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FOLIATED PLATEAU PROBLEM, PART II: HARMONIC MAPS OF FOLIATIONS

M. GROMOV

Our basic results concerning harmonic maps are parallel to those in Part I [Gro11] about minimal subvarieties. First we produce compact harmonic foliations $\mathcal{H}$ by solving in some cases the asymptotic Dirichlet problem. Then we construct transversal measures by adopting the parabolic equation method of Eells and Sampson. Finally we indicate some applications to the rigidity and the pinching problems.

1. Recollections on Harmonic Maps

Let V and X be smooth Riemannian manifolds and $f : V \to X$ a smooth map. The differential $D_f : T_v(V) \to T_x(X)$, $x = f(v)$, maps the unit ball in $T_v(V)$ to an ellipsoid in $T_x(V)$ with semi axes

$$\lambda_1 = \lambda_1(f,v) \geq \lambda_2 = \lambda(f,v) \geq \ldots \geq \lambda_k \geq 0$$

for $k = \dim V$. These can be identified with the eigenvalues of the operator

$$(D_f^* D_f)^{1/2} : T_v(V) \to T_v(V) .$$

the *energy density* e_f is then defined by

$$2e_f(v) = \sum_{i=1}^{k} \lambda_i^2(f,v) \underset{\text{def}}{=} \|D_f\|^2 ,$$

which we sometimes denote $\|df\|^2$, and the *energy* $E(f)$ is

$$E(f) = \int_V e_f(v)dv .$$

1.A. Remarks: a) One could use another symmetric function in λ_i instead of $\sum \lambda_i^2$, and thus arrive at another kind of energy. For example the product $\prod \lambda_i$ gives us the volume of f as the total energy (see 3.F_2 (d) for extra remarks on this matter).

b) The above (quadratic) energy can be (more naturally) defined using the geodesic foliation on the unit tangent bundle $Y = UT(V)$, and the map $\varphi : Y \to X$ obtained by composing f with the projection $Y \to V$. First one defines the (1-dimensional) energy density of φ along the leaves by

$$e_\varphi = \frac{1}{2} \left\| \frac{d\varphi}{dt} \right\|^2 ,$$

where $\frac{d\varphi}{dt}$ stands for the differentiations with respect to the natural parameter t on the leaves. Then one integrates to

$$E(\varphi) = \int_Y e_\varphi(y) dy$$

and observes that

$$E(\varphi) = C_k E(f)$$

for some universal (normalizing) constant $C_k > 0$. Then one can *define* $E(f)$ as $C_k^{-1} E(\varphi)$ and observe that this definition extends to certain *singular spaces* V which admit the geodesic foliations with transversal measures. A large class of interesting examples is provided by singular spaces with non-positive curvature (see [Gro3]).

1.B. DEFINITION OF Δf: First we define the Laplace operator for smooth map $f : \mathbb{R}^k \to \mathbb{R}^n$ by

$$\Delta f = \operatorname{Trace} \operatorname{Hess} f .$$

In other words Δf is the map $\mathbb{R}^k \to \mathbb{R}^n$ whose ith component for $i = 1, \dots, n$ equals the usual Laplacian of the i-th component of f,

$$\sum_{j=1}^{k} \frac{\partial^2}{\partial u_j^2} f_i .$$

Next, if f maps V to X, such that $f(v) = x$, we use the geodesic coordinates in V and X around v and x respectively and then $\Delta f(v)$ is defined as the above Euclidean Laplacian at v with respect to these coordinates. This is clearly independent of the choice of the coordinates and defines at each point $x = f(v) \in X$ a vector $\Delta f(v) \in T_x(X)$. Thus Δf is a *vector field in X along $f(V)$*, which is often called the *tension* (field) of f.

Remark: One can reduce this definition to the case $k = 1$ as in 1.A(a). Namely, one first passes to the geodesic foliation on $UT(V)$ and takes the Laplace operator of f along the leaves (i.e. geodesics). Then one observes that the original operator $\Delta f(v)$ equals the average of $\Delta_\gamma f(v)$ over the geodesics γ in V passing through v.

1.B′. Laplace operator in local coordinates. An important feature of Δ is that the principal term, which is the second order operator, depends on $v \in V$ and $f(v) \in X$ but not on the first derivatives of f at v. In other words

$$\Delta f = \sum_{ij} a_{ij} \frac{\partial^2 f}{\partial u_i \partial u_j} + \delta(f)$$

where $a_{ij} = a_{ij}\big(v, f(v)\big)$ are functions on $V \times X$ and where δ is an operator of the first order. This property is sometimes called ***semilinearity*** of Δ.

1.C. Harmonic maps. A C^2-map $f : V \to X$ is called *harmonic* if $\Delta f = 0$. One knows (and the proof is standard) that $\Delta f = 0$ is the Euler-Lagrange equation for the energy $E(f)$ and so the harmonic maps are the stationary points of $E(f)$.

Here is another simple geometric characterization of harmonic maps: f is harmonic if and only if for every locally defined convex function p on X the composed function $p \circ f$ is subharmonic on V. This is obvious with the use of the geodesic coordinates in X.

1.D. Compactness and a priori estimates. The equation $\Delta f = 0$ is (obviously) elliptic and so the C^r- topologies are equivalent on the space of harmonic maps for sufficiently large r. On the other hand the equation Δf is invariant under scale and therefore harmonic maps satisfy the classical

BLOCH PRINCIPLE. *Suppose X is compact and every harmonic map $\mathbb{R}^k \to X$ for $k = \dim V$ is constant. Then the space of harmonic maps $f : V \to X$ is compact in C^r- topology for every $r = 0, 1, 2 \ldots$ That is for each interior (i.e. not on the boundary) point $v \in V$ there exists a constant $C = C(V, v, X, r)$, such that the r-th order differential (jet) of every harmonic map $f : V \to X$ satisfies*

$$\left\| D_f^r(v) \right\| \leq C \, . \tag{$*$}$$

Proof: We recall the following trivial and well known

λ-MAXIMUM LEMMA. *Suppose that for some $\delta > 0$ the closed δ-ball $B(v,\delta) \subset V$ around v is complete (as a metric space). Then there exists $\lambda \geq 1$ depending only on δ, such that every positive locally bounded function h on V with $h(v) \geq 1$ admits a λ-maximum $w \in B(v,\delta)$, that is a point w, where*

$$h(w) \geq \max\big(h(v), \lambda^{-1}h(w')\big)$$

for all w' in the ε-ball $B(w,\varepsilon) \subset V$ for $\varepsilon = \lambda^{-1}\big(h(w)\big)^{-1/2}$.

Proof of the Lemma: First try $w_0 = v$. If it is not a λ-maximum, there exists $w_1 \in B(w_0,\varepsilon_0)$ for $\varepsilon_0 = \lambda^{-1}\big(h(w_0)\big)^{-1/2}$ where $h(w_1) \geq \lambda h(w_0)$. Then we try w_1 and if it does not work we take $w_2 \in B(w_1,\varepsilon_1)$ for $\varepsilon_1 = \lambda^{-1}h(w_1)^{-1/2}$ where $h(w_2) \geq \lambda h(w_1)$, and so on. If λ is sufficiently large, namely, if

$$\sum_{i=0}^{\infty} \varepsilon_i \leq \sum_{i=1}^{\infty} \lambda^{-i} < \delta\ ,$$

then this process necessarily stops at some w_i. Otherwise, we would get the limit point $w_\infty = \lim w_i$ at which h were unbounded. Q.E.D.

Proof of the Bloch principle: Let

$$h_f(v) = \max_{q=1,\dots,r} \left\|D_f^q(v)\right\|^{1/q}$$

and assume (*) fails to be true for arbitrarily large C. Then we have a sequence of harmonic maps f_i where $h_{f_i} \to \infty$ and we take a λ-maximum v_i of h_{f_i} for a sufficiently large λ (depending on $\delta = \operatorname{dist}(v,\partial V)$). Let $B_i \subset V$ be the ε_i-ball around v_i for $\varepsilon_i = \lambda^{-1}\big(h_{f_i}(v_i)\big)^{-1/2}$ and let us multiply the metric of B_i (induced from V) by the constant $C_i = h_{f_i}(v_i)$. The resulting Riemannian manifold, called B_i', is a ball of radius $\lambda^{-1}h_{f_i}^{1/2}(v_i)$ which goes to infinity as $i \to \infty$. Moreover B_i' converges (in an obvious sense) to the Euclidean space $\mathbf{R}^k$ for $k = \dim V$ as $i \to \infty$. On the other hand the norms $\|D^r f_i\|$ are bounded by λ^r on all B_i' for the new (scaled) metric while $h'_{f_i}(v_i) = 1$. It follows by the ellipticity of the equation $\Delta f = 0$, that if r is sufficiently large ($r \geq 4$ is good enough to use the general theory but, in fact, $r \geq 1$ suffices by remark (c) below) that the sequence $f_i : B_i' \to X$

C^r-subconverges to a harmonic map $f : \mathbb{R}^k \to X$ which is non-constant since the norm h_f at the origin of $\mathbb{R}^k$ (which is the limit of the centers v_i of B_i') equals one. Q.E.D.

Remarks: a) It is useful to think of the above argument in terms of foliations. Namely, we have the foliation of harmonic maps $cV \to X$, where cV stands for (V, cg), $c \in \mathbb{R}_+$, and as $c \to \infty$ we have limit leaves represented by harmonic maps $\mathbb{R}^k \to X$. Notice that the space of the harmonic map $\mathbb{R}^k \to X$ is acted upon by the group of similarities of $\mathbb{R}^k$ and the above rescaling amounts to applying this action.

b) The map $f : \mathbb{R}^n \to X$ obtained with Bloch principle clearly has the "norm" $h_f(v)$ bounded on $\mathbb{R}^n$. Moreover, with an obvious adjustment one may achieve such an f satisfying the following

Brody minimum property (compare [Bro]). The function $h_f(v)$ achieves the maximal value at the origin $0 \in \mathbb{R}^n$.

c) The semilinearity of Δ indicated in 1.B′ implies that C^1-bound on a harmonic map f yields the C^r-bounds. In fact, the famous theorem of Schauder says that the Hölder bound C^α, $\alpha > 0$, suffices. It follows, that we may apply the Bloch-Brody principle to the energy density e_f in place of h_f and then obtain a harmonic map $f : \mathbb{R}^n \to X$ with the maximal value of e_f at $0 \in \mathbb{R}^n$.

1.D$_1$. Bochner formula. The basic Bochner-type formula for harmonic maps f due to Eells and Sampson (see [Ee-Sa]) shows that if X has *non-positive* sectional curvature, then

$$\Delta e_f(v) \geq \|\operatorname{Hess}_f(v)\|^2 + \langle \operatorname{Ricci}_V df(v), df(v)\rangle \geq 0\ . \qquad (*)$$

In particular if $V = \mathbb{R}^n$ and e achieves the maximum at zero, then the harmonic map $f : \mathbb{R}^n \to X$ has $\operatorname{Hess}_f = 0$. It follows that f decomposes into a linear projection $\mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$, followed by an isometric totally geodesic immersion $\mathbb{R}^m \to X$.

1.D$_1'$. COROLLARY. *Let X be a compact (with boundary) manifold with $K(X) \leq 0$. Then the following three conditions are equivalent:*

1) Every harmonic map $\mathbb{R}^n \to X$ is constant;

2) There exists no double infinite geodesic $\mathbb{R}^1 \hookrightarrow X$;

3) For every Riemannian manifold V the space of harmonic map $V \to X$ is C^r-compact for every $r = 0, 1, \ldots$.

Remarks: a) One usually calls (1) the *Liouville property*, which is discussed in great detail in [Hi1].

b) It is not hard to extend 1.D$'_1$ to non-compact manifolds. For example if (the interior of) X is locally homogeneous then it satisfies (1) and (3), for maps sending a given point $v_0 \in v$ into a compact subset in X unless for every $\varepsilon > 0$ and $\ell > 0$ there exists a curve $L \subset X$ of length $\geq \ell$ and curvature everywhere $\leq \varepsilon$. (If we think of X as a leaf of a compact foliation, then this property says that there is no infinite geodesic in the leaves belonging to the closure of X.)

1.D$_2$. The second fundamental compactness theorem reads

If X is compact with $K \leq 0$ then the space of harmonic map f of finite energy,

$$E_f \leq \text{const} < \infty ,$$

is C^r-compact for all $r = 0, 1, \ldots$.

Proof: Start with the case $k = \dim V = 2$. Here the energy is scale invariant and so by the Bloch principle the non-compactness of the space $\{f \mid \Delta f = 0 \ , \ E(f) \leq \text{const}\}$ yields the existence of a non-constant harmonic map $\mathbb{R}^2 \to X$ of finite energy in so far as X is compact (with no restriction on the curvature). Then the Bochner inequality $(*)$ rules out such a map for $K(X) \leq 0$.

Now, let us give a proof which uses (*) from the very beginning and which applies to all k. We observe, that if V is compact, then Ricci_V is bounded and so the function $e = e_f$ satisfies

$$\Delta e - Ce \geq 0 \tag{+}$$

for some constant $C \in \mathbb{R}$. Then an elementary argument shows that at every interior point $v \in V$,

$$e(v) \leq C' \int_V e(v) = C' E_f \tag{+*}$$

for some constant $C' = C'(V, v, C)$. This implies the required compactness result, by Remark (c) in 1.D.

Remarks: a) The above compactness theorem is the special case of a similar result for the solution of the heat flow equation due to Eells and Sampson (see [Ee-Sa] and our §4.B).

b) There is another proof of the compactness theorem using the *integrated* Bochner inequality. For example if V is compact without boundary, then by integrating $(*)$ over V we get (compare 4.C$'_3$)

$$\int_V \| \operatorname{Hess}_f(v) \|^2 \leq - \int_V \langle \operatorname{Ricci} df(v), df(v) \rangle \leq CE(f) ,$$

where $-C$ is the lower bound on Ricci_V. Then we observe that $\| \operatorname{Hess}_f(v) \|^2$ is scale invariant for $k = \dim V = 4$ and then Bloch's principle delivers the proof for $k \leq 4$. The case $n > 4$ requires further differentiation of $(*)$ and is slightly more complicated.

1.E. Maximum principle and harmonic rigidity. Take two maps $f_1, f_2 : V \to X$ and evaluated the (usual) Laplacian of the function $\operatorname{dist}\big(f_1(v), f_2(v)\big)$ on V, where we assume the distance function $\operatorname{dist}(x_1, x_2)$ to be smooth on $X \times X$. We denote by $f : V \to X \times X$ the map given by (f_1, f_2) and let H_y for $y = (x_1, x_2)$ be the Hessian of the distance function $p(y) = \operatorname{dist}(x_1, x_2)$ on $Y = X \times X$. Then obviously

$$\Delta(p \circ f) = \operatorname{Trace} H \circ D_f + \langle \Delta f, \operatorname{grad} p \rangle \qquad (*)$$

where $H \circ D_f$ denotes the quadratic form on $T(V)$ which is the pull-back of H on $T(Y)$ under the differential $D_f : T(V) \to T(Y)$.

Example. If X has non-positive sectional curvature, then the distance function is convex on $Y = X \times X$ and so $H \geq 0$. Therefore

$$\Delta \operatorname{dist}\big(f_1(v), f_2(v)\big) \geq 0$$

for *harmonic* maps f_1 and f_2. This immediately implies the following classical

1.E$_1$. UNIQUENESS THEOREM. *Let X be a complete manifold with $K(X) \leq 0$ and V a compact connected manifold with non-empty boundary. If two harmonic maps f_1 and f_2 on V into X are equal on ∂V and homotopic relative to ∂V, then $f_1 = f_2$.*

Remarks: a) If $\partial V = \emptyset$, then a trivial additional argument shows that f_2 is a *translate* of f_1. That is the distance between the lifts of f_1 and f_2 to the universal covering $\widetilde{X}$ of X is constant, say,

$$\operatorname{dist}\left(\widetilde{f}_1(v), \widetilde{f}_2(v)\right) = c \ ,$$

and the geodesic segments (of length c) joining $\widetilde{f}_1(v)$ and $\widetilde{f}_2(v)$ are mutually *parallel* for all v in the following sense: two geodesic segments γ and γ' in $\widetilde{X}$ are called parallel if there exists an isometric totally geodesic immersion of a parallelogram $\Gamma \subset \mathbb{R}^2$ into $\widetilde{X}$, such that a pair of parallel sides goes to γ and γ'.

b) If $K(X) \leq 0$ then the energy density $e(f) = e_f$ is a convex function(al) from the space of maps $f : V \to X$ to the space of functions on V. Namely, if $f_t : V \to X$ is a homotopy of maps where the path $f_t(v)$, for every fixed v and t variable, is a *geodesic* segment in V, such that t equals a constant (depending on v) multiple of the length parameter, then $e\big(f_t(v)\big)$ is a convex function in t for every $v \in V$. It follows that the total energy $E(f) = \int_V e_f$ also is convex which leads to an alternative proof of the uniqueness theorem.

1.E$_2$. If V is a non-compact manifold on which every bounded subharmonic function is constant, then the above discussion gives us the uniqueness (or ridigity) property for harmonic maps f_1, f_2 with $\operatorname{dist}(\widetilde{f}_1, \widetilde{f}_2)$ bounded. Moreover, one can often drop "no bounded subharmonic function" condition by looking more closely at the term $\operatorname{Trace} H \circ D_f$ in the above formula $(*)$ for $\Delta \operatorname{dist}(f_1, f_2)$. Here is a typical example of such rigidity for maps into complete simply connected manifolds X with $K(X) \leq 0$.

Let V be complete manifold whose Ricci curvature is bounded from below, $\operatorname{Ricci} V \geq -R > -\infty$, *and let $f = (f_1, f_2) : V \to X \times X$ be a pair of harmonic maps such that*

$$\operatorname{dist}_X\big(f_1(v), f_2(v)\big) \leq \text{const} < \infty \ .$$

Then in the following two cases $f_1 = f_2$.

(i) X has strictly negative sectional curvature, $K(X) \leq -\kappa < 0$ and the second largest eigenvalue λ_2 of $(D_f^* D_f)^{1/2}$ is strictly positive (i.e. $\geq \varepsilon > 0$) on V.

(ii) *The Ricci curvature of* X *is strictly negative,* $\operatorname{Ricci} X \leq -\kappa < 0$ *and the* n*-th smallest eigenvalue* λ_n *of* $(D_f^* D_f)^{1/2}$ *is strictly positive on* V. (The condition $\lambda_n > 0$ implies that $k = \dim V \geq n = \dim X$ and that f is a submersion.)

Proof: Under the assumptions (i) and (ii) the Laplace of $\operatorname{dist}(f_1, f_2)$ is strictly positive on V. Namely

$$\Delta \operatorname{dist}\big(f_1(v), f_2(v)\big) \geq \varepsilon(v) \geq 0$$

where $\varepsilon(v)$ depends only on $\operatorname{dist}\big(f_1(v), f_2(v)\big)$ and is positive for dist > 0. Then the result follows from the following maximum principle of Amori and Yau (see p. 478 in [Ee-Le2]).

If a bounded positive function p *on* V *with* $\operatorname{Ricci} V \geq -R > -\infty$ *has* $\Delta p(v) \geq \varepsilon\big(p(v)\big)$, *where* $\varepsilon(p) > 0$ *for* $p > 0$, *then* $p = 0$.

Proof: Since Ricci $\geq -R$ the Laplacian of the distance function $v \mapsto \operatorname{dist}(v_0, v)$ is uniformly bounded from above for each $v_0 \in V$ and v running over the unit ball $B(v_0, 1) \subset V$. That is $\Delta \operatorname{dist}(v_0, v) \leq C_R < \infty$. (The function $d : v \mapsto \operatorname{dist}(v_0, v)$ may be non-smooth. Then one thinks of Δd as a distribution. Alternatively, one may replace the distance function by a smooth function which has $\Delta < C$ and which is arbitrarily C^0-close to the distance function.)

Since V is complete and p is bounded, there exists, for every $\varepsilon > 0$, a point $v_0 \in V$, such that $p(v_0) \geq (1 - \varepsilon)p(v)$ for all $v \in B(v_0, 1)$. Then the function

$$q(v) = (v)\big(1 - 2\varepsilon \operatorname{dist}(v_0, v)\big)$$

has a maximum point v_1 inside $B(v_0, 1)$, where necessarily $\operatorname{grad} q = 0$ and $\Delta q \leq 0$. Then a straightforward computation gives us a bound on $p(v_1)$ which shows that $p(v_1) \to 0$ for $\varepsilon \to 0$. It follows $p = 0$.

Remark: If the condition $\operatorname{Ricci} V \geq -R$ is replaced by the bound $|K(V)| \leq C$ then the above result can be reduced to the ordinary maximum principle. Namely, if $v_i \in V$ is a sequence of points such that $p(v_i) \to \sup_V p$, then we take the balls $B_i \subset T_{v_i}(V)$ of the fixed small radius $\rho > 0$ (say $\rho = C^{-1/2}$) and give each such ball the Riemannian metric induces by the exponential map $\exp_{v_i} : B_i \to V$. The bound $|K| \le C$ insures a subconvergence of B_i to a Riemannian ball B_∞ while the functions $p_i = p|B_i$ subconverge to a function on B_∞ having the maximum at the center of B_∞.

1.E$'_2$. Remarks on the harmonic rigidity: (a) It is not hard to formulate (and prove) a general proposition interpolating between the cases (i) and (ii). We leave this to the reader.

(b) The case (ii) of the theorem applies, for example, to harmonic maps $f : X \to X$ with dist $\big(x, g(x)\big)$ bounded. This shows that every such f equals the identity map in the case $K(X) \le 0$ and

$$-\infty < -R \le \operatorname{Ricci} X \le -\kappa < 0 \ .$$

(c) The conclusion of the theorem in cases (i) and (ii) remains true if the boundedness of the function $p(v) = \operatorname{dist}_X \big(f_1(v), f_2(v)\big)$ is replaced by the asymptotic bounded

$$\limsup_{v \to \infty} p(v) / \operatorname{dist}_V(v_0, v) = 0 \ .$$

In fact, there exists an $\varepsilon > 0$ (depending on the constants involved) such that the bound

$$\limsup p(v) / \operatorname{dist}(v_0, v) \le \varepsilon$$

already implies that $f_1 = f_2$.

The proof is identical to that of $\operatorname{dist}(f_1, f_2)$ bounded and is left to the reader.

2. Δ-convexity and the Asymptotic Dirichlet Problem

Let us identify maps $V \to X$ with the corresponding sections of the trivial bundle $V \times X \to X$. Then we apply to the sections all the notions we used for functions, such as the Laplace operator, harmonicity etc.

A subset $Y \subset V \times X$ is called Δ-*convex* if for every compact connected domain $U \subset V$ with smooth non-empty boundary and for every harmonic section $f : U \to V \times X$ the inclusion $f(\partial U) \subset Y$ implies $f(U) \subset Y$.

In what follows we assume as earlier that X is a complete simply connected manifold with $K(X) \le 0$. Then we have the following (well known, I believe)

2.A. EXISTENCE THEOREM. *Let $Y \subset V \times X$ be a Δ-convex set whose projection to V is proper. Then every continuous section $F_0 : V \to Y$ is homotopic to a harmonic section $f : V \to Y$.*

Proof: We shall use in our argument the solvability of the Dirichlet boundary value problem for harmonic maps $f : U \to Y$, where U is a compact Riemannian manifold with smooth boundary and Y is complete with $K(Y) \leq 0$. This is due to R. Hamilton (see [Ham]).

Let us prove Hamilton's theorem in our case where Y is simply connected. We have an easy a priori bound on all derivatives of f in terms of $\sup_{u \in U} \operatorname{dist}(y_0, f(u))$ for a fixed point $y_0 \in Y$. This supremum (obviously) is achieved on the boundary ∂U of U and so our derivatives of f in U are controlled by $f|\partial U$. Moreover, the Bloch rescaling argument used in 1.D to prove the interior a priori estimates, applies near the boundary ∂U and shows that if a harmonic map f is smooth on ∂U then the derivatives of f admit our a priori bound on all of U unless there exists a harmonic map of the half space $\mathbb{R}^k_+$ to Y which is constant on the boundary $\mathbb{R}^{k-1} = \partial \mathbb{R}^k_+$, $k = \dim U$. This provides in our case (Y is complete simply connected with $K(Y) \leq 0$) a control of C^k-norms of f on U in terms of those on ∂U.

Then we can use the standard implicit function argument and prove that the set of those C^∞-smooth (boundary value) maps $\partial U \to Y$ for which the Dirichlet problem is solvable is open in the space of all C^∞-maps $\partial U \to Y$. On the other hand, the a priori estimate shows this subset is closed and thus the universal solvability of the Dirichlet problem follows for all smooth boundary data.

Now, we exhaust V by compact domains with smooth boundaries, $U_1 \subset U_2 \subset \dots U_i \subset \dots V$ (we may assume V contains no compact component) and solve the Dirichlet problem for each U_i and $f_0|\partial U_i$. (This f_0 should be, strictly speaking, first smoothed on ∂U_i.) Thus we get harmonic sections $f_i : U_i \to Y$ which subconverge to a harmonic section $f : V \to Y$ as a properness of Y over V gives us the a priori estimate on the derivatives of f_i. This map f is homotopic to f_0 since the Δ-convexity yields the ordinary convexity of the fibers of $Y \to V$. Q.E.D.

Remark: This existence theorem can be extended to the proper Riemannian Δ-convex submersions $Y \to X$, with contractible fibers where the horizontal distribution in Y (normal to the fibers) is integrable with isometric holonomy and where the fibers have $K \leq 0$. (Probably the integrability can be replaced by some curvature type inequality on the horizontal distribution.)

2.B. Let us apply the above theorem to the following.

<u>Problem</u>. Given a smooth map $f_0 : V \to X$. Does there exist a harmonic map $f : V \to X$ which is in an appropriate sense asymptotic to f_0? For example when can one find a harmonic f within bounded distance from f_0, i.e.

$$\operatorname{dist}_X\big(f(v), f_0(v)\big) \leq \text{const} < \infty \text{ ?}$$

Our existence theorem reduces the problem to that of existence of a Δ-convex neighbourhood Y of the graph $\Gamma_{f_0} \subset V \times X$. The simplest neighbourhood to consider is *the fiberwise ρ-neighbourhood* of Γ_{f_0}, say $Y(f_0, \rho) \subset V \times X$ whose sections $V \to Y$ correspond to maps $V \to X$ within distance ρ from f_0. Now, the formula $(*)$ in 1.E shows this $Y(f_0, \rho)$ is Δ- convex for a sufficiently large ρ in the following two cases (compare 1.E$_2$).

(i) $K(X) \leq -\kappa < 0$ and the second greatest eigenvalue $\lambda_2 = \lambda_2(v)$ of $(D^*_{f_0} D_{f_0})^{1/2}$ is large enough compared to the tension field $\Delta f_0(v)$. Namely

$$\kappa^{1/2} \lambda_2^2(v) \geq 2\big\|\Delta f_0(v)\big\| \,, \tag{+}$$

for all $V \in V$. (The constant 2 here and below is not the best.)

(ii) $\operatorname{Ricci} X \leq -\kappa < 0$ and the n-th eigenvalue λ_n for $n = \dim X$ of $(D^*_{f_0} D_{f_0})^{1/2}$ is sufficiently large,

$$\kappa^{1/2} \lambda_n^2(v) \geq 2\big\|\Delta f_0(v)\big\| \tag{++}$$

for all $v \in V$.

Notice, that in both cases we assume X is complete simply connected with $K(X) \leq 0$.

2.B$_1$. COROLLARY. *In cases* (i) *and* (ii) *there exists a unique harmonic map f within bounded distance from f_0.* (The uniqueness follows from 1.E$_2$.)

Remark: Even if (+) and (or) (++) fail somewhere on V there may exist a Δ-convex neighbourhood Y of Γ_{f_0} which does not have to be a ρ-neighbourhood for any ρ. Then again we can approximate f_0 by a harmonic map f. (In fact the existence of such an f with $\operatorname{dist}(f, f_0)$ bounded is sufficient as well as necessary for the existence of Y.) For example let $\rho = \rho(v)$ be a positive function on V and let us give a sufficient condition for the Δ-convexity of the ρ-neighbourhood

$$Y(f_0,\rho) = \{(v,x) \mid \operatorname{dist}_X\big(x, f_0(v)\big) \le \rho(v)\} \subset V \times X .$$

Set

$$p(v) = \kappa^{1/2}\lambda_i^2(v) - 2\|\Delta f_0(v)\|$$

where $i = 2$ for the case $K(X) \le -\kappa$ and $i = n$ for $\operatorname{Ricci}(X) \le -\kappa$. As we already know

$$\Delta \operatorname{dist}\big(f_0(v), f(v)\big) \ge p(v)$$

whenever the distance between f_0 and f is sufficiently large. This implies the following

2.B$_2$. PROPOSITION. *There exists a constant $\rho_0 > 0$, such that the inequalities*

$$\Delta\rho(v) \le p(v)$$
$$\rho(v) \ge \rho_0$$

imply the Δ-convexity of the ρ-convexity of the ρ-neighbourhood $Y(f_0,\rho)$.

2.B$'_2$. COROLLARY. *The existence of the positive bounded function ρ on V with $\Delta\rho \le p$ implies the existence of a harmonic map $f : V \to X$ within bounded distance from f_0.*

2.B$''_2$. Examples. (a) Suppose the function $p(v) = \kappa^{-1/2}\lambda_i^2(v) - 2\|\Delta f_0(v)\|$ is positive outside a compact subset $V_0 \subset V$. If there exists a bounded positive function ρ on V, such that $\Delta\rho \le 0$ and $\Delta\rho < 0$ on V_0, then there exists a harmonic map $f : V \to X$ within bounded distance from f_0.

Notice that the existence of the above ρ is a common matter. For instance one can produce such functions ρ using *bounded* at infinity fundamental solutions of the Laplace operator on V.

(b) If V is compact connected without boundary, then the integral inequality

$$\int_V p(v) = \int_V \left(\kappa^{1/2}\lambda_i^2(v) - 2\|\Delta f_0(v)\|\right) \geq 0$$

is sufficient for the existence of ρ. Thus the above inequality insures a controlled Δ-convex neighbourhood of f_0 and by the Eells-Sampson theorem (see [Ee-Sa]) the existence of a unique harmonic map f homotopic to f_0 in this neighbourhood.

2.C. Let us apply the above existence results to specific manifolds V and X. The first interesting case is where both V and X equal the hyperbolic plane H^2. We start with a bi-Lipschitz map f_0 which means in this case

$$(1+\varepsilon)^{-1} \leq \lambda_i \leq 1+\varepsilon$$

for $i = 1,2$ and some $\varepsilon \geq 0$. If ε is sufficiently small one can slightly perturb f_0 such that the tension Δf_0 also becomes everywhere small. Then we conclude that *there exists a harmonic map f within bounded distance from f_0*.

Notice that f_0 continuously extends to the ideal boundary $\partial_\infty V = \partial_\infty H^2 = S^1$ but the resulting map

$$\partial_\infty V = S^1 \to S^1 = \partial_\infty X$$

is not, in general, smooth. As f has the same boundary values as f_0 we obtain a class of harmonic maps $f : H^2 \to H^2$ (these are, in fact, diffeomorphisms) which extend to non-smooth maps of the boundary $\partial_\infty H^2 \to \partial_\infty H^2$. (We like non-smooth maps for the same reason as in [Gro11], §1.1.A′.)

2.C′. Remarks: (a) It seems to be unknown if for every bi-Lipschitz map $f_0 : H^2 \to H^2$ there exists a harmonic map f with $\operatorname{dist}(f, f_0)$ bounded.

(b) The most interesting known harmonic maps $H^2 \to H^2$ are the lifts of those between compact surfaces, say $S_1 \to S_2$, of constant curvature -1. If such a map $S_1 \to S_2$ is an ismorphism on the fundamental groups, then the lifted map $f : H^2 \to H^2$ *continuously* extends to $\partial_\infty H^2$. In the general case the extension is a *measurable* (Fürstenberg) map. These harmonic maps f can be composed with the above f_0 with small ε and then the composed maps $f_0 \circ f$ (as well as $f \circ f_0$) can be made harmonic as earlier by bounded perturbations. Yet one has no clear idea which (measurable) maps $\partial_\infty H^2 \to \partial_\infty H^2$ appear as the boundary values of harmonic maps.

(c) A variety of general existence results for harmonic maps between open manifolds is proven in a recent paper by Li and Tam (see [L-T]). For example, they prove the following.

2.C″. THEOREM. *Let φ be a C^1-smooth map of the ideal boundary sphere $S^{k-1} = \partial_\infty H^k$ to $S^{n-1} = \partial_\infty H^n$ such that the differential D_φ does not vanish (i.e. the first eigenvalue λ_1 of $D_\varphi^* D_\varphi$ does not vanish). Then there exists a harmonic map between the hyperbolic spaces, $f : H^k \to H^n$, which equals φ on $\partial_\infty H^k$.*

Proof: Write the hyperbolic space H^k as the warped product

$$H^k = (\mathbb{R}^{k-1} \times \mathbb{R} \ , \ e^t g_0 + dt^2) \ ,$$

where g_0 is the Euclidean metric in $\mathbb{R}^{k-1}$. Similarly, let $H^n = \mathbb{R}^{n-1} \times \mathbb{R}$ and consider a map $\mu : H^k \to H^n$ given by $\mu(x,t) = \big(\lambda(x), t\big)$ where $\lambda : \mathbb{R}^{k-1} \to \mathbb{R}^{n-1}$ is a linear map. Then one sees easily that *the map μ is harmonic if and only if the Euclidean energy density of λ equals that of the identity map $\mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$*, i.e.

$$2e(\lambda) \underset{\text{def}}{=} \operatorname{Trace} \lambda^* \lambda = k - 1 \ .$$

Then we fix points $v_0 \in H^k$ and x_0 in H^n and extend the map φ to a *radial* map $f_0 : H^k \to H^n$ as follows.

(i) every geodesic ray in H^k issuing from v_0 goes to the ray in H^n starting from x_0, such that the resulting map $\partial_\infty H^k \to \partial_\infty H^n$ equals φ.

(ii) the map of every ray $r \in H^k$ to H^n is isometric outside a compact subinterval in r.

(iii) Let (s,d) be the polar coordinates of a point $v \in H^k$ where $s \in S^{k-1} = \partial H^k$ and $d = \operatorname{dist}(v, v_0)$. Then the polar coordinates of $x = f_0(v) \in H^n$ are $\big(\varphi(s), d'\big)$ where $d' = \operatorname{dist}\big(f_0(v), x_0\big)$ satisfies the relation

$$\exp(d - d') = \sqrt{2e\big(\varphi(s)\big)/k - 1} \ .$$

Clearly, the non-vanishing of $D_\varphi(s)$ implies non-vanishing of the (spherical) energy density $e(\varphi(s))$ which insures the existence of f_0 satisfying (i)-(iii). It is equally clear that f_0 is unique in a neighbourhood of infinity in H^k.

Now let us look at f_0 on the unit ball $B(v,1) \subset H^k$ for $v \to \infty$. Then we easily see that for every sequence $v_i \in H^k$ which converges to some point $s \in \partial_\infty H^k$ there exist isometries I_i of $B(v_i,1) \subset H^k$ to the fixed ball $B(v_0,1)$ and some isometries I'_i of H^n, such that the maps

$$f_i = \Gamma_i \circ f_0 \circ I_i : B(v_0,1) \to H^n$$

uniformly converge to a map μ which is in some warped (horospherical) coordinates on $H^k = \mathbb{R}^{k-1} \times \mathbb{R}$ takes the above form, $\mu(x,t) = (\lambda(x),t)$ where the linear map λ has $2e(\lambda) = k-1$ (due to (iii)). Thus the limit map μ is harmonic. It easily follows that there is a small perturbation f'_0 of f_0 which is C^2-smooth and satisfies $\|\Delta f'_0(v)\| \to 0$ for $v \to \infty$ while the second eigenvalue λ_2 of $|D^*_{f'_0} D_{f'_0}|^{1/2}$ is strictly positive on H^k. Then by the discussion in 2.B there exists a harmonic map $f : H^k \to H^n$ asymptotic to f'_0 and hence to f_0, i.e.

$$\operatorname{dist}(f(v), f_0(v)) \to 0 \quad \text{for} \quad v \to \infty\,.$$

Remarks: (a) Our proof is different from that in [Li-Ta] where the authors use the heat flow and need φ to be C^2-smooth.

(b) The above argument can be extended to a class of non-smooth maps φ with "small non-smoothness" as in the case of the hyperbolic plane.

(c) Some concrete examples of harmonic diffeomorphisms of H^2 are given in [Cho-Tr].

(d) It is easy to construct a harmonic map $f : H^2 \to H^2$ with a fold along a given geodesic $g \subset H^2$. In fact, there exists a unique harmonic map $f : H^2 \to H^2$, such that

(1) $f(v) = f(rv)$ for the reflection r of H^2 in g.

(2) Every geodesic line normal to g goes into itself.

(3) The image $f(H^2)$ is contained in the "right" component of the complement $H^2 - g$. (Notice that the image $f(H^2) \subset H^2$ is bounded by $f(g)$ which is a curve equidistant to g.)

Using this f as a model one can produce more general harmonic maps $H^2 \to H^2$ with finitely many folds (of the local type $s \mapsto |s|$) on the boundary $S^1 = \partial_\infty H^2$.

Also notice that similar symmetric harmonic maps f exist in higher dimensions. namely one can construct such a map $f : H^k \to H^n$ for $n \leq k$ which is constant on each orbit of the isometry group of H^k fixing a given $(n-1)$-dimensional subspace $H^{n-1} \subset H^k$.

2.D. If V and X are complete simply connected manifolds with *variable* negative curvature, then it seems hard, in general, to decide which boundary maps $\partial_\infty V \to \partial_\infty X$ extend to harmonic map $V \to X$. In fact, the existence of a single *proper* harmonic map $V \to X$ is an open problem.

This problem admits the positive solution for $\dim V = \dim X = 2$, assuming the curvatures of V and X are strictly negative. Indeed, Riemann's mapping theorem insures a conformal homeomorphism $V \to X$ in this case which is known to be harmonic.

Furthermore, let V_0 be a minimal surface in a higher dimensional manifold X with strictly negative curvature. Then the uniformizing conformal map $H^2 \to V_0 \subset X$ is harmonic. Thus one obtains harmonic embeddings of H^2 into manifolds X with sufficiently pinched curvature (compare [Gro11, 1.5.E_6'']). Notice that the embedding f one obtains this way is *quasi-isometric* as well as harmonic,

$$C^{-1} \leq \mathrm{dist}\big(f(v_1), f(v_2)\big) / \mathrm{dist}(v_1, v_2) \leq C \ .$$

(One does not know if every X with pinched negative curvature, $-\infty < -\kappa' \leq K(X) \leq -\kappa < 0$, receives a quasi-isometric map $f : H^2 \to X$. A similar problem arises for quasi-isometric maps $H^k \to X^n$ for $n \gg k$, but for $k \geq 3$ one rather expects the negative answer.)

More generally, let V and X have $-4 < K < -1$. Then the ideal boundaries $\partial_\infty V$ and $\partial_\infty X$ have C^1-structures (see [Hi-Pu] and one may think that the asymptotic Dirichlet problem is solvable for C^1-maps φ_i : $\partial_\infty V \to \partial_\infty X$ with sufficiently non-degenerate differentials. In fact, one may look first for a "quasi-harmonic" map $f_0 : V \to X$ where Δf_0 is small at infinity and then perturb it to a harmonic map as we did earlier. As in the constant curvature case the simplest quasi-harmonic candidate is a radial map sending geodesic rays from a given point $v_0 \in V$ to those in X issuing from $x_0 \in X$. That is, the map f_0 we look for in the polar coordinates is

$$f_0(s,d) = \big(\varphi(s), d' = d'(s,d)\big)$$

where $s \in \partial_\infty V$, $d = \operatorname{dist}(v, v_0)$ and $d' = \operatorname{dist}\big(f_0(v), x_0\big)$, and one may hope that f_0 becomes quasi-harmonic at infinity with a judicious choice of the function $d'(s,d)$, at least in the case of $-1-\varepsilon < K < -1$ for small $\varepsilon > 0$.

2.E. Let us look more closely at what happens to symmetric spaces V and X of rank one. As the first example we take the complex hyperbolic space $H^n_{\mathbb{C}}$ and recall that the boundary $S^{n-1} = \partial_\infty H^n_{\mathbb{C}}$ has a natural *contact structure*. Namely, one may think of S^{n-1} as the unit sphere in $\mathbb{C}^m$ for $m = n/2$ ($n = \dim H^n_{\mathbb{C}}$ is necessarily even) and the structure is given by the $(2m-2)$-dimensional sub-bundle $K = T \cap \sqrt{-1}\,T \subset T$ in the tangent bundle $T = T(S^{n-1})$.

Every contact diffeomorphism φ of S^{n-1} can be radially (with respect to a fixed point $v_0 \in H^n_{\mathbb{C}}$) extended to a bi-Lipschitz map $f_0 : H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$ which is asymptotically harmonic as in the case of $H^n = H^n_{\mathbb{R}}$. This yields a harmonic map $f : H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$ with a given C^1-smooth contact boundary map. Moreover, this construction applies to some *non-smooth* contact maps $S^{n-1} \to S^{n-1}$. This is interesting as the corresponding harmonic foliation $\mathcal{H}$ (see [Gro11], §0.6 and the beginning of §3) becomes quite large, but every transversal measure on this $\mathcal{H}$ is supported on the leaves corresponding to isometric maps $H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$, provided $n \geq 4$ (see §6).

The above discussion immediately extends to the general rank one case and thus we get a variety of harmonic quasi-isometric maps $V \to X$ whose boundary maps respect the relevant (generalized contact) structure at infinity. Notice that such a map exists between given V and X if and only if there exists an isometric totally geodesic embedding $V \to X$. These are $H^k_{\mathbb{R}} \to H^n_{\mathbb{R}}$ for $n \geq k$, $H^k_{\mathbb{R}} \to H^n_{\mathbb{C}}$ for $n \geq 2k$, $H^k_{\mathbb{R}} \to H^n_{\mathbb{Q}}$ for $n \geq 4k$ (here $\mathbb{Q}$ is for quatornions), $H^2_{\mathbb{R}} \to H^2_{\mathbb{K}}$ (where $\mathbb{K}$ is for Cayley numbers), $H^k_{\mathbb{C}} \to H^n_{\mathbb{C}}$ for $n \geq k$, etc.

There are further examples of asymptotically harmonic (and consequently of harmonic) maps which converge at infinity to some *standard harmonic maps*. To see these we recall that every rank one symmetric space V can be represented (in the horospherical coordinates) as a generalized warped product,

$$V = \left(N \times \mathbf{R} \ , \ e_t(g_0) + dt^2\right) \ ,$$

where N is a nilpotent Lie group (e.g. $N = \mathbf{R}^{n-1}$ for $V = H^n$ and N is the Heisenberg group for $H^n_{\mathbf{C}}$), g_0 is a left invariant metric on N and $e_t : N \to N$ is a one parameter group of dilations. We write similarly $X = N' \times \mathbf{R}$ and observe that every non-trivial homomorphism of the groups, $\lambda : N \to N'$, gives rise to a harmonic map $\mu : V \to X$ of the form $\mu(v,t) = \left(e_{t_0}\lambda(v), t\right)$ for some t_0 (compare the proof of 2.C″).

These μ are our standard maps. The simplest among them are harmonic submersions $H^n_{\mathbf{C}} \to H^{n-1}_{\mathbf{R}}$ and $H^n_{\mathbf{Q}} \to H^{n-3}_{\mathbf{R}}$ which are obtained by dividing $H^n_{\mathbf{C}}$ and $H^n_{\mathbf{Q}}$ by the center of the corresponding nilpotent group N.

3. Lower Bounds on the Energy

We return to the setting of [Gro11, §0.6], where we have a foliated space Λ whose leaves of dimension k, denoted L or V, are endowed with Riemannian metrics. We also assume that Λ carries a transversal measure and this is used to obtain the energy $E(f)$ of a map $f : \Lambda \to X$ by integrating the energy density $e_f(v)$ defined along the leaves. The question we want to address now is as follows.

What is the lower bound, denoted $E[f]$ of $E(f')$ among the maps $f' : \Lambda \to X$ homotopic to a given map f? (Here and below $[f]$ stands for the homotopy class of f.)

3.A. It seems worth-while to generalize this question by considering more general energy functionals. Namely, our map is characterized at every point $v \in \Lambda$ by k eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$ of $(D_f^* D_f)^{1/2}$ along the leaves of Λ. Then every (positive) symmetric function in λ_i gives us a function (density) say $s_f(v)$ on Λ and the corresponding "energy" $S(f)$ is the integral of $s(v)$ over Λ. The two basic examples are

(1) $s(\lambda_i) = \sum_{i=1}^{k} \lambda_i^2$. Here $s_f = 2e_f$ and $S(f)$ is twice the ordinary Dirichlet energy $E(f)$.

(2) $s = \prod_{i=1}^{n} \lambda_i$. Then s_f is the leafwise |Jacobian| of f and $S(f)$ is the "average volume" of the leaves mapped by f by X. We denote these by vol_f and $\mathrm{VOL}(f)$ respectively (which slightly differs from our notations in [Gro11, §2]).

3.A′. The lower bound on $\mathrm{VOL}[f]$ has already been considered in [Gro11, §2.1] in a slightly different setting and we shall see later on how the lower bound for $\mathrm{VOL}[f]$ is related to that of $E[f]$. Also notice that the homotopical and topological role of the functionals $S(f)$ is reasonably well understood for foliations consisting of a single (compact or not) leaf (see [Gr-El], [Gro5] and references on p. 388 in [Ee-Le2]) but, in general, the problem seems more difficult.

The typical example one may have in mind is that of the geodesic foliation of the unit tangent bundle $UT(W) = \mathrm{Gr}_1 W$ of the Riemannian manifold W. Here the leaves of the foliation correspond to (doubly infinite) geodesics in W. If one wants to have a similar higher dimensional example one may take W of constant negative curvature and then the Grassmann bundle $\mathrm{Gr}_k W$ is foliated into complete totally geodesic submanifolds of W as we have mentioned several times earlier. Here one wants to evaluate $S[f]$ for maps $\mathrm{Gr}_k W \to X$ coming from maps $W \to X$.

3.B. If two symmetric functions s and s' in $\lambda_1, \dots, \lambda_k$ satisfy $s \le s'$. Then the same inequality is valid for the corresponding energy functions,

$$S(f) \le S'(f) \quad \text{and} \quad S[f] \le S'[f] \,.$$

For example, if $k = 2$, then the energy $E(f)$ (corresponding to $\frac{1}{2}(\lambda_1^2 + \lambda_2^2)$) is bounded from below by the volume (or, rather, *area* corresponding to λ_1, λ_2),

$$E(f) \ge \mathrm{VOL}(f) \quad \text{and} \quad E[f] \ge \mathrm{VOL}[f] \,.$$

3.B$_1$. If Λ consists of a single compact leaf V and $K(X) \le 0$, then similar inequalities hold true for *all* k if the map f in question is harmonic,

$$E(f) \ge C\,\mathrm{VOL}(f) \quad \text{and} \quad E[f] \ge C\,\mathrm{VOL}[f] \,,$$

for some constant C depending only on V. In fact, the first inequality follows from the bound on e_f by $E(f)$ (see [Ee-Sa]).

Then the second inequality is immediate, as $E[f] = E(f)$ for harmonic maps f into manifolds with non-positive curvature. Furthermore, since very continuous map $V \to X$ is homotopic to a harmonic map we get the inequality $E[f] \geq C\,\mathrm{VOL}[f]$ (as well as $E[f] \geq C(VS)S[f]$ for every S) for all continuous maps $f : V \to X$.

3.B$_2$. In order to extend the above discussion to general foliations Λ one needs a bound of e by E. Unfortunately the argument in 4.C$_4$ only gives the following weaker result for the case of compact Λ or, more generally, in the case where all leaves V are complete (without boundary) and have $|K(V)| \leq \text{const} < \infty$ and $\mathrm{Inj\,Rad}\, V \geq \varepsilon > 0$. For every point $v \in V \subset \Lambda$, denote by $\overline{e}(v)$ the supremum of $e(v)$ over the unit Riemannian ball in V around v. Then we have the following average bound

$$\int_\Lambda \overline{e}_f(v) \leq CE[f] \tag{+}$$

for $C = C(\Lambda)$ and for all (leafwise, as usual) harmonic maps $f : \Lambda \to X$, assuming $K(X) \leq 0$. On the other hand the stronger bound $e_f \leq CE[f]$ is not valid in the general foliation framework. The difficulty may be already seen if one looks at harmonic maps into S^1 of cyclic coverings V_i of a fixed surface of genus ≥ 2. One can easily arrange such $f_i : V_i \to S^1$, such that $\sup\limits_{v \in V_i} e_{f_i}(v)$ is *not* bounded by $E(f_i)/\,\mathrm{Area}\, V_i$. In fact one can always make $\sup e_{f_i} \geq \text{const Area}\, V_i E(f_i)$. Yet I do not see how to produce a geometrically significant example where $E[f]$ is not controlled from below by $\mathrm{Vol}[f]$.

3.C. For a given Λ foliated into Riemannian leaves V, we take the leafwise unit tangent bundle $\Lambda' = UT(\Lambda)$ foliated into the geodesics in V. The Liouville measures in all $UT(V)$ add up with the transversal measure in Λ to a transversal measure for the (1-dimensional) foliation Λ'. Now with every map $f : \Lambda \to X$ we have $f' : \Lambda' \to X$ and

$$E(f) = c_n E(f')\ , \qquad n = \dim V\ ,$$

for a universal constant $c_n > 0$. In what follows, we renormalize the Liouville measure in order to have $c_n = 1$. Then, with this convention, we have $E(f') = E(f)$ and, consequently

$$E[f'] \geq E[f]\ .$$

3.C_1. Remark: If the leaves V of Λ have constant negative curvature, we can use the foliation of k-dimensional totally geodesic submanifolds in V. This is also possible for foliations with locally symmetric leaves which contain sufficiently many totally geodesic submanifolds. In fact one may sometimes use non-geodesic submanifolds. Also, one may apply the dimension reduction trick to bound the energy functionals $S(f)$ different from $E(f)$.

3.D. We assume here that Λ is a one-dimensional foliation and we try to evaluate $E(f)$ for $f : \Lambda \to X$ in geometric terms. Namely, we take the universal covering $\widetilde{X}$ of X and look at the lifted map, say $\widetilde{f} : \widetilde{\Lambda} \to \widetilde{X}$. For every point $v \in \Lambda$, we take the $2R$-segment of the leaf $V \ni v$ centered at v, we lift this segment to $\widetilde{\Lambda}$ and send it by $\widetilde{f}$ to $\widetilde{X}$. We observe that the distance between the ends of this segment in $\widetilde{X}$ does not depend on the lift and we denote it by $\mathrm{dist}_{\widetilde{X}}(v+R, v-R)$. Then we define the average ***covering stretch*** of f by

$$\widetilde{\mathrm{Stre}} f = \tfrac{1}{2} \limsup_{R\to\infty} R^{-1} \int_\Lambda \mathrm{dist}_{\widetilde{X}}(v+R, v-R) .$$

(Compare [Cro-Fa], where a similar invariant is called "intersection"). It is obvious that the stretch is bounded by $S(f)$ for $s = |\lambda_1|$, i.e. for $s_f(v) = \|df\|$ for the leafwise differential of f,

$$\widetilde{\mathrm{Stre}} f \le S(f) = \int_\Lambda \|df\| . \tag{$*$}$$

Therefore, if we normalize $\mathrm{Vol}\,\Lambda = 1$, we get the bound (compare [Cro-Fa])

$$\sqrt{e(f)} \ge \widetilde{\mathrm{Stre}} f . \tag{$**$}$$

Next we observe that $\widetilde{\mathrm{Stre}}$ is a homotopy invariant of f and therefore

$$\sqrt{e[f]} \ge \widetilde{\mathrm{Stre}} f . \tag{$+*$}$$

3.D_1. Example. Suppose the map f is leafwise harmonic (i.e. geodesic with $e_f = 1$. If X has no *conjugate points*, (e.g. $K(X) \le 0$) then each segment $\widetilde{f}(v-R, v+R)$ is distance minimizing in $\widetilde{X}$ and thus

$$\widetilde{\mathrm{Stre}} f = E(f) .$$

3.D_2. Remark: If the transversal measure of the closed leaves in Λ equals zero, then one may expect the equality $\sqrt{e[f]} = \widetilde{\mathrm{Stre}} f$. This is easy to show, for example, if X has no conjugate points. Another amusing case is where $\pi_1(X)$ is finite and the equality just says that $E[f] = 0$. Yet, I have not checked the general case.

3.E. Let us apply the above to the geodesic foliation Λ' associated to a k-dimensional foliated Λ. Thus we define the stretch of a map $f : \Lambda \to X$ by

$$\widetilde{\mathrm{Stre}} f = \widetilde{\mathrm{Stre}} f'$$

and observe that in the normalized case

$$\sqrt{e[f]} \geq \widetilde{\mathrm{Stre}} f \qquad (++)$$

(compare [Cro-Fa]).

In particular we have the following (foliated version of the) result of Croke and Fathi (see [Cro-Fa]).

3.E_1. ***If X has no conjugate points and f is a leafwise geodesic isometric immersion, then f is energy minimizing,***

$$E(f) = E[f] \; .$$

Another useful bound on $E[f]$ applies to those f whose lifts $\widetilde{f} : \Lambda \to X$ are *quasi-isometric* on the leaves of V of Λ. In fact, we need only a one-sided bound

$$\mathrm{dist}_{\widetilde{X}}(\widetilde{f}(\widetilde{v}_1), \widetilde{f}(\widetilde{v}_2) \geq C \, \mathrm{dist}_{\widetilde{V}}(\widetilde{v}_1, \widetilde{v}_2) \qquad (*)$$

for all leaves $\widetilde{V}$ of $\widetilde{\Lambda}$ and all pairs of points $v_1, \widetilde{V}_2$ in $\widetilde{V}$ which are sufficiently far apart, say for $\mathrm{dist}(v_1, \widetilde{v}_2) \geq R_0$ for some fixed R_0.

3.E_2. ***If all leaves $\widetilde{V}$ are simply connected manifolds without conjugate points, then***

$$\sqrt{e[f]} \geq \widetilde{\mathrm{Stre}}(f) \geq C$$

for the constant C of $(*)$.

3.F. Let us indicate a more sophisticated example of the evaluation of the stretch. Namely, we take a compact manifold W of negative curvature and consider a continuous map $f : W \to X$. Then we have the following criterion for (non)-vanishing of $\widetilde{\mathrm{Stre}} f$ which, by definition, equals the stretch of the corresponding map f' of the geodesically foliated space $UT(W)$ into X.

3.F$_1$. PROPOSITION. *The stretch of f vanishes if and only if the image of the fundamental group $f_*(\pi_1(W)) \subset \pi_1(X)$ is an amenable group.*

Idea of the proof: A general geodesic of W mapped to $\widetilde{X}$ looks very much the same as a path of a random walk with support in $f_*(\pi_1(W))$ and the random walk version of our Proposition is a well known easy fact.

3.F$_2$. Remarks: (a) Let W have constant curvature -1 and $\Lambda = \mathrm{Gr}_k W$ foliated into geodesic manifolds. Then the above Proposition yields the non-vanishing of the leafwise energy of the map $f' : \Lambda \to X$ corresponding to f. Namely, $E[f'] > 0$ if $f_*(\pi_1(W))$ is non-amenable. (Of course, for $k \geq 2$, one may easily have $E[f'] > 0$ while $f_*(\pi_1)$ is amenable).

(b) The proposition remains valid for many manifolds W with non-strictly negative curvature, such for example, as locally symmetric spaces without flat factors.

(c) Another useful generalization of 3.F$_1$ concerns sections of flat bundles over W. These correspond to homomorphisms $\pi_1(W) \to \mathrm{Iso}\,\widetilde{X}$, where the image of $\pi_1(W)$ does not have to be discrete in the isometry group $\mathrm{Iso}\,\widetilde{X}$. We leave it to the reader to work out the relevant definitions and proofs.

(d) Finally, we indicate that with every $S(f)$ associated to a given symmetric function $s = s(\lambda_1, \ldots, \lambda_k)$ one can directly associate a kind of k-dimensional stretch without using the reduction of dimension. Namely for each $v \in \Lambda$ we take the k-ball $B = B_v(R)$ in $V \ni v$ around v and denote by $\sigma(v, R)$ the lower bound of the s-energies of the maps $B \to X$ which equal f on the boundary ∂B and homotopic to $f \bmod \partial B$. Then the corresponding average stretch is

$$\widetilde{S}(f) = \limsup_{R\to\infty} \int_\Lambda \mathrm{Vol}\left(B_v(R)\right)^{-1} \sigma(v, R) \ .$$

For example, if $n = 1$ and $s = |\lambda_1|$, then this gives us our old $\widetilde{\text{Stre}}$ provided the compact leaves have measure zero. In general, this notion seems meaningful if the balls $B_v(R)$ have *subexponential growth* where one expects the equality $\widetilde{S}(f) = S[f]$, possibly under some mild restrictions on Λ.

3.G. VOL[f] and degree. Recall that every space Λ foliated into *oriented* k- dimensional manifolds with a transversal measure defines a *foliated k-cycle* in Λ whose homology class is denoted $[\Lambda] \in H_k(\Lambda; \mathbf{R})$. For example, if Λ is a smooth manifold, then for every closed k-form ω on Λ representing a cohomology glass in $H^k(\Lambda)$, the pairing $\langle[\omega], [\Lambda]\rangle$ is defined by integrating ω over Λ as follows. The restriction of ω to the (oriented!) leaves of Λ define a measure density along the leaves. This adds with the transversal measure to a measure on Λ which is then integrated over Λ. We use, as earlier the notation

$$\langle[\omega], [\Lambda]\rangle = \int_\Lambda \omega = \omega(\Lambda) .$$

3.G$_1$. Examples. a) Let $\Lambda = UT((W)$ be foliated into oriented geodesics in W. Then the class $[\Lambda] \in H_1(\Lambda; \mathbb{R})$ vanishes for $\dim W \geq 2$. In fact the involution $i : \Lambda \to \Lambda$ which sends each tangent vector $\tau \in UT(W) = \Lambda$ to $-\tau$, obviously satisfies $i_*[\Lambda] = -[\Lambda]$. On the other hand, if $\dim W \geq 2$, then, by a simple argument, $i_* = \text{Id}$ on $H_1(\Lambda)$. Q.E.D.

(b) Let W have constant negative curvature and $\Lambda = \text{Gr}_k W$ be the foliation into oriented k-dimensional totally geodesic submanifolds in W. If k is *odd* and $\dim W > k$, then again, the obvious (orientation reversion) involution on Λ is identity on $H_k(\Lambda)$ and so $[\Lambda] = 0$. On the other hand if k is even, then $[\Lambda] \neq 0$. In fact let $e \in H^k(\Lambda)$ denote the Euler class of the tangent bundle of the foliation. Then, clearly, $\langle e, [\Lambda]\rangle$ equals the integral of the Euler-Gauss-Bonnet form which is a non-zero constant on Λ.

(c) The above non-vanishing of $[\Lambda]$ is of rather general nature: *if some k-dimensional characteristic form (for a given Riemannian metric or, more generally, for leafwise connection on Λ) does not vanish on Λ, then* $[\Lambda] \neq 0$. In particular, $H_k(\Lambda; \mathbf{R}) \neq 0$.

(c$'$) Another non-vanishing criterion can be obtained with the index theorem for foliations of A. Connes: *if almost all leaves V admit harmonic L_2-forms of even dimension and no such forms of odd dimension, then* $[\Lambda] \neq 0$. (This remains true if we permute odd $\leftrightarrow$ even.) Notice that the existence (and non-existence) of a harmonic L_2-form on V of a given dimension is a quasi-isometry invariant of V.

(c″) Let us apply the above to 2-dimensional foliations and conclude: *If almost all leaves V of Λ are hyperbolic (i.e. the universal covering $\widetilde{V}$ of a.e. leaf V is conformally equivalent to the disk) then* $[\Lambda] \neq 0$. In particular, $H_2(\Lambda; \mathbb{R}) \neq 0$.

Notice, that every foliation on a compact space without hyperbolic leaves always admits a transversal measure by Alphors lemma (see [G-L-P]). On the other hand, the known techniques of building foliations seem to (?) provide an abundance of those with (2-dimensional) hyperbolic leaves. (Notice that every surface V with non-Abelian free fundamental group is necessarily hyperbolic.) Yet, getting both, hyperbolicity of the leaves and a transversal measure looks infinitely harder.

Test question. Let Λ be a foliation of a compact manifold into k-dimensional leaves with metrics of negative curvature and an ergodic transversal measure. Does the fundamental group of Λ have exponential growth?

3.G₂. Using $[\Lambda] \in H_k(\Lambda; \mathbb{R})$ one gives the following obvious criterion for non-vanishing of $\mathrm{VOL}[f]$ that is the infimum of the (averaged k-dimensional) volumes of maps $\Lambda \to X$ homotopic to a given $f : \Lambda \to X$.

If $f_*[\Lambda] \neq 0$ *then* $\mathrm{VOL}[f] \neq 0$.

If X is a connected oriented manifold of dimension k, then $H_k(X, \mathbb{R}) = \mathbb{R}$ and $f_*[\Lambda] \in H_k(X, \mathbb{R})$ is characterized by a single real number, called $\deg f$, and the above non-vanishing criterion can be (obviously) made more precise. Namely,

$$\mathrm{VOL}[f] \geq \deg f \tag{Δ}$$

for all leafwise smooth continuous maps $f : \Lambda \to X$.

Slightly less obviously is the opposite inequality,

If the (traversal) measure on Λ is ergodic, then

$$\mathrm{VOL}[f] \leq \deg f \ . \tag{∇}$$

In particular if $\deg f = 0$ (*e.g.* $[\Lambda] = 0$) *then* $\mathrm{VOL}[f] = 0$.

Proof: If $\mathrm{VOL}[f] > \deg(f)$ for some map f, then the leafwise Jacobian of f must change sign on Λ. By the ergodicity, this change of sign must take place on almost every leaf V of Λ. Now, if a *connected* manifold V is mapped to X with a "fold" along which the Jaobian changes sign, one can deform such a map at the fold, such that the mutually cancelling part of $\mathrm{VOL}(f)$ diminishes. This shows that $\mathrm{VOL}(f) > \mathrm{VOL}[f]$ and a little extra effort yields a deformation of f which diminishes $\mathrm{VOL}(f)$ by a given amount $\varepsilon < \mathrm{VOL}(f) - \deg f$. We leave the details to the reader.

Example. Let Λ be our geodesic foliation of $\mathrm{Gr}_k W$ for $K(W) = -1$ as earlier. If k is odd and $\dim W \geq k+1$, then every map $f : \Lambda \to X$ for $\dim X = k$, has $\mathrm{VOL}(f) = 0$. (Yet we know that $E[f] \neq 0$ if $f_*(\pi_1(\Lambda))$ is non-amenable.) On the other hand, if k is even, then $\mathrm{VOL}[f]$ may be positive. For example, if $\dim W = k+1$ is odd, one can always construct a map f of $\Lambda = \mathrm{Gr}_k W$ $(= UT(W))$ to S^k with $\deg f \neq 0$. But if X is an aspherical manifold (e.. $K(X) \leq 0$) then one can easily show that the f^*-image of the fundamental cohomology class of X is invariant under the orientation change involution in $\mathrm{Gr}_k W$. It follows that $\deg f = 0$ and then $\mathrm{VOL}[f]$ also vanishes.

3.H. Lower bounds on $\mathrm{VOL}_\ell[f]$ for $\ell \leq k$. Let $S = \mathrm{VOL}_\ell$ correspond to the ℓ-th elementary symmetric function,

$$\mathrm{vol}_\ell(f) = s_\ell(\lambda_1, \dots, \lambda_k) = \sum \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_\ell} .$$

Call a map $f : \Lambda \to X$ (homotopically) *ℓ-essential* if $\mathrm{VOL}_\ell[f] > 0$, and *ℓ-non-essential* otherwise. Then we say that Λ is *ℓ-essential* if the identity map $\Lambda \to \Lambda$ is such, where the homotopies in question are those preserving each leaf of V.

Example. Let Λ consist of a single leaf V (compact or not) and f can be homotoped to the $(\ell-1)$-skeleton of some triangulation of X. Then, obviously, f is ℓ-non-essential.

Conversely, suppose f is ℓ-non-essential. Then, if V and X are compact, the ℓ-skeleton V_ℓ of every triangulation of V can be sent to the $(\ell-1)$-skeleton of (some triangulation of) X by some homotopy of f (restricted to V_ℓ). In fact, if $\mathrm{VOL}_\ell(f) = \int_V (\mathrm{vol}_\ell f)(v)dv$ is small then also $\int_{V'_\ell} (\mathrm{vol}_\ell f)(v')dv'$ is small for an appropriate small perturbation $V'_\ell \subset V$ of V_ℓ (see p. 388 in [Ee-Le] and references therein).

Finally, we notice that all of V can be pushed to V_ℓ with vol_ℓ bounded. Thus the above contractibility of V_ℓ to $X_{\ell-1}$ is sufficient as well as necessary to the ℓ-non-essentiality of f.

A simple corollary of this discussion is as follows.

A compact simply connected manifold V is ℓ-essential if and only if $H_\ell(V;F) \neq 0$ for some coefficient field F.

The situation is by far more interesting for non simply connected manifolds. For instance if the universal covering $\widetilde{V}$ of V contains an ℓ-spread (see [Gro11, §1.2]) then V is ℓ-essential. In fact the lift $\widetilde{V}_\ell$ of V_ℓ to $\widetilde{V}$ cannot be moved to $\widetilde{V}_{\ell-1}$ by a *bounded* homotopy. It follows, for example that every closed manifold V of non-positive curvature is ℓ-essential for all $\ell \leq \dim V$, and that every closed aspherical manifold of dimension $k \geq 2$ is ℓ-essential for $\ell = 0, 1, 2, k-2, k-1, k$. (Probably the latter is true for all $\ell \leq \dim V$.)

3.H$_1$. Now let Λ be a foliation with transversal measure μ with finite total volume

$$\mathrm{VOL}(\Lambda) = \int_\Lambda d\mu\, dv < \infty\ ,$$

and where the leaves V are complete. We want to bound from below $\mathrm{VOL}_\ell[\mathrm{Id}]$ that is the infimum of $\mathrm{VOL}_\ell f$ for maps $f : \Lambda \to \Lambda$ which can be joined with Id by a *bounded* leafwise homotopy. That is a map

$$F : \Lambda \times [0,1] \to \Lambda$$

such that for every leaf V of Λ,

$$F\big(V \times [0,1]\big) \subset V\ ,$$

and the leafwise distance between v and $F(v,t)$ satisfies

$$\mathrm{dist}\,\big(v, F(v,t)\big) \leq C < \infty\ .$$

First we observe that for every bounded k-form Ω on the leaves the integral $\int_\Lambda f^*(\Omega)$ is invariant under the bounded homotopics (compare 4.E$_1$). It follows that

$$\mathrm{Vol}(\Lambda) \underset{\mathrm{def}}{=} \mathrm{Vol}_\ell\,\mathrm{Id} = \mathrm{Vol}_\ell[\mathrm{Id}]$$

for all (non-orientable as well as orientable) foliations.

Consequently, if the foliation is leafwise oriented and the oriented volume form Ω decomposes into the product of bounded closed forms,

$$\Omega = \Omega_1 \wedge \Omega_2 \ ,$$

Ω_1 of degree k_1 and Ω_2 of degree k_2, then

$$\mathrm{VOL}_{k_1}[\mathrm{Id}] \geq \varepsilon_1 > 0 \ ,$$

and

$$\mathrm{VOL}_{k_2}[\mathrm{Id}] \geq \varepsilon_2 > 0 \ ,$$

where ε_1 and ε_2 depend (in an obvious way) on $\mathrm{Vol}(\Lambda)$ and the implied bounds on $\|\Omega_1\|$ and $\|\Omega_2\|$.

3.H$_2$. Example. If the leaves V are complete *Kähler* manifolds, then $\mathrm{VOL}_\ell[\mathrm{Id}] > 0$ for all *even* $\ell \leq k = \dim V$.

3.H$_3$. Another interesting case where one has the sharp bound on $\mathrm{VOL}_\ell[\mathrm{Id}]$ is that where the leaves have constant negative (or zero) curvature. namely

$$\mathrm{VOL}_\ell[\mathrm{Id}] = \mathrm{VOL}_\ell \, \mathrm{Id}$$

for all $\ell \leq k$.

In fact this reduced to the special case $\ell = \dim V$ by passing to the foliation of the ℓ-dimensional totally geodesic submanifolds in the leaves.

One also expects a (non-sharp) lower bound for variable non-positive curvature and for more general foliations with *uniformly contractible* leaves (compare [Gro6]). Here is a result in this direction.

If the ℓ-th L_2-Betti number of Λ does not vanish then $\mathrm{VOL}_\ell[\mathrm{Id}] > 0$, *provided the leaves V have bounded geometry,*

$$\begin{aligned} |K(V)| &\leq C < \infty \ , \\ \mathrm{Inj\,Rad}\, V &\geq \varepsilon > 0 \ . \end{aligned} \tag{+}$$

Proof: The Betti number in question equals the trace of some integral operator on the space of ℓ-forms ω along the leaves,

$$L_2 b^\ell = \int_\Lambda H(v,v) d\mu\, dv \ ,$$

such that for every $f : \Lambda \to \Lambda$ homotopic to Id the composition of the above operator with $\omega \mapsto f^*(\omega)$ has the same trace equal $L_2 b^\ell$. On the other hand the trace of the composed operator is bounded by

$$\int \|H(v,v)\| \operatorname{VOL}_\ell(f) \leq \operatorname{VOL}_\ell(v) \sup_V \|H(v,v)\| \ .$$

The inequalities (+) imply that $\sup \|H\| < \infty$ and therefore,

$$\operatorname{VOL}_\ell(f) \geq L_2 b^\ell / \sup \|H\| \geq \varepsilon > 0 \ .$$

Example. Let $k = \dim V = 4$ and the curvatures of the leaves V are pinched between two negative constants,

$$-\infty < -\kappa_1 \leq K(V) \leq -\kappa_2 < 0 \ .$$

Then the Gauss-Bonnet integrand is positive and by the A. Connes index theorem the L_2-Euler characteristic is also positive. Hence $L_2 b^\ell \neq 0$ and so $\operatorname{VOL}_2[\mathrm{Id}] > 0$.

Remarks: (a) We did not assume here $\operatorname{Inj\,Rad} V \geq \varepsilon > 0$ as this condition can be achieved by passing to the universal covers of the leaves.

(b) Probably, $\operatorname{Vol}_\ell[\mathrm{Id}] > 0$ whenever the ℓ-dimensional L_p- cohomology does not vanish for some $p \in (1, \infty)$.

3.H₄. Now let us give two criteria for $\operatorname{VOL}_\ell[f] > 0$, where f is a map of Λ into a Riemannian manifold X.

Criterion 1. Let us left f to the universal covering $\widetilde{X}$ of X and suppose the lifted map, called $\widetilde{f}$, *embeds* the universal $\widetilde{V}$ of each leaf of Λ into $\widetilde{X}$, such that the following *strong quasi-isometry* property is satisfied. For each $\widetilde{V}$ embedded to $\widetilde{X}$ by $\widetilde{f}$ there exists a Lipschitz retraction $\widetilde{p} : \widetilde{X} \to \widetilde{V}$, such that the resulting map $\widetilde{\Lambda} \to \widetilde{V}$ is measurable and equivariant for the deck transformation group and such that the Lipschitz constant of $\widetilde{p}$ is bounded on $\widetilde{V}$. We also assume that the maps $\widetilde{f} : \widetilde{V} \to \widetilde{X}$ are uniformly Lipschitz (e.g. $\|df\|$ is bounded on Λ) and then we observe the following inequality,

$$\operatorname{VOL}_\ell[f] \geq C \operatorname{VOL}_\ell \mathrm{Id}$$

where $C > 0$ depends on the Lipschitz constants of $\widetilde{f}$ and $\widetilde{p}$.

Proof: Every homotopy f_t of f first lifts to a homotopy $\widetilde{f}_t : \widetilde{V} \to \widetilde{X}$ which is then retracted back to $\widetilde{V}$ by $\widetilde{p}$. Thus we obtain a homotopy of $\mathrm{Id} : \Lambda \to \Lambda$ whose Vol_ℓ-density is controlled by that of f_t. Q.E.D.

Example. Let Λ be a foliation consisting of totally geodesic submanifolds V in X. If X has $K(X) \geq 0$, then the normal projections $\widetilde{X} \to \widetilde{V}$ have Lipschitz constants one and the tautological map $f : \Lambda \to X$ has

$$\mathrm{VOL}_\ell[f] \geq C\,\mathrm{VOL}_\ell\,\mathrm{Id}\ , \qquad \ell = 1, 2, \ldots .$$

3.H$'_4$. Criterion 2. Suppose the map $f : \Lambda \to X$ is non-homologous to zero in dimension ℓ in the following senses. There exists a closed ℓ-form ω_1 on X and a leafwise closed continuous m-forms Ω_2 on Λ for $m + \ell = k =$ dimension of the leaves, such that $\|f^*(\omega_1) \wedge \Omega_2\| \geq \varepsilon > 0$ on Λ. Then, clearly, $\mathrm{VOL}_\ell[f] > 0$.

Remark: In both cases 1 and 2 we tacitly assumed Λ is *compact*. In the general case one should impose a certain boundedness condition on the homotopies (and differential forms) involved in the discussion.

4. Existence Theorems for Harmonic Maps of Foliations

We start with giving a foliated interpretation of the results in §2 on the asymptotic Dirichlet problem. To make our point clear we consider the following very special situation. We start with a complete manifold W of constant negative curvature and consider the foliation $\Lambda = \mathrm{Gr}_k W$ of the k-dimensional totally geodesic submanifolds in W. We denote by f_0 the projection $\Lambda \to W$ and observe that f_0 is harmonic on each leaf V of Λ. In fact, f_0 is a geodesic isometric immersion on each leaf.

Next, we consider a small C^2-perturbation g of the original metric g_0 in W (of constant curvature), where "small" refers to the C^2-norm of $g_0 - g$ measured with respect to g_0, i.e. the second jet $J^2(g_0 - g)$ must satisfy

$$\left\| J^2(g_0 - g) \right\|_{g_0} \leq \varepsilon \tag{$*$}$$

everywhere on W.

STABILITY THEOREM. *There exists an $\varepsilon_0 > 0$ depending only on* $\dim W$ *such that the above inequality* $(*)$ *with $\varepsilon \le \varepsilon_0$ implies for $n \ge 2$ that there exists a unique continuous map $f : \Lambda \to W$ with the following two properties:*

(1) f is harmonic on each leaf V of Λ with respect to the original metric g_0 on V of constant curvature (induced from (W, g_0)) and the new metric g on W.

(2) f is homotopic to f_0. Moreover, there is a (bounded) homotopy $h : \Lambda \times [0,1] \to W$ between f and f_0, where

$$\operatorname{length}_W h\big(v \times [0,1]\big) \le \delta < \infty$$

for all $v \in \Lambda$ and some $\delta > 0$. (Such a bound is, of course, automatic if Λ is compact.) Furthermore, $\delta \to 0$ as $\varepsilon \to 0$.

Proof: Pass to the universal covering $H_m = \widetilde{W}$ and use 2.B$_1$.

Remarks: (a) The above stability theorem, unlike the one for *non-parameterized* minimal varieties (see [Gro11, §0.5.A]) fails to be true for $n = 1$.

(b) The transversal measure in Λ plays no role here. Yet it becomes crucial when we allow g to be far away from g_0.

4.A. Now we turn to a general foliated space with a transversal measure ν where the leaves (of dimension k, denoted V or L) carry complete Riemannian metrics. We recall the foliated space $\mathcal{M} = \{(\lambda, \varphi)\}$ for $\lambda \in \Lambda$ and $\varphi : L_\lambda \to X$ a C^∞-map (see [Gro11, §0.6.B]) and we are interested in transversal measures μ on $\mathcal{M}$ where push-forward $p_*(\mu)$ under the projection $p = \mathcal{M} \to \Lambda$ equal ν. To make the picture clear let us fix once and for all a transversal section or *slice*, $\Sigma \subset \Lambda$ of our foliation which meets each leaf L over a non-empty discrete subset in L. Then the transversal measure ν is given by an ordinary measure on Σ, also denoted by ν which is invariant under the (holonomy) pseudogroup of Λ. Similarly, every transversal measure μ in $\mathcal{M}$ is given by an actual measure, also called μ, on the space $\widetilde{\Sigma}$ of the maps $L_\sigma \to X$ for all $\sigma \in \Sigma$. Notice that $\widetilde{\Sigma}$ is a slice of the foliated space $\mathcal{M}$ which goes onto Σ under the projection $p : \mathcal{M} \to X$.

Denote by $B(\sigma, R) \subset L_\sigma$ the Riemannian R-ball in L_σ and then for every map $f \in \widetilde{\Sigma} \subset \mathcal{M}$ we denote by $\|f\|_{C^r}\big|B(R)$ the C^r-norm of $f\big|B(\sigma, R)$ for $\sigma = p(f)$, where the C^0-component of the norm is defined as the distance from some fixed point $x_0 \in X$ and the C^s-component for $1 \le s \le r$ is the usual norm of the s-th differential of f with respect to the given Riemannian metrics in L_σ and X.

The following trivial lemma is our basic tool for constructing transversal measures in the subspace $\mathcal{H} \subset \mathcal{M}$ of *harmonic* maps $L_\lambda \to X$.

4.A$_1$. COMPACTNESS LEMMA. *Let the measure ν be finite, i.e. $\nu(\Sigma) < \infty$ and let μ_i, $i \in I$, be a family of transversal measures on $\mathcal{M}$ with the following* two *properties:*

(1) $p_*(\mu_i) = \nu$.

(2) *For arbitrary constants $\varepsilon > 0$, $R \ge 0$ and $r = 0, 1, \ldots$, there exists a constant $C > 0$ such that for each $i \in I$ the μ_i-measure of the subset $\tilde{\Sigma}_C \subset \tilde{\Sigma}$ of the maps f, such that*

$$\|f\|_{C^r}|B(R) \ge C ,$$

satisfies

$$\mu_i(\tilde{\Sigma}_C) \le \varepsilon .$$

Then the family μ_i is precompact in the weak topology, that is every sequence of measures μ_i contains a subsequence which weakly converges to a transversal measure μ on $\mathcal{M}$.

Proof: We observe that for every function $C = C(R, r)$ the space of C^∞-maps $f : L_\sigma \to X$ with

$$\|f\|_{C^r}|B(R) \le C(R, r)$$

is compact in the C^∞-topology in the space of maps. Thus for every ε there is a *compact* subset $\tilde{\Sigma}(\varepsilon) \subset \tilde{\Sigma}$, such that

$$\mu\big(\tilde{\Sigma} - \tilde{\Sigma}(\varepsilon)\big) \le \varepsilon .$$

Then the lemma follows by the standard weak compactness of measures on a compact space.

4.A$_1'$. Example. Suppose X is compact and all measures μ_i are supported on the subspace $\mathcal{H} \subset \mathcal{M}$ of the *harmonic* maps f, such that the total Dirichlet energy E with respect to μ_i is bounded by a constant independent of $i \in I$. Then the assumptions of the lemma are satisfied, as it follows from 1.D$_2$. (The bound on $e(f)$ is especially clear with $(+*)$ in 1.D$_2$ and $(+)$ in 3.B$_2$.)

4.B. The heat equation. This equation applies to maps $f : \Lambda \times \mathbf{R}_+ \to X$ and reads

$$\frac{\partial f}{\partial t} = \Delta F \ , \tag{$*$}$$

where Δ is the leafwise Laplacian of maps into X. If X is a compact manifold of negative curvature and the leaves V of Λ are complete and have

$$\mathrm{Ricci}_V \geq -R > -\infty$$

then the heat equation admits a solution f with a given initial map f_0, i.e.

$$f \big| \Lambda \times 0 = f_0 \ ,$$

provided $f_0 : \Lambda \to X$ is C^1-smooth along the leaves with the energy density bounded on Λ

$$\sup_\Lambda e(f_0) < \infty \ .$$

This is an immediate corollary of Theorem 4.1 in [Li-Ta]. Furthermore, it is easy to see that if the initial map f_0 is continuous then f is also continuous.

Remark: Compactness of X is not really necessary. Probably it is enough to have X complete and certainly the bound on the curvature of X and the first covariant derivatives i.e.

$$\big|K(X)\big| \leq \mathrm{const} < \infty$$

and

$$\big\|\nabla K(X)\big\| \leq \mathrm{const} < \infty \ ,$$

will do.

Since $K(X) \leq 0$, every solution f of $(*)$ satisfies the following two Bochner-type inequalities (see [Ee-Le1], p. 24])

$$\frac{\partial e}{\partial t} \leq \Delta e + \mathrm{Re} \tag{$+$}$$

for the energy density $e = e_f$ and

$$\frac{\partial \ell}{\partial t} \leq \Delta \ell \,, \tag{++}$$

for

$$\ell = \left\| \frac{\partial f}{\partial t} \right\|^2 = \|\Delta f\|^2 \,.$$

4.B₁. The inequality (++) shows that if $\ell(f_0)$ is bounded by a constant C, then also $\ell(f) \leq C$ for all $t \in (0, \infty)$.

To use (+) in a similar way one first observes that the function $e' = (\exp -2Rt)e$ satisfies (++), i.e.

$$\frac{\partial e'}{\partial t} \leq \Delta e' \,, \tag{+++}$$

as is immediate with (+). It follows that

$$e_f(v,t) \leq C \exp 2RT$$

for all $t \in (0,\infty)$. Furthermore (+++) shows that the function $e(v,t)$ is controlled by its average over the unit ball $B = B(v,1)$ in the leaf $L_v \subset \Lambda$. Namely for all $t \geq 1$,

$$e(v,t+1) \leq C(\operatorname{Vol} B)^{-1} \int_B e(v,t)dv \tag{*}$$

where the constant C depends on $k = \dim V$ and $R = -\inf \operatorname{Ricci} V$. This follows from Theorem 1.1 in [Li-Ta]. Also notice that (*) yields, via the Schauder estimates for parabolic equations, the uniform C^∞-compactness of the space of maps $f_t : \Lambda \to X$, for $f_t(v) = f(v,t)$ for all $t \geq 1$, provided the right hand side integrals are uniformly bounded. More precisely, the norm of the r-th order differential of f along the leaves satisfies for all $r = 1, 2, \ldots$

$$\left\|D_f^r(v,t+1)\right\| \leq C_{r,v} \int_B e(v,t)dv \,. \tag{**}$$

4.C. Decay of energy. In order to use (*) and (**) we need a bound on the total energy $E(f_t)$ for all $t \geq 0$. To evaluate this we observe the following well known infinitesimal formula for leafwise smooth maps $f : \Lambda \to X$,

$$\operatorname{div}(df \cdot \Delta f)^* = \|\Delta f\|^2 + \langle df, \nabla \Delta f \rangle$$

where $(df \cdot \Delta f)^*$ is the vector field along the leaves in Λ which is dual to the 1-form defined by $\tau \mapsto \langle df(\tau), \Delta f \rangle_X$ for all leaf-tangent vectors τ.

Here "duality" refers to the leafwise metric in Λ while the covariant derivative and the scalar product on the right hand side of the above formula are taken in X.

Next we invoke the following obvious integration rule over a foliation Λ with a transversal measure ν, such that the total volume of Λ for the measure $d\nu\, dv$ is finite, where dv denotes the leafwise Riemannian volume element.

4.C$_1$. Divergence Lemma. *If all leaves of Λ are complete and* $\operatorname{Vol}\Lambda < \infty$, *then every bounded leaf-tangent and leafwise C^1-smooth vector field, ∂ on Λ whose leafwise divergence* $\operatorname{div}\partial$ *is in $L_1(\Lambda)$ satisfies*

$$\int_\Lambda \operatorname{div}\partial = 0\ , \tag{∇}$$

Proof: The field ∂ integrates to a one-parameter transformation group of the finite measure space $(\Lambda, \mu = d\nu\, dv)$, such that $\partial\mu = \operatorname{div}\partial$. Hence,

$$0 = \int_\Lambda \partial\mu = \int_\Lambda \operatorname{div}\partial\ .$$

4.C$_2$. Corollary. *Every leafwise C^3-solution f of the heat equation $\Delta f = \frac{\partial f}{\partial t}$ satisfies*

$$\frac{dE_f}{dt} = -\int_\Lambda \|\Delta f\|^2\ ,$$

provided $\operatorname{Vol}\Lambda < \infty$ *and e_f and Δf are bounded on Δ.*

Proof: First we observe the standard formula,

$$\frac{d}{dt}\langle df, df\rangle = 2\langle df, \nabla_t df\rangle = 2\left\langle df, \nabla\frac{df}{dt}\right\rangle =$$
$$= 2\langle df, \nabla\Delta f\rangle = 2\big(\operatorname{div}(df\cdot\Delta f)^* - \|\Delta f\|^2\big) .$$

As

$$2E(f) = \int_\Lambda \langle df, df\rangle ,$$

the conclusion would follow if we had $\operatorname{div}(df\cdot\Delta f)^*$ in $L_1(\Lambda)$.

To handle the L_1-problem (i.e. the possible divergence of $\int_\Lambda \|\operatorname{div}\|$) we regularize the integral $\int \operatorname{div}$ using cut-off functions as follows. For every leaf V of L we denote $V(\varepsilon^{-1})$, $\varepsilon > 0$ the subset in V where the norm of the third differential of f is $\geq \varepsilon^{-1}$ and let $p_\varepsilon(v)$, for $v \in V$ be defined by

$$p_\varepsilon(v) = \min\big(1, \operatorname{dist}(v, V(\varepsilon^{-1}))\big) .$$

This gives a function p_ε defined on all of Λ which is the union of its leaves V.

Now we obviously have

$$\frac{d}{dt}E(f) - \int \|\Delta f\|^2 =$$
$$= \lim_{\varepsilon\to 0}\int p_\varepsilon\left(\frac{d}{dt}\langle df, df\rangle - \|\Delta f\|^2\right) =$$
$$= \lim_{\varepsilon\to 0}\int p_\varepsilon \operatorname{div}\partial ,$$

for $\partial = \frac{1}{2}(df\cdot\Delta f)^*$. Then we observe that the divergence of the field $p_\varepsilon\partial$ is in $L_1(\Lambda)$ since

$$\big\|\operatorname{div}(p_\varepsilon\partial)\big\| \leq \|\partial\| + \varepsilon^{-2} .$$

Therefore,

$$0 = \int(\operatorname{div}(p_\varepsilon\partial)) = \int p_\varepsilon \operatorname{div}\partial + \delta_\varepsilon$$

for $\delta_\varepsilon = \int\langle dp_\varepsilon, \partial\rangle \to 0$ for $\varepsilon \to 0$. This implies the needed relation $\int p_\varepsilon \operatorname{div}\partial \to 0$ for $\varepsilon \to 0$. Q.E.D.

4.C_3. Remark: The above cut-off argument allows one to extend the relation (∇),

$$\int_\Lambda \operatorname{div} \partial = 0 ,$$

to the (unbounded) fields ∂, such that $\|\partial\|$ and $\operatorname{div}\partial$ are in $L_1(\Lambda)$. In fact, one does not even need $\operatorname{Vol}(\Lambda) < \infty$ in this case.

4.C'_3. Digression: Bochner integrated. Let us use the cut-off argument to integrate the inequality

$$\|\operatorname{Hess}_f\|^2 \le -\langle \operatorname{Ricci} df, df\rangle + \Delta e_f \tag{$*$}$$

(see 4.D_1 and Remark (b) in 4.D_2). We multiply both sides of ($*$) by $q = p_\varepsilon^2$ and integrate over Λ with the notations f' for df, f'' for Hess_f, etc.,

$$\int q(f'')^2 \le R\int (f')^2 + \int \operatorname{div}(q \operatorname{grad} e_f) - \int q' f' f'' ,$$

where

$$R = -\inf_\Lambda \operatorname{Ricci}$$

(recall that "Ricci" refers to the leaves of Λ), and

$$q' = 2pp' \quad \text{for} \quad p = p_\varepsilon .$$

Then

$$\int q(f'')^2 \le R\int (f')^2 + 2\sqrt{\int (p'f')^2 \int q(f'')^2} .$$

Finally, we assume that the total energy $E(f)$ is finite,

$$E(f) = \int e_f = \int (f')^2 < \infty ,$$

and conclude that

$$\int \|(\operatorname{Hess}_f)\|^2 = \int (f'')^2 \le CE(f) . \tag{$**$}$$

4.C$_4$. Now we return to the heat equation $\frac{\partial f}{\partial t} = \Delta f$ where the initial map $f_0 = f(v,0) : \Lambda \to X$ has $e_{f_0} = \|df_0\|^2$ and $\|\Delta f_0\|$ bounded on Λ. Then $f_t = f(v,t)$ has the following four properties:

(1) e_{f_t} *remains bounded on* Λ *for every* $t \geq 0$,

$$\sup_\Lambda e_{f_t} \leq C_t$$

for some function $C_t \geq 0$.

(2) $\|\Delta f_t\|$ *is bounded on* $\Lambda \times [0,\infty)$. *Moreover,* $\sup_\Lambda \|\Delta f_t\|$ *is monotone decreasing in* t.

(3) *The total energy*

$$E(f_t) = \int_\Lambda e_{f_t}$$

is monotone decreasing in t *and converges to some limit, say*

$$E(\infty) = \lim_{t\to\infty} E(f_t) \ .$$

(4) *The integral* $\int_\Lambda \|\Delta f_t\|^2$ *is monotone decreasing in* t *and converges to zero for* $t \to \infty$.

Proof: (1) and (2) follow from 4.B$_1$ and (3) and (4) follow from 4.C.

Remark: The above argument is due to Eells and Sampson (see [Ee-Sa]), where it is used for compact manifolds V.

4.C$'_4$. Let us additionally assume that the curvature of the leaves and their covariant derivatives are bounded on Λ and also the curvature tensor of X is bounded along with its all covariant derivatives. Then we have the following two extra properties of f_t.

(5) *For every* $r \geq 1$ *the* r*-th covariant derivative* $\nabla^r f_t$ *is in* L_2,

$$\int_\Lambda \|\nabla^r f_t\|^2 \leq C_r < \infty \ .$$

(6) *For every* $s \geq 0$ *the* L_2*-norm of* $\nabla^s \Delta f_t$ *decays for* $t \to \infty$,

$$\int_\Lambda \|\nabla^s \Delta f_t\|^2 \to 0 \ .$$

Proof: (5) follows from (3) and the Schauder estimates for parabolic equations (see 4.B), and (6) is obtained by interpolation of (4) and (5) applied to $r = s + 3$.

4.D. Existence of harmonic measures. Let Λ be a foliated space with a transversal measure ν and a Riemannian metric along the leaves. We assume that all leaves are complete with Ricci curvature bounded from below,

$$\text{Ricci} \geq -R > -\infty \ .$$

We also assume that Λ admits a slice of finite measure (compare 4.A), and that $\text{Vol}\,\Lambda < \infty$. We suppose, as earlier, that the target manifold X has *non-positive* curvature, $K(X) \leq 0$. Now, to simplify the matter, we insist that X is *compact* either *without boundary* or with *convex* boundary.

Finally, we consider a continuous leafwise smooth map $f_0 = \Lambda \to X$, such that e_{f_0} and $\|\Delta f_0\|$ are bounded on Λ, and let f_t denote the solution of the heat equation. (The existence of f_t for all $t > 0$ is assured by 4.B). The map $f_t : \Lambda \to X$, for every fixed $t > 0$, tautologically defines a map of Λ to the space $\mathcal{M}$ of smooth maps of the leaves of Λ to X (see [Gro11, §0.6]) and we denote by μ_t the push-forward of ν under this map. This μ_t is a finite transversal measure on the (foliated) space $\mathcal{M}$ such that the projection of μ_t to Λ (for the obvious projection $\mathcal{M} \to \Lambda$) equals ν.

4.D$_1$. THEOREM. *There exists a finite transversal measure μ on the foliated space $\mathcal{H} \subset \mathcal{M}$ of harmonic maps of leaves of Λ to X which is a weak limit of the measures μ_{t_i} for some sequence $t_i \to 0$ such that*

(a) The projection of μ to Λ equals ν.

(b) Every energy functional $S = \int_\Lambda s$ is lower semicontinuous for $t \to \infty$,

$$S(\mu) \leq \lim_{t_i \to \infty} S(\mu_{t_i}) \ ,$$

where

$$S(\mu_t) \underset{\text{def}}{=} S(f_t) = \int_\Lambda s_{f_t} d\nu \, dv$$

and where $S(\mu)$ is defined in the same way as the integral of s over $\mathcal{H}$ for the measure $d\mu\, dv$ on $\mathcal{H}$, where dv denote the leafwise Riemannian measure of the leaves of $\mathcal{H}$ identified with the corresponding leaves in Λ under $\mathcal{H}$.

(b′) The Dirichlet energy $E(\mu) = \int_{\mathcal{H}} e\, d\mu\, dv$ *satisfies*

$$E(\mu) \le E(f_0)\ .$$

(c) If for some $\alpha > 1$ *the integral* $\int_\Lambda s^\alpha_{f_t}$ *remains bounded for* $t \to \infty$ *then*

$$S(\mu) = \lim_{t\to\infty} \left(S(f_t) = \int_\Lambda s_{f_t} \right)\ .$$

(c′) For each β in the interval, $\frac{1}{2} \le \beta < 1$, the total energy of μ for the density $e^\beta_f = \|df\|^{2\beta}$ equals *the limit of those for* μ_{t_i}, i.e.

$$\int_{\mathcal{H}} e^\beta d\mu\, dv = \lim_{t_i\to\infty} \int_\Lambda e^\beta f_{t_i} d\nu\, dv\ .$$

Proof: The weak precompactness of the measures μ_t follows from the uniform bound on the energy $E(f_t)$ (see (3) in 4.C_4) and the parabolic Schauder estimates (**) in 4.B. Then the existence of a weak sublimit μ of μ_t follows from 4.A_1. Then verifying (a)-(c') needs only recalling the definitions and topologies involved.

4.D'_1. Remark: It may, a priori, happen that the measure μ insured by 4.D has $E(f) = 0$ and then μ is supported on the constant maps (of leaves of Λ to X). This unpleasant possibility can be ruled out however if we have a lower bound on $\int_\Lambda e^\alpha_f$ for $\alpha < 1$ for the maps f homotopic to f_0. For example, if f_0 has $\widetilde{\text{Stre}} f_0 > 0$ (see 3.E), then

$$\liminf_{t\to\infty} \int_\Lambda e^\alpha_{f_t} > 0$$

for all $\alpha \ge \frac{1}{2}$ and so

$$\int_{\mathcal{H}} e^\alpha_f\, d\mu\, dv > 0$$

for $\frac{1}{2} \le \alpha < 1$ by (c′) in 4.D. It follows that there are *non-constant* harmonic maps in the support of μ. Yet, such a non-constant harmonic map may be still rather boring if its image lies in a single goedesic γ of X. (Notice that every non-compact manifold V admits plenty of harmonic maps $V \to \gamma = \mathbb{R}$.)

Now we shall give a criterion which guarantees the presence of truly interesting harmonic maps in the support of μ.

4.D_1''. *Let X have strictly negative curvature, $K(X) < 0$. Then the "energy" density*

$$\text{area}\, f \underset{\text{def}}{=} \text{vol}_2 f = \sum_{i,j} \lambda_i \lambda_j$$

(see 3.H), satisfies

$$\limsup_{t\to\infty} \int_\Lambda (\text{area}\, f_t)^2 < \infty \, .$$

Consequently, (by (c) in 4.D)

$$\text{AREA}(\mu) \underset{\text{def}}{=} \text{VOL}_2(\mu) = \lim_{t_i\to\infty} \text{AREA}(f_{t_i})$$

and

$$\text{VOL}_3(\mu) = \lim_{t_i\to\infty} \text{VOL}_3(f_{t_i}) \, .$$

Proof: We use here the general Bochner-type formula of Eells-Sampson for an arbitrary map $f : V \to X$. That is

$$\|\,\text{Hess}_v\,\|^2 - \|\Delta f\|^2 - \text{Re}_{\,f} + \kappa(\text{area}_f)^2 \le \text{div}\, \partial \, , \tag{$*$}$$

where

$$-R = \inf \text{Ricci}(V) \, ,$$
$$\kappa = -\sup K(X) \, ,$$

and ∂ is some vector field on V (whose components are linear in $f'f''$ that is an abbreviation for $\langle df, D^2 f\rangle$, compare 4.C_3').

We integrate $(*)$ over Λ and conclude

$$\int (\text{area}_f)^2 \le \left(\text{RE}_f + \int_\Lambda \|\Delta f\|^2\right)/\kappa \, . \tag{$**$}$$

This implies our assertion as $\int \|\Delta f_t\|^2$ remains bounded (in fact goes to zero) for $t \to \infty$.

4.D_1'''. Corollary. *If*

$$\text{AREA}[f_0] > 0$$

then some map f in the support of the measure μ in 4.D, has $\text{rank}(f) \ge 2$. *Similarly, if*

$$\text{VOL}_3[f_0] > 0$$

then some f has $\text{rank}(f) \ge 3$.

Recall that $[f_0]$ denotes the homotopy class of f_0 and $\text{VOL}[f_0]$ refers to the infimum of the volumes of the maps in this class.

4.D_2. Generalizations. The assumptions used in 4.D_1 are very restrictive and can be significantly relaxed. For example, one may admit non-compact complete manifolds X by adding a certain stability condition on f_0 (compare [Corl], [Don]). Also one may extend the theorem to maps of foliations to *foliations* with negative curved leaves and also one may go to sections of flat fibrations over Λ, as is needed for the super-rigidity applications. Also one may relax the assumptions on f_0 and strengthen the ties between f_0 and the limit measure μ. All that we hope to discuss in another paper.

Here, we only want to point out that one can admit *non-strictly* negative curvature, i.e. $K(X) \leq 0$ (rather than $K(X) < 0$) if instead of area f one uses the *K-area* that is

$$K\text{- area}_f(v) = \sup_{\{\tau_1,\tau_2\}} \left| \langle R(D_f\tau_1, D_f\tau_2)D_f\tau_2, D_f\tau_1\rangle \right|^{1/2}$$

where (τ_1, τ_2) runs over all orthonormal pairs of vectors at v tangent to the leaf $V \subset \Lambda$ through v and where R denotes the curvature tensor of X at $f(v)$.

This K-area may easily be non-zero even if $K(X)$ somewhere vanishes.

Examples. (a) Let $X = X_0 \times X_1$, where $K(X_1) \leq 0$ and $K(X_0)$ is *strictly* negative. Then the K-area of every map $f : \Lambda \to X$ is bounded from below by C area f_0 for the component $f_0 : \Lambda \to X_0$ of f and some $C > 0$.

(b) Let ω be a closed 2-form on X which is dominated by the curvature as follows

$$\left|\omega(a,b)\right| \leq \left| \langle R(a,b)a, b\rangle \right|^{1/2} ,$$

for all pairs of tangent vectors a and b in X. Then if $f^*(\omega)$ is leafwise non-homologous to zero (see below) then K-AREA$[f] > 0$, where "non-homologous to zero" means there exists a leafwise closed form Ω_2 on Λ of degree $k-2$ (k is the dimension of the leaves), such that

$$\int_\Lambda f^*(\omega) \wedge \Omega_2 \neq 0 ,$$

(compare 3.H_4').

4.E. Convergent heat flows and measurable homotopies. We want to describe here a situation, where the heat flow f_t *converges in measure* to a leafwise harmonic map f which is *measurably homotopic* (see below) to f_0.

4.E$_1$. DEFINITION: A measurable homotopy between f_0 and f_1 is a measurable map $F : \Lambda \times [0,1] \to X$ which is continuous on $V \times [0,1]$ for almost all leaves V of Λ and such that $F(v,0) = f_0(v)$ and $F(v,1) = f_1(v)$.

Example. Suppose a map $f : \Lambda \to X$ admits a lift to some infinite Galois coverings, say $\tilde{f} = \tilde{\Lambda} \to \tilde{X}$, such that $\tilde{f}$ is *proper* on each connected component of the lift of every leaf of Λ to $\tilde{\Lambda}$.

If Λ is compact, then f is *not homotopic* to a constant map $\Lambda \to X$.

To see why it is so, we assume for simplicity's sake that almost all leaves in Λ are simply connected. Let $f(v,t)$ be our homotopy of $f = f(v,0)$ to $f(v,1) = x_0 \in X$ and denote by $\tilde{d}(v)$ the distance between the ends of the lift to $\tilde{X}$ of the path between $f(v)$ and x_0 in X. This is a measurable function on Λ which is proper on almost every leaf. But such a function on a compact Λ may only exist if a.e. leaf is compact. Q.E.D.

4.E$'_1$. SINGULAR UNIQUENESS THEOREM. *If $K(X) < 0$ then every measurable homotopy class contains at most one harmonic map $f : \Lambda \to X$ with $\|df\|$ in $L_1(\Lambda)$, and such that* rank $f \geq 2$ *on a.e. leaf.*

Proof: Let f_t be a homotopy between harmonic maps f_0 and f_1 and let $\tilde{d}(v)$ be the distance between $\tilde{f}_0(v)$ and $\tilde{f}_j(v)$ in $\tilde{X}$ for the lift $\tilde{f}_t$ of f_t to the universal covering $\tilde{X}$ of X. If one of the maps, say f_0 has $\operatorname{rank}_v f_0 \geq 2$ and $\tilde{d}(v) > 0$, then $\Delta\tilde{d}(v) < 0$ (see 1.E). Since f_0 and f_1 have the differentials in $L_1(\Lambda)$, the same is true for the function $\tilde{d}$ and so the cut-off argument applies to the field $\partial = \operatorname{grad} \tilde{d}$ and shows that $\operatorname{div} \partial = \Delta\tilde{d} = 0$ and hence, $\tilde{d} = 0$. Q.E.D.

COROLLARY. *If* Vol $\Lambda < \infty$, *then the uniqueness conclusion holds true for maps with finite total energy $E(f)$ $(= \int e_f = \frac{1}{2}\int \|df\|^2)$.*

4.E$_2$. Now let Λ be a foliation as earlier and let us additionally assume that the transversal measure is *ergodic*. Let $f_0 : \Lambda \to X$ be a continuous map with $E(f_0) < \infty$ and Δf_0 bounded on Λ.

4.E$'_2$. EXISTENCE THEOREM. *If $K(X) \leq -\kappa < 0$ and*

$$\text{AREA}[f_0] > 0 \ ,$$

then there exists a measurable leafwise harmonic map $f : \Lambda \to X$ which is measurably homotopic to f_0 and has $E(f) = E[f_0] \leq E(f_0)$ and* rank ≥ 2 *almost everywhere on Λ.

Proof: Let f_t be the heat flow and observe that the lower bound on the area and the ergodicity of Λ imply, by the argument in 4.D$''_1$, that the second greatest eigenvalue $\lambda_2 = \lambda_2(f_t)$ of $D^*_{f_t} D_{f_t}$ is a.e. away from zero for $t \to \infty$. That is for arbitrary $\varepsilon_0 > 0$ and $\mu_0 > 0$ the measure of the set where $\Lambda_2 \leq \varepsilon_0$ is at most μ_0 for all t greater than $t_0 = t_0(\varepsilon_0, \mu_0)$.

Next we consider the distance function $\widetilde{d}(v; t_1, t_2)$ in the universal covering $\widetilde{X}$ of X between the lifts $\widetilde{f}_t$ of f_t for $t = t_1$ and $t = t_2 \geq t_1$ and recall that (see §2)

$$\Delta \widetilde{d} \leq C \| \Delta f_{t_1} \| - \delta \lambda_1^2 \ ,$$

where $\delta = \delta\big(\widetilde{d}(v)\big)$ is a certain non-negative function of $\widetilde{d}$ which is strictly positive for $\widetilde{d} > 0$. Then

$$\int_\Lambda \delta \lambda_1^2 \leq C \int_\Lambda \| \Delta f_{t_1} \|$$

which shows that δ and $\widetilde{d}$ converge in measure to zero as t_1 and t_2 go to ∞, and so f_t converges in measure to the required map f. Q.E.D.

4.E$''_2$. Remarks: (a) If Λ is non-ergodic one needs AREA > 0 on each ergodic component of Λ.

(b) Instead of AREA one could equally use VOL$_3$ or the functional $\int (\text{area})^\alpha$ for every $\alpha < 2$.

(c) if K is *non-strictly* negative, the above argument may fail only if the field of the geodesic segments $\big[\widetilde{f}_{t_1}(v), \widetilde{f}_{t_2}(v)\big]$ in $\widetilde{X}$ becomes asymptotically parallel. This leads to a generalization of the above theorem for maps to the spaces with $K \leq 0$ which is important for the (super)-rigidity results. This matter will be discussed in full somewhere else and here we only give the following

Example. Let X be the Cartesian product of several manifolds of strictly negative curvature, $X = X_1 \times X_2 \times \ldots \times X_j$ and $f_0 : \Lambda \to X$ be a map whose projection to each X_i, $i = 1, \ldots, j$, has AREA > 0. Then our proof applies to these projections and shows the convergence of the heat flow f_t.

5. Harmonic Maps of Flat Foliations

If all leaves V of Λ are Riemannian flat, then the harmonic measures μ on $\mathcal{H}$ over given transversal measure ν in V are the same thing as the measures in the space of harmonic maps $\mathbb{R}^k \to X$ which are invariant under the action of the parallel translations of $\mathbb{R}^k$.

It is worth noticing that such measures are not hard to come by, since the group $\mathbb{R}^k$ is *amenable* and so every compact $\mathbb{R}^k$-invariant subset in the space of maps $\mathbb{R}^k \to X$ admits a finite invariant measure.

5.A. If $K(X) \leq 0$, then the Eells-Sampson argument (see 4.C$'_3$) shows that every invariant measure μ on $\mathcal{H}$ is extremely special, *μ is supported on the maps $f : \mathbb{R}^k \to X$ with* $\text{Hess}_f = 0$ *a.e. with respect to μ on $\mathcal{H}$.*

It follows that every such map f is *geodesic* (or *affine*), that is a composition of a linear map $\mathbb{R}^k \to \mathbb{R}^\ell$ with a totally geodesic isometric immersion $\mathbb{R}^\ell \to X$.

5.B. The heat flow f_t on Λ with *flat* leaves has the energy density e_f *bounded* on $\Lambda \times [0, \infty)$ (compare 4.B$_1$). It follows, that *the limit measure μ* of 4.D$_1$ has

$$S(f) = \lim_{t_i \to \infty} S(f_t)$$

for every energy functional S.

In particular, if $\text{VOL}_\ell[f_0] \geq 0$ *then* $\text{supp}\,\mu$ *contains* a map f *of* rank $\geq \ell$.

Another immediate consequence of the bound on e_f is the compactness of $\text{supp}\,\mu$. Namely, *if X and Λ are compact then* $\text{supp}\,\mu$ *is compact.*

5.C. Example. Let g_1 and g_2 be two metrics of non-positive curvature on a compact manifold X and let G_1 and G_2 be the spaces of geodesic maps of $\mathbb{R}^k$ into (X, g_1) and (X, g_2) correspondingly. Then every invariant measure μ_1 in G_1 can be "homotoped" to a measure μ_2 which is harmonic with respect to g_2 and such a measure is supported in G_2. Thus we obtain a (multi-valued) correspondence between the invariant measures in G_1 and G_2.

If the above μ_1 is supported on the maps of rank $= k$, then

$$\mathrm{VOL}[\mu_1] = \mathrm{VOL}(\mu_1) > 0$$

for the metric g_1. Then we have the similar inequality with respect to g_2 which implies

$$\mathrm{VOL}(\mu_2) > 0 \, .$$

Thus we recapture the following theorem of Anderson-Schroeder (see [And-Sch]).

If (X, g_1) receives *an isometric geodesic immersion of* $\mathbb{R}^k$ *then so does* (X, g_2).

6. Maps of Kähler Foliations into Manifolds X with $K_{\mathbb{C}}(X) \leq 0$

Let

$$Q(a \wedge b, c \wedge d) = \langle R(a,b)d, c \rangle$$

be the quadratic form on $\bigwedge^2 T(X)$ corresponding to the curvature tensor R of X. Then we extend Q by complex multilinearity to the complexification $\mathbb{C}T(X)$ and look at the *Hermitian* form $Q(a \wedge b, \bar{c} \wedge \bar{d})$ on $\bigwedge^2 \mathbb{C}T(X)$. We say that $\mathbb{C}$-*curvature* $K_{\mathbb{C}}$ of X is *negative* (or non-positive) if

$$Q(a \wedge b, \bar{a} \wedge \bar{b}) \leq 0$$

for all $a \wedge b \in \bigwedge^2 \mathbb{C}T(X)$ and *strictly negative* if

$$Q(a \wedge b, \bar{a} \wedge \bar{b}) < 0$$

for all non-zero bivectors $a \wedge b$. These conditions can be expressed in terms of four *real* vectors, if we write

$$a = \alpha + i\alpha' \quad \text{and} \quad b = \beta + i\beta' \, ,$$

for α, α', β and β' in $T(X)$ and $i = \sqrt{-1}$. Namely,

$$a \wedge b = \alpha \wedge \beta - \alpha' \wedge \beta' + i(\alpha' \wedge \beta + \alpha \wedge \beta') \ ,$$
$$\overline{a} \wedge \overline{b} = \alpha \wedge \beta - \alpha' \wedge \beta' + i(\alpha' \wedge \beta + \alpha \wedge \beta') \ ,$$

and

$$Q(a \wedge b, \overline{a} \wedge \overline{b}) = Q(\alpha \wedge \beta - \alpha' \wedge \beta', \alpha \wedge \beta - \alpha' \wedge \beta')$$

$$+Q(\alpha' \wedge \beta + \alpha \wedge \beta' + \alpha' \wedge \beta + \alpha \wedge \beta') \ .$$

It follows that if Q is negative definite on $\Lambda^2\tau$ for 4- dimensional subspace $\tau \subset T(X)$, then $K_{\mathbb{C}} < 0$ and if Q is semi-negative on all $\Lambda^2\tau$ then $K_{\mathbb{C}} \leq 0$.

Now, Prop. 3.8 in [Bou-Ka] says that if a curvature tensor R on $\mathbb{R}^4$ has the sectional curvature

$$K(a,b) = \langle R(a,b)b, a\rangle / \|a \wedge b\|^2$$

pinched between $-\delta$ and $-\frac{5}{2}\delta$, then Q is a negative semi-definite on $\mathbb{R}^4$ and the strict pinching,

$$-\delta \leq K < -\frac{5}{2}\delta$$

makes Q strictly negative. Thus $\frac{5}{2}$-pinching makes $K_{\mathbb{C}}(X)$ negative for all $n = \dim X$.

Moreover, according to a private communication by D. Toledo, the precise pinching condition,

$$-\delta \leq K \leq -4\delta \Rightarrow K_{\mathbb{C}} \leq 0$$

has been just verified by L. Hernandes at the University of Chicago. It follows that *if the sectional curvature of X is locally 4- pinched, i.e.*

$$-\delta(x) \leq K_x(X) \leq -4\delta(x)$$

for some function $\delta : X \to \mathbb{R}_+$, then $K_{\mathbb{C}}(X) \leq 0$.

6.A. Examples of manifolds X with pinched negative curvature. Let X be a manifold with $K(X) < 0$, let γ be a simple closed normally orientable geodesic in X and let X' be obtained by the surgery of $X \times [0,1]$ along $\gamma \in X \times 0 \subset X \times [0,1]$. That is

$$X' = \big(X \times [0,1]\big) \cup D^2 \times D^k ,$$

for $k+2 = \dim X + 1$, and where $\partial D^2 = \gamma \subset X$.

6.A$_1$. *For every $\varepsilon' > 0$, there exists numbers $\varepsilon > 0$, $\ell > 0$ and $\pi > 0$, such that under the following three conditions (i)-(iii) the manifold X' admits a metric g' for which the boundary $\partial X'$ is convex and $-1 \leq K(X', g') \leq -1 - \varepsilon'$.*

(i) $-1 \leq K(X) - 1 - \varepsilon'$,

(ii) length $\gamma \geq \ell$,

(iii) $\operatorname{Inj}\operatorname{Rad}(X, \gamma) \geq \rho$,

where the injectivity radius is defined with the (normal) exponential map of the normal bundle of $\gamma \subset X$ to X.

The proof is achieved by a straightforward construction similar to those [Gro3] and [Gro7].

Notice that the fundamental group of X' is obtained from that of X by adding the relation $[\gamma] = 1$. One also can obtain a *orbifold* X' with $-1 \leq K(X') - 1 - \varepsilon'$ and $\pi_1' = \pi_1/[\gamma]^p$ by performing the surgery over the p-th multiple of γ.

6.A$_1'$. Remarks: (a) If the holonomy around γ is trivial (i.e. if $\dim X = 2$) then one can replace (ii) and (iii) by the following weaker condition

$$pL \exp C_1 R \geq C_2 ,$$

$L =$ length γ, $R = \operatorname{Inj}\operatorname{Ram}(X, \gamma)$, and C_1 and C_2 are positive constant depending only on $\dim X$.

(b) The above surgery construction can be performed over several geodesics in X simultaneously. Moreover, one can make surgery over certain totally geodesic submanifolds of dimension > 1, but this is slightly more delicate.

(c) There are also examples of *closed* manifolds with pinched curvature obtained by ramified coverings of constant curvature manifolds (see [GrTh]).

(d) The condition $K_{\mathbf{C}} \leq 0$ is much more amenable to surgery than the pinching condition as we shall see in another paper.

6.A$_2$. *Every symmetric space X with $K(X) \leq 0$ has $K_{\mathbf{C}}(X) \leq 0$.*

This is explained in [Sam] and [Ca-To].

6.A$_3$. Another important example of $K_{\mathbf{C}}(X) \leq 0$ is provided by the Teichmüller space X with the Weil-Peterson metric (see [Schu]). Notice that this X is non-complete. Yet it is *convex* (see [Wol]) and so one has a fully fledged theory of harmonic maps to X. (See [Jo-Ya2]).

6.B. Siu-Sampson formulas. Recall that the *complex Hessian* of a real valued function f on an almost complex manifold (V, J) is defined by

$$\mathrm{Hess}_{\mathbf{C}}\, f = dJ'df \ ,$$

where J' is the operator on the cotangent bundle of V corresponding to J on $T(V)$ (where the operator J with $J^2 = -\,\mathrm{Id}$ corresponds to the multiplication by $\sqrt{-1}$).

More generally, for a smooth map $f : V \to X$ one defines

$$\mathrm{Hess}_{\mathbf{C}}\, f = d^{\nabla} J df \ ,$$

for the antisymmetric (or exterior) part d^{∇} of the covariant derivative ∇ in X. Notice that $d^{\nabla}Jdf$ is an anti-symmetric 2-form on V with the values in $T(X)$. Also notice that $\mathrm{Hess}_{\mathbf{C}}\, f(v)$, $v \in V$, can be defined as $dJdf_e(v)$, where $f_e : V \to \mathbf{R}^n = T_x(X)$, $x = f(v)$, is a composition of f with the inverse of the exponential inverse of the exponential map of (a neighbourhood of) X and d denotes the exterior differential on the $\mathbf{R}^n$-valued forms (and functions) on V.

A map f is called *pluriharmonic* if $\mathrm{Hess}_{\mathbf{C}}\, f = 0$. Notice that this notion does not use any metric on V.

Now, let (V, J) be given a *Hermitian* metric which is the same thing as a Riemannian metric on $T(V)$ invariant under J. Then one easily sees that

A map f is pluriharmonic if and only if the restriction of f on every holomorphic curve $S \subset V$ is harmonic with respect to the metric on X induced from V.

Recall that a (real) surface $S \subset (V, J)$ is called a *holomorphic curve* if the tangent subbundle $T(S) \subset T(V)$ is J-invariant.

Now, we assume V is *Kähler*. Then for every point $v \in V$ and every tangent vector $\tau \in T_v$ there exists a holomorphic curve $S = S(\tau) \subset V$ which passes through v tangent to τ and which is *geodesic* at v, i.e. the relative curvature of S at v is zero.

This property shows that for every map f the Laplacian $\Delta f(v)$ equals the average of the Laplacian of f on $S(\tau)$ over all unit vectors $\tau \in T_v$. (Compare 1.B.) In particular *every pluriharmonic map is harmonic*.

If $\dim_{\mathbb{R}} V = 2$ then also the converse is true but for $\dim V \geq 4$ pluriharmonicity of a map is much stronger than harmonicity. In fact the relation $\mathrm{Hess}_{\mathbb{C}} f = 0$ represents N partial differential equations for

$$N = \frac{nk(k-1)}{2} ,$$

where $n = \dim X$ and $k = \dim_{\mathbb{R}} V$. Therefore, the system of these equations if *overdetermined* for $k > 2$ and we can easily show that for a *generic* metric in X every pluriharmonic map $f : V \to X$ has $\operatorname{rank} f \leq 2$ everywhere on V.

Now we have the following basic infinitesimal inequality of Sampson [Sam] which is a modification of an earlier result by Siu [Siu].

6.B$_1$. *If $K_{\mathbb{C}}(X) \leq 0$, then every harmonic map f of a Kähler manifold V into X satisfies*

$$\| \mathrm{Hess}_{\mathbb{C}} f \|^2 \leq \operatorname{div} \partial ,$$

where $\partial = \partial_f$ is a certain vector field on V built out of first and second derivatives of f. More precisely, ∂ is bilinear in df and $D^2 f$ ($= \mathrm{Hess}\, f$),

$$\partial = \Phi(df, D^2 f) ,$$

where the coefficients of the form Φ depend only on the Kähler metric in V.

6.B$_1'$. It is sometimes useful to know the error term in $(*)$. Here it is

$$\|\operatorname{Hess}_{\mathbb{C}} f\|^2 - \operatorname{div}\partial \leq \operatorname{Trace} d_{\mathbb{C}}^* Q_X \ , \qquad (**)$$

where Q is the (curvature) form on $\bigwedge^2 \mathbb{C}T(X)$ (see §6) and $d_{\mathbb{C}}^*$ denotes the dual of the complexified differential (see below) of f. Thus $d_{\mathbb{C}}^* Q$ is a Hermitian form on $\Gamma_{\mathbb{C}}^2 T(V)$ and the trace is taken with respect to the Kähler metric.

An important consequence of $(**)$ is

If $K_{\mathbb{C}} < 0$ and rank $f > 2$, *then the inequality* $(*)$ *becomes strict,*

$$\|\operatorname{Hess}_{\mathbb{C}} f\|^2 < \operatorname{div}\partial \ . \qquad (+)$$

Definition of $d_{\mathbb{C}}$. If $d : \mathbb{C}^n \to \mathbb{R}^m$ is an $\mathbb{R}$-linear map then $d_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^m = \mathbb{R}^m \oplus \sqrt{-1}\mathbb{R}^m$ is the unique $\mathbb{C}$-linear map whose composition with the projection $\operatorname{Re} : \mathbb{C}^m \to \mathbb{R}^m$ equals d. That is

$$d_{\mathbb{C}}(x) = d(x) - \sqrt{-1}d(\sqrt{-1}x) \ , \qquad x \in \mathbb{C}^n \ .$$

6.C. Sui-Sampson for foliations. Let Λ be foliated by complete Kähler manifolds whose Ricci curvatures are bounded from below by some negative constant $> -\infty$. We assume that Λ is given some transversal measure μ and let $f : \Lambda \to X$ be a measurable map which is C^3-smooth and harmonic on each leaf of Λ.

THEOREM . *If $K_{\mathbb{C}} X \leq 0$ and the map f has finite total energy*

$$E(f) \underset{\text{def}}{=} \int_\Lambda e_f \underset{\text{def}}{=} \tfrac{1}{2} \int_\Lambda \|df\|^2 < \infty \ ,$$

(where the differential d is taken along the leaves and the integration is performed with $d\mu\, dv$ for the leafwise Riemannian volume dv), then

(1) The map f is pluriharmonic on almost all leaves, i.e.

$$\operatorname{Hess}_{\mathbb{C}} f = 0 \ .$$

(2) The form Q on $\bigwedge^2 \mathbb{C}T(X)$ is isotropic on the image of the complexified differential of f. That is every bivector $a \wedge b$ tangent to the image of a leaf of Λ satisfies

$$Q_X(d_{\mathbb{C}}a \wedge d_{\mathbb{C}}b \ , \ \overline{d_{\mathbb{C}}a} \wedge \overline{d_{\mathbb{C}}b}) = 0 \ .$$

In particular, if $K_{\mathbb{C}} X < 0$, then rank $f \leq 2$ *almost everywhere on Λ.*

(3) $\operatorname{div}\partial = 0$ *almost everywhere on Λ.* (This will not be used in future.)

Proof: Integrate $(*), (**)$ and $(+)$ over Λ using the cut-off function p_ε^2 as in 4.C_3'.

Let us combine the above with the existence theorem for harmonic measures (see 4.D) and obtain the following property of leafwise smooth (non-harmonic) maps $f : \Lambda \to X$.

6.C_1. COROLLARY. *If X is compact with convex (e.g. empty) boundary and $K_{\mathbb{C}} X < 0$, then every $f : \Lambda \to X$ has*

$$\mathrm{VOL}_3[f] = 0 \, . \tag{$*$}$$

6.C_2. Let us indicate a generalization of the integrated Siu-Sampson formula for maps f with small integrals $\int \|\nabla f\|^2$ and $\int \|\nabla \Delta f\|^2$, where the basic example is the solution f_t of the heat equation for large t.

If for some family of maps f_t the L_2-norms of the first and the second (covariant) derivatives remain bounded by a fixed constant while the L_2-norms of Δf_t and $\nabla \Delta f_t$ decay for $t \to \infty$, then

$$\int_\Delta \left(\| \operatorname{Hess}_{\mathbb{C}} f_t \| - \operatorname{Trace} d_{\mathbb{C}}^* Q_X \right) \to 0$$

for $t \to \infty$. In particular, if $K_{\mathbb{C}}(X) \le \varepsilon < 0$, then

$$\mathrm{Vol}_4\, f_t \to 0 \quad \text{for} \quad t \to \infty \, .$$

Proof: It is enough to observe that the Laplace operator enters $\operatorname{div} \partial$ via some scalar product of the form $\langle \nabla f, \nabla \Delta f \rangle$. (Compare [Sam].)

*6.C_2'. **Remarks***: (a) The above applies to the heat flow f_t if the curvatures and their covariant derivatives of X and of the leaves of Λ are bounded (see 4.C_4'). Then *every* map f has

$$\mathrm{Vol}_4[f] = 0 \, . \tag{$**$}$$

(b) The estimates used in the proof of $(**)$ do not involve the dimension of X in any way. It follows, that $(**)$ remains valid for maps into *infinite dimensional* Riemannian manifolds X. In fact a great deal of the harmonic maps goes through with the infinite dimensional target space X if one allows oneself to use the following (*ultra-*) *completeness* property. Let K be a compact metric space and $f_i : K \to X$ be a sequence of uniformly Lipschitz maps, such that the distance functions d_i on K induced by f_i from X converge to some function d on K (or rather on $K \times K$). Then K admits a d-isometric map $f : K \to X$. Moreover, this (*ultra-*) *limit* map should be functorial in the category of the compact metric spaces. For example if $K' \subset K$ and we first take f' for $f_i' = f|K'$, then there should exist f for f_i such that $f|K' = f'$.

7. Hyperbolic pinching

For a Riemannian manifold X with negative curvature we define the *pinching constant* $\mathrm{pi}(X)$ with the infimum of those $\alpha \geq 1$ for which

$$-\kappa \leq K(X) \leq -\alpha\kappa \ , \qquad (*)$$

for some constant $\kappa > 0$. Then we define the *local pinching constant* $\mathrm{pilo}(X)$ which is the infimal α for which $(*)$ holds true with some *function* κ on X. Next for a class $\mathcal{X}$ of manifolds X we define $\mathrm{pi}(\mathcal{X})$ as $\inf \mathrm{pi}(X)$ over all $X \in \mathcal{X}$, and similarly we define $\mathrm{pilo}(\mathcal{X})$. If $\mathcal{X}$ consists of all complete manifolds diffeomorphic, homeomorphic, homotopy equivalent or quasi-isometric (see [G-L-P], [Gro3]) to X, then we use the notations $\mathrm{Difpi}(X)$, $\mathrm{Toppi}(X)$, $\mathrm{Hompi}(X)$, $\mathrm{Qispi}(X)$, etc.

7.A. The simplest obstruction to the (local) pinching comes from the Gauss-Bonnet-Chern-Weil theorem. Namely, every Pontryagin number p of a closed locally α-pinched manifold is bounded by the Euler characteristic as follows

$$|p(X)| \leq C(\alpha)|\chi(X)| \ ,$$

where $C(\alpha) = C_p(\alpha)$ is a continuous function in the interval $[1, \alpha_n)$ for some $\alpha_n > 0$ (here $n = \dim X$), such that $C(1) = 0$ and if $n = 4$, then $\alpha_n = \infty$.

COROLLARY. *Let X be a compact locally symmetric space of* $\mathrm{rank}_{\mathbb{R}} = 1$ *which does not have constant negative curvature. If* $\dim X$ *is a multiple of 4, then*

$$\mathrm{Toppilo}\, X \geq 1 + \varepsilon$$

for some $\varepsilon > 0$ *depending on* $n = \dim X$. (One can replace Top by Hom if the comparison manifolds have the same dimension as X.)

Remark: The above result has been known for several decades but the following evaluation of ε for $\dim X = 4$ is relatively new (see [Vi1].)

If $\dim X = 4$ *then the above* ε *equals* 3.

This result is sharp, as the relevant (locally symmetric) metric is 4-pinched to start with.

Remark: The above result by Ville has settled for $\dim X = 4$ the long standing pinching conjecture for locally symmetric spaces.

Another solution of the (non-local) 4-pinching problem for all dimension has been recently announced by U. Hamenstädt (a private communication) but her proof (based upon the study of the geodesic flow) has not appeared yet.

7.B. Another simple restriction on pinching comes from nilpotent subgroups Γ in the fundamental group $\pi_1(X)$. Namely, *if* π_1 *contains such a* Γ *of nilpotency degree* k, *then*

$$\operatorname{pi} X \geq k^2$$

provided the homological dimension of Γ *satisfies*

$$\dim \Gamma \geq \dim X - 2 \ ,$$

(compare [Gro8] and [Kan]).

Proof: Let Γ isometrically act on a complete simply connected α-pinched manifold Y. If Γ is nilpotent of degree ≥ 2 (i.e. non-Abelian) then Γ fixes a point P at the ideal boundary $\partial_\infty Y$. That is there exists a (horo)function $h : Y \to \mathbb{R}$ invariant under Γ, such that h is a convex function with $\|\operatorname{grad} h\| = 1$ and the gradient lines of h are geodesic in Y asymptotic to p and $h(y) \to -\infty$ for $y \to p$.

Now, take some non-trivial element $\gamma \in \Gamma$ and look at the displacement function $d(y) = \operatorname{dist}\big(\gamma(y), y\big)$ as $y \to p$ along some geodesic ray ℓ in Y. The lower bound on the curvature, say $K(x) \geq -\kappa$, implies that

$$\log d(y) \geq \kappa^{1/2} h(y) + \operatorname{const}_\ell$$

while the upper bound $K(x) \le -\alpha\kappa$ makes

$$\log d(y) \le (\alpha\kappa)^{1/2} h(y) + \text{const}'_\ell .$$

Therefore the displacement function d and d_1 of any two non-trivial elements in Γ satisfies for $y \to p$

$$|\log d| \le \alpha^{1/2}(1+\varepsilon)|\log d_1| , \tag{$*$}$$

where $\varepsilon \to 0$ for $y \to p$.

Now let a non-trivial element $\gamma \in \Gamma$ be a k-th order commutator,

$$\gamma = [\gamma_1[\gamma_2[\dots\gamma_k] .$$

Then, by the Margulis lemma (see [G-L-P], [Bus-Ka]) the displacement d of γ is bounded by d_i of γ_i as follows

$$d \ge C\sum_{i=1}^{k} d_i , \tag{$**$}$$

provided the *rotational part* of each γ_i at y (i.e. the holonomy around the geodesic loop in Y/Γ corresponding to γ_i) has (at most) the same order of magnitude as d_i.

Now, if $\dim\Gamma = \dim X - 1$, then the action of Γ is *cocompact* on each level $h^{-1}(t) \subset Y$ of the function h. Since these levels have bounded curvatures the quotient manifolds $h^{-1}(t)/\Gamma$ are *compact* almost flat for $t \to \infty$ and the required bound on $\operatorname{rot}\gamma_i$ is provided by the estimates in [Gro9] and in [Bus-Ka], and a similar argument (compare 7.B′ below takes care of $\dim\Gamma = \dim Y - 2$.

Finally, we play ($**$) against ($*$) for $d_i \to 0$ (and C in ($*$) bounded) and obtain the required inequality

$$\sqrt{\alpha} \ge k .$$

Example. Let X be a complete non-compact locally symmetric manifold of finite volume. If $\operatorname{rank} X = 1$ and X does not have constant curvature, then $\pi_1(X)$ contains a nilpotent subgroup of degree 2 and $\dim\Gamma = \dim X - 1$. Therefore,

$$\operatorname{Toppi} X = 4 .$$

7.B′. Let $p = \dim X - \dim \Gamma - 1$ and r be the *rank* of the orthogonal group $O(p)$, that is

$$r = \mathrm{ent}(p/2) .$$

Then we have the following estimates for the pinching, of X,

$$\mathrm{pi}\, X \geq \big(k/(r+1)\big)^2 .$$

Idea of the proof. The rotational effect of each $\gamma \in \Gamma$ is essentially restricted to p directions in $h^{-1}(t) \subset X$ "normal" to the orbits of Γ. It follows, there exists an integer N of order at most $\big(d(\gamma)\big)^{-\frac{r}{r+1}}$ such that the rotational part of γ^N is at most $\big(d(\gamma)\big)^{\frac{1}{r+1}}$. Since

$$d(\gamma^N) \leq N d(\gamma) \leq \big(d(\gamma)\big)^{\frac{1}{r+1}} ,$$

a commutator of k such elements will be of order at most $d^{\frac{k}{r+1}}$ by the Margulis lemma, and the proof is concluded as earlier.

Question. Can one improve the above estimate in order to make it non-vacuous for all p? (The above bound on $\mathrm{pi}\, X$ says nothing what-so-ever for $r + 1 \geq k$.)

Remark: Since all nilpotent groups Γ have $k \leq \dim \Gamma - 1$ the best pinching bound one can obtain with the above is

$$\mathrm{pi}\, X \geq (\dim X - 2)^2 .$$

One knows however (see [GrTh]), that for each $n \geq 4$ and every $\alpha \geq 1$ there exists a closed n-dimensional manifold X of negative curvature, such that

$$\mathrm{Toppi}\, X \geq \alpha .$$

7.C. **Quasi-isometric pinching.** A locally diffeomorphic map between Riemannian manifolds, say $f : M \to N$, is called *α-pinched* if the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \lambda_m \geq 0$ of $D_f^* D_f$ for $m = \dim M = \dim N$ satisfy with some constant $C_f \in \mathbf{R}$

$$\log \lambda_1 \leq (\alpha \log \lambda_m + C_f) ,$$

whenever $\lambda_m \geq 1$.

7.C$_1$. LEMMA. *Let Y be a complete simply connected α-pinched manifold of dimension $m + 1$. Then for every horosphere $N \subset Y$ there exists a $\sqrt{\alpha}$-pinched map $f : S^{m-1} \times \mathbf{R}_+ \to N$, such that $\lambda_m(s,t) \to \infty$ for $t \in \infty$.*

Proof: Let $r \subset Y$ be a geodesic ray in the horoball bounded by N and let M be the boundary of the ε-tube around r for some $\varepsilon > 0$, say for $\varepsilon = 1$.

Then the map f_0 of M to N, which sends each $m \in M$ to the intersection of N with the exterior normal ray to N, is α-pinched by the standard comparison theorem. On the other hand M is bi-Lipschitz equivalent to $S^{m-1} \times \mathbf{R}_+$ and the composition of f_0 with this equivalence is our f.

7.C$'_1$. VARIATION. *Let Y' be another strictly negatively curved (i.e. $K(Y') \leq -\kappa < 0$) manifold which is quasi-isometric to Y. Then for every horosphere $N' \subset Y'$ there exists a manifold N'' such that*

(i) $\dim N'' = \dim N' = m$

(ii) N'' is quasi-isometric to N'

(iii) N'' has bounded geometry, that is

$$|K(N'')| \leq C < \infty$$

and

$$\operatorname{Inj}\operatorname{Rad} N'' \geq \varepsilon > 0 \ .$$

(iv) N'' receives a $\sqrt{\alpha}$-pinched map $f' : S^{m-1} \times \mathbf{R}_+ \to N''$ with $\lambda'_M \to \infty$ as in the lemma.

Proof: Let $N'_0 \subset Y$ correspond to N' under the quasi-isometry between Y and Y'. Let N'' be the boundary of the ε-neighbourhood of the convex hull $H_0 \subset Y$ of N'_0 for some $\varepsilon > 0$, say for $\varepsilon = 1$. Then N'' lies in some δ-neighbourhood of N'_0 (see [And3]) and it is quasi-isometric to N'. It is also clear that the normal projection form a ray $r \in H_0$ delivers a α-pinched map $M \to N''$ and then our $f : S^{M-1} \times \mathbf{R}_+ \to N''$ comes along. Q.E.D.

Remark: Notice that the map f we have obtained is *proper*.

7.C$''_2$. COROLLARY. *Every horosphere N' contains arbitrarily large bounded domains Ω', such that*

$$\operatorname{Vol}_m \Omega' \geq (\operatorname{Vol}_{m-1} \partial\Omega')^{1+\beta} \tag{+}$$

for every fixed β in the interval $0 \leq \beta < \left(\sqrt{\alpha}(m-1)\right)^{-1}$.

Proof: Let $J = \big(\prod_{i=1}^{m} \lambda_i\big)^{1/2}$ and $I = \big(\prod_{i=1}^{m-1} \lambda_i\big)^{1/2}$. Then the pinching condition $\log \lambda_1 \le \sqrt{\alpha} \log \lambda_m$ for $\lambda_i \ge 1$ implies

$$J \ge Q^{1+\beta_0}$$

for $\beta_0 = \big(\sqrt{\alpha}(m-1)\big)^{-1}$.

Now for $\Omega'' = \Omega''(R) = f\big(S^{m-1} \times [0,R]\big) \subset N''$ we have

$$\mathrm{Vol}_m \Omega'' = \int_0^R dt \int_{S^{m-1}} J ds$$

and

$$\mathrm{Vol}_{m-1} \partial\Omega'' \le C + \int_{S^{m-1}} I(s,R) ds \ ,$$

for $C = \mathrm{Vol}_{n-1} f(S^{n-1} \times 0)$. It follows that for large $R \to \infty$, $\mathrm{Vol}_n \Omega''(R) \ge c \int_0^R \big(\mathrm{Vol}_{n-1} \partial\Omega(t)\big)^{1+\beta_0} dt$ for $c > 0$, which implies (+) for Ω''. This gives us what we want for N' as it is quasi-isometric to N''.

7.C$_3$. Examples. (Compare [Gro10], [Pan3]) (1) Let Y' be the complex hyperbolic space of dimension $n = m+1$. Then all large domains Ω' in the horospheres $N' \subset Y$ satisfy the following isoperimetric inequality (see [Pan1], [Var])

$$\mathrm{Vol}_m \Omega' \le (\mathrm{Vol}_{m-1} \partial\Omega')^{1+\frac{1}{m}} \ .$$

Therefore

$$\mathrm{Qispi}\, Y' \ge \left(\frac{m}{m-1}\right)^2 = \left(\frac{n+1}{n}\right)^2 \ .$$

(2) Let Y' be a quatorian hyperbolic space of dimension $n = m+1$. Then the isoperimetric inequality for N' reads

$$\mathrm{Vol}_m \Omega' \le (\mathrm{Vol}_{m-1} \partial\Omega')^{1+\frac{1}{m+2}} \ .$$

This gives

$$\mathrm{Qispi}\, Y' \ge \left(\frac{m+2}{m-1}\right)^2 = \left(\frac{n+3}{n}\right)^2 \ .$$

(3) Let Y' be the hyperbolic Cayley plane of dimension 16. Then

$$\mathrm{Vol}_{15} \Omega' \le (\mathrm{Vol}_{14} \partial\Omega')^{22/21}$$

and

$$\mathrm{Qispi}\, Y' \ge 9/4 \ .$$

Remark: There are further quasi-isometry pinching results in [Pan3] and [Pan4] but one does not know yet if $Quispi = 4$ in the above examples.

7.D. Let X be as earlier a complete Riemannian manifold which may have a non-empty convex boundary. We assume $K_{\mathbb{C}}(X)$ is strictly negative (e.g. X is locally $(4-\varepsilon)$-pinched for $\varepsilon > 0$) and we want to derive from this condition some specific geometric and topological properties of X, expressed in terms of those of continuous maps of Kähler manifolds and Kähler foliations into X. We start with the simplest case, that is of a compact connected Kähler manifold V without boundary and observe the following by now well known result essentially due to Sampson (see [Sam]).

7.D$_1$. THEOREM. *Every continuous map $f : V \to X$ is non-ℓ-essential for every $\ell \geq 3$. That is*

$$\mathrm{VOL}_\ell[f] = 0 \quad \text{for} \quad \ell \geq 3 \,. \tag{$*$}$$

In particular, if X is compact then f is homotopic to a map sending V into the 2-skeleton of a given triangulations of X.

Proof: Let f_0 be a smooth map homotopic to f and f_t the heat flow starting from f_0. Then $e(f_t) = \|df_t\|^2$ is bounded on $V \times [0, \infty)$ and so one sees as in 4.D, that

$$\int_V (\mathrm{vol}_3 f_t)^p \to 0 \,, \tag{$*+$}$$

for all $p \geq 1$, where vol_3 denotes the elementary symmetric function of degree 3 in the eigenvalues of the operator $(D_f^* Df)^{1/2}$. (The notation D_f for the differential is supposed to bring along the idea of an operator, namely $D_f(v) : T_v(V) \to T_x(X)$, while df is thought of as a 1-form on V with values in $T(X)$.) Obviously $(*+)$ implies $(*)$ which, in turn, implies the 2-contractibility of f for compact manifolds X.

7.D$_2$. ***Remarks***: (a) Notice that the proof of $(*)$ we have indicated equally applies to *infinite dimensional* manifolds X.

(b) If X is compact, the above theorem becomes a pinching result: ***If X receives a non-2-contractible map from a compact Kähler manifold, then***

$$\mathrm{Toppiloc}\, X \geq 4 \,.$$

Moreover, X admits no metric g with $K_{\mathbb{C}}(X,g) < 0$.

(c) If X is compact then f is homotopic to a harmonic map h which is more special than just 2-degenerate. Namely, there are two possibilities.

(1) There exists a compact Riemann surface S, a holomorphic map $h_1 : V \to S$, and a harmonic map $h_2 : S \to X$ such that $h = h_2 \circ h_1$.

(2) The map h sends V to a single closed geodesic in X.

The properties (1) and (2) easily follow from the pluriharmonicity of h as was pointed out in [Ca-To] and [Jo-Ya].

7.D$_3$. COROLLARY. *If $H_1(V;\mathbb{R}) = 0$, then every continuous map $V \to X$ is contractible (where X is compact with $K_{\mathbb{C}}X < 0$).*

7.D$_4$. The above conclusion for compact X remains valid in the non-compact case if the map f is homotopic to a harmonic map h. In fact, the Eells-Sampson theory insures such an h unless there exists a homotopy f_t of $f = f_0$ for $t \in [0,\infty)$ such that,

(a) the homotopy map $V \times [0,\infty) \to X$ is proper,

(b) the energy density $e(f_t) = \|df_t\|^2$ is bounded on $v \times [0,\infty)$.

Indeed the heat flow f_t satisfies (b) and then it subconverges to a harmonic map unless the whole image $f_t(V)$ goes to infinity in X for $t \to \infty$, which is exactly what (a) says. (The dichotomy between the existence of h and (a) + (b) has been pointed out in [Cor] and [Don].)

Notice that the condition (a) is purely topological (in fact, it depends on the *proper* homotopy type of X) and it implies, for example that f induces a *trivial* homomorphism on the cohomology with compact support. On the other hand (b) is a geometric condition which depends on the Lipschitz class of X.

Notice, that for every X with $K(X) \leq 0$ there exists a complete metric of negative curvature on $X \times \mathbb{R}$, such that every map f of a compact manifold V into X can be homotoped with the conditions (a) and (b) satisfied. (In fact, one can achieve $e(f_t) \to 0$ for $t \to \infty$.) This makes problematic the use of the harmonic theory for getting dimension free local pinching estimates without additional geometric assumptions on X. However, there are several pinching results under some extra conditions. Here are some of them.

(A) X has bounded geometry, i.e.

(a) $|K(X)| \leq C < \infty$

and

(b) $\operatorname{Inj}\operatorname{Rad} X \geq \varepsilon > 0$.

In this case every map f is homotopic to a composed map $h_2 \circ h_1$, where h_2 is either a holomorphic map of V onto a Riemann surface S or a map of V to the circle $S = S^1$, and where $h_1 : S \to X$ is a (non-harmonic, in general) smooth map.

Idea of the proof: Take the heat flow f_t for some sufficiently large t. Then an appropriate small perturbation of this f_t decomposes as if it were already a harmonic map.

Remark: It is not hard to figure out what happens if we forget about Inj Rad and only retain the condition $|K(X)| \leq C$. An especially easy case here is where $K(X) \leq -\kappa < 0$ and one immediately sees (compare [Don], [Cor]) that *either f is homotopic to a harmonic map and hence is decomposable as earlier or the image of the fundamental group of $\pi_1(V)$ in $\pi_1(X)$ contains a nilpotent subgroup of finite index.*

7.D$_4$. PINCHING COROLLARY. *If X receives a non-decomposable (e.g. non 2-contractible) map from a compact Kähler manifold, such that the image of the fundamental group contains no nilpotent subgroup of finite index, then the homotopy pinching constant of X is at least 4,*

$$\operatorname{Hompi} X \geq 4 .$$

7.D$'_4$. Example. *Let X be a compact manifold locally isometric to the complex hyperbolic space of dimension* ≥ 4. Then

$$\operatorname{Hompi} X = 4 \tag{i}$$

and

$$\operatorname{Toppiloc} X = 4 . \tag{ii}$$

Remarks: (a) one can replace "Top" in (ii) by "Hom" if one restricts the comparison manifolds to those of the dimension $n = \dim X$.

(b) In most of our discussions we could use *non-compact* complete Kähler manifolds V of finite volume, provided such a V admits a selfmapping φ homotopic to the identity, having finite total energy $E(\varphi) < \infty$ and having compact image $\varphi(V) \subset V$. (With such a φ every continuous map $f : V \to X$ is homotopic to another one with finite energy and then the heat flow applies.) For example, the above (i) and (ii) are valid for non-compact X of finite volume of dimension ≥ 6, where the existence of the needed $\varphi : X \to X$ is easy.

7.D$_5$. Let us try to extend the results of the previous sections to maps of Kähler foliations Λ into X. The major technical problem here is a possible unboundedness of $e(f_t)$ for $t \to \infty$ which prevents us from proving (*+) of 7.D$_1$ for all p. Yet we do have (see 6.C$_1$, 6.C$_2$) the decay of Vol_3 and Vol_4 under the heat flow and so the non-essentiality relation $\mathrm{VOL}_\ell[f] = 0$ of 7.D$_1$ remains valid for $\ell = 3$ and 4, where the case $\ell = 3$ needs an extra assumption of the compactness of X according to 6.C$_1$. (Probably, this compactness is not hard to remove.)

Now, in order to make a non-trivial conclusion, we need an example of a Kähler foliation Λ and of an essential (for $\ell = 3$ or 4) map $f : \Lambda \to X$.

Here is our basic

Example. Let X be a compact locally symmetric space which is covered either by the quatornian hyperbolic space of dimension ≥ 8 or by the Cayley plane. Then the foliation $\mathrm{Geo}_4 X$ of totally geodesic submanifolds contains a subfoliation $\Lambda \subset \mathrm{Geo}_4 X$ of Kählerian submanifolds isometric to the 4-dimensional complex hyperbolic space. This Λ carries (by an easy argument) a smooth transversal measure and it is 4-essential (see 3.G). Therefore the projection $f : \Lambda \to X$ is 4-essential (see 3.G, 3.H).

PINCHING CONCLUSION. *X admits no metric g with $K_{\mathbb{C}}(X,g) < 0$. In particular*

$$\mathrm{Toppiloc}\, X = 4 \ .$$

Remark: This result is significantly weaker than that for X covered by the complex hyperbolic space. One may expect an improvement coming from a better foliated harmonic theory or from Bochner-type formulas for non-Kählerian symmetric spaces, like those recently discovered by K. Corlette

Finally, we notice that if our $\Lambda = \mathrm{Geo}_4(X)$ contains a compact leaf, that is a compact totally geodesic Kählerian submanifold Y in X, then we may directly apply the harmonic theory to the map $f|Y$. (The existence of compact totally geodesic submanifolds of dimension ≥ 2 is an extremely difficult problem for non-arithmetic spaces X. On the other hand the harmonic theory offers a very promising approach to the arithmeticity of the quatorinian hyperbolic and Cayley spaces.)

7.E. Rigidity for quatornian and Cayley spaces. Let us improve the above results by allowing the non-strict inequality $K_{\mathbb{C}}(X) \leq 0$ but still insisting on $K(X) \leq -\varepsilon < 0$. Then we consider a compact locally symmetric space Y, quatornian hyperbolic of dimension ≥ 8 or Cayley-type, and let $\Lambda_2 \subset \mathrm{Geo}_2 Y$ denote the foliation of totally geodesic complex hyperbolic planes, and $\Lambda_4 \subset \mathrm{Geo}_4 Y$ the foliation of totally geodesic complex hyperbolic subspaces of real dimension 4.

7.E$_1$. THEOREM. *Let $f_0 : Y \to X$ be a continuous map, such that the corresponding map $g_0 : \Lambda_2 \to X$ is 2-essential (i.e.* $\mathrm{AREA}[g_0] > 0$*). Then f is homotopic to a geodesic map which is isometric up to a scalar multiple.*

Proof: Let $g : \Lambda_2 \to X$ be the leafwise harmonic map measurably homotopic to g_0 and let $h : \Lambda_4 \to X$ be such a map obtained from $h_0 : \Lambda_4 \to X$ corresponding to f_0 (see 4.E$'_2$). Then the map h is pluri-harmonic as $\mathrm{Hess}_{\mathbb{C}}\, h = 0$ (see 6.C).

Next we consider the 2-dimensional foliation $\Lambda'_2 \subset \mathrm{Geo}_2 \Lambda'_4$ of the *complex geodesic* in the leaves $V \subset \Lambda_4$ which were called earlier complex hyperbolic planes in $Y \supset V$. Since h is pluriharmonic, the corresponding map $h' : \Lambda'_2 \to X$ is harmonic. On the other hand, the composition h'' of g with the natural projection $\Lambda'_2 \to \Lambda_2$ also is (obviously) harmonic. By the uniqueness theorem for harmonic maps these two maps, h' and h'', coincide and therefore the maps g and h agree as follows.

Let V^4 and V^2 be totally geodesic submanifolds in Y representing some leaves in Λ_4 and Λ_2 respectively, such that $V^4 \supset V^2$. Then the map g on V^2 equals h on V^2. Then by elementary (projective) geometry of Y, the maps g and h come from some measurable map $f : Y \to X$, such that the corresponding maps of Λ_2 and Λ_4 to X equal g and f correspondingly.

Now, the map f is "pluriharmonic" in a very strong sense: the restriction of f to each plane $V_2 \subset Y$ from Λ_2 is harmonic. Then a trivial linear algebraic argument shows that f is geodesic and, consequently, isometric up to a scalar. Q.E.D.

RIGIDITY COROLLARY. *Every metric g on Y with $K_{\mathbb{C}}(Y,g) \leq 0$ and $K(Y,g) < 0$ is isometric up to a scaling factor to the original locally symmetric metric.*

Proof: The only point which needs checking is the relation AREA$[f] > 0$ for the projection $p : \Lambda_2 \to Y$, and this follows from the discussion in 3.G and 3.H.

Remarks: (a) It is not hard to remove the assumption $K(Y,g) < 0$.

(b) The above argument easily extends to complete non-compact Y with finite volume with a use of a self-homotopy (equivalence) $Y \to Y$ with compact image and finite energy.

(c) The condition AREA > 0 does not seem very restrictive for the map $Y \to X$ since (the fundamental group of) Y has Kazhdan's T- property.

(d) Our rigidity argument applies to those locally symmetric spaces which contain sufficiently many Kählerian subspaces.

(e) For the Kählerian case N. Mok has developed a comprehensive (super)-rigidity theory only a part of which is published in [Mok]. It seems that most (if not all) of his results extend to Kähler foliations and then can be useful for non-Kählerian symmetric spaces.

(f) Another version of harmonic (super)-rigidity theory was recently suggested by K. Corlette as we have already mentioned.

(g) Besides symmetric spaces an important example of a space with $K_{\mathbb{C}}(X) \leq 0$ is the Teichmüller space (and hence the Riemann moduli space) with the Weil-Peterson metric (see [Schu]).

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M. Gromov
IHES
35 route de Chartres
Bures sur Yvette 91440
France

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