2.2 to the noncompact case. Denote

$$\begin{aligned}\Gamma(X, \mu, A) = \{u \mid u \in \Gamma(\operatorname{Int} X - \operatorname{supp}\mu, E),\ & u \in \operatorname{Dom} A \text{ outside} \\ & \text{a neighbourhood of } \operatorname{supp}\lambda;\ j_{x_i}^{|p_i|-1}u = 0 \text{ if} \\ & x_i \in \operatorname{supp}\nu; \text{ for every } x_i \in \operatorname{supp}\lambda \text{ there exists} \\ & \text{a neighbourhood } U \text{ of } x_i \text{ and a representation} \\ & u = u_s + u_r, \text{ where } u_s \in \Gamma(U - \{x_i\}, E),\ Au_s = \\ & 0 \text{ in } U - \{x_i\},\ \operatorname{ord}_{x_i} u_s \geqslant -p_i, \text{ and } u_r \text{ can be} \\ & \text{extended to a section } \bar{u}_r \in \Gamma(U, E)\}; \\ \Gamma_\mu(X, A^*) = \{u \mid u \in \operatorname{Dom}' A^*,\ & j_{x_i}^{|p_i|-1}u = 0 \text{ if } p_i < 0\}.\end{aligned} \tag{2.22}$$

Similar definitions apply if we replace A by A^*.

The reduced divisor $\tilde{\mu}$ is defined as before and we have bilinear or sesquilinear pairings

$$\Gamma_{\tilde{\mu}}(X, A^*) \times \Gamma(X, \mu^{-1}, A^*) \to \mathbb{C}, \quad \Gamma(X, \mu, A) \times \Gamma_{\tilde{\mu}^{-1}}(X, A) \to \mathbb{C} \tag{2.23}$$

as in Definition 2.4. Also the linear operator

$$\tilde{A}: \Gamma(X, \mu, A) \to \Gamma_{\tilde{\mu}}(X, A^*) \tag{2.24}$$

is defined as in (2.15).

In the next theorem L° means the annihilator with respect to the pairings (2.23).

THEOREM 2.4. *We have*

(i) $\langle \tilde{A}u, v\rangle = \langle u, \tilde{A}^* v\rangle$; $u \in \Gamma(X, \mu, A)$, $v \in \Gamma(X, \mu^{-1}, A^*)$.

(ii) $\operatorname{Im}\tilde{A} = (\operatorname{Ker}\tilde{A}^*)^\circ$, *i.e.*, $f \in \operatorname{Im}\tilde{A}$ *iff* $f \in \Gamma_{\tilde{\mu}}(X, A^*)$ *and* $\langle f, v\rangle = 0$ *for all* $v \in \operatorname{Ker}\tilde{A}^*$.

(iii) $\dim\operatorname{Coker}\tilde{A} = \dim\operatorname{Ker}\tilde{A}^*$.

§3. Proofs of the main results

For simplicity we first prove Theorems 2.1 and 2.2. Then we shall indicate (minor) modifications that should be made to prove Theorems 2.3 and 2.4.

Theorems 2.1 and 2.2 will be proved simultaneously because these proofs intertwine.

Obviously we can assume without loss of generality that $A^* = A^t$, hence $\widetilde{E} = E^*$ and $\widetilde{F} = F^*$. So from now on the simplest canonical (bilinear) duality is used.

First we state and prove some preparatory results.

Denote by $\mathscr{D}'(X, F)$ the space of distribution sections of F (dual space to $\Gamma(X, F^*)$) and let $S(\mu, F)$ be the space of $s \in \mathscr{D}'(X, F)$ such that $\operatorname{supp} s \subset \operatorname{supp}\lambda$ and locally near the points $x_i \in \operatorname{supp}\lambda$ (i.e., the points entering in μ with positive exponents) s can be written as

$$s = \sum_{\{i|p_i>0\}} \sum_{|\alpha|\leqslant p_i-1} c_{i\alpha}\partial_x^\alpha \delta(x - x_i) \tag{3.1}$$

where δ is the Dirac measure, $c_{i\alpha} \in F_{x_i}$. So actually $S(\mu, F) = S(\lambda, F)$. Similar spaces will be used for the bundle E and other divisors.

For every $u \in \Gamma(X, \mu, A)$ we can find a "regularization" $\widetilde{u} \in \mathscr{D}'(X, E)$ such that $\widetilde{u} = u$ on $X - \operatorname{supp}\lambda$ and $A\widetilde{u} = f + s$ with $f \in \Gamma(X, F)$ and $s \in S(\mu, F)$. Denote by $\widetilde{\Gamma}(X, \mu, A)$ the space of all such regularizations. Due to the standard elliptic regularity result and the structure of fundamental solutions described in Section 1, the space $\widetilde{\Gamma}(X, \mu, A)$ can be described as a set of $\widetilde{u} \in \mathscr{D}'(X, E)$ such that $\widetilde{u} \in C^\infty$ in a neighbourhood of $\operatorname{supp}\nu$, $j_{x_i}^{|p_i|-1} u = 0$ for every $x_i \in \operatorname{supp}\nu$, and $A\widetilde{u} = f + s$ with $f \in \Gamma(X, F)$, $s \in S(\mu, F)$. Now we need

LEMMA 3.1. *The sequence*

$$0 \longrightarrow S(\widetilde{\mu}, E) \xrightarrow{i_1} \widetilde{\Gamma}(X, \mu, A) \xrightarrow{r} \Gamma(X, \mu, A) \longrightarrow 0 \tag{3.2}$$

is exact. Here i_1 and r are natural inclusion and restriction maps.

PROOF. The surjectivity of r means the existence of a regularization as mentioned before, the injectivity of i_1 is evident. So we must only prove the exactness in the middle term which actually means that if $s \in \mathscr{D}'(X, E)$, $\operatorname{supp} s \subset \operatorname{supp}\lambda$, and $As \in S(\mu, F)$ then $s \in S(\widetilde{\mu}, E)$. This is a local question and it is sufficient to consider the case $\mu = x_1^{p_1}$ with $p_1 < 0$. But then the statement easily follows from ellipticity of A. □

LEMMA 3.2. *We have*

$$\dim S(\mu, E) = \dim(\Gamma(X, E)/\Gamma_{\mu^{-1}}(X, E)) = q \sum_{p_i>0} \binom{n + p_i - 1}{n}. \tag{3.3}$$

PROOF. Evidently,

$$S(\mu, E) = \bigoplus_{p_i>0} S(x_i^{p_i}, E), \qquad \Gamma(X, E)/\Gamma_{\mu^{-1}}(X, E) = \bigoplus_{p_i>0} J_{x_i}^{p_i-1}(E),$$

where $J_{x_i}^{p_i-1}(E)$ is the space of jets of the order $p_i - 1$ of sections of E at x_i. So it is sufficient to prove that for any $p_i > 0$

$$\dim S(x_i^{p_i}, E) = \dim J_{x_i}^{p_i-1}(E) = q\binom{n+p_i-1}{n}$$

which reduces to a well-known combinatorial exercise. □

REMARK. Note that the spaces $S(\mu, E)$ and $\Gamma(X, E^*)/\Gamma_{\mu^{-1}}(X, E^*)$ are dual to each other with respect to the natural duality induced by the duality between $\mathscr{D}'(X, E)$ and $\Gamma(X, E^*)$.

PROPOSITION 3.3. *We have*

$$\operatorname{ind}\widetilde{A} = \operatorname{ind} A + qd(\mu). \tag{3.4}$$

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S(\widetilde{\mu}, E) & \xrightarrow{i_1} & \widetilde{\Gamma}(X, \mu, A) & \xrightarrow{r} & \Gamma(X, \mu, A) & \longrightarrow & 0 \\ & & \downarrow{A_S} & & \downarrow{\hat{A}} & & \downarrow{\widetilde{A}} & & \\ 0 & \longrightarrow & S(\mu, E) & \xrightarrow{i_2} & \Gamma_{\widetilde{\mu}}(X, F) \oplus S(\mu, F) & \xrightarrow{\pi_1} & \Gamma_{\widetilde{\mu}}(X, F) & \longrightarrow & 0 \end{array}$$

where the first row is as in Lemma 3.1, i_2 and π_1 are natural inclusion and projection respectively, and A_S, $\hat{A}$ are restrictions of A to the corresponding spaces of distributions. Both rows in the diagram are exact. Due to the well-known algebraic property of the Euler characteristic we have

$$\operatorname{ind}\widetilde{A} = \operatorname{ind}\hat{A} - \operatorname{ind} A_S.$$

But

$$\begin{aligned} \operatorname{ind} A_S &= \dim S(\widetilde{\mu}, E) - \dim S(\mu, F) \\ &= -q\sum_{p_i>0}\left[\binom{n+p_i-1}{n} - \binom{n+p_i-d-1}{n}\right] = -qd(\lambda). \end{aligned}$$

Hence

$$\operatorname{ind}\widetilde{A} = \operatorname{ind}\hat{A} + qd(\lambda). \tag{3.5}$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\widetilde{\mu}, E) & \longrightarrow & \widetilde{\Gamma}(X, \mu, A) & \xrightarrow{\pi_2\circ\hat{A}} & S(\mu F) & \longrightarrow & 0 \\ & & \downarrow{A_\mu} & & \downarrow{\hat{A}} & & \downarrow{\mathrm{Id}} & & \\ 0 & \longrightarrow & \Gamma_{\widetilde{\mu}}(X, F) & \xrightarrow{i_2} & \Gamma_{\widetilde{\mu}}(X, F) \oplus S(\mu, F) & \xrightarrow{\pi_1} & S(\mu, F) & \longrightarrow & 0 \end{array}$$

where the rows again are exact, A_μ is the restriction of A. From this diagram we find

$$\operatorname{ind}\hat{A} = \operatorname{ind}A_\mu. \tag{3.6}$$

Finally, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_\mu(X, E) & \xrightarrow{i_\mu} & \Gamma(X, E) & \xrightarrow{p_\mu} & J_\mu(E) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle A_\mu} & & \Big\downarrow{\scriptstyle A} & & \Big\downarrow{\scriptstyle J_\mu(A)} & & \\ 0 & \longrightarrow & \Gamma_{\widetilde{\mu}}(X, F) & \xrightarrow{i_{\widetilde{\mu}}} & \Gamma(X, F) & \xrightarrow{p_{\widetilde{\mu}}} & J_{\widetilde{\mu}}(E) & \longrightarrow & 0 \end{array}$$

where $J_\mu(E) = \Gamma(X, E)/\Gamma_\mu(X, E) = \bigoplus_{p_i<0} J_{x_i}^{|p_i|-1}(E)$, i_μ and p_μ are natural inclusion and restriction maps, $J_\mu(A)$ is the natural quotient map. Then we find

$$\operatorname{ind}A_\mu = \operatorname{ind}A - \operatorname{ind}J_\mu(A).$$

But

$$\begin{aligned} \operatorname{ind}J_\mu(A) &= \dim J_\mu(E) - \dim J_{\widetilde{\mu}}(F) \\ &= q\sum_{p_i<0}\left[\binom{n+|p_i|-1}{n} - \binom{n+|p_i|-d-1}{n}\right] = -qd(\nu), \end{aligned}$$

hence

$$\operatorname{ind}A_\mu = \operatorname{ind}A + qd(\nu)$$

and now using (3.5) and (3.6) we obtain

$$\operatorname{ind}\widetilde{A} = \operatorname{ind}A + qd(\nu) + qd(\lambda) = \operatorname{ind}A + qd(\mu). \quad \square$$

REMARK. Proposition 3.3 means that

$$\dim\operatorname{Ker}\widetilde{A} = \operatorname{ind}A + qd(\mu) + \dim\operatorname{Coker}\widetilde{A}, \tag{3.7}$$

so to prove Theorem 2.1 it suffices to prove the equality (iii) in Theorem 2.2. We start with the proof of (i) in Theorem 2.2.

LEMMA 3.4. *We have*

$$\langle\widetilde{A}u, v\rangle = \langle u, \widetilde{A^t}v\rangle, \qquad u \in \Gamma(X, \mu, A),\ v \in \Gamma(X, \mu^{-1}, A^t). \tag{3.8}$$

PROOF. Let us take a function $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) = 1$ if $|x| < 1/2$, $\chi(x) = 0$ if $|x| > 1$. For every $\varepsilon > 0$ define $\chi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\chi_\varepsilon(x) = \chi(\varepsilon^{-1}x)$, so that $\operatorname{supp}\chi_\varepsilon \subset \{x \mid |x| \leqslant \varepsilon\}$, $\chi_\varepsilon(x) = 1$ if $|x| \leqslant \varepsilon/2$, and $|\partial^\alpha\chi_\varepsilon| \leqslant C_\alpha\varepsilon^{-|\alpha|}$.

For every point $x_i \in \operatorname{supp}\mu$ introduce local coordinates in a neighbourhood U_i of x_i and then using these local coordinates for small $\varepsilon > 0$ define

$$\psi_\varepsilon(x) = 1 - \sum_{1\leqslant i\leqslant m}\chi_\varepsilon(x - x_i).$$

It follows that $\psi_\varepsilon(x) = 0$ in a neighbourhood of $\operatorname{supp}\mu$, $\psi_\varepsilon(x) = 1$ outside a small neighbourhood of $\operatorname{supp}\mu$, and $|\partial^\alpha \psi_\varepsilon| \leqslant C_\alpha \varepsilon^{-|\alpha|}$, where the derivative is taken in chosen local coordinates. Now using (2.3) and the convergence of the integrals defining both sides in (3.8) we obtain

$$\begin{aligned}\langle Au, v\rangle &= \lim_{\varepsilon\downarrow 0}\langle Au, \psi_\varepsilon v\rangle = \lim_{\varepsilon\downarrow 0}\langle u, A^t(\psi_\varepsilon v)\rangle \\ &= \lim_{\varepsilon\downarrow 0}[\langle u, \psi_\varepsilon A^t v\rangle + \langle u, [A^t, \psi_\varepsilon]v\rangle] \\ &= \langle u, A^t v\rangle + \lim_{\varepsilon\downarrow 0}\langle u, [A^t, \psi_\varepsilon]v\rangle.\end{aligned}$$

It remains to prove that the last limit vanishes. Denote $B_\varepsilon = [A^t, \psi_\varepsilon]$. Then B_ε is a differential operator of order $d-1$ with coefficients supported near $\operatorname{supp}\mu$ and near x_i:

$$B_\varepsilon = \sum_{|\alpha|\leqslant d-1} b_{\alpha,\varepsilon}(x) D^\alpha$$

with

$$\operatorname{supp} b_{\alpha,\varepsilon} \subset \{x \mid \varepsilon/2 \leqslant |x - x_i| \leqslant \varepsilon\}, \tag{3.9}$$

$$|b_{\alpha,\varepsilon}(x)| \leqslant C\varepsilon^{-d+|\alpha|}. \tag{3.10}$$

Now we must consider two cases: $p_i < 0$ and $p_i > 0$.

(a) Let $p_i < 0$. Then in U_i we have $u(x) = O(|x - x_i|^{|p_i|})$, $v(x) = o(|x - x_i|^{d-n-|p_i|})$, $D^\alpha v(x) = o(|x - x_i|^{d-n-|p_i|-|\alpha|})$. On $\operatorname{supp} b_{\alpha,\varepsilon}$ we have, due to (3.9):

$$u(x) = O(\varepsilon^{|p_i|}), \qquad D^\alpha v(x) = o(\varepsilon^{d-n-|p_i|-|\alpha|}).$$

Hence (3.10) gives

$$b_{\alpha,\varepsilon} D^\alpha v = o(\varepsilon^{-n-|p_i|}), \qquad \langle u(x), b_{\alpha,\varepsilon}(x) D^\alpha v(x)\rangle = o(\varepsilon^{-n}).$$

Now the volume of $\operatorname{supp} b_{\alpha,\varepsilon}$ is $O(\varepsilon^n)$, hence

$$\int_{U_i} \langle u(x), B_\varepsilon v(x)\rangle = o(1) \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.11}$$

(b) Let $p_i > 0$. Then on $\operatorname{supp} b_{\alpha,\varepsilon}$ we have similarly $D^\alpha v(x) = O(\varepsilon^{p_i - |\alpha|})$,

$$u(x) = o(\varepsilon^{d-n-p_i}), \quad D^\alpha v(x) = O(\varepsilon^{p_i-|\alpha|}), \quad \langle u(x), b_{\alpha,\varepsilon}(x) D^\alpha v(x)\rangle = o(1),$$

so the conclusion (3.11) is again true, which proves the lemma. □

DEFINITION 3.5. Let $\mathscr{H}$, $\mathscr{H}'$ be two complex linear spaces and $\langle\cdot,\cdot\rangle : \mathscr{H} \times \mathscr{H}' \to \mathbb{C}$ is a bilinear or sesquilinear pairing. We say that this pairing is *nondegenerate* if

$$\{u \in \mathscr{H}, \langle u, v\rangle = 0 \text{ for all } v \in \mathscr{H}'\} \quad \text{implies} \quad u = 0$$

and

$$\{v \in \mathscr{H}', \langle u, v\rangle = 0 \text{ for all } u \in \mathscr{H}\} \quad \text{implies} \quad v = 0.$$

LEMMA 3.6. *The pairings* (2.16) *are nondegenerate.*

PROOF. The statement is evident since all spaces in (2.16) contain smooth sections of the corresponding bundles supported in $X - \operatorname{supp}\mu$ and on the other hand all elements of these spaces are uniquely defined by their (smooth) restrictions to $X - \operatorname{supp}\mu$. □

Now we need the following

LEMMA 3.7. *Let* $\langle \cdot , \cdot \rangle : \mathscr{H} \times \mathscr{H}' \to \mathbb{C}$ *be a nondegenerate bilinear or sesquilinear pairing between two complex linear spaces* $\mathscr{H}$, $\mathscr{H}'$. *Let* L *be a linear subspace in* $\mathscr{H}'$, L° *its annihilator in* $\mathscr{H}$, *and* $(L^\circ)^\circ$ *the annihilator of* L° *in* $\mathscr{H}'$. *Then*

$$L \subset (L^\circ)^\circ \tag{3.12}$$

and

$$\operatorname{codim} L \geqslant \dim L^\circ. \tag{3.13}$$

Furthermore, if F *is a linear subspace in* $\mathscr{H}'$ *then*

$$\operatorname{codim} F^\circ = \dim F. \tag{3.14}$$

PROOF. The inclusion (3.12) is obvious. It gives $\operatorname{codim} L \geqslant \operatorname{codim}(L^\circ)^\circ$. Hence (3.14) implies (3.13) and we have only to prove (3.14). Clearly $\operatorname{codim} F^\circ \leqslant \dim F$, so it remains to prove that

$$\operatorname{codim} F^\circ \geqslant \dim F.$$

It is sufficient to do it in the case when $\dim F < \infty$. Consider the natural map $j : F \to (\mathscr{H}/F^\circ)'$, where L' means the space of all complex linear maps from L to $\mathbb{C}$,

$$j(f)(x + F^\circ) = \langle f , x \rangle, \qquad x \in \mathscr{H}.$$

Then j is injective due to the nondegeneracy of the pairing. Hence

$$\operatorname{codim} F^\circ = \dim \mathscr{H}/F^\circ = \dim(\mathscr{H}/F^\circ)' \geqslant \dim F$$

as required. □

LEMMA 3.8. *In the first of the pairings* (2.16)

$$(\operatorname{Im} \widetilde{A})^\circ = \operatorname{Ker} \widetilde{A}^*.$$

PROOF. Clearly

$$\operatorname{Ker} \widetilde{A}^* = \{v \mid v \in \Gamma(X, \mu^{-1}, A^*),\ A^* v = 0 \text{ on } X - \operatorname{supp}\mu\}.$$

Lemma 3.8 follows because $\operatorname{Im} \widetilde{A}$ contains all sections $\widetilde{A}u$ if $u \in \Gamma(X, E)$ and $\operatorname{supp} u \subset X - \operatorname{supp}\mu$. □

PROOF OF THEOREMS 2.1 AND 2.2. Due to Lemmas 3.7 and 3.8 we have

$$\operatorname{Im} \widetilde{A} \subset (\operatorname{Ker} \widetilde{A}^t)^\circ, \tag{3.15}$$

$$\operatorname{codim} \operatorname{Im} \widetilde{A} \geqslant \dim \operatorname{Ker} \widetilde{A}^t, \tag{3.16}$$

and we must prove that both the inclusion and the inequality are actually equalities. Furthermore, (3.14) gives

$$\operatorname{codim}(\operatorname{Ker}\widetilde{A}^t)^\circ = \dim\operatorname{Ker}\widetilde{A}^t,$$

hence equality in (3.16) implies equality in (3.15). Since $\operatorname{codim}\operatorname{Im}\widetilde{A} = \dim\operatorname{Coker}\widetilde{A}$ we must only prove that

$$\dim\operatorname{Coker}\widetilde{A} = \dim\operatorname{Ker}\widetilde{A}^t, \tag{3.17}$$

which will immediately give the proofs of Theorems 2.1 and 2.2 due to (3.7). Clearly, (3.16) and (3.7) imply

$$\begin{aligned}\dim\operatorname{Ker}\widetilde{A} &= \operatorname{ind}A + qd(\mu) + \dim\operatorname{Coker}\widetilde{A}\\ &\geqslant \operatorname{ind}A + qd(\mu) + \dim\operatorname{Ker}\widetilde{A}^t.\end{aligned} \tag{3.18}$$

Now we can apply the same results to the divisor μ^{-1} (instead of μ) and the operator $\widetilde{A}^t$ (instead of $\widetilde{A}$). Then we obtain

$$\dim\operatorname{Ker}\widetilde{A}^t \geqslant \operatorname{ind}A^t + qd(\mu^{-1}) + \dim\operatorname{Ker}\widetilde{A} = -\operatorname{ind}A - qd(\mu) + \dim\operatorname{Ker}\widetilde{A}.$$

But this is the inequality opposite to (3.18). Hence in (3.18) and (3.17) we actually have equalities. Proofs of Theorems 2.1 and 2.2 are complete. □

REMARK. An easier proof of Theorem 2.1 can be given in the case when $\mu > 0$ or $\mu < 0$ (i.e., if $p_i > 0$ for all $i = 1, \dots, m$, or, vice versa, if $p_i < 0$ for all $i = 1, \dots, m$). Of course it is sufficient to consider the case $\mu > 0$. Denote then

$$\widetilde{L}(\mu, A) = \{u \mid u \in \mathscr{D}'(X, E),\ Au \in S(\mu, F)\}.$$

Then we have an exact sequence

$$0 \longrightarrow S(\widetilde{\mu}, E) \xrightarrow{i} \widetilde{L}(\mu, A) \xrightarrow{r} L(\mu, A) \longrightarrow 0,$$

where i and r are natural inclusion and restriction maps. Hence

$$\dim\widetilde{L}(\mu, A) = r(\mu, A) + \dim S(\widetilde{\mu}, E).$$

Now consider the following exact sequence

$$0 \to \operatorname{Ker}A \longrightarrow \widetilde{L}(\mu, A) \xrightarrow{A} S(\mu, F) \xrightarrow{\partial} (\operatorname{Ker}A^t)' \longrightarrow (\operatorname{Ker}A^t)' / \operatorname{Im}\partial \longrightarrow 0$$

where ∂ is the dualization map, $\partial(f)(v) = \langle f, v\rangle$, $v \in \operatorname{Ker}A^t$. Hence

$$\begin{aligned}0 = {} & \dim\operatorname{Ker}A - \dim\widetilde{L}(\mu, A) + \dim S(\mu, F) - \dim\operatorname{Ker}A^t\\ & + \dim[(\operatorname{Ker}A^t)'/\operatorname{Im}\partial]\\ = {} & \dim\operatorname{Ker}A - r(\mu, A) - \dim S(\widetilde{\mu}, E)\\ & + \dim S(\mu, F) - \dim\operatorname{Ker}A^t + \dim[(\operatorname{Ker}A^t)'/\operatorname{Im}\partial]\\ = {} & \operatorname{ind}A + qd(\mu) - r(\mu, A) + \dim[(\operatorname{Ker}A^t)'/\operatorname{Im}\partial]\end{aligned}$$

and it remains to prove that

$$\dim[(\operatorname{Ker} A^t)'/\operatorname{Im}\partial] = r(\mu^{-1}, A^t).$$

But

$$\dim[(\operatorname{Ker} A^t)'/\operatorname{Im}\partial] = \dim(\operatorname{Im}\partial)^\circ,$$

where $(\operatorname{Im}\partial)^\circ$ is the annihilator of $\operatorname{Im}\partial$ in $\operatorname{Ker} A^t$. Clearly $(\operatorname{Im}\partial)^\circ = L(\mu^{-1}, A^t)$, which implies the required equality.

PROOF OF THEOREMS 2.3 AND 2.4. Only minor modifications are needed in the proofs of Theorems 2.1 and 2.2 to convert them into the proofs of more general Theorems 2.3 and 2.4.

In Section 2 we introduced the spaces $\Gamma(X, \mu, A)$ of smooth sections with singularities. In the noncompact case the definition takes into account domains (or boundary conditions and conditions at infinity) as well. In the compact case we used also the spaces $\Gamma_\mu(X, E)$. They should be replaced by the corresponding spaces $\Gamma_\mu(X, A)$. Also $\operatorname{Dom} A$ and $\operatorname{Dom} A^*$ should be used instead of $\Gamma(X, E)$ and $\Gamma(X, F^*)$ respectively. The space $\widetilde{\Gamma}(X, \mu, A)$ is defined as a space of distributions that are "regularizations" of functions from $\Gamma(X, \mu, A)$, so Lemmas 3.1 and 3.2 remain true. Then no change is needed in the proof of Proposition 3.3.

Using the duality, we must replace A^t with A^* and keep track of the dual domains $\operatorname{Dom}' A$ and $\operatorname{Dom}' A^*$. Lemma 3.4 is "by definition" true in our noncompact case for smooth sections from the corresponding domains (see (2.19)), and it is easily extended to sections with singularities, because further arguments are purely local and there is only a finite number of singularities. Then Lemmas 3.7 and 3.8 can be applied the same way to complete the proof. □

The easier proof given in the Remark after the end of the proofs of Theorems 2.1 and 2.2 for the case $\mu > 0$ or $\mu < 0$ can be easily extended as well.

§4. Applications

A. We begin with an obvious corollary of Theorems 2.1 or 2.3:

COROLLARY 4.1.

$$r(\mu, A) \geqslant \operatorname{ind} A + qd(\mu). \tag{4.1}$$

In particular, if $\operatorname{ind} A + qd(\mu) > 0$, then we have a nontrivial space $L(\mu, A)$ of solutions with poles allowed. So this space will be always nontrivial if we fix orders of zeros, but allow poles of sufficiently high order to make $d(\mu) > 0$ sufficiently large. For example if we fix any set of points $\{x_2, \dots, x_m\}$ and any negative integers $p_2, \dots, p_m$, but take $p_1 > 0$ sufficiently large, we obtain $r(\mu, A) > 0$, which means that there exists a nontrivial solution u of the equation $Au = 0$ on $X - \{x_1\}$ with a pole at x_1 and zeros at the points $x_2, \dots, x_m$ with multiplicities bounded from below by $|p_2|, \dots, |p_m|$. If A is a scalar operator (i.e., $q = 1$) this means that choosing a point x_1 and any finite set of points $x_2, \dots, x_m$ we can always include this set into the nodal set of a solution with a single pole at x_1. Note that the same is true for large q, where this result looks much stronger because for $q \geqslant n/2$ we can expect zeros to be generically isolated (or even absent).

REMARK. The inequality (4.1) can be proved without use of Theorem 2.1 (or 2.3). To do this let us use the space $\widetilde{L}(\lambda, A)$ that consists of all "regularizations" of functions from $L(\lambda, A)$, i.e.,

$$\widetilde{L}(\lambda, A) = \{u \mid u \in \mathscr{E}'(X, E) + \operatorname{Dom} A,\ Au \in S(\lambda, F)\}$$

(for the compact case it was used already in the Remark following the proof of Theorems 2.1 and 2.2).

Obviously, $\widetilde{L}(\lambda, A) \supset \operatorname{Ker} A$ and a solution $u \in \widetilde{L}(\lambda, A)$ of $Au = f \in S(\lambda, F)$ exists if and only if $f \in (\operatorname{Ker} A^*)^\circ$. Hence

$$\dim \widetilde{L}(\lambda, A) \geqslant \dim S(\lambda, F) + \dim \operatorname{Ker} A - \dim \operatorname{Ker} A^* = \dim S(\lambda, F) + \operatorname{ind} A.$$

But $L(\lambda, A) \cong \widetilde{L}(\lambda, A)/S(\widetilde{\lambda}, E)$, hence

$$\dim L(\lambda, A) \geqslant \dim S(\lambda, F) - \dim S(\widetilde{\lambda}, E) + \operatorname{ind} A = qd(\lambda) + \operatorname{ind} A.$$

Now

$$\begin{aligned} L(\mu, A) &= \{u \mid u \in L(\lambda, A),\ j_{x_j}^{|p_j|-1} u = 0 \text{ if } x_j \in \operatorname{supp} \nu\} \\ &= \{u \mid u \in L(\lambda, A) \cap (S(\nu, E^*))^\circ\}. \end{aligned}$$

Note that $u \in [A^* S(\widetilde{\nu}, F^*)]^\circ$ for any $u \in L(\lambda, A)$, $A^* S(\widetilde{\nu}, F^*) \subset S(\nu, E^*)$, and $\dim A^* S(\widetilde{\nu}, F^*) = \dim S(\widetilde{\nu}, F^*)$. Hence

$$\begin{aligned} \dim L(\mu, A) &\geqslant qd(\lambda) + \operatorname{ind} A - \dim S(\nu, E^*) + \dim S(\widetilde{\nu}, F^*) \\ &= qd(\lambda) + qd(\nu) + \operatorname{ind} A = qd(\mu) + \operatorname{ind} A, \end{aligned}$$

as required.

B. Now we indicate that sometimes even the equality in (4.1) can be claimed (if we have a sufficiently large number of poles, multiplicities counted).

DEFINITION 4.2. Let us say that a differential operator B on X has a *unique continuation property* if any local C^∞-solution u of the equation $Bu = 0$ having zero of infinite order at a point x_0, vanishes in a neighbourhood of x_0.

It is well known that the unique continuation property is not always true for elliptic operators. But it is true for elliptic operators B with real-analytic coefficients (because then all the solutions of $Bu = 0$ are analytic), for second-order elliptic operators that are either scalar or have a scalar principal symbol in case $n \geqslant 3$ (e.g., for the Laplacian on the differential forms on a Riemannian manifold), for elliptic operators with simple characteristics (see [4]).

PROPOSITION 4.3. *Suppose that the manifold X is connected, A^* has the unique continuation property, and the points $x_1, \dots, x_m$ are fixed. Then for every $N_0 > 0$ there exists $N > 0$ such that if $\sum_{p_i<0} |p_i| \leqslant N_0$ and $\sum_{p_i>0} p_i \geqslant N$ then*

$$r(\mu, A) = \operatorname{ind} A + qd(\mu). \tag{4.2}$$

PROOF. We have to prove that $r(\mu^{-1}, A^*) = 0$ if a positive integer N_0 is fixed and N is sufficiently large. Note first that there is only a finite number of possible choices of the points x_i with $p_i < 0$, so we can fix such a choice. After that consider the divisor $\nu_{\min} = \prod_{p_i<0} x_i^{-N_0}$. Then for any possible choice of numbers p_i with $\sum_{p_i<0} |p_i| \leqslant N_0$ we shall have

$$L(\mu^{-1}, A^*) \subset L(\nu_{\min}^{-1}, A^*).$$

Hence $L(\mu^{-1}, A^*)$ will always belong to a fixed finite-dimensional space of sections of $\tilde{F}$ over $X - \operatorname{supp} \nu_{\min}$. We have to prove that for any point $x_0 \in X - \operatorname{supp} \nu_{\min}$ there exists an integer $N > 0$ such that if $v \in L(\nu_{\min}^{-1}, A^*)$ and $j_{x_0}^N v = 0$ then $v \equiv 0$. But if it is not true then using any Hilbert norm $\|\cdot\|$ in the finite-dimensional space $L(\nu_{\min}^{-1}, A^*)$ we can choose a sequence $v_k \in L(\nu_{\min}^{-1}, A^*)$ with $\|v_k\| = 1$ and $j_{x_0}^k v = 0$, $k = 1, 2, \dots$. Now using the compactness of the unit sphere in $L(\nu_{\min}^{-1}, A^*)$ we can even suppose that $v_k \to v$ in $L(\nu_{\min}^{-1}, A^*)$ and $\|v\| = 1$. But the convergence in $L(\nu_{\min}^{-1}, A^*)$ obviously implies the convergence in C^∞-topology of sections over $X - \operatorname{supp} \nu_{\min}$. Hence $j_{x_0}^k v = 0$ for all $k = 1, 2, \dots$, contradicting the unique continuation property. □

REMARK 1. Note that in the classical Riemann-Roch theorem (see Example 2.1 in Section 2) an effective result of this sort is possible:

$$r(\mu) = 1 - g + d(\mu) \quad \text{provided} \quad d(\mu) > 2g - 2.$$

In the general case the equality conditions cannot be given in terms of $d(\mu)$ alone. For example, let us take $A = \Delta$ (the scalar Laplacian) on a compact Riemannian manifold of dimension $n \geqslant 2$. Then we can find a nonconstant real-valued meromorphic solution v of $\Delta v = 0$, i.e., a nonzero element in $L(\mu_0^{-1}, \Delta)$ for a (point) divisor μ_0 (such solutions exist provided $d(\mu_0^{-1}, \Delta) \geqslant 2$ due to Corollary 4.1). By the Sard Lemma there exists $c \in \mathbb{R}$ such that $\{x \mid v(x) = c\}$ is a hypersurface, i.e., a submanifold of codimension 1 in X. Replacing v by $v - c$ we may suppose that the nodal set $X_0 = \{x \mid v(x) = 0\}$ is a hypersurface. Now let us take $\mu_N = \mu_0 x_1 \cdots x_N$ where $x_i \in X_0$, $x_i \notin \operatorname{supp} \mu_0$, $x_i \neq x_j$, $i, j = 1, \cdots, N$. Then obviously $d(\mu_N, \Delta) \to \infty$ as $N \to \infty$ but $r(\mu_N, \Delta) \neq d(\mu_N, \Delta)$ because $L(\mu_N^{-1}, \Delta) \ni v$, hence $r(\mu_N^{-1}, \Delta) \neq 0$.

REMARK 2. A condition of the unique continuation type is necessary for the statement of Proposition 4.3 to be true. Namely, suppose that there exists a solution $v \in \Gamma_c(X, F^*)$ of the equation $A^*v = 0$ such that $v \not\equiv 0$ and $\operatorname{supp} v \neq X$. Then for any divisor μ with $\operatorname{supp} \mu \subset X - \operatorname{supp} v$ we have

$$r(\mu, A) < \operatorname{ind} A + qd(\mu),$$

since $v \in L(\mu^{-1}, A^*)$, which implies that $r(\mu^{-1}, A^*) \neq 0$. Hence the conclusion of Proposition 4.3 is not true in this case.

C. Now we turn to a local application of Theorem 2.1. First introduce necessary notations. Let $\mathscr{E}_0$ be the linear space of germs at 0 of C^∞-functions $f: U \to \mathbb{C}$, where U is a neighbourhood of 0 in $\mathbb{R}^n$. So the elements of $\mathscr{E}_0$ are equivalence classes of such functions and $f_1 \sim f_2$ for two such functions $f_i: U_i \to \mathbb{C}$ iff there exists a neighbourhood U of 0 in $\mathbb{R}^n$ such that $U \subset U_1 \cap U_2$ and $f_1 \mid U = f_2 \mid U$. Now denote

$$\mathscr{E}_0^{(k)} = \{f \mid f \in \mathscr{E}_0, j_0^k f = 0\}.$$

Suppose that we have a $(q \times q)$-matrix elliptic differential operator A of order d defined in a neighbourhood of 0 in $\mathbb{R}^n$. Then it induces a linear map

$$A: \mathscr{E}_0^{(k+d)} \otimes \mathbb{C}^q \longrightarrow \mathscr{E}_0^{(k)} \otimes \mathbb{C}^q. \tag{4.3}$$

PROPOSITION 4.4. *The map* (4.3) *is surjective for each integer* $k \geqslant 0$.

PROOF. The idea is to use Theorem 2.2 (ii). Suppose that we have a C^∞-function $f: U \to \mathbb{C}^q$ such that $j_0^k f = 0$ (here U is a neighbourhood of 0 in $\mathbb{R}^n$). We want to find a smaller neighbourhood U_1 of 0 and a C^∞-function $u: U_1 \to \mathbb{C}^q$ such that $j_0^{(k+d)} u = 0$ and $Au = f \mid U_1$. To do this we extend A and f from a neighbourhood of 0 to a compact closed manifold X (e.g., $X = S^n$ or $T^n = \mathbb{R}^n/\mathbb{Z}^n$) so that 0 becomes a point $x_1 \in X$, A an elliptic operator over X, $A: \Gamma(X, E) \to \Gamma(X, F)$, and f a C^∞-section of the vector bundle F over X (actually bundles E and F can be chosen trivial). Such an extension is obviously possible.

Now let us choose a divisor $\mu = x_1^{-(k+d+1)}$. Then $f \in \Gamma_{\tilde{\mu}}(X, F)$. We want to modify f in such a way that the modified section (coinciding with f in a neighbourhood of x_1) belongs to $(\operatorname{Ker} \tilde{A}')^\circ$. This will prove the existence of u due to Theorem 2.2.

Denote

$$N_{x_1}(X, F) = \{g \mid g \in \Gamma(X, F),\ g = 0 \text{ in a neighbourhood of } x_1\}.$$

Then we have to find $g \in N_{x_1}(X, F)$ such that $f - g \in (\operatorname{Ker} \tilde{A}')^\circ$ (hence $f - g$ will be the desired modified section).

Consider the natural map $j: \Gamma_{\tilde{\mu}}(X, F) \to (\operatorname{Ker} \tilde{A}')'$, $j(f)(v) = \langle f, v\rangle$. We want to prove that $j(f) \in j(N_{x_1}(X, F))$. But actually $j: N_{x_1}(X, F) \to (\operatorname{Ker} \tilde{A}')'$ is surjective since the dual map $j': \operatorname{Ker} \tilde{A}' \to (N_{x_1}(X, F))'$, which is defined similarly to j: $j'(v)(f) = \langle f, v\rangle$, is obviously injective. Indeed, j' is injective because $\operatorname{Ker}\widetilde{A}^t \subset \Gamma(X - \{x_1\}, F^*)$ and $N_{x_1}(X, F)$ includes all sections $g \in \Gamma(X, F)$ with $\operatorname{supp} g \subset X - \{x_1\}$. □

D. We make a few remarks about sheaves related with a given divisor and a given elliptic equation. These sheaves are actually natural localizations of the solution spaces described above.

In this subsection X is a closed compact C^∞-manifold, all other notations are the same as before. Denote by $\mathscr{L}(\mu, A)$ the sheaf of (local) solutions of $Au = 0$ with possible point singularities at the points x_j with $p_j > 0$ and zeros at the points x_j with $p_j < 0$, so that $\operatorname{ord}_{x_j} \geqslant -p_j$ for all $j = 1, \dots, k$ as in the definition of $L(\mu, A)$. So $L(\mu, A)$ becomes the space of all global sections of $\mathscr{L}(\mu, A)$, i.e.,

$$L(\mu, A) = \Gamma(X, \mathscr{L}(\mu, A)) = H^0(X, \mathscr{L}(\mu, A)).$$

Further information about the cohomologies of these sheaves is given by the following

THEOREM 4.1. (i) *$H^p(X, \mathscr{L}(\mu, A)) = 0$ for any $p \geqslant 2$.*

(ii) *The linear spaces $H^1(X, \mathscr{L}(\mu, A))$ and $H^0(X, \mathscr{L}(\mu^{-1}, A^*))$ are in a natural nondegenerate duality. In particular*

$$\dim H^1(X, \mathscr{L}(\mu, A)) = \dim H^0(X, \mathscr{L}(\mu^{-1}, A^*)).$$

PROOF. Let us introduce sheaves $\mathscr{E}(\mu, A)$ and $\mathscr{E}_{\tilde{\mu}}(F)$ that are the localizations of the spaces $\Gamma(X, \mu, A)$ and $\Gamma_{\tilde{\mu}}(X, F)$. So $\mathscr{E}(\mu, A)$ outside $\operatorname{supp}\mu$ coincides with the sheaf of all C^∞-sections of E. If $x_j \in \operatorname{supp}\mu$ and $p_j > 0$ then the local sections of $\mathscr{E}(\mu, A)$ near x_j have the form $u = u_s + u_r$, where u_s is a solution of $Au_s = 0$ defined in $\mathscr{U} - \{x_j\}$ ($\mathscr{U}$ is a neighbourhood of x_j) such that $\operatorname{ord}_{x_j} u_s \geqslant -p_j$, $u_r \in \Gamma(\mathscr{U}, E)$; if $p_j < 0$ then they are C^∞-sections u of E defined near x_j, such that $\operatorname{ord}_{x_j} u \geqslant -p_j$ (i.e., $j_{x_j}^{|p_j|-1} u = 0$). Similarly $\mathscr{E}_{\tilde{\mu}}(F)$ is the sheaf of all C^∞-sections f of F, such that $j_{x_j}^{|\tilde{p}_j|-1} f = 0$ if $x_j \in \operatorname{supp}\mu$ and $\tilde{p}_j < 0$.

Now notice that both $\mathscr{E}(\mu, A)$ and $\mathscr{E}_{\tilde{\mu}}(F)$ are fine sheaves because for any covering of X by open sets a partition of unity subordinated to this covering can be chosen on X in such a way that all functions that enter to this partition of unity are either identically zero or identically 1 near any point $x_j \in \operatorname{supp}\mu$. Hence due to Proposition 4.4 we get a fine resolution of $\mathscr{L}(\mu, A)$ as follows:

$$0 \longrightarrow \mathscr{L}(\mu, A) \longrightarrow \mathscr{E}(\mu, A) \xrightarrow{A} \mathscr{E}_{\tilde{\mu}}(F) \longrightarrow 0.$$

Now (i) immediately follows. Also we get

$$H^1(X, \mathscr{L}(\mu, A)) = \operatorname{Coker}\{A\colon \Gamma(X, \mu, A) \longrightarrow \Gamma_{\tilde{\mu}}(X, F)\}.$$

But the right-hand side here is in a nondegenerate duality with

$$\operatorname{Ker}\{A^*: \Gamma(X, \mu^{-1}, A^*) \longrightarrow \Gamma_{\tilde{\mu}^{-1}}(X, E^*)\}$$

due to Theorem 2.2. This proves (ii). □

REMARK. Obviously (i) holds for noncompact manifolds (without boundary) as well.

The statement (ii) in Theorem 4.1 is an analogue of the Serre duality (see, e.g., [6, Chapter 3, Section 7]).

E. Nadirashvili [1] used the particular case $A = \Delta$ of Theorem 2.3, that he proved (see also Example 2.4 in Section 2) to give an estimate of multiplicity of possible zeros for the Coulomb potential of k point charges in $\mathbb{R}^3$. Namely, he proved that this multiplicity cannot be more than $k-1$ (unless all charges are 0). We reproduce here his arguments in more detail and make more use of them by considering Coulomb potentials in $\mathbb{R}^n$, $n \geqslant 3$. We shall also discuss similar questions on Riemannian surfaces.

We shall begin with a very general statement about meromorphic harmonic functions on general compact Riemannian manifolds.

Let X be a compact Riemannian manifold and $x_0, x_1, \dots x_k$ a collection of $k+1$ distinct points in X. We want to find out what is the maximum possible order of zero at x_0 for a nontrivial meromorphic harmonic function on X with possible simple poles at $x_1, \dots, x_k$ (and no other poles). Denote this maximal order by $l(X; x_0 \mid x_1, \dots, x_k)$. Obviously the estimate

$$l(X; x_0 \mid x_1, \dots, x_k) \leqslant l - 1 \tag{4.4}$$

is equivalent to the equality

$$r(x_0^{-l} x_1 \cdots x_k, \Delta) = 0 \tag{4.5}$$

LEMMA 4.5. *Let l be a positive integer. Then* (4.4) (*or* (4.5)) *is true if and only if there exist functions* $u_1, \dots, u_k \in L(x_0^l, \Delta)$ *such that* $u_i(x_j) = \delta_{ij}$, $i, j = 1, \dots, k$.

PROOF. Theorem 2.1 gives

$$\begin{aligned} r(x_0^{-l} x_1 \cdots x_k, \Delta) &= d(x_0^{-l} x_1 \cdots x_k) + r(x_0^l x_1^{-1} \cdots x_k^{-1}, \Delta) \\ &= -d(x_0^l x_1^{-1} \cdots x_k^{-1}) + r(x_0^l x_1^{-1} \cdots x_k^{-1}, \Delta) \\ &= -d(x_0^l) + k + r(x_0^l x_1^{-1} \cdots x_k^{-1}, \Delta), \end{aligned}$$

so (4.5) is equivalent to

$$r(x_0^l x_1^{-1} \cdots x_k^{-1}, \Delta) \leqslant d(x_0^l) - k.$$

Note that $d(x_0^l) = r(x_0^l, \Delta)$ because $r(x_0^{-l}, \Delta) = 0$ due to the maximum principle. Hence (4.5) is equivalent to

$$r(x_0^l x_1^{-1} \cdots x_k^{-1}, \Delta) \leqslant r(x_0^l, \Delta) - k. \tag{4.6}$$

Now note that

$$L(x_0^l x_1^{-1} \cdots x_k^{-1}, \Delta) = \{u \mid u \in L(x_0^l, \Delta),\ u(x_1) = \cdots = u(x_k) = 0\}.$$

Hence (4.6) is equivalent to linear independence of the conditions $u(x_i) = 0$, $i = 1, \ldots, k$, on $L(x_0^l, \Delta)$ which immediately proves Lemma 4.5. □

This lemma reduces the proof of the multiplicity estimate to the proof of existence of harmonic functions with a single (but not necessarily simple) pole such that their restrictions to the finite set $\{x_1, \ldots, x_k\}$ give linearly independent vectors.

Lemma 4.5 is easily extended to the noncompact case as discussed in Section 2. Namely, let X be a (noncompact) connected Riemannian manifold with a compact boundary and Δ the scalar Laplacian of the given Riemannian metric. Now let A, A^* be defined both with the help of Δ but possibly with different domains $\operatorname{Dom} A$, $\operatorname{Dom} A^*$; let $\operatorname{Dom}' A$, $\operatorname{Dom}' A^*$ be also chosen. Suppose that all the conditions of Theorem 2.3 are satisfied. Suppose also that

$$\operatorname{Dom} A \subset \{u \mid u|_{\partial X} = 0,\ u(x) \to 0 \text{ as } x \to \infty\}, \tag{4.7}$$

or

$$\operatorname{Dom} A \subset \{u \mid \partial u/\partial n|_{\partial X} = 0,\ u(x) \to 0 \text{ as } x \to \infty\}, \tag{4.8}$$

i.e., $\operatorname{Dom} A$ is defined by Dirichlet or Neuman boundary conditions and some conditions at infinity, that include the requirement that u vanishes at infinity. Then the following lemma holds:

LEMMA 4.6. *$r(x_0^{-l} x_1 \cdots x_k, A) = 0$ if and only if there exist functions $u_1, \ldots, u_k \in L(x_0^l, A^*)$ such that $u_i(x_j) = \delta_{ij}$, $i, j = 1, \ldots, k$.*

The proof is obtained by the same arguments as in the proof of Lemma 4.5 except Theorem 2.3 should be applied instead of Theorem 2.1 and the maximum principle should be applied on $\operatorname{Dom} A$, which is possible due to (4.7) or (4.8).

In particular, Lemma 4.6 may be applied in the situation of Example 2.4. Here $X = \mathbb{R}^n$, $n \geqslant 3$, and $L(x_1 \cdots x_k, \Delta)$ consists of functions of the form

$$u(x) = \sum_{1 \leqslant i \leqslant k} \frac{q_i}{|x - x_i|^{n-2}}, \tag{4.9}$$

which are generalized Coulomb potentials of point charges $q_1, \ldots, q_k \in \mathbb{R}$, situated at the points $x_1, \ldots, x_k \in \mathbb{R}^n$, which are supposed to be distinct. In this case (4.5) is reduced to finding harmonic functions in $\mathbb{R}^n$ vanishing at infinity and having a single pole at x_0 of order at most l, such that their restrictions to the set $\{x_1, \ldots, x_k\}$ give linearly independent vectors. Nadirashvili could do this when $l = k$ (and $n = 3$, but his arguments are good for any $n \geqslant 3$), so his result is

THEOREM 4.2. *Let u have the form* (4.9) *and* $u \not\equiv 0$. *Then all zeros of u have orders less than or equal to* $k-1$.

PROOF. We have to check that $\operatorname{ord}_{x_0} u \leqslant k-1$ for any $x_0 \in \mathbb{R}^n$. Using the translation invariance we may assume without loss of generality that $x_0 = 0$. We shall use the Kelvin transform K, which acts on functions $u\colon \mathbb{R}^n - \{0\} \longrightarrow \mathbb{C}$ by the formula

$$Ku(x) = |x|^{2-n} u(x^*),$$

where $x \mapsto x^* = x/|x|^2$ is the inversion map on $\mathbb{R}^n - \{0\}$. The function Ku is harmonic on $\mathbb{R}^n - \{0\}$ if and only if u is harmonic there.

Now let us choose a 2-dimensional linear subspace in $\mathbb{R}^n$ such that all the points $0, x_1^*, \dots, x_k^*$ have distinct orthogonal projections to this plane. Using a rotation to change coordinates we may assume without loss of generality that this subspace is a coordinate plane. We will identify this plane with the complex line $\mathbb{C}$ and denote the orthogonal projection of $\mathbb{R}^n$ to this plane by π, so π^* will denote the corresponding map on functions.

Let us denote $z_i = \pi(x_i^*)$ and take the Lagrange interpolation polynomials

$$p_i\colon \mathbb{C} \longrightarrow \mathbb{C}, \quad \deg p_i = k-1, \quad p_i(z_j) = c_i \delta_{ij}, \quad i, j = 1, \dots, k,$$

where $c_i \in \mathbb{C} - \{0\}$, $i = 1, \dots, k$, are normalization constants to be chosen later. Now let us consider harmonic functions

$$u_i = K(\pi^* p_i), \qquad i = 1, \dots, k.$$

It is easy to check that they satisfy the requirements of Lemma 4.6 with $l = k$ after an appropriate choice of the normalization constants. □

REMARK 1. Note that the estimate of the multiplicity given by Theorem 4.2 is precise for $k = 1$ or 2. However we cannot expect it to be precise in general. In fact an easy parameter counting shows that the quantity

$$l(k) = \max\{l(\mathbb{R}^n; x_0 \mid x_1, \dots, x_k) \mid x_0, x_1, \dots, x_k \in \mathbb{R}^n\}$$

(the maximum possible order of zero over all possible configurations of charges and the zero) should be $O(k^{1/(n-1)})$. In particular we may expect an estimate by $O(\sqrt{k})$ if $n = 3$.

REMARK 2. Theorem 4.2 can be proved without use of Theorem 2.3. Namely, in this particular case Lemma 4.6 can be proved by elementary arguments from linear algebra. To do this consider (in $\mathbb{R}^n$) the following linear map

$$\mathbb{C}^k \cong L(x_1 \cdots x_k, \Delta) \longrightarrow J_{x_0}^l, \qquad u \mapsto j_{x_0}^l u,$$

where $J_{x_0}^l$ is the set of all l-jets at x_0 for scalar functions on $\mathbb{R}^n$. Here we identify the function (4.9) with the vector $(q_1, \dots, q_k) \in \mathbb{C}^k$. Now suppose that we want to prove that this map is injective. It is equivalent to the fact that the dual map is surjective. But it is easy to check that the dual map

$$S(x_0^l) \longrightarrow \mathbb{C}^k$$

maps $\sum_{|\alpha|\leqslant l} c_\alpha \delta^{(\alpha)}(x - x_0)$ to the vector, whose components are the values of the function $\sum_{|\alpha|\leqslant l} c_\alpha \partial_x^\alpha |x - x_0|^{2-n}$ at the points $x_1, \dots, x_k$. (Here $S(x_0^l) = S(x_0^l, \mathbb{C}_{\mathbf{R}^n})$.) The necessary special case of Lemma 4.6 immediately follows.

F. Now we discuss multiplicity estimates for zeros of harmonic functions on compact Riemannian surfaces X with simplest (logarithmic) singularities. This amounts to investigating the numbers $l(X; x_0 \mid x_1, \dots, x_k)$ in notations which were used in a more general context earlier in this section. Let us also denote

$$l(k; g) = \max\{l(X; x_0 \mid x_1, \dots, x_k) \mid \text{genus}(X) = g,\ x_0, x_1, \dots, x_k \in X\},$$

i.e., $l(k; g)$ is the maximum of all possible multiplicities of zeros of nontrivial harmonic functions with k simple poles on a Riemannian surface of genus g over all compact Riemannian surfaces of genus g and over all possible configurations of poles.

THEOREM 4.3. $l(k; g) \leqslant k + 2g - 1$.

PROOF. For any $u \in L(x_0^{-l} x_1 \cdots x_k, \Delta)$ consider a meromorphic $(1, 0)$-form $\omega = \partial u = (\partial u/\partial z)\, dz$ (here z is a local complex parameter on X). Then clearly $\omega \in L(x_0^{-l+1} x_1 \cdots x_k, \bar\partial^*)$ where $\bar\partial^* = \bar\partial : \Lambda^{1,0}(X) \longrightarrow \Lambda^{1,1}(X) = \Lambda^2(X)$ is the $\bar\partial$-operator on $(1, 0)$-forms on X. Using the fact that the degree of every meromorphic differential should be equal to $2g - 2$ we immediately obtain the inequality $l - 1 - k \leqslant 2g - 2$ (with the equality if ω actually has poles in all points $x_1, \dots, x_k$, has no other zeros except x_0, and the order of this zero is precisely $l - 1$). The desired result immediately follows. □

Now we will formulate a theorem that will give a more precise estimate but only for generic sets $\{x_0, x_1, \dots, x_k\}$.

THEOREM 4.4. *There exists a dense open set* $\mathfrak{A} \subset X \times \cdots \times X$ $(k + 1$ *factors), such that for any* $(x_0, x_1, \dots, x_k) \in \mathfrak{A}$ *and any nontrivial harmonic function with possible simple poles (with logarithmic singularities) at the points* $x_1, \dots, x_k$ *only, the multiplicity of a possible zero of* u *at* x_0 *is less than or equal to* $k/2$.

PROOF. We shall use Lemma 4.5 and so the degree of the divisors will be always taken with respect to the second order operator Δ. If $l \geqslant 1$ then $\dim_{\mathbb{C}} L(x_0^l, \Delta) = d(x_0^l) = 2l - 1$. Now suppose that $2l - 1 \geqslant k$ or $l \geqslant (k + 1)/2$. Then fixing x_0 we can find linearly independent functions $u_1, \dots, u_k \in L(x_0^l, \Delta)$. Hence we can find points $x_1, \dots, x_k$ such that $\det(u_i(x_j)) \neq 0$. Passing to linear combinations of the chosen functions we may even assume that $u_i(x_j) = \delta_{ij}$. Moreover we shall obviously have then $\det(u_i(x_j)) \neq 0$ for $(x_1, \dots, x_k)$ in a dense open set $\mathfrak{B}_{x_0} \subset X \times \cdots \times X$

(k factors). Note that locally $u_1, \ldots, u_k$ can be made continuous with respect to x_0 because up to an additive constant they are determined by their "singular parts" at x_0, which are in a one-one correspondence with a set of distributions $f \in S(x_0^l, \mathbb{C}_X)$ (see Section 3). It follows that the set

$$\mathfrak{A} = \{(x_0, x_1, \ldots, x_k) \mid (x_1, \ldots, x_k) \in \mathfrak{B}_{x_0}\}$$

is also open in $X \times \cdots \times X$ ($k+1$ factors).

Now Lemma 4.5 implies that $\operatorname{ord}_{x_0} u < (k+1)/2$ or, which is equivalent, $\operatorname{ord}_{x_0} u \leqslant k/2$ for any $u \in L(x_1 \cdots x_k, \Delta)$ provided $(x_0, x_1, \ldots, x_k) \in \mathfrak{A}$. □

Finally let us consider the simplest case $X = S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \infty$, i.e., the case $g = 0$.

THEOREM 4.5. *Let $g = 0$. Then*

(i) $l(k; 0) = k - 1$, *i.e., the maximal possible multiplicity of a zero for a nontrivial harmonic function with k simple poles equals $k-1$.*

(ii) *For any set of distinct points $z_0, z_1, \ldots, z_k \in X$ and any positive integer $l < k/2$ there exists a nontrivial harmonic function u on X with possible simple poles at the points $z_1, \ldots, z_k$ and a zero of order l at z_0, i.e., $l(X; z_0 \mid z_1, \ldots, z_k) \geqslant l$ for any integer $l < k/2$ and any points $z_0, z_1, \ldots, z_k$.*

(iii) $l(X; z_0 \mid z_1, \ldots z_k) = k - 1$ *if and only if*

$$\prod_{l \neq i,j} \left(\frac{z_i - z_0}{z_j - z_0} : \frac{z_i - z_l}{z_j - z_l}\right) \in \mathbb{R} \cup \infty, \qquad 1 \leqslant i, j \leqslant k, \quad i \neq j, \tag{4.10}$$

i.e., a nontrivial harmonic function with the only possible simple poles at the points $z_1, \ldots, z_k$ and with the maximal possible order of zero at z_0 exists if and only if all the products in (4.10) are real or infinite.

PROOF. Note first that (iii) obviously implies (i) since the conditions (4.10) are satisfied, e.g., if $z_i \in \mathbb{R}$, $i = 0, \ldots, k$.

Now let us prove (iii). We know already (Theorem 4.3) that $l(X; z_0 \mid z_1, \ldots, z_k) \leqslant k - 1$. Using the conformal invariance of the statement (iii), we can suppose that $z_0, z_1, \ldots, z_k \in \mathbb{C}$, i.e., that none of the points $z_0, z_1, \ldots, z_k$ coincides with ∞. Suppose that $u \in L(z_0^{-k+1} z_1, \ldots z_k, \Delta) - \{0\}$. Taking the real or imaginary part we may then suppose that u is real-valued. Then $\omega = \partial u$ will be a meromorphic $(1, 0)$-form on X with real residues, $\omega \in L(z_0^{-k+2} z_1 \cdots z_k, \bar\partial^*) - \{0\}$. Vice versa, having such a form, we can reconstruct u by the formula

$$u(z) = \operatorname{Re} \int_{z_0}^{z} \omega. \tag{4.11}$$

(The condition on the residues ensures that u is single-valued.)

So (iii) is equivalent to the statement that $\omega \in L(z_0^{-k+2} z_1 \cdots z_k, \bar{\partial}^*) - \{0\}$ with real residues exists if and only if the conditions (4.10) are satisfied. Since we suppose ω to be regular at infinity it should have the form $\omega = (P(z)/Q(z))\,dz$, where P and Q are polynomials, $\deg P \leqslant \deg Q - 2$. We can obviously assume that P and Q have no common zeros. Then we should have $\deg Q \leqslant k$ (otherwise ω will have more than k poles), hence $\deg P \leqslant k-2$. But then we should have (up to constant factors) $P(z) = (z - z_0)^{k-2}$, $Q(z) = (z - z_1) \cdots (z - z_k)$, hence

$$\omega = c \frac{(z - z_0)^{k-2}}{(z - z_1) \cdots (z - z_k)}\, dz$$

with $c \in \mathbb{C}$. The residues can be made real by a choice of the constant c here if and only if all the pairwise ratios of the residues of $(P(z)/Q(z))\,dz$ are real. These ratios are

$$\frac{(z_i - z_0)^{k-2}}{\prod_{l \neq i}(z_i - z_l)} : \frac{(z_j - z_0)^{k-2}}{\prod_{l \neq j}(z_j - z_l)} = - \prod_{l \neq i, j} \left(\frac{z_i - z_0}{z_j - z_0} : \frac{z_i - z_l}{z_j - z_l} \right),$$

which immediately leads to the desired result (iii).

To prove (ii) assume again that $z_0, z_1, \ldots, z_k \in \mathbb{C}$ and set

$$\omega = \frac{(z - z_0)^{l-1} g(z)\, dz}{(z - z_1) \cdots (z - z_k)},$$

where $g = g(z)$ is a polynomial, $\deg g \leqslant k - l - 1$. Obviously ω is then a meromorphic (1, 0)-form which is regular at infinity. Now the formula (4.11) will give us the desired function u provided ω does not vanish identically and has real residues. The resudues of ω have the form

$$\frac{(z_i - z_0)^{l-1} g(z_i)}{\prod_{l \neq i}(z_i - z_l)}, \qquad i = 1, \ldots, k.$$

Vanishing of their imaginary parts gives k real linear equations for $2(k - l)$ real numbers (the real and imaginary parts of the coefficients of g) that determine the polynomial g. If $2(k - l) > k$ or, equivalently, $2l < k$, then these equations have a nontrivial solution, and (ii) is proved. □

REFERENCES

1. N. S. Nadirashvili, *Harmonic functions with a given set of singularities*, Functional. Anal. i Prilozhen. **22** (1988), no. 1, 64–66; English transl. in Functional Anal. Appl. **22** (1988), no. 1.

2. L. Hörmander, *The analysis of linear partial differential operators*. III, Springer-Verlag, Belin, Heidelberg, New York, 1985.

3. R. Palais, *Seminar on the Atiyah-Singer index theorem*, Princeton Univ. Press, Princeton, NJ, 1965.

4. E. G. Sitnikova, *A strong zero theorem for a higher order elliptic equation*, Mat. Sb. **81** (1970), no. 3, 376–397; English transl. in Math. USSR–Sb. **10** (1970).

5. M. Gromov and M. A. Shubin, *The Riemann-Roch theorem for general elliptic operators*, C. R. Acad. Sci. Paris **314** (1992), no. 1, 363–367.

6. R. Hartshorne, *Algebraic geometry*, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

Institut des Hautes Études Scientifiques 91440, Bures-sur-Yvette, France

Institute of New Technologies, 11, Kirovogradskaya Moscow 113587, Russia and Department of Mathematics, Northeastern University, Boston, Massachusetts 02115