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# The classification of simply connected manifolds of positive scalar curvature

By MIKHAEL GROMOV and H. BLAINE LAWSON, JR.

## 0. Introduction

It is a well-known theorem of Lichnerowicz [7] that a compact spin manifold with non-vanishing $\hat{A}$-genus cannot carry a riemannian metric of positive scalar curvature. This result was generalized by N. Hitchin [6] as follows. There is a ring homomorphism

$$\hat{\mathcal{A}}: \Omega_*^{\text{Spin}} \longrightarrow KO_*(\text{pt.})$$

defined by Milnor [9] which is surjective in each dimension and is the $\hat{A}$-genus in dimensions $4k$. Using the Atiyah-Singer Index Theorem in various forms, Hitchin showed that if $X$ is a compact spin manifold with a metric of positive scalar curvature, then $\hat{\mathcal{A}}(X) = 0$. This shows, for example, that certain exotic spheres (those which do not bound spin manifolds) in dimensions 1 and 2 (mod 8), cannot carry metrics of positive scalar curvature.

Similar non-existence theorems have recently been established for manifolds with "large" fundamental groups ([3], [4], [11], [12]).

In this paper we show that these negative results are nearly sharp. The basic result is the following.

THEOREM A. *Let $X$ be a compact manifold which carries a riemannian metric of positive scalar curvature. Then any manifold which can be obtained from $X$ by performing surgeries in codimension $\geqq 3$ also carries a metric with positive scalar curvature.*

*In particular, if $X_1$ and $X_2$ are compact $n$-manifolds, $n \geqq 3$, with positive scalar curvature, then their connected sum also carries positive scalar curvature. The same is true of the "connected sum" along embedded spheres with trivial normal bundles in codimension $\geqq 3$.*

This theorem subsumes several results in Section 5 of our previous paper [3]. The simple constructions in [3] provide a useful illustration of the more

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general arguments presented here.

Theorem A has several interesting consequences. (Note: In the following, "simply-connected" means $\pi_0$ and $\pi_1$ are both zero.)

THEOREM B. *Any compact simply-connected spin manifold $X$ of dimension $\geqq 5$ which is spin cobordant to a manifold of positive scalar curvature also carries positive scalar curvature.*

In particular we see that if $X$ is spin cobordant to zero, then the conclusion holds. For example, every such manifold of dimension 5, 6, or 7 carries a metric of positive scalar curvature.

As a consequence of Theorem B and the work in [2], we see that the simply-connected spin manifolds which carry positive scalar curvature are completely determined by their Stiefel-Whitney and KO characteristic numbers. More specifically, we let $P \subset \Omega_*^{\mathrm{Spin}}$ be the set of classes containing representatives with positive scalar curvature. Then $P$ is an ideal in the ring $\Omega_*^{\mathrm{Spin}}$, and the homomorphism

$$\Pi\colon \Omega_*^{\mathrm{Spin}} \longrightarrow \Omega_*^{\mathrm{Spin}}/P$$

is, in dimension $\geqq 5$, a complete set of invariants for the existence of metrics of positive scalar curvature on simply connected spin manifolds. We conjecture that $\Omega_*^{\mathrm{Spin}}/P \simeq \mathrm{KO}_*(\mathrm{pt.})$ and that the homomorphism $\Pi$ is exactly the KO characteristic homomorphism $\hat{\mathfrak{A}}$.

If we tensor with the rational numbers, this conjecture is true.

COROLLARY B. *The homomorphism $\Pi \otimes \mathbf{Q}$ is exactly the $\hat{A}$-genus. In particular, let $X$ be a compact simply-connected spin manifold of dimension $\geqq 5$ such that $\hat{A}(X) = 0$. Then some multiple $X \# \cdots \# X$ carries a metric of positive scalar curvature.*

*Remark.* The results above together with the work in [3], [4] indicate that a classification of general spin manifolds with positive scalar curvature can be expressed using cobordisms which preserve the fundamental group. A complete set of invariants should be found using the higher $\hat{A}$-genus (analogous to the generalized Novikov higher signature).

For manifolds which are not spin, we have the following result.

THEOREM C. *Let $X$ be a compact simply connected manifold of dimension $n \geqq 5$, which is not spin. If $X$ is oriented cobordant to a manifold which carries a metric of positive scalar curvature, then $X$ also carries such a metric.*

COROLLARY C. *Every compact simply-connected $n$-manifold, $n \geqq 5$, which is not spin, carries a metric of positive scalar curvature.*

It follows in particular that every simply-connected manifold of dimension 5, 6 or 7 carries a metric of positive scalar curvature.

We would like to thank R. Stong for valuable conversations concerning the generators of cobordism rings.

We have learned since writing this paper that R. Schoen and S. T. Yau have also proved Theorem A above. Their techniques are based on proving the existence of certain singular solutions to a natural partial differential equation.

## 1. Proof of Theorem A

For clarity we begin with the case of connected sums. Suppose $X$ has dimension $n \geqq 3$ and a metric with scalar curvature $\kappa > 0$. Fix $p \in X$ and let $D = \{x \in \mathbf{R}^n : ||x|| \leqq \bar{r}\}$ be a small normal coordinate ball centered at $p$. We shall change the metric in $D$-$\{0\}$, while preserving positive scalar curvature, so that it agrees with the old one near $\partial D$ and so that near 0 it is a riemannian product $\mathbf{R} \times S^{n-1}$ where $S^{n-1}$ is a euclidean sphere of some radius. This radius can be chosen arbitrarily, provided it is sufficiently small. It follows immediately that one can add 1-handles and take connected sums while preserving positive scalar curvature.

Recall that the metric in the normal coordinates on $D$ is obtained by considering $D = \{x_1 e_1 + \cdots + x_n e_n \in T_p X : ||x|| \leqq \bar{r}\}$, where $e_1, \cdots, e_n$ is an orthonormal basis, and pulling back the metric of $X$ via the exponential map. Let $r(x) = ||x||$ be the distance to the origin in $D$, and set $S^{n-1}(\rho) = \{x \in D : r(x) = \rho\}$.

LEMMA 1. *The principal curvatures of the hypersurface $S^{n-1}(\varepsilon)$ in $D$ are each of the form $-1/\varepsilon + O(\varepsilon)$ for $\varepsilon$ small. Furthermore, let $g_\varepsilon$ be the induced metric on $S^{n-1}(\varepsilon)$ and let $g_{0,\varepsilon}$ be the standard euclidean metric of curvature $1/\varepsilon^2$. Then, as $\varepsilon \to 0$, $(1/\varepsilon^2)g_\varepsilon \to (1/\varepsilon^2)g_{0,\varepsilon} = g_{0,1}$ in the $C^2$ topology.*

*Proof.* The metric on $D$ is of the form

$$g_{ij}(X) = \delta_{ij} + O(||x||^2)$$
$$= \delta_{ij} + \sum a_{ij}^{kl} x_k x_l + O(||x||^3) .$$

The corresponding Christoffel symbols can be written:

$$\Gamma_{ij}^k(x) = O(||x||) = \sum_l \gamma_{ij}^{k,l} x_l + O(||x||^2) .$$

Consider the curve $\gamma(s) = (\varepsilon \cos(s/\varepsilon), \varepsilon \sin(s/\varepsilon), 0, \cdots, 0)$ on $S^{n-1}(\varepsilon)$. Then the covariant derivative of the velocity field along the curve has components

$$\left(\frac{D}{ds}\frac{d\gamma}{ds}\right)^k = \frac{d^2\gamma^k}{ds^2} + \sum_{i,j} \Gamma_{ij}^k \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds} .$$

Hence, the second fundamental form of $S^{n-1}(\varepsilon)$, applied to the vector $(d\gamma/ds)(0)$, is:

$$\left\langle\left(\frac{D}{ds}\frac{d\gamma}{ds}\right)(0),\ e_1\right\rangle = -\frac{1}{\varepsilon} + \Gamma^1_{22}(\varepsilon, 0, \cdots, 0)$$
$$= -\frac{1}{\varepsilon} + O(\varepsilon)\ .$$

Note that $\|d\gamma/ds\|^2 = 1 + O(\varepsilon^2)$, and so the second fundamental form on the unit vector in the direction of $(d\gamma/ds)(0)$ is again of the form $-1/\varepsilon + O(\varepsilon)$. By an orthogonal change of the coordinates $x$, this computation is valid for any tangent direction on $S^{n-1}(\varepsilon)$. This proves the first part of the lemma.

For the second part, we consider the map $f_\varepsilon\colon S^{n-1}(1) \to S^{n-1}(\varepsilon)$ given by $x \mapsto \varepsilon x$. Then at a point $x$, where $\|x\|^2 = 1$, we have

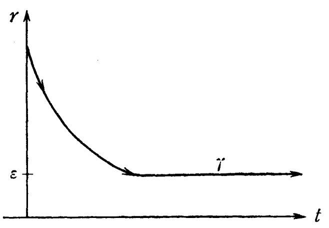
$$\frac{1}{\varepsilon^2} f_\varepsilon^*(g_\varepsilon)_x = \sum_{i,j} g_{ij}(\varepsilon x) dx_i dx_j$$
$$= \sum_{i\ j}(\delta_{ij} + \varepsilon^2 \sum a^{kl}_{ij} x_k x_l) dx_i dx_j + \varepsilon^3\ \text{(higher order terms)}\ .$$

It follows that these metrics converge to the standard metric as claimed. This completes the proof.

We now consider the riemannian product $D \times \mathbf{R}$ with coordinates $(x, t)$. We shall define a hypersurface $M \subset D \times \mathbf{R}$ by the relation

$$M = \{(x, t) \colon \big(\|x\|, t\big) \in \gamma\}$$

where $\gamma$ is a curve in the $(r, t)$-plane as pictured below:




The key point is that $\gamma$ begins along the positive $r$-axis and finishes as a straight line parallel to the $t$-axis. The metric induced on $M$ from $D \times \mathbf{R}$ extends the metric on $D$ near its boundary and finishes with a product metric of the form $S^{n-1}(\varepsilon) \times \mathbf{R}$. If $\varepsilon$ is sufficiently small, then Lemma 1 shows that we can change the metric in this tubular piece to a metric of positive scalar curvature which is a riemannian product of the standard $\varepsilon$-sphere with $\mathbf{R}$ for large time. This is accomplished by a metric of the form:

$$\sum_{i,j} g_{ij}(x, t) dx_i dx_j + dt^2$$

where $g_{ij}(x, t)$ is the induced metric on $S^{n-1}(\varepsilon)$ for early time and is the euclidean metric for large time.

The key difficulty is to choose $\gamma$ so that the metric induced on $M$ has $\kappa > 0$ at all points. To do this we begin with the following observation. Let $l$ be a line (a geodesic ray) in $D$ emanating from the origin. Then:

(i) The surface $l \times \mathbf{R}$ is totally geodesic in $D \times \mathbf{R}$.

(ii) The normal to $M$ along points of $l \times \mathbf{R}$ lies in (i.e., is tangent to) $l \times \mathbf{R}$.

It follows immediately that $\gamma_l \equiv M \cap (l \times \mathbf{R})$ is a principal curve on $M$, i.e., *its tangent lines are principal directions for the second fundamental form of $M$ in $D \times \mathbf{R}$. Furthermore, the associated principal curvature at a point corresponding to $(r, t) \in \gamma$ is exactly the curvature $k$ of $\gamma$ at that point.* The remaining principal curvatures at such a point are of the form $(-1/r + O(r)) \times \sin\theta$ where $\theta$ is the angle between the normal to the hypersurface and the $t$-axis.

We now fix a point $q \in \gamma_l \subset M$ corresponding to a point $(r, t) \in \gamma$. Let $e_1, \cdots, e_n$ be an orthonormal basis of $T_q(M)$ such that $e_1$ is the tangent vector to $\gamma_l$ and $e_2, \cdots, e_n$ (which are tangent to the $D$-factor) are principal vectors for the second fundamental form of $M$. The Gauss curvature equation states that the sectional curvature $K_{ij}$ of $M$, corresponding to the plane $e_i \wedge e_j$ is given by

$$K_{ij} = \bar{K}_{ij} + \lambda_i \lambda_j$$

where $\bar{K}_{ij}$ is the sectional curvature of $D \times \mathbf{R}$ and where $\lambda_1, \cdots, \lambda_n$ are the principal curvatures corresponding to the directions $e_1, \cdots, e_n$ respectively. As we saw above, $\lambda_1 = k$, the curvature of $\gamma_l$ in $l \times \mathbf{R}$ (which is isometric to $\gamma \subset \mathbf{R}^2$), and $\lambda_j = (-1/r + O(r)) \sin\theta$ for $j = 2, \cdots, n$. Since $D \times \mathbf{R}$ has the product metric we see that:

$$\bar{K}_{1j} = K^D_{\partial/\partial r, j} \cos^2\theta \,,$$
$$\bar{K}_{ij} = K^D_{ij} \qquad \text{for} \quad 2 \leqq i, j \leqq n \,,$$

where $K^D$ is the sectional curvature of the metric on $D$. It then follows that the scalar curvature

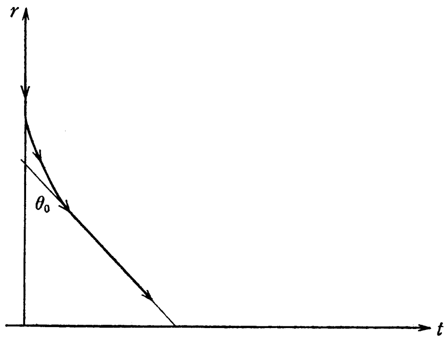
$$\kappa = \textstyle\sum_{i \neq j} K_{ij}$$

of $M$ at $(x, t)$ is given by the formula

$$\begin{aligned} \kappa = \kappa^D &- 2 \operatorname{Ric}^D\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \sin^2\theta \\ &+ (n-1)(n-2)\left(\frac{1}{r^2} + O(1)\right) \sin^2\theta \\ &- (n-1)\left(\frac{1}{r} + O(r)\right) k \sin\theta \end{aligned} \tag{1}$$

where $\kappa^D(x, t) = \kappa^D(x)$ is the scalar curvature of $D$ at $x$ and where $\mathrm{Ric}^D$ is the Ricci tensor of $D$ at $x$.

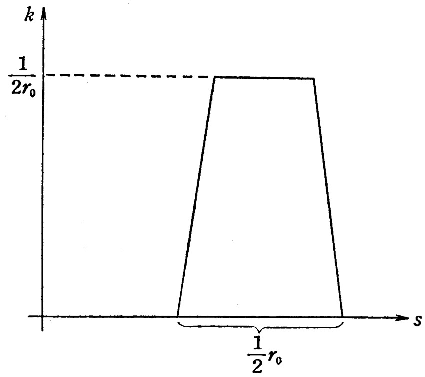
It is clear from formula (1) that there is a $\theta_0 > 0$ such that the "bent line segment" with angle $\theta_0$ pictured below gives an $M$ with $\kappa > 0$.




Let $\gamma_0$ denote the second straight line segment in this bent curve. Since $k \equiv 0$ on $\gamma_0$, we see from formula (1) that as $r$ becomes small, the scalar curvature of $M$ is of the form:

$$\kappa = \frac{(n-1)(n-2)}{r^2} + O(1) .$$

We now choose $r_0 > 0$ small and consider the point $(r_0, t_0) \in \gamma_0$. We now bend $\gamma_0$, beginning at this point, with a curvature function $k(s)$ of the following form:




(Here the variable $s$ denotes arc length along the curve.) It is clear from formula (1) that, since $n \geqq 3$, the hypersurface will continue to have $\kappa > 0$. (For this we need only assume that $r_0^{-2} > \|R^D\|_\infty$.) During this bending process, the curve will not cross the line $r = r_0/2$ since the length of the "bend" is $r_0/2$ and it begins at height $r_0$. The total amount of bend is:

$$\Delta\theta = \int k ds \cong \frac{1}{4} ,$$

an amount independent of $r_0$. Clearly, by a similar choice of the function $k$, we can produce any $\Delta\theta$, $0 < \Delta\theta \leqq 1/4$.

Our curve now continues with a new straight line segment $\gamma_1$ at an angle $\theta_1 = \theta_0 + \Delta\theta$. By repeating the process six more times, we can achieve a total bend of $\pi/2$. This completes the proof of the case of connected sums and surgeries on 0-spheres.

For the general case of surgeries on spheres of codimension $\geqq 3$, the argument is entirely similar. We shall present only an outline.

Let $S^p \subset X$ be an embedded sphere with trivial normal bundle $N$ of dimension $q \geqq 3$. Let $\nu_1, \cdots, \nu_q$ be global orthonormal sections of $N$ and identify $N \xrightarrow{\approx} S^p \times \mathbf{R}^q$ via the diffeomorphism given by $\nu_y \mapsto (y, x_1, \cdots, x_q)$ where $\nu_y = \sum x_j (\nu_j)_y$. Define $r\colon S^p \times \mathbf{R}^q \to \mathbf{R}^+$ by $r(y, x) = \|x\|$, and set $S^p \times D^q(\rho) = \{(y, x)\colon r(x) \leqq \rho\}$. Choose $\bar{r} > 0$ so that the exponential map $\exp\colon N \to X$ is an embedding on $S^p \times D^q(\bar{r}) \subset N$. Lift the metric of $X$ to $S^p \times D^q(\bar{r})$ by the exponential map. Note that $r$ is then the distance function to $S^p \times \{0\}$ in $S^p \times D^q(\bar{r})$, and curves of the form $\{y\} \times l$, where $l$ is a ray in $D^q(\bar{r})$ emanating from the origin, are geodesics in $S^p \times D^q(\bar{r})$.

We now consider hypersurfaces in the riemannian product $\big(S^p \times D^q(\bar{r})\big) \times \mathbf{R}$ of the form

$$M = \{(y, x, t)\colon \big(r(x), t\big) \in \gamma\}$$

where $\gamma$ is as before a curve in the $(r, t)$-plane. Arguments entirely analogous to those above show that $\gamma$ may be chosen passing from the $r$-axis to a line $r = \varepsilon > 0$ so that the metric on the corresponding hypersurface has $\kappa > 0$. The metric on the "tube" is a product of the metric induced on $\partial\big(S^p \times D^q(\varepsilon)\big)$ and $\mathbf{R}$. (The key to the argument is the equation corresponding to (1) above. It states that the scalar curvature $\kappa$ on $M$ can be written as

$$\kappa = \kappa^{S^p \times D^q} + O(1)\sin^2\theta + (q-1)(q-2)\frac{1}{r^2}\sin^2\theta - (q-1)\frac{k}{r}\sin\theta\,. \tag{1'}$$

By continuity one can make a small bend to angle $\theta_0 > 0$. Here $\theta_0$ is sufficiently small that the terms $\kappa^{S^p \times D^q} + O(1)\sin^2\theta_0$ are positive on $S^p \times D^q$. The curve $\gamma$ then continues as a straight line ($k \equiv 0$) until the term $(q-1)(q-2)\sin^2\theta_0/r^2$ is strongly dominating. One can then bend $\gamma$ to a line parallel to the $t$-axis as before.)

One now observes that the metric on $\partial\big(S^p \times D^q(\varepsilon)\big) = S^p \times S^{q-1}(\varepsilon)$ can be homotoped through metrics with $\kappa > 0$ to the standard product $S^p(1) \times S^{q-1}(\varepsilon)$ of euclidean spheres. (See Lemma 2.) Performing this homotopy very slowly in time $t$ gives a metric on $S^p \times S^{q-1} \times \mathbf{R}$, which is the metric constructed above for early time $t$ and is the product of euclidean metrics for late time. (See Lemma 3.) The same construction allows us to readjust the radii of the euclidean spheres in $S^p \times S^{q-1} \times \mathbf{R}$ while maintaining $\kappa > 0$. The remainder of the proof is straightforward.

LEMMA 2. *Let $ds_\varepsilon^2$ be the metric induced on $\partial(S^p \times D^q(\varepsilon)) = \{r = \varepsilon\}$. Then for all $\varepsilon > 0$ sufficiently small, $ds_\varepsilon^2$ can be homotoped through metrics of positive scalar curvature to the standard product metric on $S^p \times S^{q-1}$.*

*Proof.* As $\varepsilon \to 0$, the metric induced on $\partial(S^p \times D^q(\varepsilon))$ converges $C^2$ to the metric induced on the $\varepsilon$-sphere bundle of $N$ from the natural metric on $N$ defined using the normal connection. Hence for $\varepsilon$ sufficiently small, we may homotope $ds_\varepsilon^2$ to this metric. (The condition $\kappa > 0$ is open.) This metric is a riemannian submersion with totally geodesic fibres which carry the standard euclidean metric of curvature $1/\varepsilon^2$. One may continue to shrink $\varepsilon$. For $\varepsilon$ sufficiently small, one can deform this metric through riemannian submersions to one where the metric on $S^p$ is standard. (Here we keep the family of horizontal planes fixed.) This deformation will keep $\kappa > 0$ as one can see easily from the formulas of O'Neill [10]. One now deforms the family of horizontal planes to the standard one. Again by O'Neill, this can be done keeping $\kappa > 0$.

LEMMA 3. *Let $ds_t^2$, $0 \leqq t \leqq 1$, be a $C^\infty$ family of metrics on a compact manifold $X$. If the scalar curvature of $ds_t^2$ is positive for all $t$, then there exists an $a_0 > 0$ such that for all $a \geqq a_0$, the metric*

$$ds_{t/a}^2 + dt^2$$

*on $X \times [0, a]$ has positive scalar curvature.*

*Proof.* It is equivalent to show that there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqq \varepsilon_0$, the metric

$$d\sigma_\varepsilon^2 = \varepsilon^2 ds_t^2 + dt^2$$

on $X \times [0, 1]$ has scalar curvature $\kappa_\varepsilon > 0$. Consider local coordinates $(x_1, \cdots, x_n)$ on $X$. Then $ds_t^2 = \sum g_{ij}(x, t) dx_i dx_j$. For notational convenience we set $t = x_{n+1}$ and we write the metric $d\sigma_\varepsilon^2 = \sum \gamma_{ij}^\varepsilon dx_i dx_j$. We also fix $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_{n+1})$ and assume that $g_{ij}(\bar{x}) = \delta_{ij}$. Now the riemannian covariant derivative on $X \times I$ for the metric $d\sigma_\varepsilon^2$ is given by the Christoffel symbols

$$(\Gamma^\varepsilon)_{ij}^k = \frac{1}{2} \sum_l (\gamma^\varepsilon)^{kl} \left\{ \frac{\partial \gamma_{i,l}^\varepsilon}{\partial x_j} + \frac{\partial \gamma_{j,l}^\varepsilon}{\partial x_i} - \frac{\partial \gamma_{ij}^\varepsilon}{\partial x_l} \right\} . \tag{3}$$

Hence the second fundamental form of the hypersurface $x_{n+1} \equiv \bar{x}_{n+1}$ is given at $\bar{x}$ by

$$\begin{aligned} b^{\varepsilon}_{ij}(\bar{x}) = (\Gamma^{\varepsilon})^{n+1}_{ij} &= -\frac{1}{2}\frac{\partial \gamma^{\varepsilon}_{ij}}{\partial x_{n+1}} \\ &= -\varepsilon^2 \frac{\partial g_{ij}}{\partial x_{n+1}} . \end{aligned}$$

In particular, at $\bar{x}$ we have

$$(3) \qquad \|b^{\varepsilon}\|^2_{\varepsilon} = \sum_{i,j=1}^{n}\left[\frac{\partial g_{ij}}{\partial x_{n+1}}\right]^2 .$$

Let $K^{\varepsilon}_{ij}$ denote the sectional curvature of $X \times I$ in the $(i, j)$-direction at $\bar{x}$ with respect to the metric $d\sigma^2_{\varepsilon}$, and let $\kappa_X$ denote the scalar curvature of the (original) metric $ds^2_{\bar{x}_{n+1}}$ of the hypersurface at $\bar{x}$. Then from the Gauss curvature equation and equation (3) above, we deduce that

$$(4) \qquad \sum_{i,j=1}^{n} K^{\varepsilon}_{ij} = \frac{1}{\varepsilon^2}\kappa_X + O(1) .$$

It remains only to compute the curvatures $K^{\varepsilon}_{i,n+1}$ for $i = 1, \cdots, n$. They are given by the formulas $K^{\varepsilon}_{i,n+1} = -(1/\varepsilon^2)R^{\varepsilon}_{i,n+1,i,n+1}$ where

$$(5) \qquad \begin{aligned} R^{\varepsilon}_{i,n+1,i,n+1} &= \frac{\partial \Gamma^{n+1}_{n+1,i}}{\partial x_i} - \frac{\partial \Gamma^{n+1}_{i,i}}{\partial x_{n+1}} \\ &\quad + \sum_k [\Gamma^k_{n+1,i}\Gamma^{n+1}_{i,k} - \Gamma^k_{i,i}\Gamma^{n+1}_{n+1,k}] . \end{aligned}$$

(Here $\Gamma$ means $\Gamma^{\varepsilon}$.) Using (2) and the special form of $\gamma^{\varepsilon}$, we see easily that:

$$\begin{aligned} &\Gamma^{n+1}_{n+1,i} \equiv 0 , \\ &\Gamma^{n+1}_{i,i} = -\frac{\varepsilon^2}{2}\frac{\partial}{\partial x_{n+1}} g_{ii} \qquad \text{for} \quad i \leqq n . \end{aligned}$$

Furthermore, at $\bar{x}$ one sees that the quadric term in (5) is of order $\varepsilon^2$. It follows that $K^{\varepsilon}_{i,n+1} = O(1)$. Applying (4), we see that the scalar curvature of $d\sigma^2_{\varepsilon}$ is

$$\kappa^{\varepsilon} = \frac{1}{\varepsilon^2}\kappa_X + O(1) .$$

This completes the proof.

## 2. Proof of Theorem B

Let $W$ be a compact spin manifold with $\partial W = X_1 - X_0$ where $X_1$ is simply connected and $X_0$ carries positive scalar curvature. Assume that $W$ is of dimension $n + 1 \geqq 6$. By Theorem A we may assume $X_0$ is simply connected. Hence, by surgery we may assume $W$ is simply connected. It follows that $H_2(W, X_1) \cong \pi_2(W, X_1) = \pi_2(W)/\pi_2(X_1)$. Now the elements of $\pi_2(W)$ can be represented by smoothly embedded 2-spheres, and the second Stiefel-Whitney class $w_2: H_2(W) \cong \pi_2(W) \to \mathbf{Z}_2$ detects the non-triviality of the normal bundle of these representing spheres. Since $w_2 = 0$, we can kill $\pi_2(W)$ by surgeries. It follows by the Universal Coefficient Theorem and duality (cf. [8, page 91]), that

$$H_n(W, X_0) \cong H_{n-1}(W, X_0) \cong \text{torsion}\big(H_{n-2}(W, X_0)\big) \cong 0 .$$

Hence, by the work of Smale [13], [8], $X_1$ can be obtained from $X_0$ by a sequence of surgeries in codimension $\geqq 3$. This completes the proof of Theorem B.

Arguments for Corollary B are as follows. We recall that $\Omega_*^{\text{Spin}} \otimes \mathbf{Q} \simeq \Omega_*^{\text{SO}} \otimes \mathbf{Q}$, and therefore it suffices to consider Pontrjagin numbers. In dimension 4, there is only one such number, namely $p_1 = -24\hat{A}$. In dimension $4k$ for $k > 1$, we consider the quaternionic projective spaces $\mathbf{P}^k(\mathbf{H})$. These manifolds are spin and carry metrics of positive sectional curvature. In particular, $\hat{A}(\mathbf{P}^k(\mathbf{H})) = 0$.

We now recall that a sequence of compact oriented manifolds $\{M_k\}_{k=1}^{\infty}$, where $\dim(M_k) = 4k$, is a basis for $\Omega_*^{\text{SO}} \otimes \mathbf{Q}$ if the characteristic number $q_k(M_k) \neq 0$ for all $k$. Here $q_k$ is defined by the multiplicative sequence corresponding to the formal power series $f(z) = 1 + z^k$. (See [5].) Now the total Pontrjagin class of $\mathbf{P}^k(\mathbf{H})$ is given by the formula

$$p = (1 + \omega)^{2k+2}/(1 + 4\omega)$$

where $\omega$ is a generator of $H^4(\mathbf{P}^k(\mathbf{H}); \mathbf{Z}) \cong \mathbf{Z}$. It follows that

$$\begin{aligned} q_k &= (1 + \omega^k)^{2k+2}/\big(1 + (4\omega)^k\big) \\ &= 1 + (2k + 2 - 4^k)\omega^k , \end{aligned}$$

and therefore $q_k[\mathbf{P}^k(\mathbf{H})] \neq 0$ for $k > 1$. Hence the sequence $\mathbf{P}^2(\mathbf{C})$, $\mathbf{P}^2(\mathbf{H})$, $\mathbf{P}^4(\mathbf{H})$, $\mathbf{P}^6(\mathbf{H})$, $\cdots$ is a basis for $\Omega_*^{\text{SO}} \otimes \mathbf{Q}$. It follows that $\{\mathbf{P}^k(\mathbf{H})\}_{k=2}^{\infty}$ generates the kernel of the $\hat{A}$ homomorphism. This completes the proof of Corollary B.

## 3. Proof of Theorem C

Let $W$ be a compact oriented manifold with $\partial W = X_1 - X_0$ where $X_1$ is simply connected and not spin and where $X_0$ carries a metric with positive scalar curvature. Assume that $W$ has dimension $n + 1 \geqq 6$. By Theorem A we may assume that $X_0$ is simply connected. Hence, by surgery we may assume $W$ is simply connected. It follows that $H_2(W, X_1) \cong \pi_2(W)/\pi_2(X_1)$. Since $X_1$ is not spin, $W$ cannot be spin. In fact the map $w_2: \pi_2(W) \to \mathbf{Z}_2$ is non-zero when restricted to the image of $\pi_2(X_1)$ in $\pi_2(W)$. Now by surgery we may reduce $\pi_2(W)$ so that $w_2: \pi_2(W) \xrightarrow{\approx} \mathbf{Z}_2$ is an isomorphism. The resulting map

$$\pi_2(X_1) \longrightarrow \pi_2(W) \cong \mathbf{Z}_2 ,$$

which is essentially $w_2(X_1)$, will then be surjective, and we conclude that $H_2(W; X_1) = 0$. The proof now proceeds as before to show that $X_1$ can be obtained from $X_0$ by surgeries in codimension $\geqq 3$.

To prove Corollary C it suffices to show that there exists a set of generators for $\Omega_*^{SO}$, each of which carries a metric of positive scalar curvature. The ring $\Omega_*^{SO}$/Torsion is generated by complex projective spaces $\mathbf{P}^n(C)$ and Milnor manifolds. These latter are the hypersurfaces of degree (1, 1) in $\mathbf{P}^n(C) \times \mathbf{P}^m(C)$. They are projective space bundles over a projective space, and one can directly construct metrics of positive scalar curvature on them. (In the metric inherited from $\mathbf{P}^n(C) \times \mathbf{P}^m(C)$, the fibres are totally geodesic and carry the standard symmetric space metric. Shrinking the metric uniformly in the fibres does the trick.)

It remains only to deal with the torsion generators. There are two different constructions of generators, in [14] and [1], either of which will suffice for our purposes. We present the first. We begin with the Dold manifolds $P_{n,m} = S^n \times \mathbf{P}^m(C)/\mathbf{Z}_2$ where $\mathbf{Z}_2$ acts by $-1$ on the left and conjugation the right. This manifold carries a locally symmetric metric of positive sectional curvature. We then consider $D_{m,n} = P_{n,m} \times S^1/\mathbf{Z}_2$ where $\mathbf{Z}_2$ acts by reflection in one linear coordinate on the $S^n$ factor of $P_{n,m}$ and by $-1$ on $S^1$. This transformation is an isometry of the product metric on $P_{n,m} \times S^1$. Consider the obvious projection map $D_{m,n} \rightarrow S^1$, and let $V$ be a generic fibre of the composition

$$D_{m_1 n_1} \times \cdots \times D_{m_k n_k} \longrightarrow S^1 \times \cdots \times S^1 \xrightarrow{\ \mu\ } S^1$$

where $\mu$ denotes multiplication. Then $V$ is a bundle over the torus $T^{k-1}$. The induced metric on $V$ is locally a riemannian product of Dold manifolds (i.e., positive curvature manifolds) and flat $(k-1)$-space. This clearly has positive scalar curvature.

The Dold manifolds, together with manifolds of type $V$ above, generate the torsion of $\Omega_*^{SO}$ (cf. Wall [14]). This completes the proof.

STATE UNIVERSITY OF NEW YORK AT STONY BROOK

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