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CAT(κ)-SPACES: CONSTRUCTION
AND CONCENTRATION

To Victor Abramovich Zalgaller

We expose spaces $X$ with negative curvature having in mind applications to fractally hyperbolic groups, such as random groups and infinite Burnside groups. Originally these spaces were introduced by Alexandrov in the axiomatization spirit and a similar class (of convex spaces) was later isolated by Busemann.

Till relatively recently the major thrust of geometric research was laid on suppressing singularities, emphasizing the properties equally shared by smooth and singular spaces and proving regularization theorems claiming, under certain assumptions, that $X$ can be approximated by smooth manifolds with curvature $K \leq 0$. This was accomplished for surfaces in a famous treatise by Alexandrov and Zalgaller.

But the bulk of spaces with $K \leq 0$ is badly singular, starting from trees and most abundant among 2-polyhedra. Furthermore, almost all “natural” spaces with $K \leq 0$, such as the Bruhat–Tits buildings, are non-smooth and (unlike trees) cannot be usually approximated by smooth spaces. But geometers remained unaware of this for a stretch of time.

From another angle, the idea of negative curvature was injected into the group theory by Dehn and grew up into the small cancellation theory. In the course of the development, the geometric roots were forgotten and the role of curvature was reduced to a metaphor. (Algebraists do not trust geometry.)

It eventually turned out that the geometric language of Dehn and Alexandrov (sometimes slightly modified and/or generalized) accomplishes many needs of combinatorial group theory more efficiently than the combinatorial language.

Summing up, geometry furnishes a proper language, while the combinatorial group theory (especially random groups) provides a pool of objects for a meaningful usage of this language.
In this paper we present basic constructions of spaces $X$ with $K \leq 0$ relevant for applications in group theory (see [8]) as well as basic isoperimetric properties of maps of metric measure spaces (see [14] and [11]) into $X$. We observe, for example, that conical singularities based on expanders (with $K \leq 1$) cannot be smoothed, not even with the most generous notion of smoothing. (This will be brought into the group theoretic framework in [8].)

We furnish all necessary definitions and illustrate them by examples but refer to the textbooks for the details of standard arguments (see [2] and references therein).

§1. Metrics and geodesics

Given a metric space $X = (X, \text{dist})$ we often abbreviate and write

$$|x - y| = |x - y|_X = \text{dist}(x, y).$$

We call $X$ a geodesic space if every two points $x$ and $y$ in $X$ can be joined, albeit non-uniquely, by a shortest (geodesic) segment denoted $[x, y] \subset X$, that is an isometric embedding of a real segment of length $= \text{dist}_X(x, y)$ into $X$.

Actually, the existence of such a shortest, or minimizing, segment is not so crucial: it is enough for most purposes to have $\text{dist}(x, y)$ equal to the infimum of the length of paths in $X$ joining $x$ and $y$, where this infimum does not have to be achieved.

Also, one could use the middle point condition: the existence of $z \in X$ such that

$$\text{dist}(x, z) = \text{dist}(z, y) = \frac{1}{2} \text{dist}(x, y).$$

For complete metric spaces the last condition is equivalent to existence of a minimizing segment.

Sometimes one could require even less, the existence of $z = z_\varepsilon$ for each $\varepsilon > 0$, such that both distances $\text{dist}(x, z)$ and $\text{dist}(z, y)$ are $\leq \frac{1}{2} \text{dist}(x, y) + \varepsilon$.

From now on we assume the existence of our segments $[x, y] \subset X$ when we deal with geodesic spaces.

§2. Basic examples

(a) Every metrically complete connected Riemannian manifold $X$, possibly with a boundary, is path metric in an obvious way (where the minimizing segments may touch the boundary). In particular, every smooth
connected domain \( X \subset \mathbb{R}^n \) carries the induced path metric \( \text{dist}\_X \) which is greater than the restricted metric \( \text{dist}_{\mathbb{R}^n}|_X \), and where the equality \( \text{dist}_X = \text{dist}_{\mathbb{R}^n}|_X \) holds if and only if \( X \) is convex.

(b) Let \( X \) be a simplicial polyhedron. If we identify each simplex in \( X \) with a unit Euclidean simplex, we can speak of the length of a curve in \( X \) using the Euclidean geometry in all \( \Delta \subset X \). Then we define \( \text{dist}_X \) by taking infimum of length of paths between points \( x \) and \( y \) in \( X \). This is a true geodesic metric for locally finite polyhedra where the infimum is achieved by some \( [x, y] \in X \), while more general polyhedra sometimes need a completion in order to become geodesic in the strict sense.

The simplest polyhedra \( X \) are the 1-dimensional ones, i.e., graphs, where the above metric amounts to assigning unit length to all edges. Of course, one could live with edges of variable lengths, but when the dimension goes up, one should be careful if one assigns variable sizes and shapes to simplices in \( X \) as these must agree across common \( k \)-faces with \( k \geq 1 \).

(b') It is often necessary to assign non-Euclidean geometries to simplices in \( X \), e.g., by identifying each \( \Delta \subset X \) with a regular spherical or hyperbolic simplex of a certain size. The resulting, e.g., piecewise spherical and piecewise hyperbolic, geodesic metric in \( X \) may reveal some combinatorial properties of \( X \) invisible in the (piece-wise) Euclidean light.

\[\textbf{§3. Model-spaces}\]

The standard or model spaces of constant curvature are

(i) The round 2-sphere of radius \( R \), denoted \( S^2(R) \). This has (by definition, if you wish) curvature \( K(S^2(R)) = \frac{1}{R^2} \).

(ii) The Euclidean plane \( \mathbb{R}^2 \), where \( K(\mathbb{R}^2) = 0 \).

(iii) The hyperbolic plane \( H_\kappa \) with curvature \( -\kappa^2 \). This \( H_\kappa \) can be represented as the plane with coordinates \((t, y)\) and the Riemannian metric \( dt^2 + \frac{1}{1-\kappa^2 t^2} dy^2 \). The \( t \)-lines here are geodesic, i.e., the embeddings \( \mathbb{R} \to (\mathbb{R}, y) \subset H_\kappa \) are isometric for all \( y \in \mathbb{R} \). On the other hand the \( y \)-lines \((t, \mathbb{R})\) are curved in \( H_\kappa \) and they shrink exponentially fast as \( t \) increases.

(iv) If \( \kappa \to \infty \), then \( H_\kappa \) converges in a natural way (see [4]) to an infinite metric tree branching at all points. This serves as the model space for \( \kappa = -\infty \).
Similarly to (i)--(iii) we have $n$-dimensional spaces with curvature $\kappa$, that are $S^n(R), \mathbb{R}^n$, and $H^n_k$ (with the metric

$$dt^2 + e^{2\sqrt{-\kappa}} \sum_{i=2}^{n-1} (dy_i)^2,$$

denoted in the unified manner by $X^n_{\text{mod}}(\kappa)$ for all $\kappa$ (including $\kappa = \pm \infty$), where $X^n_{\text{mod}}(+\infty)$ is the single point space and $X^n_{\text{mod}}(-\infty)$ is the above tree for all $n = 2, 3, \ldots, \infty$.

§4. COMPARISON RELATION BETWEEN THE MODEL SPACES

Let $x_1, \ldots, x_k, x_{k+1} = x_1$ be a cyclically ordered $k$-tuple of points in $X = X^n_{\text{mod}}(\kappa)$ for some $\kappa \leq 0$. Then for every $\kappa'$ in the interval $[0, \kappa]$ there exist points $x'_i \in X' = X^n_{\text{mod}}(\kappa')$, $i = 1, \ldots, k$, $k+1 = 1$, such that

$$|x'_i - x'_j|_X \geq |x_i - x_j|_X \text{ for all } i, j = 1, \ldots, k$$

and

$$|x'_i - x'_{i+1}|_X = |x_i - x_{i+1}|_X \text{ for } i = 1, \ldots, k.$$

This is standard and elementary, where one chooses $x'_i$ making a convex $k$-gon in the plane $H^n_\kappa$ (which equals $\mathbb{R}^3$ for $\kappa = 0$). Notice that the above extends to $\kappa \geq 0$ if the points $x_i$ are contained in a sufficiently small ball in $X$.

§5. POSITIVITY RELATIONS

Recall that a symmetric matrix $d_{ij}$, $i, j = 1, \ldots, k$, can be realized by the distances between $k$ points $x_i$ in $\mathbb{R}^n$ if and only if the quadratic
form $\Phi(t_i) = -\sum_{i,j=1}^k d_{ij}^2 t_i t_j$ is positive definite on the hyperplane $H_0 = \{ \sum_{i=1}^k t_i = 0 \} \subset \mathbb{R}^k$.

Observe that this imposes infinitely many linear inequalities on the numbers $d_{ij}^2$.

If we interpret $\Phi$ as the integral of $d_{ij} = \|x_i - x_j\|^2$ over $\mathbb{R}^n \times \mathbb{R}^n$ with the weights $t_i t_j$, then the positivity of $\Phi$ generalizes as follows.

Let $\mu$ and $\nu$ be probability measures on $\mathbb{R}^n$, then

$$\Phi(\mu, \nu) \overset{\text{def}}{=} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \, d\mu d\nu - \frac{1}{2} \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \, d\mu + \iint_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \, d\nu \right) \geq 0.$$ 

In fact, $\Phi(\mu, \nu)$ obviously equals the squared distance between the centers of mass of the measures,

$$\Phi(\mu, \nu) = \left( \int_{\mathbb{R}^n} x \, d\mu - \int_{\mathbb{R}^n} y \, d\nu \right)^2,$$

as a straightforward computation shows.

**Example.** Given $x_1, \ldots, x_\ell, y_1, \ldots, y_\ell$ in $\mathbb{R}^N$, then the average of the squared distances $\|x_i - y_j\|^2$ and $\|y_i - y_j\|^2$ is bounded by the average of $\|x_i - y_j\|^2$ as follows

$$\frac{1}{\ell(\ell - 1)} \left( \sum_{i<j} \|x_i - x_j\|^2 + \sum_{i<j} \|y_i - y_j\|^2 \right) \leq \frac{\lambda}{\ell^2} \sum_{i,j=1}^\ell \|x_i - y_j\|^2, \quad (\mathbb{R}^\ell)$$

for $\lambda = \frac{1}{\ell^2}$.

The form $\Phi(\mu, \nu)$ makes sense for an arbitrary metric space $X$ with $\|x - y\|^2$ replaced by $\|x - y\|^2_X$. Clearly, positivity of $\Phi$ for all probability measures on $X$ is necessary and sufficient for the existence of an isometric embedding of $X$ into a Hilbert space.

Similarly, one can characterize the spaces $X$ embeddable into spaces of radius $R = 1/\sqrt{n}$, $\kappa > 0$, by looking at the $R$-cone $Y_k = \text{Con}_R X \subset X$. 
where
\[ |x - y|_\kappa \overset{\text{def}}{=} \begin{cases} 2R \sin \frac{|x - y|_X}{2R} & \text{for } |x - y|_X \leq \pi R \\ 2R & \text{for } |x - y|_X \geq \pi R \end{cases} \]

for all \( x, y \in X \subset Y_\kappa \). Here the relevant form \( \Phi_\kappa \) equals
\[ \Phi_\kappa(t_1, \ldots, t_k) = \sum_{i,j=1,\ldots,k} \left( R^2 - \frac{1}{2} |x_i - x_j|_\kappa^2 \right) t_i t_j \]

for \( x_i \in X \subset Y_\kappa \) and it is positive if \( X \) is embeddable into the Hilbertian \( R \)-sphere.

If \( \kappa < 0 \), then
\[ |x - y|_\kappa \overset{\text{def}}{=} 2R \sinh \frac{|x - y|_X}{2R} \]

for \( R = 1/\sqrt{-\kappa} \) and
\[ \Phi_\kappa(t_1, \ldots, t_k) = \sum_{i,j=1}^k - \left( R^2 + \frac{1}{2} |x_i - x_j|_\kappa^2 \right) t_i t_j. \]

The space \( X \) embeds into a hyperbolic space of curvature \( \kappa < 0 \) iff the form \( \Phi_\kappa \) has at most one negative square (in the diagonalizing basis).

If \( \kappa = -\infty \) and \( k = 4 \), one considers three numbers
\[ m_1 = d_{1,2} + d_{3,4}, \quad m_2 = d_{2,3} + d_{1,4}, \quad \text{and} \quad m_3 = d_{3,1} + d_{2,4}, \]

and sets \( m_+ = \max_{i=1,2,3} m_i \) and \( m_- = \min_{i=1,2,3} m_i \). Then \( d_{i,j} \) are resizable by distances in \( X_{\text{mod}}(-\infty) \) iff
\[ 3m_+ - \sum_{i=1}^3 m_i = m_+ - m_- \]

i.e., iff the second maximal among \( m_i \) equals \( m_+ \). Furthermore, if every quadruple of points in a (finite) metric space \( \{ x_i \}_{i=1,\ldots,k} \) has this property, then \( \{ x_i \} \) isometrically embeds into the tree \( X_{\text{mod}}(-\infty) \).

All of the above is well known and pretty obvious. But there are amusing corollaries.

\section{Wirtinger inequalities}

Consider cyclically ordered points
\[ x_i \in \mathbb{R}^n, \quad i = 1, \ldots, k, \quad k + 1 = 1, \]
let \( W_j \{ x_i \} = \sum_{i=1}^{n} |x_i - x_{i+j}|^2 \), and set \( W_j(k) \) to be the value of \( W_j \) on the regular \( k \)-gon in \( \mathbb{R}^2 \) inscribed into the unit circle. Then

\[
W_1 \{ x_i \} / W_j \{ x_i \} \geq W_1(k) / W_j(k) \quad (1)
\]

for all \( \{ x_1, \ldots, x_k \} \subset \mathbb{R}^n \).

\textbf{Proof.} According to Fourier (on the group \( \mathbb{Z}/k = \mathbb{Z}/k \mathbb{Z} \)) one needs to check (1) only for \( \mathbb{Z}/k \)-equivariant maps \( \{ x_i \} = \mathbb{Z}/k \to \mathbb{R}^2 \), where this is obvious. Q.E.D.

\textbf{Remarks and corollaries.} (a) The Wirtinger inequality for four points is equivalent to the above (1) for \( \ell = 2 \). Furthermore, each (1) can be algebraically derived from (1) for some \( \ell = \ell(k) \), but a direct derivation is rather messy starting from \( k = 5 \). In fact, the negative definiteness of the distance matrices \( \{ d_{ij} \} \) in \( \mathbb{R}^n \) (see §5) harbors infinitely many linear inequalities non-reducible to anything like Wir1 and their linear combinations. One exhibits particular inequalities by looking at specific arrangements of points in \( \mathbb{R}^n \), e.g., coming from expanding graphs and related combinatorial structures.

(b) There are further relations between \( W_j \)’s. In fact, in order for an inequality

\[
\sum_j c_j W_j \{ x_i \} \geq 0
\]

with given \( c_i \in \mathbb{R}, i = 1, \ldots, k - 1 \), to hold true for all \( k \)-tuples in \( \mathbb{R}^n \), one only needs this for \( \mathbb{Z}/k \)-equivariant tuples in \( \mathbb{R}^2 \). For example, if \( k = 6 \), then \( c_1 W_1 + c_2 W_2 + c_3 W_3 \geq 0 \) provided this holds for the following three \( 3 \)-tuples of numbers: \((1, 3, 2), (1, 1, 0), \) and \((1, 0, 1)\).

(c) Since the spheres \( S^\kappa(R) = X^\kappa_{\mathbb{R}^n}(\mathbb{R}^n) \) embed into \( \mathbb{R}^{n+1} \) with the spherical distance \( d \) going to \( 2R \sin \frac{dij}{2R} \) in \( \mathbb{R}^{n+1} \), the Wirtinger inequality holds for \( \{ x_i \} \subset S^\kappa(R) \) with \( W_j \) made of

\[
a_{ij} = a_{ij}(\kappa) = \sin \frac{d_{ij}}{2R} \quad i, j = 1, \ldots, k,
\]

instead of \( d_{ij} \). The extremal configurations all lie on circles in \( S^\kappa(R) \) and they make regular \( k \)-gons if \( k \) is not divisible by 2 and 3.

(c') The hyperbolic space \( \mathbb{H}^n_\kappa \) is realized by the (halves of the) sphere of radius \( R = -1/\sqrt{\kappa} \) in the Lorentz space \( \mathbb{R}^{n+1}, \sum_i y_i^2 - y_0^2 \) where the
Lorentz “distance” equals \(2R\sinh \frac{d}{2R}\) for the hyperbolic distance \(d\). One cannot apply the Euclidean Wirtinger as our \(\mathbb{R}^{n+1}\) is not Euclidean, but one can project points \(x_i \in H^n \subset \mathbb{R}^{n+1}\) to the hyperplane \(H\) normal to \(\sum_i x_i = 0\) and apply the Wirtinger inequality to the projections \(\pi_i \in H\).

The distances \(d_{ij} = |\pi_i - \pi_j|\) are expressible in terms of the hyperbolic distance \(d\). This gives us Wirtinger-type inequalities between \(d_{ij}\) which are sharp for regular \(k\)-gons in \(H^n\) but not especially elegant.

(d) One can combine the quadratic (i.e., Euclidean) Wirtinger inequalities with the comparison inequality for \(\kappa = \kappa' \geq 0\) and conclude to the quadratic inequalities for \(H^n_\kappa\). Unfortunately these are not sharp, except for some \(k\)-tuples of points lying on geodesic lines in \(H^n_\kappa\).

(e) The Wirtinger inequalities extend to an arbitrary finite group \(G\) in place of \(\mathbb{Z}_k\). Here, for a real function \(c\) on \(G\) and a map \(f: G \to X\), we set

\[
W_c(f) = \sum_{h \in G} c(h) \sum_{g \in G} |f(g) - f(gh)|^2_X
\]

and we ask for which \(c\) and \(X\) every map \(f\) has \(W_c(f) \geq Q\) (where one may try more sophisticated non-quadratic expressions). Here again we easily see for \(X = \mathbb{R}^n\) that \(W_c(f) \geq 0\) for all \(f\) if it is \(\geq 0\) for all equivariant maps \(G \to \mathbb{R}^N\), i.e., orbits of the (irreducible, if you wish) orthogonal representations of \(G\). The same equally applies to all compact groups \(G\) and Borel functions \(c\) (better measures) and maps \(f\) with the sums replaced by the corresponding integrals. In particular, we may take \(G = S^1\), where all this can be derived from the case of \(\mathbb{Z}_k\) for the obvious approximation \(\mathbb{Z}_k \to \to S^1\), which allows \(X = H^n_\kappa\) in the picture. Also observe that averaging Wirtingers inequalities (e) over \(j\) and then sending \(k \to \infty\) gives us the traditional Wirtinger inequality: every smooth map \(f: S^1 \to X = H^n_\kappa\) satisfies

\[
\frac{1}{2} \int_0^1 |f(s_1) - f(s_2)|^2 ds_1 ds_2 \leq 2\pi \int |df|^2 ds.
\]

This for \(X = \mathbb{R}\) amounts to the evaluation of the first eigenvalue (of the Laplace operator) of \(S^1\),

\[
\lambda_1(S^1) = 1.
\]

(f) One can generalize further and take a measure space \(H\) with a measure preserving action of \(G\), where one studies weighted integrals of \(|f(h) - f(gh)|^2_X\) for maps \(f: H \to X\) and where Wirtinger inequalities
for $H$ can be derived from those for $G$. Here one can allow non-compact groups $G$, especially Kazhdan's $T$-groups. Also, one may look at particular $H$, such as the unit tangent bundle of Riemannian manifold $V$ with the action of the geodesic flow in this case this flow is periodic. Thus the Wirtinger inequality for all compact symmetric spaces of rank one mapped into general CAT(0) and more general spaces $X$ follows from the classical Wirtinger for $S^1 \to \mathbb{R}$. Another example of such inequality is (5), where $H = \{x_i, y_i\}$ and $G$ is generated by the permutations of $x_i$'s of $y_i$'s and the involution $x_i = y_i$.

§7. **Cycl_k(\kappa) and Wir_k-spaces**

A metric space is called Cycl_k(\kappa) if, for each cyclically ordered $k$-tuple of points $x_i \in X$, $i = 1, \ldots, k$, there exist comparison points $x_i' \in X'' = X''_{\text{mod}}(\kappa')$ for some $\kappa' \leq \kappa$, such that

$$\|x_i' - x_{i+1}'\|_X \leq \|x_i - x_{i+1}\|_{X'}, \quad i = 1, \ldots, k,$$

and

$$\|x_i' - x_j'\|_X \geq \|x_i - x_j\|_{X'} \quad \text{for all } j \neq i + 1.$$  

(This definition is well suited for $\kappa \leq 0$, while the case $\kappa > 0$ needs a modification where the existence of comparison points is required only for “small” $k$-tuples $\{x_i\} \subset X$.)

The most important case is that of $k = 4$, where the existence of a comparison quadruple $\{x_i'\}$ implies (at least for $\kappa' \leq 0$) the existence of $\{x_i''\} \subset X'' = X''_{\text{mod}}(\kappa'')$ with $\kappa'' \leq \kappa' \leq \kappa$, such that

$$\|x_i'' - x_j''\|_{X''} = \|x_i - x_j\|_{X'} \quad \text{for all } i, j = 1, \ldots, 4,$$

as an elementary argument shows.

**Remark.** The Cycl_4(0)-property can be expressed by a family of linear inequalities between $\|x_i - x_j\|_{X'}$. Namely, $X$ is Cycl_4(0) iff the squared distance function $\|x - y\|^2$ satisfies (6) for every pair of two-point probability measures $\mu$ and $\nu$ on $X$. (A measure is called two-point if its support contains at most two points.) This can be checked by a direct computation and will be proven later on without computation.

Next we introduce Wir_k-spaces where, by definition, every $k$-tuple $\{x_i\} \subset X$, $i = 1, \ldots, k$, satisfies the Wirtinger inequalities (5); $j = 1, \ldots, k - 1$, i.e.,

$$\sum_i \|x_i - x_{i+1}\|^2_{X} / \sum_i \|x_i - x_{i+j}\|^2_{X} \geq W_i^g / W_j^g.$$
where $W_k$ denote the corresponding sums for the regular $k$-gons in $\mathbb{R}^2$ (see 1.2.C). According to our discussion in 6

$$\text{Cycl}_k(0) \Rightarrow \text{Wir}_k \quad \text{for all } k = 4, 5, \ldots$$

§8. Geometric application

Let $V$ be a compact symmetric space of rank 1 (e.g., the $n$-sphere) and let $T_\varepsilon \subset V \times V$ denote the subset of points $(v_1, v_2)$ with $|v_1 - v_2|_V = \varepsilon$. Since $V$ is two point homogeneous, the isometry group of $V$ is transitive on $T_\varepsilon$ and we give the normalized, with the total mass one, invariant (Haar) measure $\mu_\varepsilon$ to $T_\varepsilon$. Let the diameter $D$ of $V$ satisfy

$$D = \frac{1}{2}k\varepsilon$$

for some integer $k = 2, 3, \ldots$, and let $\rho \leq \text{diam} V$ be of the form $\rho = j\varepsilon$ for some $j = 1, 2, \ldots$.

If $X$ is Wir$_k$, then every (say bounded) Borel map $f: V \to X$ satisfies

$$E_\varepsilon \overset{\text{def}}{=} \iint_{T_\varepsilon} |f(v_1) - f(v_2)|_X^2 \, d\mu_\varepsilon \geq \lambda_{\varepsilon k} \int_{T_\rho} |f(v_1) - f(v_2)|_\chi^2 \, d\mu_\rho,$$

where $\lambda_{\varepsilon k} = W_k(\varepsilon)/W_k(k)$.

This is seen by integrating the Wir$_k$-inequality over the orbits of $Z_k \subset S^1$ in the unit tangent bundle $S(V)$ for $S^1 = \mathbb{R}/\mathbb{Z}$ acting on $S(V)$ by the geodesic flow.

Observe that $E_\varepsilon/\varepsilon^2$ converges, for $\varepsilon \to 0$, to the average squared partial derivation of $f$ for smooth $f$,

$$E_\varepsilon/\varepsilon \to E(f) \overset{\text{def}}{=} \int_S \|\Delta f\|^2 \, ds,$$

where the measures in the spherical fibers of the unit tangent bundle $S = S(V)$ are normalized to mass one as well as the Riemannian measure on $V$. On the other hand, a suitably weighted sum of $E_{\varepsilon_j}$, i.e., $\sum_{j=1}^k p_j E_{\varepsilon_j}$ where $kp_j$ equals the reciprocal of the Jacobian of the exponential map $\mathbb{R}^{\dim V} = T_0(V) \to V$, on the sphere of radius $j\varepsilon$ in $\mathbb{R}^{\dim V}$, converges to the mean (average) $A(f)$ of the squared oscillation $|f(v_1) - f(v_2)|_X^2$ over $V \times V$. Thus, if $X$ is Wir$_n$, i.e., Wir$_k$ for all $k$, then $E(f)$ bounds $A(f)$ by

$$2E(f) \geq \lambda^{-1}A(f),$$

where $\lambda = \lambda(V) = \frac{1}{k}A(f_0)/E(f_0)$ for the (isometric!) Veronese embedding of $V$ to $\mathbb{R}^N$, $N = N(V)$, (where $N(S^n) = n + 1$, $N(P^n) =$
\[ \lambda_1(V) = (\dim V) \lambda(V). \]

The \( \dim V \)-factor is due to the fact that the average of the square of a linear function \( \partial : \mathbb{R}^n \to \mathbb{R} \) over \( S^{n-1} \subset \mathbb{R}^n \) equals \( n^{-1} \| \partial \|^2 \) and, consequently,

\[ \int_V \| \text{grad} f \|^2 \, dv = E(f) \text{Vol} V / \dim V \]

for all Riemannian manifolds \( V \) and functions \( f : V \to \mathbb{R} \). This yields (\( + \)) since

\[ \iint_{V \times V} \| f(v_1) - f(v_2) \|^2 \, dv_1 dv_2 = 2 \text{Vol} V \int_V \| f(v) \|^2 \, dv \]

for all \( \mathbb{R}^n \)-valued \( f \) with zero mean, \( \int_V f(v) \, dv = 0 \). (This explains why we brought up this "2" earlier.)

\[ \S 9. \text{Remarks} \]

(a) Observe that

\[ \lambda(V) \geq (\text{Diam} (\text{Veronese}(V)))^{-2} \geq \frac{\pi^2}{4} (\text{Diam} V)^{-2}. \]

This makes \( \lambda_1(V) \approx \dim V \) and implies "high concentration" of functions \( f \) on \( V \) for large \( \dim V \). Such concentration persists for maps \( f : V \to X \) where \( X \) is a Riemannian \( \text{Wir}_\infty \)-space of relatively small dimension \( \dim X = \delta \dim V \). Namely,

\[ \text{Vol}(V) \int_V \| df \|^2 \, dv \geq \delta^{-1} \lambda(V) \iint_{V \times V} \| f(v_1) - f(v_2) \|^2 \, dv_1 dv_2 \quad (\ast) \]

(compare Ch. 3 \( \frac{1}{3} \) in [11]).

(b) The role of the \( \text{Wir}_\infty \)-property is minor in the above discussion; it is needed only for identifying the \textit{explicit} value of \( \lambda(V) \). For example, the inequality \( (\ast) \) holds true for all (non-Wirtinger) Riemannian manifolds \( X \) with the constant \( \lambda(V) \) replaced by a slightly smaller number

\[ \lambda_-(V) = (\text{Vol} V)^2 \iint_{V \times V} \| v_1 - v_2 \|^2 \, dv_1 dv_2, \]
where the proof is identical to (actually easier than) that of \((\ast)\) and where it applies to all \textit{piecewise} smooth Riemannian \(X\). In particular, every \(1\)-\textit{Lipschitz} (i.e., distance decreasing) map \(f: V \to X\) satisfies

\[
\int_{V \times V} |f(v_1) - f(v_2)|^2_X dV_1 dV_2 \leq \delta \int_{V \times V} |v_1 - v_2|^2_X dV_1 dV_2 \quad (\ast \ast)
\]

for \(\delta = \dim X / \dim V\), that signifies "high concentration" of \(f\) for small \(\delta\).

One can generalize further by allowing (arbitrarily singular) non-Riemannian metric spaces \(X\), where \((\ast \ast)\) can be sharpened for Banach spaces \(X\) that are far from being Euclidean. One can identify extremal families \(W \subset V\), \(\sigma \in \Sigma\), of subvarieties, such as flat tori in symmetric \(V\) and minimal geodesic segments in \(V\) with \(\text{Ricci } V \geq \rho\) (where one may allow singular \(V\) and sometimes use the Brownian orbits for \(W\)). Eventually many results on concentration of functions, e.g., various Sobolev and isoperimetric inequalities, extend to maps into rather general spaces \(X\). For example, Levi's concentration generalizes to the following

**Theorem.** Let \(f: S^n \to \mathbb{R}^{n-m}, m \geq 0\), be an arbitrary continuous map. Then there exists a point \(x \in \mathbb{R}^{n-m}\) such that the pull-back \(S_x = f^{-1}(x) \subset S^n\) is larger than an equatorial sphere \(S^m \subset S^n\) in the following sense: the volumes of the \(\varepsilon\)-neighborhoods of the two subsets satisfy

\[
\text{Vol}_{U_\varepsilon}(S_x) \geq \text{Vol}_{U_\varepsilon}(S^m) \quad (\ast)
\]

for all \(\varepsilon \geq 0\).

(More generally, one considers pairs of maps, \(f: \Sigma \to X\) and \(\varphi: \Sigma \to V\), and seeks \(x \in X\) such that the image \(S_x = \varphi(f^{-1}(x)) \subset V\) has a large \(\varepsilon\)-neighborhood. For example, if \(\Sigma\) is a closed manifold of dimension \(n\), \(X\) is a manifold of dimension \(n - m\), the map \(f\) is contractible, \(\varphi\) has non-zero degree mod2, then, in the case \(V = S^m\), there is an \(x \in X\) such that \(\text{Vol}_{U_\varepsilon}(S_x) \geq \text{Vol}_{U_\varepsilon}(S^m)\) for all \(\varepsilon \geq 0\).) We shall prove this in [9] by constructing a suitable convex partition of \(S^n\) "transversal" to the fibers of \(f: S^n \to \mathbb{R}^{n-m}\) (compare [12] and [11]).

(c) The role of negative curvature of \(X\) in the concentration of maps \(f: V \to X\) becomes more pronounced if we look at maps \(f\) with large Lipschitz constants (or, alternatively, scale \(X\) with small \(\varepsilon > 0\)). For example, if \(K(X) \leq -\kappa < 0\) (or, hyperbolic, in general), then it is approximately \textit{one-dimensional} at infinity with a logarithmic error; thus
maps $f$ to $X$ concentrate as much as maps to trees, up to a logarithmic error. Similarly, maps to non-compact symmetric spaces of rank $k$ (and to buildings) concentrate, up to certain error, as do maps to $k$-dimensional spaces.

§10

It is well known that $\lambda_1(V \times V') = \min(\lambda_1(V), \lambda_1(V'))$ for products of Riemannian manifolds. This extends to maps to arbitrary Wir4-spaces $X$ assuming there is almost everywhere defined $\int_{V \times V'} |\text{grad} f|^2 \, dv$ for our maps $f$ and the squared gradient of an $f$ on $V \times V'$ equals the sum of the two squared fiberwise gradients along the $V$- and $V'$-fibers (as obviously holds true, for instance, for smooth $V$'s, $X$'s, and $f$'s). On the other hand, if $X$ is Wir4, then every map

$$f: W = V \times V' \to X$$

satisfies

$$\|f(v_1, v'_1) - f(v_2, v'_2)\|^2 + \|f(v_1, v'_2) - f(v_2, v'_1)\|^2 \leq$$

$$\|f(v_1, v'_1) - f(v_1, v'_2)\|^2 + \|f(v_1, v'_2) - f(v_1, v'_1)\|^2 +$$

$$\|f(v_2, v'_1) - f(v_2, v'_1)\|^2 + \|f(v_2, v'_2) - f(v_1, v_2)\|^2$$

(see Figure below),

which integrates to the bound of $\iint_{W \times W} \|f(v_1, v'_1) - f(v_2, v'_2)\|^2$ by the corresponding fiberwise integrals, since our measure on the product...
$V \times V' \times V \times V'$ is symmetric under the permutations of the components.

Hence the bounds

$$\int ||\text{grad} f||^2 \geq \lambda_1 \frac{1}{2} \int \int |f(v_1) - f(v_2)|^2_X$$

for all maps $f$ of both $V$ and $V'$ to $X$ imply such a bound with the same $\lambda_1$ for the maps $V \times V' \to X$.

**Remarks.** (a) The above argument is similar to one used by S. Bobkov in his thesis for functions on metric probability spaces.

(b) One may use two different metrics on $X$, one for evaluation of $||\text{grad}||^2$ and the other for $|f(v_1) - f(v_2)|^2$.

(c) The measures used in the double integrals on each $V$ and $V'$ do not have to be product measures. Furthermore, one can use $L_p$ norms for $p \neq 2$, which may be useful for products with $L_p$-product metrics. Notice that the $\ell_1$-metric is implicit in the inequality Wir_4, where 4-tuples of points in $X$ may be thought of as maps of the Hamming square $\{0, 1\}^2 \to X$. Then one looks at the Hamming cube $\{0, 1\}^n$ (where the Hamming metric is induced from the $\ell_1$-metric on $\mathbb{R}^n \supset \{0, 1\}^n$) put to $X$ by some map $\{0, 1\}^n \to X$. If $X$ is Wir_4, then the standard (and obvious) computation (similar to our evaluation of $\lambda_1(V_1 \times V_2)$) shows that the mean of the squared great diagonals of $\{0, 1\}^n \to X$ is bounded by $n$ times squared edge length. (One should keep in mind that the sharpness of the $L_2$-estimate depends on the global Wir_4 for $X$ confronted with the inverse Wir_4 for “infinitesimal parallelograms” in $X$.)

\[\text{§11}\]

Combining Remark 9 (a) and §10, we obtain concentration for maps of products $V$ of rank 1 symmetric spaces into (mildly non-singular) Wir_4-spaces $X$, including polyhedral CAT(0)-spaces, for instance. Namely, such maps for these $V$ concentrate as much as maps to $\mathbb{R}^k$ with $k \equiv \text{dim} \ X$.

\[\text{§12}\]

Many standard tricks of the concentration theory for real-valued maps extend to general Wirtinger (and especially CAT(0))-spaces as targets. For example, concentration for a $V$ implies that for suitable (e.g., Riemannian) quotients of $V$ with low dimensional fibers. Also, mildly distorted
and sufficiently spread subvarieties $W \subset V$ of low codimension concentrate (when mapped to $X$) almost as strongly as $V$ itself (where one is additionally aided by the Lipschitz extension theorem of [13] for CAT-spaces). Thus one sees concentration of maps of Grassmann manifolds to Wir$_\infty$-spaces.

§13. Concentration in smooth $X$

If $X$ is a complete simply connected manifold with non-positive curvature (or a more general smooth CAT(0)-space), then the $L_2$-concentration of maps $f : V \to X$ is almost as good as that for maps $V \to \mathbb{R}^N$ of all manifolds $V$, due to the following simple

**Observation.** For every $f : V \to X$ there exists a map $f_0 : V \to \mathbb{R}^N$ for $N = \dim X$ (where this dimension is allowed to be $+\infty$), such that

(i) $\|df_0(v)\| \leq \|df(v)\|$ for all $v \in V$,

(ii) $\int_{V \times V} \|f_0(v_1) - f_0(v_2)\|^2 dv_1 dv_2 \geq \frac{1}{2} \int_{V \times V} \|f(v_1) - f(v_2)\|^2_X dv_1 dv_2$.

**Proof.** Take the Riemannian center of mass $x_0 \in X$ of the $f$-push-forward measure from $V$ to $X$ and observe that the map $f_0 = \exp_{x_0}^{-1} f$ has $\int_V f_0(v) dv = 0$.

The inequality (i) now follows from the contracting property of the inverse exponential map $\exp^{-1} : X \to T_{x_0}(X) = \mathbb{R}^N$ (since $K \leq 0$), while (ii) depends upon the non-decreasing (in fact isometric) feature of $\exp^{-1}$ on the rays issuing from $x_0$ and on the triangle inequality (where the latter is responsible for the unfortunate coefficient $1/2$).

**Remarks.** (a) The above remains valid for graphs in the place of manifolds $V$. For example, if $V$ is the complete bipartite graph on vertices $x_1, \ldots, x_\ell$ and $y_1, \ldots, y_\ell$ in $X$, then the above argument combined with ($\mathbb{E}_\ell$) in 5 shows that the averaged squared distances $|x_i - y_j|^2$ and $|y_i - y_j|^2$ are bounded by the averaged (over the edges) $|x_i - y_j|^2$ as follows

$$\frac{1}{\ell(\ell - 1)} \left( \sum_{i < j} |x_i - x_j|^2 + \sum_{i < j} |y_i - y_j|^2 \right) \leq \frac{\lambda}{\ell^2} \sum_{i,j=1}^{\ell} \|x_i - y_j\|^2$$

for $\lambda = 2\ell^{-1}$. 

(a') This inequality is sharp as is seen in the Cartesian product of two hyperbolic spaces, \( X = H^\kappa_{\ell} \times H^\kappa_{\ell} \) with \( \kappa < 0 \), where the ratio of the left hand side and the right hand side of \((2\eta^2)\) converges to 1 for \( \{x_i\} \) and \( \{y_i\} \) converging to the vertices of regular ideal \( \ell \)-simplices on the ideal boundaries of \( H^\kappa_{\ell} \times \partial H^\kappa_{\ell} \) and \( x_0 \times H^\kappa_{\ell} \) respectively. (Bipartite graphs can be approximated by surfaces, which shows that the extra factor 2 is unavoidable in the Riemannian category as well.)

(b) There is no Euclidean reduction of the concentration property for singular CAT(0)-spaces (defined later on). Counter-examples are provided by cones over expanders. These play an essential role in the study of random groups (see [8]). However, most elementary (local) bounds on \( \lambda_1 \) (e.g., for Ricci \( \leq -\kappa \)) are likely to extend to maps into singular CAT(0)-spaces (possibly without the 2 factor).

(c) The essential property of \( X \) in the above observation is not so much \( K \leq 0 \), but rather the existence of "many sufficiently contracting" and "sufficiently proper" maps to \( \mathbb{R}^N \), that is known as parametric hyper-Euclidean property involved in most proofs of the strong Novikov conjecture. (This property is violated by singular \( X \) with cones over arbitrary large expanders.)

§14. Diffusion, codiffusion, and harmonic maps

A prediffusion on \( V \) is a map from \( V \) to the space of \( \mathbb{R}_+ \)-paths of probability measures on \( V \), denoted
\[
V \mapsto \mu_\varepsilon(v, v')dv', \varepsilon \in (0, \infty),
\]
such that \( \mu_\varepsilon(v, v')dv' \) converges to the \( \delta \)-measure \( \delta(v)dv \) for all \( v \in V \) and \( \varepsilon \to 0 \). A prediffusion is called diffusion if the family \( \{\mu_\varepsilon\} \) makes a semigroup under the composition (convolution) of measures:
\[
\mu_{\varepsilon_1} \ast \mu_{\varepsilon_2} = \mu_{\varepsilon_1 + \varepsilon_2}
\]
(compare Chap. 7 in [3]).

A codiffusion on \( X \) is a retraction \( e \) of the space \( P(X) \) of probability measures on \( X \) back to \( X \subset P(X) \), where \( X \) is embedded to \( P(X) \) by \( x \mapsto \delta(x)dx \). (To be consistent, one should deal with homology retractions, e.g., given by contractive semigroups of maps \( P(X) \) but these do not enter the present framework.)

If \( V \) and \( X \) are endowed with prediffusion and codiffusion respectively, one defines \( \varepsilon \)-harmonic maps \( f: V \to X \) as those where \( f_*(\mu_\varepsilon(v)) \in P(X) \)
retracts to $f(v) \in X$ for all $v \in V$. Traditionally, one defines harmonic maps by the equality $c(f_\varepsilon(\mu_\varepsilon(v))) = f(v)$ for an infinitesimal $\varepsilon$, i.e., in the limit for $\varepsilon \to 0$. Another attractive possibility is passing to the limit $\varepsilon \to 1$ and augmenting the spaces $\mathcal{P}(V)$ and $\mathcal{P}(X)$ by the measures on suitable ideal boundaries of the spaces $V$ and $X$. In any case, the harmonicity amounts to $f : V \to X$ being a fixed point of the diffusion-codiffusion flow map

$$f(v) \mapsto c(f_\varepsilon(\mu_\varepsilon(v)))$$

at some $\varepsilon \in [0, \infty]$. (This brings harmonic maps on equal footing with classifying maps into spaces supporting expanding maps.)

If $X$ has negative curvature, one defines codiffusion as the center of mass: first, a measure $\nu \in \mathcal{P}(X)$ is mapped to the function on $X$ that equals the $\nu$-average of the squared distance functions on $X$,

$$\nu \mapsto d_\nu(x') \overset{\text{def}}{=} \int_X |x - x'|^2 \, d\nu(x).$$

For $K(X) \leq 0$, this function $d_\nu$ is strictly convex on $X$ and, hence, has a unique minimum point $x_{\min} = x(\nu) \in X$: this is taken for $c(\nu) \in X$. The essential feature of this $c$ is the contraction property for the $L_2$-transportation metric (see [11]). This contraction property, when confronted with the smoothing properties of the diffusion in $V$ (characteristic to curvature \( \geq -\kappa > -\infty \), compare [1]), allows “good” (e.g., Lipschitz regular) harmonic maps $V \to X$ (where in interesting cases these maps commute with a given symmetry group operating on $V$ and on $X$).

§15. Geodesic triangles and CAT(\( \kappa \))-spaces

A metric space $X$ is called CAT(\( \kappa \)) if it is geodesic and Cyd_4(\( \kappa \)). The geodesic property, essentially equivalent to the existence of a middle point $x$ between arbitrary $x_1$ and $x_2$, i.e., satisfying

$$|x_1 - x| + |x - x_2| = |x_1 - x_2|,$$

enhances the power of distance inequalities. For example, if $X$ is geodesic, then the general Cyd_4(\( \kappa \))-inequality follows (by an easy argument) from that for the special quadruples $\{x_1\} \subseteq X$ where $x_3$ lies between $x_2$ and $x_4$, i.e., on a (shortest geodesic) segment $[x_2, x_4]$, which amounts to the equality

$$|x_2 - x_3| + |x_3 - x_4| = |x_2 - x_4|.$$
If one thinks of \([x_1 - x_i], \ i = 2, 3, 4, \) as the values of the distance function \(X \mapsto d(x) = \text{dist}(x_1, x),\) then one can interpret this inequality as a *convexity* property of \(d(x)\) saying that it is “more convex” than the corresponding distance function on the model space of curvature \(\kappa.\)

Another, apparently stronger but, in fact, equivalent characterization of \(\text{CAT}(\kappa)\)-spaces \(X\) expresses the idea of *geodesic triangles* in \(X\) being narrower than the *comparison* triangles. Here a geodesic triangle \(\Delta(x_1, x_2, x_3)\) in a geodesic space \(X\) is defined as the union of the three edges \([x_i, x_j] \subset X, \ 1 \leq i < j \leq 3,\) where one allows every edge between two points if there are several of them. A *comparison triangle* \(\Delta'\) in a space \(X'\) (which will be taken of constant curvature later on) is, by definition, a geodesic triangle \(\Delta' = \Delta(x'_1) \subset X', \ x'_i \in X', \ i = 1, \ldots, 3\) such that
\[
|x_i - x_j|_X = |x'_i - x'_j|_{X'},
\]
This \(\Delta' \subset X'\) does not necessarily exist. If it does, it comes along with a canonical map \(c: \Delta' \rightarrow \Delta,\) where \(x'_i \mapsto x_i\) and each segment \([x'_i, x'_j]\) isometrically goes to \([x_i, x_j].\)

A space \(X\) is \(\text{CAT}(\kappa)\) iff each \(D\) in \(X\) admits a comparison triangle \(\Delta'\) in a model space \(X'\) with constant curvature \(\kappa' \leq \kappa\) such that the *comparison map* \(c: \Delta' \rightarrow \Delta\) is (non-strictly) distance decreasing with respect to the (non-path) metrics \(\text{dist}_{X'}|_{\Delta'}\) and \(\text{dist}_X|_{\Delta},\)
\[
|c(x') - c(y')|_X \leq |x' - y'|_{X'},
\]
for all \(x', y' \in \Delta'.\)

This is a fundamental, albeit easy to prove, result by Alexandrov.

If \(\kappa = 0,\) then a comparison triangle always exists in \(X'_0\) and is unique up to isometry; the same is true for \(\kappa > 0\) if
\[
|x_i - x_j| < \frac{\pi}{\sqrt{\kappa}} \quad \text{and} \quad \sum_{1 \leq i < j \leq 3} |x_i - x_j| \leq \frac{2\pi}{\sqrt{\kappa}}
\]
In general, the existence of \(\Delta'\) can be dropped from the definition as we allow \(\kappa' \leq \kappa.\)

**Basic examples.** (a) The standard space \(X'_n\) is \(\text{CAT}(\kappa')\) for all \(\kappa' \geq \kappa.\)

This is the essential feature of these model spaces allowing a meaningful definition of general \(\text{CAT}\)-spaces.

(a') A complete simply connected Riemannian manifold with constant (as well as variable) sectional curvature \(\leq \kappa\) is \(\text{CAT}(\kappa):\) the \(n\)-spheres of
radii $\geq \kappa^{-1/2}$ are CAT($\kappa$); $\mathbb{R}^n$ is CAT(0); the hyperbolic spaces
\[ H^n_\kappa = \left( \mathbb{R} \times \mathbb{R}^{n-1}, \, dt^2 + e^{-2\sqrt{\kappa}} \sum_{i=1}^{n-1} dy_i^2 \right), \quad \kappa \geq 0 \]
are CAT($-\kappa$) for all $n = 2, 3, \ldots, \infty$.

(b) Every (simplicial or non-simplicial) tree is CAT($-\infty$), i.e., CAT($\kappa$) for all $\kappa \in \mathbb{R}$.

(c) Every smooth domain $X$ in $\mathbb{R}^2$ is CAT(0) for the induced path metric and such domains in $H^2_\kappa$ are CAT($-\kappa$). (This is an easy but useful property which does not directly extend to higher dimensions.)

Remarks. (a) Cycl$_4$ as well as Cycl$_k$ for all $k$ are instances of concentration (of isoperimetric kind) inequalities which can be defined with an arbitrary (measuring rod) graph with the vertex set $V$ and edges $E \subseteq V \times V$ by requiring a certain bound on distances $|f(v_1) - f(v_2)|_X$ for maps $f: V \rightarrow X$ in terms of distances $|f(v_1) - f(v_2)|_X$ for $(v_1, v_2) \in E$. It is convenient to allow infinitesimal graphs $V$ where $E$ consists of pairs of infinitesimally close points. We have met such a bound for maps of Riemannian manifolds $V$ into $X$, where $E$ was represented by unit tangent vectors in $V$ and the relevant bound(s) were of the form
\[
\int_{V \times V} |f(v_1) - f(v_2)|_X^a \, dv_1 \, dv_2 \leq \left( \int_E \|df\|^b \, de \right)^{1/a}.
\]
We have also seen that smooth $X$ with $K \leq 0$ satisfy additional inequalities of this type but one does not know what is the full set of such inequalities characterizing a given class (e.g., of smooth $X$) of spaces with $K \leq 0$.

(b) The geodesic property is one logical level up from concentration inequalities as it involves the existential quantifier. It is unclear if there is a simple 3-free description of (non-geodesic!) subspaces in CAT($\kappa$)-spaces. (We shall see later on that Cycl$_4$ $\Rightarrow$ Cycl$_k$ for all $k \geq 5$ in the geodesic case but this is apparently not so in general.)

(c) Consider two probability measures $\mu$ and $\nu$ in $X$, let $c(\mu)$, $c(\nu) \in X$ be their centers of mass, and let
\[
\Phi_X(\mu, \nu) = \int_{X \times X} |x - y|^2 \, d\mu \, d\nu
\]
\[-\frac{1}{2} \left( \int \int_{X \times X} |x - y|^2 \, d\mu \, d\nu + \int \int_{X \times X} |x|^2 \, d\nu \, d\nu \right). \]

If \( X = \mathbb{R}^n \), then
\[ \Phi_X(\mu, \nu) = |c(\mu) - c(\nu)|^2, \]
(see §5), and if \( X \) is CAT(0), then
\[ |c(\mu) - c(\nu)|^2 \leq \Phi_X(\mu, \nu) \]
(\( \mathfrak{E} (0) \))

for two-point measures \( \mu \) and \( \nu \). This follows from the fact that the squared distance functions \( d \) on \( X \), and hence the convex combinations of \( d^2 \)'s, are "more convex" on geodesic lines than the function \( x^2 \) on \( \mathbb{R} \).

This means the second derivatives of \( d^2 \)'s are \( \geq 2 \), or equivalently, the difference of two squared distance functions, \( d_y^2 - d_x^2 \), is convex on each geodesic in \( X \) passing through \( x \) (and concave on geodesics through \( y \)).

(c) The inequality \( \mathfrak{E} (0) \) fails for general (non-two-point) measures in CAT(0) spaces but does hold true if the supports of \( \mu \) and \( \nu \) are contained in (possibly different) flat convex subspaces in \( X \).

(c') The general CAT(\( \kappa \)) (i.e., Cycl\( \kappa \)) property can be brought to the \( \mathfrak{E} (\kappa) \)-form. Yet, this does not (?) appear sufficiently illuminating.

§16. GEODESIC CONVEXITY AND CONVEX GLUING OF SPACES

A subset \( Y \subset X \) is called (geodesically) convex if it contains every segment with the ends in \( Y \).

**Examples.** (a) Every geodesic segment in the CAT(0)-space is convex, and every segment strictly shorter than \( \pi/\kappa \) is convex in a CAT(\( \kappa > 0 \))-space.

(b) Every subtree in a tree is convex.

(c) Every ball \( B \) in a CAT(0)-space is convex, where
\[ B = B_{\kappa \rho} (R) \overset{\text{def}}{=} \{ x \in X \mid \| x - x_0 \| \leq R \}. \]

Let \( Y_i \subset X_i, i = 1, 2 \), be non-empty convex spaces, \( \varphi: Y_1 \to Y_2 \) a bijective isometry, and denote by \( X_1 \cup_{\varphi} X_2 \) the disjoint union of \( X_1 \) and \( X_2 \) where \( Y_1 \) is identified with \( Y_2 \) via \( \varphi \).
Gluing theorem. If $X_1$ and $X_2$ are CAT($\kappa$), then so is $X_1 \vee \varphi X_2$.

The proof is straightforward, modulo elementary geometry of the model spaces (see [2]).

Examples. (a) To apply the theorem one needs $X_i$ with mutually isometric convex $Y_i \subset X_i$. One can use, for instance, segments in $X_i$ of equal lengths (which are always convex for $\kappa \leq 0$). Or, if $X_1$ happens to be isometric to $X_2$, one can use the restriction of the implied isometry $X_1 \rightarrow X_2$ to some convex subset $Y_1 \subset X_1$, e.g., to a ball $B \subset X_1$ (which is always convex for $\kappa \leq 0$).

(b) Tree-like polyhedra. A connected simplicial polyhedron $P$ is called tree-like if

$$P = \bigcup_i P_i$$

for $i$ ranging over a well-ordered set $I$, such that $P_{i+1} = P_i \cup \Delta_i$, $i \in I$, where the simplex $\Delta_i$ meets $P_i$ over a single face $\Delta'_i \subset \Delta_i$. (Clearly, trees are tree-like.) If we give to such $P$ the metric where each simplex $\Delta \subset P$ is isometric to a regular simplex of a fixed size in a simply connected space of constant curvature $\kappa$ (i.e., spherical, Euclidean, or hyperbolic), then $P$ becomes a CAT($\kappa$)-space by the Gluing theorem.

(b') Nerves of subtrees. Let $Q$ be a tree, $Q_j$, $j = 1, \ldots, k$, a finite collection of subtrees, and $P$ be the nerve of this family $\{Q_j\}$.

If $P$ is connected, then it is tree-like.

Proof. Assume there is a point $q \in Q \setminus \bigcap_j Q_j$ and let $Q_j \star$ be the farthest subtree from $q$. Then, clearly, $P$ is obtained from the nerve $P^\bullet$ of

$$(Q_1, \ldots, Q_j \star, Q_{j+1}, \ldots, Q_k)$$

by attaching a simplex to $P^\bullet$ across a single face, and an obvious induction concludes the proof.

Q.E.D.

Remark. Another significant property of this $P$ (shared by all tree-like polyhedra and possibly characterizing them) is the following sharp combinatorial isoperimetric inequality: every cyclic path of $k$-edges bounds a (possibly degenerate) disk made of at most $k - 2$ triangles.

§17. Convexity and CAT$_{\kappa}$-convexity

Take a geodesic line $\ell'$ in the model space $X'$ of curvature $\kappa$ (this line is a topological circle for $\kappa > 0$) and consider the distance function to $\ell'$,

$$d_{\ell'}(x') = \inf_{y' \in \ell'} |x' - y'|_{X'}.$$
The restriction of this function to a segment \([a', b']\) in a connected component (half-plane) of the complement \(X\setminus \rho\) is uniquely determined by the values \(d_\rho(a'), d_\rho(b')\) and the length \(|a' - b'|\) of \([a', b']\). So we can regard \(d_\rho\) as a real function, called \(\kappa\)-function. Notice that the \(\kappa\)-functions are positive and 1-Lipschitz, i.e.,
\[
|d_\rho(t_1) - d_\rho(t_2)| \leq |t_1 - t_2|.
\]

Next, a positive 1-Lipschitz function \(d\) defined on some segment in \(\mathbb{R}\) is called \(\kappa\)-convex if, for every two points \(a\) and \(b\) in this segment, there is a \(\kappa'\)-function \(d_{\kappa'}\) on \([a, b]\) with \(\kappa' \leq \kappa\), such that
\[
d_{\kappa'}(a) = d(a), \ d_{\kappa'}(b) = d(b)
\]
and
\[
d_{\kappa'}(t) \geq d(t) \text{ for } t \in [a, b].
\]

In other words, \(d\) must be more convex than \(d_{\kappa'}\). For example, if \(\kappa = 0\), then the \(\kappa\)-convexity amounts to ordinary convexity for positive 1-Lipschitz functions.

One checks elementarily that the \(\kappa\)-convexity is a local property: if \(d\) is \(\kappa\)-convex in a small subinterval around each point, then it is \(\kappa\)-convex.

About \(\kappa = -\infty\). This convexity means the \(\kappa\)-convexity, for all \(\kappa \in \mathbb{R}\), which corresponds to the behavior of the distance to a geodesic line in a tree. Clearly, every \((-\infty, 0, \kappa)\)-convex function \(d(t)\) on \([a, b]\) equals
\[
\max (d(a) - |a - t|, 0, d(b) - |b - t|).
\]

A (positive, 1-Lipschitz) function on a geodesic space is called CAT\(_{\kappa}\)-convex if its restriction to every geodesic segment is \(\kappa\)-convex. Then a space \(X\) is called CAT\(_{\kappa}\)-convex if the distance function to each segment \(Y \subset X\), i.e., \(d(x) = \inf_{y \in Y} |x - y|\), is \(\kappa\)-convex on \(X\). If \(X\) is CAT\(_{\kappa}\)-convex, then, clearly,

(i) the distance function to every convex subset \(Y \subset X\) is \(\kappa\)-convex;
(ii) the \(R\)-balls in \(X\) are convex for all \(R\) if \(\kappa \leq 0\) and for \(R \leq \pi / \sqrt{\kappa}\) for \(\kappa > 0\);

(iii) the \(p\)-neighborhood \(Y + \rho\) of each convex subset \(Y \subset X\) is convex for \(\kappa \leq 0\), where, recall
\[
Y + \rho \overset{\text{def}}{=} \{x \in X \mid \text{dist}(x, Y) \leq \rho\}.
\]
Convexity theorem. Every CAT(κ)-space is CAT_κ-convex.

This is well-known and the proof is straightforward (see [2]). Notice that the converse is true (and rather obvious) for Riemannian manifolds but not for general X. For example, Banach spaces are 0-convex; yet these are not CAT(0) unless they are Hilbertian. But for κ = −∞ the distinctions between the classes of spaces disappear:

CAT(−∞) = (−∞)-convexity.

On the topology of CAT(κ). If κ ≤ 0, then every two points in a CAT(κ)-convex space X are joined by a unique geodesic segment and so CAT(0)-convex spaces are contractible. Moreover, the balls in these spaces are convex and contractible. (If κ > 0, then convexity of the R-balls is ensured only for \(R \leq \pi/(2\sqrt{\kappa})\) and contractibility for \(R < \pi/\sqrt{\kappa}\).)

Fig. 3.

Fig. 4.

§18. CAT-(κ) AND CURVATURE

If a geodesic triangle Δ in X is subdivided into (smaller) triangles Δ_i with all vertices on Δ then the CAT(κ)-comparison inequalities for Δ_i imply (by an easy and well-known argument) that for Δ itself. Then such subdivision can be applied to all Δ_i etc., thus reducing verification of CAT(κ)-property to arbitrary small triangles.
There is a catch in this however: such subdivisions are rather special as every vertex must lie inside a geodesic edge and there is no guarantee that the new triangles will be smaller than the original $\Delta$. Yet, everything works if $X$ contains no *almost minimal* closed curves (geodesics), which amounts to requiring that the extrinsic distance $\text{dist}_X \Delta$ is “significantly” smaller than the induced path metric on $\Delta$. This means, by definition, that every geodesic $\Delta$ contains a pair of points $x$ and $y$ such that

$$\text{dist}_X(x, y) \leq (1 - \varepsilon)\text{dist}_\Delta(x, y)$$

for the path metric $\text{dist}_\Delta$ on $\Delta$, and

$$\text{dist}_\Delta(x, y) \geq \varepsilon \text{diam } \Delta$$

for some $\varepsilon = \varepsilon(X) > 0$ independent of $\Delta$.

These considerations suggest the following

**Definition.** We say that $X$ has curvature $K \leq \kappa$ at $x \in X$ if there is a neighborhood $U \subseteq X$ of $x$ such that every $\Delta$ contained in $U$ is more narrow than the model triangle $\Delta'$, i.e., the comparison map $c: \Delta' \to \Delta$ is distance decreasing. (Equivalently, one could say that a small $\varepsilon$-ball around $x$ is $\text{CAT}(\kappa)$.) Next we define spaces $X$ with $K(X) \leq \kappa$, i.e., with curvatures $\leq \kappa$, by requiring this property at every point $x \in X$.

**19. Examples**

(a) Riemannian manifolds $X$ (of finite or infinite dimension) with sectional curvature $\leq \kappa$ have $K(X) \leq \kappa$ in our (i.e., Alexandrov’s) sense.

(b) Let $X$ be a polyhedron built of (convex) simplices of constant curvature $\kappa$ (i.e., simplices from a complete simply connected space with constant curvature $\kappa$). The link $L_x$ of every vertex $x \in X$ is again a space of this kind, built of spherical simplices, i.e., those with $\kappa = 1$. Then $K(X) \leq \kappa$ if and only if every such link is $\text{CAT}(1)$.

In particular, if $\dim X = 2$ and thus every $L_x$ is a 1-polyhedron, i.e., a graph with the length of the edges measured by the angles of the corresponding triangles. Here the $\text{CAT}(1)$ condition for $L_x$ says that every cycle in $L_x$ has length $\geq 2\pi$.

More generally, $\text{CAT}(1)$ needs, besides $K \leq 1$, the uniqueness property for geodesic segments between the pairs of points with distance $< \pi$ between them. For instance, if $X$ is $\text{CAT}(\kappa)$ for $\kappa \leq 1$ and $Y = X/\Gamma$ for an isometry group $\Gamma$ with $|x - \gamma(x)| \geq 2\pi$ for all $x \in X$ and $\text{id} \neq \gamma \in \Gamma$, then $Y$ is $\text{CAT}(1)$. 
Let $X$ be a 2-polyhedron and let $f: \tilde{X} \to X$ be a ramified covering, i.e., the pull-backs $f^{-1}(x) \subset X$ are discrete for all $x \in X$ and, furthermore, there are discrete subsets $X_0 \subset X$ and $\tilde{X}_0 \subset \tilde{X}$ such that $f$ maps the complement $\tilde{X} \setminus \tilde{X}_0$ to $X \setminus X_0$ with $\tilde{X} \setminus \tilde{X}_0 \to X \setminus X_0$ being a covering map.

Every path-metric in $X$ (obviously) induces such a metric in $\tilde{X}$, and if the former had $K \leq 0$, so, obviously, does the latter. Moreover, if the metric in $X$ is flat (Euclidean) on all 2-simplices in $X$, then “most” ramified coverings $X \to X$ have $K(\tilde{X}) \leq 0$, regardless of the curvature of $X$.

For example, let $X$ be built of plane equilateral triangles and $\tilde{X} \to X$ ramified at each vertex in $X$ with order $\geq 2$. This means $X_0$ contains all vertices in $X$ and for every pair of points $\tilde{x} \in \tilde{X}_0$ and $x \in X_0$ the induced covering map $f_{\tilde{x}}$ of the link $\tilde{L}_x \subset \tilde{X}_0$ to $L_x \subset X$ non-trivially covers each simple cycle $C \subset L_x$, i.e., there is no cycle $\tilde{C} \subset \tilde{L}_x$ injectively sent by $f_{\tilde{x}}$ to $C$. Then, clearly, each cycle in $\tilde{L}_x$ has at most 6 edges and thus $K(\tilde{X}) \leq 0$.

Notice that whenever $X_0$ contains all vertices in $X$, there are plenty of ramified covers $\tilde{X} \to X$ with the above property. In fact the fundamental group $\pi'$ of the complement $X' = X \setminus X_0$ is free and therefore it contains lots of subgroups $\tilde{\pi} \subset \pi'$ such that the classes of the simple cycles $C \subset L_x$, $x \in X_0$, are not contained in $\pi$. Then the completions of the $\tilde{\pi}$-coverings of $X'$ are our $X$ with $K(X) \leq 0$.

If we are concerned with finite polyhedra $X$ and $\tilde{X}$, we need subgroups $\tilde{\pi} \subset \pi'$ of finite index in order to have finitely sheeted ramified covers $\tilde{X} \to X$. Since free groups are residually finite, we do have plenty of such $\tilde{\pi} \subset \pi'$ and, consequently, we have as many finite 2-polyhedra $\tilde{X}$ with $K(\tilde{X}) \leq 0$.

To be specific, let $X$ be the 2-skeleton of the $(n-1)$-simplex. This $X$ is simply connected and is built of $\binom{n}{3}$ triangles. If we remove the set $X_0 \subset X$ of the vertices of $X$, the complement $X \setminus X_0$ contracts to the graph $X' \subset X \setminus X_0$ spanned by the barycenters of the triangles and edges in $X$. This $X'$ has $3 \binom{n}{3}$ edges and $\binom{n}{3} + \binom{n}{2}$ vertices, where $\binom{n}{2} = \frac{n(n-1)}{2}$ is the number of edges in $X$. Thus the fundamental group of $X \setminus X_0$ is free with $m = 2 \binom{n}{3} - \binom{n}{2} + 1$ generators. The most obvious subgroup $\tilde{\pi} \subset \pi' = F_m$, which makes $K(\tilde{X}) \leq 0$ is the kernel of the
canonical homomorphism
\[ F_m = (\mathbb{Z}/2\mathbb{Z})^m \]
(while every \( m > 2 \) makes \( K(\hat{X}) < 0 \)). The multiplicity of the corresponding ramified cover \( \hat{X} \to X \) equals \( 2^m - \ell \) at \( X_0 \), where
\[ \ell = \ell_n = \frac{(n - 1)(n - 2)}{2} - (n - 1) + 1 \]
is the rank of the fundamental group of the 1-skeleton of the \((n - 1)\)-simplex.

(There are smaller ramified covers of this \( X \) with \( K \leq 0 \), and one, probably, can enlist the minimal ones. Similarly, one can ramify other symmetric 2-polyhedra, such as the 2-skeletons of the cubes and octahedra.)

§21. On construction of polyhedra
\( X \) with \( K \leq 0 \) for \( \dim X \geq 3 \)

As dimension grows, there seem to appear fewer and fewer new spaces with \( K \leq 0 \) and getting them with \( K < 0 \) is especially difficult. (In fact, all known high-dimensional hyperbolic groups are built out of “arithmetic blocks” but we are far from stating and proving any definite result in this direction.)

For example, if we sufficiently ramify 3-polyhedra \( \hat{X} \) over 1-dimensional loci \( X_1 \subseteq X \), the resulting \( \hat{X} \) will have negative curvature everywhere except the vertices \( \hat{x} \in \hat{X} \), where we can ensure curvature \( \leq 1 \) of the (2-dimensional) links \( L_x \), but not the CAT(1)-property.

(The latter could be achieved if these links had sufficiently many coherent finite coverings, i.e., if the fundamental groups of \( \hat{X} \setminus \{\text{vertices}\} \) were residually finite. In fact, a suitable residual finiteness of \( n \)-dimensional groups with \( K \leq 0 \) (or \( K < 0 \)) would lead to many examples of \((n + 1)\)-dimensional groups with \( K \leq 0 \) (or \( K < 0 \)); this indicates, in my view, that typical groups with \( K \leq 0 \) (or \( K < 0 \)) have no non-trivial finite quotients.)

The spaces like \( \hat{X} \), where the curvature is negative away from the vertices, can be modified to have \( K \leq 0 \) (or \( K < 0 \)) everywhere in two ways.

(1) Remove the vertices and replace all simplices by the ideal hyperbolic simplices. Then the resulting space \( \hat{X}' \) becomes a complete space of finite volume and \( K \leq 0 \) (or \( K < 0 \)).
(2) Suitably truncate the hyperbolic simplices and double the resulting space. This results in a compact $\tilde{X}$ with $K \leq 0$ (or $K < 0$), homeomorphic to the double of

$$\tilde{X} \setminus \{\text{small balls around the vertices}\},$$

where the “double” gluing takes place over the boundaries of these balls. In particular, one obtains in this way some (not especially exciting) 3-polyhedra with $K \leq 0$.

Finally, if we depart from a 3-dimensional pseudomanifold $X$, then we always can arrange ramified coverings with $K < 0$. In fact, such an $X$ can be obtained from a compact 3-manifold $X_\bullet$ by attaching cones to the boundary components of $X_\bullet$ (followed by some irrelevant identifications). If $X_\bullet$ happens to have constant negative curvature with mildly curved boundary, then, after passing to finite covering $\tilde{X}_\bullet$ and coning the boundary of $\tilde{X}_\bullet$, we get a compact pseudomanifold with $K < 0$.

Of course, not every $X$ gives us such an $X_\bullet$ but the desired property is satisfied by a suitable preliminary ramified cover of $X$ as can be easily derived from Thurston’s theory. So, with Thurston, we have a huge pool of compact 3-dimensional pseudomanifolds with $K < 0$.

§22. Assembling $(K \leq 0)$-spaces over geodesic graphs

It is hard to construct high dimensional spaces $X$ with $K \leq 0$ (and, especially, with $K < 0$) from scratch, but given such an $X$ one can construct many others as follows.

Let $X_1 \subset X$ be a geodesic subgraph in $X$, i.e., a union of geodesic segments $e_i$, $i \in I$, where every two segments meet, if at all, at one of their end points. Take several copies of $X$ (where, more generally, one may take various numbers of different connected components of $X$, in case $X$ was disconnected) and then glue together some among edges of equal length in the corresponding union of the copies of $X_1$, where we do not exclude gluing edges in the same connected component in (the union of copies of) $X$. (There are exactly two ways to glue together two equilong edges, where a particular gluing can be specified if we orient our graph.)

It is easy to figure out when the resulting space, say $Y$, has $K \leq 0$ (or $K < 0$). Namely, if $K(X) \leq 0$ (or $K < 0$), the same inequality holds at all points in $Y$ except, possibly, the (points coming from the) vertices of our graph $X_1$. Now, at every vertex points $y \in Y$ consider all edges $e_j$, $j \in J_x$, from the union of copies of $X_1$ adjacent to some point in (a copy of) $X$.
and call two such edges \textit{x-neighbors} if they come from edges adjacent to the same point \( x \) in \( X \). In this case there is a well (and obviously) defined \textit{angle} measured in \( X \) between these edges, denoted \( \angle_\pi(e_{j_1}, e_{j_2}) \). Observe that the same pair of adjacent edges in \( Y \) may come from different \( x \)’s in \( X \) and the resulting angle depends on which \( x \) is used. Just look at the pair of triangles glued over two pairs of edges.

Next consider cyclic chains of edges at \( y \), say \( e_1, e_2, \ldots, e_{k+1} = e_1 \), where \( e_{i+1} \) is \( x_i \)-adjacent to \( e_i \) for all \( i = 1, \ldots, k \), and where \( x_i \neq x_{i+1} \) for all \( i \).

Clearly, \( Y \) has \( K \leq 0 \) (or \( K < 0 \)) iff the total sum of angles,
\[
\angle_\pi(e_1, e_2) + \ldots + \angle_\pi(e_k, e_1),
\]
is \( \geq 2\pi \) (or \( > 2\pi \) if we want \( K < 0 \)) for all such chains of edges. This (trivially) generalizes the case of 2-polyhedra, where \( X \) equals the union of Euclidean triangles with \( X_1 \subset X \) being the union of the edges of these triangles (and where something new enters the picture if we take, for instance, 3-simplices with their edges instead of the triangles).

All of the above would be rather pointless if we had no simple way to arrange gluings satisfying the \( (\geq 2\pi)-\)condition. Fortunately, there are, roughly, as many such gluings for general \((X, X_1)\) as for triangles \((\Delta, \partial\Delta)\); in particular the ramified covering trick works for all \((X, X_1)\) as follows.

Start with an arbitrary \( Y \), e.g., obtained by doubling \( X \) across \( X_1 \). This \( Y \) has \( K \leq 0 \) everywhere except the vertices of \( X_1 \) and then we take a ramified cover \( \tilde{Y} = Y \) which ramifies at these vertices. Technically speaking, we remove \( X_1 = \{ \text{vertices of} \ X_1 \} \) from \( Y \), take a (finite if you wish) covering of the complement \( Y \setminus X_1 \) which is trivial over each of the two copies of \( X \) in \( Y \), and then metrically complete this covering by adding back the vertices.

The triviality condition says, in effect, that our covering comes from an auxiliary 2-polyhedron where each copy of \( X \) is replaced by the cone over
$X_1 \subset X$. After removing the vertices in $X_1$ the resulting 2-polyhedron contracts to a graph, and so coverings are determined by subgroups of a free group so that we have the same freedom of choosing them as for 2-polyhedra. In particular, we can construct finite ramified covers of $Y$ with $K \leq 0$ (or $K < 0$), provided all angles between the edges of $X_1$ at the vertices are strictly positive. (This is a rather mild condition; actually it takes a special effort to make up examples where it is violated.)

**Remark.** One can look at gluing across $k$-dimensional subpolyhedra $X_k \subset X$ with totally geodesic simplices but making specific examples becomes rather difficult for $k \geq 3$.

§23

Let us isolate a purely combinatorial aspect of the above construction. Say that a graph $(\tilde{V}, \tilde{E})$ is tessellated by (copies of) a graph $(V, E)$ (where $V$ and $\tilde{V}$ stand for the sets of vertices and $E$’s for the edges) if we are given embeddings $\phi_i: V \rightarrow \tilde{V}, i \in I$, such that

(a) $\bigcup_{i \in I} \phi_i(V) = \tilde{V};$

(b) $\text{card} (\phi_i(V) \cap \phi_j(V)) \leq 1$ for all $i \neq j \in I$;

(c) edges go to edges, i.e., the Cartesian squares

$$\phi_i^2: V \times V \rightarrow \tilde{V} \times \tilde{V}$$

map $E \subset V \times V$ to $\tilde{E} \subset \tilde{V} \times \tilde{V}$; furthermore, the images of $\phi_i^2(E) \subset \tilde{E}$ are mutually disjoint and

$$\bigcup_{i \in I} \phi_i^2(E) = \tilde{E}.$$ 

**Proposition.** Given finitely many finite graphs, $(V_1, E_1), \ldots, (V_k, E_k)$, there exists a finite graph $(\tilde{V}, \tilde{E})$ tessellated by each of $(V_1, E_1), \ldots, (V_k, E_k)$.

In fact, such a $(\tilde{V}, \tilde{E})$ is obtained by factoring some universal infinite graph $(\bar{V}, \bar{E})$ by a suitable cofinite subgroup of the free group operating on $(\bar{V}, \bar{E})$. To make it clear, we shall state a more precise form of the above proposition, where we assume, for simplicity’s sake that there are only two graphs, $(V_1, E_1)$ and $(V_2, E_2)$, with $\text{card} E_1 = \text{card} E_2$. We assume, moreover, that there is given a bijection $E_1 \rightarrow E_2$, where the edges $E_1 \ni e_1 \mapsto e_2 \in E_2$ are regarded as equivalent. We also fix directions on all edges and require the above correspondence to preserve the directions.
Furthermore, we assign positive lengths to the edges, thus turning $V_1$ and $V_2$ into metric spaces (assuming the graphs are connected) and assume that the corresponding (equivalent) edges have equal length. Then we shall speak of *marked directed isometric* tessilations of $(\tilde{V}, \tilde{E})$ by $(V_1, E_1)$ and $(V_2, E_2)$ meaning that

"isometric": the implied embeddings $\varphi_i$ of $V_1$ and $V_2$ into $V$ are isometries;

"directed": the graph $(\tilde{V}, \tilde{E})$ is directed and the maps $\varphi_i$ (both for $E_1$ and $E_2$) preserve the direction of edges;

"marked": if an edge $\tilde{e} \in \tilde{E}$ comes from some $e_1 \in E_1$ and $e_2 \in E_2$, then edges are equivalent, i.e., $e_1 \sim e_2$. (In other words, the tessilations agree with a marking of $E$ by equivalence classes of edges.)

**Proposition**. There exists a finite graph $(\tilde{V}, \tilde{E})$ with marked directed isometric tessilations by $(V_1, E_1)$ and by $(V_2, E_2)$.

**Proof.** Attach a copy of $(V_2, E_2)$ to $(V_1, E_1)$ at each edge in $E_1$ according to "$\sim\$". Then attach copies of $(V_1, E_1)$ to all newly created $E_2$-edges and continue ad infinitum. Thus we get a tree-like graph $(\tilde{V}, \tilde{E})$ suitably tessilated by $(V_1, E_1)$ and $(V_2, E_2)$ with an obvious cocompact action of the free group $F_t$ with $t = \frac{c(c-1)}{2}$ for $c = \text{card } E_1 = \text{card } E_2$. This is the automorphism group of the tree with 2-colored vertices and $c$-colored edges as sketched for $c = 3$ below.

![Fig. 6](image)

Then a quotient of $(\tilde{V}, \tilde{E})$ by a sufficiently small c-finite subgroup in $F_t$ is our $(V, E)$. (This can be used for construction of Enflo type expanders departing from bipartite graphs of Remark 13(a).)
§24. Effective universal coverings of spaces with $K \leq 0$

If $X$ has $K(X, x_0) \leq 0$, then small balls $B(x_0, \varepsilon) \subset X$ are convex and if $K(X) \leq 0$ everywhere, then such a ball remains locally convex in-so-far as it does not meet itself somewhere.

Fig. 7.

If this happens, we ignore the meeting points and continue to enlarge the ball, but not as a subset in $X$ but rather as an abstract metric space along with a locally isometric map to $X$. These are called over-balls $\tilde{B}(x_0, R) \rightarrow X$, $R > 0$, which all have locally convex boundaries since $K(X) \leq 0$ and so one can go from $\tilde{B}(x_0, R)$ to $B(x_0, R + \varepsilon)$ for small $\varepsilon$ (where a little extra care is needed if $X$ is not locally compact). Thus we obtain a space $\tilde{X} = \tilde{B}(x_0, R = \infty)$ along with a locally isometric map $p: \tilde{X} \rightarrow X$. This $\tilde{X}$, being locally isometric to $X$, has $K(\tilde{X}) \leq 0$ and it is exhausted by locally convex balls. It follows (by the above considerations) that, in fact, $\tilde{X}$ is CAT(0) and it is easy to see that $p: \tilde{X} \rightarrow X$ is a covering map. In particular, if a simply connected space $X$ has $K(X) \leq 0$, then it is a CAT(0)-space. This is the classical Cartan–Hadamard theorem (usually stated for non-singular spaces). Here are additional remarks clarifying the picture.

(a) If a space $X$ with $K(X) \leq \kappa \leq 0$ admits a filtration by locally convex subsets $X_t \subset X$, $t \in \mathbb{R}_+$, where $X_t \subset X_{t'}$ for $t \leq t'$ and $X_{t+\varepsilon}$ is contained in the $\varepsilon$-neighborhood $X_t + \varepsilon$ of $X_t$ for all $t \geq 0$ and some $\varepsilon = \varepsilon(t) > 0$, then $X \left(= \bigcup_{t \in \mathbb{R}_+} X_t \right)$ is CAT(\kappa) provided $X_0$ is CAT(\kappa).
(Recall that “locally convex” signifies convexity of some neighborhood of each point of the subset in question.)

(b) If $X$ is CAT(0)-convex, then every connected locally convex subset in $X$ is convex. Moreover, if $Y$ is an abstract connected metric space and
$p: Y \to X$ is a locally isometric map sending a small neighborhood of each $y \in Y$ onto a convex subset in $X$, then $p$ is one-to-one and the image $p(Y) \subset X$ is convex. (This is easy but not totally trivial even for $X = \mathbb{R}^n$.)

(c) The Cartan–Hadamard theorem remains valid for certain orbispaces with $K \leq 0$ (see [5]) and a version of this underlies the small cancellation theory (see below and [2]).

§25. Filling closed curves by disks in CAT-spaces

There is an alternative, in effect, more functorial, definition of CAT-spaces, at least for $K \leq 0$ (due to Reshetnyak [?]), which says that $X$ is CAT($\kappa$) if every closed curve in $X$ bounds a disk with curvature $\leq \kappa$. Actually one only needs Riemannian disks $D$ with metrics of constant curvature $\kappa$ (every such $D$ appears as a multi-domain over $H^2(\kappa)$). Namely, $X$ is CAT($\kappa$) if and only if for every $\ell > 0$ and every distance (non-strictly) decreasing map $\alpha$ of the circle $S = S_{\ell}$ of length $\ell$ to $X$ there exists a metric $\mu$ of constant curvature $\kappa$ on $D$ with length $(\partial D) = \ell$ and a distance decreasing map $\beta: (D, \mu) \to X$ extending $\alpha$, where the boundary $\partial D$ is naturally identified with $S_{\ell}$.

Sketch of the proof. If a geodesic triangle, viewed as a (mapped) circle in $X$, can be filled by $(D, \mu)$, then, in the case $\kappa \leq 0$, it is $\kappa$-narrow since $(D, \mu)$ is CAT($\kappa$) for every metric with curvature $\leq \kappa$ as an elementary argument shows. Conversely, every closed curve $S$ in a CAT($\kappa$)-space $X$ can be “subdivided” into “infinitesimally small” geodesic triangles as in Fig. 3 giving in the limit a filling disk $\mathcal{D}$ with curvature $\leq \kappa$, which can then be “enlarged” to $(D, \mu)$ with curvature $\equiv \kappa$.

Then one can define spaces with $K(X) \leq 0$ by requiring the existence of $(D, \mu)$ and $\beta$ for all contractible closed curves in $X$. It is not hard to show (by using, for instance, suitable minimal disks filling in curves) that the existence of $(D, \mu)$-fillings for short curves in $X$ (with shortness $\ell = \ell(x) > 0$ depending on $x \in X$ for curves contained in a ball of radius $2\ell$ around $x$) implies that for all closed curves, and so this definition is essentially local.

Reshetnyak theorem and application. The above disk $D$ of constant negative curvature $\kappa$, a priori, only immerses into the model $\kappa$-plane $H_\kappa$. But, according to Reshetnyak, one can find a convex domain $D_x \subset X_x$,
with
\[ \text{length}(\partial D_X) = \text{length}(\partial D) = \ell \]

and a distance non-decreasing homeomorphism \( D' \to D \). Thus our \( S = S_\ell \subset X \) can be filled in by a convex disk \( D' \subset H_\ell \).

Consequently \( \text{CAT}(\kappa) \)-spaces are \( \text{Cycl}_{i}(\kappa) \) for all \( i = 4, 5, \ldots \) and thus they all are \( \text{Wir}_\infty \) for \( \kappa = 0 \), and our bounds on \( \lambda_1 \) for various maps \( V \to X \) from §8-12 apply to \( \text{CAT}(0) \)-spaces.

**Question.** Does the \( \text{Cycl}_4 \)-inequality imply all \( \text{Wir}_k \), \( k = 5, 6, \ldots \), without assuming the space in question is geodesic?

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### §26. \( \text{CAT}(0) \)-families of groups

Consider a closed subset \( P \subset X \) and isometry groups \( \Gamma_p \) of \( X \) assigned to all \( p \in P \) (where one could suppress \( P \) by defining \( \Gamma_x = \{ \text{id} \} \) for \( x \in X \setminus P \)). Call \( \{ \Gamma_p \} \) a rotation family if the following two conditions are satisfied:

(i) \( \Gamma_p \) fixes \( p \) for all \( p \in P \);

(ii) each \( \Gamma_{p_0} \) maps \( P \to P \) and acts on the family \( \{ \Gamma_p \} \) by conjugation:

\[ \gamma \Gamma_p \gamma^{-1} = \Gamma_{p'} \text{ for } p' = \gamma(p) \text{ and all } \gamma \in \Gamma_{p_0}. \]

**Examples.** (a) Take a finite subset \( \mathcal{P} \subset \mathbb{C} \) and let \( X \) be the universal cover of \( \mathbb{C} \) ramified at \( \mathcal{P} \). Then the (cyclic) monodromy groups around the lifts \( p \in X \) of the points \( p \in \mathcal{P} \) make a rotation family generating the full Galois group acting on \( \mathcal{X} \).

(b) Take a union \( \overline{\mathcal{P}} \) of finitely many lines in \( \mathbb{C}P^2 \). Again the Galois group of the universal cover ramified at \( \overline{\mathcal{P}} \) is generated by “rotations” about intersections of lines. (The Galois group of \( \overline{\mathbb{Q}}/\mathbb{Q} \) is also generated by “rotations” corresponding to the Frobenius automorphisms.)

Denote by \( \Gamma_P \subset \text{Iso}X \) the rotation group generated by all \( \Gamma_p \) and let us reduce the \( \text{CAT}(0) \)-property of the quotient space \( X/\Gamma_P \) to that of the spaces \( X/\Gamma_p \), \( p \in \mathcal{P} \), under the following disjointness assumption:

(iii) The set \( \mathcal{P} \) is discrete and \( \Gamma_p \) acts freely and discretely on the complement \( X \setminus \{ p \} \) for all \( p \in \mathcal{P} \). (This is the case for the above (a) but not for (b).)
\section*{Proposition}

Let $\{\Gamma_p\}$ satisfy (i)-(iii), the space $X$ be CAT$(\kappa)$ for some $\kappa \leq 0$, and $K(X/\Gamma_p) \leq \kappa$ for all $p \in P$. Then the group $\Gamma_p$ is discrete and the quotient space $X/\Gamma_p$ is CAT$(\kappa)$. Furthermore,

(a) $\Gamma_p$ acts freely on the complement $X \setminus P$ and the isotropy subgroup of each $p \in P$ equals (exactly!) $\Gamma_p$. Moreover, if a ball $B_p \subset X$ around a point $p \in P$ contains no $p' \neq p$ in $P$, then the obvious map $B_p/\Gamma_p \to X/\Gamma_p$ is one-to-one.

(b) If $P$ is separated on bounded subsets, i.e., $|p_1 - p_2| \geq r$ for some positive monotone decreasing function $r = r_x(R) > 0$ depending on the distance $R = |p_1 - x_0|$ from a chosen point $x_0 \in X$, and for all $p_1 \neq p_2$ in $P$, then there is a subset $Q \subset P$ such that $\Gamma_p$ is freely generated by the groups $\Gamma_q$, $q \in Q$. That is, the natural homomorphism from the free product $*_{Q} \Gamma_q$ to $\Gamma_p$ is an isomorphism.

\textbf{Proof.} Take a convex subset $Y \subset X$ and see how it behaves under the projection to $X/\Gamma_p$ for some $p \in P$.

If $p$ does not lie in the closure of $Y$, our map $Y \to X/\Gamma_p$ is isometrically with locally convex boundary, and since $X/\Gamma_p$ is CAT$(0)$, our $Y$ isometrically maps onto a convex subset in $X/\Gamma_p$. In other words, the $\gamma$-translates $\gamma(Y)$ do not meet $Y$ for all $\gamma \in \Gamma_p$.

Next, let us assume $p$ lies in the boundary $\partial Y$ and suppose $Y$ is strictly convex at $p$, i.e., there is no geodesic segment in the closure of $Y$ containing $p$ as an interior point of this segment. Then every such segment, apart from one of its ends, is locally convex in $X/\Gamma_p$ and, hence, convex. Consequently, $Y$ injects to $X/\Gamma_p$ away from $p$, and thus the $\Gamma_p$-orbit $\Gamma_p(Y) \subset X$ consists of the translates $\gamma(Y)$ meeting at $p$ and nowhere else. This applies, in particular, to $\varepsilon$-neighborhoods of convex subsets in $X \setminus \{p\}$ for $\varepsilon > 0$ as these are strictly convex at all their boundary points in CAT$(0)$-spaces.

We denote the $\varepsilon$-neighborhood of $Y$ by $Y + \varepsilon$, observe that $\Gamma_p(Y + \varepsilon) = \Gamma_p(Y) + \varepsilon$, and see that $\Gamma_p(Y + \varepsilon)$ consists of disjoint translates of $Y + \varepsilon$ for $\varepsilon < \text{dist}(p, Y)$ which meet together at $p$ for $\varepsilon = \text{dist}(p, Y)$ so that $\Gamma_p(Y + \varepsilon)$ becomes convex as well as $\Gamma_p$-invariant for $\varepsilon > \text{dist}(p, Y)$.

Now see what happens when such a $\Gamma_p$-invariant growing $\varepsilon$-neighborhood hits another point $p' \in P$. More generally, let $Y \subset X$ be a strictly convex subset invariant under the group $\Gamma_Y$ generated by all $p \in P$ contained in the interior of $Y$ and let $\Gamma_Y$ be generated by $\Gamma_Y$ and all $p \in P$ contained in the boundary of $Y$. Two $\gamma$-translates of $Y$ for $\gamma \in \Gamma_Y$
may be in three kinds of mutual positions:

1. \( \gamma_1(Y) = \gamma_2(Y) \) for \( \gamma_1 \gamma_2^{-1} \in \Gamma_Y \);
2. \( \gamma_1(Y) \) meets \( \gamma_2(Y) \) at a single point, e.g., \( \gamma(Y) \) meets \( Y \) at \( p \) for every \( \gamma \in \Gamma_p \) and \( p \in P \cap \partial Y \);
3. \( \gamma_1(Y) \) is disjoint from \( \gamma_2(Y) \).

Since the local convexity is preserved at the meeting points \( p \), the orbit \( \Gamma_Y(Y) \) is convex and the map \( Y/\Gamma_Y \rightarrow X/\Gamma_Y \) is injective. Furthermore, the group \( \Gamma_Y \) is freely generated by \( \Gamma_Y \) and the groups \( \Gamma_p \) for \( p \in P \), where \( R \subset P \cap \partial Y \) intersects each \( \Gamma_Y \)-orbit of \( p \cap \partial Y \) at a single point. This is sufficient to prove (a).

Indeed, take \( R \)-balls \( B(R) \subset X \) around some point \( x_0 \in X \) (e.g., some \( p_0 \in P \)) and let

\[
\tilde{B}(R) = \Gamma_{B(R)}(B(R))
\]

(where “balls” are assumed closed,

\[
B(R) = \{ x \in X : |x - x_0| \leq R \},
\]

and \( B^0(R) \) denotes the interior where \( |x - x_0| < R \). These \( \tilde{B}(R) \) are convex as well as \( \Gamma_{B(R)} \)-invariant and their projections to \( X/\Gamma_{B(R)} \) are injective and convex. Therefore, these remain locally convex as we pass to \( X/\Gamma_P \) and so \( \tilde{B}(R) \) injectively project to balls in \( X/\Gamma_P \). This ensures the inequality \( K(X/\Gamma_P) \leq 0 \) at the (suspicious) points coming from \( p \in P \) and proves the local, and hence global, convexity of balls in \( X/\Gamma_P \) as these are isometric to \( B(R)/\Gamma_{B(R)} \).

Finally, we turn to (b) and notice that the above suffices to show that every finite subset \( P' \subset P \) contains a subset \( Q' \subset P' \) such that the group \( \Gamma' \) generated by \( \Gamma_p \), \( p \in P' \), is freely generated by \( \Gamma_q \), \( q \in Q' \). However, if, for example, points in \( P' \setminus B(R) \) accumulate to the boundary of \( B \), then we cannot (5) claim the freedom property. (Yet the injectivity of the map \( B(R)/\Gamma_{B(R)} \rightarrow X/\Gamma_P \) follows from what happens to finite subsets \( P' \subset P \).) We need at this stage the separation property of \( P \) along with the uniform convexity of the balls that yields (this is all we need) a universal upper bound on the diameter of the intersection \( (\tilde{B}(R)+\varepsilon) \cap (\tilde{B}(R)+\varepsilon) \) in terms of \( R \) and \( \varepsilon \), where \( \tilde{B}(R) \) is supposed to be disjoint from \( \gamma \tilde{B}(R) \).

It follows that for every \( R < \infty \) and \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that there is no non-trivial triple intersection between \( \gamma \)-translates of \( \tilde{B}(R) \) for \( \gamma \in \Gamma_{P \cap (B(R)+\varepsilon)} \); that is, if

\[
\gamma_1 \tilde{B}(R) \cap \gamma_2 \tilde{B}(R) \cap \gamma_3 \tilde{B}(R) \neq \emptyset,
\]
then $\gamma_i \widehat{B}(R) = \gamma_j \widehat{B}(R)$ for some $i \neq j = 1, 2, 3$, provided the subset $P \cap (\widehat{B}(R) + 1) \subset X$ is $\delta$-separated or, equivalently, $P \cap B(R + 1)$ is $\delta$-separated. Hence, the group $\Gamma_{\widehat{B}(R) + \varepsilon}$ is freely generated by $\Gamma_{\widehat{B}(R)}$ and $\Gamma_p$, $p \in R$, where $R \subset P \cap (\widehat{B}(R) + \varepsilon)$ intersects each $\Gamma_{\widehat{B}(R)}$ orbit of $P \cap (\widehat{B}(R) + \varepsilon) \setminus \widehat{B}(R)$ at a single point and (e) follows as in the case of finite sets $P \cap B(R)$.

Q.E.D.

§28. REMARKS

(a) The above argument shows that strict convex independence of points $p \in P' \subset P$ implies free independence of subgroups $\Gamma_{p'}$, $p' \in P'$, where "strict convex independence" refers to the existence of a strictly convex subset $Y \subset X$ with $P' \subset \partial Y$.

(b) Let us indicate a generalization of Proposition 27 (in the spirit of the Cartan–Hadamard theorem for non-rigid orbispaces, compare [5] and [2], where the relevant subsets (e.g., $B(R)$) may be non-convex in $X$ but project to (locally) convex subsets in $X/\Gamma_p$).

A rotation family $\{\Gamma_p\}$ is called regular if the subset $P$ is closed and the function $p \mapsto \Gamma_p$ is semicontinuous, i.e., each $p \in P$ admits a neighborhood $U_p \subset P$ such that $\Gamma_{p'} \subset \Gamma_p$ for all $p' \in U_p$. (There often exists a rather regular stratification of $P$ such that $\Gamma_p$ is constant on each stratum.)

**Generalization of Proposition 27.** If $\{\Gamma_p\}$ is regular, free away from $P$ and the quotient spaces $X/\Gamma_p$ are CAT($\kappa$) for some $\kappa \leq 0$ and all $p \in P$, then the rotation group $\Gamma_P$ is discrete and the quotient space $X/\Gamma_P$ is CAT($\kappa$).

(c) Proposition 27 in its present form has rather limited applications (see below) but it gains in significance when generalized to spaces $X$ with "approximately negative curvature" (see [6]).

§29. CONING CAT-SPACES AND THEIR SUBSPACES

Let us look at the disk of radius $r$ in the standard space of constant curvature $\kappa$ as the cone over its boundary, $D_{\kappa,r} = C_{\kappa,r}(\partial D_{\kappa,r})$, where we are mainly interested in $\kappa \leq 0$ (and where one should restrict to $r < \pi/\sqrt{\kappa}$ for $\kappa > 0$). Then, for an arbitrary geodesic (path) metric space $X$, one defines the (path) metric cone $C_{\kappa,r}(X)$ as $X \times [0, r]$ with the base $X \times 0$ shrunk to a single point: the apex, also called the center of the cone, where the metric is given by the same rule as in $D_{\kappa,r}$. Namely,
every short geodesic segment $S$ in $C_{\kappa, r} (X)$ away from the apex projects to a geodesic segment in $X$, say $\Sigma \subset X$, and the cone $C_{\kappa, r} (\Sigma) \subset C_{\kappa, r} (X)$ is isometric to the sector in $D_{\kappa, r}$ over the arc $\Sigma \subset \partial D_{\kappa, r}$ with length equal that of $\Sigma$.

The curvature $K$ of $C_{\kappa, r} (X)$ away from the apex can be evaluated in terms of $K(X)$; this curvature $K$ is $\leq \kappa$ if (and only if) the curvature $K(X)$ is bounded by the curvature $\kappa$, of the 2-sphere of radius $r$ in the standard 3-space with curvature $\kappa$. In particular, $K < 0$ if $K(X) \leq 0$. Furthermore, if $X$ is CAT($\kappa$), then $K \leq \kappa$ also at the apex of the cone. (All this is well known and rather obvious.) In particular, the unit Euclidean cone $C_{1, 1} (X)$ has $K \leq 0$ for all CAT(0)-spaces $X$. (One may think of $C_{1, 1} (X)$ as the ordinary Euclidean cone over $X$, where $X$ is isometrically immersed into the unit sphere in some $\mathbb{R}^n$.)

Next, let $U_i \subset X$, $i \in I$, be a collection of subsets in $X$ and

$$X^* \overset{\text{def}}{=} C_{\kappa, r} (X, \{U_i\})$$

be obtained by attaching the cones $U_i^* = C_{\kappa, r} (U_i)$ to $X$ across $U_i = U_i \times r$, for all $i \in I$. Notice that every two cones $U_i^*$ and $U_j^*$ in $X^*$ intersect across $U_i \cap U_j \subset X \subset X^*$. One can artificially enlarge these intersections by gluing pairs $U_i^*$ and $U_j^*$ across larger subsets in $C_{\kappa, r} (U_i \cap U_j)$. For example, given a positive function $d(x)$, we define functions $\varphi_{ij}$ on $U_i \cap U_j$, as $\varphi$ of the distance $d = d(x)$, $x \in U_i \cap U_j$, to the boundary of $U_i \cap U_j$ in $U_i \cup U_j$, i.e.,

$$d(x) = \text{dist}(x, (U_i \cup U_j) \setminus (U_i \cap U_j)).$$

Then we glue $U_i^*$ to $U_j^*$ across the subset of pairs $(x, \rho)$ where $\rho \leq \varphi_{ij}(x)$ and observe that this "gluing" defines an equivalence relation on the disjoint union of the cones $U_i^*$, and hence on $X^*$, provided the function $\varphi(d)$ is monotone increasing.

§30. Useful example

Let $X$ be a tree and $U_i$ be double infinite geodesic lines in $X$, where all intersections are segments of lengths $\ell_{ij} \leq \ell_0 < \infty$. Take $\varphi = \varphi_{\kappa, r, \ell_0}$ such that

$$W_{ij} \subset C_{\kappa, r} (U_i \cap U_j) \subset C_{\kappa, r} (U_j) = C_{\kappa, r} (U_i) = C_{\kappa, r} (\mathbb{R})$$

looks as in the picture below.

Consider the space $X^*$ obtained from $X^* = C_{\kappa, r} (X, \{U_i\})$ by gluing every $U_i^*$ to $U_j^*$ across the above $W_{ij}$. 


If the angle $\alpha = \alpha_\varphi = \alpha_{\gamma, r, \ell_0}$ in Figure 8 is $\leq 2\pi/3$, then the space $X^\varphi$ is CAT($\kappa$). In particular, if $\ell_0 \leq 2\pi r/\sigma$, then the space $X^\varphi$ for $\varphi = \varphi_{\gamma, r, \ell_0}$ is CAT(0).

**Proof.** It is clear that $X$ remains CAT($\kappa$) if we attach our cones to disjoint lines $U_i$ or, more generally, if there are no triple meeting points between $U_i$, since the intersections $W_{ij}$ are convex in $U_i^\varphi$. The problem may appear when three (or more) lines come together as three lines joining the pairs of ends in the infinite tripod do. But the condition $\alpha \geq 2\pi/3$ makes the cycles in the links of such meeting point longer than $2\pi$, which implies $K(X^\ell) \leq \kappa$ at all points. This yields CAT($\kappa$)-property since $X^\ell$ is (obviously) simply connected.

Q.E.D.

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**Corollary.** Let $\Gamma_i$, $i \in I$, be isometry groups acting on $X$, where each $\Gamma_i$ is generated by a single isometry $\gamma_i : X \to X$ mapping the line $U_i = \mathbb{R}$ into itself via a translation $x \mapsto x + R_i$; and let $\Gamma_\bullet$ be generated by the groups $\Gamma_i$, $i \in I$. If $R \geq 6\ell_0$, then the space $X^\psi/\Gamma$ is CAT(0) for a suitable $\varphi$ and if $R > 6\ell_0$, one can achieve CAT($\kappa > 0$) for $X^\psi/\Gamma$. Consequently, $\Gamma$ is freely generated by some subgroups $\Gamma_i$ among $\Gamma_i$.

**Proof.** Take $r = 3\ell_0/\pi$, apply the coning construction $C_{\ell_0, r}$. Then the corresponding $X^\varphi$ is CAT(0) for $R \geq 6\ell_0$. And if $R > 6\ell_0$, we use $C_{\kappa, r}$ with $\kappa < 0$ with $|\kappa|$ being small compared to $R - 6\ell_0$, and $r$ slightly larger than $3\ell_0/\pi$, thus getting $K(X^\psi/\Gamma_\bullet) \leq k < 0$.

Q.E.D.

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**Remarks**

(a) This Corollary shows, in particular, that the small cancellation groups $\Gamma$ with the metric $1/\ell_0$-condition are CAT(0) serving as fundamen-
t groups of 2-polyhedra with \( K \leq 0 \), while the \( \sfrac{1}{6+\varepsilon} \)-condition ensures \( \text{CAT}(\varepsilon) \)-property.

Recall, that such a \( \Gamma \) is given by finitely many relations that are just some elements, say \( \gamma_1, \ldots, \gamma_k \) in the free group \( F \) on some generators. This \( F \) acts on the standard tree \( X \), and \( \Gamma_i \) are generated by the (finitely many) \( F \)-conjugates of \( \gamma_1, \ldots, \gamma_k \). Then \( \Gamma = F/\Gamma_i \) freely and isometrically acts on our space \( X^\varphi/\Gamma_i \) that is \( \text{CAT}(0) \) in the \( \sfrac{1}{6+\varepsilon} \)-case by the above discussion, where the quotient \( (X/\Gamma_i)/\Gamma \) is obtained from the standard 2-polyhedron \( P \) representing \( \Gamma \) by little geometric tinkering (corresponding to \( X^\varphi \rightarrow X^\varphi \)) making \( K \leq 0 \) while keeping the homotopy type (and the dimension) of \( P \) intact.

(b) The above approach to small cancellation groups (which is essentially well known) will be extended in another paper to spaces \( X \) with approximately negative curvature and general “convex” groups \( \Gamma_i \).

(c) There is another way of turning \( X/\Gamma_i \) into a \( \text{CAT}(0) \)-space, consisting in taking the nerve \( Y \) of the covering of \( X \) by \( U_i \) and then dividing \( Y \) by \( \Gamma_i \) (compare §16). Unfortunately, the upper curvature bound at the fixed vertices of \( \Gamma_i \) depends, besides \( R \), on the dimension of \( Y \), i.e., the maximal multiplicity of intersection of \( U_i \), which makes the nerve construction unsuitable for most interesting \( \Gamma \). I do not exclude, however, an improvement of this making all combinatorially \( \sfrac{1}{6+\varepsilon} \)-groups \( \text{CAT}(0) \), but this seems hard to achieve for general small cancellation groups (see [7]), where the traditional approach via Dehn’s diagrams remains indispensable.

References


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