To Vladimir Igorevich Arnold

at his 60-th birthday

LAGRANGIAN INTERSECTIONS THEORY
Finite-dimensional approach

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Contents

Chapter 0

Introduction to the main results

0.0 Historical remarks

In this paper we systematically explore the possibilities of a finite-dimensional approach, known as the method of generating functions, to problems of Lagrangian and Legendrian intersections, and define invariants of spaces of Lagrangian and Legendrian embeddings which go beyond the traditional Floer homology approach.

It was known since the time Hamilton and Jacobi that Lagrangian submanifolds of symplectic manifolds can be described, at least locally, by their *generating functions*. This observation plays an important role in Hamiltonian Mechanics and Geometrical Optics and it was a motivating force for Poincaré's "last geometric theorem" (proven by G. Birkhoff by a different method) and for Arnold's conjectures about the number of fixed points of symplectomorphisms and Lagrangian intersections.

Unfortunately, in the global context, these generating functions become multi-valued (wave-fronts) and difficult to handle (although something can still be done, see, for instance [22]). On the other hand, already Hamilton considered in the context of Geometrical Optics a more general notion of a generating function (or a generating family) which depends on extra variables. By adding extra variables, one can try to trade the multi-valuedness for dimension. However, the usefulness of this notion was not understood by Hamilton's contemporaries. Jacobi, for instance, considered these extra variables to be an unnecessary complication which is nuisance for computational purposes (see [45]).* As a result, the notion was forgotten until it was rediscovered more then a century later by V. I Arnold (see [5]) in his study of singularities of Lagrangian maps, fronts and caustics, and by by L. Hörmander (see [43]), in the context of Fourier integral operators.

*We thank V. I. Arnold for this reference.

It took some time before the idea of this generating function penetrated into Symplectic topology. The difficulty is that it is unclear how to control the behavior of the generating function on extra variables at infinity, and without this control generating families were not amenable to Morse-theoretic methods. In an inexplicit form this problem was first resolved in Chaperon's (see [11]) finite-dimensional version of Conley-Zehnder's proof of Arnold's Conjecture for the $2n$-torus (see [18]). The first explicit construction of generating functions for Lagrangian submanifolds of cotangent bundles, suitable for the purposes of Symplectic topology, was done by F. Laudenbach and J.-C. Sikorav (see [47],[60]). Yu. Chekanov generalized this construction for Legendrian submanifolds of a 1-jet space, but his result (see [15]) remained unpublished until 1996.

For a while, the importance of the theory of generating functions were overshadowed by successes of theory of pseudo-holomorphic curves, Floer homology theory, and other infinite-dimensional methods. However, A. Givental (see [37] and [36]) used generating functions for proving results which were beyond the reach of these methods; C. Viterbo [65] developed a theory of symplectic invariants based on the theory of generating functions.

The development of Symplectic topology, and of its core – Lagrangian intersection theory, was greatly motivated by *Arnold conjectures* (see [1]), which roughly assert that Lagrangian intersections should abide Morse theory. In particular , the number of intersection points of Lagrangian submanifolds should be more than one can expect from pure homological considerations. However, despite all the progress achieved in Lagrangian intersections theory by infinite- and finite-dimensional methods, all the results so far were concentrated on *Morse inequalities* type estimates.† One of the goals of this paper is to show that the full strength of the *stable Morse theory,* which includes much more then just Morse inequalities, is also applicable to Lagrangian intersections theory (and its analogs in contact geometry).

It is interesting to notice that during all this period of time topologists were studying an object extremely reminiscent to the parametric generating function. It was first observed by J. Cerf [10] that topology of spaces of functions is tightly related to topology of spaces of diffeomorphisms. One of the main objects in his theory, the *Cerf diagram*, is nothing else but the wave front of a Legendrian (Lagrangian) manifold, generated by the corresponding family of functions. The theory, initiated by Cerf, known as *pseudo-isotopy theory*, was quickly algebraized by the work of Hatcher-Wagoner [41], Volodin [66], Igusa [44], Waldhausen [67] and others, and was put into the framework of the Algebraic K-theory.

In the present paper we establish a link between Symplectic topology and Pseudo-isotopy theory via the method of generating functions. In particular, we show that most algebro-K-theoretic invariants which appear in Pseudo-isotopy theory provide non-trivial information for studying the topology of spaces of Lagrangian and Legendrian embeddings, and for other problems in Symplectic topology.

The preliminary version of this work was first presented at the Floer Memorial Colloquium in Bochum in June 1992. Since then there appeared several

†Recently Fukaya ([33]) obtained more subtle estimates via holomorphic methods.

new papers exploiting the finite-dimensional approach to the Lagrangian and Legendrian intersections theory (see [64], [12], [29], [55] et al.).

About the structure of the paper

Chapter 0 of this paper contains a survey of stable Morse and Lusternik-Schnirelman theories. We also formulate there main Lagrangian and Legendrian intersection inequalities which we are dealing with in this paper, and consider the parametric analog of the theory and its relations with the Pseudo-isotopy Theory. The content of this chapter was circulated earlier as a separate paper [24].

Chapter 1 describes the geometry which makes possible the reduction of the Lagrangian intersection problem to the Stable Morse theory. As an application of this technique we give a proof of Conley-Zehnder-Chaperon theorem which provides a solution to Arnold's conjecture for the case of the n-torus. Our approach here is essentially equivalent to one of A. Givental in [35].

Chapter 2, which is devoted to cut-off and lifting constructions in Symplectic and contact geometry, is the most technical part of the paper. All arguments there are rather soft and essentially trivial. However, the cut-off in Symplectic world is quite a delicate business and can be very treacherous. These type of questions constantly arise but are mostly ignored and omitted in the related literature. Thus we decided to undertake the task of the systematic treatment of these problems, sometimes even in a more general situation than it was needed for our immediate purposes.

In Chapter 3 we revisit the geometry described in Chapter 1 from the point of view of generating functions theory. We prove Laudenbach–Sikorav's theorem [47] in a somewhat generalized form and then proceed with deducing Lagrangian intersection inequalities from 0.3 and their generalizations.

In Chapter 4 we show that the generating functions theorem in the Lagrangian case immediately implies its contact analog for Legendrian manifolds (Chekanov's theorem [15], see also [55]) and its projectivized analogs about generating hypersurfaces (comp. [29] and [55]). We use these results to prove intersection and kissing inequalities for Legendrian manifolds formulated in 0.4 and 0.5.

Chapter 5 is devoted to the parametric theory of generating functions and its relation with Pseudo-isotopy theory. In particular, we prove there Injectivity Theorem 0.6.1 from Section 0.6.

While writing this paper we benefited a lot from the information and advice which we received from specialists in many fields. In particular, V. I. Arnold provided us with several historical and contemporary references. G. Carlsson, R. Cohen, S. Gersten, A. Hatcher, K. Igusa, V. Sharko, J. Wagoner and S. Weinberger consulted us in pseudo-isotopy and stable Morse theories. We are extremely grateful to all who helped us.

0.1 Lagrangian intersection problem

0.1.1 Graphical and subgraphical Lagrangian and Legendrian submanifolds

Let us describe the classes of Lagrangian and Legendrian submanifolds for which we shall be proving our intersection inequalities.

Let U be an open subset in a smooth manifold V and $f : U \to \mathbb{R}$ be a smooth function. We introduce *the Lagrangian graph* L_f *of* f in the cotangent bundle $T^*(V)$ as

$$L_f = df(U) \subset T^*(V)\,,$$

where $df : U \to T^*(U) \subset T^*(V)$ is the differential of f. Submanifolds in $T^*(V)$ of this kind are called *graphical Lagrangian submanifolds in* $T^*(V)$ *over* U. Such a submanifold is *proper* in $T^*(V)$ if and only if $\|df(u)\| \to \infty$ for u converging to a boundary point $v \in \partial U$, where "proper" or "properly embedded" signifies here "being closed as a subset in $T^*(V)$". Sometimes we allow more general *non-exact* graphical Lagrangian submanifolds of the form $\varphi(U) \subset T^*(V)$ where φ is a *closed* but not necessarily exact 1-form on U.

Similarly we define *graphical Legendrian* submanifolds in the space $\mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$ of 1-jets of functions on V. These are images of the 1-jets $J^1_f = (df, f) : U \to \mathrm{Jet}^1(V)$ and they are denoted $\mathcal{L}_f \subset T^*(V)$ for smooth functions f on $U \subset V$.

Subgraphical varieties. Start with an example. Take a smooth submanifold $U \subset V$ of codimension $k \geq 0$ and denote by $L_U \subset T^*(V)$ the *annihilator* $(T(U))^\perp$ of $T(U) \subset T(V)$, i.e. the set of the covectors $\tau \in T^*_u(V)$ vanishing on $T_u(U)$ for all $u \in U$. Notice that this L_U has a natural structure of a k-dimensional vector bundle over U and L_U is *Lagrangian* in $T^*(V)$ for the canonical (symplectic) 2-forms ω on $T^*(V)$ (where "Lagrangian" signifies $\omega \mid L = 0$ for this $L = L_U$ and where the canonical ω may be written in local coordinates $q_1, \ldots, q_n$ on V as $\sum\limits_{i=1}^{n} dp_i \wedge dq_i$, where p_i are the dual coordinates (momenta) in the cotangent fiber).

The second basic example is that of a (parametric) *generating function* (comp. [43]). Here we have U fibered over V, and we view a smooth function $f : U \to \mathbb{R}$ as a V-parametric family of functions in the fibers, say $f_v : U_v \to \mathbb{R}$. Suppose that the set of the critical points (where $df_v = 0$) of f_v is discrete for all $v \in V$ and let $\underline{f}(v)$ be the multi-valued function on V assigning to v the set of the values of f_v on the set $\mathrm{crit}\, f_v \subset U_v$. Suppose we can separate the branches of $\underline{f}$, i.e. find a finite collection of smooth functions $f_1, f_2, \ldots$, having together the same values as $\underline{f}$. Then we may take $L_{\underline{f}} = L_{f_1} \cup L_{f_2} \cup \ldots$ in $T^*(V)$ as well as $\mathcal{L}_{\underline{f}} = \mathcal{L}_{f_1} \cup \mathcal{L}_{f_2} \cup \ldots$ in $\mathrm{Jet}^1(V)$ and call them $\underline{L}_f \subset T^*(V)$ and $\underline{\mathcal{L}}_f \subset \mathrm{Jet}^\perp(V)$ *generated by the function* f.

Now we turn to the following quite general picture where we have a smooth map $\alpha : U \to V$ and an arbitrary subset $L \subset T^*(V)$ (which will be a graphical Lagrangian submanifold in applications). We want to define some kind of a direct image $\underline{L} = \alpha_*(L) \subset T^*(U)$ which in the case of embedded $U \hookrightarrow V$ and $f = 0$ would reduce to L_U and for a fibration $U \to V$ with the above f would become $\underline{L}_f$. To do this we need the following (four) objects:

(1) $\ker D\alpha \subset T(U)$, the kernel of the differential $D\alpha : T(U) \to T(V)$,

(2) its annihilator $(\ker D\alpha)^\perp \subset T^*(U)$,

(3) the induced bundle $\alpha^*(T^*(V)) \to U$,

(4) the subset $\mathcal{A}_\alpha$ in the (total space of the) Whitney sum $T^*(U) \oplus \alpha^*(T^*(V))$ consisting of these pairs (κ_u, τ_v) where $\kappa_u \in (\ker D(\alpha))_u^\perp \subset T_u^*(U)$ and $\tau_v \in T_v^*(V)$ for $v = \alpha(u) \in V$, such that the value of the composed linear form $T_u(U) \xrightarrow[D_u\alpha]{} T_v(V) \xrightarrow[\tau_v]{} \mathbb{R}$ equals κ_u on $T_u(U)$.

Now, we define $\underline{L} \subset T^*(V)$ in three steps:

(i) intersect $L \subset T^*(U)$ with $(\ker D\alpha)^\perp$,

(ii) "restrict" $\mathcal{A}_\alpha$ to this intersection, i.e. take the intersection

$$\widetilde{L} = ((L \cap \ker D\alpha) \oplus \mathcal{L}^* T^*(V)) \cap \mathcal{A}_\alpha ,$$

(iii) project $\widetilde{L}$ to $T^*(V)$ for $(\kappa_u, \tau_v) \mapsto \tau_v$ and take the image for $\underline{L} \subset T^*(V)$.

This definition ‡ becomes more transparent in the following two cases.

I. The map α is an embedding, or more generally, an immersion $U \to V$.

II. The map α is a fibration, or more generally, a submersion $U \to V$.

Then one can visualize the operation $L \rightsquigarrow \underline{L}$ for maps $\alpha : U \to V$ of constant rank (of $D\alpha$) as these locally decompose into submersions followed by immersions. An especially pleasant case is where L is a Lagrangian (e.g. graphical) submanifold in $T^*(U)$ transversal to $(\ker D\alpha)^\perp \subset T^*(U)$ (where $(\ker D\alpha)^\perp$ itself is a smooth submanifold in $T^*(U)$ for α of constant rank). Then $\underline{L}$ is an *immersed Lagrangian* submanifold in $T^*(V)$ as a trivial argument shows.

It is useful to think of $\underline{L}$ in the above example as an abstract manifold immersed into $T^*(V)$. In fact, this "abstract manifold" is just our $\widetilde{L} \subset T^*(U) \oplus \alpha^*(T^*(V))$. But we shall use, somewhat incorrectly, the notation $\underline{L}$ hoping it will not be confused at the dangerous moments with its image under the implied immersion $\underline{L} \to T^*(V)$. (In fact, to be consistent, we should have started not with a subset but with an arbitrary map $L \to T^*(U)$ where the obvious generalization of the above constructions would assign to it another space $\underline{L}$ with a map $\underline{L} \to T^*(V)$. Then we would have a fair amount of functoriality for $L \rightsquigarrow \underline{L}$ which an interested reader may verify.)

From now on, *subgraphical Lagrangian subvariety* $\underline{L}_f \subset T^*(V)$ refers to a map $\alpha : U \to V$ of *constant rank* and is defined as $\underline{L}$ for $L = L_f$ for some smooth function $f : U \to V$. Such an $\underline{L}$ is called *regular* if $L_f \subset T^*(U)$ is transversal to $(\ker D\alpha)^\perp$ and then $\underline{L}$ is an immersed submanifold in $T^*(V)$.

Legendrian subgraphical. The above construction $L \rightsquigarrow \underline{L}$ obviously adjusts to subsets $\mathcal{L}$ in the jet space $\mathrm{Jet}^1(U) = T^*(U) \times \mathbb{R}$ and yield subsets in

‡The push-forward construction is a special case of *symplectic (or Lagrangian) correspondence*, see [43] and 2.7 below.

(or rather spaces mapped into) $\mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$. Indeed our construction is pointwise, i.e. defined on the level of points $\ell \in T^*(U)$ by assigning to them subsets $\underline{\ell} \subset T^*(V)$, such that $\underline{L} = \bigcup_{\ell \in L} \underline{\ell}$. This extends to $\mathrm{Jet}^1(U)$ by just multiplying everything by $\mathbb{R}$, i.e. by defining $\underline{(\ell, r)} = (\underline{\ell}, r)$ and then $\mathcal{L} \rightsquigarrow \underline{\mathcal{L}}$ for $\mathcal{L} \subset \mathrm{Jet}^1(U)$ is defined by adding up the pointwise operation over all point in $\mathcal{L}$. Here, as earlier, it is better to allow (non-injectively) mapped $\mathcal{L} \to \mathrm{Jet}^1(U)$ where the resulting $\underline{\mathcal{L}}$ may be mapped into $\mathrm{Jet}^1(V)$ non-injectively even if the starting $\mathcal{L}$ was honestly (injectively) embedded to $T^*(U)$.

About the terminology. A submanifold L in a symplectic manifold (M, ω) is called *Lagrangian* if the ω-orthogonal of the tangent bundle $T(L) \subset T(M)$ equals $T(L)$,

$$T^{\perp_\omega}(L) = T(L)\,.$$

This is equivalent to L being *ω-isotropic* (which amounts to $T^{\perp_\omega}(L) \supset T(L)$) as well as maximal, $\dim L = \frac{1}{2} \dim M$. Lagrangian submanifolds in $M = T^*(V)$ can be thought of as multivalued functions on V, or better as multivalued *closed* 1-forms $V \to T^*(V)$. In fact the differential relation (equation) characterizing "Lagrangian" equals the projectivization of the equation $d\varphi = 0$ on 1-forms $\varphi : V \to T^*(V)$. This means, the subset of the tangent planes to the Lagrangian submanifolds, equals the topological closure (in the Grassmann space $\mathrm{Gr}_n\, M$ for $M = T^*(V)$ of tangent n-planes in M for $n = \dim V$) of the tangent planes to submanifolds $L_f \subset T^*(V)$ for all smooth functions $f : V \to \mathbb{R}$.

A submanifold $\mathcal{L}$ in a contact manifold (N, η) is called *Legendrian* if it is a maximal integral submanifold of the hyperplane field η, i.e. $T(\mathcal{L}) \subset \eta$ and $\dim \mathcal{L} = \frac{1}{2} \dim N - 1$. The images of 1-jets $(\varphi, f) : V \to \mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$ of functions f on V are Legendrian as easily follows from the equation $\varphi = df$ satisfied by the 1-jets of functions. In fact "Legendrian" projectivizes the equation $\varphi = df$ imposed on sections $V \to T^*(V) \times \mathbb{R} = \mathrm{Jet}^1(V)$. Then it becomes clear that Legendrian submanifolds in $\mathrm{Jet}^1(V)$ projects to (immersed) Lagrangian ones in $T^*(V)$ and a trivial integration argument shows that every exact (see 2.4.1 for the precise definitions), say simply connected, Lagrangian submanifold $L \subset T^*(V)$ lifts to a Legendrian one in $\mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$, uniquely up to an $\mathbb{R}$-translation.

0.1.2 Morsification of Lagrangian intersections.

If $\underline{L}_1$ and $\underline{L}_2$ are subgraphical varieties in $T^*(V)$ then one can concoct an auxiliary function $\widetilde{f}$ with critical set homeomorphic to the intersection $\underline{L}_1 \cap \underline{L}_2$, where this intersection must be understood together with multiplicity. Namely it is defined as a subset in $\underline{L}_1 \times \underline{L}_2$ consisting of the pairs of points which one identified under the implied maps $\underline{L}_1, \underline{L}_2 \to T^*(V)$ which, recall, we do not assume to be embeddings (or even immersions) in general. For example, if $\underline{L}_1$ and $\underline{L}_2$ are graphical, say $\underline{L}_1 = L_{f_1}$ and $\underline{L}_2 = L_{f_2}$ for smooth functions on domains U_1 and U_2 in V, then clearly

$$L_{f_1} \cap L_{f_2} = \mathrm{crit}(f_1 \dot{-} f_2)$$

where $f_1 \dot{-} f_2$ means $f_1 - f_2$ on the intersection $U_1 \cap U_2$. But it is more useful to introduce another rather artificial difference, denoted $f_1 \overline{\times} f_2$ which we shall define here only in the case $V = \mathbb{R}^n$ and $T^*(V = \mathbb{R}^n) = \mathbb{R}^{2n}$. The function $f_1 \overline{\times} f_2$ is defined on $U_1 \times U_2 \times \mathbb{R}^n$ by

$$f_1 \overline{\times} f_2(u_1, u_2, v) = f_1(u_1) - f_2(u_2) + \langle v, u_1 - u_2 \rangle$$

for the Euclidean scalar product $\langle\ ,\ \rangle$ in $\mathbb{R}^n$. Furthermore, this straightforwardly generalizes to arbitrary maps $\alpha_i : U_i \to \mathbb{R}^n$, $i = 1, 2$, and functions f_i on U_i by

$$f_1 \overline{\times} f_2(u_1, u_2, v) = f_1(u_1) - f_2(u_2) + \langle v, \alpha_1(u_1) - \alpha_2(u_2) \rangle \, .$$

Thus $f_1 \overline{\times} f_2$ appears as a function on $U_1 \times U_2 \times \mathbb{R}^n$ and a simple argument shows that

0.1.1 $\operatorname{crit}(f_1 \overline{\times} f_2)$ *can be identified with* $\underline{L}_{f_1} \cap \underline{L}_{f_2}$.

One can also define $f_1 \overline{\times} f_2$ for $V \neq \mathbb{R}^n$, see 3.3.1.

0.1.3 Hamiltonian (Lagrangian) and contact (Legendrian) isotopies

Hamiltonian isotopies. Our objective is not so much the intersection between $\underline{L}_1$ and $\underline{L}_2$ themselves, but rather between suitably isotoped $\underline{L}'_1$ and $\underline{L}'_2$. The suitable isotopies here are those induced by *Hamiltonian isotopies* $I_i(t)$, $i = 1, 2$, of the ambient symplectic manifold $M = (T^*(V), \omega)$ where "Hamiltonian" for $I(t)$ means that the vector field $X_t = \frac{dI(t)}{dt}$ on M equals, for each t, to the ω*-gradient* of some function (Hamiltonian) $H_t : M \to \mathbb{R}$. The ω-gradient is uniquely defined with the (non-singular!) structure 2-form ω by the relation

$$\omega(\operatorname{grad}_\omega H, Y) = (dH)(Y)$$

for all tangent vectors $Y \subset T(M)$. (It is well known that every Hamiltonian isotopy is symplectic, i.e. $I(t) : M \to M$ preserves ω for each t. Conversely, if $H^1(M; \mathbb{R}) = 0$, then every smooth symplectic isotopy is Hamiltonian; if the cohomology $H^1(M; \mathbb{R})$ does not vanish, the Hamiltonian condition distinguishes the symplectic isotopies $I(t)$ for which the 1-form $Y \mapsto \omega(X_t, Y)$ is exact, rather than just closed.)

Contact isotopies. Recall that the jet space $N = \operatorname{Jet}^1(V)$ carries a canonical *contact structure*, i.e. a (maximally non-integrable) hyperplane field, or codimension 1 subbundle $\eta \subset T(N)$. (The canonical η on $\operatorname{Jet}^1(V) = T^*(V) \times \mathbb{R}$ is defined as $\ker(\lambda + dt)$ where λ is the canonical 1-form on $T^*(V)$ expressible in the local coordinates as $\lambda = \sum_{i=1}^{n} p_i dq_i$, so that $d\lambda = \omega$. For example if $N = \mathbb{R}^{2n+1} = \operatorname{Jet}^1(\mathbb{R}^n)$ then the 1-form λ is *globally* $\sum_{i=1}^{n} p_i dq_i$ for p_i and q_i the Euclidean coordinates in $\mathbb{R}^{2n} = \mathbb{R}^n_p \times \mathbb{R}^n_q$.)

One can easily identify Hamiltonian isotopies of $T^*(V)$ with those *contact*, i.e. η-preserving isotopies of the jet space $\operatorname{Jet}^1(V) = T^*(V) \times \mathbb{R}$ which preserve the foliation of $\mathrm{Jet}^1(V)$ into the $\mathbb{R}$-lines $m \times \mathbb{R}$ for $m \in T^*(V)$. But general contact isotopies of $\mathrm{Jet}^1(V)$ do not come from $T^*(V)$; yet they can be applied to subgraphical varieties in $T^*(V)$ since every such $\underline{L}_f$ comes from $\underline{\mathcal{L}}_f \subset \mathrm{Jet}^1(V)$ via the obvious projection $\mathrm{Jet}^1(V) \to T^*(V)$. Thus, *contact isotopy of* $\underline{L}_f$ means (comp. [53]) the projection to $T^*(V)$ of a contact isotopy of $\underline{\mathcal{L}}_f$ in $\mathrm{Jet}^1(V)$.[§] Now we can formulate the

Lagrangian intersection problem. *Study the intersection (with multiplicity) between* $\underline{L}'_1$ *and* $\underline{L}'_2$ *obtained by Hamiltonian or, more generally, contact isotopies form subgraphical varieties* $\underline{L}_1$ *and* $\underline{L}_2$ *in* $T^*(V)$. *In particular, find non-trivial (i.e. non-reducible to mere topology) low bounds on the cardinality* $\#(\underline{L}'_1 \cap \underline{L}'_2)$.

The problem makes sense for non-subgraphical, say Lagrangian, submanifolds but we have little to say about these.

We shall reduce the above problem (following in step with many people, see [1], [18], [11], [28], [31], [42], [47], [35], [37], [65], [64] et al.) to the *stable Morse theory* (of a function like $f_1 \barwedge f_2$) where the basic definitions are given below.

0.2 Recollection on the stable Morse–Lusternik–Schnirelman theory

0.2.1 Fibrations at infinity

We are interested in the stable MLS theory where we allow a *quadratic stabilization*, i.e. a passage from a smooth function $f(x)$ on X to $f(x) + Q(y)$ on $X \times \mathbb{R}^N$ where Q is a *non-singular* quadratic function (forms) on $\mathbb{R}^N$.

Recall that the MLS theory applies to functions on non-compact manifolds if the functions $f = f(x)$ in question behave as *fibrations outside a compact subset* in our X (which we assume here to be a manifold without boundary). Namely, we say that f is a *fibration at infinity*, if there exist a finite segment $[-a, a] \subset \mathbb{R}$ and a compact subset $K \subset f^{-1}[-a, a] \subset X$ such that the restriction of f to the following three subsets *fibers* them over their respective images

(i) $f^{-1}(-\infty, -a] \to (-\infty, -a]$

(ii) $f^{-1}[a, \infty) \to [a, \infty)$

(iii) $(f^{-1}[-a, a]) - K \to [-a, a]$.

(This is a particular case of a general notion of a *fibration modulo a closed subset* $K \subset X$, applied to an arbitrary smooth or continuous map $f : X \to Y$, see [38])

A practically useful (and almost obvious, comp. [38]) criterion for this is as follows

[§] Notice that a contact isotopy of Lagrangian submanifolds is not an isotopy.

Criterion 0.2.1 *f is a fibration at infinity if and only if X admits a complete Riemannian metric with respect to which $\|df(x)\| \geq \varepsilon > 0$ for x outside a compact subset in X.*

It is clear that a non-singular quadratic function is a fibration at infinity, as well as an arbitrary homogeneous polynomial f on $\mathbb{R}^N$ where the only critical point is located at zero. (Also $f+$ lower degree term is a fibration at infinity.)

Also observe the following simple but very important

0.2.2 ***Addition property.*** *If f_i on X_i are fibrations at infinity for $i = 1, 2$, then so is $f_1 + f_2$ on $X_1 \times X_2$.*

Thus "fibration at infinity" survives the quadratic stabilization $f \leadsto f + Q$.

0.2.2 Stable Morse and Lusternik-Schnirelman numbers

Let $[f]$ be a class of smooth functions where its different members f may be defined on different manifolds. We minimize the cardinality of the critical set of functions in this class and set

$$\mathrm{LuS}[f] = \inf_f \#(\mathrm{crit}\, f)\,.$$

Then minimize $\#(\mathrm{crit}\, f)$ over all *Morse* functions in the class $[f]$ where "Morse" means all critical points are non-singular and thus define $\mathrm{Mor}[f]$ (which is necessarily $\geq \mathrm{LuS}[f]$).

Now, we apply the above to the following class $[f]$ built of a single function f on X by applying the following two operations.

(1) Stabilize by passing to $f(x)+Q(y)$, for all possible non-singular quadratic functions $Q : \mathbb{R}^N \to \mathbb{R}$ and all $N = 0, 1, 2 \ldots$

(2) Add an arbitrary smooth *compact* perturbation $\varepsilon(x, y)$ to $f(x) + Q(y)$ where "compact" means "with compact support" in $X \times \mathbb{R}^N$.

We set

$$\mathrm{stabMor}(f)_{\mathrm{comp}} = \mathrm{Mor}[f + Q + \varepsilon]$$

and

$$\mathrm{stabLuS}(f)_{\mathrm{comp}} = \mathrm{LuS}[f + Q + \varepsilon]\,.$$

Notice (this is nearly obvious) that these numbers are invariant under compact perturbations $\varepsilon(x)$ of the function $f(x)$ as such an $\varepsilon(x)$ can be cut-off to a *compact* perturbation $\varepsilon(x, y)$ without introducing new critical points.

0.2.3 The stable Morse and Lusternik-Schnirelman inequalities for f

These are formulated in terms of the relative (co)homology of the levels $X_- = f^{-1}(-\infty, -a] \subset X$ for large $a \in \mathbb{R}$ and/or $X_+ = f^{-1}[a, \infty] \subset X$, also for large a, as follows

Proposition 0.2.3

$$\mathrm{stabMor}(f)_{\mathrm{comp}} \geq \operatorname{rank} H_*(X, X_-) = \operatorname{rank} H^*(X, X_+)\,, \qquad \text{(Mor)}$$

where we assume f to be a fibration at infinity (which makes the topology of the pairs X, X_- and X, X_+ independent of a for large a) and where the homology may be taken with an arbitrary coefficient field.

(Notice that $H_*(X, X_-)$ is canonically isomorphic to $H^*(X, X_+)$ by the Poincaré duality.)

Next we define cuplength $H^*(f \mid X)$ as follows. First we observe that the cup product between $H^*(X, X_+)$ and $H^*(X, X_-)$ lands in the cohomology with compact support since one may enlarge X_- to $X'_- \supset X_-$ by an isotopy of X with compact support, i.e. fixed outside a compact subset $K \subset X$, such that the union $X'_- \cup X_+$ will cover all X but a compact subset (slightly bigger than the above K). Now, the cup-length is defined as maximal $k = 0, 1, \ldots$, for which there exist cohomology classes $h_1, h_2, \ldots, h_k$ in $H^*(X)$ of degrees ≥ 1, and two classes $h_- \in H(X, X_-)$ and $h_+ \in H(X, X_+)$ (possibly, of degree zero), such that the cup-product

$$h_- \smile h_+ \smile h_1 \smile \ldots \smile h_k \neq 0$$

where this inequality is understood in the cohomology of X with compact support and where we may use an arbitrary coefficient field for H^*.

Now, the stable Lusternik-Schnirelman inequality reads

Proposition 0.2.4

$$\mathrm{stabLuS}(f)_{\mathrm{comp}} \geq \mathrm{cuplength}(f \mid X) + 1\,, \qquad \text{(LuS)}$$

where f is assumed as earlier to be a fibration at infinity.

Remark 0.2.5 The inequalities (Mor) and (LuS) appear obvious by the modern standard but their power resides in *stability* as they apply in our framework to a function on $X \times \mathbb{R}^N$ where N becomes arbitrarily large. Also notice that these inequalities remain valid if the perturbation $\varepsilon(x, y)$ in $f(x) + Q(y) + \varepsilon(x, y)$ is allowed to be a *bounded* function possibly with *infinite* support.

There are several refinements of the above inequalities in the presence of non-trivial fundamental group $\pi = \pi_1(X)$. The easiest one consists in modifying the definition of the cup-length by taking into account the cohomology of the classifying (Eilenberg-MacLane's $K(\pi, 1)$) space $B\pi$, comp. [39]. Namely, call $h \in H^*(X)$ a *π-class* if it comes from $B\pi$, i.e. equals $c^*(\underline{h})$ for some $\underline{h} \in H^*(B\pi)$, where $c : X \to B\pi$ denotes the classifying map (characterized by inducing an isomorphism $\pi_1(X) \overset{\sim}{\to} \pi = \pi_1(B\pi)$). Then we define $\mathrm{cuplength}_\pi(f \mid X)$ as the maximal number k for which there exists a π-class $h_\pi \in H^*(X)$ of degree $\ell \geq 1$, classes $h_1, h_2, \ldots, h_{k-\ell} \in H^*(X)$ of degrees ≥ 1 and h_-, h_+ as earlier such that

$$h_- \smile h_+ \smile h_\pi \smile h_1 \smile \ldots \smile h_{k-\ell} \neq 0\,.$$

Clearly, $\mathrm{cuplength}_\pi$ may become greater than cuplength whenever there is a class $\underline{h} \in H^*(B\pi)$ not decomposible into 1-dimensional classes and so the following inequality improves (LuS).

Proposition 0.2.6 (see [39] and [32])

$$\mathrm{stabLuS}(f)_{\mathrm{comp}} \geq \mathrm{cuplength}_\pi(f \mid X) + 1\,. \qquad (\mathrm{LuS}_\pi)$$

Example. If X is a closed aspherical (i.e. $K(\pi, 1)$) manifold of dimension k, then, obviously, $\mathrm{cuplength}_\pi(f \mid X) = k$ for all f on X.

0.2.4 Whitehead number

Suppose that the function f on X, which is assumed a fibration at infinity defines an *h-cobordism* $X_- \subset X \supset X_+$ which means there exists a product manifold $X' = Y \times \mathbb{R}$ and a proper homotopy equivalence $\varphi : X' \to X$ with the following four properties.

(i) φ agrees with f outside a compact subset $K \subset X'$, i.e. $\varphi \circ f : X' \to \mathbb{R}$ equals the projection $X' = Y \times \mathbb{R} \to \mathbb{R}$ outside K.

(ii) φ is a bijective diffeomorphism outside a subset $K' \subset Y \times \mathbb{R}$, such that the intersection $K' \cap (Y \times [-a, a])$ is compact for all $a \in \mathbb{R}$.

(iii) The implied (inverse) homotopy equivalence $\psi : X \to X'$ equals φ^{-1} outside $\varphi(K') \subset X$; furthermore, the homotopy from $\varphi \circ \psi$ to Id is constant outside $\varphi(K')$ and the one from $\psi \circ \varphi$ to Id is constant outside K'.

(iv) (optional) The manifold Y is diffeomorphic to the interior of a compact manifold (with boundary).

Under the above conditions the relative Morse chain complex $C^*(X, X_-)$ over the group ring of π is acyclic and one may define its *Whitehead torsion* $W(f \mid X) \in \mathrm{Wh}(\pi)$ (see [48]). Recall that every element W in the *Whitehead group* $\mathrm{Wh}(\pi)$ is represented by a $(k \times k)$-matrix over the group ring of π. The minimal possible k is denoted $\|w\|$. The following inequality easily follows from the definition:

Proposition 0.2.7

$$\mathrm{stabMor}(f)_{\mathrm{comp}} \geq 2\|W(f \mid X)\|\,. \qquad (\mathrm{Whi})$$

Notice that $\|w\| \geq 1$ for $w \neq 0$. Probably, there are examples of π's where $\|w\|$ becomes arbitrarily large. (But we did not even work out a single example with $\|w\| \geq 3$, while the examples of $w \neq 0$ are presented in [48] and examples of $\|w\| = 2$ are given in [52]). This would add value to (Whi), since every w may be realized by some h-cobordism (see [48]).

0.2.5 Sharko, Novikov-Shubin and related inequalities

The stabilization process $f \rightsquigarrow f + Q$ does not change the chain homotopy equivalence type of the Morse complex of f over the group ring of π, except for shifting the grading. In fact, the stabilization does not even change the (properly understood) simple homotopy equivalence class of this complex of $\mathbb{Z}(\pi)$-moduli. Thus *the stable Morse number of f is bounded from below by the minimal rank of a complex over $\mathbb{Z}(\pi)$ representing this (simple) chain homotopy type.* For the trivial group π this leads to (a sharpened version of) the inequality (Mor). Similar homological inequalities for general π are worked out by V. Sharko in [57]. Unfortunately the resulting inequalities involve $\mathbb{Z}(\pi)$-invariants which are quite hard to compute (especially for infinite groups π) which severely limits their applicability. The easiest among such invariants is the minimal number $g = g(\pi)$ of generators of the augmentation ideal of $\mathbb{Z}(\pi)$ as a left module over $\mathbb{Z}(\pi)$. There are examples of finite groups (pointed out to us by S. Gersten [34] and by V. Sharko [58]) where this g is strictly smaller than the number of generators of π. We have no examples of sufficiently interesting groups with arbitrarily large g but one's naive guess is that the Cartesian product of k (generic?) hyperbolic groups must have $g \geq k$.

Sometimes one may extract a non-trivial information by using cohomology with twisted coefficients, in particular L_2-cohomology of the (universal) π-covering $(\widetilde{X}, \widetilde{X}_-)$ for $\pi = \pi_1(X)$. For example the *von Neumann rank* (or dimension) of L_2H^* bounds stabMor by

$$\mathrm{stabMor}(f)_{\mathrm{comp}} \geq \mathrm{rank}_\pi\, L_2H^*(\widetilde{X}, \widetilde{X}_-) \qquad (\mathrm{Mor}_{L_2})$$

(see [51]). This becomes useful where the ordinary H^* fails to the job (see [14] and [13] for examples of groups and spaces with non-vanishing L_2-cohomology).

We conclude on a somewhat pessimistic note by observing that one has no satisfactory lower bound on stabMor for most interesting non-simply connected spaces, such for example as arithmetic varieties, i.e. symmetric spaces/congruence subgroups (albeit twisted coefficient sometimes serve unexpectedly well, see [39]).

0.2.6 The stable Morse and Lusternik-Schnirelman numbers of manifolds

If X is a closed manifolds, then all functions f on X have the same $\mathrm{stabMor}(f)_{\mathrm{comp}}$ and $\mathrm{stabLuS}(f)_{\mathrm{comp}}$ and so we may speak of $\mathrm{stabMor}(X)$ and $\mathrm{stabLuS}(X)$. If

X is an open manifold we set

$$\mathrm{stabMor}(X) = \inf_f \mathrm{stabMor}(f)_{\mathrm{comp}}$$

for "inf" taken over all positive proper functions $f : X \to \mathbb{R}_+$, and similarly, we define

$$\mathrm{stabLuS}(X) = \inf_f \mathrm{stabLuS}(f)_{\mathrm{comp}} .$$

If we allow manifolds with boundaries, we may either take the above stabLuS and stabMor of the interior $\mathrm{int}\, X = X - \partial X$, or, if we feel greedy, we restrict "inf" to those proper positive f which are *standard* near (each component of) the boundary, i.e. there is a tubular neighbourhood $N = [0,1] \times \partial X \subset X$ with $0 \times \partial X = \partial X$, such that the levels of f on $N - \partial X =]0,1] \times \partial X$ equal the slices $t \times \partial X$, $t \in]0,1]$.

Now the Morse-Lusternik-Schnirelman inequalities take the usual shape

Proposition 0.2.8

$$\mathrm{stabMor}(X) \geq \mathrm{rank}\, H_*(X) \tag{Mor}$$

and

$$\mathrm{stabLuS}(X) \geq \mathrm{cuplength}_\pi\, H^*(X) \tag{LuS$_\pi$}$$

The implied class h_+ here is of degree zero and has compact support.

0.2.7 Morse inequalities for $X \to Y$

Given a smooth map $\alpha : X \to Y$ one may define (stable) Morse and Lusternik-Schnirelman numbers using not all functions on X but only those coming from Y, i.e. of the form $f = \underline{f} \circ \alpha$, for $\underline{f} : Y \to \mathbb{R}$. A particular example is where α is an immersion and we define $\mathrm{Mor}(X \to Y \mid \mathrm{NoCrit})$ and $\mathrm{LuS}(X \to Y \mid \mathrm{NoCrit})$ by minimizing the number of (non-degenerate for Mor) critical points of $\underline{f} \circ \alpha$ over all $\underline{f} : Y \to \mathbb{R}$ which have *no critical points*. Then we stabilize by passing to $X \times \mathbb{R}^N \to Y \times \mathbb{R}^N$ and using the functions on $Y \times \mathbb{R}^N$ of the form $\underline{f} + Q + \varepsilon$ as earlier with the additional restriction of not having critical points on $Y \times \mathbb{R}^N$. This notion is most interesting for $\dim X - \dim Y = 1$ (where even the case of an immersed circle in the plane is not vacuous) and we suggest the following obvious but useful lower bound on $\mathrm{stabMor}(X \to Y \mid \mathrm{NoCrit})$ and $\mathrm{stabLuS}(X \to Y \mid \mathrm{NoCrit})$.

Proposition 0.2.9 *Suppose, there exists a manifold Z containing X and an extension of α to Z, say $\mathcal{A} : Z \to Y$, such that $X = \mathrm{crit}\, \mathcal{A}$, i.e. $X = \{z \in Z \mid \mathrm{rank}\, \mathcal{A}(z) \leq \dim Y\}$. Then*

$$\mathrm{stabLuS}(X \to Y \mid \mathrm{NoCrit}) \geq \mathrm{stabLuS}(Z) \tag{LuS $\to$}$$

and

$$\mathrm{stabMor}(X \to Y \mid \mathrm{NoCrit}) \geq \mathrm{stabMor}(Z) \tag{Mor $\to$}$$

where both manifolds X and Z (but not Y) are assumed compact without boundaries.

One generates many examples by starting with a topologically complicated Z and mapping it to Y so that it only has a *folding singularity* along a given submanifold $X \subset Z$ having simple topology, see [20]. For instance, any 3-manifold M admits a map into $\mathbb{R}^3$ with a fold along a 2-sphere which bounds a ball in M.

Morse theory on singular hypersurfaces. Let $\alpha : X \to Y$ be a hypersurface with a folding singularity $\Sigma \subset X$ of codimension one (which is non-generic) and let us try to bound from below the number of the critical points of $\underline{f} \circ \alpha$ on X where $\underline{f}$ is a *generic* (with respect to α) function on X and where we discard the critical points of $\underline{f} \circ \alpha \mid \Sigma$. If $\dim \Sigma \geq 1$, the ordinary Morse theory does not tell us much but suppose α extends to an $\mathcal{A} : Z \to Y$, such that $X \subset Z$ serves as the folding singularity $\Sigma^1_{\mathcal{A}}$ while Σ appears as $\Sigma^{11}_{\mathcal{A}}$. Then the critical points of $\underline{f} \circ \mathcal{A}$ on Z for a generic $\underline{f}$ with $\operatorname{crit} \underline{f} = \emptyset$ are located on $X \setminus \Sigma$ and coincide with the critical points of $\underline{f} \circ \alpha$ on $X \setminus \Sigma$. So the Morse theory of Z provides a non-trivial lower bound on $\# \operatorname{crit}(\underline{f} \circ \alpha \mid X \setminus \Sigma)$ (comp. 0.4.3 below).

0.2.8 Morse number of closed 1-forms

The above definitions of $\operatorname{stabLuS}(f)_{\mathrm{comp}}$ and $\operatorname{stabMor}(f)_{\mathrm{comp}}$ obviously generalize to arbitrary 1-forms φ on X which are stabilized and perturbing by

$$\varphi = \varphi(x) \rightsquigarrow \widetilde{\varphi} = \varphi(x) + dQ(y) + d\varepsilon(x, y)\,.$$

Then the stable Morse and Lusternik-Schnirelman numbers are defined via zeros of $\widetilde{\varphi}$ playing the role of critical points of functions. If φ is not closed this does not appear of any use but for closed forms φ there are non-trivial MLS inequalities. Here are some examples.

Proposition 0.2.10 *Consider all finite Abelian coverings $V_i \to V$ of orders $i = 1, 2, \ldots$, and let $\mu^{\mathrm{Ab}}(V) = \inf_{V_i} i^{-1}(\operatorname{Mor}(V_i))$ and $\mu^{\mathrm{Ab}}_{st}(V) = \inf_{V_i} i^{-1}(\operatorname{stabMor}(V_i))$. If V is a closed manifold, then these numbers bound from below the numbers of zeros of closed 1-forms φ on V by*

$$\operatorname{Mor}(\varphi) \geq \mu^{\mathrm{Ab}}(V) \qquad (\mu^{\mathrm{Ab}})$$

and

$$\operatorname{stabMor}(\varphi) \geq \mu^{\mathrm{Ab}}_{st}(V)\,. \qquad (\mu^{\mathrm{Ab}}_{st})$$

In fact, slightly perturbing φ if necessary, one reduces the problem to the case, where $\varphi = df$ for a smooth map $f : V \to S^1 = \mathbb{R}/\mathbb{Z}$. This lifts to a function $\widehat{f}$ on the cyclic covering $\widehat{V}$ of V and $\widehat{f}$ can be made $i\mathbb{Z}$-invariant by modifying $\widehat{f}$ near some non-singular level $W = \widehat{f}^{-1}(t)$. Such modification, properly performed, introduces at most $2 \operatorname{Mor}(W)$ new critical points (independently of i) and the proof follows by sending $i \to \infty$.

0.2.11 ***Remarks.***(a) One could equally pass to the maximal Abelian covering $\widetilde{V}^{\mathrm{Ab}} \to V$ and count the Morse numbers (or Betti numbers) of $\widetilde{W}^{\mathrm{Ab}}$ "per fundamental domain". Somewhat finer results are due to Novikov using homology with suitably twisted coefficients (see [49], [50], [54], [9]).

(b) The above applies to connected sums $V_0 \# V_1$ where $H_1(V_1) = 0$ and bounds the Morse numbers of forms on $V_0 \# V_1$ by $\mathrm{Mor}(V_1) - 2$ (# of components of V_1).

(c) The Morse numbers of finite coverings of V are closely related to a natural norm on the Witt-Wall group of V, (see §$8\frac{1}{2}$ in [39]).

(d) Chekanov recently proved ([17]) that for a non-exact closed form φ we have $\mathrm{LuS}(\varphi) \leq 1$. On the other hand the inequality $\mathrm{LuS}(\varphi) < 1$ implies that V fibers over S^1 and fine diffeo-obstruction for this has been extensively studied (see [46], [27]).

0.2.9 Morse inequalities for kissing numbers

Take two smooth 1-parameter families of submanifolds in Y, say $X_1(t)$ and $X_2(t)$ and observe that, generically, $X_1(t)$ is transversal to $X_2(t)$ at a given time t. Yet there may exist particular moments $t_i \in \mathbb{R}$, $i = 1, 2, \ldots$, when $X_1(t_i)$ is non-transversal to $X_2(t_i)$. Typically, such a non-transversality, a *kiss*, occurs at a single point, but there may be several at the same t. The totality of all kisses over all t (where usually t runs over $[0, 1]$) is denoted $X_1(t) \asymp X_2(t)$. For instance, when $X_1(0)$ and $X_2(0)$ are two linked embedded circles in $\mathbb{R}^3$, while $X_1(1)$ and $X_2(1)$ are unlinked then the kissing number $X_1(t) \asymp X_2(t)$ is bounded below by the *unlinking number*, i.e, the minimal number of intersections needed to unlink $X_1(0)$ and $X_2(0)$.

0.3 Lagrangian intersection inequalities

We shall state here several inequalities providing lower bounds on the cardinality of intersections $L'_1 \cap L'_2$ where L'_1 and L'_2 are obtained from graphical and subgraphical Lagrangian varieties L_1 and L_2 in $T^*(V)$ by Hamiltonian or contact isotopies. We concentrate here on illustrative examples. See Chapters 3, 4 and 5 below for the statements of general results and the proofs.

0.3.1 Deformed graphs over all of V

Take an arbitrary smooth function $f : V \to \mathbb{R}$ and let L'_f be obtained from the Lagrangian graph $L_f = df(V) \subset T^*(V)$ by a *compact* Hamiltonian isotopy, i.e. with *compact support*, or more generally, by a *compact contact* isotopy, where a diffeomorphism (or a diffeotopy of these) is said to have compact support if it equals the identity outside a compact subset. Also recall that the contact isotopy applies, strictly speaking, not to L_f but to the Legendrian graph $\mathcal{L}_f = J^1_f(V) \subset \mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$, where $J^1_f = (df, f)$, and the projection of the result of such isotopy to $T^*(V)$ appears as our L'_f. This is an *immersed* Lagrangian submanifold in $T^*(V)$ which may (and typically does) have double points (unlike the results of Hamiltonian isotopies, which being true isotopies introduce no double points). We are interested in the intersection of L'_f with the *zero-section* $\mathbb{O} = \mathbb{O}_V \subset T^*(V)$ and, as in the Morse theory, distinguish the case where L'_f intersects $\mathbb{O}$ transversally. (Notice that $L_f \cap \mathbb{O} = \mathrm{crit}\, f \subset V = \mathbb{O}$ and *transversal* intersection points correspond to *non-degenerate* critical ones.) If the intersection between L'_f and $\mathbb{O}$ is transversal, we denote it by $L'_f \pitchfork \mathbb{O}$ instead of plain $L'_f \cap \mathbb{O}$. We apply this convention, $L_1 \pitchfork L_2$ instead of $L_1 \cap L_2$, to all pairs L_1, L_2 which meet everywhere transversally.

The following lower bounds (which is a corollary of Theorem 4.1.1 below) on the cardinality of the intersection $L'_f \cap \mathbb{O}$ summarize contributions by many people (see [1], [18], [11], [31], [42], [47], [35], [65], [64] et al.).

Theorem 0.3.1

$$\#(L'_f \cap \mathbb{O}) \geq \mathrm{stabLuS}(f)_{\mathrm{comp}} \qquad (\mathrm{LuS})_\cap$$

and

$$\#(L'_f \pitchfork \mathbb{O}) \geq \mathrm{stabMor}(f)_{\mathrm{comp}}\,. \qquad (\mathrm{Mor})_\pitchfork$$

Here the notation "$\pitchfork$" indicates the *assumption* of the transversality of the intersection. Also notice that in both cases the intersections is understood with *due multiplicity* if L'_f happens to meet $\mathbb{O}$ at its own double point. In general, the intersection between immersed, or more generally mapped, manifolds $L_1 \to T^*(V)$ and $L_2 \to T^*(V)$ is understood as the subset in $L_1 \times L_2$ of pairs of points which are identified by the maps of L_1 and L_2 into $T^*(V)$.

We did not make any restrictive assumption on f so far and in fact the above inequalities hold true for all smooth f. However, if we want to combine these with the Morse theory we have to assume f is a fibration at infinity (albeit some non-trivial lower bounds on $\#\,\mathrm{crit}(f + Q + \varepsilon)$ are available for more general f).

If V is a closed manifold, one may replace the right hand sides of these inequalities by $\mathrm{stabLuS}(V)$ and $\mathrm{stabMor}(V)$ respectively and then these can be bounded from below by the traditional topological invariants of V such as $\mathrm{rank}\, H_*(V)$ and/or $\mathrm{cuplength}_\pi(V)$ (see 0.2). On the other hand, if V is non-compact, the message contained in (LuS) and (Mor) very much depends on f. For example, if $V = \mathbb{R}^n$ and f equals the product of k independent linear forms, then, as k grows, the stable Morse number of f becomes larger and larger (this is easy to see) which adds more and more juice to $(\mathrm{Mor})_\pitchfork$. On the other hand $\mathrm{stabLuS} = 1$ for all these f (which may be true for all functions on $\mathbb{R}^n$ which are fibrations at infinity) and we cannot say better than $\#(L'_f \cap \mathbb{O}_{\mathbb{R}^n}) \geq 1$.

0.3.2 Intersection $L'_f \cap \mathbb{O}_V$ for partially defined f

Let f be a smooth function on an open subset $U \subset V$. Then $L_f = df(U) \subset T^*(V) \subset T^*(U)$ is a bona fide Lagrangian submanifold in $T^*(V)$ which is, moreover, proper if $\|df(u)\| \to \infty$ for $u \to \partial U$. We may deform L_f by a compact Hamiltonian isotopy in $T^*(V)$ (i.e. with compact support) or more generally by a compact contact isotopy (applied to $\mathcal{L}_f \subset \mathrm{Jet}^1(V) \supset \mathrm{Jet}^1(U)$) and we claim (see 3.2.1 and 4.1.2 for the proof) that

Theorem 0.3.2 *The intersection of the resulting L'_f with $\mathbb{O}_V$ satisfies the above inequalities* $(\mathrm{LuS})_\cap$ *and* $(\mathrm{Mor})_\pitchfork$*, provided*

$$\|df(u)\| \geq (\mathrm{dist}(u, \partial U))^{-1} \qquad (di^{-1})$$

for $u \in U$ close to ∂U, where the distance in U to the boundary is measured with some smooth Riemannian metric in V.

Remarks. (a) The condition (di^{-1}) is satisfied, for example if the boundary ∂U is smooth and $f(u)$ equals $(\mathrm{dist}(u, \partial U))^\beta$ near ∂U for some $\beta < 0$.

(b) It may seem at first sight that the above follows from the inequalities $(\mathrm{LuS})_\cap$ and $(\mathrm{Mor})_\pitchfork$ of the previous section applied to U instead of V. But one should note here that the isotopy in $T^*(V) \supset T^*(U) \subset L_f$ may take (a compact part of) L_f out of $T^*(U)$ and so the direct reduction of the case $U \neq V$ to $U = V$ is impossible. (But a somewhat indirect reduction of this kind *is* possible in the framework of generating functions over V, see Section Section 3 below).

Corollary 0.3.3 *If f on U is positive then in the above two cases*

$$\#(L'_f \cap \mathbb{O}_V) \geq \mathrm{cuplength}_\pi(U) \qquad (\mathrm{cup}_\pi)$$

and

$$\#(L'_f \pitchfork \mathbb{O}_V) \geq \mathrm{rank}\, H_*(U)\,. \qquad (\mathrm{rank} H)$$

Notice, that the topology of the ambient V plays no role in these inequalities.

Intersection $L'_f \cap L'_{f_0}$ for $f_0 : V \to \mathbb{R}$. Let us replace the zero function on V by an arbitrary smooth f_0 and let us deform L_{f_0} to L'_{f_0} by a Hamiltonian or contact isotopy with compact support, where this isotopy may be different from the one moving L_f to L'_f. Then we have the following

0.3.4 Generalization. *If f and U satisfy the above (di^{-1}) then (with no extra assumptions on f_0)*

$$\#(L'_f \cap L'_{f_0}) \geq \mathrm{stabLuS}(f - f_0)_{\mathrm{comp}} \qquad (\mathrm{LuS})_\cap$$

and

$$\#(L'_f \pitchfork L'_{f_0}) \geq \mathrm{stabMor}(f - f_0)_{\mathrm{comp}}\,. \qquad (\mathrm{LuS})_\pitchfork$$

About the proof. If L'_{f_0} was obtained by a Hamiltonian isotopy $I_0(t)$ in $T^*(V)$ then everything reduces to the previous case (where $f_0 = 0$ and $I_0(t) = \mathrm{Id}$) by applying $I_0^{-1}(t) I_{f_0}^{-1}$ to the whole picture, where $I_{f_0} : \tau_v \underset{\mathrm{def}}{\mapsto} \tau_v + df_0(v)$. But in the contact case, where we have $I_0(t)$ acting on $\mathcal{L}_{f_0} \subset \mathrm{Jet}^1(V)$, one needs an additional argument (see Section 4.1.2 below).

0.3.3 Intersection $L'_{f_1} \cap L'_{f_2}$ for f_1 and f_2 defined on different domains

Here we have two open subsets U_1 and U_2 in V and smooth functions f_1 on U_1 and f_2 on U_2. We apply as earlier compact contact (e.g. Hamiltonian) isotopies to L_{f_1} and L_{f_2}, intersect the resulting L'_{f_1} and L'_{f_2} and try to bound from below the cardinality of $L'_{f_1} \cap L'_{f_2}$ in terms of the Morse theoretic properties of the function $f_1 \underset{\cdot}{-} f_2$ which just means $f_1 - f_2$ on $U_1 \cap U_2$. Of course, this is obvious for the original unperturbed submanifolds as $L_{f_1} \cap L_{f_2} = \mathrm{crit}(f_1 \underset{\cdot}{-} f_2)$. But after the deformation one must use the Morse data of a more complicated auxiliary function defined on $U_1 \times U_2 \times \mathbb{R}^N$ and denoted $f_1 \overline{\times} f_2$. (If $V = \mathbb{R}^n$ one may take $N = n$ and

$$f_1 \overline{\times} f_2(u_1, u_2, x) = f(u_1) - f(u_2) + \langle x, u_1 - u_2 \rangle$$

and for general V one needs an embedding of V into some Euclidean space, see 3.3. Then one may extract the topological information concerning $f_1 \overline{\times} f_2$ from that for $f_1 \underset{\cdot}{-} f_2$ and obtain lower bounds on the intersection $L'_{f_1} \cap L'_{f_2}$ in terms of the Morse theory of $f_1 \underset{\cdot}{-} f_2$ on $U_1 \cap U_2$. This is indicated below under rather restrictive assumptions on $f_i : U_i \to \mathbb{R}$, $i = 1, 2$ while the general case is treated in 3.4.3.

Preliminary definitions. Let U_1 and U_2 be domains in V with smooth boundaries ∂U_1 and ∂U_2. A tangency point of the boundaries, i.e. a $v \in \partial U_1 \cap \partial U_2$ where $T_v(\partial U_1) = T_v(\partial U_2)$ is called *positive (negative)* if the tangent spaces $T_v(\partial U_1)$ and $T_v(\partial U_2)$ have equal (respectively opposite) coorientations with respect to the external normal fields on ∂U_i, $i = 1, 2$.

A function f on $U \subset V$ with a smooth boundary ∂U is called *very nice* if it equals $\varphi\, \mathrm{dist}^\beta(u, \partial U)$ near ∂U, where "dist" refers to a smooth Riemannian metric in V, where $\beta \leq -1$ and φ is a smooth function on the closure of U such that $\varphi \mid \partial U$ has no zero critical value.

Theorem 0.3.5 ***Lagrangian Morse inequality for $L'_1 \cap L'_2$.*** *Let U_1 and U_2 be relatively compact domains in V with smooth boundaries and let f_i be very nice functions on U_i, such that the implied φ_1 and φ_2 do not vanish on $\partial U_1 \cap \partial U_2$ and $\varphi_1(v) - \varphi_2(v) \neq 0$ at the positive tangency points while $\varphi_1 + \varphi_2 \neq 0$ at the negative tangency points of ∂U_1 and ∂U_2. Then*

$$\#(L'_{f_1} \pitchfork L'_{f_2}) \geq \mathrm{rank}\, H_*(U_1 \cap U_2, (U_1 \cap U_2)_-)\,, \qquad (H\mathrm{Mor})_{\pitchfork}$$

where L'_{f_i} are obtained from L_{f_i} by compact contact (e.g. Hamiltonian) isotopies and $(U_1 \cap U_2)_- \subset U_1 \cap U_2$ denotes the subset where $f_1 \underset{\cdot}{-} f_2 \leq -a$ for some *sufficiently large a. (In particular, if $f_1 \dot{-} f_2 \geq 0$, one may use the absolute homology $H_*(U_1 \cap U_2)$.)*

Remark. If we do not assume the transversality of the intersection, all we can say is

$$\#(L'_{f_1} \cap L'_{f_2}) \geq 1 \quad \text{for} \quad H_*(U_1 \cap U_2, (U_1 \cap U_2)_-) \neq 0\,,$$

since the multiplicative structure on H^* and/or π_1 of $U_1 \cap U_2$ may dissolve in $U_1 \times U_2 \times \mathbb{R}^N$ where the function $f_1 \overline{\times} f_2$ lives. (For example, both U_1 and U_2 may be contractible). An important case where this *does not* happen is for $U_1 \subset U_2$, e.g. $U_1 = U_2$, and then the Morse theory of the function $f_1 - f_2$ on U_1 regains some control over the intersection $L'_{f_1} \cap L'_{f_2}$. For example, we show in Chapter 3:

0.3.6 *Suppose $U_1 = U_2 = U$, and f_1, f_2 are very nice functions (or functions which satisfies the condition (di^{-1}) on U such that $f_1 - f_2 : U \to \mathbb{R}$ is a fibration at infinity. Then the following inequality holds:*

$$\#(L'_{f_1} \cap L'_{f_2}) \geq \text{cuplength}_\pi(f_1 - f_2 \mid U_1) + 1\,, \qquad (\text{LuS}_{\pi_1})\,.$$

0.3.4 Intersection between deformed conormal bundles

Let U_1 and U_2 be smooth submanifolds of positive codimension in V and $L_{U_1}, L_{U_2} \subset T^*(V)$ be their conormal bundles (i.e. the annihilators of their tangent subbundles in $T(V)$, see 0.1.1). We assume U_1 and U_2 intersect transversally in V and then $L_{U_1} \cap L_{U_2}$ is contained in $\mathbb{O}_V \subset T^*(V)$ where it identifies with $U_1 \cap U_2 \subset V = \mathbb{O}_V$.

Let L'_1 and L'_2 be obtained from L_{U_1} and L_{U_2} respectively by contact (e.g. Hamiltonian) isotopies with contact support where $L_U \subset T^*(V)$ lifts to $\mathcal{L}_U \subset \text{Jet}^1(V) = T^*(V) \times \mathbb{R}$ by $\mathcal{L}_U = L_U \times 0$. Then we have the following

Theorem 0.3.7 *If submanifolds U_1 and U_2 are proper in V (i.e. closed as subsets) and $U_1 \cap U_2$ is compact, then*

$$\#(L'_1 \pitchfork L'_2) \geq \text{rank}\, H_*(U_1 \cap U_2)\,. \qquad (H\text{Mor})_{\pitchfork}$$

This can be derived from the similar inequality in 0.3.3 applied to suitable functions f_1 and f_2 defined in tubular neighbourhoods on U_1 and U_2 respectively.

0.3.5 Fiber product $U_1 \otimes U_2$

The above inequality $(H\,\mathrm{Mor})_\pitchfork$ remains valid for properly *immersed* U_1 and U_2 in V where the intersection $U_1 \cap U_2$ must be counted with multiplicity, i.e. replaced by the following fiber product $U_1 \otimes U_2$.

Given maps $\alpha_i : U_i \to V$, $i, 1, 2$, consider $\alpha_1 \times \alpha_2 : U_1 \times U_2 \to V \times V$ and take the pull-back of the diagonal $\Delta \subset V \times V$ for $U_1 \otimes U_2 \subset V \times V$,

$$(\alpha_1 \times \alpha_2)^{-1}(\Delta) = U_1 \otimes U_2\,. \qquad (\otimes)$$

Furthermore, let us consider arbitrary smooth functions f_1 on U_1 and f_2 on U_2, take $\underline{L}_{f_1}$ and $\underline{L}_{f_2}$ (see 0.1.1) instead of L_{U_1} and L_{U_2} (corresponding to $f_1 = 0$ and $f_2 = 0$), and contactly isotope them to some L'_1 and L'_2.

Generalization 0.3.8 *The above* $(H\,\mathrm{Mor})_\pitchfork$ *remains valid as it stands for immersions if* $U_1 \cap U_2$ *is understood as* $U_1 \otimes U_2$.

See 3.4.3 for the proof. Notice that we do not need any assumption on f_1 and f_2 since U_1 and U_2 sit in V properly. In the general, non-proper case we must impose some conditions on the behaviour of f_i on U_i, $i = 1, 2$ at infinity (of U_i) in order to keep $(H\,\mathrm{Mor})_\pitchfork$ valid.

0.3.6 Whitehead bound on Lagrangian intersection

We have the full-fledged inequalities bounding $\#(\underline{L}'_f \cap \mathbb{O}_V)$ from below in terms of $\mathrm{stabLuS}(f)$ and $\mathrm{stabMor}(f)$. Here is an example.

Let $\overline{U}$ be an h-cobordism between two closed manifolds, say ∂_1 and ∂_2 serving as the two components of the boundary $\partial\overline{U}$, i.e. $\overline{U}$ is a compact manifold with the boundary $\partial\overline{U} = \partial_1 \sqcup \partial_2$. Let f be a smooth function on $U = \mathrm{int}\overline{U} = \overline{U} - \partial\overline{U}$, which equals $-(\mathrm{dist}(u, \partial_1))^{-1}$ near ∂_1 and $(\mathrm{dist}(u, \partial_2))^{-1}$ near ∂_2. Smoothly embed $\overline{U}$ to some V, e.g. to $\mathbb{R}^m$. Let $\underline{L}_f \subset T^*(V)$ be the pushforward of $L_f \subset T^*(U)$ under this embedding restricted to U and take some $\underline{L}'$ obtained from $\underline{L}_f$ by a compact contact (e.g. Hamiltonian) isotopy.

Theorem 0.3.9 *If the Whitehead invariant (torsion)* W *of the cobordism* $\partial_- \subset \overline{U} \supset \partial_+$ *is non-zero, then* $\underline{L}'$ *necessarily intersects* $\mathbb{O}_V$*, and in the transversal case the cardinality of the intersection is bounded from below by the Whitehead number of (the Whitehead invariant of)* $\overline{U}$*,*

$$\#(\underline{L}' \pitchfork \mathbb{O}_V) \geq 2\|W(\overline{U})\|\,. \qquad (\mathrm{Whi})_\pitchfork$$

Notice, that if $\dim U \geq 6$ then U is diffeomorphic to $\partial_- \times \mathbb{R}$ as well as to $\partial_+ \times \mathbb{R}$ and for $\dim V = \dim U$ the submanifold $\underline{L}_f \subset T^*(V)$ is also diffeomorphic to the cylinder. The asymptotic geometry of $\underline{L}'_f$ is also rather standard, the same as for the trivial cobordism $\overline{U} = \partial_- \times [0, 1]$, provided ∂_+ is diffeomorphic to ∂_- (which is often the case, see [48]). Yet the non-triviality of the cobordism is recorded by how $\underline{L}'_f$ must meet $\mathbb{O}_V$.

0.3.7 A refined intersection inequality for immersed hypersurfaces

We want to indicate an application of the Morse theory for maps $X \to Y$ (see 0.2.7) which we do here in the simplest case of a closed immersed hypersurface $\alpha : U \to \mathbb{R}^n$. We look at the intersection of $L_U = (T(U))^\perp \subset \mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ with the submanifold $L'_f \subset \mathbb{R}^{2n}$ obtained by a Hamiltonian isotopy (with, possibly, infinite support) from L_f for an arbitrary smooth $f : \mathbb{R}^n \to \mathbb{R}$, e.g. $f = 0$. Now we have the following improvement of our earlier intersection inequalities.

Theorem 0.3.10 *If L'_f does not meet the zero section $\mathbb{O}_{\mathbb{R}^n} \subset T^*(\mathbb{R}^n)$, then*

$$\#(L'_f \cap L_U) \geq \mathrm{stabLuS}(U \to \mathbb{R}^n \mid \mathrm{NoCrit}) \qquad (\cap \to)$$

and

$$\#(L'_f \pitchfork L_U) \geq \mathrm{stabMor}(U \to \mathbb{R}^n \mid \mathrm{NoCrit})\,. \qquad (\pitchfork \to)$$

Notice that if U has *transversal* selfintersection, then $(T(U))^\perp$ has no double points outside $\mathbb{O}_{\mathbb{R}^n}$ and so the above inequalities regard (embedded!) Lagrangian submanifolds in the complement $T^*(\mathbb{R}^n) \setminus \mathbb{O}_{\mathbb{R}^n}$.

0.3.8 Intersection between subgraphical varieties defined for proper maps $U_i \to V$

Let U_1 and U_2 be smooth manifolds properly mapped to V by smooth maps of constant rank, say $\alpha_i : U_i \to V$, $i = 1, 2$, and f_i be smooth functions on U_i. Then we have the subgraphical varieties $\underline{L}_{f_i}$ mapped to $T^*(V)$, $i = 1, 2$, and we contactly isotope them with compact supports to $\underline{L}'_i \subset T^*(V)$. We claim the following Morse inequality

Theorem 0.3.11

$$\#(\underline{L}'_1 \pitchfork \underline{L}'_2) \geq \mathrm{rank}\, H_*(U_1 \cap\!\!\!\!\times\, U_2)\,. \qquad (H\mathrm{Mor})_\pitchfork$$

Here the (implied by $\pitchfork$) transversality of the intersection is understood as follows. If $\underline{L}_{f_i}$ are regular, i.e. $L_{f_i} \subset T^*(U_i)$ are transversal to $(\ker D\alpha_i)^\perp$ (as in 0.1.1) then $\underline{L}'_i$, as well as $\underline{L}_{f_i}$ are immersed *manifolds* and the transversality makes the usual sense. In general, we *assume* that for each intersection point $(\underline{\ell}'_1, \underline{\ell}'_2) \in \underline{L}'_1 \times \underline{L}'_2$ (recall that the intersection between $\underline{L}'_1$ and $\underline{L}'_2$ is understood as $\underline{L}'_1 \cap\!\!\!\!\times \underline{L}'_2$) the corresponding (under the isotopies) points $\underline{\ell}_1 \in \underline{L}_{f_1}$ and $\underline{\ell}_2 \in \underline{L}_{f_2}$ are regular, i.e. they come from *transversal* intersection points between L_{f_i} and $(\ker D\alpha_i)^\perp$. In this case $\underline{L}'_1$ and $\underline{L}'_2$ are local submanifolds in $T^*(V)$ at these points and the transversality is understood as usual. In other words, we assume that the (possible) singularities of $\underline{L}_1$ and $\underline{L}_2$ lie away from their intersection points and the intersection points are transversal.

If α_1 and/or α_2 are non-proper, one should add, as earlier, some asymtotic assumptions on f_i in order to have $(H\mathrm{Mor})_\pitchfork$ hold true (see 3.4.3).

0.3.9 Beyond $T^*(V)$

Let L_1 and L_2 be graphical or subgraphical Lagrangian submanifolds in $T^*(V)$ and let $T^*(V) \hookrightarrow (M, \omega)$ be an equidimensional symplectic embedding. We apply compact Hamiltonian isotopies of M to L_1 and L_2 and we want to bound from below the intersection between the resulting L'_1 and L'_2 in M. A case where this can be done with our present techniques is for $M = T^*(W)$ and our embedding being induced by an (equidimensional!) embedding $V \hookrightarrow W$. Furthermore, even if M itself is not a (part of a) cotangent bundle it may stabilize to such, i.e. $M \times \mathbb{R}^{2N}$ for ($\mathbb{R}^{2N} = \mathbb{R}^N_p \times \mathbb{R}^N_q$, $\sum dp_i \wedge dq_i$) may be of the form $T^*(W)$. Then the above applies to the stabilizations of L_1 and L_2 that are $L_1 \times \mathbb{R}^N_p$ and $L_2 \times \mathbb{R}^N_q$ in $T^*(V \times \mathbb{R}_q) \hookrightarrow M$. One knows in this regard (see [25]) that *convex* symplectic manifolds M do stabilize to $T^*(W)$ under the following

Reality condition. *The tangent bundle $T(M)$ with its $\mathbb{C}$-vector bundle structure is real, which amounts in symplectic terms to the existence of a Lagrangian subbundle in $T(M)$.*

Remark 0.3.12 Probably, there is a stabilization process with non-trivial moreover, non-real, vector bundle over M instead of $M \times \mathbb{R}^{2n} \to M$ which would make the reality conditions on M redundant.

Thus the manageable embeddings $T^*(V) \to M$ are those where $M \times \mathbb{R}^{2N}$ embeds further to some $T^*(W)$ such that $L_1 \times \mathbb{R}^N_p$ and $L_2 \times R^N_q$ become subgraphical in $T^*(W)$. Unfortunately, we do not yet see how to use this approach to cover all cases where we expect non-trivial intersection Morse inequalities for $L'_1 \cap L'_2$ in M. Our expectation is justified in some cases by an alternative *holomorphic* method, namely the Floer-Morse homology and Hofer-Lusternik-Schnirelman theory.

0.4 Legendrian submanifolds

0.4.1 Intersection and selfintersection invariants of Legendrian submanifolds

Given a subset $\mathcal{L}$, e.g. a Legendrian submanifold, in a contact manifold N we want to assign to it a number, invariant under contact isotopies with compact supports. An apparently cheap way to do it is to take another subset $\mathcal{M} \subset N$, (e.g. a submanifold of complementary dimension to $\mathcal{L}$) and set

$$\cap_{\mathcal{M}}(\mathcal{L}) = \inf_{\mathcal{L}'} \#(\mathcal{L}' \cap \mathcal{M}) \qquad (\cap_{\mathcal{M}})$$

where $\mathcal{L}'$ runs over all subsets obtained from $\mathcal{L}$ by contact isotopies with compact supports. In the case where $\mathcal{L}$ and $\mathcal{M}$ are smooth submanifolds we may also define

$$\pitchfork_{\mathcal{M}}(\mathcal{L}) = \inf_{\mathcal{L}'} \#(\mathcal{L}' \pitchfork \mathcal{M}) \qquad (\pitchfork_{\mathcal{M}})$$

where we limit to those $\mathcal{L}'$ (contact isotopic to $\mathcal{L}$ with compact support) which are everywhere transversal to $\mathcal{M}$. In what follows, we deal with $N = \mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$ where we take two subgraphical varieties $\mathcal{L} = \underline{\mathcal{L}}_f$ and $\underline{\mathcal{L}}_g$ and use $\mathbb{R}\underline{\mathcal{L}}_g$ for $\mathcal{M}$, where $\mathbb{R}\underline{\mathcal{L}}_g$ denotes the $\mathbb{R}$-orbit of $\underline{\mathcal{L}}_g$ for the obvious $\mathbb{R}$-action on $T^*(V) \times \mathbb{R}$. Then the intersection between $\mathcal{L}'$ and $\mathcal{M}$ identifies with our old friend, Lagrangian intersection between $\underline{L}'_f$ and $\underline{L}_g$ in $T^*(V)$. This gives us in many cases non-trivial Morse theoretic lower bounds on $\cap_{\mathcal{M}}(\mathcal{L})$ and $\pitchfork_{\mathcal{M}}(\mathcal{L})$ under suitable assumptions on f and g, defined on some $U \to V$ and $W \to V$. (Comp. also [26], [15], [53]).

Theorem 0.4.1 *Let V be a closed manifold, $U \to V$ a proper submersion* (which amounts here to U being a closed manifold fibered over V) *and let $W \subset V$ be a closed submanifold. Take $\mathcal{M} = L_W \times \mathbb{R} \subset T^*(V) \times \mathbb{R} = \mathrm{Jet}^1(V)$ and $\mathcal{L} = \underline{\mathcal{L}}_f$ for some smooth function f on U. Then*

$$\cap_{\mathcal{M}}(\mathcal{L}) \geq \mathrm{stabLuS}(U \mathbin{\cap\!\!\!\!\times} W) \qquad (\mathrm{LuS} \cap_{\mathcal{M}})$$

and

$$\pitchfork_{\mathcal{M}}(\mathcal{L}) \geq \mathrm{stabMor}(U \mathbin{\cap\!\!\!\!\times} W)\,, \qquad (\mathrm{Mor} \pitchfork_{\mathcal{M}})$$

where $(U \mathbin{\cap\!\!\!\!\times} W)$ in this case is just the total space of the restricted fibration $U \mid W$. In particular, if $W = V$, this is U itself and for $W = \{v\}$ this is the fiber $U_v \subset U$.

Take for instance, $U = V \times T^N$ for the N-torus T^N with large $N = 1, 2, \ldots$ eventually going to infinity (and unrelated to the above manifold N). Then $\cap_{\mathcal{M}}(\mathcal{L})$ grows (at least) as N and $\pitchfork_{\mathcal{M}}$ as 2^N. This is not especially exciting for a standard function $f_0 = f_0(v,t) = f_1(v) + f_2(t)$ where $\mathcal{L}$ consists of many components corresponding to critical points of $f_2(t)$ on T^N, but one modify such f_0 with a suitable *parametric Morse surgery* (comp. [23]

to make $\underline{\mathcal{L}}_f$ *connected* and thus having zero degree (equal to $\chi(T^N)$) for the projection $\underline{\mathcal{L}}_f \to V$. Yet no contact isotopy can bring $\underline{\mathcal{L}}_f$ to the complement of a fiber $T^*_{v_0}(V) \times \mathbb{R} = \mathrm{Jet}^1_v(V) \subset \mathrm{Jet}^1(V)$. In fact, one can thus achieve almost any given topological type of $\underline{\mathcal{L}}_f$ and of its (Legendrian!) embedding to $\mathrm{Jet}^1(V)$. For example, $\underline{\mathcal{L}}_f$ can be made diffeormorphic to the sphere S^n, $n = \dim V$, which, for $n \geq 2$, can be topologically isotoped to a single point in $\mathrm{Jet}^1(V)$. Yet every contact isotopy $\underline{\mathcal{L}}'_f$ of this spherical $\underline{\mathcal{L}}_f$ necessarily meets our $\mathcal{M}$ at more than N points (and 2^N under the transversality assumption). It follows, that for growing $N \to \infty$ we obtain an infinite sequence of such spheres $\underline{\mathcal{L}}_i$ for some f_{N_i} which are mutually contact non isotopic. (Here we can drop "compact support" as $\underline{\mathcal{L}}_i$ are compact.)

Remark 0.4.2 One can add to these $\underline{\mathcal{L}}_i$ yet another $\mathcal{L}$ having no-intersection with $\mathcal{M}$ at all and thus being contact non-isotopic to any of $\underline{\mathcal{L}}_i$. Namely, one should recall the *h-principle for Legendrian immersions* (see [38]). This particular h-principle allows a regular Legendrian isotopy of $\underline{\mathcal{L}}_f$ to some $\mathcal{L}^{\#}$ which does not meet $\mathcal{M}$, provided $\underline{\mathcal{L}}_f$ admits a homotopy in $\mathrm{Jet}^1(V)$ to $\mathrm{Jet}^1(V) - \mathcal{M}$. In the course of such regular homotopy $\mathcal{L}(t)$ from $\mathcal{L}(0) = \underline{\mathcal{L}}_f$ to $\mathcal{L}(1) = \mathcal{L}^{\#} \subset \mathrm{Jet}^1(V) - \mathcal{M}$ there necessarily appear self-crossing points of the moving $\mathcal{L}$, i.e. double points of $\mathcal{L}(t_j)$ for some moments $t_j \in [0,1]$ after which we loose any control over the contact invariants of $\mathcal{L}$. In particular no *lower bound* on $\cap_{\mathcal{M}}\mathcal{L}$ is known for such an $\mathcal{L}$.

0.4.2 Selfintersection of L under $\mathcal{L}$

The stable Morse theory also provides lower bounds to the intersection $\underline{\mathcal{L}}'_f \cap \mathbb{R}\underline{\mathcal{L}}'_g$ where both $\underline{\mathcal{L}}_f$ and $\underline{\mathcal{L}}_g$ were contactly isotoped in $\mathrm{Jet}^1(V)$ (and $\mathbb{R}\mathcal{L}$ denotes the $\mathbb{R}$-orbit of $\mathcal{L}$). In particular, we may apply this to the case where $f = g$ and define

$$\cap(\mathcal{L}\mathbb{R}\mathcal{L}) = \inf_{\mathcal{L}', \mathcal{L}''} \#(\mathcal{L}' \cap \mathbb{R}\mathcal{L}'') \qquad (\cap\mathbb{R})$$

where $\mathcal{L}'$ and $\mathcal{L}''$ run over all submanifolds contact isotopic to $\mathcal{L}$ (where we should add "with compact support" if $\mathcal{L}$ is non-compact as required by our present framework of the Lagrangian intersection theory). Next we bring in a closely related invariant of $\mathcal{L}$, denoted

$$\cap\!\!\!\downarrow(\mathcal{L}) = \inf_{L'} \#(\text{double points of } L') \qquad (\cap\!\!\!\downarrow)$$

where $L' \subset T^*(V)$ run over the projection to $T^*(V)$ of all $\mathcal{L}'$ obtained from $\mathcal{L}$ by contact isotopies. Similarly we define $\pitchfork\!\!\!\downarrow(\mathcal{L})$ by only allowing those $\mathcal{L}'$ where all selfintersection points of $L' \subset T^*(V)$ under $\mathcal{L}'$ are *transversal*. Then we observe the following simple

Proposition 0.4.3 *Suppose $\mathcal{L}$ is a closed manifold. Then*

$$\cap\!\!\!\downarrow(\mathcal{L}) \geq \cap(\mathcal{L}\mathbb{R}\mathcal{L}) - \mathrm{LuS}(\mathcal{L}) \qquad (\cap\!\!\!\downarrow \gtrsim \cap)$$

and

$$\pitchfork\!\!\!\downarrow(\mathcal{L}) \geq \pitchfork(\mathcal{L}\mathbb{R}\mathcal{L}) - \mathrm{Mor}(\mathcal{L}), \qquad (\pitchfork\!\!\!\downarrow \gtrsim \pitchfork)$$

Here we define the non-stable Lusternik-Schnirelman and Morse number in the obvious way, and $\pitchfork(\mathcal{L}\mathbb{R}\mathcal{L})$ is defined as $\cap(\mathcal{L}\mathbb{R}\mathcal{L})$ by just restricting inf to transversal pairs $(\mathcal{L}', \mathcal{L}'')$. Once we have a strong enough lower bound on $\#(\underline{\mathcal{L}}'_f \cap \mathbb{R}\mathcal{L}''_g)$ we may combine the above with the following upper bounds on LuS and Mor of $\mathcal{L}$,

Proposition 0.4.4 *(i)* $\mathrm{LuS}(\mathcal{L}) \leq \dim \mathcal{L} + 1$ *for all connected manifolds* $\mathcal{L}$*;*

(ii) $\mathrm{LuS}(\mathcal{L}) \leq \frac{1}{2} \dim \mathcal{L} + 1$ *for all simply connected manifolds of dimension* ≥ 6*;*

(iii) $\mathrm{Mor}\,\mathcal{L} \leq \mathrm{rank}\, H_*(\mathcal{L})$ *if* $\mathcal{L}$ *is a simply connected,* $\dim \mathcal{L} \geq 5$*, and* $H_*(\mathcal{L})$ *has no torsion.*

The corresponding stable inequalities are valid with no restriction on $\dim \mathcal{L}$*.*

The part (i) is obvious, part (iii) follows from Smale's theorem (see [61]), while part (ii) is a partial case of Theorem 5.1 in Takens' paper [62].

Next, we bound $\cap(\mathcal{L}\mathbb{R}\mathcal{L})$ from below as earlier in the case where $\mathcal{L} = \underline{\mathcal{L}}_f$ for an f defined on a closed manifold U fibered over V.

Theorem 0.4.5

$$\cap(\mathcal{L}\mathbb{R}\mathcal{L}) \geq \mathrm{cuplength}_\pi(U) + 1 \qquad (\mathrm{LuS}_\pi)$$

and

$$\pitchfork(\mathcal{L}\mathbb{R}\mathcal{L}) \geq \mathrm{rank}\, H_*(U) \qquad (H\,\mathrm{Mor})$$

We combine this with $(\cap\!\!\downarrow \gtrsim \cap)$ in the case $U = V \times T^N \to V$ and $\underline{\mathcal{L}}_f$ being diffeomorphic to S^n (with LuS = Mor = 2) and obtain the bounds

Corollary 0.4.6

$$\cap\!\!\downarrow(\underline{\mathcal{L}}_f) \geq \mathrm{cuplength}_\pi(V) + N - 1) \qquad (\cap\!\!\downarrow \mathrm{LuS})$$

and

$$\pitchfork\!\!\downarrow(\underline{\mathcal{L}}_f) \geq \mathrm{rank}\, H_*(V) + 2^N - 2\,. \qquad (\pitchfork\!\!\downarrow \mathrm{Mor})$$

These show that no exact (e.g. Hamiltonian) Lagrangian regular homotopy $L(t)$ of $\underline{L}_f$ can eliminate much of double points of $\underline{L}_f$ if its Legendrian lift $\mathcal{L}(t)$ has no self-crossings. (One does not know if one can erase IF.)

0.4.3 Morse-theoretic lower bounds on $\cap_{\mathcal{M}}(\mathcal{L})$ for general Legendrian $\mathcal{L}$

Take an arbitrary (immersed or embedded) Legendrian submanifold $\mathcal{L} \subset \mathrm{Jet}^1(V)$, and denote by $\alpha : \mathcal{L} \to V$ its projection to V and by $f : \mathcal{L} \to \mathbb{R}$ the projection to $\mathbb{R}$ for $\mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R} \to \mathbb{R}$. The subgraphical construction applied to these α and f yields another Legendrian variety, say $\mathcal{L}^* = \underline{\mathcal{L}}_f \to \mathrm{Jet}^1(V)$ which is usually larger than $\mathcal{L}$. For example, if $\mathcal{L}$ is generic, then $\mathcal{L}^*$ consists of several irreducible components corresponding to the *equisingular* strata of α where $\mathcal{L} \subset \mathcal{L}^*$ equals the closure of the *regular part* ${\underline{\mathcal{L}}_f}^{\mathrm{reg}}$ of $\underline{\mathcal{L}}_f$. Here $\underline{\mathcal{L}}_f^{\mathrm{reg}}$ is defined by the restrictions of α and f to the part $\mathcal{L}^{\mathrm{reg}} \subset \mathcal{L}$ where α has the maximal rank. The critical points of the function f and $\mathcal{L}$ correspond in this picture to the intersection of $\underline{L}_f \hookrightarrow T^*(V)$ with $\mathbb{O}_V \subset T^*(V)$ (or of $\underline{\mathcal{L}}_f$ with $\mathbb{O}_V \times \mathbb{R}$ in $\mathrm{Jet}^1(V)$) while the (most interesting for us) intersections of the Lagrangian submanifold $L \subset T^*(V)$ under $\mathcal{L}$ with $\mathbb{O}_V$ correspond to the critical points of $f \mid \mathcal{L}^{\mathrm{reg}}$. Notice that we can also consider the map $\hat{\alpha} = \alpha \times f : \mathcal{L} \to V \times \mathbb{R}$ which is a hypersurface (front of the Legendrian submanifold $\mathcal{L}$) in $V \times \mathbb{R}$ with non-generic folding, and possibly higher order singularities, and view the function f as the composiion $t \circ \hat{\alpha}$, where $t : V \times \mathbb{R} \to \mathbb{R}$ is the projection to the second factor. Now, define the stable Morse number μ_{st} of f on $\mathcal{L}^{\mathrm{reg}} \to V$, by considering the stabilized maps

$$\hat{a}^N = \hat{\alpha} \times \mathrm{Id} : \mathcal{L} \times \mathbb{R}^N \to V \times \mathbb{R} \times \mathbb{R}^N\ , \quad N = 1, 2, \ldots\ ,$$

the functions

$$F = Q(z) + \varepsilon(v, z) + t \quad \text{on} \quad V \times \mathbb{R} \times \mathbb{R}^N, \quad v \in V, t \in \mathbb{R}, z \in \mathbb{R}^N,$$

for all non-singular quadratic forms (functions) Q on $\mathbb{R}^N$ and all smooth compact (i.e. with compact supports) ε. Then we set

$$\mu_{st}(\mathcal{L}) = \inf_{N,F} \# \operatorname{crit}(F \circ \hat{\alpha}^N \mid \mathcal{L}^{re} \times \mathbb{R}^N),$$

where $N = 1, 2, \ldots$, and F runs over functions as above with the extra restriction of $F \circ \hat{\alpha}^N$ being *Morse* on $\mathcal{L}^{\text{reg}} \times \mathbb{R}^N$ (comp. 0.2.7 above). Finally we claim the following lower bound on the number of transversal zeros of Lagrangian submanifolds $L' \subset T^*(V)$ obtained from $\mathcal{L}$ by a compact contact isotopy followed by the projection to $T^*(V)$. (We use here the notations from 0.4.1).

Theorem 0.4.7

$$\pitchfork_{\mathbb{R}\mathbb{O}}(\mathcal{L}) \geq \mu_{st}(\mathcal{L}), \qquad (\pitchfork_{\mathbb{O}})$$

where $\mathbb{O} = J^1_{f=0}(V) \subset \operatorname{Jet}^1(V)$.

This, and similar bounds for $\cap_{\mathcal{M}}(\mathcal{L})$, $\pitchfork_{\mathcal{M}}(\mathcal{L})$ and for some $\mathcal{M} \neq \mathbb{R}\mathbb{O}$, are proven in Section 4.2.2 below. See also the discussion in 0.5.3.

0.5 Legendrian submanifolds in $PT^*(V)$

The *Projectivized cotangent bundle* $P = PT^*(V)$, i.e. the space of all tangent hyperplanes to V, carries a natural contact structure, denoted $\eta \subset T(P)$, which is characterized by the following property: the tangential, also called *Legendrian*, lift of each hypersurface $W \subset V$ to $PT^*(V)$, denoted $\mathcal{L}_W \subset PT^*(V)$ is a *Legendrian*, submanifold for η, i.e. everywhere tangent to η.

***P*-graphical manifolds.** More generally, let $W \subset V$ be a smooth submanifold in V of positive codimension. Then the submanifold $\mathcal{L}_W \subset P = PT^*(V)$ consisting of the hyperplanes $H_w \subset T_w(V)$, $w \in W$, tangent to W is also Legendrian, as it is known since the 18^{th} century. Such Legendrian submanifolds are called P-graphical.

***P*-subgraphical $\mathcal{L} = \underline{\mathcal{L}}^\alpha$.** In fact, the Legendrian graph (or lift) is defined for an arbitrary smooth map $\alpha : W \to V$ as follows. Denote by $\mathcal{L}$ the space of those pairs (w, H_v) for $v = \alpha(w)$ and H_v a hyperplane in $T_v(V)$, where H_v contains the image of the differential $D_\alpha(T_w(W))$ in $T_v(V)$. Then we have the natural map $\mathcal{L} \to PT^*(V)$ called *Legendrian subgraph* $\underline{\mathcal{L}}^\alpha$ of α in $PT^*(V)$. Notice that if $\alpha : W \to V$ is an equidimensional map with a fold along a submanifold $\Sigma \subset W$ then the Legendrian subgraph $\underline{\mathcal{L}}^\alpha$ coincides with the P-graphical manifold $\mathcal{L}_{W'} \subset P = PT^*(V)$ for the (immersed) hypersurface $W' = \alpha(W)$.

***Embeding* $\operatorname{Jet}^1(V) \subset PT^*(V \times \mathbb{R})$.** The *horizontal* hyperplanes in $V \times \mathbb{R}$ naturally correspond to 1-jets on V which define a natural embedding $\operatorname{Jet}^1(V) \subset$

$PT^*(V \times \mathbb{R})$. This corresponds to the usual interpretation of functions f on V as *horizontal* hypersurfaces in $V \times \mathbb{R}$ which are graphs $\Gamma_f : V \to V \times \mathbb{R}$ so that

$$\mathcal{L}_f = \underline{\mathcal{L}}^{\Gamma_f} \subset PT^*(V) \times \mathbb{R} \supset \mathrm{Jet}^1(V)$$

where, recall, $\mathcal{L}_f$ denotes the Legendrian graph of f, i.e. $J^1_f(V) \subset \mathrm{Jet}^1(V)$. Furthermore, if $W \subset V \times \mathbb{R}$ is a generic non-horizontal hypersurface, then, obviously, $\mathcal{L}_W \subset PT^*(V)$ equals the closure of $\underline{\mathcal{L}}_f$ where f denotes the projection of W to $\mathbb{R}$ and where the implied map α is the projection $W \to V$.

0.5.1 Legendrian linking problem

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Legendrian submanifold (or, possibly singular Legendrian subvarieties) in a contact manifold N. We want to define some *Legendrian linking number* measuring the minimal number of crossings between (compact) contact isotopies $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ moving $\mathcal{L}_1 = \mathcal{L}_1(0)$ and $\mathcal{L}_2 = \mathcal{L}_2(0)$ to the new positions $\mathcal{L}_1(1)$ and $\mathcal{L}_2(1)$ in N which are *disjoint* in a suitable sense. For example, if $N = \mathbb{R}^{2n+1} = \mathrm{Jet}^1(\mathbb{R}^n)$, the disjointness may mean being contained in disjoint Euclidean balls. More generally, we want to evaluate the infimum of the crossing number,

$$\inf \#(\mathcal{L}_1(t) \times \mathcal{L}_2(t))$$

where the infimum is taken from (compact) contact isotopies from a given class (e.g. those eventually moving $\mathcal{L}_1$ and $\mathcal{L}_2$ to disjoint positions).

It is convenient to think of the crossing set $\mathcal{L}_1(t) \times \mathcal{L}_2(t)$ as the pull-back of the diagonal $\Delta \subset N \times N$ under the map $\mathcal{L}_1 \times \mathcal{L}_2 \times \mathbb{R} \to N \times N$, for

$$(\ell_1, \ell_2, t) \mapsto (\ell_1(t), \ell_2(t)) .$$

The crossing is called *regular* if this map is transversal to Δ and then as earlier we often want to limit "inf" to the regular crossing which is expressed in writing by $\mathcal{L}_1(t) \underset{\text{reg}}{\times} \mathcal{L}_2(t)$, where this notation *implies* the regularity assumption (in the same manner $L_1 \pitchfork L_2$ implies transversality).

Now we observe that all our previous intersection inequalities for Legendrian and Lagrangian submanifolds could be reformulated in terms of lower bounds on the cardinalities of $\mathcal{L}_1(t) \times \mathcal{L}_2(t)$ and $\mathcal{L}_1(t) \underset{\text{reg}}{\times} \mathcal{L}_2(t)$ in $N = \mathrm{Jet}^1(V)$ for $\mathcal{L}_1(t) = \mathcal{L}_1$, $t \in \mathbb{R}$, and $\mathcal{L}_2(t) = \mathbb{R}\mathcal{L}_2$, where $\mathbb{R}\mathcal{L}_2$ is the $\mathbb{R}$-orbit of $\mathcal{L}_2$ in $\mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$. Presently we shall extend these inequalities in two directions.

1. We shall allow more general contact isotopies, not only those coming from some $\mathbb{R}$-action.

2. We shall pass from $\mathrm{Jet}^1(V)$ to $PT^*(V)$. (This is a true generalization when we speak of $PT^*(V \times \mathbb{R}) \supset \mathrm{Jet}^1(V)$.)

The crossing numbers between $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ will be bounded from below in terms of the kissing numbers (see 0.2.9) of certain pairs of smooth homotopies usually in $V \times \mathbb{R}^N$ or in $V \times V \times \mathbb{R}^N$ where one of the homotopies will be an *isotopy of hypersurfaces.* (Beware! $N \neq N$: the dimension of $\mathbb{R}^N$ has nothing to do with the above contact manifold N.) This will bring along certain asymmetry between $\mathcal{L}_1$ and $\mathcal{L}_2$ which will not be apparent, however, in many final theorems.

Notice that crossing = kissing for Legendrian isotopies as Legendrian submanifolds $\mathcal{L} \subset N$ have $\dim \mathcal{L} = \frac{1}{2} \dim N - 1$. But our lower bounds for Legendrian crossings will be significantly stronger than the purely topological kissing estimates not taking into account the particular *contact* nature of the isotopies in question.

0.5.2 Examples of linking inequalities

Let W_1 and W_2 be two submanifolds properly immersed into V, such that they intersect *transversally.* In this case their Legendrian lifts $\mathcal{L}_1 = \mathcal{L}_{W_1}$ and $\mathcal{L}_2 = \mathcal{L}_{W_2}$ in $PT^*(V)$ do not intersect. Let $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ be compact contact isotopies of $\mathcal{L}_1$ and $\mathcal{L}_2$, such that the resulting $\mathcal{L}_1(1)$ and $\mathcal{L}_2(1)$ have disjoint (i.e. non-intersecting) projections to V. Then the number of (regular) crossing between these isotopies admits the following lower bounds.

I. Suppose the intersection $W_1 \cap W_2$ is compact.

Theorem 0.5.1 *The following bound on the number of crossings holds provided the crossings are regular*

$$\#(\mathcal{L}_1(t) \underset{\text{reg}}{\times} \mathcal{L}_2(t)) \geq \frac{1}{2} \operatorname{rank} H_*(W_1 \boxtimes W_2)\,. \qquad (\tfrac{1}{2} H_*)$$

Recall, that $\boxtimes$ denotes the fiber product which equals $\cap$ for *embedded* manifolds.

II. Let $V = W \times \mathbb{R}$ and the projection $W_1 \to W$ has non-zero degree. Here we assume W and W_1 equidimensionally connected and, furthermore, orientable, if we work with coefficients different from $\mathbb{Z}/2\mathbb{Z}$. Let $W_2 \subset V$ be compact and lie *one the left of* W_1, which means, the *negative* $\mathbb{R}$-*orbit* of W_2, i.e. the set of the pairs $(w_2, t_2 + t)$ for all $(w_2, t_2) \in W_2$ and $t \leq 0$, does not intersect W_1.

Theorem 0.5.2 *If the projection of $\mathcal{L}_2(1)$ to V lies on the right of such projection of $\mathcal{L}_1(1)$, then*

$$\#(\mathcal{L}_1(t) \times \mathcal{L}_2(t)) \geq \operatorname{cuplength} H^*(W_2)\,, \qquad (\times\ \text{cup})$$

and

$$\#(\mathcal{L}_1(t) \underset{\text{reg}}{\times} \mathcal{L}_2(t)) \geq \operatorname{rank} H^*(W_2)\,. \qquad (\times\ H_*)$$

Furthermore, if $W_1 = W \times t_0$, for some $t_0 \in \mathbb{R}$, then

$$\#(\mathcal{L}_1(t) \underset{\text{reg}}{\times} \mathcal{L}_2(t)) \geq \operatorname{stabMor}(W_2) \qquad (\times\ \text{Mor})$$

Probably, there is a similar LuS-inequality for non-regular crossings.

This ($\diagup\!\!\!\!\searrow$ Mor) seems non-trivial even in the case where $\mathcal{L}_1(t) = \mathcal{L}_{W_1(t)}$ for an isotopy $W_1(t)$ of $W_1 = W_1(0) = W \times t_0$ moving W_1 from the left to the right of W_2. (We could not find a direct topological proof of this inequality albeit the homological Morse inequality is readily available for an arbitrary regular homotopy $W_1(t)$.) In fact the full-fledged inequality ($\diagup\!\!\!\!\searrow$ Mor) remains valid for regular homotopies *without cooriented self-kisses*, (comp. [3]). This means we allow pairs of tangent hyperplanes to $W_1(t)$, for certain moments t, to become equal in $T(V)$, but the coorientations in these planes induced from W_1 (which is naturally cooriented in V) must be opposite. Our proof of this (see 4.2) is based on

The contact geometry in $\ddot{P}T^*(V)$. This $\ddot{P} = \ddot{P}T^*(V)$ denotes the bundle of cooriented hyperplanes in $T(V)$ with the contact structure $\ddot{\eta}$ on $\ddot{P}$ induced from η on $P = PT^*(V)$ by the obvious double cover $\ddot{P} \to P$. Every $\mathcal{L} \subset P$ lifts to $\ddot{\mathcal{L}} \subset \ddot{P}$ double covering $\mathcal{L}$. On the other hand, every *cooriented* hypersurface $W \subset V$ admits a *cooriented lift* $\dot{\mathcal{L}}_W \subset \ddot{P}$ which diffeomorphically project on the non-cooriented lift $\mathcal{L}_W \subset P$. Here we deal with contact isotopies $\mathcal{L}_1(t)$ of $\dot{\mathcal{L}}_1 = \dot{\mathcal{L}}_{W_1}$. The projection $\mathcal{L}_1(t)$ of $\dot{\mathcal{L}}_1(t)$ to P is not, in general, an isotopy; yet the above inequalities remain valid and ($\diagup\!\!\!\!\searrow$ Mor) for such $\mathcal{L}_1(t)$ under $\dot{\mathcal{L}}_1(t)$ applies to regular homotopies $W_1(t)$ without cooriented self-kisses since these lifts to *isotopies* $\dot{\mathcal{L}}_{W_1(t)} \subset \ddot{P}$.

0.5.3 Crossing and linking invariants of Legendrian submanifolds of $PT^*(V)$ and $\ddot{P}T^*(V)$

We recall the numbers $\cap_{\mathcal{M}}(\mathcal{L})$ and $\pitchfork_{\mathcal{M}}(\mathcal{L})$ defined in 0.4.1 for an arbitrary (embedded, immersed, or just smoothly mapped) manifold $\mathcal{M} \to N$, where $N = (N, \eta)$ is a contact manifold and $\dim \mathcal{M} + \dim \mathcal{L} = \dim N$ (and $\dim \mathcal{L} = \frac{1}{2}\dim N - 1$ as $\mathcal{L}$ is assumed contact). A particular case of $\mathcal{M}$ is a Legendrian isotopy $\mathcal{L}_0(t)$, with $t \in T \in \mathbb{R}$, thought of as a map $\mathcal{L}_0 \times T \to N$, where the above intersection numbers can be interpreted as the minimal possible numbers of (regular for $\pitchfork$) crossings between $\mathcal{L}_0(t)$ and the constant isotopies $\mathcal{L}'(t) = \mathcal{L}'$ for all $\mathcal{L}'$ contact isotopic to $\mathcal{L}$. A particularly useful class of "test" manifolds $\mathcal{M} \to N$ is contituted by *sub-Lagrangian*¶ ones, i.e. the projections of Lagrangian submanifolds from $M = (T(N)/\eta)^* - \mathbb{O}_N$ to N, where M, called the *symplectization* of N, carries the natural symplectic structure induced by the embedding $M \to T^*(N)$ assigning to each linear form $T(N)/\eta \to \mathbb{R}$ the corresponding form $T(N) \to T(N)/\eta \to \mathbb{R}$.

Another interesting invariant may be defined for a cooriented structure η, i.e. admitting a transversal vector field ∂ in N. One can always choose such ∂ *contact*, i.e. with the corresponding flow $\mathcal{I}_\partial(t) : N \to N$ preserving η, which allows a small contact perturbation of each Legendrian $\mathcal{L}$ to a nearby Legendrian $\mathcal{L}' \subset N$ not intersecting $\mathcal{L}$. Then a suitable Legendrian linking number, i.e. the minimal possible number of crossings between $\mathcal{L}$ and an isotopy $\mathcal{L}'(t)$ disengaging $\mathcal{L}'$ from $\mathcal{L}$, serves as an invariant of $\mathcal{L}$.

¶In [26] these submanifolds were called *pre-Lagrangian*

We shall indicate below how to evaluate such invariants for some $\mathcal{L}$ in $N = P = PT^*(V)$ and in $N = \ddot{P} = \ddot{P}T^*(V)$.

Crossing over cylinders. Let $V = V_0 \times \mathbb{R}$ and $\mathcal{M} = \bigcup_t \mathcal{L}_0(t) = \bigcup_t \mathcal{L}_{V_0 \times t}$. Take a smooth map $\alpha : W \to V$ and let $\mathcal{L} = \underline{\mathcal{L}}^\alpha$, where the present cases of interest are the following two.

I. The map α is an immersion and so $\underline{\mathcal{L}}^\alpha = \mathcal{L}_W$ for $W_\alpha \hookrightarrow V$.

II. The map α is a submersion (i.e. of rank $= \dim V$) at almost all points in V with a generic singularity $\mathcal{L} \subset W$ where $\operatorname{rank} \alpha = \dim V - 1$. Here $\underline{\mathcal{L}}^\alpha$ equals $\mathcal{L}$ immersed to $PT^*(V)$ by the projectivized differential of α restricted to $\mathcal{L} \subset W$. (Notice, that if $\alpha \mid \mathcal{L}$ has a fold of its own, then the Legendrian variety $\underline{\mathcal{L}}^\beta$ associated to $\beta = \alpha \mid \mathcal{L} : \mathcal{L} \to V$ is different form $\underline{\mathcal{L}}^\alpha$; yet it contains $\underline{\mathcal{L}}^\alpha$ as an irreducible component, where other components come from the further $\Sigma^{11\cdots}$-strata of α.)

Now we recall the stable Morse number of α with respect to the (linear) function $(v_0, t) \mapsto t \in \mathbb{R}$ on V defined in the present context as follows. Introduce *admissible* functions F on $V \times \mathbb{R}^N$, $N = 1, 2, \ldots$, of the form

$$F = F(v_0, t, z) = t + Q(z) + \varepsilon(v, z)$$

where Q is non-singular quadratic on $\mathbb{R}^N$, where ε has compact support, and such that F has no critical points on $V \times \mathbb{R}^N$. Then define the stable Lusternik-Schnirelman number

$$\ell s_{st}(\alpha) = \inf_F \# \operatorname{crit}(F \circ \alpha^N)$$

where $\alpha^N = \alpha \times \mathrm{id} : W \times \mathbb{R}^N \to V \times \mathbb{R}^N$, and where the infimum is taken over all admissible F. Then we define the stable *Morse* number $\mu_{st}(\alpha)$ by taking the infimum over all admissible *Morse* function.

0.5.3 Crossing inequalities in PT^***.** *If W is a closed manifold, then the numbers $\mu_{st}(\alpha)$ and $\ell s_{st}(\alpha)$ are invariant under contact isotopies of $\mathcal{L} = \underline{\mathcal{L}}^\alpha$ in $P = PT^*(W)$. Moreover,*

$$\cap_{\mathcal{M}}(\mathcal{L}) \geq \ell s_{st}(\alpha) \qquad (\cap \ell s)$$

and

$$\pitchfork_{\mathcal{M}}(\mathcal{L}) \geq \mu_{st}(\alpha)\,. \qquad (\pitchfork \mu)$$

Observe that $\ell s_{st}(\alpha) \geq \mathrm{stabLuS}(W)$ and $\mu_{st}(\alpha) \geq \mathrm{stabMor}(W)$. Moreover, $\ell s_{st}(\alpha)$ and μ may become arbitrarily large for $\mathcal{L}$ having fixed intrinsic topology. This may be arranged either with a complicated immersion $\alpha : \mathcal{L} = W \to V$, or by taking a topologically complicated W mapped to V with the singularity diffeomorphic to a given $\mathcal{L}$ (compare [20]). Thus one obtains, for example, infinitely many compact Legendrian submanifolds in $PT^*(\mathbb{R}^n)$ diffeomorphic to S^n which are not mutually contact isotopic while having, for $n \geq 2$, equal ordinary topological invariants.

Remark on general (non-subgraphical) $\mathcal{L}$. the above may be extended, up to some degree, to an arbitrary (immersed or embedded) Legendrian submanifold $\mathcal{L} \subset P = PT^*(V)$. For example, one may bound from below the intersection number $\#(\mathcal{L}' \cap \mathcal{M})$ for all $\mathcal{L}'$ contact isotopic to $\mathcal{L}$ in terms of the stable Lusternik-Schnirelman (Morse for $\pitchfork$) numbers of suitable functions restricted to the non-singular locus $\mathcal{L}^{\mathrm{reg}}$ of the projection $\alpha : \mathcal{L} \to V$ as we did in 0.4.3.

On general Lagrangian "tests" $\mathcal{M} \in T^*(V) - \mathbb{O}_V$**.** Every such $\mathcal{M}$ projects to a sub-Lagrangian variety in $PT^*(V)$ but evaluation of $\cap_{\mathcal{M}}(\mathcal{L})$ and $\pitchfork_{\mathcal{M}}(\mathcal{L})$ seems beyond reach for general (immersed) $\mathcal{M}$ and even the simplest $\mathcal{L}$ such as $\mathcal{L}_W$. One particular case where the above applies without complications is for the Lagrangian graph $\mathcal{M}$ of a non-vanishing closed 1-form on V. Another workable possibility is where V possesses an infinite covering, say $\widetilde{V} \to V$, such that $\mathcal{L}$ admits a *compact* lift $\overline{\mathcal{L}}$ to the corresponding covering $\widetilde{P}$ of $P = PT^*(V)$. Since $\widetilde{V}$ admits function $\widetilde{f}$ without critical points, we can use their Lagrangian graphs $\widetilde{\mathcal{M}} = L_{\widetilde{f}} \subset T^*(\widetilde{V})$ to define and evaluate intersections (crossing) invariants of $\overline{\mathcal{L}}$ and attribute them to $\mathcal{L}$.

Crossing invariants of $\mathcal{L} \subset \ddot{P}T^*(V)$**.**
Suppose $\mathcal{L} = \dot{\mathcal{L}}_W$ for an immersed *cooriented* hypersurface W in V, e.g. a W bounding an equidimensionally immersed $U \hookrightarrow V$. Then the above invariants for $V = V_0 \times \mathbb{R}$ (or more generally, in the presence of a closed non-vanishing 1-form on some covering of V) make sense for $\mathcal{L}$ where in the definition of $\dot{\mu}_{st}$ and $\dot{\ell s}_{st}$ replacing μ_{st} and ℓs_{st} one should count only "one half" of the critical points of f, namely those having "coorientation" compatible to that of W, i.e. the gradient of f in V at the critical points of $f \mid W$ must agree with the coorientation of W at these points and the similar selection rule is applied to the critical points of F on $W \times \mathbb{R}$.

Examples. The figure ∞ in $\mathbb{R}^2$ has $\dot{\mu}_{st} = \dot{\ell s}_{st} = 0$ (while $\mu_{st} = \ell s_{st} = 2$). But if a closed hypersurface bounds an immersed U, then

$$\dot{\mu}_{st}(W) \geq \operatorname{rank} H_*(U) \qquad (\mathrm{Mor})^{\cdot}$$

by the standard Morse inequalities for manifolds with boundaries, and similarly,

$$\dot{\ell s}_{st}(W) \geq \operatorname{cuplength}_{\pi} H^*(U) \qquad (\mathrm{LuS})^{\cdot}$$

for $\pi = \pi_1(U)$. Notice that W may bound several quite different U's, see [20], and one should take the one with the "largest" topology to make (Mor) and (LuS) the strongest.) And what we find interesting here is that

0.5.4 *The numbers $\dot{\mu}_{st}(W)$ and $\dot{\ell s}_{st}(W)$ are invariant under contact isotopies of $\mathcal{L} = \mathcal{L}_W$ in $\ddot{P}T^*(V)$.*

See 4.2 for the proof and further examples of $\mathcal{L} \subset \ddot{P}T^*(V)$.

Legendrian selflinking in $\ddot{P} = \ddot{P}T^*(V)$**.** Since the contact structure $\ddot{\eta}$ in $\ddot{P}$ is cooriented, we may slightly move a Legendrian $\mathcal{L}$ along a field ∂ transversal to $\ddot{\eta}$ as mentioned earlier and define (non-ambiguously for *embedded* $\mathcal{L}$)

$$\operatorname{slink} \mathcal{L} = \operatorname{link}(\mathcal{L}, \mathcal{L}')$$

where the Legendrian linking number between $\mathcal{L}$ and its slightly moved copy $\mathcal{L}'$ is defined as the minimal number of (regular, if we want so) crossings between contact isotopies $\mathcal{L}(t)$ and $\mathcal{L}'(t)$ moving $\mathcal{L} = \mathcal{L}(0)$ and $\mathcal{L}' = \mathcal{L}'(0)$ into positions $\mathcal{L}(1)$ and $\mathcal{L}'(1)$ in $\ddot{P} = \ddot{P}T^*(V)$, such that their projections to V lie in two small disjoint balls.[‖] And if we insist on *regular* crossings, we denote the corresponding number

$$\text{slink}_{\pitchfork}\ \mathcal{L} = \text{link}_{\pitchfork}\ (\mathcal{L}, \mathcal{L}')\,.$$

If $\mathcal{L} = \dot{\mathcal{L}}_W$ for a closed cooriented hypersurface $W \subset V$, then $\mathcal{L}' = \dot{\mathcal{L}}_{W'}$, where W' is obtained by a small normal move of W'. In this case the Legendrian linking number can be bounded from below in terms of the kissing number for isotopies disengaging W, from W', i.e. moving them to small disjoint balls in V. For example,

Proposition 0.5.5 *if W bounds an immersed U as above, then*

$$\text{slink}_{\pitchfork}\ \mathcal{L} \geq \text{rank}\, H_*(U)\,.$$

This allows, for every V, of dimension $n \geq 3$, and infinity of mutually contact non-isotopic Legendrian embeddings $S^{n-1} \to \ddot{P}T^*(V)$ with equal topological invariants.

0.6 Pseudo-isotopies and the homotopy groups of the spaces Lag and $\mathcal{L}$eg

Let V_0 be a compact manifold, possibly with a boundary, let $V = V_0 \times [0,1]$ and denote by $\text{Diff}_0 = \text{Diff}(V/V_0, t \mid \partial V)$ the group of C^∞-diffeomorphisms φ of V such that

(a) φ fixes some neighbourhood of $V_0 \times 0$ in V, say $V_0 \times [0,\varepsilon] \subset V$ for some $\varepsilon = \varepsilon(\varphi) > 0$.

(b) φ preserves the function $(v,t) \mapsto t$ on some neighbourhood of ∂V in V.

(Notice that (b) is a purely cosmetic condition which does not affect the homotopy type of the group.)

The group Diff_0 is called the *pseudo-isotopy* group of V. It was introduced and first sudied by J. Cerf, see [10]. A. Hatcher ([40]) and K. Igusa ([44]) proved a stabilization theorem for homotopy groups of Diff_0. The computation of the stable homotopy type of pseudo-isotopy groups were reduced by F. Waldhausen ([67]) to the Algebraic K-theory.

Next, let M be the symplectic manifold $T^*(V) \setminus \mathbb{O}_V$ and N be the contact manifold $\text{Jet}^1(V) \setminus (\mathbb{O}_V \times \mathbb{R})$, where, recall, $\text{Jet}^1(V) = T^*(V) \times \mathbb{R}$. Denote by Lag the space of smooth embedded Lagrangian submanifold $L \subset M$ which

[‖] Note that the invariant slink has very little to do with the self-linking invariant for Legendrian curves, also called Thurston-Bennequin invariant $\text{tb}(\mathcal{L})$, defined in [7] (however, one has the inequality $\text{slink}_{\pitchfork} \geq |\text{tb}(\mathcal{L})|$ when the invariant $\text{tb}(\mathcal{L})$ is defined).

equal to the Lagrangian graph of the function $(v,t) \mapsto t$ near the boundary, i.e. $L = L_t$ over some neighbourhood of $\partial V \subset V$. Similarly, define $\mathcal{L}$eg, the space of smooth embedded Legendrian submanifolds in N which are equal $\mathcal{L}_t$ near the boundary. Notice that Lag naturally embeds into $\mathcal{L}$eg, such that $L \mapsto \mathcal{L}$ amounts to L being equal to the projection of $\mathcal{L}$ under $\mathrm{Jet}^1(V) \to T^*(V)$. The group $\mathrm{Diff}_0(V)$ naturally (and symplectically) acts on $T^*(V)$ and (contactly) on $\mathrm{Jet}^1(V)$ keeping M and N invariant. Then $\mathrm{Diff}_0 \subset \mathrm{Diff}\, V$ acts on the spaces Lag and $\mathcal{L}$eg, since the boundary condition (a) and (b) on Diff_0 match the requirements $L = L_t$ and $\mathcal{L} = \mathcal{L}_t$ near ∂V (and since the action of $\mathrm{Diff}(V)$ is symplectic on $T^*(V)$ and contact on $\mathrm{Jet}^1(V)$).

We consider the orbit maps $\mathrm{Diff}_0 \to \mathrm{Lag} \hookrightarrow \mathcal{L}\mathrm{eg}$ sending $\varphi \in \mathrm{Diff}_0$ to $\varphi(L_t) \in \mathrm{Lag}$ and to $\varphi(\mathcal{L}_t)$ in $\mathcal{L}$eg.

Theorem 0.6.1 ***Injectivity Theorem.*** *The orbit maps are injective on the homotopy groups of* Diff_0*,*

$$0 \to \pi_i(\mathrm{Diff}_0) \to \pi_i(\mathrm{Lag}) \to \pi_i(\mathcal{L}\mathrm{eg}) \qquad \text{(inj)}$$

provided $\dim V$ *is sufficiently large compared to* i *(say* $\geq 3i+6$*).*

The above (inj) can be combined with the known results on the (stable) homotopy type of Diff_0. of V_0. For example, if V_0 equals the $(n-1)$-ball B^{n-1}, then Theorem 0.6.1 together with Waldhausen's theorem [67] and A. Borel's computations of $K(\mathbb{Z})$ (see [8]) give

Corollary 0.6.2 *Suppose that* $i = 3(\mathrm{mod} 4)$ *and* $n \gg i$*. Then*

$$\mathrm{rank}(\pi_i(\mathcal{L}\mathrm{eg}) \otimes \mathbb{Q}) \geq 1 = \mathrm{rank}(\pi_i(\mathrm{Diff}_0) \otimes \mathbb{Q}) \qquad (\star)_{\mathcal{L}},$$

as well as

$$\mathrm{rank}(\pi_i(\mathrm{Lag}) \otimes \mathbb{Q}) \geq 1 = \mathrm{rank}(\pi_i(\mathrm{Diff}_0) \otimes \mathbb{Q}) \qquad (\star)_{L}$$

Here the above $\mathcal{L}$eg can be viewed as the space of Legendrian submanifolds in $\mathbb{R}^{2n+1} - \mathbb{R}^{n+1} = \mathrm{Jet}^1(\mathbb{R}^n) - \mathbb{R}^{n+1}$ standard at infinity, and Lag can be similarly interpreted in $\mathbb{R}^{2n} - \mathbb{R}^n$. The above group Diff_0 is homotopically the same in this case as the group of diffeomorphisms of B^n fixing a small neighbourhood of a boundary point in B^n.

Chapter 1

Reduction of the Lagrangian intersection problem to the stable Morse theory

When we Hamiltonian deform a graphical submanifold $L_f \subset T^*(V)$ for some $f : V \to \mathbb{R}$ it (obviously) remains graphical in-so-far as the projection $I(t)(L_f) \to V$ remains a diffeomorphism and then the intersection $I(t)(L_f) \cap \mathbb{O}_V$ is governed by the Morse theory. And when $I(t)(L_f)$ fails being graphical after some moment t, one may, amazingly, compensate for this by passing to a suitable graphical Lagrangian submanifold $L_{\widetilde{f}}(t) \subset T^*(V \times \mathbb{R}^{N(t)})$ having the same intersection with $\mathbb{O}$ as $I(t)(L_f)$. Thus we exhibit the intersection $I(t)(L_f) \cap \mathbb{O}$ as crit $\widetilde{f}$ for a function $\widetilde{f}$ on $V \times \mathbb{R}^n$ stabilizing f. This idea and the construction of $\widetilde{f}$ emerged from the work of Rabinowitz [56], Conley–Zehnder [18], Chaperon [11], Laudenbach–Sikorav [47], Chekanov [15] and Givental [35]. We give in this section a rendition of this construction which, as we hope, makes it as clear as possible.

1.1 Monitoring ε-parallel Lagrangian pairs by affine polarizations in $\mathbb{R}^{2n}$

1.1.1 Polarizations

A Lagrangian linear subspace $\mathcal{P}_0$ in the symplectic space $\mathbb{R}^{2n} = \mathbb{R}^n_p \times \mathbb{R}^n_q = \left(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dp_i \wedge dq_i\right)$ defines an *affine* (or linear) *polarization* $\mathcal{P}$ of $\mathbb{R}^{2n}$ which is, by definition, the partition of $\mathbb{R}^{2n}$ into affine subspaces parallel to $\mathcal{P}_0$. We use sometimes the same letter $\mathcal{P}$ for the polarization as well as the original linear subspace itself and denote by $\mathcal{Q}$ the quotient space $\mathbb{R}^{2n}/\mathcal{P}$. Then $\mathbb{R}^{2n}$ can be identified with the cotangent bundle $T^*(\mathcal{Q})$ where the affine spaces of $\mathcal{P}$ serve as the cotangent fibers. The cotangent bundle structure is not unique. To make it canonical one should pick up a smooth Lagrangian section $\mathcal{Q} \overset{\sim}{\to} L_0 \subset \mathbb{R}^{2n}$ and declare it the zero section $\mathbb{O}_\mathcal{Q} \subset \mathbb{R}^{2n}$. Then, indeed, there exists a unique symplectic diffeomorphism of $T^*(\mathcal{Q})$ with its canonical symplectic structure onto $(\mathbb{R}^{2n}, \omega)$ sending each fiber $T^*_q(\mathcal{Q})$ onto the affine space $\mathcal{P}_q \subset \mathbb{R}^{2n}$ such that $O_q \in T^*_q(\mathcal{Q})$ goes to $L_0 \cap \mathcal{P}_q$. And this diffeomorphism $T^*_q(\mathcal{Q}) \to \mathcal{P}_q$ is necessarily an affine map for every $q \in \mathcal{Q}$. (All this is trivial and well known).

Furthermore, if L_0 is a linear subspace, then the diffeomorphism $T^*(\mathcal{Q}) \to \mathbb{R}^{2n}$ is linear for the natural linear structure in $T^*(\mathcal{Q})$ coming from that of $\mathcal{Q}$ identified with L. The latter may be seen directly as for every pair of Lagrangian subspaces $\mathcal{P}$ and L_0 in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ there obviously exists a linear symplectic automorphism of $\mathbb{R}^{2n}$ sending $\mathcal{P} \mapsto \mathbb{R}^n \times 0$ and $L_0 \mapsto O \times \mathbb{R}^n$.

If L is another (besides L_0) Lagrangian submanifold in $\mathbb{R}^{2n}$ transversal to $\mathcal{P}$, i.e. to *all* affine subspaces constituting $\mathcal{P}$, then it defines in an obvious way a multivalued *closed* 1-form on $\mathcal{Q}$ denoted

$$\varphi = [L - L_0]_\mathcal{P}$$

which lives on the multidomain $L \to \mathcal{Q}$ as L is projected to $\mathcal{Q}$ by $\mathcal{P}$. If $L \to \mathcal{Q}$ is one-to-one, this is an ordinary single-valued 1-form on the image $L \subset \mathcal{Q}$.

The above $\varphi = [L - L_0]_\mathcal{P}$ makes sense for an arbitrary pair of Lagrangian submanifolds transversal to $\mathcal{P}$, without assuming L_0 is a *section*, i.e. its projection to $\mathcal{Q}$ is a diffeomorphism of L_0 onto $\mathcal{Q}$. In the general case however, one should note that φ is defined over the fiber product (or intersection) of L and L_0 over $\mathcal{Q}$, denoted $L \otimes L_0$ as in 0.3.5. And the (obvious) key observation is that

1.1.1 *the zeros of φ one-to-one correspond to the intersection points between L and L_0. In particular, if φ is exact and thus integrates to a function f on $L \otimes L_0$, then*

$$\operatorname{crit} f = L \cap L_0 \,,$$

where non-degenerate critical points correspond to transversal intersection points.

1.1.2 ε-stable transversality and ε-parallelism

Define the *minimal angle* $\alpha = \alpha(L, \mathcal{P})$ between a smooth submanifold $L \subset \mathbb{R}^{2n}$ and a polarization $\mathcal{P}$, as

$$\alpha = \inf_{\tau, p} \|\tau - p\| ,$$

where τ runs over all unit tangent vectors to L, i.e. $\tau \in T_\ell(L) \subset T_\ell(\mathbb{R}^{2n})$, $\ell \in L$, and p over all unit tangent vectors to $\mathcal{P}$ at all $\ell \in L$ (where "tangent to $\mathcal{P}$" means tangent to an affine space in $\mathcal{P}$). We say that $\mathcal{P}$ is *ε-stably transversal to* L if $\alpha > \varepsilon$. This is equivalent to $\mathcal{P}$ being transversal to every *ε-perturbation* L' which is obtained from L by a smooth map $I : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with the differential ε-close to the identity on L, i.e.

$$\|D_I(\ell) - \mathrm{Id}\,\| \leq \varepsilon$$

for all $\ell \in L$. (Recall that the norm of a linear operator A is defined as $\inf_{\|x\| \leq 1} \|A(x)\|$).

Finally, we say that L and L_0 in $\mathbb{R}^{2n}$ are *ε-parallel*, if there exists a polarization $\mathcal{P}$ transversal to both of them which is, moreover, ε-stably transversal to L.

$\sqrt{2}$-Example. Every *linear* Lagrangian subspace $L \subset \mathbb{R}^{2n}$ is ε-parallel to itself for each $\varepsilon \subset \sqrt{2}$. Indeed, the Euclidean normal $L^\perp \subset \mathbb{R}^{2n}$ is Lagrangian and has the minimal angle $\sqrt{2}$ with L.

$\sqrt{2}/2$-Example. Two arbitrary linear Lagrangian subspaces L and L_0 in $\mathbb{R}^{2n}$ are ε-parallel for each $\varepsilon < \sqrt{2}$ as we can use an arbitrarily small Lagrangian perturbation of $L^\perp$ for $\mathcal{P}$ since such a perturbation makes $\mathcal{P}$ transversal to L_0. But if we want a symmetric picture, where $\mathcal{P}$ remains transversal to ε-perturbations of both L and L_0, then we can guarantee it only with $\varepsilon < 2\sin\pi/8 > \sqrt{2}/2$. (We suggest the reader to construct a suitable $\mathcal{P}$ and then to generalize this to k Lagrangian subspaces in $\mathbb{R}^{2n}$.)

Alternating polarization. Consider the Cartesian product of $2k$-copies of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^n_p \times \mathbb{R}^n_q$

$$\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n} = \mathbb{R}^{2n}_1 \times \mathbb{R}^{2n}_{\bar{1}} \times \mathbb{R}^{2n}_2 \times \mathbb{R}^{2n}_{\bar{2}} \times \ldots \times \mathbb{R}^{2n}_{\bar{k}}$$

with the symplectic form $\Omega = \omega \oplus -\omega \oplus \omega \oplus -\omega \oplus \ldots \oplus -\omega$ for $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. There are two distinguished Lagrangian subspaces in this $\mathbb{R}^{2nk}$,

$$\Delta_{\text{even}} = \Delta_{1\bar{1}} \times \Delta_{2\bar{2}} \times \ldots \times \Delta_{k\bar{k}}$$

for $\Delta_{i\bar{i}}$ being the diagonal in $\mathbb{R}^{2n} \times \mathbb{R}^{2n} = \mathbb{R}^{2n}_i \times \mathbb{R}^{2n}_{\bar{i}}$, and

$$\Delta_{\text{odd}} = \Delta_{\bar{1}2} \times \Delta_{\bar{2}3} \times \ldots \times \Delta_{\bar{k}1} .$$

The product of the $\mathbb{R}^n_p$-polarizations, i.e. $\mathbb{R}^n_p \times \mathbb{R}^n_p \times \ldots \times \mathbb{R}^n_p \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n}$ for $\mathbb{R}^n_p = \mathbb{R}^n_p \times O \subset \mathbb{R}^{2n} = \mathbb{R}^n_p \times \mathbb{R}^n_q$ is *not* transversal to Δ_{even}. However, consider the following *alternating polarization*

$$\mathcal{P}_{\text{alt}} = \mathbb{R}^n_p \times \mathbb{R}^n_q \times \mathbb{R}^n_p \times \ldots \times \mathbb{R}^n_q ,$$

for $\mathbb{R}^n_p = \mathbb{R}^n_p \times O \subset \mathbb{R}^{2n} = \mathbb{R}^{2n}_1$, $\mathbb{R}^n_q = O \times \mathbb{R}^n_q \subset \mathbb{R}^{2n} = \mathbb{R}^{2n}_{\bar{1}}$, followed by $\mathbb{R}^n_p = \mathbb{R}^n_p \times O \subset \mathbb{R}^{2n} = \mathbb{R}^{2n}_2$, etc. (where subscripts p and q play quite different roles from the sub-indices i and $\bar{i}$). Then we have, as an obvious argument shows, the following

Proposition 1.1.2 *The alternating polarization $\mathcal{P}_{\text{alt}}$ is transversal to both Δ_{even} and Δ_{odd}. Moreover this transversality is ε-stable in the above sense for $\varepsilon < \sqrt{2}/2$.*

Quadratic function $\int[\Delta_{\text{even}} - \Delta_{\text{odd}}]_{\mathcal{P}}$**.** If a polarization $\mathcal{P}$ is transversal to both, Δ_{even} and Δ_{odd} (as is the case with $\mathcal{P}_{\text{alt}}$) then we get the closed 1-form $\Phi = [\Delta_{\text{even}} - \Delta_{\text{odd}}]_{\mathcal{P}}$, defined on all of $\mathcal{Q} = \mathbb{R}^{4nk}/\mathcal{P}$, which is exact and integrates to a non-singular quadratic function (form) Q on $\mathcal{Q} = \mathbb{R}^{2nk}$. This form Q is indeed non-singular since Δ_{even} is transversal to Δ_{odd} and it diagonalizes in an obvious basis to $\sum_{i=1}^{n}\sum_{j=1}^{k} x_{ij}y_{ij}$.

1.1.3 Partial ε-stability and ε-parallelism

Our Lagrangian submanifolds will often be of the form

$$\widetilde{L} = L_f \times \Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$$

where $L_f \subset \mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ is the Lagrangian graph of a function f defined on a domain $U \subset \mathbb{R}^n$ and $\Delta \subset \mathbb{R}^{2N}$ is a linear Lagrangian subspace. If, for example, U is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $f(u) = (\text{dist}(u, \partial U))^{-1}$ near the boundary, then $\mathbb{R}^n_p \times O$ is the *only* linear polarization in $\mathbb{R}^{2n}$ transversal to L_f and so the transversality $L_f \pitchfork (\mathbb{R}^n_p \times O)$ is highly unstable. (In our language, L_f is not ε-parallel to itself for $\varepsilon > 0$). But, on the other hand, there are plenty of polarizations $\widetilde{\mathcal{P}}$ in $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$ transversal to $\widetilde{L}$, e.g. those of the form $\mathbb{R}^n_p \times \mathcal{P}$, for linear polarizations $\mathcal{P}$ in $\mathbb{R}^{2m}$ transversal to Δ. Such a $\mathcal{P}$ can be chosen ε-stably transversal to Δ, say with $\varepsilon = \sqrt{2}/2$, which we express by saying that $\widetilde{\mathcal{P}} = \mathbb{R}^n_p \times \mathcal{P}$ is *ε-stably transversal to $\widetilde{L}$ in the Δ-direction.* If this is the case, the transversality persists under ε-perturbations

$$\widetilde{L} \rightsquigarrow \widetilde{L}' = L_f \times \Delta'$$

for ε-perturbations $\Delta' \subset \mathbb{R}^{2m}$ of Δ.

Making $f + Q + \varepsilon'$.

Suppose we have another, now linear Lagrangian subspace $\widetilde{L}_0 \subset \mathbb{R}^{2n} \times \mathbb{R}^{2N}$ transversal to $\widetilde{\mathcal{P}}$. then the function $\widetilde{f} = \int[\widetilde{L} - \widetilde{L}_0]_{\widetilde{\mathcal{P}}}$ is defined over $U \times \mathbb{R}^{2N}$ and equals $f(u) + Q(u, y)$, for a quadratic function $Q(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^{2N}$ restricted to $U \times \mathbb{R}^{2N}$. In fact, this Q is determined by the linear Lagrangian subspaces $\Delta \subset \mathbb{R}^{2N}$ and $\widetilde{L}_0 \subset \mathbb{R}^{2n} \times \mathbb{R}^{2N}$ and is defined by $Q = \int[\mathbb{O} \times \Delta - \widetilde{L}_0]_{\widetilde{\mathcal{P}}}$ for $\mathbb{O} = \mathbb{O}_{\mathbb{R}^n} = O \times \mathbb{R}^n_q \subset \mathbb{R}^{2n} = \mathbb{R}^n_p \times \mathbb{R}^n_q = T^*(\mathbb{R}^n = \mathbb{R}^n_q)$. We want to isolate an important case for our applications where $\operatorname{rank} Q = N$ and so Q can be written as $Q(z)$ for $z \in \mathbb{R}^N$.

Quasisplit $\widetilde{L}_0$**.** We say that $\widetilde{L}_0$ is Δ-*quasisplit* if $\dim(\widetilde{L}_0 \cap \Delta) = 0$ and $\dim(\widetilde{L}_0 \cap (\mathbb{O} \times \Delta)) = n$. This is equivalent to the intersection $\widetilde{L}_0 \cap (\mathbb{R}^{2n} \times \Delta)$ being transversal with the projection $\mathbb{O} \subset \mathbb{R}^{2n}$ of this intersection to $\mathbb{R}^{2n}$.

If $\widetilde{L}_0$ is Δ-quasisplit, then, obviously, rank $Q = N$ and $\ker Q \subset \mathbb{R}^n \times \mathbb{R}^N$ is transversal to $\mathbb{R}^N = O \times \mathbb{R}^N$. Thus we have a projection, say $z : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ with $\ker z = \ker Q$, and we can write $Q = Q(z)$ and $\widetilde{f} = \widetilde{f}(x) + Q(z)$ for the new splitting $\mathbb{R}^{n+N} = \mathbb{R}^n_x + \mathbb{R}^N_z$.

About $\varepsilon = \varepsilon(y) = \varepsilon(x,z)$**.** Now we assume that $\mathcal{P}$ is ε-stably transversal to Δ and take an ε-small smooth perturbation $\Delta \rightsquigarrow \Delta'$. Then Δ' remains transversal to $\mathcal{P}$ and $\widetilde{L}' = L \times \Delta'$ transversal to $\widetilde{\mathcal{P}}$. The perturbed function $\widetilde{f}' = \int [\widetilde{L}' - \widetilde{L}_0]_{\widetilde{\mathcal{P}}}$ obviously split into the sum,

$$\widetilde{f}' = f(x) + Q(z) + \varepsilon'(y)$$

where y is the coordinate projection to $\mathbb{R}^N$ for the original (x,y)-splitting $\mathbb{R}^n \times \mathbb{R}^N$. And if we want to use only the (x,z)-splitting, we write $f(x) + Q(z) + \varepsilon'(x,z)$. Notice that the Euclidean norm of $d\varepsilon'$ is bounded roughly by $(N+n)\varepsilon$ for our perturbation bounded in the operator norm by ε. Furthermore, if the perturbation $\Delta \rightsquigarrow \Delta'$ has compact support, then so does $\varepsilon(y)$ for $y \in \mathbb{R}^N$ (but not $\varepsilon(x,z)$, since the pull-backs of compact subsets for $y : \mathbb{R}^n_x \times \mathbb{R}^N_z \to \mathbb{R}^N_y$ are not compact).

Summarizing all the above we obtain

Proposition 1.1.3 *There exists a one-to-one correspondence between the Lagrangian intersection $\widetilde{L}' \cap \widetilde{L}_0$ and the critical set of $\widetilde{f}' = f + Q + \varepsilon'$*

$$\widetilde{L}' \cap \widetilde{L}_0 \leftrightarrow \operatorname{crit} \widetilde{f}' ,$$

where transversal intersection points correspond to non-degenerate critical ones.

We shall see in the Section 1.4 below how this applies to the Lagrangian intersections $L'_f \cap L_0$ in $\mathbb{R}^{2n}$ where L'_f is obtained from L_f by a Hamiltonian isotopy which is by no means small. Also we take an additional case of ε' by suitably cutting it off at infinity and making it *compact* (i.e. with compact support) without changing $\operatorname{crit}(f + Q + \varepsilon')$ (see 1.4). Thus *the intersection theory for $L'_f \cap L_0$ will be reduced to the stable Morse theory of the function f* (where the required non-degeneracy of Q will come from transversality between Δ and Δ_0 in $\mathbb{R}^{2n}$).

Remark The ε-perturbed Δ', with our current notion of ε-perturbation, does not have to be a Lagrangian graph of a function over $\mathbb{R}^N$, as the projection $\Delta' \to \mathbb{R}^N = \mathbb{R}^{2N}/\mathcal{P}$ may be non-bijective albeit *locally* diffeomorphic because of the transversality $\Delta' \pitchfork \mathcal{P}$. However in all our examples, this perturbation is *bounded* (even ε-small) in the ordinary sense, i.e. $\Delta \to \Delta' \subset \mathbb{R}^{2N}$ moves every point in Δ by a distance $\leq \delta < \infty$. This, together with the transversality, implies that $\Delta' \to \mathbb{R}^N$ is a diffeomorphism.

1.2 Making Lagrangian pairs ε-parallel in $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$

Let $I(t) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $t \in [0,1]$, be a smooth isotopy with *a uniform C^1-bound on the derivative* $\frac{dI(t)}{dt}$, e.g. $I(t)$ is C^2-smooth and *compact*, i.e. $I(t) = I(0) = \mathrm{Id}$ outside a compact subset in $\mathbb{R}^{2n}$. Then $I = I(1)$ can be decomposed into diffeomorphisms which are ε-close to the identity in the C^1-norm, $I = I_k \circ I_{k-1} \circ \ldots \circ I_1$ where $I_i = I(t_i) \circ I^{-1}(t_{i-1})$ and where ε can be made arbitrarily small with larger and larger k. Now we want to study the intersection $I(L) \cap L_0$ for given L and L_0 in $\mathbb{R}^{2n}$ by looking at the graph of the (chain of the) diffeomorphisms I_i as follows. First express the intersection $I(L) \cap L_0$ by the schematic equation

$$L \ni x \xrightarrow{I} y \in L_0$$

or equivalently, by

$$L \ni x = \overline{x} \xrightarrow{I} y \in L_0 \,.$$

This decomposes along with $I = I_k \circ I_{k-1} \circ \ldots \circ I_1$ as

$$L \ni x_1 = x_{\overline{1}} \xrightarrow{I_1} x_2 = x_{\overline{2}} \xrightarrow{I_2} \ldots = x_{\overline{k}} \xrightarrow{I_k} x_{k+1} \in L_0 \,,$$

which amounts to the two rows of equations

$$L \ni x_1, x_2 = I_1(x_{\overline{1}}), \ldots, x_{k+1} = I_k(x_{\overline{k}})$$

and

$$x_1 = x_{\overline{1}}, x_2 = x_{\overline{2}}, \ldots, x_{k+1} \in L_0 \,.$$

The first row defines the product of L by the graphs of I_i, denoted

$$\begin{array}{ccccccccc} L & \times & \Gamma_{\overline{1}2} & \times & \Gamma_{\overline{2}3} & \times & \ldots & \times & \Gamma_{\overline{k},k+1} \\ \cap & & \cap & & \cap & & & & \cap \\ \mathbb{R}^{2n}_1 & \times & (\mathbb{R}^{2n}_{\overline{1}} \times \mathbb{R}^{2n}_2) & \times & (\mathbb{R}^{2n}_{\overline{2}} \times \mathbb{R}^{2n}_3) & \times & \ldots & \times & (\mathbb{R}^{2n}_{\overline{k}} \times \mathbb{R}^{2n}_{k+1}) \end{array}$$

where $\Gamma_{\overline{i},i+1} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n} = \mathbb{R}^{2n}_{\overline{i}} \times \mathbb{R}^{2n}_{i+1}$. And the second row defines the product of the diagonals $\Delta = \Delta_{i\overline{i}} \subset \mathbb{R}^{2n}_i \times \mathbb{R}^{2}_{\overline{i}}$ with L_0,

$$\begin{array}{ccccccccc} \Delta_{1\overline{1}} & \times & \Delta_{2\overline{2}} & \times & \ldots & \times & \Delta_{k\overline{k}} & \times & L_0 \\ \cap & & \cap & & & & \cap & & \cap \\ (\mathbb{R}^{2n}_1 \times \mathbb{R}^{2n}_{\overline{1}}) & \times & (\mathbb{R}^{2n}_2 \times \mathbb{R}^{2n}_{\overline{2}}) & \times & \ldots & \times & (\mathbb{R}^{2n}_k \times \mathbb{R}^{2n}_{\overline{k}}) & \times & \mathbb{R}^{2n}_{k+1} \end{array} \,.$$

Then the intersection $I(L) \cap L_0$ is identified with

$$(L \times \Gamma_{\mathrm{odd}}) \cap (\Delta_{\mathrm{even}} \times L_0) \qquad (*)$$

for

$$\Gamma_{\mathrm{odd}} = \Gamma_{\overline{1}2} \times \Gamma_{\overline{2}3} \times \ldots \times \Gamma_{\overline{k},k+1}$$

and

$$\Delta_{\mathrm{even}} = \Delta_{1\overline{1}} \times \Delta_{2\overline{2}} \times \ldots \times \Delta_{k\overline{k}} \,.$$

Namely, the projection

$$\mathbb{R}^{2n}_1 \times \mathbb{R}^{2n}_{\bar{1}} \times \ldots \times \mathbb{R}^{2n}_{\bar{k}} \times \mathbb{R}^{2n}_{k+1} \to \mathbb{R}^{2n}_{k+1}$$

sends the intersection

$$(L \times \Gamma_{\text{odd}}) \cap (\Delta_{\text{even}} \times \mathbb{R}^{2n}_{k+1}) \qquad (**)$$

diffeomorphically onto $I(L)$ in $\mathbb{R}^{2n} = \mathbb{R}^{2n}_{k+1}$, such that the intersection $(*) \subset (**)$ goes onto $I(L) \cap L_0$.

Now we observe that since all I_i are ε-close to Id in the C^1-norm, their graphs $\Gamma_{\bar{i},i+1}$ are ε-close to the diagonals $\Delta_{\bar{i},i+1} \subset \mathbb{R}^{2n}_{\bar{i}} \times \mathbb{R}^{2n}_{i+1}$ and therefore the product of $\Gamma_{\bar{i},i+1}$ for $i = 1, 2, \ldots, k$ is ε-close to

$$\Delta_{\text{odd}} \overset{\text{def}}{=} \Delta_{\bar{1}2} \times \Delta_{\bar{2}3} \times \ldots \times \Delta_{\bar{k},k+i}$$

in our big Euclidean space $\mathbb{R}^{2n(2k+1)} = \mathbb{R}^{2n}_1 \times \mathbb{R}^{2n}_{\bar{1}} \times \ldots \times \mathbb{R}^{2n}_{k+1}$. (Here we use the fact that the norm of Cartesian sum of operators satisfies

$$\|A_1 \oplus A_2 \oplus \ldots \oplus A_k\| = \max_{1 \le i \le k} \|A_i\|,$$

which is quite different from the additive behavior of the Euclidean norm on vectors.) Thus $\widetilde{L}' = L \times \Gamma_{\text{odd}}$ appears as an ε-perturbation of $\widetilde{L} = L \times \Delta_{\text{odd}}$ where ε can be made arbitrarily small if we do not mind large k.

We did not use so far the symplectic and/or linear structure of $\mathbb{R}^{2n}$ but now we are going to do it. Namely, we assume that the isotopy $I(t)$ in question is Hamiltonian and therefore the diffeomorphisms I_i are symplectic. Then their graphs $\Gamma_{i,\overline{i+1}}$ are all Lagrangian for the form $\omega \oplus -\omega$ in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and so their product $\Gamma_{\text{odd}} \subset \mathbb{R}^{2n}_{\bar{1}} \times \mathbb{R}^{2n}_2 \times \mathbb{R}^{2n}_{\bar{2}} \times \ldots \times \mathbb{R}^{2n}_{\bar{k}}$ is Lagrangian for the form $-\omega \oplus \omega \oplus -\omega \oplus \ldots \oplus \omega$, as well as Δ_{odd}. Similarly, Δ_{even} is Lagrangian in $\mathbb{R}^{2n}_1 \times \mathbb{R}^{2n}_{\bar{1}} \times \ldots \times \mathbb{R}^{2n}_{\bar{k}}$ with the form $\omega \oplus -\omega \oplus \omega \oplus \ldots \oplus -\omega$. Thus the submanifolds $\widetilde{L} = L \times \Delta_{\text{odd}}$, $\widetilde{L}' = L \times \Gamma_{\text{odd}}$ and $\widetilde{L}_0 = \Delta_{\text{even}} \times L_0$ are *Lagrangian* in $\mathbb{R}^{2n(2k+1)} = \mathbb{R}^{2n}_1 \times \mathbb{R}^{2n}_{\bar{1}} \times \ldots \times \mathbb{R}^{2n}_{k+1}$ for $\Omega = \omega \oplus -\omega \oplus \omega \oplus \ldots \oplus \omega$.

Next we observe that the intersection $(\mathbb{O} \times \Delta_{\text{odd}}) \cap (\Delta_{\text{even}} \times \mathbb{O})$ for $\mathbb{O} = O \times \mathbb{R}^n_q$ consists of the strings $(O, x, O, x \ldots, O, x)$, i.e. equals the full diagonal in $\mathbb{O} \times \mathbb{O} \times \ldots \times \mathbb{O} = (\mathbb{R}^n_q)^{2k+1}$ and so it bijectively goes to $\mathbb{O}$ under the projection $\mathbb{O} \times \Delta_{\text{odd}} \to \mathbb{O}$. Thus $\widetilde{L}_0 = \mathbb{O} \times \Delta_{\text{odd}}$ is Δ_{even}-quasisplit in the sense of 1.1.2.

Also note that for $L = L_0 = O \times \mathbb{R}^n_q$ $\widetilde{L}$ and $\widetilde{L}_0$ are ε_0-*parallel* with any positive $\varepsilon_0 < \sqrt{2}$ as these are linear subspaces. It follows that if the above ε measuring the C^1-distance from I_i to id is $< \varepsilon_0$, we can find a polarization $\widetilde{\mathcal{P}}$ in $\mathbb{R}^{2n(2k+1)}$ transversal to both and $\widetilde{L}_0$, where it is convenient to have (split) $\widetilde{\mathcal{P}} = \mathbb{R}^n_p \times \mathcal{P}$. (Such a $\widetilde{\mathcal{P}}$, ε_0-transversal to $\widetilde{L}$ and $\widetilde{L}_0$, obviously exists, e.g. take the alternating one, see 1.1.3)

Now we are in a position to define the function

$$\widetilde{f}' = \int (\widetilde{L}' - \widetilde{L}_0)_{\widetilde{\mathcal{P}}}.$$

This has

$$\operatorname{crit} \widetilde{f}' = \widetilde{L}' \cap \widetilde{L}_0 = L'_f \cap L_0\,, \tag{$\star$}$$

where non-degenerate critical points of $\widetilde{f}'$ correspond exactly to transversal intersection points. According to 1.1.3 we also know that,

$$\widetilde{f}' = f(x) + Q(z) + \varepsilon'(x,z) \tag{$\star\star$}$$

for $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{2kn}$, the function Q being quadratic non-singular and $\|d\varepsilon'\|$ bounded. Recall (see 1.1.3) that the (x,z)-splitting of $\mathbb{R}^{(2k+1)n}$ is not quite the original one. Also note that the Euclidean norm $\|d\varepsilon'\|$ does not have to be small (unlike the ℓ_∞-norm corresponding to the operator norm) but the bound is independent of N (which does not seem to play any role).

Remark. Everything of what we have done but the last step applies to an arbitrary symplectic manifold M instead of $\mathbb{R}^{2n}$. Namely, the intersection $I(L)\cap L_0$ can be adequately represented by $(L\times\Gamma_{\text{odd}})\cap(\Delta_{\text{even}}\times L_0)$ in $M\times M\times\ldots\times M$ where Γ_{odd} equals a small perturbation of Δ_{odd}. What is special about the linear structure in $\mathbb{R}^{2n}$ is the abundance of polarizations in $\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\ldots\times\mathbb{R}^{2n}$.

1.3 Solution of the Arnold conjecture for the torus

We disgress for the moment to show how the above (seemingly trivial) discussion leads to an immediate proof of the following

Theorem 1.3.1 (Chaperon-Conley-Zehnder, see [11], [18]) *Let a Lagrangian submanifold $\underline{L}'$ in the cotangent bundle of the n-torus T^n be obtained by a Hamiltonian isotopy $\underline{I}(t)$ of $\underline{\mathbb{O}} = \mathbb{O}_{T^n}$ in $T^*(T^n)$. Then*

$$\#(\underline{L}' \cap \underline{\mathbb{O}}) \geq \operatorname{stabLuS}(T^n) = n+1 \tag{stL}$$

and

$$\#(\underline{L}' \pitchfork \underline{\mathbb{O}}) \geq \operatorname{stabMor}(T^n) = 2^n\,. \tag{stM}$$

Proof. Pass to the universal covering $\mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ of $T^*(T^n)$ for $T^n = \mathbb{R}^n/\mathbb{Z}^n$ and identify geometric objects over T^n (e.g. functions, Lagrangian manifolds in $T^*(T^n)$ etc.) with $\mathbb{Z}^n$-invariant objects over $\mathbb{R}^n$. In particular $\underline{L}' \subset T^*(T^n)$ lifts to $L' \subset T^*(\mathbb{R}^n)$ which is obtained by a $\mathbb{Z}^n$-invariant Hamiltonian isotopy $I(t)$ in $\mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ from $O \times \mathbb{R}^n_q \subset \mathbb{R}^{2n} = \mathbb{R}^{2n}_p \times \mathbb{R}^{2n}_q$. Since $\mathbb{O}_{T^n} = T^n$ is compact, one can (compare 2.2.3) choose $\underline{I}(t)$ also compact (i.e. with compact support in $T^*(T^n)$) without changing the action of $\underline{I}(t)$ on $\mathbb{O}_{T^n}$. In this case all derivatives of $I(t)$ and of $\frac{dI(t)}{dt}$ are bounded in $\mathbb{R}^{2n}$. Therefore the intersection $L' \cap \mathbb{O}_{\mathbb{R}^n}$ can be identified with $\operatorname{crit} \widetilde{f}'$ for the above $\widetilde{f}'(= f + Q + \varepsilon')$ which appears as the primitive (integral) of the 1-form $\widetilde{\Phi}' = [\widetilde{L}' - \widetilde{L}_0]_{\widetilde{\mathcal{P}}}$ on $\mathbb{R}^{2n(2k+1)}$. Now we use the diagonal action of $\mathbb{Z}^n$ on $\mathbb{R}^{2n(2k+1)} = \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n}$ where each $\mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ is acted upon by $\mathbb{Z}^n$ via the above action of $\mathbb{Z}^n$ on $\mathbb{R}^n = \mathbb{R}^n_q$. The quotient space

$$(\mathbb{R}^{2n(2k+1)}, \Omega = \omega \oplus -\omega \oplus \omega \oplus \ldots \oplus -\omega)/\mathbb{Z}^n$$

naturally identifies with $T^*(T^n \times \mathbb{R}^{2nk}) = T^*(T^n) \times \mathbb{R}^{4nk}$, where $T^*(T^n)$ comes from the full diagonal $\Delta_{\text{full}} = \{x, x, \ldots, x\}$ in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n})$ and the $\mathbb{R}^{4nh}$-factor is Ω-orthogonal to Δ_{full}. It is also clear that $\widetilde{L} = \mathbb{O}_{\mathbb{R}^n_p} \times \Delta_{\text{odd}}$ and $\widetilde{L}_0 = \Delta_{\text{even}} \times \mathbb{O}_{\mathbb{R}^n_p}$ are invariant under this (diagonal!) action and $\widetilde{L}' = \mathbb{O}_{\mathbb{R}^n_p} \times \Gamma_{\text{odd}}$ is also invariant because the isotopies $I_i = I(t_i)\, I^{-1}(t_{i-1})$ (whose graphs make Γ_{odd}) come from the corresponding symplectomorphisms $\underline{I}_i$ of $T^*(T^n)$. Since $\widetilde{\mathcal{P}}$ is also $\mathbb{Z}^n$-invariant (as $\mathbb{Z}^n$ acts by translations) the 1-form $\widetilde{\Phi}'$ is $\mathbb{Z}^n$-invariant as well. Furthermore, since the isotopy $\underline{I}(t)$ is *Hamiltonian* the (necessarily exact) form $\widetilde{\Phi}'$ is *equivariantly exact*, i.e. equals $d\widetilde{f}'$ for a $\mathbb{Z}^n$-*invariant* function $\widetilde{f}'$ on $\mathbb{R}^{2n(2k+1)}$.

Now it is time to go back to the torus, or rather to $T^*(T^n) \times \mathbb{R}^{4nk} = \mathbb{R}^{2n(2k+1)}/\mathbb{Z}^n$. The function $\widetilde{f}'$ descends to a function $\underline{\widetilde{f}'} = Q(z) + \underline{\varepsilon}'(x,z)$ on $T^n \times \mathbb{R}^{4nk}$ (with no $\underline{f}(x)$ term for $x \in T^n$ because our $\underline{L}_f = \mathbb{O}_{T^n}$) where Q is a non-singular quadratic form on $\mathbb{R}^{4nk}$ and $\underline{\varepsilon}'$ has $\|d\underline{\varepsilon}'\|$ bounded. Since $\operatorname{crit} \widetilde{f}' = \widetilde{L}' \cap \mathbb{O} = L' \cap \mathbb{O}$ over $\mathbb{R}^n$, we have the same relation descended to T^n (as everything over $\mathbb{R}^n$ is equivariant) and so

$$\operatorname{crit} \underline{\widetilde{f}'} = \underline{L}' \cap \underline{\mathbb{O}} \subset T^*(T^n)$$

where non-degenerate (Morse) critical points of $\underline{\widetilde{f}'}$ correspond to transversal intersection points between $\underline{L}'$ and $\underline{\mathbb{O}}$ in $T^*(T^n)$.

Finally we *cut-off* $\underline{\varepsilon}'$ at infinity, i.e. we take a large compact subset $K \subset T^n \times \mathbb{R}^{4nk}$ and a smooth function ε'' on $T^n \times \mathbb{R}^{4nk}$, such that

(i) ε'' equals $\underline{\varepsilon}'$ on K,

(ii) ε'' has compact support,

(iii) $\|d\varepsilon''(x,z)\| \leq \text{const}$,

where this const depends on $\sup \|d\underline{\varepsilon}'\|$ (which we know $< \infty$) but does not depend on K. The existence of such ε'' is obvious (see 2.1.6 for an extra discussion) and $Q + \varepsilon''$ has the same critical points as $Q + \underline{\varepsilon}'$ if K is sufficiently large, as $\|dQ(x,z)\| \to \infty$ for $z \to \infty$.

Thus we related our Lagrangian intersection $L' \cap \mathbb{O}_{T^n}$ to $\operatorname{crit}(Q + \varepsilon'')$ where $Q(x,z) = Q(z)$ is a non-singular quadratic form on $\mathbb{R}^{4nk}$ and $\varepsilon'' = \varepsilon''(x,z)$ on $T^N \times \mathbb{R}^{4nk}$ has compact support. Thus (stL) and (stM) follow from the definitions of stabLuS and stabMor (see 0.2.2). QED

1.4 Evaluation of $\#(I(L_f) \cap \mathbb{O}_{\mathbb{R}^n})$ in terms of d-bounded MLS theory

Let us bring the discussion in 1.2 preceding the toral digression to its logical conclusion. We consider the case where $L = L_f \subset \mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ for a smooth function f on an open subset $U \subset \mathbb{R}^n$ or, more generally, where $U \to \mathbb{R}^n$ is a multidomain (i.e. an equidimensional immersion) and $L = \underline{L}_f$ (see Section 0.1.1 for the notations). Then we apply a Hamiltonian isotopy $I(t)$ to L with C^1-bounded derivative $\frac{dI(t)}{dt}$ and we want to bound from below the cardinality of the intersection $I(L) \cap \mathbb{O}_{\mathbb{R}^n}$ for $I = I(1)$ in terms of the stable Morse and Lusternik-Schnirelman numbers of f. Here we shall need the following version of these numbers.

MLS-***numbers defined with*** d***-bounded*** ε***.***
Let us modify the definitions

$$\text{stabMor}(f)_{\text{comp}} = \text{Mor}[f + Q + \varepsilon]$$

and

$$\text{stabLuS}(f)_{\text{comp}} = \text{LuS}[f + Q + \varepsilon]$$

of 0.2.2, where the function $\varepsilon = \varepsilon(x, z)$ was required to be *compact* by relaxing this condition and allowing an arbitrary *d-bounded* function ε, which means $\|d\varepsilon\| \leq \text{const} < \infty$. Here we assume the manifold X in question where $f = f(x)$ is defined comes along with a Riemannian metric with respect to which we measure the norm of the differential of ε. (In fact $\varepsilon = \varepsilon(x, z)$ is defined on $X \times \mathbb{R}^N$ where we use the product metric.) We denote the resulting numbers

$$\text{stabMor}(f)_{dbd} \quad \text{and} \quad \text{stabLuS}(f)_{dbd}$$

and observe that in many cases these are equal to their "compact" counterparts. For example, if X is a compact manifold, every d-bounded $\varepsilon(x, z)$ can be (obviously) cut-off to a compact one without affecting the critical points of $f + Q + \varepsilon$. This is also true (and equally obvious) if X is complete with respect to the back-ground metric and $f(x)$ satisfies $\|df(x)\| \to \infty$ for $x \to \infty$ (compare 2.1.5 below). Now we are ready to state and prove the

Theorem 1.4.1 ***d-bounded MLS-nequalities for*** $I(\underline{L}_f) \cap \mathbb{O}_{\mathbb{R}^n}$***.*** *Let I and $\underline{L}_f$ be as above. Then*

$$\#(I(\underline{L}_f) \cap \mathbb{O}_{\mathbb{R}^n}) \geq \text{stabLuS}(f)_{dbd} \qquad (\text{LuS}_{dbd})$$

and

$$\#(I(\underline{L}_f) \pitchfork \mathbb{O}_{\mathbb{R}^n}) \geq \text{stabMor}(f)_{dbd} \,. \qquad (\text{Mor}_{dbd})$$

Proof. Actually, the proof is furnished by $(\star)$ and $(\star\star)$ in 1.2, where we need some caution in choosing the polarization $\widetilde{\mathcal{P}}$ in $\mathbb{R}^{2n(2k+1)}$. We cannot expect it to be ε-stably transversal to $\widetilde{L}$, but we achieve *the ε-stability in the Δ_{odd}-direction* with

$$\begin{array}{ccccccccccccc} \widetilde{\mathcal{P}} & = & \mathcal{P}_{\text{alt}} & = & \mathbb{R}^n_p & \times & \mathbb{R}^n_q & \times & \mathbb{R}^n_p & \times & \dots & \times & \mathbb{R}^n_p \\ \cap & & & & \cap & & \cap & & \cap & & & & \cap \\ \mathbb{R}^{2n(2k+1)} & & = & & \mathbb{R}^{2n} & \times & \mathbb{R}^{2n} & \times & \mathbb{R}^{2n} & \times & \dots & \times & \mathbb{R}^{2n} \end{array}$$

and this is all we need.

The role of X is played here by U with the Riemannian metric induced from $\mathbb{R}^n$ and, clearly, $\varepsilon'(x, y)$ of $(\star\star)$ is d-bounded for this metric. Thus $(\star\star)$ matches the d-bounded MLS numbers and the proof is concluded. QED

Remark and Corollaries. (i) If $U = \mathbb{R}^n$ and $\|df(x)\| \to \infty$ for $x \to \infty$, then, as we know, we may pass back to the "compact" MLS numbers. This is also possible for general U if we require, besides the condition $\|df(u)\| \to \infty$ with $\|u\| \to \infty$, the lower bound $\|df(u)\| \geq \text{dist}^{-1}(u, \partial U)$. This is done with a suitable cut-off of ε' discussed at length in Section 2.1.

(ii) ***Bounded perturbations and Hofer metric.*** One can further modify the definition of the MLS numbers by allowing *bounded* ε (i.e. with $|\varepsilon| \leq \text{const} < \infty$) rather than d-bounded one. This still leads to a meaningful MLS theory for many functions f and suggests, on the symplectic side, perturbations L' of L which are bounded in the *Hofer metric*. Recently Yu. Chekanov (see [16] used infinite-dimensional approach to prove homological estimates for $L \otimes L'$.

(ii) ***About more general L and L_0.*** The ε-parallelization trick in 1.2 reducing the intersection between $I(L)$ and L_0 to that between $L \times \Gamma_{\text{odd}}$ and $\Delta_{\text{even}} \times L_0$ applies to arbitrary L and L_0 in $\mathbb{R}^{2n}$ which reduces the intersection problem between $I(L)$ and L_0 for subgraphical L and L_0 to the Morse theory of some auxiliary function $\widetilde{f}$. This also works for L and L_0 in $T^*(V)$ for all V via an embedding $V \subset \mathbb{R}^m$. But we shall follow a different route where we interpret the reduction

$$I(L) \cap \mathbb{O} \rightsquigarrow (L \times \Gamma_{\text{odd}}) \cap (\Delta_{\text{even}} \times \mathbb{O})$$

in terms of the *generating functions* which allow a quicker way of proving general intersection inequalities (see Chapters 3 and 4 below).

Chapter 2

Cutting off Morse functions and Hamiltonian isotopies with applications to monotonicity of Lagrangian intersection numbers under standard constructions

We state and prove in this section several technical results allowing us to reduce certain perturbations of functions of the form $f + \varepsilon$ to those where ε is compact (i.e. with compact support) without affecting $\mathrm{crit}(f + \varepsilon)$. We also address a similar problem for perturbations of Lagrangian submanifolds $L \rightsquigarrow L^\varepsilon$ where we want to keep unchanged some intersection $L^\varepsilon \cap L_0$ playing the role of $\mathrm{crit}(f+\varepsilon)$. This is subsequently used for reducing lower bounds on $\#(L^\varepsilon \cap L_0)$ to similar bounds for some auxiliary manifolds $\underline{L}^\varepsilon$ by showing that the number $\#(\underline{L}^\varepsilon \cap L_0)$ in the worst (i.e. least) case is no smaller than $L^\varepsilon \cap L_0$. Similar considerations are used to reduce contact isotopies to Hamiltonian ones.

All arguments we use in this section are rather soft and essentially trivial but we failed in our prior attempts to shovel them under the rug.

2.1 Lipschitz cut-off and extension of functions

It is well known and obvious that every Lipschitz function defined on a subset in a metric space, say φ_0 on $X_0 \subset X$, admits a Lipschitz extension φ to all of X with the same Lipschitz constant, $\mathrm{Lip}(\varphi) = \mathrm{Lip}(\varphi_0)$. If X is a smooth Riemannian manifold, and φ is smooth, then

$$\mathrm{Lip}(\varphi) = \sup_{x \in X} \|d\varphi(x)\| .$$

Conversely, every Lipschitz function φ on X can be *smoothed*, i.e. approximated by a C^∞-function φ_δ in the fine C^0-topology such that

$$\sup_{x \in X} \|d\varphi_\delta(x)\| \leq \mathrm{Lip}(\varphi) + \delta$$

for an arbitrary $\delta > 0$ given beforehand.

2.1.1 Cut-off of d-bounded functions

Let φ be a smooth *d-bounded* function on a Riemannian manifold X such that the support $\mathrm{supp}\,\varphi \subset X$ is *complete* with respect to the induced metric $\mathrm{dist}_X\,|_{\mathrm{supp}\,\varphi}$ which is equivalent to every closed *bounded* subset in $\mathrm{supp}\,\varphi$ being compact (since X is locally compact). Then φ *admits a d-bounded cut-off.* Recall, that "d-bounded" means $\|d\varphi\| \leq \mathrm{const} < \infty$ the existence of the above cut-off spells out as follows,

Lemma 2.1.1 *For each compact subset $K \subset X$ and every* $\mathrm{const}^+ > \mathrm{const}$ *there exists a smooth d-bounded function $\underline{\varphi}$ on X with compact support, such that $\underline{\varphi}\,|_K = \varphi\,|_K$ and, moreover, having* $\mathrm{supp}\,\underline{\varphi} \subset \mathrm{supp}\,\varphi$ *and* $\|d\underline{\varphi}\| \leq \mathrm{const}^+$.

Proof. First make φ bounded as well as d-bounded by taking a large constant C and setting $\overline{\varphi} = (\mathrm{sign}\,\varphi)\min(|\varphi|, C)$. Then smooth $\overline{\varphi}$ and cut it off by multiplying by a (cut-off) function $c(x)$ with compact support, such that $c\,|_K = 1$ and $\|dc(x)\| \leq \delta$ for $x \in \mathrm{supp}\,\varphi$ for some small $\delta > 0$. (The existence of such c obviously follows from the completeness of $\mathrm{supp}\,\varphi$). QED

Alternative proof. Fix a point $x_0 \in X$, take a large R and let $X_0 \subset X$ be the union of K, the zero set O_φ of φ and the complement to the ball $B = B(x_0\,R)$,

$$X_0 = K \cup O_\varphi \cup B^c .$$

Define φ_0 on X_0 by being equal φ on K and zero on $O_\varphi \cup B^c$ and observe that φ_0 has almost the same Lipschitz constant as $\varphi \mid K$ for large $R = R(K)$. Then Lipschitz extends φ_0 to $\underline{\varphi}$ on X and smooth it. This smoothed $\underline{\varphi}$ has compact support (as well as the original non-smoothed $\underline{\varphi}$) since $B \cap \mathrm{supp}\,\varphi$ is compact due to the completeness of $\mathrm{supp}\,\varphi$. QED

Morse cut-off of d-bounded perturbations

Lemma 2.1.2 *Let f and ε be smooth functions on a Riemannian manifold X, such that*

(i) $\|df\|$ blows up at infinity on $\operatorname{supp}\varepsilon$.

(This means, that for each $C > 0$ there exists a *compact* subset $S \subset \operatorname{supp}\varepsilon$ such that $\|df(x)\| \geq C$ for all $x \in (\operatorname{supp}\varepsilon) \setminus S$.)

(ii) ε is d-bounded on X and $\operatorname{supp}\varepsilon \subset X$ *is complete.*

Then the d-bounded perturbation $f + \varepsilon$ can be replaced by a compact one, $f + \underline{\varepsilon}$, i.e. where $\operatorname{supp}\underline{\varepsilon}$ *is compact, such that*

$$\operatorname{crit}(f + \underline{\varepsilon}) = \operatorname{crit}(f + \varepsilon)\,.$$

Proof. Cut-off ε (in the place of the above φ) with a *large* $K \subset X$ such that $\|df(x)\| > \|d\varepsilon(x)\|$ outside K.

Invariance of $\operatorname{stabMor}(f)_{\text{comp}}$ ***under perturbations of*** f

Recall that $\operatorname{stabMor}(f)_{\text{comp}}$ and $\operatorname{stabLuS}(f)_{\text{comp}}$ are defined with functions $f(x)+Q(y)+\varepsilon(x,y)$ where ε as $X\times\mathbb{R}^N$ is required to have a compact support. This does not allow functions $\varepsilon = \varepsilon(x)$ with compact support on X; however, these are d-bounded on X, for every Riemannian metric on X, and they remain d-bounded on $X \times \mathbb{R}^N$ with the product metric. Thus, by applying the Morse cut-off we see that

Lemma 2.1.3 *The stable Morse and Lusternik-Schnirelman numbers of a function f on X do not change if we compactly perturb f on X.*

Quadratic stabilization with non-trivial bundles $Z \to X$ Let $Z \to X$ be a vector bundle over X with a fiberwise non-singular quadratic form (function) $Q(z)$ on Z. Then one may speak about the quadratic stabilization $f(x) + Q(z) + \varepsilon(z)$ and about the corresponding MLS numbers of f. We claim that

the result will be the same as for $Z = X \times \mathbb{R}^N$ with $Q = Q(y)$, $y \in \mathbb{R}^N$.

In fact, one can complement (Z,Q) by some (Z',Q'), such that the bundle $Z \oplus Z'$ becomes trivial and, moreover, $(Z \oplus Z', Q \oplus Q')$ becomes isomorphic to $X \times \mathbb{R}^N$ with an untwisted quadratic form, i.e. depending only on $y \in \mathbb{R}^N$. The above cut-off discussion allows an insertion of compact perturbations ε before (as well as after) adding Z' which is all we need to justify our claim. QED

2.1.2 Expanding functions and their perturbations

A function $f(x)$ on a Riemannian manifold $X = (X,g)$ is called *expanding at infinity* if there exists a *complete* Riemannian metric g' on X such that

(i) g' *dominates* g, i.e. $g' - g \geq 0$ (which means "non-negative definite").

(ii) $\|df(x)\|_{g'} \to \infty$ for $x \to \infty$, where the latter "$\to \infty$" means "eventually going out of every compact subset in X".

Lemma 2.1.4 *Every d-bounded perturbation $f+\varepsilon$ of such an f on $X=(X,g)$ can be cut-off to a compact one with the same critical set.*

This follows from 2.1.2 applied to (X,g'), as $\|d\varepsilon\|_{g'} \le \|d\varepsilon\|_g$ for $g' \ge g$. In fact, the expanding property of f is needed only on $\operatorname{supp}\varepsilon$ which suggests the following notions expressing the idea of partial expanding.

Let $\overline{X}$ be the metric completion of $X=(X,g)$ and $\partial_g X = \overline{X} - X$. Take a point $\overline{x} \in \partial_g X$ and say that f is *expanding at* $\overline{x}$ if there exists a closed δ-ball in X around $\overline{x}$ for some $\delta > 0$,

$$B_\delta = \{x \in X \mid \operatorname{dist}(x,\overline{x}) \le \delta\}$$

such that $f \mid B_\delta$ is expanding for the induced Riemannian metric $g \mid B_\delta$. (Notice, that the presence of an ordinary boundary does not prevent a Riemannian manifold from being *metrically* complete.) Furthermore, f is called *locally expanding* at $Y \subset \partial_g X$ if it expands at all $\overline{x} \in Y$. It is called (globally) *expanding at* Y if there exists a closed neighbourhood $B \subset X$ of Y (i.e. B contains an open subset $B^0 \subset X$ whose completion contains Y) such that $f \mid B$ is expanding. (If $\overline{X}$ is locally compact the two notions coincide.)

Variation. Let X be an open subset in a complete Riemannian manifold (V,g). Then the metric associated to $g \mid X$ is *greater* than $\operatorname{dist}_V \mid X$. Thus the boundary $\partial_g X$ maps to the ordinary topological boundary $\partial X \subset V$. Here we say f *expands at* ∂X if it expands at some closed neighbourhood $B \subset V$ of ∂X intersected with X and similarly we define expansion at a subset Y in ∂X. This is a stronger condition than the expansion at the pullback $\overline{Y} \subset \partial_g X$ of Y under the map $\partial_g X \to \partial X$ but in most cases the difference between the two notions will be ignored. In fact, the two notions (obviously) coincide if each point $x \in \partial X \subset V$ can be reached by a curve in X of finite length, e.g. if ∂X is smooth.

Proposition 2.1.5 ***(Morse theoretic role of expansion.)*** *Let X be a Riemannian manifold and f be a smooth function on X expanding at $\partial_g X$. If a smooth d-bounded function ε on X has bounded support, i.e.* $\operatorname{diam}_X \operatorname{supp}\varepsilon < \infty$ *for* dist_g *on X, then there exists a smooth function $\underline{\varepsilon}$ on X with a compact support, such that*

$$\operatorname{crit}(f+\underline{\varepsilon}) = \operatorname{crit}(f+\varepsilon)\,.$$

Furthermore, if X appears as an open subset in the Riemannian manifold V, this conclusion remains valid if f is expanding at $\partial X \subset V$ and $\operatorname{diam}_V \operatorname{supp}\varepsilon < \infty$.

The proof must be clear after all has been said.

Expanding functions on X mapped to V non-injectively

The notion of expansion with respect to g makes sense for degenerate semidefinite quadratic forms g on X, in particular for those induced by smooth maps $\alpha : X \to V$ from Riemannian metrics on V. In what follows we shall need the following version of expansion. Say that f is *expanding relative to a given smooth map* $\alpha : X \to V$, if there exists a Riemannian metric g in V and a *complete* Riemannian metric g' on X, such that the differential $D\alpha$ has bounded norm on X with respect to g and g' and $\|df(x)\|_{g'} \to \infty$ whenever $x \to \infty$ with $\alpha(x)$ being contained to a compact subset in V (i.e. for every compact subset $K \subset V$ and $C > 0$ there exists a compact subset $K' \subset X$, such that $\|df(x)\|_{g'} \geq C$ for $x \in \alpha^{-1}(K) - K'$).

In fact, what we need of this expansion, is the following obvious

Lemma 2.1.6 ***(Cut-off property.)*** *If f expands with respect to α, then for every compact $\varepsilon = \varepsilon(v)$ on V there exists a compact cut-off $\underline{\varepsilon} = \underline{\varepsilon}(x)$ of $\varepsilon \circ \alpha$ on X, such that the differential of $\underline{\varepsilon}$ in the region where $\underline{\varepsilon} \neq \varepsilon \circ \alpha$ is smaller than df in the sense of the g'-norm. In particular, the functions $f + \varepsilon \circ \alpha$ and $f + \underline{\varepsilon}$ have the same critical points.*

Examples of expanding functions. Let $\alpha : X \to V$ be an embedding onto an open subset $U \subset V$. If f satisfies the $(di)^{-1}$-condition in 0.3.2, i.e. $\|df(x)\| \geq (\mathrm{dist}_V(\alpha(x), \partial U))^{-1}$, then f is expanding at the boundary. In fact one may use any positive function $\varphi(\mathrm{dist})$ with $\int_0^1 \varphi(t)dt = \infty$ instead of $(\mathrm{dist})^{-1}$ as the (relevant) metric g' on $X = U$, obtained by the conformal sealing the background metric g on V with the factor $\varphi^2(\mathrm{dist}(u, \partial U))$ on U, is complete whenever g is complete.

Notice, that such functions f are abundant on domains $U \subset V$ with smooth boundaries ∂U, e.g. $(\mathrm{dist}(u, \partial U))^\beta$ for $\beta < 0$ and very nice functions defined in 0.3.3. And, in general, every proper smooth function f_0 on U which is a fibration at infinity can be modified to an $f = \varphi \circ f_0$ with some $\varphi(t)$, such that f satisfies $(di)^{-1}$.

2.2 Separation and divergence in symplectic manifolds

2.2.1 Separation in a metric space

Two subsets A and B in a metric space X are called *asymptotically* 2δ*-separated* or 2δ*-separated at infinity* if their δ-neighbourhoods have bounded intersection,

$$\mathrm{Diam}(\mathrm{Nb}_\delta\, A) \cap (\mathrm{Nb}_\delta\, B) < \infty\,. \qquad (*)_{\delta\delta}$$

We say A and B are *separated at infinity*, if $(*)_{\delta\delta}$ holds with *some* $\delta > 0$. They are called *(asymptotically) divergent*, if this holds for *all* $\delta > 0$. More generally, A and B are called *separated (divergent) away* from some subset C in X (usually $C \subset A \cap B$, e.g. $C = A \cap B$) if $(\mathrm{Nb}_\delta\, A) \cap (\mathrm{Nb}_\delta\, B)$ is contained is some δ'-neighbourhood of C for *some* (respectively, *all*) $\delta > 0$.

Graphical examples in $\mathbb{R}^{2n} = T^*(\mathbb{R}^n)$***.*** (i) Let A and B in $\mathbb{R}^{2n}$ be Lagrangian graphs of two smooth functions f and g on $\mathbb{R}^n$. Then they are separated at infinity, if and only if

$$\|\operatorname{grad} f(x) - \operatorname{grad} g(y)\| \geq \delta > 0 \qquad (\bigtriangledown)_0$$

for all $x, y \to \infty$ satisfying $\|x - y\| \to 0$. And A and B diverge iff

$$\|\operatorname{grad} f(x) - \operatorname{grad} g(y)\| \to \infty \qquad (\bigtriangledown)_\infty$$

for all $x, y \to \infty$ satisfying $\|x - y\| \leq \text{const} < \infty$.

(ii) Let A and B in $\mathbb{R}^{2n}$ be the Lagrangian graphs of smooth function f and g defined on a *bounded* domain $U \subset \mathbb{R}^n$. Assume $\|\operatorname{grad} f\| = \|df\|$ and $\|\operatorname{grad} g\| = \|dg\|$ blow up at ∂U which is equivalent to $A = L_f$ and $B = L_g$ being properly embedded (i.e. closed) in $\mathbb{R}^{2n}$. *Then A and B are separated at infinity iff* $(\bigtriangledown)_0$ holds whenever x and y converge to a point $\overline{u} \in \partial U$.

2.2.2 Adapted metrics in symplectic manifold (M, ω)

A Riemannian metric g on M is called *adapted* (to the symplectic form ω) if $g + \sqrt{-1}\omega$ is a *Hermitian metric* with respect to some almost complex structure $J : T(M) \to T(M)$ preserving g and ω. This is equivalent to the existence of a g-*orthonormal* coframe x_i, y_i, $i = 1, \ldots, n = \dim M/2$, at each point in M, such that ω equals $\sum_{i=1}^{n} x_i \wedge y_i$ at this point. Yet another equivalent definition reads,

$$\|dH\|_g = \|\operatorname{grad}_\omega H\|_g ,$$

for all smooth functions H on M, where, recall, $\operatorname{grad}_\omega H$ is the (Hamiltonian) vector field which is ω-dual to dH.

Subsets A and B of a symplectic manifold (M, ω) are called *symplectically* (asymptotically) separated (divergent) if they are (asymptotically) separated (divergent) with respect to a *complete adapted metric.*

The notions of the asymptotic separation and divergence between subsets (especially Lagrangian submanifolds) in $M = (M, \omega)$ will always be applied to some adapted (and usually complete) metric g. In our examples such a metric will be close at hand, e.g. the ordinary Euclidean metric in $\mathbb{R}^{2n} = T^*(\mathbb{R}^n)$. On the other hand, the *symplectically invariant* notion of separation (or divergence) referring to *the existence of some complete adapted metric g* with respect to which given A and B in M are asymptotically separated (divergent) does not seem so trivial. In fact, showing that certain pairs of disjoint properly embedded Lagrangian submanifolds are *not* symplectically separated requires an application of the pseudo-holomorphic techniques (at least at the present stage of knowledge).

Let us show that a complete adapted metric always exists.

Lemma 2.2.1 ***Existence of complete adapted metrics*** *Every symplectic manifold $M = (M, \omega)$ admits a complete adapted metric g.*

Proof. The required metric will be constructed starting with an arbitrary adapted metric g_0 and applying a certain symplectic automorphism A of $T^*(M)$ to it. This A is constructed with an exhaustion of M by compact domains with smooth boundaries S_i where A expands g_0 transversally to all S_i. Namely, we take small ε_i-neighbourhoods $N_i \subset M$ of S_i, normally (with respect to g_0) decomposed as $N_i = S_i \times [-\varepsilon_i, \varepsilon_i]$. We denote by $\Sigma_i \subset T(N_i)$ and $\nu_i \in T(N_i)$ the subbundles tangent and normal to the slices $S_i \times t$, $t \in [-\varepsilon_i, \varepsilon_i]$, respectively and take some symplectic automorphisms $A_i : T(N_i) \hookleftarrow$ preserving the decomposition $T(N_i) = \Sigma_i \oplus \nu_i$ and acting on ν_i by $A_i(\nu) = 2\nu$. Then A is taken equal Id outside all N_i and $A \mid T(N_i) \underset{\text{def}}{=} A^{\varphi_i}$ where $\varphi_i = \varphi_i(s,t) = \varphi_i(t)$ is a suitable sequence of positive functions on $[-\varepsilon_i, \varepsilon_i]$, such that φ_i vanish at the ends $\pm\varepsilon_i$ and are large and fast growing with i on the subsegments $[-\varepsilon_i/2, \varepsilon_i/2]$. Clearly $g = Ag_0$ is complete (as well as adapted) for suitable φ_i. QED.

2.2.3 The cut-off problem for Hamiltonian isotopies

Consider two closed subsets L_0 and L in $M = (M, \omega)$ and a Hamiltonian isotopy $I(t)$ of M with t running over some segment $T \subset \mathbb{R}$. We want to *cut-off* $I(t)$ to a Hamiltonian isotopy $\underline{I}(t)$ with the following three properties.

(i) $\underline{I}(t)$ is *compact*, i.e. $\underline{I}(t) = \underline{I}(0)$ outside a compact subset in M.

(ii) $\underline{I}(t)(L) \cap L_0 = I(t)(L) \cap L_0$ for all $t \in T$.

(iii) $\operatorname{supp} \underline{I}(t) \subset \operatorname{supp} I(t)$ where $\operatorname{supp} I(t)$ is the complement to

$$\operatorname{fix} I(t) = \{x \in X \mid I(t)(x) = x \quad \text{for all} \quad t \in T\}.$$

This is similar to cutting-off the perturbation $f + \varepsilon$ of a function f on V where $L = L_f$, $L_0 = \mathbb{O}_V \subset M = T^*(V)$ and $I(t)L_f = L_{f+t\varepsilon}$ for $t \in [0,1]$ (compare 2.1.2).

2.2.2 *Hamiltonian cut-off Lemma Let (M, ω) admit an adapted metric g such that*

(a) the implied Hamiltonian $H(t)$ defining $I(t)$ is d-bounded, i.e. has $\|dH(t)\|_g <$ const $\leq \infty$;

(b) The support $S = \operatorname{supp} I(t) \subset M$ is complete with respect to $\operatorname{dist}_{M,g}$ associated to g;

(c) The subsets $L \cap S$ and $L_0 \cap S$ diverge with respect to $\operatorname{dist}_{M,g}$.

Then there exists a cut-off $\underline{I}(t)$ of $I(t)$ satisfying (i)–(iii).

Proof. Apply the cut-off to each function $H(t)$, $t \in T$ in place of φ in 2.1, where $K \subset M$ is chosen sufficiently large (as in 2.1.2). Clearly, the isotopy $\underline{I}(t)$ with the resulting $\underline{H}(t)$ does the job. QED

2.2.4 Cutting off homogeneous isotopies

Let $M = (M, \omega)$ comes along with a free action of the multiplicative group $\mathbb{R}_+^x$, denoted $m \mapsto sm$ for $m \in M$ and $s \in \mathbb{R}_+^*$, and suppose this action *scales* ω, which means $\omega \mapsto s\omega$ for $m \mapsto sm$. If we use the vector field ∂ on M defining this action, then the scaling property reads $\partial\omega = \omega$ where $\partial\omega$ denotes the Lie derivative of ω by ∂.

A diffeomorphism of M is called *homogeneous* if it commutes with the action or, equivalently, if it preserves the field ∂. An isotopy is called *homogeneous* if it consists of homogeneous diffeomorphisms.

Lemma 2.2.3 *Every homogeneous symplectic isotopy is Hamiltonian.*

Indeed, let φ be a closed 1-form for which its ω-dual vectorfield, say $\varphi^{\perp_\omega}$, commutes with ∂. Then the relations $[\partial, \varphi^{\perp_\omega}] = 0$ and $\partial\omega = \omega$ imply $\partial\varphi = \varphi$ and so φ is *exact* as $\partial\varphi = d(\varphi(\partial))$. QED.

Fix a section $N \hookrightarrow M$ of the (trivial principle) fibration $M \to N = M/\mathbb{R}_+^x$ and write $M = N \times \mathbb{R}_+^x$ with $N = N \times 1$. Now every Hamiltonian (isotopy) on M may be cut-off by multiplying with a function $\mathrm{Tr}(s)$, $s \in \mathbb{R}_+^x$ which vanish for large s and equals one near zero, and we may also use the cut-off at $s = 0$ with $\underline{h} = 1 - \overline{h}$. Then we can combine the two, i.e. use $\overline{h}\underline{h}$. Next such a cut-off can be moved far away from $N \times 1 \subset M$ as follows. Let an isotopy $\overline{I}(t)$ be obtained from $I(t)$ by cutting off (the Hamiltonians $H(t)$) with $\overline{h}$. Then define $I^{\overline{s}}(t) = \overline{s} \circ \overline{I}(t) \circ \overline{s}^{-1}$ for some large $\overline{s} \in \mathbb{R}_+^x$ acting on M. Clearly $I^{\overline{s}}(t)$ is still a Hamiltonian isotopy with the fixed point set moved by $\overline{s}$ toward infinity. Similarly, by conjugating the $\underline{h}$-cut-off $\underline{I}(t)$ with the diffeomorphisms $\underline{s} \in \mathbb{R}_+^x$ near zero, we move the cut-off close to zero. As the isotopies $I^{\overline{s}}(t)$ and $I^{\underline{s}}(t)$ match near $N \times 1$, they define together a family of cut-off's, say $I^{\overline{\underline{s}}}(t)$ of $I(t)$, where eventually $\overline{s} \to \infty$ and $\underline{s}$ is chosen equal $\overline{s}^{-1} \to 0$, such that $I^{\overline{\underline{s}}}(t)$ *equals* $I(t)$ *in the band* $N \times [\underline{s}, \overline{s}] \subset M = N \times \mathbb{R}_+^x$ and the support of $I^{\overline{\underline{s}}}(t)$ lies in a somewhat wider band, say $N \times [t^{-1}, t]$ for some (finite!) $t > \overline{s}$.

All this will usually be applied to homogeneous isotopies with *compact supports in* N, i.e. of the form $K \times \mathbb{R}_+^x$ for a compact $K \subset N$. Then the above $I^{\overline{\underline{s}}}(t)$ *have compact support in* M (and, in fact, without the compactness assumption the definition of the cut-off of a Hamiltonian isotopy is sometimes dubious as it may destroy the global integrability of a Hamiltonian field).

Finally we observe that

Proposition 2.2.4 *Such a cut-off does not change the intersection of* $N \times 1 \subset M$ *with* $I(t)(L)$ *for* $\mathbb{R}_+^x$*-invariant subsets* $L \subset M$*, and thus it does not change intersections between* $I(L)$ *and* $L_0 \subset N \times 1$.

All this can be summarized in the following

Proposition 2.2.5 *A homogeneous isotopy with compact support in N satisfies the conclusion of the above Lemma 2.2.2, namely it admits a cut-off satisfying* (i)–(iii) *in 2.2.2 for all $L_0 \subset N \times 1 \subset M$ and all $\mathbb{R}_+^x$-invariant $L \subset M$.*

Dividing Hamiltonian isotopies into local steps We want to modify our arbitrary Hamiltonian isotopy $I(t)$ to another one, say $I_\varepsilon(t)$, such that $I_\varepsilon(t)$ decomposes into isotopies with ε-small supports. Here is the precise statement.

Lemma 2.2.6 ***(Partition of unity for Hamiltonian isotopies)*** *Let $I(t)$, $t \in [0,1]$ be a Hamiltonian isotopy with a compact support and $\{M_i\}$, $i = 1, \ldots,$ be a covering of M open subsets M_i. Thus, for an arbitrary $\varepsilon > 0$, there exists a Hamiltonian isotopy $I_\varepsilon(t)$ with the following properties*

- *(i) $I_\varepsilon(0) = I(0) = \mathrm{id}$ and $I_\varepsilon(1) = I(1)$;*
- *(ii) The restriction of $I_\varepsilon(t)$ to each subsegment $[t, t+\varepsilon] \subset [0,1]$ has its support contained in some subset $M_i = M_i(t)$, $i = 1, 2 \ldots$;*
- *(iii) The isotopy $I_\varepsilon(t)$ is ε-close to $I(t)$, meaning that every diffeomorphism $I_\varepsilon(t)$, $t \in [0,1]$ is ε-close to $I(t)$ in the C^2-topology in the space of maps $M \to M$ which is, in fact, given a metric s associated to some Riemannian metric in M;*
- *(iv) One can have $I_\varepsilon(t)$, defined for $(\varepsilon, t) \in \mathbb{R}_+ \times [0,1]$, continuous as a map $\mathbb{R}_+ \times [0,1] \to C^2(M,M)$ with $I_0(t) = I(t)$ for all t.*

This is done with an obvious modification of the Hamiltonian function $H(m,t)$ on $M \times [0,1]$ or by just referring to the general non-linear partition of unity theorem on p. 90 in [38]. Notice that either of the two proofs delivers an extra property of I_ε which is sometimes useful.

2.3 Stable intersection numbers $\mathrm{stab} \cap_{L_0}(L)_{\mathrm{comp}}$ and $\mathrm{stab} \pitchfork_{L_0}(L)_{\mathrm{comp}}$

Let L and L_0 be two (usually Lagrangian) subsets in $M = (M, \omega)$. We *stabilize* the pair (L_0, L) to the pair $(\widetilde{L}_0 = L_0 \times \mathbb{R}_q^n, \widetilde{L} = L \times \mathbb{R}_p^n)$ in $M \times \mathbb{R}^{2N}$ for $\mathbb{R}^{2N} = \mathbb{R}_p^N \times \mathbb{R}_q^N$ and we apply a compact Hamiltonian isotopy $\widetilde{I}(t)$ to $\widetilde{L}$. Then we define

$$\mathrm{stab} \cap_{L_0}(L)_{\mathrm{comp}} = \inf_{\widetilde{I}(t), N} \#(\widetilde{L}_0 \cap \widetilde{I}(\widetilde{L}))$$

where $\widetilde{I} = \widetilde{I}(1)$.

This notation will also be used in the case where L_0 and L are not subsets but rather maps (e.g. immersions) $L_0, L \to M$ where the intersection $\widetilde{L}_0 \cap \widetilde{I}(\widetilde{L})$ must be replaced by the fibered product (over $M \times \mathbb{R}^{2N}$), i.e. $\widetilde{L}_0 \otimes \widetilde{I}(L)$, but where we keep the notation $\cap_{L_0}(L)$ instead of the more accurate $\otimes_{L_0}(L)$.

If L and L_0 are (possibly immersed) *submanifolds* and we may speak of transversality, we define

$$\operatorname{stab} \pitchfork_{L_0}(L)_{\mathrm{comp}} = \inf_{\widetilde{I},N} \#(\widetilde{L}_0 \cap \widetilde{I}(L))$$

where now $\widetilde{I}(t)$ runs over those isotopies for which $\widetilde{I}(\widetilde{L})$ is *transversal* to $\widetilde{L}_0$.

2.3.1 Invariance of the stable intersection numbers under compact Hamiltonian isotopies

Lemma 2.3.1

$$\operatorname{stab} \cap_{L_0}(L)_{\mathrm{comp}} = \operatorname{stab} \cap_{L_0}(I(L))_{\mathrm{comp}} \qquad (\cap_{L_0})$$

and

$$\operatorname{stab} \pitchfork_{L_0}(L)_{\mathrm{comp}} = \operatorname{stab} \pitchfork_{L_0}(I(L))_{\mathrm{comp}} \qquad (\pitchfork_{L_0})$$

where $I = I(1)$ for a compact Hamiltonian isotopy $I(t)$ in M.

The proof of $(\cap_{L_0})$ and $(\pitchfork_{L_0})$ parallels that of 2.1.3 with 2.1.2 replaced by 2.2.2.

Remarks. a) It is also clear, that these intersection invariants $\operatorname{stab} \cap_{L_0}$ and $\operatorname{stab} \pitchfork_{L_0}$ of L are also invariant under compact Hamiltonian isotopies of L_0 and, in fact,

$$\operatorname{stab} \cap_{L_0}(L) = \operatorname{stab} \cap_L(L_0)$$

as well as

$$\operatorname{stab}_{L_0} \pitchfork(L) = \operatorname{stab} \pitchfork_L(L_0)\,.$$

(b) *Non-stabilized numbers* $\cap_{L_0}(L)_{\mathrm{comp}}$ *and* $\pitchfork_{L_0}(L)_{\mathrm{comp}}$. These are defined by minimizing $\#(L_0 \cap I(L))$ and $\#(L_0 \pitchfork I(L))$ over all compact Hamiltonian isotopies $I(t)$ of M itself without stabilizing by $M \rightsquigarrow M \times \mathbb{R}^{2N}$. Clearly (because of the cut-off), these can be only greater than their stable counterparts but there is no single (known) example of the strict inequality.

(c) There is nothing special about the (Lagrangian!) submanifolds $\mathbb{R}^N_p = \mathbb{R}^N_p \times 0$ and $\mathbb{R}^N_q = 0 \times \mathbb{R}^N_q$ in $(\mathbb{R}^{2N}, dp \wedge dq)$. We could equally stabilize (L, L_0) to $(\widetilde{L} = L \times \overline{L}, \widetilde{L}_0 = L_0 \times \overline{L}_0)$ for an arbitrary pair of transversal linear Lagrangian subspaces $\overline{L}$ and $\overline{L}_0$ in $\mathbb{R}^{2N}$ as every such pair is (linearly) symplectomorphic to $(\mathbb{R}^N_p, \mathbb{R}^N_q)$ in $\mathbb{R}^{2N}$. In fact every (non-linear) Lagrangian pair $(\overline{L}, \overline{L}_0)$ in $\mathbb{R}^{2N}$ would give the same stabilization if $(\overline{L}, \overline{L}_0)$ is symplectomorphic to $(\mathbb{R}^N_p, \mathbb{R}^N_q)$ in $\mathbb{R}^{2N}$.

2.3.2 Monotonicity of the Lagrangian intersection numbers under diagonalization

The standard way to treat an intersection $L \cap L_0$ in M is to identify it with $(L \times L_0) \cap \Delta$ in $M \times M$ where Δ denotes the diagonal in M. As we want to stick to *Lagrangian* submanifolds we give $M \times M$ the structure $\omega \oplus -\omega$ with respect to which Δ is Lagrangian. And if $M = T^*(V)$ we work in $T^*(V \times V) = (M \times M, \omega \oplus \omega)$ with Δ replaced by $\Delta^- \underset{\text{def}}{=} L_{\underline{\Delta}}$ where $\underline{\Delta}$ is the diagonal in $V \times V$. We observe that the involution $(\tau, \tau_0) \mapsto (-\tau, \tau_0)$ on $T^*(V) \times T^*(V) = T^*(V \times V)$ sends Δ^- to Δ and so the intersection $L \cap L_0$ in $T^*(V)$ identifies with $(-L \times L_0) \cap \Delta^-$ in $T^*(V \times V)$. The passage

$$L \cap L_0 \leadsto ((-L) \times L_0) \cap \Delta^-$$

keeps us in the category of *subgraphical* Lagrangian submanifolds since Δ^- is subgraphical and $-L$ is subgraphical whenever L is subgraphical.

Now we claim

Lemma 2.3.2 *A lower bound on $\cap_\Delta(L \times L_0)_{\text{comp}}$ in $(M \times M, \omega \oplus -\omega)$ implies such a bound on $\cap_{L_0}(L)_{\text{comp}}$ and the same is true for the stable intersection numbers as well as their $\pitchfork$-counterparts. In other words,*

$$\cap_\Delta(L \times L_0)_{\text{comp}} \leq \cap_{L_0}(L)_{\text{comp}}$$

and similar inequalities hold true for $\text{stab} \cap_\Delta$, $\text{stab} \pitchfork_\Delta$ *and* $\pitchfork_\Delta$.

Proof. The only point to settle is to make the Hamiltonian isotopy $I(t) \times \text{id}$ in $M \times M$ *compact*, for compact $I(t)$, without affecting the intersection $(I(t)(L) \times L_0) \cap \Delta$. This is done using the cut-off $H(t) \mapsto H(t)\varphi_0$ for a suitable compact function φ_0 on the second M in $M \times M$. This can be seen directly, or with a use of a complete adapted metric g on M (see 2.2.2) so that 2.2.3 applies to $(M \times M, g \oplus g)$ and the isotopy $I(t) \times \text{id}$ in $M \times M$ acting on $L \times L_0$ and having the support S $(g \oplus g)$-divergent from Δ.

Remark. It is not hard to show that more sophisticated *algebraic* quantities associated to Lagrangian pairs (L, L_0), such as the Floer homology (whenever defined), are invariant (rather than merely monotone) under $(L, L_0) \leadsto (L \times L_0, \Delta)$. Probably, the intersection numbers $\cap_{L_0}(L)$, $\pitchfork_{L_0}(L)$ etc. are also *invariant* for $(L, L_0) \leadsto (L \times L_0, \Delta)$ for most (all?) Lagrangian pairs (L, L_0).

2.3.3 Monotonicity for the $\overline{\vee}$-diagonalization

Let us extend the operation $f \overline{\vee} f_0$ from (partially defined) functions in $\mathbb{R}^n$ (see 0.3.3) to (Lagrangian) submanifolds L and L_0 in $\mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ as follows

$$L \overline{\vee} L_0 = L \times (-L_0) \times \mathbb{O}_{\mathbb{R}^n} + d(z(x-y)) \subset T^*(\mathbb{R}^n = \mathbb{R}^n_x) \times T^*(\mathbb{R}^n = \mathbb{R}^n_y) \times T^*(\mathbb{R}^n = \mathbb{R}^n_z) = T^*(\mathbb{R}^{3n})$$

where adding df to a subset in $T^*(V)$ means applying the symplectomorphism $\tau \mapsto \tau + df$, and where $-L_0$ in $T^*(V = \mathbb{R}^n)$ has the same meaning as earlier. The intersection $(L \overline{\times} L_0) \cap \mathbb{O}_{\mathbb{R}^{3n}}$ is given by the equation

$$\begin{aligned} \tau_x + z &= 0 \\ -\tau_y - z &= 0 \\ x - y &= 0 \end{aligned}$$

which are equivalent to

$$x = y\ , \quad \tau_x = \tau_y \quad \text{and} \quad z = -\tau_x$$

where the first two define the intersection $L \cap L_0$. Thus we write

$$(L \overline{\times} L_0) \cap \mathbb{O}_{\mathbb{R}^{3n}} = L \cap L_0$$

and observe that this equality (obviously) respect the transversality of the intersection and extends to non-embedded $L, L_0 \to T^*(\mathbb{R}^n)$.

Every Hamiltonian isotopy $I(t)$ of $L \subset T^*(\mathbb{R}^n = \mathbb{R}^n_x)$ defines the isotopy $I(t) \times \mathrm{id} \times \mathrm{id}$ in $T^*(\mathbb{R}^{3n})$ which has the support divergent from $\mathbb{O}_{\mathbb{R}^{3n}}$, provided L_0 is proper in $T^*(\mathbb{R}^n = \mathbb{R}^n_y)$. In fact, if we limit (x, τ_x) to a compact subset in $T^*(\mathbb{R}^n = \mathbb{R}^n_x)$ and send $(y, \tau_y, z) \to \infty$, then either $z \to \infty$ and then $\tau_x + z \to \infty$, or $y \to \infty$ and then $x - y \to \infty$, or z and y stay bounded and $\tau_y \to \infty$ which makes $-\tau_y - z \to \infty$. And in any case $(L \overline{\times} L_0) \cap \mathrm{supp}(I(t) \times \mathrm{id} \times \mathrm{id})$ goes away from $\mathbb{O}_{\mathbb{R}^{3n}}$ as

$$L \overline{\times} L_0 = \{(x, \tau_x + z), (y, -\tau_y - z), (z, x - y)\} \subset \mathbb{R}^{6n}$$

for $\{x, \tau_x\} = L$ and $\{y, \tau_y\} = L_0$. It follows that $I(t) \times \mathrm{id} \times \mathrm{id}$ admits a compact cut-off non-affecting the intersection $(L \overline{\times} L_0) \cap \mathbb{O}_{\mathbb{R}^{3n}}$ and so we have

Lemma 2.3.3 *The stable and unstable intersection numbers* $\cap_{\mathbb{O}}(L \overline{\times} L_0)_{\mathrm{comp}}$, $\pitchfork_{\mathbb{O}}(L \overline{\times} L_0)_{\mathrm{comp}}$ *etc. (for* $\mathbb{O} = \mathbb{O}_{\mathbb{R}^{3n}}$*) do not exceed* $\cap_{L_0}(L)_{\mathrm{comp}}$, $\pitchfork_{L_0}(L)_{\mathrm{comp}}$, *etc.*

2.3.4 Bipolar symplectic extensions

A *symplectic extension* of $M = (M, \omega)$ is, by definition, a larger symplectic manifold $\widetilde{M} \subset M$, such that the implied symplectic form $\widetilde{\omega}$ of $\widetilde{M}$ restricts to ω on M. A *bipolar structure* is given by a pair of properly embedded *coisotropic* submanifolds $\widetilde{Z}$ and $\widetilde{Z}_0$ in $\widetilde{M}$ such that

(i) $\widetilde{Z} \cap \widetilde{Z}_0 = M$, and the intersection is transversal.

(ii) There exists a smooth retraction $\widetilde{\Pi} : \widetilde{Z} \to M$ where every fiber is a maximal connected $\widetilde{\omega}$-isotropic submanifold (leaf) in $\widetilde{Z}$. (Notice that such a projection, if it exists, is uniquely determined by $\widetilde{\omega}$ restricted to $\widetilde{Z}$ as its fibers are the maximal isotropic leaves in $\widetilde{Z}$.) We also require the existence of similar $\widetilde{\Pi}_0 : \widetilde{Z}_0 \to M$ with connected isotropic fibers.

Notice that bipolar extensions are (obviously) composable, i.e. if $\widetilde{M} \supset M$ is a bipolar extension of M and $\widetilde{\widetilde{M}} \supset \widetilde{M}$ is such an extension of $\widetilde{M}$ then $\widetilde{\widetilde{M}} \supset \widetilde{M}$ has a natural bipolar structure built of those of $\widetilde{M} \supset M$ and $\widetilde{\widetilde{M}} \supset \widetilde{M}$.

The *standard (split) extension* of M is $M \times \mathbb{R}^{2N}$ for $\mathbb{R}^{2N} = \mathbb{R}^N_p \times \mathbb{R}^N_q = T^*(\mathbb{R}^N_q)$ where $\widetilde{Z} = M \times \mathbb{R}^N_p$ and $\widetilde{Z}_0 = M \times \mathbb{R}^N_q$. *A split extension* $(\widetilde{M}, \widetilde{Z}, \widetilde{Z}_0)$ is a one isomorphic to $(M \times \mathbb{R}^{2N}, M \times \mathbb{R}^N_p, M \times \mathbb{R}^N_q)$. A bipolar extension $\widetilde{M}$ of M is called *subsplit* if there exists a bipolar extension $\widetilde{\widetilde{M}}$ of $\widetilde{M}$ such that the composed extension $\widetilde{\widetilde{M}}$ of M is split (where we systematically suppress Z's to keep the notations bearable).

Cotangent example. Let $M = T^*(V)$ and $Z \to V$ be a Euclidean vector bundle with the implied norm called $Q : Z \to \mathbb{R}$. Let $\widetilde{Z} = \widetilde{Z}_Q \subset \widetilde{M} = T^*(Z)$ consist of the 1-forms on Z which are equal to dQ on the fibers $Z_v \subset Z$, $v \in V$. Similarly, define $\widetilde{Z}_0 = \widetilde{Z}_{Q_0}$ for $Q_0 = a_0 Q$ and a real number a_0. If $a_0 \neq 1$, then these $\widetilde{Z}$ and $\widetilde{Z}_0$ transversally intersect along $M = T^*(V)$, obviously embedded into $\widetilde{M} = T^*(Z)$, and form a bipolar extension of M, called $(Q, a_0 Q)$-*extension*.

Such an extension $\widetilde{M} \supset M$ is *split* if $Z \to V$ is a *trivial* bundle, and it has *always subsplit* since every Z can be *complemented* to a trivial bundle.

2.3.5 H-divergency

Let $(\widetilde{M}, \widetilde{Z}, \widetilde{Z_0})$ be a bipolar extension of M. Every (Hamiltonian) function K on M lifts to $H \circ \widetilde{\Pi}$ on $\widetilde{Z} \xrightarrow{\widetilde{\Pi}} M$ and then it can be (non-uniquely) extended to a function $\widetilde{H}$ on $\widetilde{M} \supset M$. Notice that the symplectic gradient $\operatorname{grad}_{\tilde{\omega}} \widetilde{H}$ is necessarily (and obviously) tangent to $\widetilde{Z}$ which allows a (non-unique) lift of Hamiltonian isotopies from M to $\widetilde{Z}$ provided the field $\operatorname{grad}_{\tilde{\omega}} \widetilde{H}$ is *integrable*, i.e. has no orbits going to infinity (in $\widetilde{Z}$) in finite time.

In order to make this lift of isotopies canonical, it suffices to choose a subbundle $\nu \subset T(\widetilde{M}) \mid \widetilde{Z}$ complementary to the tangent bundle $T(\widetilde{Z}) \subset T(\widetilde{M})$. Using such ν we may restrict to those $\widetilde{H}$ on $\widetilde{M}$, whose differential on $\widetilde{Z}$ equals the pull-back of $d(H \circ \widetilde{\Pi})$ under the ν-factorization $T(\widetilde{M}) \mid \widetilde{Z} \to T(\widetilde{Z})$, and this normalizes the symplectic gradient of $\widetilde{H}$ on $\widetilde{Z}$. Equivalently, one may choose a subbundle $\mu \in T(\widetilde{Z})$ complementary to $\ker D\widetilde{\Pi}$ and lift Hamiltonian fields ∂ from M to μ-horizontal ones on $\widetilde{Z}$, i.e. contained in μ and project to ∂ by $D\widetilde{\Pi}$. (This μ reappears in 2.5 under the name of "connection $\bigtriangledown$".) Now, once we know how to lift fields from M to $\widetilde{Z}$ with ν or with μ, we may also lift Hamiltonian isotopies $I(t)$ from M to $\widetilde{Z}$ where they may be denoted either $I^\nu(t)$ or $I^\mu(t)$ depending on whether we use ν or μ for the lift of fields.

Say that $(\widetilde{M}, \widetilde{Z}, \widetilde{Z}_0)$ is H-*divergent*, if for every compact Hamiltonian isotopy $I(t)$, $t \in [0,1]$, of $M = \widetilde{Z} \cap \widetilde{Z}_0$ and every compact subset $\widetilde{K} \subset \widetilde{Z}$ there exists a *compact* Hamiltonian isotopy $\widetilde{I}(t)$ of $\widetilde{M}$ such that

(i) The action of $\widetilde{I}(t)$ on $\widetilde{K}$ equals that of $I^\nu(t)$ where the implied subbundle $\nu \in T(\widetilde{M}) \mid \widetilde{Z}$ must be independent of $I(t)$ and $\widetilde{K}$. (Notice that this implies $\widetilde{I}(t)(\widetilde{K}) \subset \widetilde{Z}$ for all t.)

(ii) If $H(t) \circ \widetilde{\Pi}$ vanishes at some $z \in \widetilde{Z}$ for all $t \in [0,1]$ (i.e. the point $m \in M$ under $\widetilde{Z}$ remains fixed under $I(t)$) then $\widetilde{H}(t)$ also vanishes at z for all t.

(iii) There exists a neighbourhood U of $\widetilde{Z}_0$ in $\widetilde{M}$ independent of $I(t)$ and K, such that $\widetilde{H}(t)(\widetilde{Z} \setminus \widetilde{K})$ does not intersect U for all $t \in [0,1]$.

Lemma 2.3.4 ***(Local criterion of*** H***-divergency)*** *Let M admit a cover by open subsets $M_i \subset M$, $i = 1, \ldots,$ such that the above $\widetilde{I}(t)$ exists whenever* $\operatorname{supp} I(t)$ *is contained in one among the subsets M_i, where ν and U may depend on i. Then $\widetilde{I}(t)$ exists for all $I(t)$ with compact supports.*

Proof. Replace $I(t)$ by $I_\varepsilon(t)$ of 2.2.6 where $I_\varepsilon(t)$ for $\varepsilon = k^{-1}$ decomposes into a finite sequence of isotopies, $I_\varepsilon(t) = I_1 * I_2 * \ldots * I_k$ and where $I_j(t)$ is defined on the segment $[j-1/k, j/k] \subset [0,1]$ and has its support in some M_i for $i = i(j)$. Then we choose compact subsets $\widetilde{K}_j \subset \widetilde{Z}$ with large $\widetilde{K}_k$, depending on I_k, with $\widetilde{K}_{k-1}$ much larger than $\widetilde{K}_k$ and so on, where $\widetilde{K}_1$ is the largest of all. Then the lifts $\widetilde{I}_j$ neatly compose to an allowable isotopy $\widetilde{I}_\varepsilon(t)$ on $\widetilde{M}$. Furthermore, since this isotopy is ε-close to $I^\nu(t)$ on $\widetilde{K}_k$ (as ν is independent of I_j and K_j, albeit may depend on i), it may be slightly perturbed to become equal to $I^\nu(t)$ on a slightly smaller $\widetilde{K}$. QED.

2.3.5 Metric criterion for H-divergency. *Let $\widetilde{M}$ admit a complete adapted metric $\widetilde{g}$, such that*

(i) the projection $\widetilde{\Pi} : \widetilde{Z} \to M$ is Lipschitz, i.e. has $\operatorname{Lip} \widetilde{\Pi} < \infty$ *with respect to* $\operatorname{dist}_{\widetilde{M}} \mid \widetilde{Z}$ *and some (non-adapted background) Riemannian metric g on M.*

(ii) $\widetilde{Z}$ and $\widetilde{Z}_0$ diverge away from $M = \widetilde{Z} \cap \widetilde{Z}_0$.

Then $\widetilde{M}$ is H-divergent.

Proof. The lift of $H(t)$ to $H(t) \circ \widetilde{\Pi}$ on $\widetilde{Z}$ is Lipschitz for $\operatorname{dist}_{\widetilde{M}}$ by (i) and so it admits a Lipschitz extension to all of $\widetilde{M}$ where the desired cut-off follows from 2.1.1. (Notice that one may use the normal bundle for ν in this case.) QED

2.3.6 H-divergence for cotangent extensions. *The cotangent extension $\widetilde{M} = T^*(Z)$ of $M = T^*(V)$ for a Euclidean vector bundle $Z \to V$ is H-divergent.*

Proof. This is obvious for trivial bundles where $\widetilde{M} = M \times \mathbb{R}^{2N}$ and one may apply 2.3.5 to a product metric. Then the general case follows from 2.3.5 since every fibration Z is locally trivial over V and one may use the pull back of a fine covering of V to $M = T^*(V) \to V$ for M_i. QED

Remark. One could avoid the localization Lemma 2.3.4 by exhibiting a complete adapted metric $\widetilde{g}$ on $T^*(Z)$ satisfying (i) and (ii) of 2.3.5.

2.3.6 Invariance of the stable intersection numbers under bipolar extensions

If $(\widetilde{M}, \widetilde{Z}, \widetilde{Z}_0)$ is a bipolar extension, every pair of (Lagrangian) subvarieties L and L_0 in $\widetilde{M}$ lifts to a (Lagrangian) pair $(\widetilde{L}, \widetilde{L}_0)$ in $\widetilde{M}$ for $\widetilde{L} = \widetilde{\Pi}^{-1}(L) \subset \widetilde{Z} \widetilde{\subset} M$ and $\widetilde{L}_0 = \widetilde{\Pi}_0^{-1}(L_0) \subset \widetilde{Z}_0 \subset \widetilde{M}$.

Lemma 2.3.7 *Let $\widetilde{M}$ be H-divergent and subsplit, where moreover, the implied extension $\widetilde{\widetilde{M}}$ of $\widetilde{M}$ is also H-divergent. Then*

$$\operatorname{stab} \cap_{\widetilde{L}_0} (\widetilde{L})_{\mathrm{comp}} = \operatorname{stab} \cap_{L_0} (L)_{\mathrm{comp}} \qquad (\widetilde{\cap})$$

and

$$\operatorname{stab} \pitchfork_{\widetilde{L}_0} (\widetilde{L})_{\mathrm{comp}} = \operatorname{stab} \pitchfork_{L_0} (L)_{\mathrm{comp}} \qquad (\widetilde{\pitchfork})$$

Proof. The H-divergency allows a cut-off of isotopies as in the case of functions (comp 2.1.6).

Corollary 2.3.8 *The above $(\widetilde{\cap})$ and $(\widetilde{\pitchfork})$ hold true for the $(Q, a_0 Q)$-extension $\widetilde{M} = T^*(Z)$ of $M = T^*(V)$ for all Euclidean vector bundles $Z \to V$ and $a_0 \neq 1$.*

Indeed these extensions satisfy the H-divergency and subsplitness conditions (see 2.3.4 and 2.3.6).

Remarks. (a) Recall that "subvariety L in M" allows immersed (and more generally mapped) manifolds $L \to M$.

(b) We did not work out meaningful examples of $(\widetilde{\cap})$ and/or $(\widetilde{\pitchfork})$ for $M \neq T^*(V)$.

(c) Let Q and Q_0 arbitrary be fiberwise quadratic functions on $Z \to V$, such that $Q - Q_0$ is *fiberwise non-singular* then the pair $(\widetilde{Z}_Q, \widetilde{Z}_{Q_0})$ provides a bipolar structure to $\widetilde{M} = T^*(Z)$. It is not hard to see that this pair is symplectomorphic to $(\widetilde{Z}_{Q-Q_0}, \widetilde{Z}_0)$ which yields the H-divergency and the subsplitness property for $(\widetilde{Z}_Q, \widetilde{Z}_{Q_0})$. This, in turn, *implies $(\widetilde{\cap})$ and $(\widetilde{\pitchfork})$ for this extension.*

2.3.7 Monotonicity of the stable intersection numbers in $T^*(V)$ under Euclidean embeddings

Let us reduce the Lagrangian intersection problem in $T^*(V)$ to that in $\mathbb{R}^{2m} = T^*(\mathbb{R}^m)$ for some $m \geq n = \dim V$ by smoothly embedding $V \to \mathbb{R}^m$. Recall that such an embedding necessarily exists if $m \geq 2n$ and it can be chosen proper if we wish so. An embedded $V \subset \mathbb{R}^m$ admits a tubular neighbourhood $Z \subset \mathbb{R}^m$ which carries a (non-unique) structure of a Euclidean vector bundle over V. Then Lagrangian pairs (L, L_0) in $T^*(V)$ are first stabilized (or suspended) to $(\widetilde{L}, \widetilde{L}_0)$ in $T^*(Z)$ with the above (Q, a_0Q)-extension and then $\widetilde{L}$ and $\widetilde{L}_0$ go to $T^*(\mathbb{R}^m)$ via the embedding $T^*(Z) \subset T^*(\mathbb{R}^m)$ induced by the *equidimensional* embedding $Z \subset \mathbb{R}^m$. Clearly, the (stable) intersection numbers between $\widetilde{L}$ and $\widetilde{L}_0$ in $T^*(\mathbb{R}^m) \supset T^*(Z)$ are no greater that these numbers in $T^*(Z)$ since $\widetilde{L}$ has more compact Hamiltonian isotopies in $T^*(\mathbb{R}^m)$ than in $T^*(Z)$. Then 2.3.7 and 2.3.8 imply

Proposition 2.3.9 *Stable intersection numbers may only decrease as we go from L and L_0 in $T^*(V)$ to $\widetilde{L}$ and $\widetilde{L}_0$ in $T^*(\mathbb{R}^n) \supset T^*(Z)$.*

About $a_0 = 0$**.** If we choose $a_0 = 0$ then $\widetilde{L}_0 \subset T^*(\mathbb{R}^m)$, which is independent of Q, will be not proper as the embedding $Z \subset \mathbb{R}^m$ is non-proper. So if we want to keep properness of $\widetilde{L}_0$ (as well as of $\widetilde{L}$) in $T^*(\mathbb{R}^m)$ we better take $a_0 \neq 0$, for example $a_0 = -1$. On the other hand, if $L_0 = \mathbb{O}_V \subset T^*(V)$, then $\widetilde{L}_0 = \mathbb{O}_Z \in T^*(Z)$ for $a_0 = 0$ and this non-proper $\widetilde{L}_0 \subset T^*(\mathbb{R}^m)$, which is just the Lagrangian graph of $f = 0$ on $Z \subset \mathbb{R}^m$, happily extends to $\mathbb{O}_{\mathbb{R}^m} \subset T^*(\mathbb{R}^m)$, i.e. the graph of $f = 0$ on all of $\mathbb{R}^m$. Thus we see the desired inequality between the stable intersection numbers between L and $\mathbb{O}_V$ in $T^*(V)$ and $\widetilde{L}$ and $\mathbb{O}_{\mathbb{R}^m}$ in $T^*(\mathbb{R}^m)$.

About $a_0 = \infty$**.** In this case $\widetilde{Z}_0$ equals $T^*(\mathbb{R}^m) \mid V$ and $\widetilde{L}_0$ is just $\alpha_*(L_0)$ for the embedding $V \to \mathbb{R}^m$.

About properness of $V \subset \mathbb{R}^m$**.** This may or may not be needed to ensure the properness of $\widetilde{L} \subset T^*(\mathbb{R}^m)$. For example, if $L = L_f$ for a function defined in a *relatively compact* domain $U \subset V$, then $\widetilde{L} \subset T^*(\mathbb{R}^m)$ is proper whenever $L_f \subset T^*(V)$ is proper. On the other hand if $L = L_{V_0}$ for a *properly embedded* submanifold $V_0 \subset V$, then $\widetilde{L}$ may become non-proper in $T^*(V)$ for certain non-proper embeddings $V \subset \mathbb{R}^m$. So, to avoid unnecessary troubles, we shall assume our embeddings $V \subset \mathbb{R}^m$ proper.

About the choice of Z**.** A particularly nice Z is the *ε-neighbourhood* of V in $\mathbb{R}^m$ where $\varepsilon = \varepsilon(v)$ is a small positive fast decaying function on V. Let $\Pi : Z = Z_\varepsilon \to V$ be the normal projection onto V, such that $\Pi^{-1}(v)$, $v \in V$, is a Euclidean ε-ball $B(v, \varepsilon)$ in $\mathbb{R}^m$ of dimension $n - m$ for $n = \dim V$, centered at v and normal to V at v. Then the fibration $\Pi : Z \to V$ can be given a Euclidean vector bundle structure by just choosing a suitable quadratic function Q on Z. In fact, such a Q can be constructed as follows. Start with a smooth convex function ψ on the open segment $(-1, 1)$ which equals t^2 near zero and which blows up to $+\infty$ for $t \to \pm 1$. Notice that there exists a unique monotone diffeomorphism $(-1,1) \to \mathbb{R}$ sending ψ to the function t^2 on $\mathbb{R}$. Then ψ can be uniquely written as $\varphi(t^2)$ on $(-1,1)$ and we define

$$Q(z) = \varphi(\varepsilon^{-2}(v)\|z - v\|^2) \quad \text{for} \quad v = \Pi(z)\,.$$

Clearly, there exists a unique Euclidean vector bundle structure on $Z \to V$, such that the Euclidean structure in each fiber $Z_v = \Pi^{-1}(v)$ is characterized by the two following conditions:

(i) the straight segments in the ball $\Pi^{-1}(v) = B(v,\varepsilon) \subset \mathbb{R}^m$ issuing from v remain straight for the linear structure in $Z_v = \Pi^{-1}(v)$ underlying the Euclidean structure;

(ii) the function Q serves as the Euclidean norm on Z.

This choice of Z and the Euclidean vector bundle structure in Z will be helpful for our applications when we want certain properties of L and L_0 (such as being divergent) to pass unharmed to $\widetilde{L}$ and $\widetilde{L}_0$ in $T^*(\mathbb{R}^m)$.

About the meaning $L_W \subset T^*(V)$**.** The Lagrangian submanifold L_W associated to a submanifold $W \subset V$ is not graphical in our sense but it may be perturbed to such L', e.g. to L_Q for the above $Q = Q_\varepsilon$ on a neighbourhood Z_ε of W in V. Clearly L_Q converges to L_W in the obvious sense for $\varepsilon \to 0$ and the intersection theory for L_W thus reduces to that of L_Q. This is nowhere explicitly exploited in our present paper as it stands but we use a similar meaning of the Legendrian lift $\mathcal{L}_W \subset PT^*(V)$, see Section 0.5 above.

2.3.8 Invariance of the stable intersection numbers between L and $\mathbb{O}_V$ under embedings $V \to W$

If $L_0 = \mathbb{O} = \mathbb{O}_V$ and $\widetilde{L}_0 = \widetilde{\mathbb{O}} = \widetilde{\mathbb{O}}_{\mathbb{R}^m}$ then, in fact, one has the following *equalities* (instead of inequalities in 2.3.9)

Proposition 2.3.10

$$\operatorname{stab} \cap_{\widetilde{\mathbb{O}}}(\widetilde{L}) = \operatorname{stab} \cap_{\mathbb{O}}(L)$$

and

$$\operatorname{stab} \pitchfork_{\widetilde{\mathbb{O}}}(\widetilde{L}) = \operatorname{stab} \pitchfork_{\mathbb{O}}(L)$$

for $\widetilde{L}$ *"suspending"* L *via* Q.

The proof relies on the basic construction $(L \cap L_0) \leadsto (L \times \Gamma_{\text{odd}}) \cap (\Delta_{\text{even}} \times L_0)$ of 1.2 (see Section 3.1 below for more details). Then one generalizes by considering an embedding of V to an arbitrary manifold W and defining $\widetilde{L} \subset T^*(W)$ as earlier with some regular neighbourhood $Z \subset W$ of $V \subset W$. It is not hard to see (using embeddings $V \subset W \subset \mathbb{R}^m$) that the above equalities remain valid for all W instead of $\mathbb{R}^m$. The details of the proof are left to the reader.

2.4 Contact invariance and monotonicity of the stable Lagrangian intersection numbers in Liouville manifolds

2.4.1 Liouville structures

A Liouville structure on a symplectic manifold (M, ω) is given by a 1-form λ, defined up to addition of differentials of functions, such that $d\lambda = \omega$. And if we start with a smooth manifold M without any background symplectic structure, then a Liouville structure is an element λ in the space $\{$1-forms$/\{d$(functions)$\}$ with non-singular differential $\omega = d\lambda$. (Sometimes we do not distinguish between a Liouville structure and a 1-form representing it.) A Liouville structure on (M, ω) (obviously) exists iff ω is exact; it is unique iff $H^1(M; \mathbb{R}) = 0$.

Cotangent Example. The cotangent bundle $(T^*(V), \omega = dp \wedge dq)$ comes along with the canonical Liouville (structure) form $\lambda = dpq$.

If (M, λ) is Liouville, then $N = M \times \mathbb{R}$ carries a natural contact structure, namely

$$\eta = \eta_\lambda = \ker(\lambda + dt),$$

which is independent of the choice of the Liouville form λ up to a canonical isomorphism. Namely, if we replace λ by $\lambda' = \lambda + d\varphi$, then the diffeomorphism $(m, t) \mapsto (m, t + \varphi(t))$ of $M \times \mathbb{R}$ sends η_λ to $\eta_{\lambda'}$.

Warning. But this is radically false if we add a *non-exact* closed 1-form μ to λ. The resulting contact manifold $(N, \ker(\lambda + \mu + dt))$ does not have to be contactomorphic to $(N, \eta = \ker(\lambda + dt))$, as the following example (see [21]) shows. Let M is the annulus $S^1 \times (1, 2)$ with the Liouville form $\lambda_a = (z + a)\delta\theta$, where $\theta \in S^1 = \mathbb{R}/\mathbb{Z}, z \in (0, 2)$ and a is a constant ≥ 0. *Then the contact manifolds* $(N = M \times \mathbb{R}, \ker(\lambda_a + dt))$ *and* $N, \ker(\lambda_{a'} + dt)$ *are not isomorphic unless* $a = a'$.

Next, let $N = (N, \eta)$ be an arbitrary *cooriented* contact manifold, i.e. $\eta = \ker \nu$ for some 1-form ν on N and denote by N^+ the *positive symplectization* of N, i.e. the space of positive linear forms on the bundle $T(N)/\eta$ where the symplectic structure ω^+ on N^+ is induced from $T^*(N)$ by the obvious embedding $N^+ \subset T^*(N)$. This N^+ has a natural structure of a (necessarily trivial) principal $\mathbb{R}_+$-bundle over N whose sections correspond to choices of positive 1-forms ν representing η. Given such a section (form) ν on N, we may split $N^+ = N \times \mathbb{R}_+$ where $\omega^+ = d(\sigma\nu)$ for $\sigma \in \mathbb{R}_+$.

Clearly, the form ω^+ scales under the (principal) action of $\mathbb{R}_+$ on N which can be written as $\omega^+ \mapsto \sigma\omega^+$ for $x \mapsto \sigma x$ in N. If we denote by $\sigma\frac{\partial}{\partial\sigma}$ the vector field corresponding to this action, then the scaling property is expressed by $\sigma\frac{\partial}{\partial\sigma}(\omega^+) = \omega^+$ (compare 2.2.3).

2.4.2 Contact stabilization of integrable Liouville manifolds

A Liouville manifold M is called *integrable* if there is a representative 1-form λ with $d\lambda$ called ω for which the ω-dual vector field denoted $\partial = \partial_\lambda$ is *integrable*, i.e. integrates to a 1-parameter group of diffeomorphisms of M which necessarily scale the form ω.

Proposition 2.4.1 *If M is integrable then the symplectization of the contactization N of $M = (M, \omega = d\lambda)$, say $M^* = N^+$, is symplectomorphic to $(M,\omega) \times (\mathbb{R}^2, dp \wedge dq)$.*

Proof. The symplectic plane $\mathbb{R}^2$ is obviously symplectomorphic to $\mathbb{R}^2_+ = \{t, s \mid s > 0\}$ where the field $s\frac{\partial}{\partial s}$ acts and scales the form $dt \wedge ds$. Then the field $\partial_0 = \partial + s\frac{\partial}{\partial s}$ acts on the manifold $M_0 = M \times \mathbb{R}^2_+ = M \times \mathbb{R}^2$ where it integrates to an (obviously) free action of $\mathbb{R} = \mathbb{R}^x_+$ which scales the form $\omega_0 = \omega + dt \wedge ds$. Then the quotient space, say $N_0 = M_0/\mathbb{R}^x_+$ for this action has a natural cooriented contact structure, such that the symplectization N_0^+ of N_0 is isomorphic to M_0. What remains to do is to identify this N_0 with $N = (M \times \mathbb{R}, \lambda + dt)$. The implied contact structure η_0 on N_0 can be reconstructed as follows. Take the (Liouville) 1-form λ_0 which is ω_0-dual to the field ∂_0 and restrict λ_0 to N_0 embedded to M_0 as a section of the fibration $M_0 \to N_0$. Then the kernel of this $\lambda_0 \mid N_0$ is our η_0. Now we notice that $N = M \times \mathbb{R}$, embedded to $M_0 = N \times \mathbb{R}^x_+$ as $N \times 1$ meets every $\mathbb{R}^x_+$-orbit of the action defined by ∂_0 at a single point and so this $N \times 1$ can be viewed as one of our sections $N_0 \to M_0$. Finally we observe that the form λ_0 on $M_0 = M \times \mathbb{R}^2_+$ equals the sum of λ, which is ω-dual to ∂ in M, and sdt which is $ds \wedge dt$-dual to $s\frac{\partial}{\partial s}$ in $\mathbb{R}^2_+$. Thus λ_0 restricts to the required $\lambda + dt$ on $N_0 = N \times 1$ defined by the equation $s = 1$ in $M \times \mathbb{R}^2_+$. QED.

Example: Cotangent space** $M = T^*(V)$ **is integrable. In fact, the vector field generating the (scaling) action of $\mathbb{R}^*_+$ on $T^*(V)$ for $\tau \mapsto \sigma\tau$, is Liouville. Thus M^* is symplectomorphic to $M \times \mathbb{R}^2$ for this $M = T^*(V)$.

Remark. One can iterate this stabilization, $M \rightsquigarrow M^* \rightsquigarrow (M^*)^* \rightsquigarrow \ldots$. Since M^* is *always* integrable, this is equivalent to the obvious stabilization $M \times \mathbb{R}^{2i}$ from M^* on.

Also, one could start from a cooriented contact manifold N and stabilize either to Liouville manifolds N^+, $(N^+)^*$, $((N^+)^*)^*$ or to the contact ones, $N^+ \times \mathbb{R}$, $(N^+)^* \times \mathbb{R}$, etc. In fact, Liouville and contact manifolds appear on equal footing in such stabilization chains where odd dimensional numbers are contact and even dimensional are Liouville.

2.4.3 Monotonicity of the stable intersection numbers $\operatorname{stab}\pitchfork_{L_0}(L)$ and $\operatorname{stab}\cap_{L_0}(L)$ under contact isotopies of focal submanifolds L

A submanifold L in a Liouville manifold (M, λ) is called *exact*, if $\lambda \mid L$ is an exact form. For example every Lagrangian submanifold L with $H^1(L;\mathbb{R}) = 0$ is exact. If $L \subset M$ is exact, it admits a natural lift $L \to \mathcal{L} = \mathcal{L}_\lambda \subset N = M \times \mathbb{R}$ by $\ell \mapsto (\ell, \varphi(\ell))$ where φ is the primitive (integral) of the 1-form λ on L. There is a minor ambiguity here as $\varphi(\ell)$ is defined only up to an additive constant (assuming L is connected). Furthermore, one can eliminate the dependence of $\mathcal{L}_\lambda$ on the representative form λ by extending φ to all of M and replacing λ by $\lambda' = \lambda - d\varphi$. This λ' restricts to zero on L, i.e. $\lambda' \mid T(L) = 0$, and one can use $\mathcal{L} = L \times 0 \subset M \times \mathbb{R}$ corresponding to $\varphi'(\ell) \equiv 0$.

Warning. If $f : L \to M$ is a Lagrangian *immersion* then there is a difference between the exactness of the restriction of the Lioville form λ to $L' = f(L)$ and the exactness of the induced form $f^*(\lambda)$ on L.

Liouville fields. A vector field ∂ on $M = (M, \lambda)$ is called *Liouville* if it is ω-dual (for $\omega = d\lambda$) to some 1-form in the (Liouville) class of λ.

Focality. $L \subset M$ is called *focal* if there exists an integrable Liouville field ∂ *tangent* to L. For example, if L is a Lagrangian submanifold, this is equivalent to the vanishing of the ω-dual 1-form $\partial^{\perp_\omega}$ on L.

For instance, let $M = N_+ = N \times \mathbb{R}_+$ be the symplectization of a contact manifold (N, η) with Liouville form $\sigma\nu$, see 2.4.1. Then Lagrangian submanifolds of N_+ which are focal with respect to the Liouville *form $\sigma\nu$* are exactly the cones $\Lambda \times \mathbb{R}_+$, where Λ is a Legendrian (i.e. tangent to η) submanifold of (N, η). See next section for more examples of focal Lagrangian submanifolds.

Notice that any focal Lagrangian submanifold is exact.

Contact isotopy. A (possibly immersed) submanifold $L' \subset M$ is called *contact isotopic* to an exact submanifold L if it is obtained by the following three operations.

1. Lift L to an $\mathcal{L} \subset N = M \times \mathbb{R}$.

2. Apply a contact isotopy to $\mathcal{L}$ in $N = (N, \eta = \ker(\lambda + dt))$ and call the resulting submanifold $\mathcal{L}' \subset M \times \mathbb{R}$.

3. Take the projection of $\mathcal{L}'$ back to M for L'.

The implied projection $\mathcal{L}' \to M$ does not have to be an embedding but it is necessarily (and obviously) an immersion. If it *is* an embedding, then the contact isotopy is a symmetric relation between L and L'. But if L' requires double points, we do not regard L as contact isotopic to L' and the passage $L \rightsquigarrow L'$ becomes irreversible.

2.4.2 Contact Monotonicity. *If L is a focal submanifold and L' is contact isotopic to L with the implied isotopy of $M \times \mathbb{R}$ being compact, then*

$$\operatorname{stab}\cap_{L_0}(L')_{\rm comp} \geq \operatorname{stab}\cap_{L_0}(L)_{\rm comp} \qquad (\cap \geq)$$

and

$$\operatorname{stab} \pitchfork_{L_0}(L')_{\mathrm{comp}} \geq \operatorname{stab} \pitchfork_{L_0}(L)_{\mathrm{comp}} \qquad (\pitchfork \geq)$$

where $L_0 \subset M$ *is an arbitrary submanifold.*

Proof. Recall that $(\mathbb{R}^2, dp \wedge dq)$ is symplectomorphic to $(\mathbb{R}^2_+, dt \wedge ds)$ and one can obviously choose a symplectomorphism $\mathbb{R}^2 \leftrightarrow \mathbb{R}^2_+$ such that the $\mathbb{R}^1_q$-line in $\mathbb{R}^2$ (given by $p = 0$) goes to the $(s = 1)$-line ℓ_t, given by the equation $s = 1$ in $\mathbb{R}^2_+$, and the p-line goes to the s_+-line ℓ_0 defined as $\{t, s \mid t = 0, s > 0\} \subset \mathbb{R}^2_+$. Thus the pair $(L_0 \times \ell_t, L \times \ell_s)$ in $M \times \mathbb{R}^2_+ = M \times \mathbb{R}^2$ stabilizes (L_0, L) in the sense of 2.3. Next we recall that $M \times \mathbb{R}^2_+$ with the field $\partial_0 = \partial + s\frac{\partial}{\partial s}$ is isomorphic to the symplectization N^+ of $N = M \times \mathbb{R}$ with the field ∂_0 corresponding to the canonical "scaling" field in N^+. Since λ vanishes on L, the ω-dual field $\partial = \partial_\lambda$ is tangent to L, and so $L \times \ell_s$ is invariant under the field $\partial_0 = \partial + s\frac{\partial}{\partial s}$ in $N^+ = M \times \mathbb{R}^2_+$. Hence each contact isotopy in N is given by a homogeneous, i.e. ∂_0-preserving, Hamiltonian isotopy of $M \times \mathbb{R}^2_+ = N^+$. Such an isotopy of N^+ moves $\widetilde{L} = \mathcal{L} \times \ell_s$ to some ∂_0-invariant $\widetilde{L}'$ in $M \times \mathbb{R}^2_+$ corresponding to $\mathcal{L}' \subset N$ obtained from $\mathcal{L} = L \times 0 \subset N = M \times \mathbb{R}$ by the corresponding contact isotopy in N. This $\widetilde{L}'$ does not equal $\mathcal{L}' \times \ell_s$ anymore; yet the intersection $\widetilde{L}' \cap N$, for $N = N \times 1 \subset N \times \mathbb{R}^x_+ = (M \times \mathbb{R}) \times \mathbb{R}_+ = M \times \mathbb{R}^2_+$, still equals $\mathcal{L}'$ as each orbit of ∂_0 transversally meets N at a single point. Consequently, $\widetilde{L}' \cap (L_0 \times \ell_t) = \mathcal{L}' \cap (L_0 \times \ell_t)$, as $L_0 \times \ell_t \subset N = N \times 1 = M \times \ell_t \subset M \times \mathbb{R}^2_+$. It follows that the intersection $L' \cap L_0$ for $L' \subset M$ under $\mathcal{L}' \subset N = M \times \ell_t$, which identifies with the intersection $\mathcal{L}' \cap (L_0 \times \ell_t)$ in $N = M \times \ell_t$, equals the (stabilized) intersection $\widetilde{L}' \times \widetilde{L}_0$ in $M \times \mathbb{R}^2 = M \times \mathbb{R}^2_+$ for $\widetilde{L}_0 = L_0 \times \ell_t$. This does not quite match the definition of the stable intersection numbers (see 2.3) as $\widetilde{L}'$ is obtained from $\mathcal{L}$ by a *non-compact* Hamiltonian isotopy but the cut-off in 2.2.5 takes care of the non-compactness problem. QED.

L' ***as a symplectic reduction of*** $\widetilde{L}'$ ***in*** $M \times \mathbb{R}^2$**.** Since the intersection $\widetilde{L}' \cap (M \times \mathbb{R}) \times 1 \subset (M \times \mathbb{R}) \times \mathbb{R}_+ = M \times \mathbb{R}^2_+$ equals $\mathcal{L}' \subset M \times \mathbb{R}$ which projects to L', we conclude that L' equals the symplectic reduction (see 2.5) of $\widetilde{L}' \subset M \times \mathbb{R}^2_+$ for the coisotropic submanifold $M \times \mathbb{R} \times 1$ in $M \times \mathbb{R}^2_+$. Since we can identify $M \times \mathbb{R}^2_+$ and $M \times \mathbb{R}^2$, we may say that L' *is the symplectic reduction of* $\widetilde{L}' \subset M \times \mathbb{R}^2$ *with the coisotropic submanifold* $M \times \mathbb{R}^1_q \subset M \times \mathbb{R}^2 = M \times \mathbb{R}^1_p \times \mathbb{R}^1_q$, *where* $\widetilde{L}'$ *is obtained by a compact Hamiltonian isotopy from* $\widetilde{L} = L \times \mathbb{R}_p$.

On monotonicity for non-embedded and/or singular L ***and*** L_0 ***in*** M**.** First we notice that our argument requires nothing of L_0 whatsoever and so the monotonicity holds true for all L_0 (with our usual convention of counting multiple intersection points for non-embedded $L, L_0 \to M$ and defining transversality for the $\pitchfork$-inequality for singular L and L_0). On the other hand, if L is an immersed Lagrangian submanifold which is exact in the sense of having the 1-form induced from λ on the source manifold exact (see Warning above) and thus admitting (generically embedded) Legendrian lift $\mathcal{L} \subset M \times \mathbb{R}$, we cannot, in general, modify λ in order to make this 1-form zero on L. In fact this is possible only if the lift $\mathcal{L}$ of L has the same double points as L under it which imposes a strong limitation on L and which prevents elimination of double points of L by the contact isotopies we deal with (which agrees with the irreversibility of $L \rightsquigarrow L'$ mentioned earlier). On the other hand, we may define *focality* of a *smooth map* $L \to M$ by insisting on the existence of an *integrable Liouville* field ∂ on M *tangent* to L, which is equivalent to the vanishing of the 1-form on L induced from $\partial^{\perp_\omega}$. (The field δ here is unique being equal ∂ lifted to L.) In general, this tangency condition is not sufficient for the focality. But if ∂ vanishes on the image of $L \to M$, then ∂ lifts to $\partial = 0$ on L and if, furthermore, such a ∂ is integrable on M, then $L \to M$ is focal.

It is perfectly clear, that the above contact monotonicity holds true for all smooth manifolds $L \to M$ satisfying the focality condition as well as for singular L in M with a naturally generalized focality property (compare Section 2.3.2 above).

Contact isotopies in non-Liouville manifolds. If M is non-Liouville but the support of a Lagrangian regular isotopy if contained in an open subset $M_0 \subset M$ which *is Liouville*, then one may distinguish contact isotopies. If, moreover, M_0 is *integrable* and $L \cap M_0$ is *focal* in M_0 one may apply the above construction but, unfortunately these assumptions are rarely satisfied (never, if M is compact).

Another possibility is where ω has integral periods and therefore serves as the curvature of a principal S^1-bundle $N \to M$ with an S^1-*connection* which is just an S^1-invariant contact structure $\eta \subset T(N)$ transversal to the S^1-fibers. If this connection splits over $L \subset M$ (e.g. L is Lagrangian with $H^1(L) = 0$), then L lifts to a Legendrian $\mathcal{L} \subset N$ (which is unique up to S^1-rotation) and the contact isotopies of $\mathcal{L}$ in N project to what we may call *contact isotopies of L in M*. Some intersection properties for such isotopies are obtained in [35], [53] and [26].

2.4.4 Examples of focal submanifolds

(A) *If M is an integrable Liouville submanifold, e.g. a cotangent space, then every closed exact Lagrangian submanifold $L \subset M$ is focal.*

To see this write $\lambda \mid L = d\varphi$ and extend φ to a function φ' on M with compact support. Then the field ∂' corresponding (i.e. ω-dual) to the form $\lambda' = \lambda - d\varphi'$ is tangent to L and is integrable as it equals the integrable field $\partial = \lambda^{\perp_\omega}$ at infinity.

(B) *Let $L = \underline{L}_f$ for a submanifold U properly embedded to V and an arbitrary smooth function f on U. Then L is focal in $M = T^*(V)$.*

In fact, this is immediate for $L = L_U$ corresponding to $f = 0$ on V as the Liouville form $\lambda = pdq$ of V vanishes on L_U which implies the tangency of $\partial = \lambda^{\perp_\omega}$ to L_U. Then we observe that $\underline{L}_f = I_{f'}(L_U)$, where $I_{f'}$ is the symplectomorphism $\tau \mapsto \tau + df'$ of $T^*(V)$ constructed with some extensions f' of f to V. Hence $\underline{L}_f$ is also focal since focality is symplectoinvariant.

(C) *Let $L \subset \mathbb{R}^{2n}$ be a properly embedded Lagrangian submanifold which has bounded distortion in $\mathbb{R}^{2n}$. This means, there is a constant $C > 0$ such that* *every two points x_1 and x_2 in L can be joint by a curve in L of length $\leq C\,\mathrm{dist}_{\mathbb{R}^{2n}}(x_1, x_2)$. Then L is focal.*

Proof. Take the primitive (integral) φ of the Liouville form $\lambda = pdq$ restricted to L. This form has linear growth, i.e. $\|\lambda_x\| \leq \mathrm{const}\,\|x\|$ and so the Lipschitz constant of φ growth linearly on L for the intrinsic geometry of L. Since L is boundedly distorted, the same is true for the Lipschitz constant of φ in the ambient Euclidean geometry and then φ can be extended to a function φ' on all of $\mathbb{R}^{2n}$ with $\|d\varphi'(x)\| \leq \mathrm{const}'\,\|x\|$ as a trivial argument shows. (Exhaust $\mathbb{R}^{2n}$ by the balls $B_i = B(R = 2^i)$, $i = 0, 1, 2, \ldots$ and extend φ successfully from $L \cap B_i$ to B_i with $\mathrm{Lip} = \mathrm{const}\,2^i$.) Then take $\lambda' = \lambda - d\varphi'$, observe that this form has (at most) linear growth on $\mathbb{R}^{2n}$ and so the corresponding field ∂', also having linear growth, is integrable on $\mathbb{R}^{2n}$.

Exercise. Show that a vector field ∂ on a *complete* Riemannian manifold X is integrable provided it grows at most linearly. Moreover, the growth $\|\partial_x\| \lesssim |x| \log |x|$ is also sufficient for the integrability, where $|x| \underset{\mathrm{def}}{=} \mathrm{dist}_X(x, x_0)$. Then generalize (C) to submanifolds $L \subset \mathbb{R}^{2n}$ with logarithmic distortion.

(D) *Let $U \subset \mathbb{R}^{2n}$ be a relatively compact open subset with smooth boundary and f be a function on U which equals near the boundary* $(\mathrm{dist}(u, \partial U))^\alpha$ *for some $\alpha < 0$. Then the Lagrangian graph $L_f \subset \mathbb{R}^{2n}$ is exactly focal.*

This can be proven in two ways. Firstly one may observe that L_f has bounded distortion. Secondly, one may notice that L_f is asymptotic to $L_{\partial U}$ and there is a symplectomorphism of $\mathbb{R}^{2n}$ moving the infinity of L_f to $L_{\partial U}$. We leave the details to the reader who is also invited to look at non-compact U as well as U's in general manifolds V.

(D$'$) *If f is a very nice function on U* (see Section 0.3.3) *then L_f is focal.*

This is also left to the pleasure of the reader.

It is possible *all* properly embedded subgraphical Lagrangian submanifolds in $T^*(V)$ are focal. We cannot prove this but we shall show in the next section that such manifolds satisfy the monotonicity property for contact isotopies anyway.

2.5 Isotopy lifting property and monotonicity of intersections numbers for symplectic reductions

2.5.1 Isotopy lifting property for symplectic reductions

A *symplectic reduction* from a symplectic manifold $(\widetilde{M}, \widetilde{\omega})$ to (M, ω) is a particular kind of a *Lagrangian correspondence*, i.e. a Lagrangian submanifold $\widetilde{Z} \subset (\widetilde{M} \times M, \widetilde{\omega} \oplus -\omega)$ such that the projection of $\widetilde{Z}$ to $\widetilde{M}$ is a proper embedding and the projection to M is a smooth fibration. We write it as $\widetilde{M} \supset \widetilde{Z} \to M$, where clearly, $\widetilde{Z}$ is coisotropic in $\widetilde{M}$ and the fibration $\widetilde{\Pi} : \widetilde{Z} \to M$ has $\widetilde{\omega}$-isotropic fibers. Take a subset $\widetilde{L} \subset \widetilde{M}$ and let $L = \widetilde{\Pi}(\widetilde{L} \cap \widetilde{Z}) \subset M$. We apply a compact

Hamiltonian isotopy $I(t)$, $t \in [0,1]$, to L and we want to *lift* it to a compact Hamiltonian isotopy $\widetilde{I}(t)$ in $\widetilde{M}$ where "lift" means that $\widetilde{\Pi}(\widetilde{I}(t)(\widetilde{L}) \cap \widetilde{Z}) = I(L)$ for all $t \in [0,1]$. In fact, we want slightly more of our lift. For example, if $\widetilde{L}$ was a smooth submanifold in $\widetilde{M}$ *transversal* to $\widetilde{Z}$ we wish to keep the intersection transversal for all t. Also, we may deal with $\widetilde{L}$ non-injectively mapped to $\widetilde{M}$ and the notion of the lift should be adjusted in the obvious way.

Example: Split reduction. Let $\widetilde{M} = M \times \overline{M}$ for a symplectic manifold $\overline{M}$ and $\widetilde{Z} = M \times \overline{L}$ for a Lagrangian submanifold $\overline{L} \subset \overline{M}$. Then every Hamiltonian H on M lifts to $\widetilde{M}$ via the projection $\widetilde{M} = M \times \overline{M} \to M$ and thus every Hamiltonian isotopy $I(t)$ lifts to $\widetilde{I}(t) = I(t) \times \mathrm{id}$ on $\widetilde{M}$. It has all the desired property except for the compactness (of its support) which is taken care of with the cut-off $\widetilde{H} \rightsquigarrow \underline{\widetilde{H}} = \varphi \widetilde{H}$ where φ is a smooth function on $\widetilde{M}$ which equals one in a small neighbourhood $\widetilde{U} \subset \widetilde{M}$ of $\widetilde{Z}$ and vanishes outside a slightly greater neighbourhood of $\widetilde{Z}$. The cut-off isotopy $\underline{\widetilde{I}}$ obtained by such procedure is as good as $\widetilde{I} = I \times \mathrm{id}$ near $\widetilde{Z}$. It does not have the compact support yet (as it may infinitely spread along $(m \times \overline{L})$'s in $\widetilde{Z}$) but this will be remedied presently.

Now we return to the general reduction $\widetilde{M} \supset \widetilde{Z} \to M$ and prove the following

Lemma 2.5.1 *If the subset $\widetilde{L} \subset \widetilde{M}$ is closed and the projection of $\widetilde{L} \cap \widetilde{Z}$ to M is proper, then every compact Hamiltonian isotopy $I(t)$ in M lifts to a compact Hamiltonian isotopy $\widetilde{I}(t)$ in $\widetilde{M}$ which maps some neighbourhood $\widetilde{Z}_0 \subset \widetilde{Z}$ of $\widetilde{L} \cap \widetilde{Z} \subset \widetilde{Z}$ into $\widetilde{Z}$ for all $t \in [0,1]$.*

The last property ensures the preservation of the transversality of the intersection $\widetilde{L} \cap \widetilde{Z}$ with a good margin.

Proof. Lift the Hamiltonian $H(t)$ defining $I(t)$ to $H(t) \circ \widetilde{\Pi}$ on $\widetilde{Z}$ and extend it to some $H'(t)$ on $\widetilde{M} \supset \widetilde{Z}$. Notice that the $\widetilde{\omega}$-gradient $\partial'(t)$ of $H'(t)$ is tangent to $\widetilde{Z}$ and thus the corresponding isotopy $I'(t) \mid \widetilde{Z}$ covers $I(t)$. This property obviously holds for every extension $H'(t)$ of $H(t) \circ \widetilde{\Pi}$ whenever $\widetilde{\Pi}$ is a submersion. On the other hand the field $\partial'(t)$ on $\widetilde{Z}$ is globally integrable which follows from the fact that $\widetilde{\Pi}$ is a fibration. Now we want to correct the extension H' using the following lemma.

Lemma 2.5.2 ***(Complete connection.)*** *If $\widetilde{\Pi} : \widetilde{Z} \to M$ is a smooth fibration, then there exists a subbundle (connection) $\nabla \in T(\widetilde{Z})$ complementary to $\ker d\widetilde{\Pi} \subset T(\widetilde{Z})$, such the ∇-horizontal lift of each integrable field in M is integrable in $\widetilde{Z}$.*

Proof. Since $\widetilde{\Pi}$ is a locally trivial fibration, such ∇ obviously exists over each small open subset in M and then such local ∇'s are patched together over some locally finite covering of M. (This is standard and the details are left to the reader.)

Next observe that for every ∇ complementary to $\ker d\widetilde{\Pi}$ there exists a smooth retraction of some neighbourhood of $\widetilde{Z}$ to Z, say $R : \widetilde{U} \to \widetilde{Z}$, such that the $\overline{\omega}$-gradient ∂' of $H' = H \circ \Pi \circ R$ equals the ∇-horizontal lift of $\partial = \mathrm{grad}_\omega H$ for all $H : M \to \mathbb{R}$. Take this H' and cut it off to $\underline{H}'(t)$ outside some (smaller) neighborhood by $\widetilde{U}$. The support of the resulting Hamiltonian $\underline{H}'(t)$ may be not compact, but its restriction to the intersection of the support with $\widetilde{L}$ is compact, and thus it can be further cut off to $\underline{\widetilde{H}}(t)$ without affecting the action of the corresponding isotopy $\widetilde{I}(t)$ on $\widetilde{L}$. QED

Lift of families. The above lift construction is rather canonical and obviously applies to compact families of Hamiltonian isotopies. This can be regarded as a kind of *Serre fibration property* for the map $\widetilde{L} \mapsto L$.

2.5.2 Monotonicity of the intersection numbers under symplectic reduction

Let $\widetilde{\widetilde{M}} \supset \widetilde{Z} \to M$ be a symplectic reduction. Take $\widetilde{L} \subset \widetilde{\widetilde{M}}$ and $L_0 \subset M$ as in 2.5.1.

Proposition 2.5.3 *The (stable) intersection numbers between $\widetilde{L}$ and $\widetilde{L}_0 = \Pi^{-1}(L_0)$ for the projection $\Pi : \widetilde{Z} \to M$ are (non-strictly) smaller than these for $L = \Pi(\widetilde{L})$ and L_0, i.e.*

$$\cap_{\widetilde{L}_0}(\widetilde{L})_{\text{comp}} \leq \cap_{L_0}(L)_{\text{comp}} \ , \ \pitchfork_{\widetilde{L}_0}(\widetilde{L})_{\text{comp}} \leq \pitchfork_{L_0}(L)_{\text{comp}}$$

and similarly for the stable $\cap_{\widetilde{L}_0}$ and $\pitchfork_{\widetilde{L}_0}$ with our usual conventions concerning counting multiple intersections (where $L = \widetilde{L} \cap \widetilde{Z} \to M$ may be non-embedded even for embedded $\widetilde{L}$) *and the transversality implied by the symbol $\pitchfork$.*

Proof. The projection $\widetilde{\Pi}$ bijectively sends $\widetilde{L} \cap \widetilde{L}_0$ to $L \cap L_0$ (by the definition of $L \cap L_0$ understood as $L \otimes L_0$) and then the proof trivially follows from the Lemma 2.5.1.

Remark. Since

$$\cap_{L_0}(L)_{\text{comp}} = \cap_L(L_0)_{\text{comp}}$$

one gets

$$\cap_{\widetilde{L}}(\widetilde{L}_0)_{\text{comp}} \leq \cap_L(L_0)_{\text{comp}},$$

etc. This can also be proven directly with a suitable notion of divergence between $\widetilde{L}$ and $\widetilde{L}_0$ (compare Section2.2). In fact one can show the monotonicity of the (stable) intersection numbers under a suitable class of (H-divergent but not necessarily subsplit) bipolar extensions.

2.6 Lifting isotopies and monotonicity of intersection numbers for contact and Liouville reductions

2.6.1 Contact reductions

A (contact) reduction of a contact manifold $(\widetilde{N}, \widetilde{\eta})$ to (N, η) is given, by definition, by a submanifold $\widetilde{\mathcal{Z}} \subset \widetilde{N}$ and a fibration $\widetilde{\Pi} : \widetilde{\mathcal{Z}} \to N$ such that the pull-back of the subbundle $\eta \subset T(N)$ under the differential of $\widetilde{\Pi}$ equals $\widetilde{\eta} \cap T(\widetilde{\mathcal{Z}})$. Then $\mathcal{L} \subset N$ is called a *reduction* of $\widetilde{\mathcal{L}} \subset \widetilde{N}$ via $\widetilde{\mathcal{Z}}$ if $\mathcal{L}$ equals the projection of the intersection $\widetilde{\mathcal{L}} \cap \widetilde{\mathcal{Z}}$ to N under the projection $\widetilde{\Pi}$.

Lemma 2.6.1 *If $\widetilde{\mathcal{L}}$ is closed in $\widetilde{N}$ and the projection $\widetilde{\mathcal{L}} \cap \widetilde{N}$ to N is proper, then for every compact contact isotopy $\mathcal{I}(t)$, $t \in [0,1]$ in N there exists a compact contact isotopy $\widetilde{\mathcal{I}}(t)$ in $\widetilde{N}$ such that*

(a) $\widetilde{\Pi}(\widetilde{\mathcal{I}}(t)(\widetilde{\mathcal{L}}) \cap \widetilde{\mathcal{Z}}) = \mathcal{I}(t)$ for all $t \in [0,1]$,

(b) $\widetilde{\mathcal{I}}(t)$ maps some neighbourhood of $\widetilde{L} \cap \widetilde{\mathcal{Z}} \subset \widetilde{\mathcal{Z}}$ into $\widetilde{\mathcal{Z}}$ for all $t \in [0,1]$.

Proof. We may proceed exactly as in the symplectic case since contact vector fields display the same extension (and, hence cut-off) flexibility as Hamiltonian ones. Here, as earlier, we can construct such lifts for families of contact isotopies (and the details of all this we leave to the reader).

2.6.2 Liouville reductions

Let M and $\widetilde{M}$ be Liouville manifolds. Reduction in the Liouville category is given by $\widetilde{Z} \subset \widetilde{M}$ with a fibration $\Pi : \widetilde{Z} \to M$ such that the Liouville structures on M and $\widetilde{M}$ induce the same structure on $\widetilde{Z}$, where "structure" may mean an actual 1-form or a 1-form modulo d (functions). It is convenient here to stick to forms and write the above condition as $\Pi^*(\lambda) = \widetilde{\lambda} \mid \widetilde{Z}$. If this is the case, the Liouville reduction $\widetilde{Z}$ induces a contact reduction from $\widetilde{N} = \widetilde{M} \times \mathbb{R}$ with $\widetilde{\eta} = \ker(\widetilde{\lambda} + dt)$ to $N = N \times \mathbb{R}$ with $\eta = \ker(\lambda + dt)$, that is $\widetilde{\mathcal{Z}} = \widetilde{Z} \times \mathbb{R}_\Delta$ where $\mathbb{R}_\Delta$ denotes the diagonal $\mathbb{R}$ in $\mathbb{R} \times \mathbb{R}$.

2.6.2 ***Examples.*** (a) Let $\alpha : U \to V$ be a smooth fibration. Then $\widetilde{Z} \subset T^*(U)$, consisting of 1-forms vanishing on the fibers of α, is (obviously) a Liouville reduction from $\widetilde{M} = T^*(U)$ to $M = T^*(V)$.

(b) Let $\alpha : U \to V$ be an embedding. Then $\widetilde{Z} = T^*(V) \mid U \subset T^*(V)$ is a Liouville reduction from $\widetilde{M} = T^*(V)$ to $M = T^*(U)$.

(c) Let N be a cooriented contact manifold and N^+ the positive symplectization of N. Recall that N^+ comes along with an embedding $N^+ \subset T^*(N)$

and so we have a canonical Liouville *form* λ^+ on N^+ coming from that in $T^*(N)$. Then we take $\widetilde{N} = N^+ \times \mathbb{R}$ with $\widetilde{\lambda} = \lambda^+ + dt$ and observe that the zero level of the projection $\widetilde{N} \to \mathbb{R}$, say $\widetilde{\mathcal{Z}} \subset \widetilde{N}$, is an instance of a reduction of $(\widetilde{N}, \widetilde{\eta} = \ker \widetilde{\lambda})$ to N for the obvious projection $\widetilde{\mathcal{Z}} \to N$. An important special case is $N = \ddot{P}T^*(V)$, i.e. the bundle of the cooriented tangent hyperplanes in V, where $\widetilde{N} = \mathrm{Jet}^1(V) \setminus \mathbb{O}_V \times \mathbb{R}$. If we apply the reduction procedure to $\widetilde{\mathcal{L}} = \mathcal{L}_f \subset \widetilde{N} \subset \mathrm{Jet}^1(V)$, where the function f on V has no zero critical values, then the reduction of $\widetilde{\mathcal{L}}$ to N equals the (projective) Legendrian lift of the (cooriented!) zero level $W = f^{-1}(O)$, i.e. $\dot{\mathcal{L}}_W \subset \ddot{P}T^*(V)$. (This will be used in 4.2.1 for the construction of *generating hypersurfaces* for Legendrian subvarieties in $\ddot{P}T^*(V)$.)

2.6.3 Contact monotonicity for Liouville reductions

Let $\widetilde{M}$ and M be Liouville manifolds with the respective forms $\widetilde{\lambda}$ and λ and let $\widetilde{M} \supset \widetilde{Z} \xrightarrow{\Pi} M$ be a Liouville reduction. Then for every (say Lagrangian) subvariety $\widetilde{L} \subset \widetilde{M}$ we have its $\widetilde{Z}$-reduction, namely $L = \Pi(\widetilde{L} \cap \widetilde{Z})$. Furthermore, if $\widetilde{L}$ is an exact submanifold, it lifts to a Legendrian $\widetilde{\mathcal{L}} \subset \widetilde{N} = \widetilde{M} \times \mathbb{R}$ which reduces to $\mathcal{L}$ in $N = M \times \mathbb{R}$ via $\widetilde{\mathcal{Z}} = \widetilde{Z} \times \mathbb{R}_\Delta$, for $\mathbb{R}_\Delta = \Delta \subset \mathbb{R} \times \mathbb{R}$. Thus we may speak of a *contact isotopy* of $L = \Pi(\widetilde{L} \cap \widetilde{Z})$ in M as a contact isotopy of $\mathcal{L}$ in $N = M \times \mathbb{R}$ followed by the projection to M.

Proposition 2.6.3 *If $\widetilde{L}$ is focal and the projection Π is proper on $\widetilde{L} \cap \widetilde{Z}$, then the stable intersection numbers between L', obtained by a contact isotopy from L, and an arbitrary $L_0 \subset M$ satisfy*

$$\mathrm{stab} \cap_{L_0}(L')_{\mathrm{comp}} \geq \mathrm{stab} \cap_{\widetilde{L}_0}(\widetilde{L})_{\mathrm{comp}} \qquad (\widetilde{\cap})$$

and

$$\mathrm{stab} \pitchfork_{L_0}(L')_{\mathrm{comp}} \geq \mathrm{stab} \pitchfork_{\widetilde{L}_0}(\widetilde{L})_{\mathrm{comp}} \qquad (\widetilde{\pitchfork})$$

where $\widetilde{L}_0 = \Pi^{-1}(L_0) \subset M$.

Proof. Compact contact isotopies of $\mathcal{L}$ lift (see 2.6.1) to such isotopies of $\widetilde{\mathcal{L}}'$ where 2.4.2 applies. QED

2.7 Symplectic, contact and Liouville correspondences

Recall that a correspondence between manifolds M_1 and M_2 is a submanifold $L_{12} \subset M_1 \times M_2$ which can be viewed as (the graph of) a partially defined multivalued map from M_1 to M_2 or from M_2 to M_1. A correspondence $L_{12} \subset M_1 \subset M_2$ is called a *reduction* from M_1 to M_2 if the projection $L_{12} \to M_1$ is an embedding and M_2 is a fibration. (Here one also says that M_2 is a *subquotient* of M_1.)

Every correspondence L_{12} from M_1 to M_2 transforms subsets $L_1 \subset M_1$ to subsets $L_2 \subset M_2$ by the following rule. Take $L_1 \times L_{12} \subset M_1 \times M_1 \times M_2$, intersect it with $\Delta_1 \times M_2$ for the diagonal $\Delta_1 \subset M_1 \times M_1$ and project the intersection $(L_1 \times L_{12}) \cap (\Delta_1 \times M_1)$ to M_2. The resulting subset L_2 can be thought of as the image $L_{12}(L_1)$, also denoted $L \circ L_{12}$. There is another option, where we take not the image of the projection but the intersection itself together with the map to M_2. We still call it L_2 in M_2 and think of this as a *parametrized subset* in M_2. Actually L_{12} applies to parametrized subsets $L_1 \to M_1$ and defines parametrized subsets $L_2 \to M_2$ by the same rule: take L_2 consisting of the triples $(\ell_1, \ell_{\bar{1}}, \ell_2)$ with $\ell_1 \underset{M_1}{=} \ell_{\bar{1}}$ and $(\ell_{\bar{1}}, \ell_2) \in L_{12}$ and map it to L_2 via ℓ_2. Here $\ell_1 \underset{M_1}{=} \ell_{\bar{1}}$ signifies the equality of the (images) of the corresponding points in M_1. (To be truly consistent we should use *parametrized* correspondences $L_{12} \to M_1 \times M_2$ which compose by the above rule to parametrized correspondences again

$$L_{12} \circ L_{23} = \{(\ell_1, \ell_2), (\ell_{\bar{2}} \ell_3) \mid \ell_2 = \ell_{\bar{2}}\}.$$

If the manifolds in question come along with contravariant structures, such as exterior forms or subbundles in the tangent bundles, one may speak of correspondences in the structured category by requiring the induced structures on $L_{12} \subset M_1 \times M_2$ coming from the projections to M_1 and M_2 to be equal. In our cases, where the structure is symplectic, Liouville or contact, one usually requires such L_{12} to be *maximal*, i.e. having $\dim L_{12} = \frac{1}{2}(\dim M_1 + \dim M_2)$. Thus symplectic (Liouville) correspondences between M_1 and M_2 are just Lagrangian (respectively exact Lagrangian) submanifolds in $M_1 \times (-M_2)$ where "$-$" refers to the reversal of the sign of structure form. A pleasant feature of Lagrangian correspondences is the absence of singularities for composition in general position. Namely the composition of generic immersed Lagrangian $L_{12} \subset M_1 \times (-M_2)$ and $L_{23} \subset M_2 \times (-M_3)$ is an immersed Lagrangian submanifold in M_3. In particular $L_{12}(L)$ is immersed Lagrangian in M_2 for generic Lagrangian submanifolds $L \subset M_1$.

Every correspondence $L_{12} \subset M_1 \times M_2$ gives rise to a (split) reduction, namely from $\widetilde{M} = M_1 \times M_2 \times M_2$ to M_2 via $\widetilde{Z} = \Delta_1 \times M_1 \subset \widetilde{M}$ with the (obvious) projection $\Pi = \Pi_2 : \widetilde{Z} \to M_2$, such that

$$L_{12}(L) = \Pi(\widetilde{Z} \times (L \times L_{12}))$$

for all $L \subset M_1$. (As correspondence from $\widetilde{M}$ to M this $\widetilde{Z}$ sets in $\widetilde{M} \times M$ as the graph of Π over $\widetilde{Z} \subset \widetilde{M}$).

2.7.1 Contact monotonicity for Liouville correspondences

The above remark allows a generalization of 2.6.3 to arbitrary Liouville correspondences L_{12}. Namely, every such L_{12} from M_1 to M_2 induces the contact correspondence $\mathcal{L}_{12} = L_{12} \times \mathbb{R}_\Delta$ from $N_1 = M_1 \times \mathbb{R}$ to $N_2 = M_2 \times \mathbb{R}$, where $\mathbb{R}_\Delta$ denotes the diagonal in $\mathbb{R}\times\mathbb{R}$. Thus the image $L_{12}(L) \subset M_2$ lifts to $\widetilde{\mathcal{L}}_{12}(\mathcal{L}) \subset N_2$ for every exact $L \subset M_1$ liftable to a Legendrian $\mathcal{L} \subset N_1$. So we may speak of an L' being *contact isotopic to* L, and the following proposition holds:

Proposition 2.7.1 *If L is focal and the map from $L_{12}(L) = \widetilde{Z} \times (L \times L_{12})$ to M_2 is proper, then*

$$\operatorname{stab} \cap_{L_0}(L')_{\text{comp}} \geq \operatorname{stab} \cap_{\widehat{L}_0}(L \times L_{12}) \qquad (\cap_\Delta)$$

and

$$\operatorname{stab} \pitchfork_{L_0}(L')_{\text{comp}} \geq \operatorname{stab} \pitchfork_{\widehat{L}_0}(L \times L_{12}) \qquad (\pitchfork_\Delta)$$

for $\widehat{L}_0 = \Delta \times L_0$ and all $L_0 \subset M_2$.

2.7.2 $\overline{\times}$-Reduction and generating functions for subgraphical varieties

Let us modify the above construction for special correspondences between *cotangent bundles* in order to reduce the right hand sides of $(\cap_\Delta)$ and $(\pitchfork_\Delta)$ to the stable intersection numbers between L_0 and L themselves.

Observe that every smooth correspondence $\Gamma \subset U \times V$, e.g. the graph Γ_α of a smooth map $\alpha : U \to V$, gives rise to a Liouville correspondence between $T^*(U)$ and $T^*(V)$ denoted $T^\star(\Gamma)$ which is obtained from $L_\Gamma \subset T^*(U \times V) = T^*(U) \times T^*(V)$ (i.e. the annihilator of $T(\Gamma)$ in $T(U \times V)$) by the involution $(\tau_u, \tau_v) \mapsto (\tau_u, -\tau_v)$ (which is done to have it (exact) Lagrangian for $\omega_U + (-\omega_V)$ rather than for the symplectic form $\omega_U + \omega_V$ of $T^*(U \times V)$). Clearly $(V \leadsto T^*(V), \Gamma \leadsto T^\star(\Gamma))$ is a functor from smooth correspondences to Liouville correspondences and hence to symplectic and contact ones (where we write $T^\star$ instead of T^* not to confuse it with the cotangent bundle of Γ viewed as an abstract manifold rather than a correspondence).

Now we can give a better definition of $\underline{L} = \alpha_*(L) \subset T^*(V)$ (see Section 0.1.1) by observing that

$$\alpha^*(L) = T^\star(\Gamma_\alpha)(L)$$

for all $L \subset T^*(U)$. Eventually we shall be dealing with graphical $L = L_f$ and then we want to represent $\underline{L}_f$ by generating functions. We do this first for $V = \mathbb{R}^n$ with the following

Definition of $L^{\overline{\times}}$. This $L^{\overline{\times}} \subset T^*(U \times \mathbb{R}^n \times \mathbb{R}^n)$ for $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^n_y \times \mathbb{R}^n_z$ is given by

$$L^{\overline{\times}} = (L \times \mathbb{O}_{\mathbb{R}^n \times \mathbb{R}^n}) + d(\langle y - \alpha(u), z \rangle) \qquad (\overline{\times})$$

where $\alpha : U \to \mathbb{R}^n = \mathbb{R}^n_x$ and $\langle , \rangle$ is the Euclidean scalar product in $\mathbb{R}^n = \mathbb{R}^n_x = \mathbb{R}^n_y = \mathbb{R}^n_z$. We claim that

Lemma 2.7.2

$$\underline{L}^{\overline{\times}} \underset{\text{def}}{=} \alpha_*^{\overline{\times}}(L^{\overline{\times}}) \subset \mathbb{R}^n \quad \textit{equals} \quad \underline{L} = \alpha_*(L)\,. \qquad (\overline{\times}\overline{\times})\,,$$

where $\alpha^{\overline{\times}} : U \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is the* y*-projection.*

Proof. Recall that $\underline{L}$ consists of the pairs $\ell_u \in L \cap T_n^*(U)$, $\zeta_x \in T_x^*(\mathbb{R}^n)$, such that $\alpha(u) = x$ and $\zeta_x \circ D(\alpha) = \ell_u$ where ζ_x is a linear function on $T_x(\mathbb{R}^n)$, and this $\underline{L}$ is mapped to $T^*(\mathbb{R}^n)$ by $(\ell_u, \zeta_x) \mapsto (x, \zeta_x)$. On the other hand, $L^{\overline{\times}}$ consists of the covectors with the following (u, y, z)-components,

$$\ell_u - z_y^{\perp} \circ D_u(\alpha)\ , \quad z_y^{\perp} \quad (y - \alpha(u))_z^{\perp}\ ,$$

where $\ell_u \in L \cap T_u^*(U)$ and where $z^{\perp}$ and $(y - \alpha(u))^{\perp}$ denote the covectors of the scalar multiplication with the corresponding vectors. Since the kernel of $D\alpha^{\overline{\times}}$ is given by vanishing of x and z components, the intersection $L^{\overline{\times}} \cap \ker D(\alpha^{\overline{\times}})$ equals

$$\{\ell_u - z_y^{\perp} \circ D_u(\alpha) = 0,\ z_y^{\perp},\ y - \alpha(u) = 0\}\,,$$

i.e. it consists of the pairs $\ell_u \subset L \cap T^*(U)$, $z_y^{\perp} \in T^*(\mathbb{R}_y^n)$, such that $y = \alpha(u)$ and $\ell_u = z_y^{\perp} \circ D_u(\alpha)$. QED

2.7.3 Generating functions for subgraphical varieties

If $\underline{L} \subset T^*(\mathbb{R}^n)$ is a subgraphical variety, $\underline{L} = \underline{L}_f$ for $f : U \to \mathbb{R}$ and $\alpha : U \to \mathbb{R}^n$, then one sees with 2.7.2 that $\underline{L}_f$ *is generated by the function*

$$f^{\overline{\times}} \underset{\text{def}}{=} f(u) + \langle y - \alpha(u), z\rangle$$

on $U \times \mathbb{R}_y^n \times \mathbb{R}_z$*, where the implied fibration is the* y*-projection* $U \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$*.*

Notice that $f^{\overline{\times}} = f + \langle y', z\rangle$ for $y' = y - \alpha(u)$ and it is our old friend, quadratic stabilization of.

If $V \neq \mathbb{R}^n$, we do not expect such generating function but we can embed $V \subset \mathbb{R}^m$, construct $f^{\tilde{\overline{\times}}}$ corresponding to $L^{\tilde{\overline{\times}}} \subset U^{\sim} \times \mathbb{R}^m \times \mathbb{R}^m$ and then restrict it to $f^{\overline{\times}}$ on $U^{\sim} \times V \times \mathbb{R}^m$. This $f^{\overline{\times}}$ does generate $\underline{L}_f$ (for the projection $U^{\sim} \times V \times \mathbb{R}^m \to V$) but it is by no means a quadratic stabilization of f, and we did not check whether the stable Morse theory of $f^{\overline{\times}}$ dominates that of f (and/or of $f^{\tilde{\overline{\times}}}$).

2.7.4 Contact monotonicity for subfocal varieties

Subfocal varieties in $T^*(V)$**.** Call $\underline{L} \subset T^*(V)$ *subfocal* if it equals $\alpha_*(L)$ for a focal Lagrangian submanifold $L \subset T^*(U)$ and some smooth $\alpha : U \to V$. As L lifts to a Legendrian $\mathcal{L} \subset \mathrm{Jet}^1(U)$, our $\underline{L}$ also lifts to $\underline{\mathcal{L}} \subset \mathrm{Jet}^1(V)$, namely to $\underline{\mathcal{L}} = \alpha_*(\mathcal{L})$. Now we may speak of Legendrian isotopies of L (i.e. those of $\underline{\mathcal{L}}$ followed by the projection from $\mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$ to $T^*(V)$ and we claim the following

Proposition 2.7.3 ***(Subfocal monotonicity.)*** *If $\underline{L}'$ is obtained by compact contact isotopy of $\underline{L}$, then*

$$\mathrm{stab} \cap_{\mathbb{O}}(\underline{L}')_{\mathrm{comp}} \geq \mathrm{stab} \cap_{\mathbb{O}}(L)_{\mathrm{comp}} \tag{$\cap_{\mathbb{O}}$}$$

and

$$\mathrm{stab} \pitchfork_{\mathbb{O}}(\underline{L}')_{\mathrm{comp}} \geq \mathrm{stab} \pitchfork_{\mathbb{O}}(L)_{\mathrm{comp}} \tag{$\pitchfork_{\mathbb{O}}$}$$

where $\mathbb{O} = \mathbb{O}_V \subset T^*(V)$, provided $\underline{L}$ is proper, i.e. the implied map $\underline{L} \to T^*(V)$ is proper.

Proof. We start with the case $V = \mathbb{R}^n$ where according to $(\overline{\times}\overline{\times})$ and $(\widetilde{\cap})$ and $(\widetilde{\pitchfork})$ in we get such inequalities with $L^{\overline{\times}}$ on the right hand side instead of L. But the operation $L \rightsquigarrow L^{\overline{\times}}$ in this case amounts to multiplying L by the Lagrangian graph L_Q of the quadratic form $Q = \langle y', z \rangle$, where we change the variable y to $y' = y - \alpha(u)$. Therefore the stable intersection numbers of $L^{\overline{\times}}$ with $\mathbb{O}$ equal those for L. Thus the case $V = \mathbb{R}^n$ is concluded.

Now, for general V, we use, as earlier, an embedding $V \subset \mathbb{R}^m$ and procede as follows. First, we extend $\underline{L}$ to $T^*(\mathbb{R}^m)$ using a quadratic form (norm) Q in some normal neighbourhood $Z \subset \mathbb{R}^m$ of V (see 2.3.7). We call this extension $\underline{L}^{\sim}(= \underline{L} + dQ) \subset T^*(Z) \subset T^*(\mathbb{R}^m)$. The role of the manifold U is taken here by $U^{\sim} = U \widetilde{\times} Z$ where Z is mapped to V via the normal projection, (so that $U \widetilde{\times} Z$ equals the total space of the induced bundle $\alpha^*(Z)$ over U), and U embeds into $U^{\sim} = U \widetilde{\times} Z$ as the zero section. The form Q lifts to $Q^{\sim}$ on $U \widetilde{\times} Z$ via the tautological map $U \widetilde{\times} Z \to Z$ and $L \subset T^*(U)$ extends to $L^{\sim} = L + dQ^{\sim} \subset T^*(U \overline{\times} Z)$. Clearly, $\underline{L}^{\sim} = \alpha^{\sim}_*(L^{\sim})$ where $\alpha^{\sim} : U^{\sim} \to \mathbb{R}^m$ is the composition of the projection $U^{\sim} = U \widetilde{\times} Z \to Z$ and the embedding $Z \subset \mathbb{R}^m$. Then we apply the $\overline{\times}$-construction to $L^{\sim}$ and $\underline{L}^{\sim}$ and obtain, what we denote $L^{\tilde{\overline{\times}}} \subset T^*(U^{\sim} \times \mathbb{R}^m_y \times \mathbb{R}^m_z)$ which goes (i.e. reduces) to $L^{\sim}$ under the y-projection $U^{\sim} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, called here $\alpha^{\tilde{\overline{\times}}}$,

$$\alpha^{\tilde{\overline{\times}}}_*(L^{\tilde{\overline{\times}}}) = \alpha^{\sim}_*(L^{\sim})\,. \tag{$\tilde{\overline{\times}}$}$$

Now, in $\mathbb{R}^m$, we do have our inequalities for $(\underline{L}^{\sim})'$ obtained by a contact isotopy of $\underline{L}^{\sim}$, namely,

$$\mathrm{stab} \cap_{\mathbb{O}}(\underline{L}^{\sim})'_{\mathrm{comp}} \geq \mathrm{stab} \cap_{\mathbb{O}}(L^{\tilde{\overline{\times}}})_{\mathrm{comp}} \tag{$\cap_{\widetilde{\mathbb{O}}}$}$$

and

$$\mathrm{stab} \pitchfork_{\mathbb{O}}(\underline{L}^{\sim})'_{\mathrm{comp}} \geq \mathrm{stab} \pitchfork_{\mathbb{O}}(L^{\tilde{\overline{\times}}})_{\mathrm{comp}}\,. \tag{$\pitchfork_{\widetilde{\mathbb{O}}}$}$$

Then we observe that the right hand sides of the inequalities $(\cap_{\widetilde{\mathbb{O}}})$ and $(\pitchfork_{\widetilde{\mathbb{O}}})$ are, in fact, equal to those of $(\cap_{\mathbb{O}})$ and $(\pitchfork_{\mathbb{O}})$. Indeed, the pair $(L^{\tilde{\overline{\times}}}, \mathbb{O})$ in

$T^*(U^{\sim}) \times \mathbb{R}^m \times \mathbb{R}^m)$ clearly is a bipolar extension of $(L, \mathbb{O})$ associated to the Euclidean vector bundle $U^{\sim} \times \mathbb{R}^{2m}$ over U, where, recall, $U^{\sim} = U \boxtimes Z = \alpha^*(Z) \to U$ (compare the cotangent example in 2.3.4).

What remains to do is to bound the left hand sides of the inequalities $(\cap_{\widetilde{\mathbb{O}}})$ and $(\pitchfork_{\widetilde{\mathbb{O}}})$ by these in $(\cap_{\mathbb{O}})$ and $(\pitchfork_{\mathbb{O}})$. If $(\underline{L}^{\sim})'$ were obtained from $\underline{L}^{\sim}$ by a *Hamiltonian* (rather than contact) isotopy, then this would follow from the monotonicity of the stable intersection numbers under Euclidean embeddings. The (obvious) extension of this to the contact isotopies is discussed in the next section (see Lemma 2.4.2).

Remark about $L^{\overline{\times}}$ ***for*** $V \neq \mathbb{R}^n$***.*** If we want to have $\underline{L}$ as a reduction, we may restrict $L^{\tilde{\overline{\times}}}$ to $U^{\sim} \times V \times \mathbb{R}^m$ in $U^{\sim} \times \mathbb{R}^m \times \mathbb{R}^m$, where "restriction" means "symplectic reduction" via $T(U^{\sim} \times \mathbb{R}^m \times \mathbb{R}^m) \mid U^{\sim} \times V \times \mathbb{R}^m$. This restricted $L^{\tilde{\overline{\times}}}$, call it $L^{\overline{\times}} \subset T^*(U^{\sim} \times V \times \mathbb{R}^m)$, reduces to our original $\underline{L} \subset T^*(V)$ via the projection $U^{\sim} \times V \times \mathbb{R}^m \to V$. Unfortunately this $L^{\overline{\times}}$ is not the standard extension of $L \subset T^*(U)$ (and we certainly do not expect any *vector* bundle over U to be a fibration over V) and it is unclear whether the stable intersection numbers of $L^{\overline{\times}}$ with $\mathbb{O}$ bound these numbers for L. (But one may obviously bound $\mathrm{stab} \cap_{\mathbb{O}} L^{\overline{\times}}$ from below by the stable intersection number of $L^{\tilde{\overline{\times}}}$ with $L_{U^{\sim} \times V \times \mathbb{R}^m}$ rather than with $\mathbb{O} = \mathbb{O}_{U^{\sim} \times \mathbb{R}^m \times \mathbb{R}^m}$.)

2.8 Extending, lifting and cutting-off contact isotopies

Everything we have done to Hamiltonian isotopies in the symplectic case can be performed with an equal ease in the contact category.

2.8.1 Contact monotonicity for Euclidean embeddings

Let $V \subset \mathbb{R}^m$ be a smooth submanifold with a normal neighbourhood $Z \subset \mathbb{R}^m$ of V and normal projection $\Pi : Z \to V$. We give a Euclidean vector bundle structure to Z as in 2.3.7 and denote the Euclidean norm in Z by $Q : Z \to \mathbb{R}$. Then every $\mathcal{L} \subset \mathrm{Jet}^1(V)$ extends to $\mathcal{L}^{\sim} \subset \mathrm{Jet}^1(\mathbb{R}^m)$ as follows. First, denote by $\widetilde{\mathcal{Z}}_0 \subset \mathrm{Jet}^1(Z) \subset \mathrm{Jet}^1(\mathbb{R}^m)$ the space of the 1-jets of functions which are constant on the fibers of the projection $\Pi : Z \to V$. Thus $\widetilde{\mathcal{Z}}_0 = \widetilde{Z}_0 \times \mathbb{R}$, where $\widetilde{Z}_0 \subset T^*(Z)$ is the coisotropic submanifold responsible for the symplectic reduction associated to the fibration $Z \to V$, and this $\widetilde{\mathcal{Z}}_0$ naturally projects to V, say by $\widetilde{\Pi}_0 : \widetilde{\mathcal{Z}}_0 \to V$. The 1-jet of Q naturally (and contactly) acts in $\mathrm{Jet}^1(Z)$ by $j \mapsto j + J^1_Q$. We set

$$\widetilde{\mathcal{Z}}_Q = \widetilde{\mathcal{Z}}_0 + J^1_Q$$

and we denote by $\widetilde{\Pi}_Q$ the projection of $\widetilde{\mathcal{Z}}_Q$ to V. Now we define *the Q-extension of $\mathcal{L} \subset \mathrm{Jet}^1(V)$* as

$$\mathcal{L}^{\sim} = \widetilde{\mathcal{L}}_Q = \widetilde{\Pi}_Q^{-1}(\mathcal{L}) \subset \mathrm{Jet}^1(Z) \subset \mathrm{Jet}^1(\mathbb{R}^m)\,.$$

Next, given $\mathcal{L} \subset \mathrm{Jet}^1(V)$, we apply all compact contact isotopies to $\mathcal{L}$, project the resulting $\mathcal{L}'$ to $L' \subset T^*(V)$ and take

$$\mathrm{stab} \cap_{\mathbb{R}\mathbb{O}}(\mathcal{L})_{\mathrm{comp}} \overset{\mathrm{def}}{=} \inf \mathrm{stab} \cap_{\mathbb{O}}(L')_{\mathrm{comp}}$$

where the infimum is taken over all ($\mathcal{L}'$ obtained from $\mathcal{L}$ by) compact contact isotopies in $\mathrm{Jet}^1(V)$ and where $\mathbb{O} = \mathbb{O}_V \subset T^*(V)$ (while $\mathbb{R}\mathbb{O} = \mathbb{O}_V \times \mathbb{R} \subset \mathrm{Jet}^1(V) = T^*(V) \times \mathbb{R}$). Similarly we define $\mathrm{stab} \pitchfork_{\mathbb{R}\mathbb{O}}(\mathcal{L})$ and then claim the following inequalities (compare 2.3.9)

Lemma 2.8.1

$$\mathrm{stab} \cap_{\mathbb{R}\mathbb{O}}(\mathcal{L})_{\mathrm{comp}} \geq \mathrm{stab} \cap_{\mathbb{R}\mathbb{O}}(\mathcal{L}^{\sim})_{\mathrm{comp}} \tag{$\cap_{\mathbb{R}\mathbb{O}}$}$$

and

$$\mathrm{stab} \pitchfork_{\mathbb{R}\mathbb{O}}(\mathcal{L})_{\mathrm{comp}} \geq \mathrm{stab} \pitchfork_{\mathbb{R}\mathbb{O}}(\mathcal{L}^{\sim})_{\mathrm{comp}}\,. \tag{$\pitchfork_{\mathbb{R}\mathbb{O}}$}$$

Clearly, these inequalities are exactly what is needed to complete the proof of the subfocal monotonicity in the previous section.

2.8.2 The proof of Lemma 2.8.1

We proceed following the symplectic case step by step.

(1) ***Cut-off for split extensions.*** Let N be a (cooriented) contact manifold with the contact structure $\eta = \ker \lambda$, take $\widetilde{\lambda} = \lambda + pdq$ on $\widetilde{N} = N \times \mathbb{R}^{2\ell}$ for the form $pdq = \sum\limits_{i=1}^{\ell} p_i dq_i$ on $\mathbb{R}^{2\ell} = \mathbb{R}^\ell_p \times \mathbb{R}^\ell_q$ and let $\widetilde{\mathcal{Z}}_q = N \times \mathbb{R}^\ell_q$ for $\mathbb{R}^\ell_q = O \times \mathbb{R}^\ell_q$. Observe that $\widetilde{\eta} = \ker \widetilde{\lambda}$ is a contact structure on $\widetilde{N}$, such that the restriction of $\widetilde{\eta}$ to $\widetilde{\mathcal{Z}}_q$, i.e. the intersection $\widetilde{\eta} \cap T(\mathcal{Z}_q)$ equals the pull-back of η under the differential of the projection $\widetilde{\Pi}_q : \widetilde{\mathcal{Z}}_q \to N$. Take a contact vector field ξ on N and let $\bar{\xi}$ be the horizontal lift of ξ to $\widetilde{\mathcal{Z}}_q$, i.e. with the components (ξ, O) in $T(\widetilde{\mathcal{Z}}_q) = T(N) \times T(\mathbb{R}^\ell_q)$.

Lemma 2.8.2 *If ξ has compact support in N then for every compact subset $\widetilde{K} \subset \widetilde{\mathcal{Z}}_q$ there exists a contact field $\widetilde{\xi}$ on $\widetilde{N}$ with the following two properties.*

(a) The restriction of $\widetilde{\xi}$ to $\widetilde{K}$, i.e. to $T(\widetilde{N}) \mid \widetilde{K}$ equals $\bar{\xi}$.

(b) The norm of $\widetilde{\xi}$ is bounded by a constant independent of $\widetilde{K}$, where the norm is taken for the product Riemannian metric $g \oplus$ Euclidean on $\widetilde{N} = N \times \mathbb{R}^{2\ell}$, where g is a given Riemannian metric on N.

Proof. The translations of $\mathbb{R}_q$ act on $\widetilde{N}$ by *contact* transformations and this action commutes with the field $\overline{\xi}$ on $\widetilde{\mathcal{Z}}_q$. Then one can easily extend $\overline{\xi}$ to a field $\widetilde{\xi}_0$ on $\widetilde{N}$ which also commutes with this action and has its support contained in an invariant neighbourhood having *compact* projection to $\mathbb{R}^\ell_p = \widetilde{N}/\text{action}$. So what remains is to cut off $\widetilde{\xi}'_0$ in the $\mathbb{R}^\ell_q$-direction where again the action comes along handily. Actually, it is the easiest to proceed by induction in ℓ which immediately reduces the problem to the case $\ell = 1$. The support of $\widetilde{\xi}_0$ in this case is a cylinder, say $S \times \mathbb{R}^1_q$, with a *compact* slice $S \subset N \times \mathbb{R}^1_p$ and $\widetilde{\xi}_0$ is $\mathbb{R}^1_q$-invariant in this cylinder. Then we cut off $\overline{\xi}$ "on the right" to make it zero for $q \geq 0$ and next move the cut as far to the right as we want to so that the resulting field, say $\overline{\xi}^+$, remains equal to $\widetilde{\xi}_0$ for $q \leq q_+$ for a given q_+ in $\mathbb{R}^1_q$ (comp. 2.2.3). Since the action of $\mathbb{R}^1_q$ is isometric for our $g\oplus$ Euclidean, the metric shape of the cut does not depend on q_+ and so the norm of $\widetilde{\xi}^+$ does not grow as q_+ grows in $\mathbb{R}^1_q (= \mathbb{R})$. The we similarly cut $\widetilde{\xi}_0$ on the left and the resulting $\widetilde{\xi} = \widetilde{\xi}^+_-$ has compact support and it agrees with $\widetilde{\xi}_0$ in $S \times [q_-, q_+]$ for given q_- and q_+. QED

Remarks.

(a) We used all along in the above argument the existence of *some* cut-off for contact fields. This is possible as the sheaf of contact fields is flexible.

(b) One could avoid mentioning any metric in the statement and the proof of the cut-off property by emphasizing the role of the $\mathbb{R}^\ell$-action.

(2) ***Dividing contact isotopies into local steps.*** One can replace an arbitrary compact contact isotopy $\mathcal{I}$ by $\mathcal{I}_\varepsilon$ decomposable as $\mathcal{I}_1 * \mathcal{I}_2 * \ldots * \mathcal{I}_k$ where all $\mathcal{I}_j$ have small supports (compare Lemma 2.2.6).

(3) ***Lifting of compact contact isotopies of*** $\mathcal{L} \subset \mathrm{Jet}^1(V)$ ***to such isotopies of*** $\widetilde{\mathcal{L}} \subset \mathrm{Jet}^1(Z) \subset \mathrm{Jet}^1(\mathbb{R}^m)$***.*** Before lifting an isotopy we divide it into local steps (compare 2.3.4) and thus reduce the problem to the case where $Z = V \times \mathbb{R}^\ell \to V$ for $\ell = m - \dim V$. Then, obviously, $\widetilde{\mathcal{Z}}_0 \subset \mathrm{Jet}^1(Z) = \mathrm{Jet}^1(V) \times \mathbb{R}^{2\ell}$ identifies with $\mathrm{Jet}^1(V) \times \mathbb{R}^\ell_q$. Furthermore, since $\widetilde{\mathcal{Z}}_Q$ is obtained from $\widetilde{\mathcal{Z}}_0$ by a contactomorphism, we may also treat $\widetilde{\mathcal{Z}}_Q$ as $\mathrm{Jet}^1(V) \times \mathbb{R}^\ell_q$. Since contact vector fields ξ (horizontally) lift to $\widetilde{\mathcal{Z}}_Q$, our isotopy $\mathcal{I}$ of $\mathcal{L}$ lifts to an isotopy $\overline{\mathcal{I}}$ of $\widetilde{\mathcal{L}} \subset \mathrm{Jet}^1(Z)$ which keeps $\widetilde{\mathcal{L}}$ within $\widetilde{\mathcal{Z}}_Q$. Then the cut-off of contact fields $\overline{\xi}$ in $\mathrm{Jet}^1(V) \times \mathbb{R}^{2\ell}$ allows a cut-off $\widetilde{\mathcal{I}}$ of $\overline{\mathcal{I}}$, such that $\widetilde{\mathcal{I}}$ agrees with $\overline{\mathcal{I}}$ on a large compact subset $\widetilde{K}$ in $\widetilde{\mathcal{Z}}_Q$ and such that $\widetilde{\mathcal{I}}$ remains bounded by a constant independent of $\widetilde{K}$ (where the implied metric is $g\oplus$ Euclidean moved by the contactomorphism $j \mapsto j + J^1_Q$). It follows that the isotoped $\widetilde{\mathcal{L}}$ does not develop undesirable intersections with $\widetilde{\mathcal{Z}}_0$ and hence with $\mathbb{O}_Z \times \mathbb{R}$ sitting in $\widetilde{\mathcal{Z}}_0$, since $\widetilde{\mathcal{Z}}_0$ and $\widetilde{\mathcal{Z}}_Q$ diverge for our metric. This implies the above inequalities $(\cap_{\mathbb{R}\mathbb{O}})$ and $(\pitchfork_{\mathbb{R}\mathbb{O}})$ by the same (trivial) logic we used in the Hamiltonian case. QED

Lament. What a pain to write down and to follow such argument void of any mathematical idea! The authors are as frustrated as the reader undoubtly is but, apparently, there is no short-cut across this morass of lifts, cut-offs and extensions permitting all of Chapter 2.

Chapter 3

Lower bounds for Lagrangian intersection by the Morse theory applied to generating functions

We shall interpret the ε-parallelization procedure

$$I(L) \cap \mathbb{O} \rightsquigarrow (L \times \Gamma_{\text{odd}}) \cap (\Delta_{\text{even}} \theta s \mathbb{O})$$

of 1.2 as a construction of the generating function $Q + \varepsilon$ for the Lagrangian submanifold $I^{-1}(\mathbb{O}) \subset \mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ and then we reduce the Lagrangian intersection problem $I(L) \cap L_0$ for subgraphical varieties in $T^*(V)$ to the Morse theory.

3.1 Geometry of Δ_{even}, Δ_{odd} and Γ_{odd} revisited

3.1.1 The ε-parallelization procedure as a construction of a generating function

We return to the symplectomorphisms I_i, $i = 1, \dots, k$, of $\mathbb{R}^{2n}$ which (usually) are of the form $I_i = I(t_i)I^{-1}(t_{i-1})$ for a Hamiltonian isotopy $I(t)$ in $\mathbb{R}^{2n}$. What is important here, is that $\|DI_i - \text{id}\| \leq \varepsilon_0$, $i = 1, \dots, k$ for some small fixed $\varepsilon_0 > 0$ (e.g. for $\varepsilon_0 = \sqrt{2}/3$). The discussion in 1.2 and 1.4 associates to these I_i the following data.

(1) ***Non-singular quadratic form Q on $\mathbb{R}^N$ for $N = 2nk$.*** This form $Q = Q_N(z)$ is completely canonical, it does not depend on I_i. In fact $Q = Q(z)$

appears on $\mathbb{R}^n \times \mathbb{R}^N = \mathbb{R}^n_x \times \mathbb{R}^N_z$ as

$$Q = \int [\mathbb{O} \times \Delta_{\text{odd}} - \Delta_{\text{even}} \times \mathbb{O}]_{\mathcal{P}}$$

where $\mathcal{P} = \mathcal{P}_{\text{alt}}$ is the alternating polarization in

$$\mathbb{R}^{2n} \times \mathbb{R}^{2N} = \underbrace{\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n}}_{2k+1},$$

where $\mathbb{O}$ stands for $\mathbb{O}_{\mathbb{R}^n} \subset \mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ and $\mathbb{O} \times \Delta_{\text{odd}}$ and $\Delta_{\text{even}} \times \mathbb{O}$ are the linear Lagrangian subspaces in $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n}$ defined in 1.2. Here we identify $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$ with $T^*(\mathbb{R}^n_x \times \mathbb{R}^N_z)$ via the isomorphisms $(\mathbb{R}^{2n} \times \mathbb{R}^{2N})/\mathcal{P} \approx \mathbb{R}^n_x \times \mathbb{R}^N_y \approx \mathbb{R}^n_x \times \mathbb{R}^N_z$, where (x, y)-splitting of $\mathbb{R}^n \times \mathbb{R}^N$ corresponds to $\mathbb{R}^{2n} \times \mathbb{R}^{2N} = \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n}$ while the (x, z)-splitting is enforced by Q (see Sections 1.2 and 1.4). Here we stick to the (x, z)-splitting $\mathbb{R}^n \times \mathbb{R}^N = \mathbb{R}^n_x \times \mathbb{R}^N_z$ and think of y as an additional linear map, $y : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$, where one should keep in mind the following properties of this y which are immediate with the discussion in Sections 1.2 and 1.4.

The map y is linearly independent from the projections $x : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ and $z : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$. This means that $x^{-1}(0) = 0 \times \mathbb{R}^N$ is sent by y bijectively onto $\mathbb{R}^N$ and $z^{-1}(0) = \mathbb{R}^n \times 0$ is sent by y injectively into $\mathbb{R}^N$.

In fact it is worth remembering that $\mathbb{R}^N = \underbrace{\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n}}_{k}$ and so y splits into k linear maps $y_i : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^{2n}$ corresponding to the k projections $\mathbb{R}^N \to \mathbb{R}^{2n}$. It is clear (from the definition of Q, y and z, see Sections 1.2 and 1.4) that *each* y_i is independent from x, i.e. y_i is injective on $x^{-1}(0)$.

(2) ***Smooth function*** $\varepsilon = \varepsilon(x, z)$ ***on*** $\mathbb{R}^n \times \mathbb{R}^N$**.** This ε is actually of the form $\varepsilon = \varepsilon(y)$ for the above linear map $y : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$. In fact $\varepsilon(y) = \varepsilon_1(y_1) + \varepsilon_2(y_2) + \cdots + \varepsilon_k(y_k)$ for the splitting $\mathbb{R}^N = \underbrace{\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \ldots \times \mathbb{R}^{2n}}_{k}$. Every ε_i (and hence ε as well) have $\|d\varepsilon_i\| \le \varepsilon_0$ and if the isotopy $I(t)$ in question is compact then all ε_i are also compact (i.e. have compact supports).

The role of ε is the passage from Δ_{odd} to Γ_{odd}. Namely $\Gamma_{\text{odd}} = I_\varepsilon(\Delta_{\text{odd}})$ where I_ε is the symplectomorphism of $\mathbb{R}^{2N} = T^*(\mathbb{R}^N_y)$ given by $\tau \mapsto \tau + d\varepsilon$. We also write $\Gamma_{\text{odd}} = \Delta_{\text{odd}} + d\varepsilon$ and, with the same notations in $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$,

$$L \times \Gamma_{\text{odd}} = L \times \Gamma_{\text{odd}} + d\varepsilon \underset{\text{def}}{=} I_\varepsilon(L \times \Gamma_{\text{odd}}),$$

where one may think of $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$ as $T^*(\mathbb{R}^n \times \mathbb{R}^N = \mathbb{R}^n_x \times \mathbb{R}^N_z)$ and $\varepsilon = \varepsilon(x, z)$.

It is clear from the definition of Γ_{odd} and Δ_{even} (see 1.2) that $\mathbb{R}^{2n} \times \Gamma_{\text{odd}}$ is *transversal* to $\Delta_{\text{even}} \times \mathbb{O}$. In fact $\mathbb{R}^{2n} \times \Gamma_{\text{odd}}$ is transversal to $\Delta_{\text{even}} \times x$ for every $x \in \mathbb{R}^{2n}$. This is equivalent to the transversality of the Lagrangian graph $L_{-(Q+\varepsilon)}$ to $\mathbb{R}^{2n} \times \mathbb{O}_{\mathbb{R}^N}$ in $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$ as we shall see presently.

(3) ***Interpretation of the equality*** $I(L) \cap \mathbb{O} = (L \times \Gamma_{\text{odd}}) \cap (\Delta_{\text{even}} \times \mathbb{O})$ ***in terms of*** $Q + \varepsilon$**.** This equality (see 1.2) means, in effect, that *the projection*

of the intersection $(L \times \Gamma_{\text{odd}}) \cap (\Delta_{\text{even}} \times \mathbb{O})$ *in* $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$ *onto the first factor equals* $I(L) \cap \mathbb{O} \subset \mathbb{R}^{2n}$.

Now we think of $\mathbb{R}^{2n} \times \mathbb{R}^{2N}$ as the cotangent bundle of $\mathbb{R}^n_x \times \mathbb{R}^N_z$ where the cotangent structure comes from the alternating polarization and where we declare $\Delta_{\text{even}} \times \mathbb{O}$ the zero section. Then $L \times \Gamma_{\text{odd}}$ identifies with $I_\varepsilon(L \times L_Q) = I_{\varepsilon+Q}(L \times \mathbb{O}_{\mathbb{R}^N})$ and $(L \times L_{\text{odd}}) \cap (\Delta_{\text{even}} \times \mathbb{O})$ identifies with $I_{\varepsilon+Q}(L \times \mathbb{O}_{\mathbb{R}^N}) \cap \mathbb{O}_{\mathbb{R}^n \times \mathbb{R}^N}$. Finally, we notice that the application of the diffeomorphism $I^{-1}_{\varepsilon+Q}$ does not effect the projection to $\mathbb{R}^{2n}$ because $\varepsilon = \varepsilon(y)$ and $Q = Q(z)$ do not depend on x and, since $I^{-1}_{\varepsilon+Q}(\mathbb{O}_{\mathbb{R}^n \times \mathbb{R}^N}) = L_{-(Q+\varepsilon)}$, we conclude that *the projection of the intersection* $(L \times \mathbb{O}_{\mathbb{R}^N}) \cap L_{-(Q+\varepsilon)}$ *to* $\mathbb{R}^{2n}$ *equals* $I(L) \cap \mathbb{O}_{\mathbb{R}^n}$ *for all* $L \subset \mathbb{R}^{2n}$.

This applies, in particular, to each point $\ell \in \mathbb{R}^{2n}$ and implies the following

3.1.1 Reduction property. *The projection of the intersection*

$$(\mathbb{R}^{2n} \times \mathbb{O}_{\mathbb{R}^N}) \cap L_{-(Q+\varepsilon)}$$

to $\mathbb{R}^{2n}$ *equals* $I^{-1}(\mathbb{O}_{\mathbb{R}^n}) \subset \mathbb{R}^{2n} = T^*(\mathbb{R}^n)$ *where the intersection between* $\mathbb{R}^{2n} \times \mathbb{O}_{\mathbb{R}^N}$ *and* $L_{-(Q+\varepsilon)}$ *is transversal. In other words,* $I^{-1}(\mathbb{O}) \subset T^*(\mathbb{R}^n)$ *equals the symplectic reduction of the Lagrangian graph of* $-Q-\varepsilon$ *in* $T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^N)$ *via the coisotropic subspace* $T^*(\mathbb{R}^n) \times \mathbb{O}_{\mathbb{R}^N} \subset T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^N)$*. Or, equivalently,* $I^{-1}(\mathbb{O})$ *is generated by the function* $Q(z) + \varepsilon(x,z)$ *on* $\mathbb{R}^n \times \mathbb{R}^N$.

Cutting-off ε**.** The above function ε does not have compact support even if $I(t)$ does, but it decomposes into the sum

$$\varepsilon(x,z) = \varepsilon(y) = \varepsilon_1(y_1) + \varepsilon_2(y_2) + \cdots + \varepsilon_k(y_k)$$

where all ε_i are compact for compact $I(t)$. Since all projection $y_i : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^{2n}$ are linearly independent from z, the function $\varepsilon = \sum_{i=1}^{k} \varepsilon_i$ has compact support on $\mathbb{R}^n \times O$ in $\mathbb{R}^n \times \mathbb{R}^N$ as well as on $\mathbb{R}^n \times B \subset \mathbb{R}^n \times \mathbb{R}^N$ for every compact subset $B \subset \mathbb{R}^N$. Thus if we cut off ε to zero outside a sufficiently large B without enlarging $\|d\varepsilon\|$ (see 2.1.1), the Lagrangian graph of $-Q - \underline{\varepsilon}$ for the cut-off $\underline{\varepsilon}$ will have the same intersection with $\mathbb{R}^{2n} \times \mathbb{O}_{\mathbb{R}^N}$ as $L_{-Q-\varepsilon}$, i.e. $d_z(-Q-\underline{\varepsilon})$ will have the same zeros as $d_z(-Q-\varepsilon)$, since $\|d_z Q\|$ is larger than $\|d\varepsilon\|$ outside B.

3.1.2 Existence of generating functions

3.1.2 Generating conclusion in $T^*(\mathbb{R}^n)$. *Let* $I(t)$ *be a compact Hamiltonian isotopy in* $T^*(\mathbb{R}^n)$ *for* $t \in [0,1]$ *with* $I(t=0) = \text{id}$ *(as we always assume). Then the Lagrangian submanifold* $I(\mathbb{O}) \subset T^*(\mathbb{R}^n)$ *for* $I = I(t=1)$ *can be transversally* *generated by a smooth function* $\widetilde{f}$ *on* $\mathbb{R}^n \times \mathbb{R}^N$ *for some* $N = N(I(t))$ *where* $\widetilde{f} = Q(z) + \varepsilon(x, z)$ *for a non singular quadratic function* Q *on* $\mathbb{R}^N$ *and some smooth* ε *with compact support.*

Here, "transversally generated" means that the Lagrangian graph $L_{\widetilde{f}}$ is *transversal* to $\mathbb{R}^n \times \mathbb{O}_{\mathbb{R}^N}$ in $\mathbb{R}^{2n} \times \mathbb{R}^{2N} = T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^N)$ and the (subgraphical) image $\underline{L}_{\widetilde{f}} = x_*(L_{\widetilde{f}})$ for the projection $x : \mathbb{R}^n \times \mathbb{R}^{2N} \to \mathbb{R}^n$, equals $I(L)$.

This follows from the above where we simplified the notations by switching from I^{-1} to I and from $-Q - \varepsilon$ to $Q + \varepsilon$.

Generating functions for $I(\mathbb{O})$***,*** $I(L_f)$ ***and*** $I(\underline{L}_f)$ ***in*** $T^*(V)$***.*** Now we are ready to prove the full Generating Function Theorem due to Laudenbach and Sikorav (see [47]).

Theorem 3.1.3 *For every compact Hamiltonian isotopy* $I(t)$ *in* $T^*(V)$ *the image* $I(\mathbb{O}_V)$ *can be transversally generated by* $Q(z) + \varepsilon(x, z)$ *on* $V \times \mathbb{R}^N$ *where* Q *is a non-singular quadratic function on* $\mathbb{R}^N$ *and* ε *is smooth and compact.*

Proof. Properly embed V to $\mathbb{R}^m$ and take a compact Hamiltonian isotopy $\widetilde{I}(t)$ in $\mathbb{R}^{2m}$, such that $\widetilde{I}(t)(\mathbb{O}_{\mathbb{R}^m})$ remains transversal to the (coisotropic) submanifold $\widetilde{Z} = T^*(\mathbb{R}^m) \mid V \subset T^*(\mathbb{R}^m)$ and $I(t)(\mathbb{O}_{\mathbb{R}^m}) \cap \widetilde{Z}$ goes to $I(t)(\mathbb{O}_V)$ under the obvious projection $\widetilde{Z} \to V$ for all $t \in [0, 1]$ (the existence of such $\widetilde{I}$ follows from 2.5.1). Then the restriction of the generating function for $\widetilde{I}(\mathbb{O}_{\mathbb{R}^m})$ to $V \times \mathbb{R}^N \subset \mathbb{R}^m \times \mathbb{R}^N$ (obviously) serves as the generating function for $I(\mathbb{O}_V)$. QED.

Corollary 3.1.4 *If our* I *applies to some graphical submanifolds* $L_f \subset T^*(V)$ *for* $f \neq 0$*, then* $I(L_f)$ *can be transversally generated by a function on* $V \times \mathbb{R}^N$ *of the form* $f(v) + Q(z) + \varepsilon(v, z)$ *with* Q *and* ε *as earlier.*

Proof. Let $I^f(t) = I_f^{-1} \circ I(t) \circ I_f$ for $I_f : \tau \mapsto \tau + df$. Generate $I^f(\mathbb{O}_V)$ by $Q + \varepsilon$ and observe that $f + Q + \varepsilon$ generates $I(L_f)$ in this case. QED.

Theorem 3.1.3 can be further generalized for an isotopy of subgraphical varieties as follows (comp. [30] and [55]).

Proposition 3.1.5 *Let* $\alpha : U \to V$ *be a smooth fibration and* $\underline{L} \subset T^*(V)$ *be generated by a function* f *on* U*, i.e.* $\underline{L} = \underline{L}_f = \alpha_*(L_f)$*. If* $\underline{L}$ *is proper in* $T^*(V)$*, then every compact Hamiltonian isotopy* $I(t)(\underline{L})$ *lifts to such an isotopy of* L_f*, say* $\widetilde{I}(t)(L_f)$*, such that* $I(t)(\underline{L})$ *equals the reduction of* $\widetilde{I}(t)(L_f)$ *for all* t *(see 2.5.1).*

Proof. It follows from 3.1.3 that each $\widetilde{I}(t)(L_f)$ is generated by some function $\widetilde{f}_t$ on $U \times \mathbb{R}^{N(t)}$ of the form

$$\widetilde{f}_t = f(u) + Q(z) + \varepsilon_t(u, z)$$

where Q is a non-singular quadratic form on $\mathbb{R}^{N(t)}$ and ε is smooth compact. Then, *the same* $\widetilde{f}_t$ *generates* $I(t)(\underline{L})$ *for the obvious fibration* $U \times \mathbb{R}^{N(t)} \to V$. QED

The homotopy lifting property for generating functions. Notice that the construction of ε is rather canonical in terms of $I(t)$ and the only discontinuity in t comes from the dimension $N = N(t)$ (which is taken care of below). Also observe that the generation of $\widetilde{I}(t)(L_f)$ by $\widetilde{f}_t$ can be assumed *transversal*, i.e. with $L_{\widetilde{f}_t}$ transversal to $T^*(U) \times \mathbb{O}_{\mathbb{R}^{N(t)}} \subset T^*(U \times \mathbb{R}^{N(t)})$. Hence, if f was generating $\underline{L}$ *transversally*, then so does $\widetilde{f}_t$ for $I(t)(\underline{L})$.

The above applies, in particular, to $U = V \times \mathbb{R}^{N_0}$ and shows that *the class of properly immersed Lagrangian submanifolds* $\underline{L}$ *in* $T^*(V)$ *transversally generated by functions of the form* $Q(z) + \varepsilon(v, z)$ *for quadratic non-singular* Q *and compact* ε *is invariant under compact Hamiltonian isotopies of* $\underline{L}$. In fact, here we think of the generating function $\widetilde{f}_t$ for $I(t)(\underline{L})$ as continuous in t, where N also depends on t. Of course, the latter cannot be made continuously but this can be amended by stabilizing to $N = \infty$ and using Q and ε in infinitely many variables $x_1\, x_2, \ldots, x_N \ldots$, where Q is a standard non-singular form on $\mathbb{R}^\infty = \bigcup_{N=0}^{\infty} \mathbb{R}^N$, say $Q = x_1\, x_2 + x_3\, x_4 + \ldots$, and ε is of the form

$$\varepsilon = \varepsilon_1(v, x_1, \ldots, x_N)\, \varepsilon_0(\frac{x_{N+1}}{E_1})\, \varepsilon_0(\frac{x_{N+2}}{E_1}) \ldots,$$

where ε_1 is a compact function on $V \times \mathbb{R}^N$ as earlier, $E_1 = \mathrm{Max}|\varepsilon_1|$, and ε_0 is a standard cut-off function $\mathbb{R} \to [0, 1$ which equals 1 on $[-1, 1]$, vanishes at infinity and has small derivative (say $|\varepsilon_0'| \leq 1/2$.*

Generating function for families $L_b \subset T^*(V)$***.*** Suppose we have a family of (say, immersed) Lagrangian submanifolds L_b generated by a family $\widetilde{f}_b$ of functions on $V \times \mathbb{R}^N$ for b running over a compact space B. Then if we deform L_b by a family $I_b(t)$ of Hamiltonian isotopies, we may also modify $\widetilde{f}_b$ (enlarging N if we use $N < \infty$) so that the modified functions $\widetilde{f}_b^1$ generate the family $I_b(1)(L_b)$. This can be done by the above construction as it is continuous in L and $I(t)$. Alternatively, if the families in question are smooth in b for a smooth manifold B, we notice that the family $\widetilde{f}_b$ can be viewed as a function say $\widehat{f}$ on $V \times B \times \mathbb{R}^N$ generating a subvariety in $\widehat{L} \subset T^*(V \times B)$. This $\widehat{L}$ reduces to L_b on each fiber $V = V \times b$ (or rather in $T^*(V) \times b$). Conversely, every smooth family $L_b \subset T^*(V = V \times b)$ defines such an $\widehat{L}$, which is unique up to adding a closed 1-form in the b-variable. Thus the above homotopy lifting property for an isotopy of an individual submanifold, namely of $\widehat{L}$, yields such a result for families. See also Section 5.1.3 below.

*Here the cut-off function ε_0 is chosen in such a way that the critical points of the function $\varepsilon + Q$ contain in $V \times \mathbb{R}^N \subset V \times \mathbb{R}^\infty$.

3.2 Bounds for Lagrangian intersections

3.2.1 Lower bound for intersections of deformed subgraphical Lagrangian varieties with the zero-section

Let $\alpha : U \to V$ be a smooth map and $\underline{L}_f = \alpha_*(L_f)$ be the subgraphical variety associated to a smooth function f on U. The intersection $I(\underline{L}_f) \cap \mathbb{O}$ for $\mathbb{O} = \mathbb{O}_V$ in $T^*(V)$ is the same thing as $\underline{L}_f \cap I^{-1}(\mathbb{O})$ and if $I^{-1}(\mathbb{O})$ is generated by $\widetilde{f}(v,z) = Q(z) + \varepsilon(v,z)$ on $V \times \mathbb{R}^N$ then this intersection corresponds (with our usual multiplicity convention for $\underline{L}_f$ *non-injectively* mapped to $T^*(V)$) to the critical set of the function $\overline{f} = \overline{f}(u,z)$ on $U \times \mathbb{R}^N$ which is $\overline{f}(u,z) = f(u) - Q(z) - \widetilde{\alpha} \circ \varepsilon$ for $\widetilde{\alpha} : (u,z) \mapsto (\alpha(u), z)$. This correspondence matches transversal intersection points with non-degenerate critical points and thus provides lower bounds on the cardinalities $\#(I(\underline{L}_f) \cap \mathbb{O})$ and $\#(I(\underline{L}_f \pitchfork \mathbb{O}))$ in terms of suitable stable Lusternik–Schnirelman and Morse invariants of α and f. Here one should exercise some caution for non-proper maps $U \to V$ (e.g. equidimensional embeddings) as the function $\widetilde{\alpha} \circ \varepsilon$ becomes non-compact on $U \times \mathbb{R}^N$ for compact ε on $V \times \mathbb{R}^N$.

However, *if the function f on U is expanding at the boundary of U (mapped) in V* (see Section 2.1.2) *then one can cut off* $\widetilde{\alpha} \circ \varepsilon$ and arrive at friendlier Morse theoretic inequalities. Thus one immediately obtains the intersection inequalities in 0.3.1, 0.3.2, 0.3.6, 0.3.7, where the contact case follows by 2.4.2. Moreover such inequalities apply not only to the original Lagrangian pairs $(\underline{L}_f, \mathbb{O} = \mathbb{O}_V)$ but also to their Euclidean stabilization which yields lower bounds on $\mathrm{stab} \cap_{\mathbb{O}} (\underline{L}_f)_{\mathrm{comp}}$ and $\mathrm{stab} \pitchfork_{\mathbb{O}} (\underline{L}_f)_{\mathrm{comp}}$ by the stable Lusternik-Schnirelman and Morse numbers. In fact, the opposite inequalities are also valid (and obvious) in the stable framework and so we obtain *equalities* between the stable intersection numbers and stable MLS numbers. For example, we have

Proposition 3.2.1 *Let $\alpha : U \to V$ be a (possibly non-proper) smooth embedding and $f : U \to \mathbb{R}$ be expanding at the boundary of U in V. Then*

$$\mathrm{stab} \cap_{\mathbb{O}} (\underline{L}_f)_{\mathrm{comp}} = \mathrm{stabLuS}(f)_{\mathrm{comp}}$$

and

$$\mathrm{stab} \pitchfork_{\mathbb{O}} (\underline{L}_f)_{\mathrm{comp}} = \mathrm{stabMor}(f)_{\mathrm{comp}} .$$

In fact, the above applies to all smooth maps α, but for non-injective α it is worthwhile to use the refined Morse theory taking α into account (compare Section 0.3.7 above).

3.2.2 Bounding $\mathrm{stab} \cap_{\mathbb{O}} (L)$ from below by $\mathrm{stab} \cap_{\mathbb{O}} (L)_{+d\varepsilon}$

Let us modify the definition of stable intersection numbers in $M = T^*(V)$ as follows (compare Section 2.3). Given L_0 and L in M, we first stabilize to $\widetilde{L}_0 = L_0 \times \mathbb{O}_{\mathbb{R}^N} \subset M \times \mathbb{R}^{2N}$ and $\widetilde{L} = L \times L_Q \subset M \times \mathbb{R}^{2N}$ where $\mathbb{O}_{\mathbb{R}^N}$ is just another name for $\mathbb{R}^N_q$ of 2.3 and where L_Q for a non-singular quadratic Q on $\mathbb{R}^N$, albeit different from $\mathbb{R}^N_p$, is equivalent to $\mathbb{R}^N_p$ (see (c) in 2.3.1). But now we deviate from the definition of $\operatorname{stab}\cap_{L_0}(L)_{\text{comp}}$ by setting

$$\operatorname{stab}\cap(L)_{+d\varepsilon} = \inf_{\varepsilon,N} \#(\widetilde{L}_0 \cap \widetilde{I}_\varepsilon(\widetilde{L}))$$

where ε runs over all smooth compact (i.e. with compact supports) functions on $V\times\mathbb{R}^N$ and $\widetilde{I}_\varepsilon$ denotes the symplectomorphism $\tau\mapsto\tau+d\varepsilon$ of $T^*(V\times\mathbb{R}^N) = T^*(V)\times\mathbb{R}^{2N}$ so that $\widetilde{I}_\varepsilon(\widetilde{L})$ can be written as $\widetilde{L}+d\varepsilon$. Here as earlier, L may be non-injectively mapped into $M = T^*(V)$ and then the intersection points are counted in the sense of $\wp$. Furthermore, we may limit inf to those ε for which $\widetilde{I}_\varepsilon(\widetilde{L}_0)$ and $\widetilde{L}$ meet only at regular points and transversally. Then we denote the resulting transversal intersection number by $\operatorname{stab}\pitchfork_{L_0}(L)_{+d\varepsilon}$.

Notice that the symplectomorphisms $\widetilde{I}_\varepsilon$ constitute a rather small part among all $\widetilde{I}$. Their advantage is the preservation of the graphical and subgraphical classes of L. For example if $L = L_f$ then, obviously

$$\operatorname{stab}\cap_{\mathbb{O}}(L)_{+d\varepsilon} = \operatorname{stabLuS}(f)_{\text{comp}}$$

and

$$\operatorname{stab}\pitchfork_{\mathbb{O}}(L)_{+d\varepsilon} = \operatorname{stabMor}(f)_{\text{comp}}$$

for $\mathbb{O} = \mathbb{O}_V$. On the other hand, the discussion in 3.1 shows that we do not lose much by restricting to $\widetilde{I}_\varepsilon$. In particular, we have the following extension of the results of the previous section to arbitrary (non-subgraphical) subvarieties $L\subset T^*(V)$ (which at the present moment has no serious applications).

3.2.2 Intersection (in)equalities for $I(L)\cap\mathbb{O}$ in $T^*(V)$. *An arbitrary $L\subset T^*(V)$ satisfies*

$$\operatorname{stab}\cap_{\mathbb{O}}(L)_{\text{comp}} = \operatorname{stab}\cap_{\mathbb{O}}(L)_{+d\varepsilon} \qquad (\cap_{+d\varepsilon})$$

and

$$\operatorname{stab}\pitchfork_{\mathbb{O}}(L)_{\text{comp}} = \operatorname{stab}\pitchfork_{\mathbb{O}}(L)_{+d\varepsilon'}\,. \qquad (\pitchfork_{+d\varepsilon})$$

Before engaging into the proof we make several clarifying remarks. First, we recall that in most applications L is a *Lagrangian submanifold* immersed or embedded to $T^*(V)$. But the above relations hold true, as we shall see below, for an arbitrary subset $L\subset T^*(V)$. In fact L may be non-injectively mapped to $T^*(V)$ where the intersection is understood in the $\wp$-sense (see 0.3.4) and the transversality notation $\pitchfork$ is understood as earlier (see 0.3.8). Next we observe that the intersection aspect of $(\cap_{+d\varepsilon})$ and $(\pitchfork_{+d\varepsilon})$ are the inequalities $\operatorname{stab}\cap_{\mathbb{O}}(L)_{\text{comp}} \geq \operatorname{stab}\cap_{\mathbb{O}}(L)_{+d\varepsilon}$ and $\operatorname{stab}\pitchfork_{\mathbb{O}}(L)_{\text{comp}} \geq \operatorname{stab}\pitchfork_{\mathbb{O}}(L)_{+d\varepsilon}$ while the opposite inequalities are rather obvious. Finally we point out the right hand sides of $(\cap_{+d\varepsilon})$ and $(\pitchfork_{+d\varepsilon})$ for (sub)graphical manifolds L are (obviously) expressible in terms of the stable Morse theory thus providing the lower Morse theoretic bounds for Lagrangian intersections indicated in the previous section.

Proof. Replace, as earlier, $I(L) \cap \mathbb{O}$ by $L \cap I^{-1}(\mathbb{O})$ and generate $I^{-1}(\mathbb{O})$ by $-Q-\varepsilon$ on $V \times \mathbb{R}^n$. Then the intersection $L \cap I^{-1}(\mathbb{O})$ identifies with $(L \times \mathbb{O}_{\mathbb{R}^N}) \cap L_{-(Q+\varepsilon)}$ which equals

$$I^{-1}_{(Q+\varepsilon)}(I_\varepsilon(L \times L_Q) \cap \mathbb{O}_{\mathbb{R}^n \times \mathbb{R}^N}) .$$

Thus we bound $\#(I(L) \cap \mathbb{O})$ from below by (the worst case of) $I_\varepsilon(L \times L_Q) \cap \mathbb{O}_{\mathbb{R}^n \times \mathbb{R}^N}$. This gives us the (interesting) inequalities "$\geq$" while the (trivial) inequalities "$\leq$" are left to the reader. QED

Corollaries. We have already mentioned in Section 2.3.8 that the above equalities imply *the invariance of* $\operatorname{stab} \cap_{\mathbb{O}}(L)$ and $\operatorname{stab} \pitchfork_{\mathbb{O}}(L)$ *under embeddings* $V \subset W$. Another consequence is *the symplectic invariance* of $\operatorname{stab} \cap_{\mathbb{O}}(L)_{+d\varepsilon}$ and $\operatorname{stab} \pitchfork_{\mathbb{O}}(L)_{+d\varepsilon}$. (This raises the question of such invariance for $\cap_{L_0}(L)$ where $L_0 \neq \mathbb{O}$.)

3.3 Intersection inequalities for $I_1(L_1) \cap I_2(L_2)$

Our strategy here, for subgraphical L_1 and L_2, is to pass to $(L_1 \tilde{\barwedge} L_2) \cap \mathbb{O}$ and use $f_1 \tilde{\barwedge} f_2$ as follows.

3.3.1 Definition of $f_1 \tilde{\barwedge} f_2$

Let $\alpha_i : U_i \to V$ be smooth maps and $f_i : U_i \to \mathbb{R}$ smooth functions for $i = 1, 2$. First we define $f_1 \barwedge f_2 : U_1 \times U_2 \times \mathbb{R}^n \to \mathbb{R}$ for $V = \mathbb{R}^n$ by

$$f_1 \barwedge f_2(u_1, u_2, z) = f_1(u_1) - f_2(u_2) + \langle z, \alpha_1(u_1) - \alpha_2(u_2) \rangle$$

and observe that this agrees with the corresponding operation on Lagrangian subvarieties in $T^*(\mathbb{R}^n)$ (see 2.3.3). Namely the subgraphical variety defined by $f_1 \barwedge f_2$ in $T^*(\mathbb{R}^{3n})$ via the map $(u_1, u_2, z) \mapsto (\alpha_1(u_1), \alpha_2(u_2), z)$ equals

$$\underline{L}_{f_1 \barwedge f_2} = \underline{L}_{f_1} \barwedge \underline{L}_{f_2} .$$

Next we adapt this definition to an arbitrary smooth manifold V using a smooth proper embedding $V \subset \mathbb{R}^m$ and a normal neighbourhood $Z \subset \mathbb{R}^m$ of V with the normal projection denoted $\Pi : Z \to V$. We denote by $\Pi_i : Z_i \to U_i$ the induced fibrations $\alpha_i^*(Z)$, $i = 1, 2$ and we take some (quadratic norm) function Q on Z as in 2.3.7 which lifts to Q_i on Z_i, $i = 1, 2$. Now we extend f_i to $\widetilde{f}_i$ on $\widetilde{Z}_i$ by

$$\widetilde{f}_i(z_i) = f_i \circ \Pi_i(z_i) + (-1)^{i-1} Q_i(z_i) .$$

Clearly, this corresponds to the $(Q, -Q)$-extension of the corresponding pair $(\underline{L}_{f_1}, \underline{L}_{f_2})$ (see 2.3.4).

Finally *define*

$$f_1 \tilde{\barwedge} f_2 = \widetilde{f}_1 \barwedge \widetilde{f}_2 . \qquad (\tilde{\barwedge})$$

Of course, this depends on the embedding $V \to \mathbb{R}^m$ and a choice of Z and Q, but we just assume these are being fixed in a certain way.

Let us make a remark which will be used below. Supose that the functions f_i, $i = 1, 2$ are expanding at the boundaries for the maps $\alpha_i : U_i \to V$, (see2.1.2) then $f_1 \tilde{\bar{\times}} f_2$ also has this expanding property.

3.3.2 $\tilde{\bar{\times}}$-Theorem

Theorem 3.3.1 *The cardinality of the intersection between compactly Hamiltonian isotoped subgraphical manifolds $\underline{L}_{f_1}$ and $\underline{L}_{f_2}$ satisfies*

$$\#(I_1(\underline{L}_{f_1}) \cap I_2(\underline{L}_{f_2})) \geq \mathrm{stabLuS}(f_1 \tilde{\bar{\times}} f_2)_{\mathrm{comp}}$$

and

$$\#(I_1(\underline{L}_{f_1}) \pitchfork I_2(\underline{L}_{f_2})) \geq \mathrm{stabMor}(f_1 \tilde{\bar{\times}} f_2)_{\mathrm{comp}} \,,$$

provided the functions f_i on U_i are expanding at the boundaries of the maps $\alpha_i : U_i \to V$.

Proof. Since the intersection numbers are monotone for Euclidean embeddings $V \subset \mathbb{R}^m$ (see 2.3.7), we may assume $V = \mathbb{R}^n$ and use $\bar{\times}$ instead of $\tilde{\bar{\times}}$. Then we use the $\bar{\times}$-monotonicity in 2.3.3 and reduce the intersection problem for $(\underline{L}_{f_1}, \underline{L}_{f_2})$ to that for $(\underline{L}_{f_1} \bar{\times} \underline{L}_{f_2}, \mathbb{O})$ which is covered by the results in 3.2. QED.

Remarks. (a) If $V = \mathbb{R}^n$ one can use $\bar{\times}$ instead of $\tilde{\bar{\times}}$ which gives us, a priori, different inequalities. In fact, since $\tilde{\bar{\times}}$ depends on the embedding $V \subset \mathbb{R}^m$, the neighbourhood $Z \subset \mathbb{R}^m$ of V and $Q : Z \to \mathbb{R}$, we may try to choose these in a most efficient way making stabLuS and stabMor of $f_1 \tilde{\bar{\times}} f_2$ as large as possible. But it is unlikely we can gain anything essential this way.

(b) The Morse theoretic invariants of $f_1 \tilde{\bar{\times}} f_2$ can be refined by taking into account the topology of the maps α_1 and α_2 as in 0.2.7.

3.4 Morse theory for $f_1 \tilde{\bar{\times}} f_2$

3.4.1 Criterion for $f_1 \tilde{\bar{\times}} f_2$ to be a fibration at infinity

Let us give a criterion for $f_1 \tilde{\bar{\times}} f_2$ to be a fibration at infinity and thus amenable to the ordinary Morse inequalities. As we know, it suffices to have a complete Riemannian metric on the manifold $\widetilde{Z}_1 \times \widetilde{Z}_2 \times \mathbb{R}^m$ (where $f_1 \tilde{\bar{\times}} f_2$ is defined) such that $\|d(f_1 \tilde{\bar{\times}} f_2)\| \geq \varepsilon > 0$ with respect to this metric. We shall look for such a metric in the form

$$\widetilde{g}_1 + \widetilde{g}_2 + \text{Euclidean}$$

and start with the simplest case of $V = \mathbb{R}^n$ where $\tilde{\bar{\times}}$ reduces to $\bar{\times}$. So we have to compute the differential of $f_1 \bar{\times} f_2$ which splits into u_1, u_2 and z components as follows

$$\begin{gathered} df_1(u_1) + z^\perp \circ D(\alpha_1(u_1)) \in T^*_{u_1}(U_1) \\ -df_2(u_2) - z^\perp \circ D(\alpha_2(u_2)) \in T^*_{u_2}(U_2) \\ \alpha_1(u_1) - \alpha_2(u_2) \in \mathbb{R}^n \in T^*_z(\mathbb{R}^n)\,. \end{gathered}$$

Then the inequality $\|d(f_1 \bar{\times} f_2)\| \le \varepsilon$, implies

(i) $\|\alpha_1(u_1) - \alpha_2(u_2)\|_{\mathbb{R}^n} \le \varepsilon$

(ii) $\|d_{\mathrm{vert}} f_1(u_1)\|_{g_1} \le \varepsilon$ and $\|d_{\mathrm{vert}} f(u_2)\|_{g_2} \le \varepsilon$ where $d_{\mathrm{vert}} f_i$ means the restriction of df_i to $\ker D\alpha_i$.

(iii) $\|d_{\mathrm{hor}} f_1(u_1) + z^\perp \circ D(\alpha_1(u_1))\|_{g_1} \le \varepsilon$ and $\|d_{\mathrm{hor}} f_2(u_2) + z^\perp \circ D(\alpha_2(u_2))\|_{g_2} \le \varepsilon$, where $T^{\mathrm{hor}}_i = (\ker D\alpha_i)^\perp$ and d_{hor} is the restriction of d to T^{hor}.

Intersect the images

$$D\alpha_1(T_{u_1}(U_1)) = D\alpha_1(T^{\mathrm{hor}}_{u_1}) \quad \text{and} \quad D\alpha_2(T_{u_2}(U_2)) = D\alpha_2(T^{\mathrm{hor}}_{u_2})$$

in

$$\mathbb{R}^n = T_{\alpha_1(u_1)}(\mathbb{R}^n) = T_{\alpha_2(u_2)}(\mathbb{R}^n)$$

and lift this intersection to $T^{\mathrm{hor}}_{u_i}$ via $D^{-1}\alpha_2(u_i)$, $i = 1, 2$. Thus we obtain subspaces, say $S_{u_i} \subset T^{\mathrm{hor}}_{u_i}$, and an isomorphism $\delta : S_{u_1} \overset{\sim}{\to} S_{u_2}$, that is $D^{-1}(\alpha_2(u_2)) \circ D(\alpha_1(u_1))$. Denote by σ_i the restriction of df_i (or equivalently of $d_{\mathrm{hor}} f_i$) to S_i and define

$$\delta^-(u_1, u_2) = \inf_\zeta \max(\|\sigma_2 + \zeta\|_{g_2}\ ,\ \|\sigma_1 + \delta^*(\zeta)\|_{g_1})$$

where ζ runs over all covectors in S_2 and δ^* denotes the adjoint to δ.

Now, we say that (α_1, f_1) and (α_2, f_2) are *separated at infinity* if

(1) the map $\alpha_1 \times \alpha_2 : U_1 \times U_2 \to \mathbb{R}^n \times \mathbb{R}^n$ is transversal to the diagonal outside a compact subset in $U_1 \times U_2$,

(2) there exist complete Riemannian metrics g_i on U_i and $\varepsilon > 0$ such that the set of pairs $(u_1, u_2) \in U_1 \times U_2$ satisfying (i), (ii) and

(iii) $\delta^-(u_1, u_2) \le \varepsilon$,

is compact.

It easily follows from (i)-(iii), that this condition implies $\|d(f_1 \bar{\times} f_2)\| \ge \varepsilon > 0$ at infinity and thus we proved that

3.4.1 *If (α_1, f_1) and (α_2, f_2) are separated at infinity, then $f_1 \bar{\times} f_2$ is a fibration at infinity.*

Example. Let α_1 and α_2 be equidimensional immersions, such that f_i expand at the boundaries of α_i, $i = 1, 2$, and the Lagrangian (sub)graphs $\underline{L}_{f_1}$ and $\underline{L}_{f_2}$ are separated at infinity in the sense of 2.2. Then (α_1, f_1) and (α_2, f_2) are separated at infinity, as is immediately seen from the definition.

Now we extend all this to general $V \subset \mathbb{R}^m$ by just imposing the separation condition on $(\widetilde{\alpha}_1, \widetilde{f}_1)$ and $(\widetilde{\alpha}_2, \widetilde{f}_2)$.

3.4.2 From $f_1 \tilde{\bar{\times}} f_2$ to $f_1 \underset{\cdot}{-} f_2$

It is preferable to work with the function $f_1 \underset{\cdot}{-} f_2$ on $U_1 \otimes U_2$ (which reduces to $f_1 - f_2$ on $U_1 \cap U_2$ for embeddings $\alpha_i : U_i \to V$). One can either pass from the homological invariants of $f_1 \tilde{\bar{\times}} f_2$ to those of $f_1 \underset{\cdot}{-} f_2$, whenever this is possible, by a direct algebro-topological computation or one may use a *deformation argument* of the following kind.

3.4.2 *Suppose we have a family of smooth manifolds, X_μ, $\mu \in [0,1]$, i.e. a submersion $X \to [0,1]$ with the fibers X_μ such that X is a fibration over the semi-open interval* $]0,1]$. *Let F_μ be a family of smooth functions on X_μ, i.e. a smooth function $F : X \to \mathbb{R}$ with the following properties*

- $(*)$ *F_μ is a fibration at infinity on X_μ for all $\mu \in [0,1]$;*
- $(**)$ *the fibration at infinity structure of F_μ is continuous in μ for $\mu > 0$. This means, for each closed subinterval $I \in]0,1]$ the map of $X_I = F^{-1}(I)$ to $I \times \mathbb{R}$ for $x \mapsto (\mu, F(x))$ is a fibration at infinity;*
- $(***)$ *the critical values of F_μ are bounded in the absolute values for $\mu \to 0$.*

Then

$$\mathrm{stabMor}(F_1)_{\mathrm{comp}} \geq \mathrm{rank}\, H_*(X_0, F_0^{-1}(-\infty, -a])\,.$$

In other words, *the homological structure of F_μ for $\mu = 0$ bounds the stable Morse number of F_1.*

Proof. Every relative cycle C_0 in X_0 can be moved to C_μ in a nearby X_μ. If C_0 is non-homologous to zero, it non-trivially intersects with some C'_0 of complementary dimension. This goes to C'_μ with $C'_\mu \cap C_\mu = C'_0 \cap C'_0 \neq 0$ and so the Morse homology of F_0 *injects* to this of F_μ. But the latter is independent of $\mu > 0$ which concludes the argument. QED

Now we want to apply the above to $f_1 \bar{\times} f_2$ by deforming it to a stabilized $f_1 \underset{\cdot}{-} f_2$. Let $X_\mu \subset U_1 \times U_2 \times \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^n_z$ be given by the equations

$$\begin{aligned} \alpha_1(u_1) &= x \\ \alpha_2(u_2) &= (1-\mu)x + \mu y \end{aligned}$$

and F_μ on X_μ be

$$F_\mu(u_1, u_2, x, y, z) = f_1(u_1) - f_2(u_2) + \langle z, x - y \rangle\,.$$

Assume that the map $\alpha_1 \times \alpha_2 : U_1 \times U_2 \to \mathbb{R}^n \times \mathbb{R}^n$ is transversal to the diagonal. Then $U_1 \otimes U_2 = (\alpha_1 \times \alpha_2)^{-1}(\Delta)$ and $X = \underset{0 \leq \mu \leq 1}{\cup} X_\mu$ are smooth manifolds where $X \xrightarrow{\mu} [0,1]$ is a submersion with the exceptional fiber $X_0 = (U_1 \otimes U_2) \times \mathbb{R}^n_y \times \mathbb{R}^n_z$

while the fibers X_μ for $\mu > 0$ are all diffeomorphic to $U_1 \times U_2 \times \mathbb{R}^n_z$. The functions F_μ for $\mu > 0$ are equal to

$$f_1(u_1) - f_2(u_2) + \mu^{-1}\langle z, \alpha_1(u_1) - \alpha_2(u_2)\rangle$$

while

$$F_0 = f_1(u_1) - f_2(u_2) + \langle z, \alpha_1(u_1) - y\rangle$$

which is

$$f_1(u_1) \dot{-} f_2(u_2) + \langle z, y'\rangle$$

on $U_1 \boxtimes U_2 \times \mathbb{R}^n_{y'} \times \mathbb{R}^n_z$ for $y' = \alpha_1(u_1) - y$. The critical values of F_μ are, clearly, independent of μ for $\mu > 0$ and if (α_1, f_1) and (α_2, f_2) are separated at infinity, then F_μ are fibrations at infinity with this structure continuous in $\mu > 0$. What remains for an application of 3.4.2 is to assume fibration at infinity property for $f_1 \bar{\times} f_2$ and thus for F_0. Summing up we have the following

3.4.3 *Homological* ($\dot{-}$)*-theorem.* *Let $\alpha_1 : U_i \to V$, $i = 1, 2$, be smooth maps, such that $\alpha_1 \times \alpha_2$ is transversal to the diagonal $\Delta \subset V \times V$ and f_i be smooth functions on U_i satisfying the following three conditions*

(1) *the functions f_i are expanding at the boundaries of the maps $\alpha_i : U_i \to V$ in the sense of 2.1.2;*

(2) *The function $f_1 \dot{-} f_2$, i.e. $f_1(u_1) - f_2(u_2)$ on $U_1 \boxtimes U_2 = (\alpha_1 \times \alpha_2)^{-1}(\Delta)$, is a fibration at infinity;*

(3) *The pairs $(\widetilde{\alpha}_1, \widetilde{f}_1)$ and $(\widetilde{\alpha}_2, \widetilde{f}_2)$ are separated at infinity, where $\widetilde{\alpha}_i : \widetilde{Z}_i \to \mathbb{R}^m$ and $\widetilde{f}_i$ on $\widetilde{Z}_i$ are defined with some embedding $V \subset \mathbb{R}^m$ and a normal neighbourhood $Z \subset \mathbb{R}^m$ of V (see 2.3.7, 3.2.2);*

Then the subgraphical varieties $\underline{L}_{f_1}$ and $\underline{L}_{f_2}$ in $T^*(V)$ subjected to compact Hamiltonian isotopies admit, under the transversality assumption, the following lower bound on the cardinality of their intersection by the homology of the pair $(U_1 \boxtimes U_2, (f_1 \dot{-} f_2)^{-1}(-\infty, -a))$ for some sufficiently large a,

$$\#(I_1(\underline{L}_{f_1}) \pitchfork I_2(\underline{L}_{f_2})) \geq \operatorname{rank} H_*(U_1 \boxtimes U_2, (f_1 \dot{-} f_2)^{-1}(-\infty, -a))\,.$$

Proof. This follows from the above discussion applied to $(\widetilde{\alpha}_i, \widetilde{f}_i)$ and the $\tilde{\bar{\times}}$-theorem 3.3.1.

3.4.4 Remarks.

(a) The condition (1)-(3) are not hard to verify for concrete examples. In particular, they are clearly satisfied for f_i in 0.3.5 and 0.3.7 which implies the claims made in those theorems for Hamiltonian isotopies, while the contact case follows by 2.4.3 (compare 3.2). In fact the above intersection inequalities may hold for compact contact isotopies $I_i(t)$, $i = 1, 2$ but we cannot prove the needed focality without extra restrictions on (α_i, f_i) such as "niceness" of 0.3.3, etc.

(b) The $(\bar{\cdot})$-theorem (and its proof) is close in spirit to the Floer homology argument. In fact, a result of this nature can be obtained holomorphically for certain pairs of Lagrangian submanifolds in an arbitrary convex symplectic manifold M. For example, such an argument applies to graphical submanifolds $L_i = L_{f_i}$ in $M = T^*(V)$ for smooth functions f_i on open subsets $U_i \subset V$, $i = 1, 2$. However, the assumptions needed for the Floer theory for *non-compact* L_i (which should prevent certain infinite holomorphic discs and bands in M with finite energy) seem somewhat different from our condition "expanding at infinity". It would be interesting to find a general statement including the cases covered by generating and holomorphic techniques simultaneously.

Case $U_1 = U_2 = U$**.** In this case we prove the full-fledged stable MLS inequalities in terms of the function $f_1(\bar{\cdot})f_2$

Lemma 3.4.5 *Let $\alpha : U \to V$ be a smooth map and $f_1, f_2 : U \to \mathbb{R}$ be smooth functions such that $f_1 - f_2 : U \to \mathbb{R}$ is a fibration at infinity. Then the following inequalities holds*

$$\text{stabMor}(f_1 \tilde{\barwedge} f_2) \geq \text{stabMor}(f_1 - f_2)\,; \qquad \text{(Mor)}$$

$$\text{stabLuS}(f_1 \tilde{\barwedge} f_2) \geq \text{stabLuS}(f_1 - f_2)\,. \qquad \text{(LuS)}$$

Proof. We will assume that $V = \mathbb{R}^n$, U is a domain in $\mathbb{R}^n$ and $\alpha : U \hookrightarrow \mathbb{R}^n$ is just the inclusion map and leave to a reader the treating of the general case according to the scheme of Section 3.3.

In this case the function $f_1 \tilde{\barwedge} f_2$ reduces to the function

$$f_1 \barwedge f_2(u_1, u_2, z) = f_1(u_1) - f_2(u_2) + \langle z, (u_1 - u_2)\rangle\,,$$

defined on $U \times U \times \mathbb{R}^n$. Let us show that, given a non-degenerate quadratic form $Q : \mathbb{R}^N \to \mathbb{R}$ and a compact function $\varepsilon(u_1, u_2, z, y)$ on $U \times U \times \mathbb{R}^n \times \mathbb{R}^N$, one can find a compact function $\delta : U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ such that there is a one-to one correspondence between the critical points of the function

$$F = f_1 \barwedge f_2(u_1, u_2, z) + Q(y) + \varepsilon(u_1, u_2, z, y)$$

defined on $U \times U \times \mathbb{R}^n \times \mathbb{R}^N$ and the function

$$G(u, w, z, y) + \delta(u, w, z, y)$$

on $U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N$, where

$$G(u, w, z, y) = f_1(u) - f_2(u) + \langle z, w\rangle + Q(y).$$

This would prove the lemma because the function G is just the quadratic stabilization of the difference $f_1 - f_2 : U \to \mathbb{R}$.

We have

$$F(u_1, u_2, z, y) = f_1(u_1) - f_2(u_1) + \langle z, (u_1 - u_2)\rangle + Q(y) + \varepsilon(u_1, u_2, z, y) + (f_1(u_1) - f_2(u_2)) = G(u_1, u_1 - u_2, z, y) + \gamma(u_1, u_2, z, y)\,,$$

where

$$\gamma(u_1, u_2, z, y) = \varepsilon(u_1, u_2, z, y) + (f_1(u_1) - f_2(u_2))\,.$$

Let K be a compact set in U and C a large constant such that $K\times K\times B^n_C\times B^N_C \supset \operatorname{supp}(\varepsilon)$, where $B^n_C \subset \mathbb{R}^n$, $B^N_C \subset \mathbb{R}^N$ are balls of radius C. Let $\theta : U \to [0, 1]$ be a cut-off function such that $\theta(u) = 1$ for $u \in K$, and $\theta(u) = 0$ if $u \in U \setminus K'$ for a compact set $K' \subset U$ such that $\operatorname{Int} K' \supset K$ and such that the difference $f_1 - f_2$ has no critical points in $U \setminus K'$. Let $\eta : \mathbb{R}_+ \to [0, 1]$ be another cut-off function, equal 1 on $[0, 1]$, equal 0 outside $[0, 2]$, and such that $|\eta'| < 2$. Define a compact function $\delta : U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ by the formula

$$\delta(u, w, z, y) = \varepsilon(u, u-w, z, y)+(f_2(u)-f_2(u-w))\theta(u)\theta(u-w)\eta(||z||/C')\eta(||y||/C'),$$

where the large constant C' will be chosen later. The function

$$\widetilde{G}(u, w, z, y) = G(u, w, z, y) + \delta(u, w, z, y)$$

coincides with $F(u, u - w, z, y)$ if $(u, u - w, z, y) \in K \times K \times B^n_{C'} \times B^N_{C'}$ and coincides with $G(u, w, z, y)$ at infinity. Let us show that all the critical points of the function G are contained in the set

$$\widetilde{K} = \{(u, w, z, t)|\ (u, u - w, z, y) \in K \times K \times B^n_{C'} \times B^N_{C'}\}.$$

This would prove that the critical points of the fuctions $\widetilde{G}$ and F are in one-to-one correspondence.

Take a point $A = (u, w, z, y) \in (U \times U \times \mathbb{R}^n \times \mathbb{R}^N) \setminus \widetilde{K}$. Then the following cases can occur:

a) $||z|| > C'$.

In this case we have

$$\frac{\partial\widetilde{G}}{\partial w}(A) = z + h(u, u - w)\eta(||z||/C')\eta(||y||/C')\,,$$

where $h : U \times U \to \mathbb{R}$ is a *compact*, and therefore bounded function. Thus if the constant C' is chosen to be bigger than $\max|h|$ then

$$\frac{\partial\widetilde{G}}{\partial w}(A) \neq 0.$$

b) $||y|| > C'$.

Then we have

$$\left|\left|\frac{\partial\widetilde{G}}{\partial y}(A)\right|\right| \geq ||dQ(A)|| - \frac{H(u,w,z)||d\eta||}{C'} > aC' - 2C^{''}C' > 0,$$

if the constant C' is sufficiently large. Here H is a bounded function, $C^{''} = \max H$, and a is a constant (equal to the minimal absolute eigenvalue of the quadratic form Q).

c) $(u, u - w) \in (U \times U) \setminus (K' \times K')$ and $||z|| < C'$.

In this case we have

$$\frac{\partial\widetilde{G}}{\partial z}(A) = w.$$

Thus if A is a critical point of $\widetilde{G}$ then we should have $w = 0$. Hence

$$\frac{\partial\widetilde{G}}{\partial u}(A) = f_1'(u) - f_2'(u)\,.$$

By our assumption the difference $f_1 - f_2$ has no critical points outside of the compact set K'. Hence A cannnot be a critical point of the function $\widetilde{G}$.

QED

Proof of Proposition 0.3.6. In view of 2.1.2 very nice functions (or functions which satisfy the condition (di^{-1})) extend at the boundary of U. Thus Theorem 3.3.1 gives us the estimate

$$\#(L'_{f_1} \cap L'_{f_2}) \geq \mathrm{stabLuS}(f_1 \barwedge f_2)\,.$$

Hence combining with the estimates 3.4.5(LuS) and 0.2.6(LuS_π) we obtain the required inequality 0.3.6($\mathrm{LuS}_{\pi)}$. QED

3.5 Intersection inequalities for closed non-exact 1-forms

Let $L' \subset T^*(V)$ be obtained by a compact Hamiltonian isotopy $I(t)$ from the graph $L_\varphi = \varphi(\mathbb{O}) \subset T^*(V)$ of a closed 1-form φ. One can write $L_\varphi = I_\varphi(\mathbb{O})$ for the symplectomorphism $\tau \mapsto \tau + \varphi$ of $T^*(V)$ and

$$L' = I(1)(L_\varphi) = I_\varphi \circ I(1)(\mathbb{O})\,.$$

Thus the generating function $\widetilde{f}$ for $I(1)(\mathbb{O})$ gives us the generating form $\varphi + d\widetilde{f}$ for L'. One may treat similarly more general subgraphical subvarieties $\alpha_*(L_\varphi) \subset T^*(V)$ for closed 1-forms φ on U and then all our results in the previous sections 3.1–3.4 generalize from functions to forms. Combining this with 0.2.8 one can obtain intersection inequalities for non-exact Lagrangian submanifolds. For instance, applying 0.2.11(b) we get

3.5.1 *Let V_0, V_1 be closed equidimensional manifolds and $H_1(V_1) = 0$. Suppose that a Lagrangian submanifold L' is Lagrangian isotopic to the 0-section $\mathbb{O} \subset T^*(V)$, where $V = V_0 \# V_1$. Then*

$$L' \pitchfork \mathbb{O} \geq \operatorname{stabMor}(V_1) - 2(\#\text{of components of } V_1)\,.$$

Contact isotopies. If $L \subset T^*(V)$ is a non-exact submanifold, it admits no lift to $\operatorname{Jet}^1(V)$ and so the notion of a contact isotopy is not directly applicable. But we may correct the matter with I_φ by calling a regular Lagrangian homotopy $L(t)$ in $T^*(V)$ a contact isotopy if there exists a family of closed 1-forms $\varphi(t)$ on V, such that $I^{-1}_{\varphi(t)}(L(t))$ is a contact isotopy in the earlier sense. Then the above Hamiltonian results (obviously) extend to this contact situation.

3.6 Lagrangian intersection inequalities for symplectic foliations

Consider a smooth manifold $\mathcal{M}$ foliated into $2n$-dimensional leaves M carrying a symplectic structure given by a 2-form ω on the tangent bundle of the foliation, denoted $T(\mathcal{F}) \subset T(\mathcal{M})$, where $\mathcal{F}$ refers to the foliation structure. A smooth submanifold $L \subset \mathcal{M}$ of dimension $n + \operatorname{codim} \mathcal{F}$ is called *Lagrangian* if the intersection $T(L) \cap T(\mathcal{F})$ is ω-isotropic (i.e. ω vanishes there).

The intersection problem for pairs of Lagrangian submanifolds L_1 and L_2 concerns lower bounds on a suitable size of $L_1 \cap L_2$. For example, one wants to know when a leafwise symplectic diffeomorphism $\mathcal{M} \to \mathcal{M}$ sending each leaf M into itself necessarily has a fixed point in (almost) every leaf. (Actually, this question makes sense and remains interesting for Poisson manifolds of variable rank.) It seems the Floer holomorphic homology theory can be transported to the foliated world. What we indicate below is how the things work with generating functions.

Transversal measures. We assume our foliations come along with smooth transversal measures. Such a measure, say μ assign a non-negative number to each Borel subset $A \subset \mathcal{M}$ having (at most) countable intersection with each leaf M, and this number, denoted $\mu(A)$, is invariant under Borel transformation of $\mathcal{M}$ preserving the leaves (i.e. mapping each M into itself). Now we can measure $L_1 \cap L_2$ by taking $\mu(L_1 \cap L_2)$ (which we assume $= \infty$ if $L_1 \cap L_2$ meets some M uncountably).

Morse theory for foliations. Let $\mathcal{V}$ be a closed smooth manifold foliated into smooth n-dimensional leaves V. Then smooth functions f on $\mathcal{V}$ admit a meaningful leaf-wise Morse theory as was pointed out by Thom (see [63]). Namely, one takes some leaf-wise Riemannian metric g on $\mathcal{V}$ and defines the *leaf-wise gradient flow* $\operatorname{grad}_g f$. This is an actual smooth flow on $\mathcal{V}$ preserving each leaf V and increasing f. It follows that $\mathcal{V}$ decomposes into stable (or unstable) subvarieties of the fixed point set $\Sigma \subset \mathcal{V}$ where $\operatorname{grad}_g f = 0$. If f is *generic*, then the subset $\Sigma' \subset \Sigma$ of the *degenerate* leaf-wise critical points of f has $\dim \Sigma' < \operatorname{codim} \mathcal{F}$, and the above decomposition $\mathcal{V}$ is cellular on almost all leaves V. In fact all this remains true for non-compact manifolds and asymptotically "nice" functions f on $\mathcal{V}$, for examples for *quadratic stabilization* of functions on compact manifolds.

Now, if the foliation $\mathcal{F}$ on $\mathcal{V}$ comes along with a smooth transversal measure μ, we may measure Σ by $\mu(\Sigma)$ and define the stable and unstable Morse and Lusternik-Schnirelman numbers as earlier.

Discouraging example. Let $\mathcal{V}$ be the 2-torus with the standard irrational 1-foliation $\mathcal{F}$. Then one can easily construct a generic smooth function f on this $\mathcal{V} = T^2$ with Σ equal a given, arbitrarily short, closed curve in T^2. It follows the Morse number of this $(\mathcal{V}, \mathcal{F})$ is zero for the natural transversal measure μ.

However, as it was observed by Connes, *one does have a non-trivial lower bound on stabMor* $(\mathcal{V}, \mathcal{F})$ *if the* L_2*-cohomology of* $\mathcal{F}$ *does not vanish.* In fact, if $\mathcal{V}$ is compact, the von Neumann rank (dimension) of $L_2H^*(\mathcal{F})$ is bounded in terms of the (properly transversally defined) measure of the cells in a suitable cell decomposition of a μ-generic leaf M. In particular one immediately has (if one knows the definition of L_2H^*)

$$\mu(\Sigma) \geq \operatorname{rank} L_2H^*(\mathcal{F})\,, \tag{Con}$$

which gives, after stabilization, the inequality

$$\operatorname{stabMor}(\mathcal{F}) \geq \operatorname{rank} L_2H^*(\mathcal{F})\,.$$

(See [19], where an analytic proof is indicated in a more general context of transversally measurable foliations.)

Now we turn to the symplectic foliation on $\mathcal{M} = T^*(\mathcal{F})$ for $\mathcal{F}$ foliating $\mathcal{V}$. This $\mathcal{M}$ is foliated into the symplectic leaves $T^*(V)$ and is denoted by $\mathcal{F}^*$ on $\mathcal{M}$. Hamiltonian isotopies $I(t)$ in $\mathcal{M}$ are defined via smooth time dependent Hamiltonians H on $\mathcal{M}$ restricted to the leaves $T^*(M)$ of $\mathcal{F}^*$. As the (full) cotangent bundle $T^*(\mathcal{V})$ naturally maps to $T^*(\mathcal{F})$ the Hamiltonian H lift from $T^*(\mathcal{F})$ to $T^*(\mathcal{V})$ and, in particular, every Hamiltonian isotopy $I(t)$ of the zero section $\mathbb{O}$ of the bundle $T^*(\mathcal{F}) \to \mathcal{V}$ comes from such an isotopy of $\mathbb{O} = \mathbb{O}_{\mathcal{V}} \subset T^*(\mathcal{V})$. In other words, the leafwise Lagrangian submanifolds $L \subset T^*(\mathcal{F})$ come from our old Lagrangian submanifolds, say $\overline{L} \subset T^*(\mathcal{V})$ by "restricting" these $\overline{L}$ to the leaves of $\mathcal{F}^*$ which in the case of a graphical $\overline{L} = \overline{L}_f \subset T^*(\mathcal{V})$ corresponds to actual restriction of f to the leaves V of $\mathcal{F}$ on $\mathcal{V}$. Thus generating functions for $\overline{L}$, whenever they exist, restrict to generating function for $L \subset T^*(\mathcal{F})$ and therefore

3.6.1 *The Hamiltonian isotoped zero section in* $T^*(\mathcal{F})$ *satisfies the intersection inequality,*

$$\mu(\mathbb{O} \pitchfork I(1)(\mathbb{O})) \geq \operatorname{stabMor}(\mathcal{F}) \geq \operatorname{rank} L_2H^*(\mathcal{F})\,. \tag{μL_2}$$

Here we assume $\mathcal{V}$ to be a compact manifold, and the sign "$\pitchfork$" contains the assumption of the transversality of the intersection $\mathbb{O} \cap I(1)(\mathbb{O}) \cap T^*(M)$ for almost all M.

Remarks. (a) Most of the results of the present paper extend along the above lines to the framework of smooth foliations with smooth transversal measures. Probably one may drop (or at least relax) the smoothness assumption on the measure and/or generalize the above to transversally measurable (and/or continuous) foliations. Actually, to get the Morse-Thom theory, all one needs is a continuous function f on $\mathcal{V}$ which is smooth along the leaves and is Morse μ-almost everywhere, i.e. having μ-measure zero of the leaves where f has a *degenerate* critical point, and then Morse-Connes inequalities apply.

(b) Connes' original argument elaborating Witten's proof of Morse inequalities is technically more difficult than the one indicated above and using Thom's idea. But the analytic approach by Connes yields extra information of geometric and analytic nature. (It would be interesting to apply Witten-Connes method in the holomorphic framework of Floer homology.)

(c) It would be interesting to refine the Morse inequality for $\mathcal{F}$ by bringing in a suitable (norm on) some K-theory of $\mathcal{F}$, where, of course, the major problem is finding convincing examples. (Recall, that the basic examples of foliations are these on G/Γ, for a lattice Γ in a Lie group G, where G/Γ is foliated into the orbit of some $H \subset G$ acting on G/Γ. And one gains extra examples by dividing $G/\Gamma \leadsto K\backslash G/\Gamma$ for a compact subgroup $K \subset G$.)

(d) The Lusternik-Schnirelman theory, probably, does not survive the L_2-atmosphere; yet a possibility of foliated LS invariants is suggested by the discussion in §$2\frac{2}{3}$ of [39].

(e) The refinements of the ordinary Morse theory indicated in 0.3.7 also make sense for foliations where we may speak, for example, of (stable) Morse numbers of a particular function on a non-compact foliated manifold $\mathcal{V}$. Also we may have a foliated map $\alpha : \mathcal{U} \to \mathcal{V}$ and restrict the (stable) Morse theory on $\mathcal{U}$ to functions coming from $\mathcal{V}$. All this extends to the Lagrangian geometry of $T^*(\mathcal{F})$ via the generating functions.

Also the Morse theoretic approach to kissings between moving submanifolds extends to the foliated framework. The general set up here is of two foliated submanifolds $\mathcal{W}_1$ and $\mathcal{W}_2$ moving in time inside $\mathcal{V}$. For example, $\mathcal{V}$ may consists of a single leaf V and then we have foliated $\mathcal{W}_i$, $i = 1,2$ moving in V. The kisses in question are those between the leaves W_1 of $\mathcal{W}_1$ and W_2 of $\mathcal{W}_2$ the totality of which form a k-dimensional subvariety $\mathcal{K}$ in $\mathcal{W}_1 \times \mathcal{W}_2 \times [0,1]$, where we limit the motions $\mathcal{W}_i(t)$ of $\mathcal{W}_i$ to $t \in [0,1]$, and where $k = \operatorname{codim} \mathcal{F}_1 + \operatorname{codim} \mathcal{F}_2$ for the implied foliations $\mathcal{F}_i$ on $\mathcal{W}_i$, $i = 1,2$, provided $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$ are generic in V. If $\mathcal{F}_i$ are equipped with transversal measures μ_i, one defines the kissing number between $\mathcal{W}_i(t)$ as $\mu_1 \times \mu_2(\mathcal{K})$ and observes that it can be bounded from below in L_2-homological terms as in the usual case. Then one can bring in the generating hypersurfaces (see Chapter 4) and evaluate the crossing (or linking) numbers of $\mathcal{L}_{\mathcal{W}_i} \subset T^*(V)$, where the implied Legendrian lifts of $\mathcal{W}_i$ apply to the leaves $W_i \subset \mathcal{W}_i$ thus giving us Legendrian foliations in $T^*(V)$. Working out the precise statements and proofs is left to the reader.

Justification. The above may appear as a mindless generalization for generalization sake. Thus foliation people may resent the symplectization of the Morse theory while a symplectically oriented reader could be disheartened by foliations and the von Neumann dimension. But we rejoice to see several great ideas playing together on the common ground: Morse theory, symplectization, foliations (starting the road to Connes' non-commutative geometry), Floer theory, etc.

3.7 Generating functions in the algebraic category

If V is an affine algebraic manifolds over some field K of characteristic zero one asks for a polynomial generating function $\widetilde{f} : V \times K^N \to K$ for a given Lagrangian submanifold $L \subset T^*(V)$ where one may additionally require certain degree of non-singularity of $\widetilde{f}$, e.g. having the principal (highest degree) homogeneous term of $\widetilde{f}$ on $v \times K^N = K^N$ *non-degenerate* for all $v \in V$, i.e. vanishing only at $O \in K^N$. Such $\widetilde{f}$ look interesting, for example, for the graphs of symplectic maps $I : K^{2n} \to K^{2n}$, already for $n = 1$. In fact, since generating functions go along with composition of maps one finds plenty of I's with "generated" graphs of the form $I = I_1 \circ I_2 \circ \ldots \circ I_k$ where $I_i : K^{2n} \to K^{2n}$ are some elementary symplectomorphisms of K^{2n}, e.g. those preserving the fibers of linear maps $K^{2n} \to K$. Much of the formal constructions of the present paper extend to the algebraic framework, say over $\mathbb{C}$ where the complex analytic theory is also available. But we see no useful applications at this stage.

The contact geometry, at least on the formal level, also can be grown on the algebraic sail. Here one even has an abundance of projective examples, namely $PT^*(V)$ for projective varieties V. These N's are full of interesting Legendrian subvarieties, e.g. the lifts of subvarieties in V, and it may be interesting to decide which of them can (and which can not) be generated by *non-singular* hypersurfaces in $V \times V'$, e.g. in $V \times P^N$.

Chapter 4

Generating functions and generating hypersurfaces for Legendrian submanifolds

We start here with reinterpreting our earlier contact isotopy results in Legendrian language of $\mathrm{Jet}^1(V)$. Then we turn to the Legendrian linking problem in $PT^*(V)$.

4.1 Generating functions for $\mathcal{I}(t)(\mathcal{L}_f)$ in $\mathrm{Jet}^1(V)$

4.1.1 Homotopy covering property for generating functions

A subvariety $\underline{\mathcal{L}} \subset \mathrm{Jet}^1(V)$ is called *generated by a function* f on U, if U comes along with a *fibration* $\alpha : U \to V$ and $\underline{\mathcal{L}} = \alpha_*(\mathcal{L}_f) = \underline{\mathcal{L}}_f$ where, recall $\mathcal{L}_f = J^1_f(V) \subset \mathrm{Jet}^1(V)$ and α_* is the pushforward associated to the contact correspondence suspending the Lagrangian one $T^\star(\Gamma_\alpha) \subset T^*(U) \times (-T^*(V))$, (see 0.1.1, 2.7). We say that $\underline{\mathcal{L}}$ is *transversally generated* by f if $\mathcal{L}_f$ is transversal to $\mathcal{Z} \subset \mathrm{Jet}^1(U)$ consisting of the 1-jets of functions constant on the fibers of α, i.e. if L_f is transversal to the corresponding coisotropic submanifold $Z \subset T^*(U)$. Clearly, if the reduction is transversal, then $\underline{\mathcal{L}}$ is an immersed Legendrian submanifold in $\mathrm{Jet}^1(V)$.

Theorem 4.1.1 Chekanov [15], see also [12],[55], [30]) *Let $\mathcal{L}$ be generated by $f : U \to \mathbb{R}$ and $\mathcal{I}(t)$ be a compact contact isotopy in* $\mathrm{Jet}^1(V)$*. Then $\mathcal{I}(t)(\underline{\mathcal{L}})$ is generated, for each $t \geq 0$, by a function $\widetilde{f_t}$ on $U \times \mathbb{R}^{N(t)}$ of the form*

$$\widetilde{f_t} = f(u) + Q(z) + \varepsilon_t(u, z)\,,$$

where $Q = Q_{N(t)}$ is a non-singular quadratic form on $\mathbb{R}^{N(t)}$ and ε_t is a smooth compact function. Furthermore, if f was generating $\underline{\mathcal{L}}$ transversally, then so does $\widetilde{f_t}$ to $\mathcal{I}(t)(\underline{\mathcal{L}})$.

Proof. Everything follows from the symplectic case with the aid of 2.6.1 and 2.2.4. First, let us recall that the Legendrian manifold $\mathcal{L} = \underline{\mathcal{L}}_f \subset \mathrm{Jet}^1(V)$ is obtained from the graphical Legendrian manifold $\tilde{\mathcal{L}} = \mathcal{L}_f \subset \mathrm{Jet}^1(U)$ by a contact reduction (see 2.6.1) above. Thus the isotopy lifting property for contact reductions (see Lemma 2.6.1) provides us with a compact Legendrian isotopy $\tilde{\mathcal{L}}(t)$ in $\mathrm{Jet}^1(U)$ which covers the isotopy $\mathcal{L}(t) \subset \mathrm{Jet}^1(V)$.

Set $M = T^*(U)$, $N = \mathrm{Jet}^1(U) = T^*(U \times \mathbb{R})$, and denote by ∂ the Liouville vector field $p\frac{\partial}{\partial p}$ in M. The symplectization N^+ of the contact manifold $N = \mathrm{Jet}^1(U)$ is symplectomorphic to $T^*(U \times \mathbb{R})$. with the symplectic form $d(pdq + tds)$, $q \in U$, $s \in \mathbb{R}_+$ (see 2.4.1). The canonical scaling in N^+ corresponds to the action

$$(p, q, t, s) \mapsto (\lambda p, q, t, \lambda s),\ \lambda \in \mathbb{R}_+,$$

and the manifold $N = N \times 1 \subset N_+$ corresponds to the hypersurface

$$\Sigma = \{s = 1\} \subset T^*(U \times \mathbb{R}_+).$$

The Legendrian isotopy $\tilde{\mathcal{L}}(t) \subset N$ symplectizes to a Lagrangian isotopy $\hat{L}(t) \subset T^*(U \times \mathbb{R}_+)$, invariant under this action. In particular, the symplectization of the graphical Legendrian manifold $\mathcal{L}_f = \mathcal{L}(0) \subset N$ is the graphical Lagrangian submanifold $\hat{L}(0)$ generated by the function

$$F(u, s) = sf(u)\,, u \in U, s \in \mathbb{R}_+.$$

Next we conclude (comp. Section 2.4.3) that since the intersection $\hat{L}(t) \cap \Sigma$ coincides with $\tilde{\mathcal{L}}(t)$ then the symplectic reduction of $\hat{L}(t)$ with the coisotropic submanifold $\Sigma = M \times \mathbb{R}$ equals the (immersed) Lagrangian submanifold $\tilde{L}(t) \subset M$, the projection of $\tilde{\mathcal{L}}(t) \subset M \times \mathbb{R}$ to M. Using the cut-off adjustment, as in 2.2.4, we can make the Lagrangian isotopy $\hat{L}(t)$ compact without affecting the intersection $\hat{L}(t) \cap \Sigma$, and, therefore without changing the symplectic reduction. Hence we are in the position to apply to the family $\hat{L}(t) \subset T^*(U \times \mathbb{R}_+)$ the theory of generating functions in the Lagrangian case (see Section 3.1.2) to construct a family of functions $F(t) : U \times \mathbb{R}_+ \times \mathbb{R}^{N(t)} \to \mathbb{R}$, $t \in [0, 1]$, such that

- $F(t)(u, z) = sf(z) + \varepsilon(t)(s, u, z) + Q(z)$, $u \in U, z \in \mathbb{R}^{N(t)}, s \in \mathbb{R}_+$ where $\varepsilon(t)$ is a family of compact functions and $Q : \mathbb{R}^{N(t)} \to \mathbb{R}$ is a non-degenerate quadratic form;
- $\varepsilon(0) = 0$;

- the function $F(t)$ generates $\hat{L}(t)$, i.e $\hat{L}(t) = \underline{L(t)}_{F(t)}$.

Thus the restriction of $F(t)$ to $(U \times 1) \times \mathbb{R}^{N(t)} \subset U \times \mathbb{R}_+ \times \mathbb{R}^{N(t)}$ generates the family of immersed Lagrangian manifolds $\tilde{L}(t) \subset T^*(U)$, and the Legendrian isotopy $\tilde{\mathcal{L}}(t) \subset \mathrm{Jet}^1(U)$ as well. QED

The above covering property (by now obviously) extends to families, as in the symplectic case, and shows that "the (transversal) generation map" from functions on $V \times \mathbb{R}^\infty$ to Legendrian subvarieties in $\mathrm{Jet}^1(V)$ is a Serre fibration.

In Section 5.1.3 we will take a closer look at this fibration.

4.1.2 The proof of the Legendrian intersection inequalities of 0.4.1

The above generating function theorem can be put to use very much the same way as its Hamiltonian counterpart in Chapter 3. First, all the intersection inequalities proven in 3 for Lagrangian submanifolds deformed by a Hamiltonian isotopy immediately can be generalized to the similar inequalities when one allows contact isotopy of Lagrangian submanifolds.* Then one obtains the intersection results of 0.4.1 as follows.

Proof of Theorem 0.4.1 Suppose that $\mathcal{L}'$ is Legendrian isotopic to $\underline{\mathcal{L}}_f$. According to Theorem 4.1.1 the deformed Legendrian submanifold $\mathcal{L}'$ is generated by a function $F = f(u) + \varepsilon(u,z) + Q(z)$ on $U \times \mathbb{R}^N$, where ε is a compact function and Q is a non-degenerate quadratic form on $\mathbb{R}^N$. Intersection points of $\mathcal{L}'$ and $\mathcal{M} = L_W \times \mathbb{R}$ correspond to critical points of F restricted on $(U \pitchfork W) \times \mathbb{R}^N$. Thus the inequalities $\mathrm{LuS}\cup_{\mathcal{M}}$ and $\mathrm{Mor}\pitchfork_{\mathcal{M}}$ follow from the definition of the stable Lusternik-Schnirelman and Morse numbers. QED

Proof of Theorem 0.4.5 If $\mathcal{L}'$ is Legendrian isotopic to $\mathcal{L} = \underline{\mathcal{L}}_f$ then it can be defined, as above, by the generating function $\tilde{f} = Q(z) + \varepsilon(u,z)$. the intersection points of $\mathcal{L}'$ and $\mathbb{R}\mathcal{L}$ correspond to the critical points of the function $\tilde{f} - f = Q(z) + \varepsilon(u,z) - f(u)$. Hence the inequalities (LuS_π) and $(H\mathrm{Mor})$ follow from 0.2.6 and 0.2.3 . QED

4.1.3 Contact isotopies of Lagrangian and Legendrian graphs of closed 1-forms on V

The intersection theory for these was already indicated in Section 3.5 and now we want to generalize the idea of the generating function. The first possibility coming to one's mind is to replace $\mathbb{R}$ by $S^1 = \mathbb{R}/\mathbb{Z}$ and look at Legendrian submanifolds in $\mathrm{Jet}^1(V, S^1) = \mathrm{Jet}^1(V)/\mathbb{Z}$. Every smooth map $f : V \to S^1$ gives

*Alternatively the contact generalization can be obtained using 2.4.2, comp. 3.2.1

rise to $\mathcal{L}_f \subset \mathrm{Jet}^1(V, S^1)$ in an obvious way (where df is a closed form which is not exact unless f is contractible). And every smooth map $\alpha : U \to V$ pushes forward submanifolds from $\mathrm{Jet}^1(U, S^1)$ to those in $\mathrm{Jet}^1(V; S^1)$ as earlier. In particular, if $U \to V$ is a fibration, we may speak of $\underline{\mathcal{L}} \subset \mathrm{Jet}^1(V; S^1)$ generated by $f : U \to S^1$. We know already how to generate the projection $\underline{L}$ of $\underline{\mathcal{L}}$ to $T^*(V)$ (see 3.5) by $\varphi + d(Q + \varepsilon)$, where $\varphi = df$ in the present case. Since $\varphi + d(Q + \varepsilon)$ integrates to a smooth map $\underline{L} \times \mathbb{R}^N \to S^1$, we obtain a generating map $\widetilde{f} : U \times \mathbb{R}^N \to S^1$ for $\underline{\mathcal{L}}$. We suggests the reader would list the properties of this f and, even better, would find new applications (which we ourselves do not have).

More generally, every closed 1-forms φ on V gives a (non-trivial) flat structure to the line bundle $V \times \mathbb{R} \to V$. We denote this flat bundle by $\mathcal{R}_\varphi \to V$ and look at the space $\mathrm{Jet}^1_\varphi(V) = \mathrm{Jet}^1(V \to \mathcal{R}_\varphi)$ of the 1-jets of flat sections $V \to \mathcal{R}_\varphi$. Then for every $\alpha : U \to V$ one may induce $\mathcal{R}^*_\varphi = \alpha^*(\mathcal{R}_\varphi)$ over U and speak of $\alpha_*(\mathcal{L})$ for $\mathcal{L} \subset \mathrm{Jet}^1_\varphi(U) = \mathrm{Jet}^1(U \to \mathcal{R}^*_\varphi)$. In particular, it makes sense to speak of sections of $\mathcal{R}^*_\varphi$ on U generating $\mathcal{L} \subset \mathrm{Jet}^1_\varphi(V)$ and what we have done for $\varphi = df$, where $f : V \to S^1$, easily extends to the present case. But we do not know what good may come from these generating sections.

4.2 Generating hypersurfaces in $\ddot{P}T^*(V)$

4.2.1 Construction of generating hypersurfaces

Recall the space $\ddot{P}T^*(V)$ of cooriented hyperplanes in $T(V)$ where the natural contact structure is denoted by $\ddot{\eta}$. Every smooth map $\alpha : U \to V$ induces a contact correspondence between $\ddot{P}T^*(U)$ and $\ddot{P}T^*(V)$ and thus the pushforward map $\alpha_*(\mathcal{L}) \subset \ddot{P}T^*(V)$ for all $\mathcal{L} \subset \ddot{P}T^*(U)$. We are particularly interested here in the situation where α is a fibration and $\mathcal{L} = \mathcal{L}_W$ for a cooriented hypersurface $W \subset U$. In this case we say that $\underline{\mathcal{L}} = \alpha_*(\mathcal{L})$ is *generated* by W (comp. [29]). It is called *transversally* generated if $\mathcal{L} = \mathcal{L}_W$ is transversal to the submanifold $\widetilde{\mathcal{Z}} \subset \ddot{P}T^*(U)$ defining the implied *reduction* of $\ddot{P}T^*(U)$ to $\ddot{P}T^*(V)$. Namely $\widetilde{\mathcal{Z}}$ consists of the hyperplanes in $T(U)$ tangent to the fibers of α and it fibers over V for the obvious map $\widetilde{\Pi} : \widetilde{\mathcal{Z}} \to \ddot{P}T^*(V)$, so that $\alpha_*(\mathcal{L}) = \widetilde{\Pi}(\mathcal{L} \cap \widetilde{\mathcal{Z}})$. If $\mathcal{L}$ is a Legendrian submanifold in $\ddot{P}T^*(V)$, e.g. $\mathcal{L} = \mathcal{L}_W$ and the intersection $\mathcal{L} \cap \mathcal{Z}$ is transversal (which is the case for a fixed fibration α and $\mathcal{L} = \mathcal{L}_W$ with generic W) then $\underline{\mathcal{L}} = \alpha_*(\mathcal{L})$ is an immersed Legendrian submanifold in $\ddot{P}T^*(V)$.

We are going to prove now that the property of $\underline{\mathcal{L}}$ being generated is invariant under compact contact isotopies of $\underline{\mathcal{L}}$. We assume here that the hypersurface $W \subset U$ is given as a *non-critical* level of a smooth function $f : U \to \mathbb{R}$,

$$W = \{u \in U \mid f(u) = 0\}\,.$$

Then we take a smooth compact contact isotopy $\mathcal{I}(t)$ for $t \in [0, 1]$ in $\ddot{P}T^*(V)$ and let $\underline{\mathcal{L}}(t) = \mathcal{I}(t)(\underline{\mathcal{L}})$ where $\underline{\mathcal{L}} = \underline{\mathcal{L}}(0) = \alpha_*(\mathcal{L}_W)$. We impose the *properness* assumption on $\underline{\mathcal{L}}$, i.e. the properness of the above map $\mathcal{L} \cap \widetilde{\mathcal{Z}} \to \ddot{P}T^*(V)$ and then we have (see also [29]) the following

Theorem 4.2.1 *There exists an integer N and a smooth function $\widetilde{f} = \widetilde{f}(u, z, t)$ on $U \times \mathbb{R}^N \times [0, 1]$ with the following two properties:*

(1) For every fixed $t \in [0, 1]$, the function $\widetilde{f}(u, z, t)$ on $U \times \mathbb{R}^N$ has no zero critical value and the hypersurface

$$\widetilde{W}(t) = \{u, z \mid \widetilde{f}(u, z, t) = 0\} \subset U \times \mathbb{R}^N$$

generates $\underline{\mathcal{L}}(t)$ for the map $(u, z) \mapsto \alpha(u)$; furthermore, if the generation of $\underline{\mathcal{L}}$ by W is transversal, then so is the generation of $\underline{\mathcal{L}}(t)$ by $\widetilde{W}(t)$:

(2) The function $\widetilde{f}$ is asymptotically quadratic non-singular in z. More precisely,

$$\widetilde{f} = f(u) + Q(z) + \varepsilon(u, z, t)\,,$$

where Q is a non-singular quadratic form on $\mathbb{R}^N$ and ε is a smooth function with compact support, such that $\varepsilon(u, z, 0) = 0$.

Proof. Recall (see (c) in 2.6.2) that $\underline{\mathcal{L}}$ equals the contact reduction of $\underline{\mathcal{L}}_f = \alpha_*(\mathcal{L}_f) \subset \mathrm{Jet}^1(V)$ where $\mathcal{L}_f \subset \mathrm{Jet}^1(U)$ is the Legendrian graph of f. The isotopy $\underline{\mathcal{L}}(t)$ of $\underline{\mathcal{L}}$ lifts to a compact contact isotopy $\underline{\mathcal{L}}_f(t)$ of $\underline{\mathcal{L}}_f$ by the homotopy lifting property for contact reduction (see 2.6.1) and then the homotopy covering property from Theorem 4.1.1 applies. QED

4.2.2 *Remarks.*

(a) The above theorem remains valid for families of hypersurfaces (see similar discussion for the Lagrangian case in Section 3.1.2, as well as Section 5.1.3 below).

(b) Let us ask ourselves what happens to non-dividing hypersurfaces W, e.g. to such $W \subset V$. If W is coorientable, it comes as a level of a smooth map $f : V \to S^1$, $W = f^{-1}(s_0)$ and f lifts to a real function $\widehat{f}$ on the cyclic covering $V^{\mathrm{cy}} \to V$ corresponding to $\mathbb{R} \to S^1$. Then the above applies to $\widehat{f}$ where we may assume $0 \in \mathbb{R}$ goes to our $s_0 \in S^1$ and W lifted to V^{cy} identifies with the zero level of $\widehat{f}$. Thus we generate the Legendrian submanifolds $\mathcal{I}(t)(\dot{\mathcal{L}}_W)$ by hypersurfaces in $V^{\mathrm{cy}} \times \mathbb{R}^N$ which allows sufficient applications to the crossing (linking) problem as in 4.2.2 (An alternative and essentially equivalent approach is suggested by contact isotopies of closed forms in 3.5) Finally, we briefly look at the case where $W \subset V$ (or more generally $W \subset U$) is *not* coorientable. In this case, there may exist a flat (non-orientable) line bundle $\mathcal{R} \to V$ so that W appears as the zero of a smooth section $f : V \to \mathcal{R}$. Then, in the spirit of the discussion at the beginning of this section, we may generate $\mathcal{I}(t)\mathcal{L}_W$ by hypersurfaces in $V \times \mathbb{R}^N$ which are zeros of sections $V \times \mathbb{R}^N \to \mathcal{R}^*$ for the induced bundle $\mathcal{R}^* \to V \times \mathbb{R}^N$ where such a section is non-singular quadratic at infinity along the Euclidean slices $v \times \mathbb{R}^N$ over which our $\mathcal{R}^*$ is trivial.

4.2.2 Lower bounds on crossing numbers between Legendrian submanifolds

Proofs of Theorems 0.5.1, 0.5.2(⤩ cup) and 0.5.2(⤩ H_*)

Take two submanifolds W_1 and W_2 in V and let $\mathcal{L}_1(t)$ and $\mathcal{L}(t) = \mathcal{L}_2(t)$ be compact isotopies of their Legendrian lifts $\mathcal{L}_{W_1} = \mathcal{L}_1(0)$ and $\mathcal{L}_{W_2} = \mathcal{L}(0)$ to $PT^*(V)$. We want to find a Morse theoretic lower bound on the crossing number between these isotopies and we proceed as follows. First, we reduce the general case to the one where $\mathcal{L}_1$ does not move, i.e. $\mathcal{L}_1(t) = \mathcal{L}_1$ for all t. This is done by just replacing the isotopy $\mathcal{I}(t)$ moving $\mathcal{L}$ by $\mathcal{I}_0^{-1}(t)\mathcal{I}(t)$ where $\mathcal{I}_0(t)$ is the isotopy responsible for $\mathcal{L}_1(t)$. Next, as we need the above theorem applied to $\mathcal{L}(t)$, we need W_2 to be a properly embedded hypersurface bounding some domain, say $V_+ \subset V$. If this is the case, we take some function f defining W_2 which is positive on V_+ and negative on $V_- = V \setminus V_+$. If $\operatorname{codim} W_2 > 1$, we are still not lost as we may take a small normal neighbourhood $W_2^\delta \subset V$ of W_2 and replace W_2 by the boundary $W' = \partial W_\delta$. This W' is naturally cooriented and its cooriented Legendrian lift $\dot{\mathcal{L}}' = \dot{\mathcal{L}}_{W'} \subset \ddot{P}T^*(V)$ equals small (of the order δ) perturbation of the lift $\ddot{\mathcal{L}}_{W_2} \subset \ddot{P}T^*(V)$. (Recall that $\ddot{\mathcal{L}}_{W_2}$ equals the double cover of the non-cooriented $\mathcal{L}_{W_2} \subset PT^*(V)$; but if W_2 is *a cooriented hypersurface*, we also have *the cooriented* lift denoted $\dot{\mathcal{L}}_{W_2} \subset \ddot{P}T^*(V)$ which consists of the tangent hyperplanes to W_2 cooriented by W_2.) Notice that we need W_2 to be properly embedded to V whether it is hypersurface or not, while W_1 may be an arbitrary immersed submanifold.

Now, we take the hypersurfaces $\widetilde{W}_2(t) \subset V \times \mathbb{R}^N$ generating $\mathcal{L}(t)$ and observe that their kisses with $W_1 \times \mathbb{R}^N$ (staying still) correspond to crossings of $\mathcal{L}(t)$ with $\mathcal{L}_1 = \mathcal{L}_{W_1}$. Thus we may apply the kissing Morse theory and obtain the inequalities stated in 0.5. In fact, everything reduces here to the ordinary Morse theory. To see this we restrict the function $\widetilde{f} = \widetilde{f}(v, z, t)$ to $W_1 \times \mathbb{R}^N \times [0, 1]$ and let

$$\overline{W}_0 = \{w, z, t \mid \widetilde{f}(w, z, t) = 0\} \subset W_1 \times \mathbb{R}^N \times [0, 1] \,.$$

We may assume (slightly perturbing $\widetilde{f}$ if necessary) that the function $\widetilde{f} \mid W_1 \times \mathbb{R}^N \times [0, 1]$ has no zero critical value and so $\overline{W}_1$ is smooth. Then the kisses correspond to the critical points of the function $(w, z, t) \mapsto t$ on $\overline{W}_1$, and so we back to the old good Morse theory for functions. Of course, we have to check in our cases that this t-function on $\overline{W}_1$ is amenable to the Morse theory and compute the relevant homological invariants of $\overline{W}_1$, but this is a routine exercise in the situations indicated in Theorem 0.5.1 and Theorem 0.5.2(⤩ cup) and 0.5.2(⤩ H_*). In fact, everything follows from the fact that the projection

$$(w, z, t) \mapsto w$$

maps $\overline{W_1}$ with a non-zero degree onto W_1 in the case of Theorem 0.5.2, and onto $W_1 \boxtimes W_2$ in the case of Theorem 0.5.1. QED

Remark 4.2.3 ***On the asymmetry between W and W_0.*** Besides the extra requirement that W, unlike W_0, must be embedded, the isotopies $\mathcal{L}(t)$ and $\mathcal{L}_0(t)$ are somewhat different. Namely, we lift W to $\ddot{P}T^*(W)$ and $\mathcal{L}(t)$ is the isotopy of such lift in $\ddot{P}T^*(V)$ while W_0 should be lifted to $PT^*(V)$ and isotoped there. As $\ddot{P}T^*(V)$ double cover $PT^*(V)$, every isotopy in $PT^*(V)$ lifts to $\ddot{P}T^*(V)$. But not conversely, when we project an isotopy $\mathcal{L}(t)$ from $\ddot{P}T^*(V)$ to $PT^*(V)$ it may develop double points which are not present in $\ddot{P}T^*(V)$. Thus the kind of isotopy we apply to $\mathcal{L}_0$ is more restrictive than the one used for $\mathcal{L}$.

Crossing numbers for isotopies of P-subgraphical submanifolds. The argument from the proof of Theorem 0.5.2 obviously generalizes to the case where $\mathcal{L}_0 \subset PT^*(V)$ is an arbitrary P-subgraphical subvariety, i.e. of the form $\mathcal{L}^\alpha$ for an arbitrary smooth map $\alpha : W \to V$ (see 0.1.1). In fact, one can even benefit from the complexity of the map α by appealing to the corresponding Morse theory (see Section 0.2.7).

If both $\mathcal{L}$ and $\mathcal{L}_0$ are subgraphical, one may pass to $V_\times = V \times V$ and take the P-subgraph of the map $\alpha \times \alpha_0 : W \times W_0 \to V_\times$ where $\alpha : W \to V$ and $\alpha_0 : W_0 \to V$ are the maps defining $\mathcal{L}$ and $\mathcal{L}_0$. The resulting subvariety, say $\mathcal{L}_* = \mathcal{L} * \mathcal{L}_0 = \mathcal{L}^{\alpha\times\alpha_0} \subset PT^*(V_\times)$, can be indeed defined in terms of $\mathcal{L}$ and $\mathcal{L}_0$, thus extending the definition of "$*$" to *all* $\mathcal{L}$ and $\mathcal{L}_0$ in $PT^*(V)$. Then the crossings between $\mathcal{L}(t)$ and $\mathcal{L}_0(t)$ appear as these between $\mathcal{L}(t) * \mathcal{L}_0(t)$ and $\mathcal{L}_\Delta$ in $PT^*(V_\times)$ for the diagonal $\Delta \subset V_\times = V \times V$. Thus we arrive at the situation where $\mathcal{L}_*$ plays the role of $\mathcal{L}_0$ and $\mathcal{L}_\Delta$ the role of $\mathcal{L}$ in $V_\times$ and we may study the problem with a generating hypersurface for $\mathcal{L}_\Delta(t)$ where the relevant isotopy of $\mathcal{L}_\Delta(t)$ in $\ddot{P}T^*(V_\times)$ is the one which compensate for $\mathcal{I}(t) * \mathcal{I}_0(t)$, i.e. $\mathcal{I}_*^{-1}$, such that $\mathcal{I}_*^{-1}(t)(\mathcal{L}(t) * \mathcal{L}_0(t)) = \mathcal{L} * \mathcal{L}_0$ for all $t \in [0,1]$. This takes care of the propositions 0.4.7, 0.5.3, 0.5.4 and 0.5.5.

Remark. This diagonalization trick, as well as the thickening of Δ to $\partial\Delta_\delta$ (which has been already used for general $W \subset V$) need little cut-off adjustments as mentioned in 2.3.2.

4.2.3 Pleating

We develop in this section additional tools needed for the proof of Theorem 0.5.2($\nearrow$ $\searrow$ Mor).

Let N be a contact manifold with a co-oriented contact structure η given by a 1-form μ. Then the cooriented symplectization $\tilde{N}_+$ of N splits:

$$\tilde{N}_+ = N \times \mathbb{R}_+,$$

and the canonical 1-form on $\tilde{N}_+$ can be identified with $s\mu$, $s \in \mathbb{R}_+$.

Given a Legendrian isotopy $\Phi_t : \mathcal{L} \to N$, $t \in [0,1]$, in N, we say that it admits a *Lagrangian lift* to $\tilde{N}_+$ if there exists a Lagrangian *embedding* $\tilde{\Phi} : \mathcal{L} \times [0,1] \to \tilde{N}_+$ such that

$$\pi \circ \tilde{\Phi}(l,t) = \Phi_t(l),\ t \in [0,1], l \in \mathcal{L},$$

where π is the projection $\tilde{N}_+ \to N$. In other words, $\tilde{\Phi}(l,t) = (\Phi_t(l), \sigma(l,t))$ for a positive function σ on the cylinder $\mathcal{L} \times [0,1]$. Let $\bar{\mu}$ denotes the 1-form induced on $\mathcal{L} \times [0,1]$ from μ by the map

$$(l,t) \mapsto \Phi_t(l),\ t \in [0,1],\ l \in \mathcal{L}.$$

Notice that that the fact that Φ_t is Legendrian just means that $\bar{\mu} = \varphi dt$ for a function $\varphi = \varphi(l,t)$ on $\mathcal{L} \times [0,1]$. Then the form induced from $\tilde{\omega} = d(s\mu)$ by the map $\tilde{\Phi}$ equals $d(\sigma\bar{\mu}) = d(\varphi\sigma dt)$.

This form vanishes iff the function $\varphi\sigma$ depends only on t, i. e. it is constant on each slice $\mathcal{L} \times t$, $t \in [0,1]$. On the other hand, given any positive function σ one can always find a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that the map

$$(l,t) \mapsto= (\Phi_t(l), \gamma(t)\sigma(l,t))$$

is an embedding. Thus we have

Criterion 4.2.4 *The isotopy Φ_t admits a Lagrangian lift iff there exists a function $\sigma > 0$ such that the function $\varphi\sigma : V \times [0,1] \to \mathbb{R}$ depends only on t.*

Notice that such function σ do not always exist. If, for instance, φ has a zero point in every slice $\mathcal{L} \times t$ without being identically zero on $\mathcal{L} \times t$ then σ does not exists (provided that $\mathcal{L}$ is connected.)

On the other hand, suppose that $\varphi = \psi\rho$, where ψ does not vanish and ρ depends only on t. Then every function $\sigma = \psi^{-1}(\alpha(t))$ will do the job. Choosing α to have a large positive derivative, which is achieved choosing $\alpha(t) = \exp(\lambda t)$ with a large λ, we ensure that the map $\tilde{\Phi}$ is an embedding.

The next lemma shows that any Legendrian isotopy can be C^0-approximated by a Legendrian isotopy admitting a Legendrian lift.

4.2.5 Pleating Lemma. *Let $\Phi_t : \mathcal{L} \to V$, $t \in [0,1]$, be a Legendrian isotopy, where $\mathcal{L}$ is compact. The isotopy Φ_t can be C^0-approximated by a (smooth) Legendrian isotopy Φ_t^ε such that:*

(1) For each $\varepsilon > 0$ the induced form $\bar{\mu}^\varepsilon$ on $\mathcal{L} \times [0,1]$ decomposes as $\bar{\mu}^\varepsilon = \psi^\varepsilon \rho^\varepsilon dt$, where $\psi^\varepsilon > 0$ and $\rho^\varepsilon = \rho^\varepsilon(t)$, with both ψ^ε and ρ^ε smooth.

(2) $\Phi_t^\varepsilon \underset{\varepsilon \to 0}{\to} \Phi_t$. This means a C^∞-convergence for each $t \in [0,1]$, uniformly in t.

(3) If the original form $\bar{\mu}$ does not vanish on some slices $\mathcal{L} \times t_i$, $i = 1, \dots, k$, as well as in finite number of points $x_1, \dots, x_m$ in $\mathcal{L} \times [0,1]$, then Φ_t^ε can be chosen equal Φ_t near $\bigcup\limits_1^k (\mathcal{L} \times t_i) \cup \{x_1, \dots, x_m\}$.

The Pleating Lemma holds also in the following parametric form.

4.2.6 Pleating for families. *Suppose ${}_\kappa\Phi_t$ is a continuous family of Legendrian isotopies, as in the Pleating lemma, depending on the parameter κ running over a compact space K.*

(1′) *Then the approximation ${}_\kappa\Phi_t^\varepsilon$ satisfying properties* $(1)-(3)$ *from the Pleating lemma can be made simultaneously for all $\kappa \in K$.*

(2′) *If $K = [0,1]$ and the starting isotopy ${}_0\Phi_t$ has $\bar{\mu} = {}_{\kappa=0}\bar{\mu}$ non-vanishing then one can choose the approximation ${}_\kappa\Phi_t^\varepsilon$ equal ${}_\kappa\Phi_t$ for $\kappa = 0$, as well as at the end points $t = 0, 1$ for all $\kappa \in [0,1]$.*

4.2.4 Proof of Theorem 0.5.2 ($\times\!\!\!\!\nearrow\!\!\!\!\searrow$ Mor)

As in the proof for the other cases of this theorem we can reduce to the case when one of the isotopies, say $\mathcal{L}_2$, is constant, i.e $\mathcal{L}_2(t) = \mathcal{L}_{W_2}$ for all $t \in [0,1]$. It still can be arranged that $\mathcal{L}_1(t)$ projects to the right side of $\mathcal{L}_{W_2}$, and, moreover, that $\mathcal{L}_1(1) = W \times t_1$ for some $t_1 > t_0$. We will assume that $t_0 = 0$, $t_1 = 1$. Let us extend the isotopy $\mathcal{L}_1(t)$ from $t \in [0,1]$ to the whole real line $\mathbb{R}$ by setting

$$\mathcal{L}_1(t+1) = \mathcal{L}_1(1-t) + 2t,\ t \in [0,2],$$

$$\mathcal{L}_1(t) = \mathcal{L}_{W_1} \times t,\ t \notin [0,2].$$

Here we denote by $+a$ the translation of $(x,t) \mapsto (x, t+a)$ of V. It is important to notice that $\mathcal{L}_1(t) \cap \mathcal{L}_{W_1} = \emptyset$ for $t \notin [0,2]$.

Observe now that the (positive) symplectization of the standard contact structure $\ddot{\eta} = \{\mu = 0\}$ on $N = \ddot{P}(V)$ equals $T^*(V) \setminus V$ with the canonical symplectic structure. Let $\Phi_t : \mathcal{L} \to \ddot{P}(V)$ be a parametrization of the isotopy $\mathcal{L}_1(t)$, $t \in \mathbb{R}$. It can be easily achieved by a C^∞-small perturbation of the isotopy Φ_t, that the form $\bar{\mu}$ on $\mathcal{L} \times \mathbb{R}$, induced from μ by the map

$$(l,t) \mapsto \Phi_t(l),\ t \in \mathbb{R},\ l \in \mathcal{L},$$

does not vanish on slices $\mathcal{L} \times t$ for $t \notin (0,2)$, and near the intersection points $(l_i, t_i) \in \mathcal{L}_2(t) \cap \mathcal{L}_1(t)$.

Now we apply Pleating Lemma 4.2.5 in order to C^0-approximate the isotopy $\Phi_t\, t \in [0,2]$ by an isotopy, still denoted by Φ_t, which coincides with the original isotopy for $t \notin (0,2)$ and near the intersection points (l_i, t_i), and which admits a Lagrangian lift $\widetilde{\Phi} : \mathcal{L} \times [0,1] \to T^*(V) \setminus V$. The intersection points of $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t) = \Phi_t(\mathcal{L})$ remain unchanged as a result of this perturbation. On the other hand, the constant isotopy $\mathcal{L}_2(t)\,, t \in \mathbb{R}$, lifts to $T^*(V)$ as the conormal bundle L_W of W, and the intersection points between the Lagrangian manifolds L_W and $\widetilde{L} = \widetilde{\Phi}(\mathcal{L} \times \mathbb{R})$ correspond to crossing points between the isotopies $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ (for $t \in [0,1]$). Thus it remains to notice that outside a compact set the Lagrangian submanifold $\widetilde{L}$ coincides with the graphical Lagrangian submanifold

L_0 which is generated by the projection $V = W \times \mathbb{R} \to \mathbb{R}$, and isotopic to L_0 via a compactly supported isotopy. Thus the required inequality $\rlap{$\nearrow$}\searrow$ StabMor follows from 0.3.1(Mor)$_{\pitchfork}$. QED

Remark 4.2.7 It would be desirable to find the proof of the above thorem without the Pleating Lemma. In particular, the application of this lemma does not allow treating of the case of non-transversal kissings. However, so far we were unable to find a way around the pleating argument.

Chapter 5

Family of Lagrangian submanifolds and Pseudo-isotopies

5.1 Pseudo-isotopies

5.1.1 The space $\mathcal{P}(V_0)$

For a possibly non-compact manifold V_0 we set $V = V_0 \times \mathbb{R}$, denote by t the projection $V \to \mathbb{R}$ and consider the space $\mathcal{P}(V_0)$ of functions $V \to \mathbb{R}$ without critical points, which coincide with t at infinity.

Notice that the space $\mathcal{P}(V_0)$ is homotopy equivalent to the space of functions $h : V \to \mathbb{R}$ (comp. 5.1.2 below) without critical points, and which can written as

$$h = t + \varepsilon(t, x), \quad x \in V_0, t \in \mathbb{R},$$

where the function ε has compact support with respect to the variable x, and satisfies the inequality

$$\frac{\partial \varepsilon(t, x)}{\partial t} > -1 \quad \text{for} \quad t \notin [0, 1].$$

Indeed, the function ε can be cut-off to 0 for large values of $|t|$ without affecting critical points of the function $h = t + \varepsilon$.

The space $\mathcal{P}(V_0)$ is called the space of *pseudo-isotopies* of V_0. This terminology is explained by the following

Lemma 5.1.1 (J. Cerf, [10]) *The space $\mathcal{P}(V_0)$ is homotopy equivalent to the space* Diff_0 *of diffeomorphisms $V \to V$ which coincide with the identity on $V_0 \times (-\infty, 0)$ and on $\partial V_0 \times \mathbb{R}$, and which preserve the function t at infinity.*

We denote here by ∂V_0 the complement of a compact set in V_0.

5.1.2 Stabilization

Let us denote by Q_N the quadratic form

$$x_1x_2 + x_3x_4 + \ldots x_{2N-1}x_{2N}$$

on $\mathbb{R}^{2N}$, and by $Q = Q_\infty$ the quadratic form

$$x_1x_2 + x_3x_4 + \ldots$$

on $\mathbb{R}^\infty$.

Let $\mathcal{Q}_N(V_0)$ be the space of functions $V \times \mathbb{R}^{2N}$ *without critical points* which have the form

$$t + \varepsilon(t, x, y) + Q_N(y),\ t \in \mathbb{R},\ x \in V_0,\ y \in \mathbb{R}^N,$$

where the function ε, $t \in \mathbb{R}$ has compact support with respect to x, and

$$\frac{\partial\varepsilon(t, x, y)}{\partial t} > -1 \quad \text{for} \quad t \notin [0, 1].$$

Thus $\mathcal{Q}_0(V_0)$ is homotopy equivalent to $\mathcal{P}(V_0)$. Moreover, we have

Proposition 5.1.2 (see [41]) *The space $\mathcal{Q}_N(V_0))$ is homotopy equivalent to the pseudo-isotopy space $\mathcal{P}(V_0 \times \mathbb{R}^{2N})$.*

The space $\mathcal{Q}_N(V_0)$ embeds into $\mathcal{Q}_{N+1}(V_0)$ by the formula

$$t + \varepsilon(t, x, y_1, y_2, \ldots, y_{2N}) + Q_N(y_1, \ldots, y_{2N}) \mapsto$$

$$t + \varepsilon(t, x, y_1, y_2, \ldots, y_{2N})\varepsilon_0(\frac{y_{2N+1}}{E})\varepsilon_0(\frac{y_{2N+2}}{E}) + Q_{N+1}(y_1, \ldots, y_{2N+2}),$$

where $E = \max|\varepsilon|$ $\varepsilon_0 : \mathbb{R} \to [0, 1]$ is a fixed cut-off function which is equal to 1 on $[-1, 1]$, equal to 0 outside a compact set and has a small derivative, say $|\tilde{\varepsilon}|\mathbb{1}/2$ (comp. 3.1.2 above).

The direct limit $\mathcal{Q}_\infty(V_0) = \underset{\rightarrow}{lim}\, \mathcal{Q}_N(V_0)$ can be interpreted as a space of functions on $\mathbb{R}^\infty$ of the form

$$t + \varepsilon(t, x, y_1, y_2, \ldots, y_{2N})\varepsilon_0(\frac{y_{2N+1}}{E})\varepsilon_0(\frac{y_{2N+2}}{E}) \ldots + Q_\infty(y_1, \ldots).$$

The following Hatcher-Igusa stabilization theorem is a fundamental fact of the pseudo-isotopy theory.

Theorem 5.1.3 (see [40],[44]) *The stabilization map*

$$\text{Stab} : \mathcal{P}(V_0) \to \mathcal{P}_\infty(V_0)$$

induces isomorphism of i-th homotopy groups for $3i + 5 < n$.

Consider now the space $\mathcal{M}_\infty$ which consists of functions on $\mathbb{R}^\infty$ with exactly one critical point which correspond to the critical value 0 and which have the form

$$\varepsilon(y_1,\dots,y_N)\varepsilon_0(\frac{y_{N+1}}{E})\dots + Q_\infty(y),\ y=(y_1,\dots,y_N,\dots)\in\mathbb{R}^\infty,$$

for some N, and where ε has compact support and E and ε_0 is as above. Let $\mathrm{Map}(V_0,\mathcal{M}_\infty)$ be the space of maps $V_0\to\mathcal{M}_\infty$ which take the constant value $Q_\infty\in\ \mathcal{M}_\infty$ outside a compact set. Let us denote by $\mathrm{Path}(\mathrm{Map}(V_0,\mathcal{M}_\infty))$ the space of paths $[0,1]\to\mathrm{Map}(V_0,\mathcal{M}_\infty)$ which begin at the constant map $V_0\mapsto Q_\infty\in\mathrm{Map}(V_0,\mathcal{M}_\infty)$.

Exercise 5.1.4 Show that the space $\mathcal{M}_\infty$ is homotopy equivalent to $\mathcal{P}_\infty(\cdot)$, the stable pseudo-isotopy space of the point.

The Proposition 5.1.2 enables us to define a map

$$\mathrm{Path}(\mathrm{Map}(V_0,\mathcal{M}_\infty))\to\mathcal{Q}_\infty(V_0)$$

as follows. Given a path $\gamma=\gamma(t)(x)(y)\,t\in[0,1]$ we first extend it for $t\in\mathbb{R}$ as independent from t outside $[0,1]$ (this needs slight smoothing near $t=0,1$), and then associate with the extended path, still denoted by γ, the function

$$t+\gamma(t)(x)(y),\ t\in\mathbb{R},\ x\in V_0\,, y\in\mathbb{R}^\infty.$$

This function has no critical points and thus, according to 5.1.2, belongs to the stable pseudo-isotopy space $\mathcal{P}_\infty(V_0)=\mathcal{Q}_\infty(V_0)$.

5.1.3 Spaces of generating functions

Given a fibration $\alpha:U\to V$ and a function $f:U\to\mathbb{R}$ we denote by $\mathcal{L}eg(f)$ the space of Legendrian submanifolds in $\mathrm{Jet}^1(V)$ which coincide with $\underline{\mathcal{L}}_f$ at infinity. Let $\mathcal{F}(f)$ be the space of functions $f:U\times\mathbb{R}^\infty\to\mathbb{R}$ of the form

$$f(u)+Q_\infty(z)+\varepsilon(u,z),\ u\in U,\ z\in\mathbb{R}^\infty,$$

where $Q_\infty:\mathbb{R}^\infty\to\mathbb{R}$ is the quadratic form

$$Q=z_1z_2+z_3z_4+\dots,$$

and ε is of the form

$$\varepsilon=\varepsilon_1(u,z_1,z_2,\dots,z_N)\varepsilon_0(\frac{z_{N+1}}{E})\varepsilon_0(\frac{z_{N+2}}{E})\dots$$

where ε_1 is a compact function on $U\times\mathbb{R}^N$ for some N, $E=\max|\varepsilon_1|$, and ε_0 is a standard cut-off function as in 5.1.

Let $\widetilde{\mathcal{F}}(f)$ be the subspace of those functions $g\in\mathcal{F}(f)$ which transversally generate an *embedded* Legendrian submanifold in $\mathrm{Jet}^1(V)$. Let $\mathcal{G}:\widetilde{\mathcal{F}}(f)\to\mathcal{L}eg(f)$ be the "generating map" which associates with a function $g\in\widetilde{\mathcal{F}}(f)$ the submanifold $\underline{\mathcal{L}}_g\subset\mathrm{Jet}^1(V)$ which it generates.

The following theorem is a reformulation of Theorem 4.1.1 (comp. [64])

Theorem 5.1.5 ***Serre fibration property for generating functions.*** *The map*

$$\mathcal{G} : \widetilde{\mathcal{F}}(f) \to \mathcal{L}\text{eg}(f)$$

is a Serre fibration.

The fibers of the fibration $\mathcal{G}$ over different components of $\mathcal{L}$eg are not, necessarily homotopy equivalent (some of them are, actually, empty!). The fiber over the component of $\underline{\mathcal{L}}_f$ is homotopy equivalent to the space of maps

$$(U, \partial U) \to (\mathcal{M}_\infty, Q_\infty).$$

In view of the claim in the Exercise 5.1.4 and known homotopical information about the stable pseudo-isotopy space of a point (see 0.6 above and [67] and [8]) the homotopy type of the fiber is quite well understood.

5.2 Proof of Injectivity Theorem 0.6.1

Let us specify the above discussion to the situation of 5.1.1. Suppose that V has the form $V = V_0 \times \mathbb{R}$, the function f is the projection $t : V_0 \times \mathbb{R} \to \mathbb{R}$, U coincides with V, and the fibration $U \to V$ is the identity map. Notice that the Legendrian submanifold $\mathcal{L} = \mathcal{L}_f \subset \text{Jet}^1(V) = \text{Jet}^1(V_0) \times T^*(\mathbb{R})$ is the product of the 0-section in $\text{Jet}^1(V_0)$ with a line in the plane $T^*(\mathbb{R})$, parallel to the 0-section.

Let us denote by $\mathcal{L}$eg the connected component of $\mathcal{L}\text{eg}(t)$ which contains $\mathcal{L}$, by $\mathcal{L}\text{eg}_0$ the subset of $\mathcal{L}$eg which consists of Legendrian submanifolds which do not intersect the 0-section in $\text{Jet}^1(V)$, and by $\widetilde{\mathcal{F}}_0$ the pre-image $\mathcal{G}^{-1}(\mathcal{L}\text{eg}_0)$.

All the fibers of the Serre fibration

$$\mathcal{G}_0 = \mathcal{G}|_{\widetilde{\mathcal{F}}_0} : \widetilde{\mathcal{F}_0} \to \mathcal{L}\text{eg}_0$$

are homotopy equivalent to the space $\mathcal{G}^{-1}(\mathcal{L})$, which can be canonically identified with $\Omega V_0(\text{Map}(V_0, \mathcal{M}_\infty))$, the space of based loops of the space $\text{Map}(V_0, \mathcal{M}_\infty)$ Here we denote by $\text{Map}(V_0, \mathcal{M}_\infty)$ the space of maps $(V_0, \partial V_0) \to (\mathcal{M}_\infty, Q_\infty)$ with the origin at the point q_∞ corresponding to the constant map $V_0 \mapsto Q_\infty$.

The exact homotopy sequence of the fibration $\mathcal{G}_0 : \widetilde{\mathcal{F}}_0 \to \mathcal{L}\text{eg}_0$ has the form:

$$\ldots \to \pi_k(\Omega(\text{Map}(\mathcal{M}_\infty, V_0))) \xrightarrow{\gamma^*} \pi_k(\widetilde{\mathcal{F}}_0) \xrightarrow{\mathcal{G}_0^*} \pi_k(\mathcal{L}\text{eg}_0) \to \ldots \quad ,$$

where γ is the inclusion map $\Omega(\text{Map}(\mathcal{M}_\infty, V_0)) \to \widetilde{\mathcal{F}}_0$.

Let us recall that according to Cerf Lemma 5.1.1 the pseudo-isotopy space $\text{Diff}_0 = \mathcal{P}(V_0)$ can be viewed as the space of functions on V which coincide with t at infinity and which have no critical points. The total space $\widetilde{\mathcal{F}}_0$ is a subspace of the stable pseudo-isotopy space $\mathcal{Q}_\infty(V_0) = \mathcal{P}_\infty(V_0)$ defined in 5.1 above. According to 5.1.2 there is an inclusion $l : \mathcal{F}_0 \to \mathcal{P}_\infty(V_0)$, and the image of the stabilization map $\text{Stab} : \mathcal{P}(V_0) \to \mathcal{P}_\infty(V_0)$ is contained in $l(\widetilde{\mathcal{F}}_0) \subset \mathcal{P}_\infty(V_0)$. Thus the map Stab can be factored as

$$\text{Diff}_0 = \mathcal{P}(V_0) \xrightarrow{k} \widetilde{\mathcal{F}}_0 \overset{l}{\hookrightarrow} \mathcal{P}_\infty(V_0)\,.$$

According to Theorem 5.1.3 we have an isomorphism

$$\mathrm{Stab}_* = l_* \circ k_* : \pi_i(\mathcal{P}(V_0)) \to \pi_i(\mathcal{P}_\infty(V_0))$$

for $3i + 5 < n = \mathrm{dim} V_0$. In particular, in this range of dimensions the homomorphism k_* is injective and we have

$$\mathrm{Im} k_* \cap \mathrm{Ker} l_* = \{0\}.$$

On the other hand the composition

$$\Omega(\mathrm{Map}(V_0, \mathcal{M}_\infty)) \xrightarrow{\gamma} \widetilde{\mathcal{F}}_0(V_0) \xrightarrow{l} \mathcal{P}_\infty(V_0))$$

can be factored as

$$\Omega V_0(\mathrm{Map}(V_0, \mathcal{M}_\infty)) \xrightarrow{m} \mathrm{Path}(\mathrm{Map}(V_0, \mathcal{M}_\infty)) \xrightarrow{n} \mathcal{P}_\infty(V_0)\,,$$

where $\mathrm{Path}(\mathrm{Map}(V_0, \mathcal{M}_\infty))$ is the space of paths in $\mathrm{Map}(V_0, \mathcal{M}_\infty))$ beginning at the constant map $q_\infty : V_0 \to Q_\infty \in \mathcal{M}_\infty$.

Since the space $\mathrm{Path}(\mathrm{Map}(V_0, \mathcal{M}_\infty))$ is contractible, the composition $l \circ \gamma$ is homotopic to a point, and thus we have

$$\mathrm{Im}\gamma_* \subset \mathrm{Ker} l_*.$$

Thus,

$$\mathrm{Im} k_* \cap \mathrm{Im}\gamma_* = \mathrm{Im} k_* \cap \mathrm{Ker}(\mathcal{G}_0)_* = \{0\},$$

i.e. the composition $\mathcal{G}_0 \circ k : \mathcal{P}(V_0) \to \mathcal{L}\mathrm{eg}$ induces an injective homomorphism on homotopy groups in the required range of dimensions. QED

Concluding remark

The relations between the topology of spaces of Lagrangian and Legendrian embeddings and the pseudo-isotopy theory goes far beyond the Injectivity theorem 0.6.1 and it is not yet properly understood. Here is an example.

Suppose that a (finitely presented) group π has a non-trivial second Whitehead group $\mathrm{Wh}_2(\pi)$, see [41]. Given a manifold V_0 with $\pi_1(V_0) = \pi$ there exists (see [40]) a surjective homomorphism

$$H : \pi_0(\mathcal{P}(V_0)) \to \mathrm{Wh}_2(V_0)\ .$$

Let a function $f \in \mathcal{P}(V_0)$ represent an element $[f] \in \pi_0(\mathcal{P}(V_0))$ with non-trivial invariant $H([f]) \in \mathrm{Wh}_2(V_0)$. Let $f_s : V = V_0 \times \mathbb{R} \to \mathbb{R}$, $s \in \mathbb{R}$ be a family of functions which coincide with the projection $t : V \to \mathbb{R}$ on $V_0 \times (\mathbb{R} \setminus [-1, 1))$, and such that $f_s = t$ for $s < -1$ and $f_s = f$ for $s > 1$. The function $F : \mathbb{R} \times (V_0 \times \mathbb{R})$, given by the formula

$$F(s, v) = f_s(v)\ , s \in \mathbb{R}\,, v \in V,$$

generates a Legendrian submanifold $\mathcal{L} \in \mathrm{Jet}^1(\mathbb{R}) = \mathbb{R}^3$. In other words the front-projection of $\mathcal{L}$ is the Cerf diagram of the family f_s, $s \in \mathbb{R}$. It can be easily arranged that $\mathcal{L}$ is conected. Then Theorem 5.1.5 implies that

The Legendrian knot $\mathcal{L}$ is not trivial, i. e. it is not Legendrian isotopic to the standard Legendrian knot $\mathcal{L}_0$ in $\mathbb{R}^3$ represented by the simplest front with 2 *cusps and without self-intersections.*

Indeed, if there exists a Legendrian isotopy $\mathcal{L}_\theta, \theta \in [0,1]$, connecting $\mathcal{L}_0$ with $\mathcal{L}_1 = \mathcal{L}$, then according to Theorem 4.1.1 there exists a covering homotopy $F_\theta : \mathbb{R} \times V \times \mathbb{R}^N$ with $\mathcal{L}_\theta = \underline{\mathcal{L}}_{F_\theta}$ of the form

$$F_\theta(s,v,z) = F(s,v) + Q(z) + \varepsilon_\theta(s,v,z),\ s \in \mathbb{R}, v \in V, z \in \mathbb{R}^N,$$

where $Q : \mathbb{R}^N \to \mathbb{R}$ is a non-degenerate quadratic form and ε_θ is a family of compact functions. For $|s| \geq 1$ the functions ${}^sF_\theta(v,z) = F_\theta(s,v,z)$ have no critical points. According to 5.1.2 these functions can be viewed as elements of the space $\mathcal{Q}_N(V_0) = \mathcal{P}(V_0 \times \mathbb{R}^N)$, and with the help of Theorem 5.1.3 we conclude that $H([{}^1F_1]) \neq 1$, while $H([{}^{-1}F_1]) = 1$. But according to [40] these two functions cannot be joined by a homotopy $H([{}^tF_1])$ whose Cerf diagram is the the front of the trivial Legendrian knot $\mathcal{L}_0$. QED

Similarly, non-trivial elements of higher Whitehead groups serve as obstructions for the triviality of Legendrian knots of dimension > 1.

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