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Spin and scalar curvature in the presence of a fundamental group. I

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0. Introduction

It is a generally accepted principle that the fundamental group of a positively curved manifold must be "small". There has been little progress until recently in applying this principle to manifolds of positive* *scalar* curvature. This is not surprising since any manifold of the form $X = X_0 \times S^2$ carries a metric of positive scalar curvature no matter how large the fundamental group is.

A breakthrough in the problem was achieved in a recent sequence of papers [16], [17] of R. Schoen and S. T. Yau. They established, in particular, that the tori of dimension $\leqq 7$ support no metric of positive scalar curvature. Their techniques employ the regularity of certain minimal hypersurfaces which fails in dimensions $\geqq 8$.

Another approach to positive scalar curvature is suggested by the work of Lichnerowicz [13]. He proved that a spin manifold with non-vanishing $\hat{A}$-genus carries no metric of positive scalar curvature. One of the ingredients in his proof is the Atiyah-Singer Index Theorem applied to the Dirac operator.

We show in this paper how the spin argument gains additional power when the universal covering of a manifold X (and hence the fundamental group $\pi_1(X)$) is "large". In particular we establish new obstructions to the existence of positive scalar curvature metrics on manifolds which are not simply-connected. In a companion paper [19] we show that for simply-connected manifolds the previously known obstructions essentially form a complete set of invariants.

Definitions. A compact orientable riemannian manifold X of dimension n is called ε^{-1}-*hyperspherical* if there exists a short map of positive degree from X onto the euclidean n-sphere of radius ε^{-1}. (A short map is one which

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* It is known that on a compact connected manifold X of dimension $\leqq 3$, any function which is negative somewhere is the scalar curvature function of some metric on X. (See [10].)

does not increase distance.) This is equivalent to the condition that there exists an ε-contracting map of positive degree onto the unit sphere. (An ε-contracting map is one which multiplies all distances by a factor less than or equal to ε.)

If for every $\varepsilon > 0$, there exists a finite covering of X which is ε^{-1}-hyperspherical and spin, then X is called *enlargeable*.

Note that enlargeability is a homotopy invariant. Indeed, if $X \to Y$ is a map of positive degree and Y is enlargeable, then clearly X is also enlargeable provided it has a finite spin covering. The simplest example of an enlargeable manifold is the torus. Observe that the product of enlargeable manifolds is enlargeable, and that the connected sum of any spin manifolds with an enlargeable manifold is again enlargeable. We shall prove in Sections 3, 4 and 6 that all compact hyperbolic manifolds, all solvmanifolds and all sufficiently large 3-manifolds are enlargeable.

THEOREM A. *Let X be an enlargeable manifold. Then X carries no riemannian metric of positive scalar curvature. In fact any metric of non-negative scalar curvature on X is flat.*

This gives the general answer to a question of Kazdan and Warner [10].

COROLLARY A. *Any metric of non-negative scalar curvature on the torus T^n is flat.*

A smooth map $f\colon X \to Y$ between compact, connected, oriented manifolds is said to have *non-zero $\hat{A}$-degree* if $\hat{A}(f^{-1}(p)) \neq 0$ for some regular value p of f. The map is said to be *spin* if $f^*w_2(Y) = kw_2(X)$ for $k = 0$ or 1. (Here w_2 denotes the second Stiefel-Whitney class.)

THEOREM B. *Let X be a compact manifold which admits a spin map of non-vanishing $\hat{A}$-degree onto an enlargeable manifold. Then X carries no metric of positive scalar curvature. In fact any metric of non-negative scalar curvature on X is Ricci flat.*

COROLLARY B. *Let $X = T^n \# X_0$ where X_0 is a spin manifold. Then X carries no metric of positive scalar curvature. Any metric of non-negative scalar curvature on X is flat* (*and X must be the standard torus*).

Note that when X is a compact spin manifold with $\hat{A}(X) \neq 0$ and $f\colon X \to \{\text{pt.}\}$, Theorem B reduces to the Lichnerowicz Theorem [13]. Theorem B gives a family of results which interpolate between Theorem A and the Lichnerowicz result. For example, if X_0 is spin and $\hat{A}(X_0) \neq 0$, and if X_1 is enlargeable, then $X_0 \times X_1$ carries no metric with positive scalar curvature.

This paper also contains some positive results.

THEOREM C. *The following manifolds carry metrics of positive scalar curvature.*

1. *Any n-manifold* ($n \geqq 3$) *of the form*

$$X = (S^n/\Gamma_1) \# \cdots \# (S^n/\Gamma_k) \# (S^1 \times S^{n-1}) \# \cdots \# (S^1 \times S^{n-1})$$

where each (S^n/Γ_j) *is an elliptic space form.*

2. *Any manifold of the form* $X = \partial(X_0 \times D^2)$ *where* X_0 *is a compact connected manifold with non-empty boundary.*

3. *The double of* X *where* X *is a compact manifold which carries a metric such that both the scalar curvature of* X *and the mean curvature of its boundary are positive.*

By use of Theorem C(3), the following result will be proved in the second part of this paper.

COROLLARY C. *Any compact manifold* X *which carries a metric with sectional curvature* $\leqq 0$, *cannot carry a metric with scalar curvature* $\kappa > 0$. *Moreover, any metric with* $\kappa \geqq 0$ *on* X *is flat.*

In this paper and its sequel we treat the case of 3-manifolds. Modulo certain questions concerning manifolds with finite fundamental group, we obtain a classification of 3-manifolds with positive scalar curvature.

The paper is organized as follows. In Section 1 we present the Bochner-Weitzenbock formula for the Dirac operator on a twisted spin bundle. In Section 2 we define a topological invariant $\hat{\mathfrak{A}}(X)$ for non-simply-connected spin manifolds, which is analogous to the Novikov higher signature. This invariant is realized as the index of a family of elliptic operators. We show that it vanishes if the manifold carries a metric of positive scalar curvature.

In Section 3 we introduce a geometric construction which allows us to generalize the results of Section 2. The idea is roughly dual to the Kodaira vanishing arguments. In place of twisting by a positive bundle, we "untwist" over negative coverings.

Section 4 treats the case of solvmanifolds; Section 5 contains the proof of Theorem C; and Section 6 concerns 3-manifolds.

The results of this paper have a purely topological consequence. We recall the following result.

THEOREM ([11]). *If a compact manifold admits a (non-trivial)* S^3*-action, then it carries a metric of positive scalar curvature.*

Atiyah and Hirzebruch proved that if a spin manifold X admits an

S^1-action, then $\hat{A}(X) = 0$. This is not true of the invariant $\hat{\mathfrak{A}}$ since $\hat{\mathfrak{A}}(T^n) \neq 0$. However, from the above result and Theorem 2.1 (§ 2), we have:

THEOREM D. *If a compact spin manifold X admits an S^3-action, then* $\hat{\mathfrak{A}}(X) = 0$.

Final Remark. The circle of ideas presented here suggested that one should be able to prove the generalized Novikov conjecture for manifolds whose universal covering space is hypereuclidean. We have learned that T. Farrell and W. C. Hsiang recently proved a result of this type [6].

1. The vanishing theorem for twisted spin bundles

In this section we shall briefly enunciate the general Bochner-Weitzenböck formula for a Dirac operator. Our setting is the following. Let X be a riemannian manifold and let $\mathrm{Cl}(X)$ denote its associated bundle of Clifford algebras. This is the bundle over X whose fibre at each point x is the Clifford algebra of the tangent space $T_x(X)$ with its given norm. There is a canonical embedding $T(X) \subset \mathrm{Cl}(X)$. This bundle carries a natural orthogonal connection, extending the one on T, and characterized by the fact that covariant differentiation is a derivation on the algebra of sections, i.e.,

$$\nabla(\varphi \cdot \psi) = (\nabla\varphi) \cdot \psi + \varphi \cdot (\nabla\psi)$$

for all $\varphi, \psi \in \Gamma(\mathrm{Cl}(X))$.

We now consider a bundle of modules S over the bundle of algebras $\mathrm{Cl}(X)$. We assume S is equipped with a metric and an orthogonal connection so that: (i) For each unit tangent vector $e \in T_x(X)$, $x \in X$, the module multiplication $e\colon S_x \to S_x$ is orthogonal, (ii) covariant differentiation is a derivation with respect to module multiplication, i.e.,

$$\nabla(\varphi \cdot \sigma) = (\nabla\varphi) \cdot \sigma + \varphi \cdot (\nabla\sigma)$$

for all $\varphi \in \mathrm{Cl}(X)$ and all $\sigma \in \Gamma(S)$. Under these assumptions there is a self-adjoint elliptic first order differential operator $D\colon \Gamma(S) \to \Gamma(S)$ defined by setting

$$D = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}$$

where $e_1, \cdots, e_n$ is locally a basis of pointwise orthonormal vector fields on X. The square of D is given by the formula

$$D^2 = \sum_{j,k} e_j \cdot e_k \cdot \nabla_{e_j, e_k}$$

where $\nabla_{v,w} \equiv \nabla_v \nabla_w - \nabla_{\nabla_v w}$ is the hessian operator. When X is compact, the operators D and D^2 have the same kernel.

There is another second order operator $\nabla^*\nabla\colon \Gamma(S) \to \Gamma(S)$ having the same symbol as D^2. It is given locally by

$$\nabla^*\nabla = -\sum_{j=1}^{n} \nabla_{e_j, e_j} .$$

This operator is non-negative and self-adjoint. Its kernel is the space of parallel sections of S.

The following general formula yields most of the known vanishing theorems in riemannian geometry (see [12] for example). Let $R_{v,w} = \nabla_{v,w} - \nabla_{w,v}$ denote the curvature tensor of the connection on S, and define a global section $\mathcal{R} \in \Gamma(\mathrm{Hom}(S, S))$ by the formula:

$$\mathcal{R} = \frac{1}{2} \sum_{j,k} e_j \cdot e_k \cdot R_{e_j, e_k} .$$

THEOREM 1.1.

$$D^2 = \nabla^*\nabla + \mathcal{R} . \tag{1.1}$$

Proof. Let $e_1, \cdots, e_n$ be local orthonormal vector fields on X. Then

$$\begin{aligned} D^2 &= \sum_{j,k} e_j \cdot e_k \cdot \nabla_{e_j, e_k} \\ &= \sum_j e_j \cdot e_j \cdot \nabla_{e_j, e_j} + \sum_{j<k} e_j \cdot e_k \cdot (\nabla_{e_j, e_k} - \nabla_{e_k, e_j}) \\ &= \nabla^*\nabla + \mathcal{R} . \end{aligned}$$

Note that $\mathcal{R}_x \colon S_x \to S_x$ is a symmetric transformation. We say $\mathcal{R}$ is *non-negative* if $\mathcal{R}_x \geqq 0$ for all x. We say $\mathcal{R}$ is *positive* if it is non-negative and $\mathcal{R}_x > 0$ at some point x.

COROLLARY 1.2. *Suppose X is compact and oriented. If $\mathcal{R}$ is positive, then* $\ker(D) = 0$. *If $\mathcal{R}$ is non-negative, then* $\ker(D)$ *is the space of parallel sections.*

For the bundles of modules to be considered in this paper we shall require a spin structure. Recall (cf. [15]) that an orientable manifold X is called a *spin manifold* if its second Stiefel-Whitney class $w_2(X)$ is zero. Suppose X is equipped with a riemannian metric and let $P_{\mathrm{SO}_n}(X)$ be the bundle of oriented orthonormal tangent frames. A *spin structure* on X is a principal Spin_n-bundle $P_{\mathrm{Spin}_n}(X)$ together with a Spin_n-equivariant map $\xi \colon P_{\mathrm{Spin}_n}(X) \to P_{\mathrm{SO}_n}(X)$ which commutes with the projection maps onto X. The condition $w_2(X) = 0$ is necessary and sufficient for the existence of a Spin structure.

Suppose now that X is an oriented riemannian manifold of dimension $2n$ with a Spin structure. Let Cl_{2n} denote the Clifford algebra of $\mathbf{R}^{2n}$ with its standard inner product. Then $\mathrm{Cl}_{2n} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathrm{End}(\mathbf{C}^{2^n})$, and so Cl_{2n} has a unique irreducible complex representation (cf. [1]). Restricting to $\mathrm{Spin}_{2n} \subset \mathrm{Cl}_{2n}$ we obtain a unitary representation Δ of Spin_{2n}. The associated complex vector bundle

$$S(X) = P_{\mathrm{Spin}_{2n}} \times_{\Delta} \mathbf{C}^{2^n}$$

is called the *bundle of (complex) spinors* over X. This bundle is naturally a bundle of modules over $\mathrm{Cl}(X) = P_{\mathrm{Spin}_{2n}} \times_{\mathrm{Ad}} \mathrm{Cl}_{2n}$. Lifting the riemannian connection on $P_{\mathrm{SO}_{2n}}(X)$ to $P_{\mathrm{Spin}_{2n}}(X)$ determines a connection satisfying the hypotheses in the discussion above. The curvature of this bundle is given by the formula

$$R^S_{v,w} = \frac{1}{2} \sum_{j<k} \langle R_{v,w}(e_j), e_k \rangle e_j \cdot e_k \ ,$$

where $R_{v,w}$ denotes the Riemann curvature tensor of X, and where the "$\cdot$" denotes Clifford multiplication. A straightforward computation now shows that the operator $\mathcal{R}$ above reduces to a scalar operator

$$\mathcal{R} = \frac{1}{4}\kappa$$

where κ is the scalar curvature of X. We conclude that $\kappa > 0$ implies $\ker(D) = 0$. This is the well known theorem of Lichnerowicz [13].

Suppose now that E is any hermitian vector bundle over X with a unitary connection. Consider the bundle $S(X) \otimes E$ with the canonical tensor product connection. This is again a bundle of modules over X satisfying the above hypotheses. It will be called a *twisted spin bundle over* X. The corresponding operator $\mathcal{R}$ is of the form

$$\mathcal{R} = \frac{1}{4}\kappa + \mathcal{R}_0 \tag{1.2}$$

where

$$\mathcal{R}_0(\sigma \otimes e) = \frac{1}{2} \sum_{j,k} (e_j \cdot e_k \cdot \sigma) \otimes R^E_{e_j, e_k}(e) \tag{1.3}$$

and where R^E denotes the curvature tensor of the connection on E. From the discussion above we have the following result.

THEOREM 1.3. *Let X be a compact riemannian spin manifold and let $S(X) \otimes E$ be a twisted spin bundle over X. If $\kappa > 4\mathcal{R}_0$, then* $\ker D = 0$.

From the bundle $S(X) \otimes E$ one can construct a natural elliptic complex. Suppose $\dim(X) = 2n$ and let ω be the parallel section of $\mathrm{Cl}(X) \otimes_{\mathbf{R}} \mathbf{C}$ given locally by the formula $\omega = i^n e_1 \cdots e_{2n}$ where $e_1, \cdots, e_{2n}$ are pointwise orthonormal vector fields. Then $\omega^2 = 1$, and $e_j \omega = -\omega e_j$ for all j. There is a decomposition

$$S(X) \otimes E = S^+ \oplus S^-$$

into the $+1$ and -1 eigenbundles for Clifford multiplication by ω. Clifford multiplication by e_j gives isometries $e_j\colon S_x^{\pm} \xrightarrow{\approx} S_x^{\mp}$. Since $\nabla\omega = 0$, we see that restriction of the Dirac operator gives an elliptic operator

$$D^+\colon \Gamma(S^+) \longrightarrow \Gamma(S^-) \tag{1.4}$$

whose adjoint, denoted D^-, is also the restriction of D. From the Atiyah-Singer Theorem [2] we have that

$$\operatorname{Index}(D^+) = \{\operatorname{ch} E\cdot \hat{\mathbf{A}}(X)\}[X] \tag{1.5}$$

where $\hat{\mathbf{A}}$ denotes the total $\hat{\mathbf{A}}$-class of X. We can now state the main result of this section.

THEOREM 1.4. *Let X be a compact riemannian spin manifold of even dimension, and let $S(X)\otimes E$ be a twisted spin bundle over X. If $\kappa > 4\mathcal{R}_0$, then* $\ker(D^+)$ *and* $\operatorname{coker}(D^+)$ *are zero. In particular, if $\kappa > 4\mathcal{R}_0$, then* $\{\operatorname{ch} E\cdot \hat{\mathbf{A}}(X)\}[X] = 0$.

2. The higher $\hat{\mathbf{A}}$-genus and families of Dirac operators

In this section we present an invariant for spin manifolds which is analogous to the Novikov higher signature. We show that this invariant arises as the index of a family of elliptic operators obtained by twisting the fundamental spin complex with a family of flat hermitian line bundles. The results of the previous section will imply that if there exists a metric with $\kappa > 0$, this invariant must vanish.

We point out that a more elementary proof of stronger results will be given in the next section. The argument presented below has, in our opinion, some independent methodological interest. However, the reader unfamiliar with the Index Theorem for families can skip this section.

Our constructions here closely follow those of Lusztig [14], so our presentation will be brief.

Let X be a compact spin manifold of dimension $2n$. The *higher-$\hat{A}$-genus* is an element $\hat{\mathfrak{A}} \in \Lambda^*\operatorname{Hom}(H^1(X;\mathbf{Z}),\mathbf{Z})$ given as follows. Let $x_1,\cdots,x_N$ be a basis of $H^1(X;\mathbf{Z})$ and let $x_1^*,\cdots,x_N^*$ be the dual basis in $\operatorname{Hom}(H^1(X;\mathbf{Z}),\mathbf{Z})$. For each multi-index $I=\{i_1,\cdots,i_p\}$ where $i_1<\cdots<i_p$ and $2n-p\equiv 0(4)$, let $X_I\subset X$ be a compact submanifold with trivial normal bundle dual to the cohomology class $x_{i_1}\cup\cdots\cup x_{i_p}$. Let $\hat{A}(X_I)\in\mathbf{Z}$ be its $\hat{A}$-genus. Then

$$\hat{\mathfrak{A}}(X) = \textstyle\sum_I \hat{A}(X_I)x_I^* \,. \tag{2.1}$$

Note that $\hat{\mathfrak{A}}(X)$ is a topological invariant. It is furthermore an invariant of spin cobordisms which preserve the fundamental group.

Suppose now that $T = V/\Gamma$ is an r-dimensional torus and let $T^* = V^*/\Gamma^*$ be the dual torus. Let E_0 be the hermitian line bundle over $T\times T^*$ given as the quotient of $V\times V^*\times \mathbf{C}$ by the action of $\Gamma\times\Gamma^*$ which associates to (γ, γ^*) the transformation $(v, v^*, z)\to(v+\gamma, v^*+\gamma^*, e^{2\pi i v^*(\gamma)}z)$. This bundle is flat on the factors $T\times\{v^*\}$. A straightforward computation (see [14]) shows that

$$c_1(E_0) = \omega \cong \mathrm{Id.} \in \mathrm{Hom}(\Gamma, \Gamma) \cong \Gamma^* \otimes \Gamma \cong H^1(T; Z) \otimes H^1(T^*; Z) \subset H^2(T\times T^*; Z) .$$

If $x_1, \cdots, x_n$ is a basis for Γ^* and $x_1^*, \cdots, x_n^*$ a dual basis for Γ, then

$$\frac{\omega^k}{k!} = (-1)^{k(k+1)/2} \sum\nolimits_{|I|=k} x_I x_I^* . \tag{2.2}$$

Consider now a compact riemannian spin manifold X of dimension $2n$ and suppose $f\colon X\to T$ is any smooth map. Then $E = (f\times \mathrm{id.})^*(E_0)$ is a hermitian line bundle over $X\times T^*$ which carries a canonical flat connection on each of the factors $X\times\{v^*\}$. The construction presented at the end of Section 1 gives a family of twisted spin bundles $S(X)\otimes E$, and a family of elliptic operators $D^+\colon \Gamma(S^+)\to\Gamma(S^-)$ parameterized by the torus T^*.

Applying [3] we see that the index of the family is

$$\hat{\mathfrak{A}}_f(X) = \{\mathrm{ch}(E)\cdot\hat{\mathbf{A}}(X)\}[X] \in H^{\mathrm{even}}(T^*; \mathbf{Z}) .$$

To simplify matters one considers the universal case where

$$T = \mathrm{Alb}(X) = H_1(X; \mathbf{R})/H_1(X; \mathbf{Z})_{\mathrm{mod\ torsion}}$$

is the Albanese variety of X and $f\colon X\to \mathrm{Alb}(X)$ is the canonical map(defined up to homotopy). The dual torus $\mathrm{Alb}(X)^* \equiv \mathrm{Pic}(X)$ is called the Picard variety of X. There is a natual isomorphism

$$H^*(\mathrm{Pic}(X); \mathbf{Z}) \cong \Lambda^*\mathrm{Hom}(H^1(X; \mathbf{Z}), \mathbf{Z}) \cong \Lambda^*(H_1(X; \mathbf{Z})/\mathrm{torsion}) .$$

An easy computation using (2.2) shows that the index of this family of operators over $\mathrm{Pic}(X)$ is exactly the higher $\hat{A}$-genus of X. Given any map $f\colon X\to T$, one has that $\hat{\mathfrak{A}}_f(X) = f_*\hat{\mathfrak{A}}(X)$.

Suppose now that X admits a metric of positive scalar curvature, and consider the family of operators $D^+\colon\Gamma(S^+)\to\Gamma(S^-)$ over $\mathrm{Pic}(X)$ constructed above. Since E is flat on each fibre, we conclude from Theorem 1.4 that the kernel and cokernel of each operator in this family is zero. This implies that the index of the family is zero. We have therefore proved the following result.

THEOREM 2.1. *Let X be a compact spin manifold of even dimension. If X admits a metric of positive scalar curvature, then*

$$\hat{\mathfrak{A}}(X) = 0 .$$

COROLLARY 2.2. *Let X be a compact spin manifold of any dimension n. If X admits a map of positive degree onto the torus T^n, then X admits no metric of positive scalar curvature.*

Proof. If n is even, apply Theorem 2.1. (Note that the component of $\hat{\mathfrak{A}}$ in degree n is not zero.) If n is odd, apply the theorem to $X \times S^1$.

Theorem 2.1 can be sharpened somewhat by applying the following results. The first theorem generalizes earlier work of J.P. Bourguignon.

THEOREM 2.3 (Kazdan and Warner [10]). *Let X be a compact riemannian manifold with scalar curvature $\kappa \geqq 0$. If X is not Ricci flat, then X carries a conformally equivalent metric with $\kappa > 0$.*

THEOREM 2.4 (Cheeger and Gromoll [5]). *Let X be a compact riemannian manifold which is Ricci flat. Then the universal covering $\tilde{X}$ of X splits as a riemannian product $\tilde{X} = E \times X_0$ where E is flat euclidean space and where X_0 is a compact, simply-connected (Ricci flat) manifold.*

Combining the results above gives the following general answer to a question of Kazdan and Warner. (The case $n \leqq 7$ has been proved by Schoen and Yau.)

COROLLARY 2.5. *Any riemannian metric of non-negative scalar curvature on the torus T^n is flat.*

Theorem 2.1 actually gives a family of results which "interpolate" between the Lichnerowicz Theorem and Corollary 2.2 above. To state these results, we introduce the following concept. Let X and Y be compact, connected, oriented manifolds, and consider a smooth map $f\colon X \to Y$. Then for any regular value p of f, the set $f^{-1}(p)$ is an oriented manifold whose oriented cobordism class is independent of the choice of p.

Definition 2.6. The *$\hat{A}$-degree* of the smooth map $f\colon X \to Y$ is the number $\hat{A}(f^{-1}(p))$ where p is any regular value of f. (We set $\hat{A}(\varnothing) = 0$.)

COROLLARY 2.7. *Let X be a compact spin manifold of any dimension n. Suppose X admits a smooth map $f\colon X \to T^{n-4k}$ of non-zero $\hat{A}$-degree. Then X carries no metric of positive scalar curvature.*

Note that Corollary 2.2 implies that any manifold of the form

$$X = T^n \# Y ,$$

where Y is a compact spin manifold, cannot carry a metric with $\kappa > 0$. In fact, if $T \# Y$ is not diffeomorphic to the standard torus, $T \# Y$ cannot carry a metric with $\kappa \geqq 0$.

Note also that from Corollary 2.7 any compact manifold of the form

$$X = T^n \times Y$$

where Y is spin and $\hat{A}(Y) \neq 0$, cannot carry a metric with $\kappa > 0$. In fact any metric with $\kappa \geqq 0$ is a twisted riemannian product of a flat metric on T^n with a Ricci flat metric on Y.

3. An untwisting trick

In this section we shall present a different proof of the results of Section 2. This proof generalizes to a substantially larger class of manifolds.

The basic idea here is that a riemannian manifold X with $\kappa > 0$ cannot be too large in the following specific sense. Let S^m denote the euclidean m-sphere of curvature 1. A smooth map $f: X \to S^m$ is *c-contracting* (for some $c > 0$) if $||f_* V|| \leqq c$ for all unit tangent vectors V on X.

PROPOSITION 3.1. *For each constant $\kappa_0 > 0$ and each $n \in \mathbf{Z}^+$ there is a constant $c = c(\kappa_0, n) > 0$ with the following property. Let X be any compact riemannian spin manifold of dimension $2n$ with $\kappa \geqq \kappa_0$. Then there exist no c-contracting maps $f: X \to S^{2n}$ of positive degree.*

More generally, for any such X of dimension $2n + 4k$ there exist no c-contracting maps $f: X \to S^{2n}$ of non-zero $\hat{A}$-degree. (Here c depends also on k.)

Proof. Suppose $f: X \to S^{2n}$ is a smooth map and set

$$c(f) = \sup\{||f_* V||: V \text{ is a unit tangent vector on } X\}\ .$$

Let E_0 be a hermitian vector bundle over S^{2n} such that $c_n(E_0) \neq 0$. Fix a hermitian connection in E_0 and let R^{E_0} denote its curvature tensor. Let $E = f^* E_0$ and give E the induced connection. We now consider the twisted spin bundle $S(X) \otimes E$ with the tensor product connection, and we construct the associated elliptic complex $D^+: \Gamma(S^+) \to \Gamma(S^-)$ presented in Section 1. We wish to apply Theorem 1.4. Let $\mathcal{R}_0$ be the curvature expression given by (1.3) and set $||\mathcal{R}_0|| = \sup\{\langle \mathcal{R}_0(\varphi), \varphi \rangle: ||\varphi|| = 1\}$. It is clear from (1.3) that there is a constant α_0 depending only on dimension such that $||\mathcal{R}_0|| \leqq \alpha_0 ||R^E||$. However, since the connection on E was induced from the one on E_0, we have $R^E_{v,w} = R^{E_0}_{f_* v, f_* w}$ (under the obvious identification of the fibres over x and $f(x)$). Hence, $||R^E|| \leqq c^2 ||R^{E_0}||$. It follows that

$$||\mathcal{R}_0|| \leqq \alpha c^2$$

where α depends only on the data fixed on S^{2n}.

Now since $\kappa \geqq \kappa_0$, we have from Theorem 1.4 that if $c < \sqrt{\kappa_0/\alpha}$, then $\{\mathrm{ch}(E) \cdot \hat{\mathbf{A}}(X)\}[X] = 0$. Now $\mathrm{ch}(E) = 1 + \omega$ where

$$\omega = \frac{1}{(n-1)!} f^* c_n(E_0) \ .$$

Since $\kappa > 0$, we have $\hat{A}(X) = 0$, and therefore $\{\omega \cdot \hat{\mathrm{A}}(X)\}[X] \cong \hat{A}(f^{-1}(p)) = 0$. This completes the proof.

Note that the condition $\kappa \geqq \kappa_0$ is local and is preserved under the process of taking coverings, i.e., of "unwrapping" the manifold. If there exist unwrappings of the manifold which are uniformly large in all directions, then there exist contracting maps to the sphere. A fundamental collection of manifolds with this property consists of certain manifolds with non-positive sectional curvature.

Definition 3.2. A group π is said to be *residually finite* if

$$\bigcap \{N \colon N \lhd \pi \text{ and } |\pi/N| < \infty\} = \{1\} \ .$$

PROPOSITION 3.3. *Let X be a compact riemannian n-manifold of non-positive sectional curvature such that $\pi_1(X)$ is residually finite. Then given any $c > 0$ there exists a finite covering $X' \to X$ such that X' admits a c-contracting map to S^n of degree* 1.

Proof. Fix a point x in the universal covering $\tilde{X}$ of X and consider the diffeomorphism $e^{-1} \colon \tilde{X} \to T_x(\tilde{X}) \cong R^n$ where e is the exponential map. The theory of Jacobi fields shows that e^{-1} is everywhere 1-contracting. We choose a degree-1 map $\phi \colon R^n \to S^n$ which is constant outside the euclidean ball B_1 of radius 1. Then there is a constant $\alpha > 0$ such that for all $r > 0$, the map $\phi_r \colon \tilde{X} \to S^n$, given by $\phi_r(x) = \phi(r \cdot e^{-1}(x))$, is αr-contracting and constant outside $e(B_r)$.

Let $F \subset \tilde{X}$ be a fundamental domain for the action of $\pi = \pi_1(X)$ on $\tilde{X}$. Then for each r there is a finite set of elements $g_1, \cdots, g_N \in \pi$ such that $e(B_r) \subset \bigcup_{i=1}^{N} g_i(F)$. Since π is residually finite, there exists a subgroup $\pi' \subset \pi$ of finite index, such that $g_i \notin \pi'$ for any i. Let $X' \to X$ be the finite covering corresponding to π'. Then there is a fundamental domain $F' \subset \tilde{X}$ for the action of π' such that $e(B_r) \subset$ interior (F). The map ϕ_r now descends to an αr-contracting map $\phi'_r \colon X' \to S^n$ of degree 1. This completes the proof.

This last result can be substantially generalized.

Definition 3.4. A compact riemannian manifold Y is said to be *enlargeable in dimension n* if for each constant $c > 0$, there exists a finite covering $Y' \to Y$ such that Y' is spin and Y' admits a c-contracting map $f \colon Y' \to S^n$ of non-zero $\hat{A}$-degree. If $n = \dim Y$, Y is simply called *enlargeable*.

Definition 3.5. A smooth map $f \colon X \to Y$ between connected manifolds is called a *spin map* if $w_2(X) = kf^* w_2(Y)$ for $k = 0$ or 1.

We now have the following.

PROPOSITION 3.6. *Let Y be a compact, orientable, n-dimensional manifold which is enlargeable. Suppose X is any compact riemannian manifold which admits a spin map $f: X \to Y$ of non-zero $\hat{A}$-degree. Then X is enlargeable in dimension n.*

Proof. Let $\alpha = \sup\{\|f_* v\|: v \in TX \text{ and } \|v\| = 1\}$. Given $c > 0$, let $q: Y' \to Y$ be the covering such that Y' is spin and admits a (c/α)-contracting map $F: Y' \to S^n$ of non-zero degree. Let $p: X' \to X$ be the covering corresponding to the subgroup $(f_*)^{-1}(q_*\pi_1(Y'))$. Then there is a map $f': X' \to Y'$ such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. X' is spin since

$$w_2(X') = p^* w_2(X) = kp^* f^* w_2(Y) = kf'^* q^* w_2(Y) = kf'^* w_2(Y') = 0 .$$

The map $F' = F \circ f'$ is clearly c-contracting and has non-zero $\hat{A}$-degree. This completes the proof.

Combining the results above gives our main result.

THEOREM 3.7. *A compact manifold which is enlargeable in some dimension $n \geqq 0$ cannot carry a metric of positive scalar curvature.*

In particular, we have from Proposition 3.3 the following result.

THEOREM 3.8. *Let Y be a compact riemannian n-manifold with sectional curvatures $\leqq 0$ and such that $\pi_1(Y)$ is residually finite and some finite covering is spin. Then any compact manifold X which admits a spin map $f: X \to Y$ of non-vanishing $\hat{A}$-degree, carries no (non-flat) riemannian metric with scalar curvature $\kappa \geqq 0$.*

Note. If n is odd, the above theorem is proved by taking products with S^1.

COROLLARY 3.9. *Let Y_0 be homeomorphic to a locally homogeneous space $\Gamma \backslash G/H$ of non-positive sectional curvature, and let Y_1 be any compact spin manifold of the same dimension. Suppose some finite covering of X is spin. Then the connected sum*

$$X = Y_0 \# Y_1$$

carries no metric of positive scalar curvature.

The same conclusion holds for $Y_0 \times Y_2$ where Y_2 is any compact spin manifold with $\hat{A}(Y_2) \neq 0$.

That Γ is residually finite follows from Selberg [18]. We note that if Y_0 is hyperbolic, then there is always some finite covering which is spin [20]. This may be true of all manifolds considered in Corollary 3.9.

In the second part of this paper we shall prove Theorem 3.8 without the hypotheses of spin and residual finiteness.

4. Generalizations to solvmanifolds

The arguments of the last section can be applied to manifolds other than those of non-positive curvature. The key concept in these arguments is that of enlargeability. Let $B^n(r)$ denote the euclidean ball of radius r. Also for a smooth map $e: B^n(r) \to X$, where X is riemannian, we set

$$|||e||| = \inf\{||e_*(v)||: v \in TB^n(r) \text{ and } ||v|| = 1\} .$$

The arguments used to establish Proposition 3.3 show the following.

PROPOSITION 4.1. *Suppose X is a compact riemannian n-manifold such that:*

(1) $\pi_1 X$ is residually finite.

(2) Some finite covering of X is spin.

(3) For each $r > 0$, there is a smooth embedding $e_r: B^n(r) \to \tilde{X} =$ the universal covering manifold of X, such that $|||e_r||| \geqq 1$.
Then X is enlargeable.

Definition 4.2. A compact manifold which satisfies the hypothesis (3) of Proposition 4.1 will be called *expandable*.

Note that the condition of expandability is independent of the metric on the manifold X. This is seen as follows. Suppose that for some metric on X we have the family of mappings $e_r: B^n(r) \hookrightarrow \tilde{X}$ with $|||e_r||| \geqq 1$. Given another metric on X, we have $|||e_r|||' \geqq 1/\alpha$ for some $\alpha > 0$ (independent of r). We now replace e_r by $e'_r(x) = e_{\alpha r}(\alpha x)$. Then $|||e'_r|||' = \alpha |||e'_{\alpha r}||| \geqq 1$. Thus the notion of expandability is a property of the diffeomorphism class of X.

Note that an expandable manifold is a $K(\pi, 1)$.

PROPOSITION 4.3. *Suppose X_0 and X_1 are compact expandable manifolds and let Y be a manifold of the form*

$$Y = X_1 \times_\rho X_0$$

where $\rho: \pi_1(X) \to \mathrm{Diff}(X_0)$ is a homomorphism. Then Y is expandable.

Proof. Let $\tilde{X}_k$ denote the universal covering of X_k and let $n_k = \dim(X_k)$

for $k = 0, 1$. Now $Y = \tilde{X}_1 \times X_0/\pi_1(X_1)$ where $\pi_1(X_1)$ acts jointly by deck transformations on the left and by the representation ρ on the right. Clearly Y is a fibre bundle over X_1 with fibre X_0 and with a foliation $\mathcal{F}$ of dimension n_1 (coming from the product structure above) transverse to the fibres.

We introduce a metric on Y with the property that the tangent spaces to the fibres are perpendicular to $\mathcal{F}$ and such that the metric on $T\mathcal{F}$ is lifted from some metric g_1 on X_1. (This is always possible.) Then on $\tilde{X}_1 \times X_0$ this metric has the form

$$g = \tilde{g}_1 \oplus g_0$$

where $\tilde{g}_1$ is the lift of g_1 to $\tilde{X}_1$ and where g_0 is a family of metrics on X_0 parameterized by points of $\tilde{X}_1$.

Fix $r > 0$ and let $e_{1,r}: B^{n_1}(r) \hookrightarrow \tilde{X}_1$ be an embedding such that $|||e_{1,r}||| \geqq 1$. Set $z = e_{1,r}(0)$. Then there is a constant $\alpha > 0$ such that $(g_0)_{z'} \geqq (1/\alpha^2)(g_0)_z$ for all $z' \in \text{image}(e_{1,r})$. Choose $e_{0,\alpha r}: B^{n_0}(\alpha r) \to \tilde{X}_0$ such that $|||e_{0,\alpha r}||| \geqq 1$ in the metric $(\tilde{g}_0)_z$, and define $e_{0,r}: B^{n_0}(r) \hookrightarrow \tilde{X}_0$ by $e_{0,r}(x) = e_{0,\alpha r}(\alpha x)$. Then $|||e_{0,r}||| \geqq 1$ in each metric $(\tilde{g}_0)_{z'}$ for $z' \in \text{image}(e_{1,r})$. It follows easily that

$$e_r = (e_{1,r}, e_{0,r}): B^{n_1}(r) \times B^{n_0}(r) \longrightarrow \tilde{X}_1 \times \tilde{X}_0$$

satisfies $|||e_r||| \geqq 1$. This proves the proposition.

COROLLARY 4.4. *Any compact solvmanifold is enlargeable.*

Proof. A compact solvmanifold X is parallelizable and has residually finite fundamental group. Furthermore, there is a fibration $X \to S^1$ whose fibre is again a compact solvmanifold. Hence, by induction X is expandable.

THEOREM 4.5. *Let X be a compact spin manifold which admits a map of non-zero $\hat{A}$-degree onto a compact solvmanifold. Then X carries no metric of positive scalar curvature.*

5. Some constructions of manifolds with positive scalar curvature

As mentioned in the introduction, any manifold which admits a non-trivial S^3-action carries a metric with $\kappa > 0$ [11]. This covers most examples known to date. We present here some other constructions of metrics with $\kappa > 0$ which will prove useful in the second part of this paper. We begin with a simple case to illustrate the ideas.

PROPOSITION 5.1. *The manifold $(S^1 \times S^2)\# \cdots \#(S^1 \times S^2)$ carries a metric of positive scalar curvature.*

Proof. Consider the domain $D \subset \mathbf{R}^2$ given in Figure 1. (All curves are euclidean circles.)

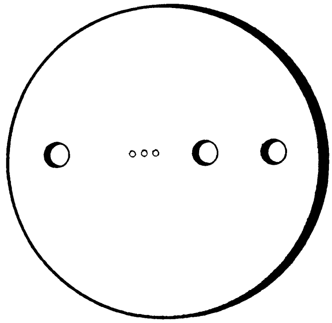

FIGURE 1

Embed $\mathbf{R}^2 \subset \mathbf{R}^4$ linearly and let Ω be an ε-neighborhood of $D \subset \mathbf{R}^4$ where ε is small. Then $X \equiv \partial\Omega \cong (S^1 \times S^2)\# \cdots \#(S^1 \times S^2)$, and X has circular "C^2 creases" which we smooth in a circularly symmetric way.

We claim that for ε sufficiently small, the metric induced from the euclidean metric has $\kappa \geqq 0$ and $\kappa_x > 0$ for some x. (By a theorem of Kazdan and Warner [10], one can then change the metric conformally to one with $\kappa > 0$.)

The only points that need to be checked are the non-convex points, i.e., the pieces of X near the inner circles of ∂D. Here the surface can be generated explicitly by rotation in the (x_1, x_4) plane of the surface $\Sigma = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3\colon \text{dist.}(x, l) = \varepsilon\}$ where $l = \{(x, 0, 0)\colon x \geqq 1\}$.

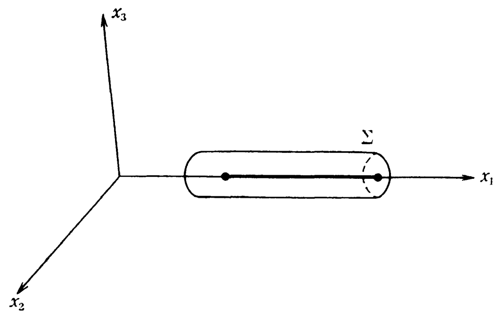

FIGURE 2

The surface Σ is itself obtained by rotating a curve $\gamma \subset \mathbf{R}^2$.

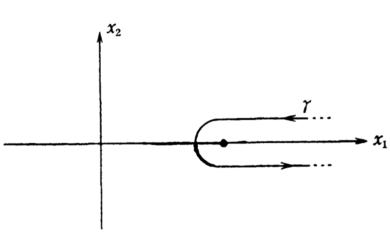

FIGURE 3

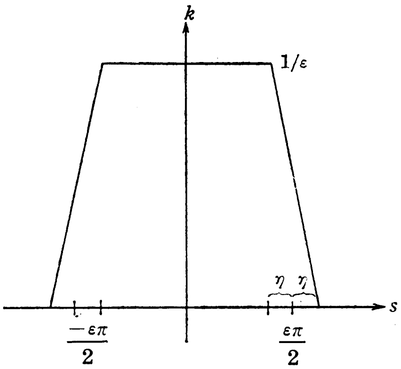

FIGURE 4

This curve must be smoothed at the points $(1, \pm\varepsilon)$. This can be done explictly as follows. Let $k(s)$ be the even function given in Figure 4, where $0 < \eta \ll \varepsilon$. Then $k(s)$ is the curvature function of a unique curve $\gamma(s)$, with arc length s, such that $\gamma(0) = (1 - \varepsilon, 0)$ and $\gamma'(0) = (0, -1)$. Since $\int k ds = \pi$, we see that γ is the desired C^2 smoothing.

By symmetry it will suffice to compute the scalar curvature at points of the hypersurface along the curve $\gamma(s)$, $s \geqq 0$, in the (x_1, x_2)-plane. At such a point $x = (x_1, x_2, 0, 0)$, the principal curvatures of the hypersurface are k_1, k_2, and $-(1/x_1)\cos\theta$, where k_1 and k_2 are the principal curvatures of $\Sigma \subset R^3$ and where θ is the angle between the normal to γ and the x_1-axis. Now for $0 \leqq s \leqq \varepsilon\pi/2 - \eta$ we have $k_1 = k_2 = 1/\varepsilon$ and $x_1 \approx 1$. Hence, the scalar curvature $\kappa = 4/\varepsilon^2 - 2\cos\theta/x_1 > 0$. For $s \geqq \varepsilon\pi/2 + \eta$, one sees easily that $\kappa = 0$. For $s \in I = [\varepsilon\pi/2 - \eta, \varepsilon\pi/2 + \eta]$ we have that $k_1 = k$, $k_2 \approx 1/\varepsilon$ and $x_1 \approx 1$. In particular, $2/\varepsilon > k_2 > 1/2\varepsilon$ and $x_1 > 1/2$. It follows that in this region,

$$\begin{aligned}\eta &= 2k_1k_2 - 2(k_1 + k_2)\cos\theta/x_1 \\ &\geqq \frac{k}{\varepsilon} - 4(k + 2/\varepsilon)\cos\theta\ .\end{aligned}$$

Let $\gamma' \equiv T = (a, b)$. Then $T' = kN$ where $N = (-b, a)$ is the unit normal to γ, and $\cos\theta = b > 0$. We are interested in the function $f(s) = k(s)/\varepsilon - 4[k(s) + 2/\varepsilon]b(s)$. This function is > 0 at the left endpoint of I, $=0$ at the right endpoint, and is differentiable in I^0. Now in I we have: $k' = -1/2\varepsilon\eta$, $b' = ka$, $0 \leqq k \leqq 1/\varepsilon$, and $a^2 + b^2 = 1$. Therefore,

$$\begin{aligned}f' &= k'(1/\varepsilon - 4b) - 4(k + 2/\varepsilon)ka \\ &= (1/2\varepsilon\eta)(1/\varepsilon - 4b) + 0(1/\varepsilon^2) < 0\ .\end{aligned}$$

Hence $\kappa \geqq f \geqq 0$ on I. This completes the proof.

REMARK 5.2. The above construction applied to the plane figure:

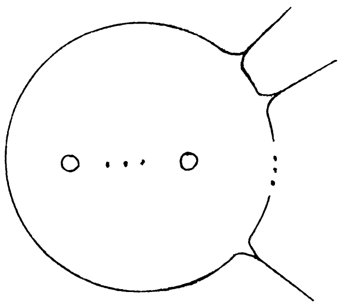

FIGURE 5

gives a metric on $(S^1 \times S^2)\#\cdots\#(S^1\times S^2)$, punctured at k points, which has $\kappa \geqq 0$ and is the product metric $\mathbf{R} \times S^2$ on the ends.

PROPOSITION 5.3. *Let X be a compact n-manifold, $n \geqq 3$, of constant positive sectional curvature. Then X, punctured at a finite number of points, carries a metric which has $\kappa > 0$ and which, on each end $\mathbf{R} \times S^{n-1}$, is the standard product metric.* (*The radius of the sphere on each end can be arbitrarily prescribed.*)

Proof. The construction is local. We consider a piece of the euclidean sphere S^n embedded in the usual way in $\mathbf{R}^{n+1}$. Consider a point p close to S^n and on the "outside". Let Σ^n be the boundary of the convex hull of $S^n \cup \{p\}$. (We change S^n by replacing a small cap with a cone of tangent segments from p.)

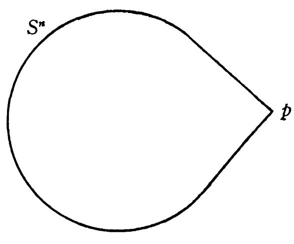

FIGURE 6

The C^2 "crease" along the small sphere $S^{n-1}_\varepsilon \subset S^n$ can be locally smoothed so that the metric has $\kappa > 0$.

We now observe that the cone can be pulled out to a cylinder preserving positive scalar curvature. The cone is obtained by rotating a line segment l, in the (x_0, x_1)-plane, about the x_0-axis in $\mathbf{R}^{n+1}$. We may renormalize the picture by a homothety so that l is the segment joining $(0, 1)$ to $(\varepsilon, 0)$. We replace l with a bent segment l'. The curvature

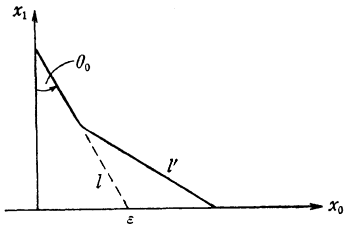

FIGURE 7

of the segment is a non-negative function $k \in C_0^\infty(\mathbf{R}^+)$ whose integral is the total bend. The resulting manifold of rotation will have scalar curvature

$$\kappa = \frac{(n-1)}{x_1}\sin\theta\left[\frac{(n-2)}{x_1}\sin\theta - 2k\right]$$

where θ is the angle of the normal with the x_0-axis. Hence we may bend through some angle β while preserving $\kappa > 0$.

The vertex of our new figure is again a cone. We renormalize and repeat the procedure. Since the cone is less acute, we may bend through an angle $\geqq \beta$. In a finite number of steps we arrive at the cylinder. This completes the proof.

Combining the above results gives the following.

THEOREM 5.4. *Any 3-manifold of the form*

$$M^3 = (S^3/\Gamma_1)\,\#\cdots\#\,(S^3/\Gamma_n)\#(S^1\times S^2)\,\#\cdots\#\,(S^1\times S^2)\ ,$$

where each Γ_i *is a finite subgroup of* SO_4 *acting freely and orthogonally on* S^3, *carries a metric of positive scalar curvature.*

The construction in Proposition 5.1 can be considerably generalized.

THEOREM 5.5. *Let* X_0 *be any compact manifold with non-zero boundary. Then the manifold* $X = \partial(X_0 \times D^2)$ *carries a metric of positive scalar curvature.*

COROLLARY 5.6. *Let* X_0 *be any compact manifold without boundary. Then the manifold obtained from* $X_0 \times S^1$ *by surgery on a finite number of circles* $\{p_1\}\times S^1, \cdots, \{p_n\}\times S^1$ *carries a metric of positive scalar curvature.*

Proof of Theorem 5.5. Introduce on X_0 a metric of positive sectional curvature (cf. [7]) and consider the riemannian product $X_0 \times D^2$ with the flat 2-disk. Let $X_1 \subset X_0$ be the complement of a thin collar of ∂X_0. Then for ε sufficiently small, the boundary of $\{p \in X_0 \times D^2\colon \mathrm{dist}(p, X_1) \leqq \varepsilon\}$ will have $\kappa > 0$ where this manifold is of class C^2. Examination of the non-C^2 points shows that they can be smoothed while preserving the condition $\kappa > 0$. The argument is a straightforward application of the Gauss-curvature formulas for a hypersurface.

THEOREM 5.7. *Let X be a compact riemannian manifold with boundary such that*:

(i) $\kappa > 0$ *on* X;

(ii) $H > 0$ *on* ∂X *where* H *is the mean curvature of* ∂X *with respect to the exterior normal.*

Then the double of X *carries a metric with* $\kappa > 0$.

Proof. Let $X_1 = X \backslash C$ where C is a thin collar of ∂X. We can choose X_1 so that the mean curvature H of ∂X_1 is still positive. We then consider the riemannian product $X \times I$ and define $D(X) = \{p \in X \times I\colon \text{dist.}(p, X_1) = \varepsilon\}$ where $0 < \varepsilon \ll 1$. This manifold is homeomorphic to the double of X. It contains certain obvious C^2 creases. We claim that for ε sufficiently small, these creases can be smoothed in $X \times I$ so that the induced metric has $\kappa > 0$.

On the regions of $D(X)$ which are parallel to X in $X \times I$ we clearly have $\kappa > 0$. The difficulty comes at the bending points.

Fix $x \in \partial X_1$ and let σ be the geodesic segment in X_1 emanating orthogonally from ∂X_1 at x. Then $\sigma \times I$ is totally geodesic in $X \times I$. Let $\gamma = (\sigma \times I) \cup D(X)$ be the intersection of this surface with $D(X)$. It will be of the form pictured below.

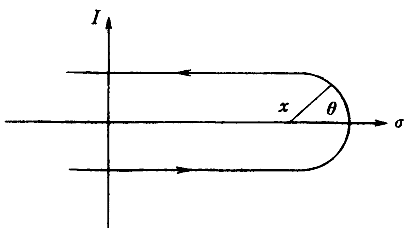

FIGURE 8

Let $\mu_1, \cdots, \mu_n$ be the principal curvatures of ∂X_1 at x. At a point corresponding to angle θ (see Figure 8), the principal curvatures of $D(X)$ will be of the form $\lambda_0 = (1/\varepsilon)\cos\theta + O(\varepsilon)$ and $\lambda_k = \big(\mu_k + O(\varepsilon)\big)\cos\theta + O(\varepsilon^2)$ for $k = 1, \cdots, n$. It follows from the Gauss curvature equation, that the scalar curvature κ of $D(X)$ is of the form:

$$\kappa = \kappa_X + \Big(\frac{2}{\varepsilon} H + O(1)\Big)\cos^2\theta + O(\varepsilon)$$

where κ_X is the scalar curvature function of X (and of $X \times I$). For θ close to $\pi/2$ one can construct a C^2 smoothing of $D(X)$ so that the condition $\kappa > 0$

is preserved. This completes the proof.

REMARK 5.8. It is clear from the proof that in Theorem 5.7 we can choose the metric on $D(X)$ to agree with the one on X outside a neighborhood of $\partial X \subset D(X)$. Moreover, we may assume that if the metric on X satisfies $\kappa \geqq \kappa_0 > 0$, then this metric on $D(X)$ satisfies $\kappa \geqq \kappa_0/2$.

6. Some results for 3-manifolds

The techniques above can be applied to a class of $K(\pi, 1)$-manifolds which includes many manifolds of dimension 3.

THEOREM 6.1. *Let X be a compact $K(\pi, 1)$-manifold of dimension n such that π is residually finite and some finite covering of X is spin. Suppose there exists a compact expandable $(n-1)$-manifold X_0 and a map $\varphi: X_0 \to X$ such that $\varphi_*: \pi_1(X_0) \to \pi_1(X)$ is injective. Then X is an enlargeable manifold. In particular, no compact manifold which admits a spin map of non-vanishing $\hat{A}$-degree onto X can carry a metric of positive scalar curvature.*

Proof. Let $X\hat{} \to X$ be the covering space corresponding to the subgroup $\pi\hat{} = \varphi_*(\pi_1 X_0) \subset \pi_1(X)$. Then the map φ lifts to a map $\varphi\hat{}: X_0 \to X\hat{}$ which is a homotopy equivalence. In particular, there is a map $\rho: X\hat{} \to X_0$ such that $\rho \circ \varphi\hat{}$ is homotopic to the identity map on X_0.

Fix a metric on X and lift it to $X\hat{}$. By passing to a finite cover, we may assume X_0 and $X\hat{}$ are oriented. Then $X\hat{}$ is a complete manifold with two ends. We replace $\varphi\hat{}(X_0)$ with a compact oriented embedded hypersurface $H \subset X\hat{}$ which is homologous to $\varphi\hat{}(X_0)$. Since the class $[\varphi\hat{}(X_0)] \in H_{n-1}(X\hat{}; Z)$ is a generator, we may assume H is connected. Then $X\hat{}\text{-}H$ has two connected components X^+ and X^-. We define $d: X\hat{} \to R$ by

$$d(x) = \begin{cases} -\text{dist.}\,(x, H) & \text{if } x \in \bar{X}^- \\ \text{dist.}\,(x, H) & \text{if } x \in \bar{X}^+ . \end{cases}$$

Since $X\hat{}$ is complete, the function d is proper. For each $r > 0$ we set $X\hat{}_r = d^{-1}([-r, r])$ and define

$$\|\rho_*\|_r = \sup\{\|\rho_* V\|: V \text{ a unit tangent vector on } X\hat{}_r\} .$$

Let $\tilde{X}$, $\tilde{X}_0$ denote the universal covering manifolds of X and X_0, and consider the map

$$F: \tilde{X} \longrightarrow \tilde{X}_0 \times \mathbf{R}$$

given by $F(x) = (\tilde{\rho}(X), d(x))$ where $\tilde{\rho}$ is the lifting of ρ. This map is proper and of positive degree in the following sense. Let $B \subset \tilde{X}_0$ be an embedded

$(n-1)$-ball and consider the map $s\colon \tilde{X}_0 \times \mathbf{R} \to S^n$ given by the composition:

$$\tilde{X}_0 \times \mathbf{R} \longrightarrow (\tilde{X}_0/(\tilde{X}_0 - B)) \times (\mathbf{R}/(\mathbf{R}-[-1, 1])) = S^{n-1} \times S^1 \longrightarrow S^{n-1} \wedge S^1 = S^n .$$

Then $s \circ F\colon \tilde{X} \to S^n$ is constant outside a compact set and thereby defines maps $s \circ F\colon X' \to S^n$ on sufficiently large compact orientable coverings X' of X. (Recall that $\pi_1(X)$ is residually finite.) These maps will be of positive degree. This can be seen by a straightforward homology argument.

It remains to show that for any given $c > 0$, we can construct such a map which is c-contracting. Choose $r > 0$ so that $1/r < c$ and let $R = ||\rho_*||_r \cdot c^{-1}$. Since X is expandable there exists an embedding $e\colon B^{n-1}(2R) \to \tilde{X}_0$ such that $|||e||| \geqq 1$. This gives a map $s_0\colon \tilde{X}_0 \to S^{n-1}$ which is constant outside $e(B^{n-1}(2R))$ and is $||\rho_*||_r^{-1} \cdot c$-contracting. Let $s_1\colon \mathbf{R} \to S^1$ be given by

$$\mathbf{R} \xrightarrow{1/r} \mathbf{R} \longrightarrow \mathbf{R}/(\mathbf{R}-[-1, 1]) .$$

Then $s\colon \tilde{X}_0 \times \mathbf{R} \to S^n$ is defined by the composition

$$\tilde{X}_0 \times \mathbf{R} \xrightarrow{(s_0, s_1)} S^{n-1} \times S^1 \longrightarrow S^{n-1} \wedge S^1 .$$

One can easily see that the map $s \circ F$ is c-contracting. This completes the proof.

This theorem has the following immediate consequence (viz. Schoen and Yau [16]).

THEOREM 6.2. *Let X be a compact 3-manifold of the form $X = X_0 \# X_1$ where X_1 is a $K(\pi, 1)$. If π is residually finite and contains an infinite subgroup which is isomorphic to the fundamental group of a compact surface, then X carries no metric of positive scalar curvature. In fact any metric of non-negative scalar curvature on X is flat.*

In the second part of this paper we shall remove the restrictions on π in Theorem 6.2.

Recall that any compact orientable 3-manifold X can be decomposed as a connected sum

$$X = \Sigma_1 \# \cdots \# \Sigma_n \# k \cdot (S^1 \times S^2) \# K_1 \# \cdots \# K_m$$

where each Σ_i is covered by a homotopy 3-sphere and each K_i is a $K(\pi, 1)$ manifold [8]. It is a question whether each Σ_i is of the form S^3/Γ where $\Gamma \subset \mathrm{SO}_4$ is a finite group whose orthogonal action on S^3 is free. If so, this will give a classification of 3-manifolds of positive scalar curvature.

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