Morse Spectra, Homology Measures, Spaces of Cycles and Parametric Packing Problems.

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Contents

Abstract

An "ensemble" $\Psi = \Psi(X)$ of (finitely or infinitely many) particles in a space X, e.g. in the Euclidean 3-space, is customary characterised by the set function

$$U \mapsto ent_U(\Psi) = ent(\Psi_{|U}),\ U \subset X,$$

that assigns the *entropies* of *the U-reductions* $\Psi_{|U}$ *of* Ψ, to all bounded open subsets $U \subset X$. In the physicists' parlance, this entropy is

"the logarithm of the number of the states of $\mathcal{E}$
that are effectively observable from U",

This "definition", in the context of mathematical statistical mechanics, is translated to the language of the measure/probability theory.[1]

But what happens if *"effectively observable number of states"* is replaced by

"the number of effective/persistent degrees of freedom
of ensembles of moving particles"?

We suggest in this paper several mathematical counterparts to the idea of *" persistent degrees of freedom "* and formulate specific questions, many of which are inspired by Larry Guth's results and ideas on the Hermann Weyl kind of asymptotics of *the Morse (co)homology spectra* of the *volume energy function* on the spaces of *cycles* in balls.[2] And often we present variable aspects of the same idea in different sections of this paper.

Hardly anything that can be called "new theorem" can be found in our paper but we reshuffle many known results and expose them from a particular angle. This article is meant as an introductory chapter to something yet to be written with much of what we present here extracted from my yet unfinished manuscript *Number of Questions*.

1 Overview of Concepts and Examples.

We introduce below the idea of *"parametric packing"* and of related concepts which are expanded in detail in the rest of the paper.

A. Let X be a topological space, e.g. a manifold, and I is a countable index set that may be finite, especially if X is compact.

A collection of I-tuples of non-empty open (sometimes closed) subsets $U_i \subset X$, $i \in I$, is called *a packing* or an *I-packing* of X if these subsets *do not intersect.*

Denote by $\Psi(X; I)$ the space of these packings with some natural topology, where, observe there are several candidates for such a topology if X is *noncompact.*

[1]See: Lanford's *Entropy and equilibrium states in classical statistical mechanics, Lecture Notes in Physics, Volume 20, pp. 1-113, 1973* and Ruelle's *Thermodynamic formalism : the mathematical structures of classical equilibrium statistical mechanics*, 2nd Edition, Cambridge Mathematical Library 2004, where the emphasis is laid upon (discrete) *lattice* systems. Also a *categorical rendition* of Boltzmann-Shannon entropy is suggested in *"In a Search for a Structure, Part 1: On Entropy"*, www.ihes.fr/~gromov/PDF/structre-serch-entropy-july5-2012.pdf

[2]*Minimax problems related to cup powers and Steenrod squares*, Geometric and Functional Analysis, 18 (6), 1917-1987 (2009).

B. *Homotopies Constrained by Inradii and Waists.* We are interested in the homotopy, especially (co)homology, properties of subspaces $\mathcal{P} \subset \Psi(X; I)$ defined by imposing *lower bounds* on the sizes of U_i. where the two such invariants of $U \subset X$ we shall be often (albeit sometimes implicitly) use in this paper are the *the inradius of* U and the *the* k*-waist of* U, $k = 1, 2, ..., dim(X) - 1$, defined later on in the metric and the symplectic categories.

C. *Metric Category and Packings by Balls.* Let X be a metric space, let $r_i \geq 0$ be non-negative numbers and take metric balls in X of radii $R_i \geq r_i$ for U_i. Packings by such balls are traditionally called *sphere packings* where one is especially concerned with packing homogeneous spaces (e.g. spheres and Euclidean spaces) by *equal* balls.

The corresponding space $\mathcal{P} = \mathcal{P}(X; \{\geq r_i\}_{i \in I})$ naturally embeds into the *Cartesian power space*

$$X^I = \underbrace{X \times X \times ... \times X}_{I}$$

of I-tuples of points X where it is distinguished by the inequalities

$$dist(x_i, x_j) \geq d_{ij} = r_i + r_j, \ \ i, j \in I, i \neq j,$$

and where, observe all these spaces with $d_{ij} > 0$ lie in the Cartesian power space X^I minus diagonals,

$$X^I \smallsetminus \bigcup_{i,j \in I} Diag_{ij} \text{ for } Diag_{ij} \subset X^I \text{ defined by the equations } dist(x_i, x_j) = 0.$$

And if X is Riemannian manifold with *the convexity (injectivity?) radius* $\geq R$, then, clearly, the inclusion

$$\mathcal{P}(X; \{\geq r_i\}_{i \in I}) \hookrightarrow X^I \smallsetminus \bigcup_{i,j \in I} Diag_{ij}$$

is *a homotopy equivalence* for $\sum_{i \in I} r_i < R/2$; moreover, if all r_i are mutually equal, this homotopy equivalence is *equivariant* for the permutation group that acts on I and thus on X^I and on $X^I \smallsetminus \bigcup_{i,j \in I} Diag_{ij} \subset X^I$ and $\mathcal{P}(X; \{\geq r_i = r\}) \subset X^I \smallsetminus \bigcup_{i,j \in I} Diag_{ij}$.

The packings spaces *covarinatly functorially* behave under *expanding* maps between metric spaces $X \to Y$.

But *contravariant* functoriality under *contracting*, i.e. *distance decreasing*, maps $f : X \to Y$ needs the following extension of the concept of ball packings to I-tuples of *subsets* (rather than points) $V_i \subset X$ instead of points $x_i \in X$.

D. *Packings by Tubes.* These are I-tuples of closed subsets $V_i \in X$, such that mutual distances[3] between them satisfy $dist(V_i, V_j) \geq d_{ij}$.

E. *Packing by Cycles.* The above becomes interesting if all $V_i \subset X$ support given *nonzero* homology classes h_i of dimension k, $k = 0, 1, ...,$ in X, or if they support k-cycles some of which are linked in X, either individually or "parmetrically" (compare **P** below and section 18)

F. *Packings by Maps.* This is yet another variation of the concept of "packing". Here *subsets* $U_i \subset X$ are replaced by *maps* $\psi_i : U \to X$, where in general, the domain U of ψ_i may depend on $i \in I$.

[3]Recall that $dist(V_1, V_2)$ between two subsets in a metric space X is defined as the *infimum* of the distances between points $x_1 \in V_1$ and $x_2 \in V_2$.

Now, "packing by ψ_i" means packing by the images of these maps, i.e. these images should not intersect with specific "packing conditions" expressed in terms of geometry of U and of these maps.

For instance, if X and U are equidimensional Riemannian manifolds on may require the maps ψ_i to be expanding. Or ψ_i may belong to a particular category (pseudogroup) of maps.

G. *Symplectic Packings by Balls.* Here $X = (X, \omega)$ is a $2m$-dimensional symplectic manifold, U_i are balls of radii R_i in the standard symplectic space $\mathbb{R}^{2n} = (\mathbb{R}^2)^n$ and symplectic packings are given by I-tuples of symplectic embeddings $U_i \to X$ with disjoint images.

Another attractive class of symplectic packings is that by polydiscs and by *R-tubes around Lagrangian submanifolds* in X.

H. *Holomorphic Packings.* These make sense and look interesting for packing by holomorphic maps $U_i \to \mathbb{C}^n$, $n \geq 2$, with *Jacobians one*, in particular, by *symplectic holomorphic* maps for n even, but I have not thought about these.

I. *Essential Homotopy.* This is the part of the homotopy, e.g. cohomology, structure of a *geometrically* defined subspace $\mathcal{P} \subset \Psi$ that comes from the ambient space Ψ, where Ψ itself is defined in a purely *topological* terms.

One think of Ψ as the background that supports the *geometric information* on $\mathcal{P}$ written in the *homotopy theoretic* language of Ψ.

This information concerns the *relative homotopy size* of $\mathcal{P}$ in Ψ, often expressed by particular (quasi)numerical invariants, such, for instance, as *homotopy hight, cell numbers, cohomology valued measures.*

J. *Example: Packings by Two Balls.* The space $\mathcal{P}(R) \subset X \times X$, $R \geq 0$, of packings of a metric space X by two R-balls is defined by the inequality

$$X \times X \supset \mathcal{P}(R) =_{def} \{x_1, x_2\}_{dist(x_1,x_2) \geq 2R},$$

where, observe the distance function d on $X \times X$ is related to the distance in $X \times X$ to the diagonal $X_{dia} = Diag_{12} \subset X \times X$ by

$$d = dist_X(x_1, x_2) = \sqrt{2} \cdot dist_{X \times X}((x_1, x_2), X_{dia}).$$

The *algebraic topology* of the distance function d on $X \times X$, more specifically the (co)homologies of the *inter-levels*

$$d^{-1}[R_1, R_2] \subset X \times X,$$

(that carries, in general, more *geometric* information about X than what we call "essential homotopy" of these subsets) can be thought of (be it essential or non-essential) as *a cohomology valued "measure-like" set function* on the real line, namely

$$[a, b] \mapsto H^*(d^{-1}[a, b]) \text{ for all segments } [a, b] \subset \mathbb{R}.$$

Exceptionally, e.g. if X is a symmetric space, the distance function is "*Morse perfect*"[4] : all of homotopy topology (e.g. homology) of f is "essential", quite transparent instances of which are *projective spaces over* $\mathbb{R}, \mathbb{C}$ *and* $\mathbb{H}$ as well

[4] "Morse perfect" functions do not have to be "textbooks Morse": they may be *non-smooth* and have *positive dimensional sets of critical points.*

flat tori, where these functions obviously induced from (Morse perfect distance) functions $d_0 : X \to \mathbb{R}$ for $d_0(x) = dist(x, x_0)$.

On the other hands much of geometrically significant *geometric information* carried by *the topology* of the distance functions on manifolds of *negative curvature,* does not quite conform to such concept of "essential". (The simplest part of the information encoded by d, that is represented by the set of the lengths of *undistorted*[5] closed geodesics in X, is homotopically, but not necessarily homologically, essential.)

K. *Permutation Symmetries.* The Cartesian power space X^I is acted upon by the symmetric group $\mathbb{S}_N = Sym(I)$, $N = card(I)$ and that this action is *free* in the complement to the diagonals.

Let G be a subgroup in $\mathbb{S}_N = Sym(I)$, $N = card(I)$, e.g. $G = \mathbb{S}_N$, and let $\mathcal{P} \subset$ be a subspace invariant under this action. We are especially interested in those homotopy characteristics of $\mathcal{P}$ that are encoded by the kernel of the cohomology homomorphism $\kappa^* : H^*(BG) \to H^*(\mathcal{P}/G)$ for the classifying map κ from the *quotient space*[6] $\mathcal{P}/G$ to the classifying space BG of the group G,

$$\kappa : \mathcal{P}/G \to BG.$$

Some of the questions we want to know the answer to are as follows:[7]

L. Let F_0 be a G-equivariant map from a topological space S with an action of a group $G = G(I) \subset \mathbb{S}_N$, $N = card(I)$, on it, to the Cartesian power space X^I,

$$F_0 : S \to X^I,$$

and let $[F_0]_G$ denote the equivarinat homotopy class of this map.

What is the maximal radius $R = R(S, [F_0])$, such that F_0 admits an equivariant homotopy to an equivariant map F from S to the space $\mathcal{P}(X; I, R) \subset X^I$, of I-packings of X by balls of radii R,

$$F : S \to \mathcal{P}(X; I, R) \subset X^I, \ F \in [F_0]_{\mathbb{S}_N}?$$

What is the supremum $R_{max}(k)$ of these $R = R(S, [F_0])$, over all $(S, [F_0])$ with $dim(S) = k$?

In other words, what is the maximal $R = R_{max}(k) = R(X, N, k)$, such that every k-dimensional G-invariant subset in the complement of the diagonals in X^I (where the action of $\mathbb{S}_N \supset G$ is free) admits an equivariant homotopy to $\mathcal{P}(X; I, R) \subset X^I$?

M. Let

$$K^* = K^*(N, \varepsilon) \subset H^*(BG; \mathbb{F}_p), \ \mathbb{F}_p = Z/p\mathbb{Z}, \ \text{for } G = G(N), N = card(I)$$

[5]A submanifold Y in a Riemannian manifold X is called *undistorted* if the distance in Y associated with the *induced Riemannian* structure coincide with the *restriction of the distance function* from $X \supset Y$. For instance the shortest non-contractible curve in X is undistorted.

[6]Our $\mathcal{P}$ is contained in the complement to the diagonals in X and, hence, the action of G is free on $\mathcal{P}$; otherwise, we would replace the quotient space $\mathcal{P}/G$ by the *homotopy quotient* $\mathcal{P}/\!\!/G$.

[7]The cohomology of $\mathbb{S}_N$ are well understood, (e.g. see [1]) but I shamefully failed to extract a rough estimate of the ranks $rank(H^i(\mathbb{S}_N, \mathbb{F}_p)$ from what I read. But even if these ranks are bound by $const^N$, the minimal number of cells in the optimal cell decorposition of the classifying space $B\mathbb{S}_N$ must(?) grow roughly as $N!$ for $N \to \infty$.

be the kernels of the homomrphisms

$$\kappa\star = \kappa^*_{\mathbb{F}_p} : H^*(BG;\mathbb{F}_p) \to H^*(\mathcal{P}/G;\mathbb{F}_p),$$

where $\mathcal{P} = \mathcal{P}(X;I,\varepsilon)$ is the space of packings of a given Riemannian manifold X, e.g. of the unit sphere S^l or the torus $\mathbb{T}^l$, by N balls of radius ε.

($\mathbf{M}_1$) *What is the behaviour of the (graded) ranks of these kernels* $K^*(N,\varepsilon)$ *as functions of* ε*?*

($\mathbf{M}_2$) *What is the asymptotic behaviour of these* $rank(K^*(N,\varepsilon))$ *for* $N\to\infty$ *and* $\varepsilon\to 0$, *where a particular case of interest is that of*

$$\varepsilon = \varepsilon_N = const\cdot N^\alpha \text{ for some } \alpha < 0?$$

N. *Packing Energies and Morse Packing Spectra.* The space $\mathcal{P} = \mathcal{P}_\varepsilon = \mathcal{P}(X;I,\varepsilon)$ can be seen as *the sublevel* of a suitable "*energy function*" E on the ambient space $\Psi = X^I \supset \mathcal{P}_\varepsilon$, where any monotone decreasing function in

$$\rho(\psi) \text{ for } \psi = \{x_i\} \text{ and } \rho(\psi) = \min_{x_i\neq x_j} dist(x_i,x_j)$$

will do[8] for $E(\psi_1,\psi_2) = E(\psi_1)+E(\psi_2)$ and where simple candidates for such functions are

$$E(\psi) = \frac{1}{\rho(\psi)} \text{ or } E(\psi) = -\rho(\psi),$$

or, that seems most appropriate from a certain perspective,

$$E(\psi) = -\log\rho(\psi).$$

Notice that $\rho(\psi)$ equals $\sqrt{2}$ times the distance from $\{x_i\}\in X^I$ to the union of the diagonals $Diag_{ij}\subset X^I$ that are defined by the equations $x_i = x_j$,

$$\rho(\psi) = \sqrt{2}\cdot\min_{ij} dist_{X^I}(\psi, Diag_{ij}),\ \ \psi = \{x_i\}.$$

$\mathbf{N}^\star$**.** We are predominantly interested in *the homotopy significant (Morse) spectra* of such energy functions $E:\Psi\to\mathbb{R}$, on topological spaces Ψ, where such a spectrum is the set of those values $e\in\mathbb{R}$ where the homotopy type of the sublevel $E^{-1}(-\infty,e]$ undergoes an *irreversible change* (precise definitions are given in section 4) and the above ($\mathbf{M}_1$) concerns such changes that are recored by the variation of the kernels $K^*(N,\varepsilon)$.

O. *"Duality" between Homology Spectra of Packings and of Cycles.* Evaluation of the homotopy (or homology) spectrum of packings, in terms of the above **A**, needs establishing two opposite geometric inequalities, similarly how it goes for the spectra of *Laplacians* associated to *Dirichlet's energies.*

$\mathbf{O_I}$:*Upper Bounds on Packing Spectra.* Such a bound for packings a manifold X by R-balls, means an inequality $R\le\rho_{up}$ for some $\rho_{up} = \rho_{up}(S,F_0)$ (or several

[8]The role of real numbers $\mathbb{R}$ here reduces to indexing the subsets $\Psi_r\subset\Psi$, $r\in\mathbb{R}$, according to their order by inclusion: $\Psi_{r_1}\subset\Psi_{r_2}$ for $r_1\le r_2$.
In fact, our "spectra" make sense for functions with values in an arbitrary *lattice* (that is a partially ordered set that admits inf and sup), while *additivity*, that is the most essential feature of the physical energy, becomes visible only for spaces Ψ that split as $\Psi = \Psi_1\times\Psi_2$.

such inequalities for various S and F_0), that would guarantee that a map $F_0 : S \to X^I$ *is homotopic (or at least homologous)* to a map with image in the packing space $\mathcal{P}(X; I, R) \subset X^I$.

This, in all(?) known examples, is achieved by *explicit constructions* of specific "homotopically (or homologically) significant" packings families $P_s \in \mathcal{P}(X; I, R)$, $s \in S$, for $R \geq \rho_{up}$.

$\mathbf{O_{II}}$:*Lower Spectral Bounds.* Such a bound $R \geq \rho_{low} = \rho_{low}(S, F_0)$ is supposes to signify that $F_0 : S \to X^I$ is *not homotopic (or not even homologous)* to a map $S \to \mathcal{P}(X; I, R) \subset X^I$.

All (?) known bounds of this kind are obtained by *(parametric) homological localisation* that is by confronting such maps F_0, think of them as *families of $N = card(I)$ moving balls in X parametrised by S*, with *families of cycles* in X where the two families have a nontrivial (co)homology pairing between them.

A simple, yet instructive, instance of this is where:

there is *a (necessarily non-zero) homology class $h_\circ \in H_k(X)$* for some $k = 1, 2, ..., n = dim(X)$, such that

the image $h_S \in H_*(X^I)$ of some homology class from $H_*(S)$ under the induced homomorphism

$$(F_0)_* : H_*(S) \to H_*(X^I)$$

have *non-zero homology intersection with the power class $h_\circ^{\otimes I} \in H_*(X^I)$*,

$$h_S \frown h_\circ^{\otimes I} \neq 0.$$

Obviously, in this case,

if a map $F_0 : S \to X^I$ is homotopic to a map into the packing space $\mathcal{P}(X; I, R) \subset X^I$, then every closed subset $Y \subset X$ that supports the class $h_\circ$ admits a packing by N-balls $U_i \subset Y$, $i \in I$, $N = card(I)$, of radii R for the restriction of the distance function from X to Y. Consequently,

$$N \cdot R^k \leq const_X < \infty.$$

Example. Let X be a Riemannian product of two closed connected Riemannian manifolds, $X = Y \times Z$, let $S = Z^I$ and $F_0 : Z^I = (Z \times y_0)^I \subset X^I$, $y_0 \in Y$, be the tautological embedding. If this $S \subset X^I$ can be moved by a homotopy in X^I to $\mathcal{P}(X; I, R) \subset X^I$, then Y can be packed by N-balls of radius R.

Notice that the converse is also true in this case. In fact, if balls $U_{y_i}(R) \subset Y$ pack Y, then the Cartesian product

$$S = \bigtimes_{i \in I} (Z \times y_i) \subset X^I$$

is contained in $\mathcal{P}(X; I, R)$.

In general, when cycle moves, this kind of argument, besides suitable nontrivial (co)homology pairing, needs lower bounds on spectra of the *volume-energies* in spaces of k-cycles, in particular lower bounds on *k-waists* of our manifolds that correspond to the bottoms of such spectra.

In particular, such bounds on *symplectic waists*, are used in the symplectic geometry for proving *non-existence of individual packings, as well as of multi parametric families of certain symplectic packings.*

P. *Packings by Tubes around Cycles.* The concepts of spaces of packings and those of cycles can be brought to the common ground by introducing the space of I-tuples of *disjoint k-cycles* V_i in X and of (the homotopy spectrum of) the function $E\{V_i\}$ that would somehow incorporate $vol_k(V_i)$ along with $\log dist_X(V_i, V_j)$.[9]

(One may replace the distances $dist_X(V_i, V_j)$ in X by distances in *the flat metric* in the space of cycles, where $dist_{flat}(V_i, V_j)$ is defined as the $(k+1)$-volume of *the minimal* $(k+1)$*-chain* between V_i and V_j in X, but this would lead to a quite different picture.)

Q. *Spaces of Infinite Packings.* If X is a non-compact manifold, e.g. the Euclidean space $\mathbb{R}^n$, then there are many candidates for THE SPACE OF PACKINGS, all of which are infinite dimensional spaces with infinite dimensional (co)homologies, where this infinities may be (partly) offset by actions of infinite groups on these spaces.

For instance, spaces $\mathcal{P}$ of packings of $\mathbb{R}^n$ by countable sets of R-balls $U_i(R) \subset \mathbb{R}^n$ are acted upon by the isometry group $iso(\mathbb{R}^n)$ that, observe , commute with the action of the group $Sym(I)$ of bijective transformations of the (infinite countable) set I.

The simplest(?) instance of an interesting infinite packing space $\mathcal{P} = \mathcal{P}(\mathbb{R}^n; I, R)$ is where $I = \mathbb{Z}^n \subset \mathbb{R}^n$ is the integer lattice, where the ambient space Ψ equals the space of *bounded displacements* of $\mathbb{Z}^n \subset \mathbb{R}^n$ that are maps $\psi : i \mapsto x_i \in \mathbb{R}^n$, such that

$$dist(i, x_i) \le C = C(\psi) < \infty \text{ for all } i \in \mathbb{Z}^n$$

and where $\mathcal{P} \subset \Psi$ is distinguished by the inequalities $dist(x_i, x_j) \ge 2R$.

The essential part of the infinite dimensionality of this Ψ comes from the infinite group $\Upsilon = \mathbb{Z}^n \subset iso(\mathbb{R}^n)$, that acts on it. In fact, Ψ naturally (and Υ-equivarinatly) imbeds into the union of compact infinite product spaces

$$\Psi \subset \bigcup_{C>0} B(C)^{\Upsilon},$$

where $B(C) \subset \mathbb{R}^n$ is the Euclidean ball of radius C with the centre at the origin, and where several entropy-like topological invariants, such as the *mean dimension* $dim(\mathcal{P}) : \Upsilon$ and *polynomial entropy* $H^*(\mathcal{P}) : \Upsilon$, are available.

The quotient space of the above space Ψ, or rather of this Ψ minus the diagonals, by the infinite "permutation group"[10] $Sym(I)$ consists of the set of certain discrete subsets $\Delta \subset \mathbb{R}^n$. These Δ has the property that the intersections of it with all bounded open subsets $V \subset \mathbb{R}^n$ satisfy *the uniform density condition*,

$$\text{✽} \qquad card(V \cap \Delta) - card(V \cap \mathbb{Z}^n) \le const_\Delta \cdot vol(U_1(\partial V))$$

where $U_1(\partial V) \subset \mathbb{R}^n$ denotes the union of the unit balls with their centres in the boundary of V.

[9]One may think of these V_i as images of k-manifolds mapped to X that faithfully corresponds to cycles with $\mathbb{Z}_2$-coefficients.

[10]One only needs here the subgroup of $Sym(I = \mathbb{Z}^n)$ that consists of *bounded* bijective displacements $\mathbb{Z}^n \to \mathbb{Z}^n$, i.e. where these "displacements" $i \mapsto j$ satisfy $dist(i, j) \le C < \infty$.

But there are by far more uniformly dense (i.e. that satisfy ✽) subsets than the images of $\mathbb{Z}^n$ under bounded displacement.

In fact, it is far from clear what topology (or rather homotopy) structure should be used in the space of uniformly dense subsets, that, for instance, would render this space *(path)connected.*

R. *On Stochasticity.* The traditional probability may be brought back to this picture, if, for instance, packing spaces are defined by the inequalities

$$dist(x_i, x_j) \geq \rho(i, j),$$

where $\rho(i, j)$ assumes two values, $R_1 > 0$ and $R_2 > R_1$, taken independently with given probabilities $p(i, j) = p(i - j)$, $i, j \in I = \Upsilon = \mathbb{Z}^n$.

S. *From Packings to Partitions and Back.* An I-packing P of a metric space X by subsets U_i can be canonically extended to the corresponding *Dirichlet-Voronoi partition*[11] by subsets $U_i^+ \supset U_i$, where each $U_i^+ \subset X$, $i \in I$, consists of the points $x \in X$ nearest to U_i, i.e. such that

$$dist(x, U_i) \leq dist(X, U_j),\ j \neq i.$$

If, for instance, X a convex[12] Riemannian space with constant curvature and if all U_i are convex then U_i^+ are *convex polyhedral sets.*

Conversely, convex subsets $U \subset X$ can be often *canonically shrunk* to single points $u \in U$ by families of *convex* subsets, $U_t \subset U$, $0 \leq t \leq 1$,

$$\text{where } U_0 = U,\ U_1 = u \text{ and } U_{t_2} \subset U_{t_1} \text{ for } t_2 \geq t_1\ .$$

For instance, if X has non-positive curvature, such a shrinking can be accomplished with the *inward equidistante deformation* of the boundary ∂U.

This shows, in particular, that the space of convex I-partitions (as well as of convex I-packings) of a convex space X of constant curvature is $\mathbb{S}_N$-*equivarinatly,* $N = card(I)$, *homotopy equivalent* to the space of I-tuples of distinct points $x_i \in X$. (The case of negative curvature reduces to that of the positive one via projective isomorphisms between bounded convex spaces of constant curvatures.)

Partitions of metric spaces, especially convex ones whenever these are available, reflects finer aspects the geometry of X than sphere packings. For instance, families of convex partitions obtained by consecutive division of convex sets by hyperplanes are used for sharp evaluation of *waists* of spheres as we shall explain later on.

On the other hand, a typical Riemannian manifold X of dimension $n \geq 3$ admits only approximately convex partitions (along with convex packings), where the geometric significance of these remains problematic.

T. *Composition of Packings, (Multi)Categories and Operads.* If U_i pack X then packings of U_i, $i \in I$, by U_{ij}, $j \in J_i$, define a packing of X by all these U_{ij}.

Thus, for instance, in the case of X and all U being Euclidean balls, this composition defines a *topological/homotopy operad* structure in the space of packings of balls by balls.

[11]Here, "*I-Partition*" means a covering of X by closed subsets $V_i \subset X$, $i \in I$, with non-empty non-intersecting interiors, where we often tacitly assume certain regularity of the boundaries of these U_i.

[12]"*Convex*" means that every two points are joint by a unique geodesic.

The significance of such a structure is questionable for round ball packings, especially for those of *high density*, since composed packings constitute only a small part of the space of all packings.

But packing *of cubes by smaller cubes* and *symplectic* packings of balls by smaller balls seem more promising in this respect, since even quite dense packings in these cases, even partitions, may have a significant amount of (persistent) degrees of freedom.[13]

U. *On Faithfulness of the "Infinitesimal Packings" Functor.* The (multi)category structures in spaces of packings define, in the limit, similar structures in spaces of packings of spaces X by "infinitely many infinitesimally small" subsets $U_i \subset X$.

*Question **U1**.* How much of the geometry of a (compact) space X, say with a metric or symplectic geometry, can be seen in the homotopies of spaces of packings of X by such U_i?

*Question **U2**.* Is there a good category of "abstract packing-like objects", that are not, a prori, associated to actual packings of geometric spaces?

Concerning Question **U1**, notice that the above mentioned pairing between "cycles" and packings, shows that the volumes of certain *minimal subvarieties* in a Riemannian manifold X, can be reconstructed from the homotopies of packings of X by arbitrarily small balls.

For instance,

if X is a complete Riemannian manifold with *non-negative sectional curvatures*, then *the lengths of its closed geodesics* are (easily) seen in the homotopy spectra of these packing spaces. (see section 9)

And if X is an *orientable surface*, then this remains true with *no assumption on the curvatures* for the geodesics that are *length minimising in their respective homotopy classes.*

Similarly, much of the geometry of *waists* of a convex set X, say in the sphere S^n, may be seen in the homotopies of spaces of partition of X into convex subsets, see [22].

*Question **U3**.* Would it be useful to enhance the homotopy structure of a packing space of an X, say by (infinitesimally) small balls, by keeping track of (infinitesimal) geometric sizes of the homotopies in such a space?

V. *Limited Intersections.* A similar to packings (somewhat less interesting?) space is that of N-tuples of balls with *no k-multiple intersections* between them, (this space contains the space of $(k-1)$-tuples of packings) can be seen with the distance function to the union of the *k-diagonals* – there are $\binom{N}{k}$ of them – that are the pullbacks of the principal diagonal $\{x_1 = x_2 = ... = x_k\}$ in X^J, where $card(J) = k$, under maps $X^I \to X^J$ corresponding to $N!/(N-k)!$ imbedding $J \to I$.

W. *Spaces of Coverings.* Individual packing often go together with coverings, say, with minimal covering of metric spaces by r-balls. Possibly, this companionship extends to that between *spaces* of packings and *spaces* of coverings.

[13]Besides composition, there are other operations on (nested) packings. For instance, (close to each other) large balls (cubes) may "exchange" small balls (cubes) in them.

2 A few Words on Non-Parametric Packings.

Classically, one is concerned with *maximally dense* packings of spaces X by *disjoint balls*, rather than with the homotopy properties of families of moving balls in X.

Recall, that *a sphere packing* or, more precisely, *a packing of a metric space X by balls of radii r_i, $i \in I$, $r_i > 0$*, for a given *indexing set* I of finite or countable cardinality $N = card(I)$ is, by definition, a collection of (closed or open) balls $U_{x_i}(r_i) \subset X$, $x_i \in X$, with mutually non-intersecting interiors.

Obvioulsy, points $x_i \in X$ serve as centres of such balls if and only if

$$dist(x_i, x_j) \geq d_{ij} = r_i + r_j.$$

Basic Problem. What is the *maximal radius* $r = r_{max}(X; N)$ such that X admits a packing by N balls of radius r?

In particular,

what is the asymptotics of $r_{max}(X; N)$ for $N \to \infty$?

If X is a compact n-dimensional Riemannian manifold (possibly with boundary), then the principal term of this asymptotics depends only on the volume of X, namely, one has the following (nearly obvious)

Asymptotic Packing Equality.

$$\lim_{N \to \infty} \frac{N \cdot r_{max}(X; N)^n}{vol_n(X)} = \circledcirc_n,$$

where $\circledcirc_n > 0$ is a universal (i.e. independent of X) *Euclidean packing constant* that corresponds in an obvious way to the *optimal density* of the sphere packings of the Euclidean space $\mathbb{R}^n$.

(Probably, the full asymptotic expansion of $r_{max}(X; N)_{N \to \infty}$ is expressible in terms of the derivatives of the curvature of X and derivatives the curvature similarly to Minakshisundaram-Pleijel formulae for spectral asymptotics.)

The explicit value of $\circledcirc_m$ is known only for $n = 1, 2, 3$. In fact, the optimal, i.e. maximal, packing density of $\mathbb{R}^n$ for $n \leq 3$ can be implemented by a $\mathbb{Z}^m$-*periodic* (i.e. invariant under some discrete action of $\mathbb{Z}^n$ on $\mathbb{R}^n$) packing, where the case $n = 1$ is obvious, the case $n = 2$ is due to Lagrange (who proved that the optimal packing is the hexagonal one) and the case of $n = 3$, conjectured by Kepler, was resolved by Thomas Hales.

(Notice that $\mathbb{R}^3$, unlike $\mathbb{R}^2$ where *the only* densest packing is the hexagonal one, admits *infinitely many* different packings; most of these are *not $\mathbb{Z}^3$-periodic*, albeit they are $\mathbb{Z}_2$-periodic.

Probably, none of densest packings of $\mathbb{R}^n$ is $\mathbb{Z}^n$-periodic for large m, possibly for $n \geq 4$. Moreover, *the topological entropy* of the action of $\mathbb{R}^n$ on the space of optimal packings may be non-zero.

Also, there may be infinitely many algebraically independent numbers among $\circledcirc_1, \circledcirc_2, ...$; moreover, the number of algebraically independent among $\circledcirc_1, \circledcirc_2, ..., \circledcirc_n$ may grow as $const \cdot n$, $const > 0$.)

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3 Homological Interpretation of the Dirichlet-Laplace Spectrum.

Let Ψ be a topological space and $E : \Psi \to \mathbb{R}$ a continuous real valued function, that is thought of as an energy $E(\psi)$ of states $\psi \in \Psi$ or as a Morse-like function on Ψ.

The subsets

$$\Psi_e = \Psi_{\leq e} = E^{-1}(\infty, e] \subset \Psi, \ r \in \mathbb{R},$$

are called the (closed) *e-sublevels of E*.

A number $e_\circ \in \mathbb{R}$ is said *to lie in the homotopy significant spectrum of E* if the homotopy type of Ψ_r undergoes a *significant*, that is *irreversible*, change as e passes through the value $e = e_\circ$, that may be understood as *non-existence of a homotopy* of the subset $\Psi_{e_\circ}$ in Ψ that would bring it to the sublevel $\Psi_{e<e_\circ} \subset \Psi_{e_\circ}$.

Basic Quadratic Example. Let Ψ be an infinite dimensional projective space and E equal the ratio of two quadratic functionals. More specifically, let E_{Dir} be the Dirichlet function(al) on differentiable functions $\psi = a(x)$ normalised by the L_2-norm on a compact Riemannian manifold X,

$$E_{Dir}(\psi) = \frac{||d\psi||^2_{L_2}}{||\psi||^2_{L_2}} = \frac{\int_X ||d\psi(x)||^2 dx}{\int_X \psi^2(x)dx}.$$

The eigenvalues $e_0, e_1, e_2, ..., e_N, ...$ of E_{Dir} (i.e. of the corresponding Laplace operator) are *homotopy significat* since the rank of the inclusion homology homomorphism $H_*(\Psi_r; \mathbb{Z}_2) \to H_*(\Psi; \mathbb{Z}_2)$ *strictly* increases (for $* = N$) as e passes through e_N.

An essential feature of Dirichlet energy that, as we shall see, is shared by many other examples is *homological localisation.*

Let X be partitioned by closed subsets U_i, $i \in I$, $card(I) = n$, with piecewise smooth boundaries. Then

the N-th eigenvalue $e_N = e_N(X)$ is bounded from below by the minimum of the first eigenvalues of U_i,

$$e_N(X) \geq \min_{i \in I} e_1(U_i).$$

Indeed, by linear algebra, every N-dimensional projective space of functions on X, say $S = P^N \subset \Psi = P^\infty$, contains a (necessarily non-zero) function $\psi_\star(x)$ such that

$$\star_i \qquad \int_{U_i} \psi_\star(x)dx = 0 \text{ for all } i \in I.$$

Therefore,

$$\sup_{a \in P^n} E_{Dir}(\psi) \geq \max_{i \in I} E_{Dir}(\psi_{\star|U_i}) \geq e_i,$$

where the key feature of this argument.

– *simultaneous solvability of the equations* $\star_i$ –

does not truly need the linear structure in P^n. but only the fact that

$S \subset \Psi$ supports a cohomology class $h \in H^1(\Psi; \mathbb{Z}_2) = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, with non vanishing ⌣power $h^{\smile N} \in H^N(\Psi; \mathbb{Z}_2)$.

If U_i are small approximately round subsets, then

$$e_1(U_i) \geq \varepsilon_n \left(\frac{1}{vol(U_i)}\right)^{\frac{2}{n}}, n = dimX,$$

and, with suitable partitions into such subsets, one bounds $e_N(X)$ from below[14] by

$$e_N(X) \geq \varepsilon_X \cdot \left(\frac{N}{vol(X)}\right)^{\frac{2}{n}}.$$

This can be quantified in terms of the geometry of X.

For instance, if $Ricci(X) \geq -n\delta$, $\delta \geq 0$, and $diam(X) \leq D$, then the *homology localisation* for $(n-1)$*-volume energy*

$$E_{vol}(\psi) = vol_{n-1}(\psi^{-1}(0))$$

in conjunction with Cheeger's spectral inequality implies that

$$e_N(X) \geq \varepsilon_n^{1+D\sqrt{\delta}} D^{-2} N^{\frac{2}{n+1}}.$$

(See section 7 and "Paul Levy Appendix" in [23].)

4 Induced Energies on (Co)Homotopies on (Co)Homologies.

On Stable and Unstable Critical points. If E is a Morse function on a smooth manifold Ψ, then the homotopy type of the energy sublevels $\Psi_e = E^{-1}(-\infty, e] \subset \Psi$ does change at all critical values e_{cri} of E. However, only exceptionally rarely, for the so called *perfect Morse functions*, such as for the above quadratic energies, these changes are irreversible. In fact, every value $r_0 \in \mathbb{R}$ can be made critical by an *arbitrarily small* C^0*-perturbation*[15] E' of a smooth function $E(\varphi)$, such that E' equals E outside the subset $E^{-1}[r_0 - \varepsilon, r_0 + \varepsilon] \subset \Psi$; thus, the topology change of the sublevels of E' at r_0 is insignificant.

But the spectra of Morse-like functions introduced below have such homotopy significance built into their very definitions.

$\mathcal{H}_\circ(\Psi)$*,* $E_\circ$ *and the Homotopy Spectrum.* Let $\mathcal{S}$ be a class of topologicl spaces S and let $\mathcal{H}_\circ(\Psi) = \mathcal{H}_\circ(\Psi; \mathcal{S})$ be the category where the objects are homotopy classes of continuous maps $\phi : S \to \Psi$ and morphisms are homotopy classes of maps $\varphi_{12} : S_1 \to S_2$, such that the corresponding triangular diagrams are (homotopy) commutative, i.e. the composed maps $\phi_2 \circ \varphi_{12} : S_1 \to \Psi$ are homotopic to ϕ_1.

Extend functions $E : \Psi \to \mathbb{R}$ from Ψ to $\mathcal{H}_\circ(\Psi)$ as follows. Given a continuous map $\phi : S \to \Psi$ let

$$E(\phi) = E_{max}(\phi) = \sup_{s \in S} E \circ \phi(s),$$

[14] It is obvious that $e_N(X) \leq C_X \cdot \left(\frac{N}{vol(X)}\right)^{\frac{2}{n}}$. In fact, the numbers $N_{sp \leq e}(U)$ of the eigenvalues $e_i(U) \leq e$ of open subsets $U \subset X$ satisfy *Hermann Weyl's asymptotic formula* $N_{sp \leq e}(U) \asymp D_n vol(U)^{\frac{n}{2}}$, $n = dim(X)$, where the existence of the limits $N_\asymp(U)$ of $N_{sp \leq e}(U) e^{\frac{2}{n}}$ for $e \to \infty$ and additivity of the set function $U \to N_\asymp(U)$ follows from the locality of the $\smile$product, while the evaluation of D_n, that happens to be equal $2\pi^{-n} vol(B^n(1))$ for $B^n(1)$ being the unit Euclidean ball, depends on the (Riemannian) geometry of the Dirichlet energy.

[15] "C^0" refers to the uniform topology in the space of continuos functions.

denote by $[\phi] = [\phi]_{hmt}$ the homotopy class of ϕ. and set

$$E_\circ[\phi] = E_{mnmx}[\phi] = \inf_{\phi \in [\phi]} E(\phi).$$

In other words,

$E_\circ[\phi] \leq e \in \mathbb{R}$ if and only if the map $\phi = \phi_0$ admits a homotopy of maps $\phi_t : S \to A$, $0 \leq t \leq 1$, such that ϕ_1 sends S to the sublevel $\Psi_e = E^{-1}(-\infty, e] \subset \Psi$.

The covariant (homotopy) $\mathcal{S}$-spectrum of E is the set of values $E_\circ[\phi]$ for some class $\mathcal{S}$ of (homotopy types of) topological spaces S and (all) continuous maps $\phi : S \to \Psi$.

For instance, one may take for $\mathcal{S}$ the set of homemorphism classes of countable (or just finite) cellular spaces. In fact, the set of sublevels Ψ_r, $r \in \mathbb{R}$, themselves is sufficient for most purposes.

Lower and Upper Bounds on Spectra. Lower bounds on homotopy spectra say, in effect, that "homotopically *complicated/large*" maps ϕ (that may need complicated parameter spaces S supporting them) necessarily have *large energies* $E_\circ[\phi]$.

Conversely, upper bounds depend on construction of *complicated* $\phi : S \to \Psi$ with *small energies.*

On Topology, Homotopy and on Semisimplicial Spaces. The topology of a space Ψ per se is not required for the definition of homotopy (and cohomotopy below) spectra. What is needed is a *"homotopy structure" in* Ψ defined by distinguishing a class of maps from "simple spaces" S into Ψ.

If such a structure in Ψ is associated with polyhedra taken for "simple S", then Ψ is called a *semisimlicial (homotopy) space* with its "homotopyy structure" defined via the contravariant functor $S \rightsquigarrow maps(S \to \Psi)$ from the category of simplicial complexes and simplicial maps to the category of sets.

$\mathcal{H}^\circ(\Psi)$, E° and the Cohomotopy $\mathcal{S}$-Spectra. Now, instead of $\mathcal{H}_\circ(\Psi)$ we extend E to the category $\mathcal{H}^\circ(\Psi)$ of homotopy classes of maps $\varphi : \Psi \to T$, $T \in \mathcal{S}$, by defining $E^\circ[\varphi]$ as the supremum of those $e \in \mathbb{R}$ for which the restriction map of φ to the energy sublevel $\Psi_e = E^{-1}(-\infty, e] \subset \Psi$,

$$\varphi_{|\Psi_e} : \Psi_e \to T,$$

is *contractible.*[16] Then the set of the values $E^\circ[\varphi]$, is called *the contravariant homotopy (or cohomotopy) $\mathcal{S}$-spectrum of E.*

For instance, if $\mathcal{S}$ is comprised of the *Eilenberg-MacLane $K(\Pi, n)$-spaces*, $n = 1, 2, 3, ...$, then this is called *the* Π*-cohomology spectrum of E.*

Relaxing Contractibility via Cohomotopy Operations. Let us express "contractible" in writing as $[\varphi] = 0$, let $\sigma : T \to T'$ be a continuous map and let us regard the (homotopy classes of the) compositions of σ with $\varphi : \Psi \to T$ as an *operation* $[\varphi] \overset{\sigma}{\mapsto} [\sigma \circ \varphi]$.

Then define $E^\circ[\varphi]_\sigma \geq E^\circ[\varphi]$ by maximising over those e where $[\sigma \circ \varphi_{|\Psi_e}] = 0$ rather than $[\varphi_{\Psi_e}] = 0$.

[16] In some cases, e.g. for maps φ into discrete spaces T such as Eilenberg-MacLane's $K(\Pi; 0)$, *"contractible"*, must be replaced by *"contractible to a marked point serving as zero"* in T that is expressed in writing as $[\varphi] = 0$.

Pairing between Homotopy and Cohomotopy. Given a pair of maps (ϕ, φ), where $\phi : S \to \Psi$ and $\varphi : \Psi \to T$, write

$[\varphi \circ \phi] = 0$ *if the composed map* $S \to T$ *is contractible,*

$[\varphi \circ \phi] \neq 0$ *otherwise.*

Think of this as a function with value "0" and "$\neq 0$" on these pairs.[17]

E_*, E^* and the (Co)homology Spectra. If h is a homology class in the space Ψ then $E_*(h)$ denotes *the infimum* of $E_\circ[\phi]$ over all (homotopy classes) of maps $\phi : S \to \Psi$ such that h is *contaned in the image* of the homology homomorphism induced by ϕ.

Dually, the energy $E^*(h)$ on a cohomology class $h \in H^*(\Psi;\Pi)$ for an Abelian group Π, is defined as $E^\circ[\varphi_h]$ for the h-inducing map from Ψ to the product of Eilenberg-MacLane spaces:

$$\varphi_h : \Psi \to \bigtimes_n K(\Pi, n),\ n = 0, 1, 2, \dots .$$

In simple words, $E^*(h)$ equals the supremum of those e for which h vanishes on $\Psi_e = E^{-1}(\infty, e] \subset \Psi$.[18]

Then one defines the (co)homology spectra as the sets of values of these energies E_* and E^* on homology and on cohomology.

Homotopy Dimension (Height) Growth. The roughest invariant one wishes to extract from the (co)homotopy spectra of an energy $E : \Psi \to \mathbb{R}$ is *the rate of the growth of the homotopy dimension* of the sublevels $\Psi_e = E^{-1}(-\infty, e] \subset \Psi$, where the homotopy dimension a subset $B \subset A$ is the minimal d such that B, or at least every polyhedral space P mapped to B, is contractible in A to a subset $Q \subset A$ of dimension d.[19]

In many cases, this dimension is known to satisfy a polynomial bound $homdim(\Psi_e) \leq ce^\delta$ for some constants $c = c(E)$ and $\delta = \delta(E)$, where such an inequality amounts to *a lower* bound on the spectrum of E.

In the simplest case of Ψ homotopy equivalent to P^∞, this dimension as function of e carries *all* spectral information about E.

For instance if E is the Dirichlet energy of function on a Riemannian n-manifold X where the eigenvalues are bounded from below by $e_N(X) \geq \varepsilon_X N^{\frac{2}{n}}$, one has $homdim(\Psi_e) \leq c \cdot e^{\frac{n}{2}}$ for $c = \varepsilon_X^{-\frac{n}{2}}$.

Multidimensional Spectra. Let $\mathcal{E} = \{E_j\}_{j \in J} : \Psi \to \mathbb{R}^J$ be a continuous map. Let h be a cohomology class of Ψ and define the *spectral hypersurface* $\Sigma_h \subset \mathbb{R}^J$ in the Euclidean space $\mathbb{R}^J = \mathbb{R}^{l=card(J)}$ as the boundary of the subset $\Omega_h \subset \mathbb{R}^J$ of the J-tuples of numbers (e_j) such that the class h vanishes on the subset $\Psi_{<e_j} \subset \Psi$ defined by the inequalities

$$E_j(\psi) < e_j,\ j \in J.$$

$$\Sigma_h = \partial \Omega_h,\ \Omega_h = \{e_j\}_{h|\bigtimes_{j \in J} \Psi_{e_j} = 0}.$$

[17]If the space T is *disconnected*, it should be better endowed with a marking $t_0 \in T$ with "contractible" understood as "contractible to t_0".

[18]The definitions of energy on homology and cohomology obviously extend to generalised homology and and cohomology theories.

[19]This is called *essential dimension* in [19] and it equals the homotopy height of (the homotopy class of) the inclusion $B \hookrightarrow A$.

(This also make sense for general cohomotopy classes h on Ψ with $h = 0$ understood as contractibility of the map $\psi : \Psi \to T$ that represent h to a marked "zero" point in T where marking is unnecessary for connected spaces T.)

More generally, given a continuous map $\mathcal{E} : \Psi \to Z$, one "measures" open subsets $U \subset Z$ according to the sizes of "the parts" of the homology of Ψ that are "contained" in $\mathcal{E}^{-1}(U) \subset \Psi$, that are the images of the homology homomorphism $H_*(U) \to H_*(\Psi)$; similarly, kernels of the cohomology homorphisms $H^*(\Psi \setminus U) \to H^*(\Psi)$ serve as a measure-like function $U \mapsto \mu^*(U)$ on Z, (see section 11).

If $\Psi \to Z$ are smooth manifolds, and $\mathcal{E}$ is a proper smooth map, then the the set $\Sigma_\mathcal{E} \subset Z$ of critical values of $\mathcal{E}$ "cuts" Z into subsets where the measure μ^* is (nearly) constant. (It is truly constant on the connected components of $Z \setminus \Sigma_\mathcal{E}$ but may vary at the boundaries of these subsets, since these boundaries are contained in $\Sigma_\mathcal{E}$.) Here, "the cohomology spectrum of $\mathcal{E}$" should be somehow defined via "coarse-graining(s)" of the "partition" of Z into these subsets according to the values of μ^*.

Packing Example. Take Ψ equal the I-Cartesian power of a Riemannian manifold, $\Psi = X^I$, let J consist of unordered pairs (i_1, i_2) $i_1 \neq i_2$, thus $card(J) = l = N(N-1)/2$, $N = card(I)$, and let $\mathcal{E}$ be given by the reciprocals of the l functions $E_j = dist_X(x_{i_1}, x_{i_2})$. (This map is equivariant for the natural actions of the permutation group $Sym_N = aut(I)$ on Ψ and on $\mathbb{R}^J$ and the most interesting aspects of the topology of this $\mathcal{E}$ that are indicated below become visible only in the equivariant setting of section 12)

Spectral Families. A similar (dual?) picture arises when one has a family of functions $E_z : \Psi_z \to \mathbb{R}$ parametrised by a topological space $Z \ni z$, where the family $homospec_z \subset \mathbb{R}$ of homotopy spectra of E_z is seen as the *spectral hypersurface* Σ of $\{E_z\}_{z \in Z}$,

$$\Sigma = \bigcup_{z \in Z} homospec_z \subset Z \times \mathbb{R}.$$

On Positive and Negative Spectra. Our definitions of homotopy and homology spectra are best adapted to functions $E(\psi)$ bounded from below but they can be adjusted to more general functions E such as $E(x) = \sum_k a_k x_k^2$ where there may be infinitely many negative as well as positive numbers among ψ_k.

For instance, one may define the spectrum of a E unbounded from below as the limit of the homotopy spectra of $E_\sigma = E_\sigma(\psi) = \max(E(\psi), -\sigma)$ for $\sigma \to +\infty$.

But often, e.g. for the action-like functions in the symplectic geometry, one needs something more sophisticated than a simple minded cut-off of "undesirable infinities".[20]

On Continuous Homotopy Spectra. There also is a homotopy theoretic rendition/generalisation of *continuous spectra* with some Fredholm-like notion of homotopy,[21] such that, for instance, the natural inclusion of the projectivised Hilbert subspace $PL_2[0,t] \subset PL_2[0,1]$, $0 < t < 1$, would not contract to any $PL_2[0, t-\varepsilon]$.

[20] It seems, however, that neither a general theory nor a comprehensive list of examples exit for the moment.

[21] See *On the uniqueness of degree in infinite dimension* by P. Benevieri and M. Furi, http://sugarcane.icmc.usp.br/PDFs/icmc-giugno2013-short.pdf.

5 Families of k-Dimensional Entities, Spaces of Cycles and Spectra of k-Volume Energies.

The *"k-dimensional size"* of a metric space X, e.g. of a Riemannian manifold, may be defined in terms of the (co)homotopy or the (co)homology spectrum of the *k-volume* or a similar energy function on a *space of "virtually k-dimensional entities"* Y in X. These spectra and related invariants of X can be defined by means of families of such "entities" as in the following examples.

(A) If X and S are topological spaces of dimensions $n = dim(X)$, and $m = n - k = dim(S)$ then continuous maps $\varsigma : X \to S$ define S-families of the fibres $Y_s = \varsigma^{-1}(s) \subset X$.

(B) Given a pair of spaces T and $T_0 \subset T$, where $dim(T_0) = dim(T) - m$, and an S-family of maps $\phi_s : X \to T$, $s \in S$, one arrives at an S-family of "virtually m-codimensional" subsets in X by taking the pullbacks $Y_s = \phi_s^{-1} \subset X$.

(C) In the case of *smooth manifolds* X and S and *generic smooth maps* $\varsigma : X \to S$, the families from (A) and similarly for (B) can be seen geometrically as maps from S to the *space of k-manifolds* with the natural (homotopy) semisimplicial structure defined by bordisms.

And for general continuous maps ς we think of $Y_s = \varsigma^{-1}(s) \subset X$ as an S-families of *virtual* k-(sub)manifolds in X.

(D) More generally, if X and S are *pseudomanifolds* of dimensions n and $m = n - k$ and $\varsigma : X \to S$ is a *simplicial* map, then the fibres $Y_s \subset X$ are k-dimensional pseudomanifods for all s in S away from the $(m-1)$-skeleton of S with the map $s \mapsto Y_s$ being semisimlicial for the natural semisimplicial structure in the space of pseudomanifolds.

(E) In order to admit families of *mutually intersecting* subsets $Y_s \subset X$ we need an auxiliary space Σ mapped to S by $\varsigma : \Sigma \to S$. Then we let $\tilde{Y}_s = \varsigma^{-1}(s) \subset \Sigma$ and define S-families $Y_s \subset X$ via maps $\chi : \Sigma \to X$ by taking images $Y_s = \chi(\tilde{Y}_s) \subset X$.

($\tilde{\text{E}}$) Such a Σ may be constructed starting from a family of subsets $Y_s \subset X$ as the subset $\Sigma = \tilde{X} \subset X \times S$ that consists of the pairs (x, s) such that $x \in Y_s$. However, this $\tilde{X} \to X$, unlike more general $\Sigma \to X$, does not account for possible self-intersections of $\tilde{Y}_s$ mapped to X.)

(F) If we want to keep track of multiplicities of maps $\tilde{Y}_s \to Y_s \subset X$ it is worthwhile to regard the maps

$$\chi_{|\tilde{Y}_s} : \tilde{Y}_s \to X$$

themselves, rather tan their images, as our (virtual) "k-dimensional entities in X".

(G= C+F) The space of smooth k-submanifolds in an n-dimensional manifold X can be represented by the space of continuous maps σ from X to the Thom space T of the inversal m-dimensional vector bundle ofr $m = n - k$, where "virtual k-submanifolds" in X come as the σ-pullbacks of the zero section of this bundle.

Then *the space of bordisms* associated to arbitrary topological space X may be defined with n-manifolds Σ, maps $\sigma : \Sigma \to T$ and maps $\chi : \Sigma \to X$.

Space $\mathcal{C}_k(X; \Pi)$ of k-Cycles in X.

There are several homotopy equivalent candidates for *the space of k-cycles with* Π-*coefficients* of a topological space X.

For instance, one may apply the construction of a *semisimplicial space of k-cycles* associated to a chain complex of Abelian groups (see section 2.2 of [21]) to the complex $\{C_i \overset{\partial_i}{\to} C_{i-1}\}_{i=0,1,...,k,...}$ of *singular* Π-*chains* that are $\sum_\nu \pi_\nu \sigma_\nu$ for $\pi_\nu \in \Pi$, where Π is an Abelian group and where $\sigma_\nu : \triangle^i \to X$ are continuous maps of the i-simplex to X.

For instance, k-dimensional psedomanifolds mapped to X define singular $\mathbb{Z}_2$-cycles in X ($\mathbb{Z}$-cycles if Σ and S are oriented) where the above S-families agree with the semisimplicial structure in $\mathcal{C}_k(X;\mathbb{Z}_2)$.

More generally, l-chains in the space of k-cycles in X can be represented by l-dimensional families of k-cycles in X that are $(k+l)$-chains in X. This defines a map between the corresponding spaces of cycles

$$\top : \mathcal{C}_l(\mathcal{C}_k(X;\Pi);\Pi)) \to \mathcal{C}_{k+l}(X;\Pi \otimes_{\mathbb{Z}} \Pi)$$

for all Abelian (coefficient) groups Π, as well as a natural homomorphism of degree $-k$ from the cohomology of a space X to the cohomology of the space of k-cycles in X with Π-coefficients, denoted

$$\top^{[-k]} : H^n(X;\Pi) \to H^{n-k}(\mathcal{C}_k(X;\Pi);\Pi), \text{ for all } n \geq k \ ,$$

provided $\Pi = \mathbb{Z}$ or $\Pi = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.

Almgren-Dold-Thom Theorem for the Spaces of Cycles. Let X be a triangulated space and $f : X \to \mathbb{R}^m$ a generic piece-wise linear map. Then the "slicings" of generic $(m+k)$-cycles $V \subset X$ by the fibres of f, define homomorphisms from the homology groups $H_{m+k}(X;\Pi)$ to the homotopy groups $\pi_m(\mathcal{C}_k(X;\Pi))$ of the spaces $\mathcal{C}_k(X;\Pi)$ of k-dimensional Π-cycles in X for all Abelian groups Π.

In fact, the map $r \mapsto f^{-1}(r) \cap V$, $r \in \mathbb{R}^m$, sends $\mathbb{R}^m$ to $\mathcal{C}_k(X;\Pi)$, where the complement to the (compact!) image $f(V) \subset \mathbb{R}^m$ goes to the the zero cycle.

The Almgren-Dold-Thom Theorem claims that these homomorphisms

$$H_{m+k}(X;\Pi) \to \pi_m(\mathcal{C}_k(X;\Pi))$$

are isomorphisms. This is easy for the above semisimlicial spaces (see [21]) and, as it was shown by Almgren, this remains true for spaces of *rectifiable cycles with flat topology.*

(Recall, that the *flat/filling distance* between two homologous k-cycles C_1, C_2 in X with $\mathbb{Z}$- or $\mathbb{Z}_p$-coefficients is defined as the infimum of $(k+1)$-volumes (see below) of $(k+1)$-chains D such that $\partial D = C_1 - C_2$.

Also notice, that If Π is a field, then $\mathcal{C}_k(X;\Pi)$ *splits into the product* of *the Eilenberg-MacLane spaces* corresponding to the homology groups of X,

$$\mathcal{C}_k(X;\Pi) = \bigtimes_{n=0,1,2,...} K(\Pi_n, n), \text{ where } \Pi_n = H_{k+n}(X;\Pi),$$

where, observe, $H^n(K(\Pi,n);\Pi)$ is *canonically* isomorphic to Π for the cyclic groups Π.)

k-Volume and Volume-like Energies.

• *Hausdorff Measures of Spaces and Maps.* The k-dimensional Hausdorff measure of a *semimetric*[22] space Y is defined for all positive real numbers $k \geq 0$ as

$$Haumes_k(Y) = \beta_k \cdot \inf_{\{r_i\}} \sum_{i \in I} r_i^{-k},$$

where I is a countable set, $\beta_k = \frac{\pi^{k/2}}{\Gamma(\frac{k}{2}+1)}$ is the normalising constant that, for integer k, equals the volume of the unit ball B^k and where the infimum is taken over all I-tuples $\{r_i\}$ of positive numbers, such that Y admits an I-covering by balls of radii r_i.

The corresponding *Hausdorff measure of a map* $f : Y \to X$, where X is a metric space, is, by definition, the Hausdorff measure of Y with the semimetric induced by f from X.

If f is one-to-one then $Haumes(f) = Haumes(f(Y))$ but, if f has a "significant multiplicity" then $Haumes(f) > Haumesf(Y)$.

δ-Neighbourhoods $U_\delta(Y) \subset X$ and Minkowski Volume. Minkowski k-volume of a subset Y in an n-dimensional Riemannian manifold X and/or in a similar space with a distinguished measure regarded as the n-volume, is defined as

$$Mink_k(Y) = \liminf_{\delta \to 0} \frac{vol_n(U_\delta(Y))}{\delta^{n-k} vol_{n-k}(B^{n-k})}.$$

In general, the the Minkowski k-volume may be much smaller than the Hausdorff k-measure but the two are equal for "regular" subsets $Y \subset X$ where "regular" includes

• compact smooth and piecewise smooth submanifolds in smooth manifolds;
• compact real analytic and semianalitic subspaces in real analytic spaces;
• compact minimal subvarieties in Riemannian manifolds.

Besides Minkowski volumes themselves,

the normalised n-volumes of the δ-neughbourhoods of subsets $Y \subset X$,

$$\delta\text{-}Mink_k(Y) =_{def} \frac{vol_n(U_\delta(Y))}{\delta^{n-k} vol_{n-k}(B^{n-k})},$$

also can be used as "volume-like energies" that have interesting homotopy spectra.

The pleasant, albeit obvious, feature of the volume δ-$Mink_k(Y)$ for $\delta > 0$ is its *continuity* with respect to *the Hausdorff metric* in the space of subsets $Y \subset X$.

On δ-Covers and δ-Packings of Y. The *minimal* number of δ-balls *needed to cover* Y provides an alternative to δ-$Mink_k(Y)$ and the *maximal* number of *disjoint* δ-balls in X with *centers in* Y plays a similar role.

The definitions of these numbers *does not depend* on vol_n; yet, they are closely (and obviously) related to δ-$Mink_k(Y)$.

[22]"Semi" allows a possibility of $dist(y_1, y_2) = 0$ for $y_1 \neq y_2$.

6 Minmax Volumes and Waists.

Granted a space Ψ of "*virtually k-dimensional entities*" in X and a volume-like energy function $E = E_{vol_k} : \Psi \to \mathbb{R}_+$, "the first eigenvalue" – the bottom of the homotopy/(co)homology spectrum of this E is called the *k-waist of X*.

To be concrete, we define waist(s) below via the two basic operations of producing S-parametric families of subsets – taking pullbacks and images of maps represented by the following diagrams

$$\mathcal{D}_X = \{X \overset{\chi}{\leftarrow} \Sigma \overset{\varsigma}{\rightarrow} S\}$$

where S and Σ are simplicial (i.e. triangulated topological) spaces of dimensions $m = \dim(S)$ and $m + k = dim(\Sigma)$ and where our "entities" are the pullbacks

$$\tilde{Y}_s = \varsigma^{-1}(s) \subset \Sigma, \ s \in S,$$

that are mapped to X by χ.

Definitions.

[**A**] *The maximal k-volume* – be it Hausdorff, Minkowski, δ-Minkowski, etc,– of such a family is defined as the supremum of the of the corresponding volumes of the image restrictions of the map χ to Y_s, that is

$$\sup_{s \in S} vol_k(\chi(\tilde{Y}_s)).$$

(It is more logical to use the *volumes of the maps* $\chi_{|Y_s} : \tilde{Y}_s \to X$ rather than of their images but this is not essential at this point.)

[**B**] *The minmax k-volume* of the pair of the *homotopy classes* of maps ς and χ denoted $vol_k[\varsigma, \chi]$ is defined as

$$\inf_{\varsigma, \chi} \sup_{s \in S} Haumes_k(\chi(Y_s)),$$

where the infimum is taken over all pairs of maps (ς, χ) in a given homotopy class $[\varsigma, \chi]$ of (pairs of) maps.

[**C**] *The k-Waist* of a Riemannian Manifold X, possibly with a boundary, is

$$waist_k(X) = \inf_{\mathcal{D}_X} \sup_{s \in S} vol_k(\chi(Y_s)),$$

where the infimum (that, probably, leads to the same outcome as taking *maximum*) is taken over all diagrams $\mathcal{D}_X = \{X \overset{\chi}{\leftarrow} \Sigma \overset{\varsigma}{\rightarrow} S\}$ that represent "*homologicaly substantial*" families of subsets $Y_s = \chi(\tilde{Y}_s = \varsigma^{-1}(s)) \subset X$ that support k-cycles in X.

What is Homologically Substantial? A family of subsets $Y_s \subset X$ is regarded as homologically substantial if it satisfy some *(co)homology condition* that insures that the subsets $S_{\ni x} \subset S$, $x \in X$, that consist of $s \in S$ such that $Y_s \ni x$, is non-empty for all (some?) $x \in X$.

In the setting of smooth manifolds and smooth families, the simplest such condition is *non-vanishing* of the "*algebraic number of points*" in $S_{\ni x}$ for generic $x \in X$ that make sense if $dim(Y_x) + dimS = dimX$ for these x.

More generally, if $dim(Y_s) + dim(S) \geq dimX$, then the corresponding condition in the *bordisms homology theory* asserts *non-vanishing of the cobordism class of submanifold* $S_{\ni x} \subset X$ (for cobordisms regarded as homologies of a point in the bordism homology theory).

$\mathbb{Z}_2$*-waists.* One arrives at a particular definition, namely, of what we call $\mathbb{Z}_2$*-waist* if "homologically" refers to homologies with $\mathbb{Z}_2$*-coefficients* (that is the best understood case), and, accordingly, the above inf is taken over all diagrams $D_X = \{X \overset{\chi}{\leftarrow} \Sigma \overset{\varsigma}{\rightarrow} S\}$ where S and Σ are *pseudumanifolds of dimensions* n *and* $n-k$ *with boundaries*[23] (probably, using only *smooth manifolds* S and Σ in our diagrams would lead to the same $\mathbb{Z}_2$-waist) and where ς and χ are continuous maps, such that $\chi : \Sigma \to X$ *respects the boundaries,*[24] *i.e.* $\partial\Sigma \to \partial X$ and where χ *has non-zero* $\mathbb{Z}_2$ degree that exemplifies the idea of "homological substantionality".[25]

One may render this definition more algebraic by

• admitting an *arbitrary* (decent) topological space for the role Σ (that is continuousy mapped by ς to an m-dimensional *pseudomanifold* S);

and

• replacing the *"non-zero degree condition"* by requiring that the fundamental $\mathbb{Z}_2$-homology class $[X]$ of X should lie in the image of homology homomorphism $\chi_* : H_*(\Sigma; \mathbb{Z}_2) \to H_*(X; \mathbb{Z}_2)$, or equivalently, that the cohomology homomorphism $\chi^* : H^*(X; \mathbb{Z}_2) \to H^*(\Sigma; \mathbb{Z}_2)$, does not vanish on the fundamental *cohomology* class of X.

(This naturally leads to a definition of the *the waists of an* n*-dimensional homology and cohomology class* h of dimension $N \geq n$, where one may generalize/refine further by requiring non-vanishing of some cohomology operation applied on $\chi^*(h)$, as in [30].)

Examples of Homologically Substantial Families. (i) The simplest, yet significant, instances of such families of (virtually k-dimensional) subsets in n-manifolds X are the pullbacks $Y_s = \varsigma^{-1}(s) \subset X$, $s \in \mathbb{R}^{n-k}$, for continuous maps $\varsigma : X \to \mathbb{R}^{n-k}$. (The actual dimension of some among these Y_s may be strictly greater than k.)

(ii) Let S be a subset in the projectivized space[26] P^∞ of continuous maps $X \to \mathbb{R}^m$ and $Y_s = s^{-1}(0) \subset X$, $s \in S$, be the zero sets of these maps.[27] Let $P^\infty_{reg} \subset P^\infty$ consist of the maps $X \to \mathbb{R}^m$ *the images of which linearly span all of* $\mathbb{R}^m$, where, observe, the inclusion $P^\infty_{reg} \subset P^\infty$ is a *homotopy equivalence* in the present (infinite dimensional) case.

[23] An "n-psedumanifold with a boundary" is understood here as a simpicial polyhedral space, where all m-simplices for $m < n$ are contained in the boundaries of n-simples and where the boundary $\partial\Sigma \subset \Sigma$ is comprised of the $(n-1)$-simplifies that have *odd* numbers of n-simplices adjacent to them.

[24] Σ has non-empty boundary only if X does and $\varsigma : \Sigma \to S$, unlike $\chi : \Sigma \to X$, *does not have* to send $\partial\Sigma \to \partial S$.

[25] If one makes the definition of waists with the volumes of maps $\chi_{|Y_s} : Y_s \to X$ instead of the volumes of their images $\chi(Y_s) \subset X$ that would leads, a priori, to *larger* waists. However, in the $\mathbb{Z}_2$-case, the waists defined with the volumes of images, probably, equal the ones, defined via the volumes of maps, even for non-manifold targets X. This is obvious under mild regularity/genericity assumptions on the maps ς and χ, but needs verification in our setting of general continuous maps.

[26] *Projectivization* $P(L)$ *of a linear space* L (e.g. of maps $X \to \mathbb{R}^m$) is obtained by removing zero and dividing $L \setminus 0$ by the action of $\mathbb{R}^\times$.

[27] This P^∞ *is* an infinite dimensional projective space, unless X is a finite set.

Observe that there is a natural map, say G, from P^∞_{reg} to the Grassmnian Gr^∞_m of m-planes in the linear space of functions $X \to \mathbb{R}$, where the linear subspace $G(p)$ in the space of functions $X \to \mathbb{R}$ for a (projectifized) map $p : X \to \mathbb{R}^m$ consists the compositions $l \circ p : X \to \mathbb{R}$ for all linear functions $l : \mathbb{R}^m \to \mathbb{R}$.

If the cohomology homomorphism $\mathbb{Z}_2 = H^m(P^\infty;\mathbb{Z}_2) \to H^m(S;\mathbb{Z}_2)$ does not vanish, then the S-family $Y_s \subset X$ is $\mathbb{Z}_2$-homologically substantial, provided $S \subset P^\infty_{reg}$ i.e. if the images of $s : X \to \mathbb{R}^m$, $s \in S$, linearly spans all of $\mathbb{R}^m$ for all $s \in S$.

Indeed, if S is an m-dimensional psedomanifold, for $m = n-k$, then the number of points in the subset $S_{\ni x} \subset$ for which $Y_s \ni x$, (that is defined under standard genericity conditions) does not vanish $\mod 2$. In fact, it is easy to identify this number with the value of the Stiefel-Whitney class of the complementary bundle to the canonical line bundle over the projective space $P^\infty \approx P^\infty_{reg}$), where this "complementary bundle" is unduced from the the canonical $(\infty - m)$-bundle over the Grassmnian Gr^∞_m by the above map $G : P^\infty_{reg} \to Gr^\infty_m$.

(iii) In the smooth situation, the above $Y_s = s^{-1}(0) \subset X$ are, generically, submanifolds of codimension m in X with *trivial normal bundles.*

General submanifolds and families of these are obtained by mappings ϕ from X to the Thom space T_m of a universal $\mathbb{R}^m$ bundle V by taking the pullbacks of the zero $\mathbf{0} \subset V \subset T_m$.

And if $\{\phi_s\} : X \to T_m$ is a family of maps parametrised by the m-sphere $S \ni s$ that equals the closure of a fibre of V in $T_m \supset V$, then the family $Y_s = \phi_s^{-1}(\mathbf{0}) \subset X$ is homologically substantial, if the map $S \to T_m$ defined by $s \mapsto \phi_s(x_0) \in T_m$, for some point $x_0 \in X$, has non-zero intersection index with the zero $\mathbf{0} \subset V \subset T_m$.

(iv) The above, when applied to maps between the suspensions of our spaces, $\phi^{\wedge k} : X \wedge S^k \to T_m \wedge S^k$, delivers families of (virtual) submanifods of dimensions $n - m$, $n = dim(X)$, mapped to X via the projection $X \wedge S^k \to X$, where these families are nomologically substantial under the condition similar to that in (iii).

(v) There are more general (non-Thom) spaces, T, where ($(n - m)$-volumes of) pullbacks of m-codimensional subspaces $T_0 \subset T$ are of some interest.

For instance, since the space of m-dimensional Π-cocycles of (the singular chain complex of) X (see [21]) is homotopy equivalent to space of continuous maps $X \to K(\Pi, m)$, it may be worthwhile to look from this perspective at (e.g. cellular) spaces T that represent/approximate *Eilenberg Maclane's* $K(\Pi, m)$ with the codimension m skeletons of such T taken of $T_0 \subset T$.

Positivity of Waists. *The k-dimensional $\mathbb{Z}_2$-waists of all Remanning n-manifolds X are strictly positive for all $k = 0, 1, 2, ..., n$:*

$[\geq_{nonsharp}]_{\mathbb{Z}_2}$ $$\mathbb{Z}_2\text{-}waist_k(X) \geq w_{\mathbb{Z}_2} = w_{\mathbb{Z}_2}(X) > 0.$$

About the Proof. The waists defined with all of the above "k-volumes" are *monotone under inclusions*,

$$waist_k(U_1) \leq waist_k(U_2) \text{ for all open subsets } U_1 \subset U_2 \subset X,$$

and they also properly behave under λ-Lipschitz maps $f : X_1 \to X_2$ of non-zero degrees,

$$waist_k(X_2 = f(X_1)) \leq \lambda^k waist_k(X_1) \text{ for maps } f : X_1 \to X_2 \text{ with } deg_{\mathbb{Z}_2}(f) \neq 0.$$

Thus, $[\geq_{nonsharp}]_{\mathbb{Z}_2}$ for all X follows from that for the unit (Euclidean) n-ball B^n.

Then the case $X = B^N$ reduces to that of the unit n-sphere S^n,[28] while the lower bound $[\geq_{nonsharp}]_{\mathbb{Z}_2}$ for the k-waist of S^n defined with the *Hausdorff measure* of $Y \subset X = S^n$ is proven in [21] by a reduction to a *combinatorial filling inequality.*

On the other hands the *sharp values* of $\mathbb{Z}_2$-waists of spheres are known for the *Minkowski volumes* and, more generally, for all δ-$Mink_k$, $\delta > 0$. Namely

$[waist]_{sharp}$, $$\delta\text{-}Mink_k\text{-}waist_{\mathbb{Z}_2}(S^n) = \delta\text{-}Mink_k(S^k),$$

where $S^k \subset S^n$ is an equatorial k-sphere and

$$\delta\text{-}Mink_k(S^k \subset S^n) =_{def} \frac{vol_n(U_\delta(S^k))}{\delta^{n-k} vol_{n-k}(B^{n-k})}$$

for the δ-neighbourhood $U_\delta(S^k) \subset S^n$ of this sphere in S^n and the unit ball $B^{n-k} \subset \mathbb{R}^{n-k}$.

Every $\mathbb{Z}_2$-homologically substantial S-family of "k-cycles" $Y_s \subset S^n$ has a member say $Y_{s_\circ}$ such that

$$\delta\text{-}Mink_k(Y_{s_\circ}) \geq \delta\text{-}Mink_k(S^k) \textit{ for all } \delta > 0 \textit{ simultaneously.}$$

In particular,

given an arbitrary continuous map $\varsigma : S^n \to \mathbb{R}^{n-k}$*, there exist a value* $s_\circ \in \mathbb{R}^{n-k}$*, such that the* δ*-neighbourhoods of the* $s_\circ$*-fiber* $Y_{s_\circ} = \varsigma^{-1}(s_\circ) \subset S^n$ *are bounded from below by the volumes of such neighbourhoods of the equatorial* k*-subspheres* $S^k \subset S^n$*,*

$$vol_n(U_\delta(Y_{s_\circ})) \geq vol_n(U_\delta(S^k)) \textit{ for all } \delta > 0.$$

Consequently the Minkowski volumes of this $Y_{s_\circ}$ is greater than the volume of the equator,

$[\circ]_k$. $$Mink_k(Y_{s_\circ}) \geq Mink_k(S^k) \text{ for this very } s_\circ \in \mathbb{R}^{n-k}.$$

This is shown in [22] by a *parametric homological localisation argument.* (See section 7 below; also see section 19 for further remarks, examples and conjectures.)

7 Low Bounds on Volume Spectra via Homological Localization.

Start with a simple instance of homological localisation for *the* $(n-1)$*-volumes of zeros* of families *of real functions* on Riemannian manifolds.

[**I**] *Spectrum of* $E_{vol_{n-1}}$. Let L be an $(N+1)$-dimensional linear space of functions $s : X \to \mathbb{R}$ on a compact N-dimensional Riemannian manifold X and let $U_1, U_2, ...U_N \subset X$ be disjoint balls of radii ρ_N, such that

$$\rho_N^n \sim \circledcirc_n \frac{vol_n(X)}{N} \text{ for large } N \to \infty ,$$

[28] In fact, the sphere S^n is bi-Lipschitz equivalent to the double of the ball B^n.

Also the ball $B^n = B^n(1)$ admits *a radial k-volume contracting map onto* the sphere $S^n(R)$ of radius R, such that $vol_k(S^k(R)) = vol_k(B^k(1)$; this allows *a sharp evaluation* of certain "regular k-waists" of B^n, see [22], [21], [30].

where $\circledcirc_n > 0$ is a universal constant (as in the "packing section" 2). By the *Borsuk-Ulam theorem*, there exists a non-zero function $l \in L$ such that the zero set $Y_l = s^{-1}(0) \subset X$ cuts all U_i into equal halves and the Hausdorff measures (volumes) of the intersections of Y_l with U_i and *the isoperimetric inequality* implies the lower bound

$$vol_{n-1}(Y_l \cap U_i) \geq \beta_{n-1}\rho_N^{n-1} - o(\rho_N^{n-1}),\ N \to \infty,$$

for

$$\beta_{n-1} = vol_{n-1}(B^{n-1}(1)) = \frac{\pi^{n-1/2}}{\Gamma(\frac{n-1}{2}+1)}.$$

Therefore,

the supremum of the the Hausdorff measures of the zeros $Y_l \subset X$ of non-identically zero functions $l : X \to \mathbb{R}$ from an arbitrary $(N+1)$-dimensioanl linear space L of functions on X is bounded from below for large $N \geq N_0 = N_0(X)$ by

$$\sup_{l\in L\setminus 0} vol_{n-1}(Y_l) \geq \delta_n N^{\frac{1}{n}} vol_n^{\frac{n-1}{n}}(X)$$

for a universal constant $\delta_n > 0$.

[**I***] *Homological Generalisation.* This inequality remains valid for all *non-linear* spaces L (of functions on compact Romanian manifolds X) that are invariant under scaling $l \mapsto \lambda l$, $\lambda \in \mathbb{R}^\times$, provided the projectivisations $(L\setminus 0)/\mathbb{R}^\times \subset P^\infty$ of L in the projective space P^∞ of all continuous functions on X *support the (only) nonzero cohomology class* from $H^N(P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2$.

[**II**] $E_{vol_{n-m}}$*-Spectra of Riemannian manifolds X.* Let L be an $(mN+1)$-dimensional space of continuous maps $l : X \to \mathbb{R}^m$ that is invariant under scaling $l \to \lambda l$, $\lambda \in \mathbb{R}^\times$, and let the projectivized space $S = (L\setminus 0)/\mathbb{R}^\times \subset P^\infty$ in the projective space P^∞ of continuous maps $X \to \mathbb{R}^m$ modulo scaling support non-zero cohomology class from $H^{mN}(P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2$, e.g. L is a liner pace of maps $X \to \mathbb{R}^m$ of dimension $mN+1$.

Then the zero sets $Y_s = Y_l = l^{-1}(0) \subset X$, for $s = s(l) \in S \subset P^\infty$ being non-zero maps $l : X \to \mathbb{R}^m$ mod $\mathbb{R}^\times$-scaling,

$$[*]_{n-m} \qquad \sup_{s\in S} vol_{n-m}(Y_s) \geq \delta_n N^{\frac{m}{n}} vol_n^{\frac{n-m}{n}}(X)$$

for large $N \geq N_0 = N_0(X)$ and a universal constant $\delta_n > 0$, and where vol_{n-m} stands for the Minkowski $(n-m)$-volume.

$$vol_{n-m}(Y_s \cup U) \leq (n-m)\text{-}waist(U) - \varepsilon$$

Then, by the definition of waist, the cohomology restriction homomorphism $H^m(P^\infty;\mathbb{Z}_2) \to H^m(S^\backprime(U);\mathbb{Z}_2)$ vanishes for all $\varepsilon > 0$

Apply this to N open subsets $U_i \subset X$, $i = 1,...,N$, and observe that the *non-zero* cohomology class that comes to our $S \subset P^\infty$ from $H^{Nm}(P^\infty;\mathbb{Z}_2)(=\mathbb{Z}_2)$ equals the $\smile$-product of *necessarily non-zero* m-dimensional classes coming from $H^m(P^\infty;\mathbb{Z}_2)$.

This, interpreted as the "simultaneous $\mathbb{Z}_2$-homological substantionality" of the families $Y_s(i) = Y_s \cup U_i \subset U_i$, for all $i = 1,...,N$, shows that *there exists an $s \in S$, such that*

$$vol_{n-m}(Y_s \cup U_i) \geq (n-m)\text{-}waist(U) - \varepsilon,$$

for all $i = 1, ..., N$, in agreement with the definition of waists in the previous section.

Finally, take an efficient packing of X by N balls U_i as in the above [**I**] and [**I***] and derive the above $[*]_{n-m}$ from the lower bound on waists (see $[\geq_{nonsharp}]_{\mathbb{Z}_2}$ in the previous section) of U_i. QED.

(The lower bound $[\geq_{nonsharp}]_{\mathbb{Z}_2}$ on waists was formulated under technical, probably redundant, assumption on L saying that the restrictions of spaces L to $U_i \subset X$ lie in the subspaces $P^\infty_{reg}(U_i)$ (such that the images of the "regular" $l : U_i \to \mathbb{R}^m$ span $\mathbb{R}^m$) of the corresponding projective spaces $P^\infty = P^\infty(U_i)$ of maps $U_i \to \mathbb{R}^m$. This assumption can be easily removed for $vol_{n-m} = Mink_{n-m}$ by a simple approximation argument applied to δ-$Mink_{n-m}$-waists and, probably, it also seems not hard(?) to remove for $vol_{n-m} = Haumes_{n-m}$ as well.[29]

Further Results, Remarks and Questions.

(a) The (projective) space of the above Y_s can be seen, at least if Y_s are "regular", as a (tiny for $n-k > 1$) part of the space $\mathcal{C}_k(X;\mathbb{Z}_2)$ of all k-dimensional $\mathbb{Z}_2$-cycles in X, say for the n-ball $X = B^n$, where the full space $\mathcal{C}_k(B^n;\mathbb{Z}_2)$ of the relative k-cycles mod 2 is Eilenberg-MacLane's $K(\mathbb{Z}_2, n-k)$ by the Dold-Thom-Almgren theorem, see [2] and section 2.2 in [21].

If $n-k-1$, then $\mathcal{C}_k(B^n;\mathbb{Z}_2) = P^\infty$ and $H^*(\mathcal{C}_k(B^n;\mathbb{Z}_2);\mathbb{Z}_2)$ is the polynomial algebra in a single variable of degree one, but if $n-m \geq 2$, then the cohomology algebra of $\mathcal{C}_k(X;\mathbb{Z}_2)$ is freely generated by infinitely many monomials in *Steenrod squares* of the generator of $H^{n-k}(\mathcal{C}_k(B^n;\mathbb{Z}_2)) = \mathbb{Z}_2$.

Thus, if $n-k > 2$, the cohomology spectrum of E_{vol_k} is indexed not by integers (that, if $n-k = 1$, correspond to graded ideals of the polynomial algebra in a single variable), but by the graded ideals in a more complicated algebra $H^{n-k}(\mathcal{C}_k(B^n;\mathbb{Z}_2)) = \mathbb{Z}_2$ with the Steenrod algebra acting on it.

(b) *Guth' Theorem.* The asymptotic of this "*Morse-Steenrod spectra"* of the spaces $\mathcal{C}_k(B^n;\mathbb{Z}_2)$ of k-cykles in the n-balls were evaluated, up to a, probably redundant, lower order term, by Larry Guth (see [30] where a deceptively simple looking corollary of his results is the following

Polynomial Bound on the Spectral Homotopy Dimension for the Volume Energy.

The homotopy dimensions (heights) of the sublevels $\Psi_e = E^{-1}(-\infty.e] \subset \Psi$ of the vol_k-energy $E = E_{vol_k} : \Psi \to \mathbb{R}_+$ on the space $\Psi = \mathcal{C}_k(X;\mathbb{Z}_2)$ of k-dimensional rectifiable $\mathbb{Z}_2$-cycles in a compact Riemannian manifold X satisfies

$$hom dim(\Psi_e) \leq ce^\delta$$

where the constant $c = c(X)$ depends on the geometry of X while $\delta = \delta(n)$ depends only on the dimension n of X.

This, reformulated as a lower spectral bound on E_{vol_k}, reads.

[29] This was stated as a problem in section 4.2 of [19] but I do not recall if the major source of complication was the issue of regularity.

Let X be a compact Riemannian manifold and let $Y_s \subset X$, $s \in S \subset \mathcal{C}_k(X;\mathbb{Z}_2)$, be an S-family of k-cycles (one may think of these as k-pseudo-submanifolds in X) that is ***not*** *contractible to any N-dimensional subset in $\Psi = \mathcal{C}_k(X;\mathbb{Z}_2)$. Then*

$$\sup_{s\in S} vol_k(Y_s) \geq \varepsilon \cdot N^\alpha,$$

where $\varepsilon = \varepsilon(X) > 0$ depends on the geometry of X, while $\alpha = \alpha(n)(= \delta^{-1})$, $n = dim(X)$, is a universal positive constant.

In fact, Guth's results yield a nearly sharp bound

$$\sup_{s\in S} vol_k(Y_s) \geq \varepsilon(X,\alpha) \cdot N^\alpha,$$

for all $\alpha < \frac{1}{k+1}$, where, conjecturally, this must be also true for $\alpha = \frac{1}{k+1}$.

(c) *Is there a direct simple proof of the above inequaliy with some, let it be non-sharp, α that would bypass fine analysis (due to Guth) of the Morse-Steenrod cohomology spectrum of E_{vol_k} on the space of cycles?*

Does a polynomial lower bound hold for (k-volumes of) $\mathbb{Z}_p$- and for $\mathbb{Z}$-cycles?

Apparently, compactness of spaces of (quasi)minimal subvarieties in X implies discreteness of homological volumes spectra via the Almgren-Morse theory, but this does not seem(?) to deliver even logarithmic lower spectral bounds due to the absence(?) of corresponding bounds on Algren-Morse indices of minimal subvarities in terms of their volumes.

(d) The lower bounds for the k-volumes of zeros of families maps $X \to \mathbb{R}^m$ (see [**II**] above) can be, probably, generalized in the spirit of Guth' results, at least in the $\mathbb{Z}_2$-setting, to spaces of maps ψ from X to the total spaces of $\mathbb{R}^m$-bundles V and to the Thom spaces of such bundles where $E_{vol_k}(\psi) =_{def} vol_k(\psi^{-1}(\mathbf{0}))$ for the zero sections $\mathbf{0} \subset V$ of these bundles[30] where the Steenrod squares should be replaced by the bordism cohomology operations that are in the $\mathbb{Z}_2$-case amount to taking Stefel-Whitney classes.

Here, as well as for spaces of maps ψ from X to more general spaces T where $E_{vol_k}(\psi) = vol_k(\psi^{-1}(T_0))$ for a given $T_0 \subset V$, one needs somehow to factor away the homotopy classes of maps ψ with $E_{vol_k}(\psi) = 0$ (compare [37]).

Also it may be interesting to augment the k-volume by other (integral) invariants of $Y = \psi^{-1}(T_0)$, where the natural candidates in the case of k-dimensional (mildly singular) $Y = \psi^{-1}(\mathbf{0})$ would be curvature integrals expressing the k-volumes of the tangential lifts of these $Y \subset X$ to the Grassmann spaces $Gr_k(X)$, $k = dimY$, of target k-planes in X (compare section 3 in[26]).

(e) One can define spaces of subsets $Y \subset X$ that support " k-cycles (or rather *cocycles*), where these Y do not have to be regular in any way, e.g. rectifiable as in Guth' theorem, or even geometrically k-dimensional. But Guth's parametric homological localization along with the bounds on waists from the previous section yield the same lower bounds on the volume spectra on these space as in the rectifiable case.

[30] A similar effect can be achieved by replacing $\mathbb{R}^\times$-scaling by the action of the full linear group $GL_m(\mathbb{R})$ on $\mathbb{R}^m$ and working with the equivaruant cohomology of the space of maps $X \to \mathbb{R}^m$ with the corresponding action of $GL_m(\mathbb{R})$ on it.

8 Variable spaces, Homotopy Spectra in Families and Parametric Homological Localisation.

Topological spaces Ψ with (energy) functions E on them often come in families.

In fact, the proofs of the sharp lower bound on the Minkowski waists of spheres (see section 6) and of Guth' lower bounds on the full vol_k-spectra (see the previous section) depend on localizing *not* to (smallish) *fixed* disjoint subsets $U_i \subset X$ but to variable or "*parametric*" ones that may change/move along with the subsets Y_s, $s \in S$, in them.

In general, families $\{\Psi_q\}$ are constituted by "fibres" of continuous maps F from a space $\Psi = \Psi_Q$ to Q where the "fibers" $\Psi_q = F^{-1}(q) \subset \Psi$, $q \in Q$, serve as the members of these families and where the energies E_q on Ψ_q are obtained by restricting functions E from Ψ to $\Psi_q \subset \Psi$.[31]

Homotopy spectra in this situation are defined with continuous families of spaces S_q that are "fiberes" of continuous maps $S \to Q$ and where the relevant maps $\phi : S \to \Psi$ send $S_q \to \Psi_q$ for all $q \in Q$ with these maps denoted $\phi_q = \phi_{|S_q}$.

Then the energy of the fibered homotopy class $[\phi]_Q$ of such a fiber preserving map ϕ is defined as earlier as

$$E[\phi]_Q = \inf_{\phi \in [\phi]_Q} \sup_{s \in S} E \circ \phi(s) \leq \sup_{q \in Q} E_q[\phi_q],$$

where the latter inequality is, in fact, an equality in many cases.

Example 1: k-Cycles in Moving Subsets. Let U_q be a Q-family of open subsets in a Riemannian manifold X. An instance of this is the family of the ρ-balls $U_x = U_x(\rho) \in X$ for a given $\rho \geq 0$ where X itself plays the role of Q.

Define $\Psi = \Psi_Q$ as the space $C_k\{U_q; \Pi\}_{q \in Q}$ of k-dimensional Π-cycles[32] $c = c_q$ in U_q for all $q \in Q$, that is $\Psi = \Psi_Q$ equals the space of pairs $(q \in Q, c_q \in C_k(U_q; \Pi))$, where, as earlier, Π is an Abelian (coefficient) group with a norm-like function; then we take $E(c) = E_q(c_q) = vol_k(c)$ for the energy.

Example 2: Cycles in Spaces Mapped to an X. Here, instead of subsets in X we take locally diffeomorphic maps y from a fixed Riemannian manifold U into X and take the Cartesian product $C_k(U; \Pi) \times Q$ for $\Psi = \Psi_Q$.

Example 1+2: Maps with variable domains. One may deal families of spaces U_q (e.g. "fibers" $U_q = \psi^{-1}(q)$ of a map between smooth manifolds $\psi : Z \to Q$) along with maps $y_q : U_q \to X$.

On Reduction 1 ⇒ 2. There are cases, where the spaces $\Psi_Q = C_k\{U_q; \Pi\}_{q \in Q}$ of cycles in moving subsets $U_q \subset X$ topologically split:

$$\Psi_Q = C_k(B; \Pi) \times Q, \text{ for a fixed manifold } U.$$

A simple, yet representative, example is where $Q = X$ for the m-torus, $X = \mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$, where $B = U_0$ is an open subset in $\mathbb{T}^m$ and where $Y = X = \mathbb{T}^m$ equals the space of translates $U_0 \mapsto U_0 + x$, $x \in \mathbb{T}^m$.

[31] Also one may have functions with the range also depending on q, say $a_q : \Psi_q \to R_q$ and one may generalise further by defining families as some (topological) sheaves over Grothendieck sites.

[32] "k-Cycle" in $U \subset X$ means a *relative* k-cycle in $(U, \partial U)$, that is a k-chain with boundary contained in the boundary of U. Alternatively, if U is an open non compact subset, "k-cycle" means an *infinite* k-cycle, i.e. with (a priori) *non-compact support.*

For instance, if U_0 is a ball of radius $\varepsilon \leq 1/2$, then it can be identified with the Euclidean ε-ball $B = B(\varepsilon) \subset \mathbb{R}^m$.

Similar splitting is also possible for *parallelizable* manifolds X with injectivity radii $> \varepsilon$ where moving ε-balls $U_x \subset X$ are obtained via the exponential maps $exp_q : T_q = \mathbb{R}^m \to X$ from a fixed ball $B = B(\varepsilon) \subset \mathbb{R}^m$.

In general, if X is *non-parallelizable*, one may take the space of the tangent orthonormal frames in X for Q, where, the product space $C_k(B;\Pi) \times Q$, where $B = B(\varepsilon) \subset \mathbb{R}^m$, makes a principle $O(m)$- fibration, $m = dim(X)$, over the space $C_k\{U_x(\varepsilon);\Pi\}_{x\in X}$ of cycles of moving ε-balls $U_x(\varepsilon) \subset X$.[33]

Waists of "variable metric spaces" needed for homological localization of the spectra of the volume energies on *fixed spaces* can be defined as follows.

Let $\mathcal{X} = \{X_q\}_{q\in Q}$ be a family of metric spaces seen as the fibres, i.e. the pullbacks of points, of a continuous map $\varpi : \mathcal{X} \to Q$ and consider S-families of subsets in X_q that are $Y_s \subset X_{q(s)}$ defined with some maps $S \to Q$ for $s \mapsto q = q(s)$.

The k-waist of such a family is defined as

$$waist_k\{X_q\} = \inf \sup_{s\in S} vol_k(Y_s)$$

where vol_k is one of the "volumes" from section 6, e.g. Hausdorff's k-measure and where "inf" is taken over all *"homologically substantial"* families Y_s.

In order to define the latter and to keep the geometric picture in mind, we

• fix a section $Q_0 \subset \mathcal{X}$, that is a continuous family[34] of points $x(q) \in X_q$;

• assume that X is an n-manifold and that the family Y_s is given by fibres of a map $\Sigma = \cup_{s\in S} Y_s \to S$ for Σ being an n-pseudomanifold of dimensions n.

Then the family $Y_s \subset X_{q(s)}$ comes via a map $\Sigma \to \mathcal{X}$ and "homologically substantial" is understood as *non-vanishing* of the intersection index of $Q_0 \subset \mathcal{X}$ with Σ mapped to $\mathcal{X}$.

For instance, if $\varpi : \mathcal{X} \to Q$ is a fibration with *contractible* fibres X_s, the section $Q_0 \subset \mathcal{X}$ exits and homotopically unique that makes our index non-ambiguously defined.

Notice that $waist_k\{X_q\}$ may be strictly smaller than $\inf_{q\in Q} waist_k(X_q)$; yet, the argumnt(s) used for individual X show that the waists of *compact* families of (connected) Riemannian manifolds are *strictly positive.*

Example: "Ameba" Penetrating a Membrane. Let a domain $U_t \subset \mathbb{R}^3$, $0 \leq t \leq 1$, be composed of a pair of disjoint balls of radii t and $1-t$ joint by a δ-thin tube. The 2-waist of U_t is at least $\pi/4$, that is the waist of the ball of radius $t/2$ for all $t \in [0,1]$. But the waist of the "variable domain" U_t equals the area of the section of the tube that is $\pi\delta^2$.

9 Restriction and Stabilisation of Packing Spaces.

There is no significant relations between *individual* packings of manifolds and their submanifolds, but such relations do exist for *spaces of packings.*

[33]Vanishing of Stiefel-Whitney classes seems to suffice for (homological) splitting of this vibration in the case $\Pi = Z_2$ as in section 6.3 of my article *"Isoperimetry of Waists..."*) in GAFA

[34]In fact, one only needs a distinguished "horizontal (co)homology class" in $\mathcal{X}$.

For instance, let $X_0 \subset X$ be a closed n_0-dimensional submanifold in an n-dimensional, Riemannian manifold X and let

$$\frown_I : H_*(X^I) \to H_{*-*'}(X_0^I),\ *' = N(n-n_0),\ \ N = card(I),$$

be the homomorphism corresponding to the (generic) intersections of cycles in the Cartesian power X^I with the submanifold $X_0^I \subset X^I$, where the homology groups are understood with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$-coefficients, since we do not assume that X_0 is orientable. Then

the homological packing energies E_* of X and X_0 (obviously) *satisfy*

$$E_*(\frown_I (h)) \le E_*(h),$$

for all homology classes $h \in H_*(X^I) = H_*(X)^{\otimes I}$, where packings of X_0 are understood with respect to the metric, i.e. distance function, induced from $X \supset X_0$.

Corollary. Let $P(X; I, r) \subset X^I$ be the space of I-packings of X by balls of radii r and let $S \subset P(X; I, r)$ be a K-cycle, $K = N(n-n_0)$ that has a non-zero intersection index with $X_0^I \subset X^I$.

Then X_0 admits a packing by N-balls of radius r.

Next, let us invert the intersection homomorphism $\frown_I$ in a presence of a *projection* also called *retraction* $p : X \to X_0$ of X to X_0, i.e. where p fixes X_0.

If p is a fibration or, more generally it is a generic smooth p, then the pullbacks $Y_i = p_{-1}(x_i)X$ are k-cycles, $k - n - n_0$, that transversally meet X_0 at the points $x_i \in X_0$. It follows, that the Cartesian product $N(n-n_0)$-cycle

$$S = \bigtimes_{i \in I} Y_i \subset X^I$$

has non-zero intersection index with $X_0^I \subset X_I$.

And – now geometry enters – if p is a (non-strictly) *distance decreasing* map, then, obviously, this S is positioned in the space $P(X; I, r) \subset X^I$ of I-packing of X by r-balls $U_r(x_i)$ and multiplication of cycles in $C \subset P(X_0; I, r)$ with S, that is $C \mapsto C \times S$, *imbeds*

$$H_*(P(X_0; I, r)) \underset{\times S}{\to} H_{*+Nk}(P(X; I, r)),$$

such that the composed map

$$H_*(P(X_0; I, r)) \underset{\times S}{\to} H_{*+Nk}((P(X; I, r)) \underset{\frown_I}{\to} H_*(P(X_0; I, r))$$

equals the identity.

Thus,

the homology packing spectrum of X fully determines such a spectrum of X_0.[35]

[35]If p is *a homotopy retraction*, e.g. $p : X \to X_0$ is a vector bundle, then the above homomorphisms come from the the Thom *isomorphisms* between corresponding spaces. Yet, there is more to packings of X than what comes from X_0, it already seen in the case where X is a ball and $X_0 = \{0\}$.

However, ball packings of X_0 in this case properly reflect properties of packing of X by r-neighbourhoods of k-cycles $Y_i \subset X$ the intersection indices of which with X_0 equal one.

Example. Let $\underline{X}$ be a compact manifold of negative curvature and $X_0 \subset \underline{X}$ be a closed geodesic. Then the above applies to the covering $X = X(X_0)$ of $\underline{X}$ with the cyclic fundamental group generated by the homotopy class of X_0.

Therefore,

the lengths of all the closed geodesics in $\underline{X}$ are determined by the homotopy packing spectrum of $\underline{X}$.

Questions. How much of the geometry of minimal varieties V in X, that are critical points of the volume energies, can be seen in terms of the above families of packings of X by balls (or by non-round subsets as in [22]) "moving transversally to" V?

(Minimal subvarieties V can be approximated by sets of centres of small δ-balls densely packing these V; this suggests looking at spaces of packings of $c\delta$-neihgbourhoods $U_{c\delta}(V) \subset X$ by $(1-\varepsilon)\delta$-balls for some $c > 1$.

Symplectic Remark. The above relation between, say, individual packings of an $X_0 \subset X$ by N balls and $N(n - n_0)$-dimensional families of packings of X by balls "moving transversally to X_0" is reminiscent of *hyperbolic stabilisation of Morse functions* as it used in the study of *generating functions* in the symplectic geometry, see []viterbo, eliashberg-gromov and references therein.

Is there something more profound here than just a simple minded similarity?

10 Homotopy Height, Cell Numbers and Homology.

The homotopy spectral values $r \in \mathbb{R}$ of $E(\psi)$ are "named" after (indexed by) the homotopy classes $[\phi]$ of maps $\phi : S \to \Psi$, where $r = r_{[\phi]}$ is, by definition, the minimal r such that $[\phi]$ comes from a map $S \to \Psi_r \subset \Psi$ for $\Psi_r = E^{-1}(-\infty, r]$. In fact, such a "name" depends only on the partially ordered set, cal it $\mathcal{H}_{\gtrless}(\Psi)$, that is *the maximal partially ordered reduction* of $\mathcal{H}_{\circ}(\Psi)$ defined as follows.

Write $[\phi_1] \prec [\phi_2]$ if there is a morphism $\psi_{12} : [\phi_1] \to [\phi_2]$ in $\mathcal{H}_{\circ}(\Psi)$ and turn this into a partial order by identifying objects, say $[\phi]$ and $[\phi']$, whenever $[\phi] \prec [\phi']$ as well as $[\phi'] \prec [\phi]$.

Perfect Example. If X is (homotopy equivalent to) the real projective space P^∞ then the partially ordered set $\mathcal{H}_{\gtrless}(\Psi)$ is isomorphic to the set of nonnegative integers $\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$. This is why spectral (eigen) values are indexed by integers in the classical case.

In general the set $\mathcal{H}_{\gtrless}(\Psi)$ may have undesirable(?) "twists". For instance, if Ψ is homotopy equivalent to the circle, then $\mathcal{H}_{\gtrless}(\Psi)$ is isomorphic to set $\mathbb{Z}_+$ with the *division order*, where $m \succ n$ signifies that m divides n. (Thus, 1 is the maximal element here and 0 as the minimal one.)

Similarly, one can determine $\mathcal{H}_{\gtrless}(\Psi)$ for general Eilenberg-MacLane spaces $\Psi = K(\Pi, n)$. This seems transparent for Abelian groups Π. But if a space Ψ, not necessarily a $K(\Pi, 1)$, has a non-Abelian fundamental group $\Pi = \pi_1(\Psi)$, such as the above space $\Psi_N(X)$ of subsets $\psi \subset X$ of cardinality N, then keeping track of the conjugacy classes of subgroups $\Pi' \subset \Pi$ and maps $\phi : S \to \Psi$ that send $\pi_1(S)$ to these Π' becomes more difficult.

If one wishes (simple mindedly?) to remain with integer valued spectral values, one has to pass to some numerical invariant that that takes values in a quotient of $\mathcal{H}_{\gtrless}$ isomorphic to $\mathbb{Z}_+$, e.g. as follows.

Homotopy Height. Define *the homotopy (dimension) height* of a homotopy class $[\phi]$ of continuous map $\phi : S \to \Psi$ as the minimal integer n such that the $[\phi]$ factors as $S \to K \to \Psi$, where K is a cell complex of dimension (at most) n.

"Stratification" of Homotopy Cohomotopy Spectra by Hight. This "hight" or a similar hight-like function defines a partition of the homotopy spectrum into the subsets, call them $Hei_n \subset \mathbb{R}$, $n = 0, 1, 2, ...,$ of the values of the energy $E[\phi] \in \mathbb{R}$ on the homotopy classes $[\phi]$ with homotopy heights n, where either the *supremum* or the *infimum* of the numbers $r \in Hei_n$ may serve as the *"n-th HH-eigenvalue of ψ".*

One also may "stratify" cohomotopy spectra by replacing "*contractibility condition* of maps $\psi_{|\Psi_r} : \Psi_r \to T$ by $\psi_{|\Psi_r} \le n$.

In the classical case of $\Psi = P^\infty$ any such "stratification" of homotopy "eigenvalues" lead the usual indexing of the spectrum. where, besides the homotopy hight, among other hight-like invariant invariants we indicate the following.

Example 1: Total Cell Number. Define $N_{cell}[\phi]$ as the minimal N such that $[\phi]$ factors as $S \to D \to \Psi$, where D is a cell complex with (at most) N cells in it.

What are, roughly, the total cell numbers of the classifying maps from packing spaces of an X by N balls to the classifying space $B\mathbb{S}_N$?

What are these numbers for the maps between classifying spaces of "classical" finite groups G corresponding to standard injective homomorphisms $G_1 \to G_2$?

Example 2: Homology Rank. Define $rank_{H_*}[\phi]$ as the maximum over all fields $\mathbb{F}$ of the the $\mathbb{F}$-ranks of the induced homology homomorphisms
$[\phi]_* : H_*(S; \mathbb{F}) \to H_*(\Psi; \mathbb{F})$.

On Essentiality of Homology. There are other prominent spaces, X, besides the infinite dimensional projective spaces $X = P^\infty$, and energy functions on them, such as

spaces Ψ of loops $\psi : S^1 \to X$ in simply connected Riemannian manifolds X with length(ψ) taken for $E(\psi)$[36],

where the cell numbers and the homology ranks spectra for $E(\psi) = lenght(\psi)$ are "essentially determined" by the homotopy height. (This is why the homotopy height was singled out under the name of "essential dimension" in my paper *Dimension, Non-liners Spectra and Width.*)

However, the homology carries significantly more information than the homotopy hight for the k-volume function on the *spaces of k-cycles of codimensions* ≥ 2 as it was revealed by Larry Guth in his paper *Minimax problems related to cup powers and Steenrod squares*.

On Height and the Cell Numbers of Cartesian Products. If the homotopy heights and/or cell numbers of maps $\phi_i : S_i \to \Psi_i$, $i = 1, ..., k$, can be expressed in terms of the corresponding homology homomorphisms over some filed $\mathbb{F}$ independent of i, then, according to *Künneth formula,* the homotopy hight of the

[36]This instance of essentiality of the homotopy heights is explained in my article *Homotopical Effects of Dilatation*, while the full range of this property among "natural" spaces Ψ of maps ψ between Riemannian manifolds and energies $E(\psi)$ remains unknown.

Cartesian product of maps,

$$\phi_1 \times ... \times \phi_k : S_1 \times ... \times S_k \to \Psi_1 \times ... \times \Psi_k,$$

is additive

$$height[\phi_1 \times ... \times \phi_k] = height[\phi_1] + ... + height[\phi_k]$$

and the cell number is multiplicative

$$N_{cell}[\phi_1 \times ... \times \phi_k] = N_{cell}[\phi_1] \times ... \times N_{cell}[\phi_k].$$

What are other cases where these relation remain valid?

Specifically, we want to know what happens in this regard to the following classes of maps:

(a) *maps between spheres* $\phi_i : S^{m_i+n_i} \to S^{m_i}$,

(b) *maps between locally symmetric spaces*, e.g. compact manifolds of *constant negative curvatures*,

(c) *high Cartesian powers* $\phi^{\times N} : S^{\times N} \to A^{\times N}$ of a single map $\phi : S \to \Psi$.

When do, for instance, the limits

$$\lim_{N\to\infty} \frac{height[\phi^{\times N}]}{N} \text{ and } \lim_{N\to\infty} \frac{\log N_{cell}[\phi^{\times N}]}{N}$$

not vanish? (These limits exist, since the the hight and the logarithm of the cell number are sub-additive under Cartesian product of maps.)

Probably, the general question for *"rational homotopy classes* $[...]_{\mathbb{Q}}$*"* (instead of "full" homotopy classes" $[...] = [...]_{\mathbb{Z}}$) of maps into *simply connected* spaces Ψ_i is easily solvable with *Sullivan's minimal models*.

Also, the question may be more manageable for *homotopy classes* mod p.

Multidimensional Spectra Revisited. Let $h = h^T$ be a cohomotopy class of Ψ, that is a homotopy class of maps $\Psi \to T$, and let υ be a function on homotopy classes of maps $U \to T$ for open subsets $U \subset \Psi$, where the above height-like functions, such as the *homology rank*, are relevant examples of such a υ.

Then the values of υ on h restricted to open subsets $U \subset \Psi$ define a numerical (set) function, $U \mapsto \upsilon(h_{|U})$ and every continuous map $\mathcal{E} : \Psi \to Z$ pushes down this function to open subsets in X.

For instance, if $\upsilon = 0, 1$ depending on whether a map $U \to T$ is contractible or not and if $Z = \mathbb{R}^l$, then this function on the "negative octants" $\{x_1 < e_1, ..., x_k < e_k, ..., , x_l < e_l\}$ in $\mathbb{R}^l$ carries the same message as Σ_h from section 4.

11 Graded Ranks, Poincare Polynomials, Ideal Valued Measures and Spectral $\smile$-Inequality.

The images as well as kernels of (co)homology homomorphisms that are induced by continuous maps are *graded* Abelian groups and their ranks are properly represented not by individual numbers but by *Poincaré polynomials.*

Thus, sublevel $\Psi_r = E^{-1}(-\infty, r] \subset \Psi$ of energy functions $E(\psi)$ are characterised by the *polynomials* Poincaré$_r(t; \mathbb{F})$ of the the inclusion homomorphisms $\phi_i(r) : H_i(\Psi_r; \mathbb{F}) \to H_i(\Psi; \mathbb{F})$, that are

$$\text{Poincaré}_r\text{=Poincaré}_r(t; \mathbb{F}) = \sum_{i=0,1,2,...} t^i rank_{\mathbb{F}} \phi_i(r).$$

Accordingly, the homology spectra, that are the sets of those $r \in \mathbb{R}$ where the ranks of $\phi_*(r)$ change, are indexed by such polynomials with positive integer coefficient. (The semiring structure on the set of such polynomials coarsely agrees with basic topological/geometric constructions, such as taking $E(\psi) = E(\psi_1) + E(\psi_2)$ on $\Psi = \Psi_1 \times \Psi_2$.)

The set function $D \mapsto \text{Poincaré}_D$ that assigns these Poincaré polynomials to subsets $D \subset \Psi$, (e.g. $D = \Psi_r$) has some measure-like properties that become more pronounced for the set function

$$\Psi \supset D \mapsto \mu(D) = \mu^*(D;\Pi) = \mathbf{0}^{\smallsetminus *}(D;\Pi) \subset H^* = H^*(\Psi;\Pi),$$

where Π is an Abelian (homology coefficient) group, e.g. a field $\mathbb{F}$, and $\mathbf{0}^{\smallsetminus *}(D;\Pi)$ is the *kernel* of the cohomology restriction homomorphism for the complement $\Psi \smallsetminus D \subset \Psi$,

$$H^*(\Psi;\Pi) \to H^*(\Psi \smallsetminus D;\Pi).$$

Since the cohomology classes $h \in \mathbf{0}^{\smallsetminus *}(D;\Pi) \subset H^* = H^*(\Psi;\Pi)$ are representable by cochains with the support in D, [37]

the set function

$$\mu^* : \{subsets \subset \Psi\} \to \{subgroups \subset H^*\}$$

is additive for the sum-of-subsets in H^* and super-multiplicative[38] *for the the $\smile$-product of ideals in the case where Π is a commutative ring*:

$$[\cup +] \qquad \mu^*(D_1 \cup D_2) = \mu^*(D_i) + \mu^*(D_2)$$

for *disjoint* open subsets D_1 and D_2 in Ψ, and

$$[\cap \smile] \qquad \mu^*(D_1 \cap D_2) \supset \mu^*(D_1) \smile \mu^*(D_2)$$

for all open $D_1, D_2 \subset \Psi$.[39]

The relation $[\cap \smile]$, applied to $D_{r,i} = E_i^{-1}(r,\infty) \subset \Psi$ can be equivalently expressed in terms of cohomomoly spectra as follows.

Spectral $[\min \smile]$-*Inequality.*[40] Let $E_1, ..., E_i, .., E_N : \Psi \to \mathbb{R}$ be continuous functions/energies and let $E_{min} : \Psi \to \mathbb{R}$ be the minimum of these,

$$E_{min}(\psi) = \min_{i=1,...,N} E_i(\psi), \ \psi \in \Psi.$$

Let $h_i \in H^{k_i}(\Psi;\Pi)$ be cohomology classes, where Π is a commutative ring, and let

$$h_\smile \in H^{\sum_i k_i}(\Psi;\Pi)$$

[37] This property suggests an extension of μ to multi-sheated *domains D over Ψ* where D go to Ψ by non-injective, e.g. locally homeomorphic finite to one, maps $D \to A$.

[38] This, similarly to *Shannon's subadditivity inequality*, implies the existence of *"thermodynamic limits"* of *Morse Entropies*, see [7].

[39] See section 4 of my article *Singularities, Expanders and Topology of Maps. Part 2.* for further properties and applications of these "measures" .

[40] This inequality implies the existence of *Hermann Weyl limits* of energies of cup-powers in infinite dimensional projective (and similar) spaces, see [19].

be the $\smile$-product of these classes,

$$h_\smile = h_1 \smile \ldots \smile h_i \smile \ldots \smile h_N.$$

Then

$$[\min \smile] \qquad E^*_{min}(h_\smile) \geq \min_{1=1,\ldots,N} E^*_i(h_i).$$

Consequently, the value of the "*total energy*"

$$E_\Sigma = \sum_{i=1,\ldots,N} E_i : \Psi \to \mathbb{R}$$

on this cohomology class $h_\smile \in H^*(\Psi;\Pi)$ is bounded from below by

$$E^*_\Sigma(h_\smile) \geq \sum_{i=1,\ldots,N} E^*_i(h_i).$$

(This has been already used in the homological localisation of the volume energy in section 7.) kk

On Multidimensional Homotopy Spectra. These spectra, as defined in sections 4, 10,11 represent the values of the pushforward of the "measure" μ^* under maps $\mathcal{E} : \Psi \to \mathbb{R}^l$ on special subsets $\Delta \subset \mathbb{R}^l$; namely, on complements to $\times_{k=1,\ldots,l}(-\infty, e_k] \subset \mathbb{R}^l$ and the spectral information is encoded by $\mu^*(\mathcal{E}^{-1}(\Delta)) \subset H^*(\Psi)$.

On may generalise this by enlarging the domain of μ^*, say, by evaluating $\mu^*(\mathcal{E}^{-1}(\Delta))$ for some class of simple subsets Δ in $\mathbb{R}^l$, e.g. convex sets and/or their complements.

On $\wedge$-Product. The (obvious) proof of $[\cap \smile]$ (and of $[\min \smile]$) relies on locality of the $\smile$-product that, in homotopy theoretic terms, amounts to factorisation of $\smile$ via $\wedge$ that is the *smash product* of (marked) Eilenberg-MacLane spaces that represent cohomology, where, recall, the *smash product* of spaces with marked points, say $T_1 = (T_1, t_1)$ and $T_2 = (T_2, t_2)$ is

$$T_1 \wedge T_2 = T_1 \times T_2 / T_1 \vee T_2$$

where the factorisation "$/T_1 \vee T_2$" means "*with the subset* $(T_1 \times t_2) \cup (t_1 \times T_2) \subset T_1 \times T_2$ *shrunk to a point*" (that serves to mark $T_1 \wedge T_2$).

In fact, general cohomotopy "measures" (see 1.9) and spectra defined with maps $\Psi \to T$ satisfy natural (obviously defined) counterparts/generalizations of $[\cap \smile]$ and $[\min \smile]$, call them $[\cap \wedge]$ and $[\min \wedge]$ that are

On Grading Cell Numbers. Denote by $N_{i\_cell}[\phi]$ the minimal number N_i such that homotopy class $[\phi]$ of maps $S \to \Psi$ factors as $S \to K \to \Psi$ where K is a cell complex with (at most) N_i cells *of dimension* i and observe that the total cell number is bounded by the sum of these,

$$N_{cell}[\phi] \leq \sum_{i=0,1,2,\ldots} N_{i\_cell}[\phi].$$

Under what conditions on ϕ does the sum $\sum_i N_{i\_cell}[\phi]$ (approximately) equal $N_{cell}[\phi]$?

What are relations between the cell numbers of the covering maps ϕ between (arithmetic) locally symmetric spaces Ψ besides $N_{cell} \leq \sum_i N_{i\_cell}$?[41]

[41] The identity maps $\phi = id : \Psi \to \Psi$ of locally symmetric spaces Ψ seem quite nontrivial

12 Symmetries, Equivariant Spectra and Symmetrization.

. If the energy function E on Ψ is invariant under a continuous action of a group G on Ψ – this happens frequently – then the relevant category is that of *G-spaces* S, i.e. of topological spaces S acted upon by G, where one works with *G-equivariant* continuous maps $\phi : S \to \Psi$, *equivariant* homotopies, equaivariinat (co)homologies, decompositions, etc.

Relevant examples of this are provided by symmetric energies $E = E(x_1, ..., x_N)$ on Cartesian powers of spaces, $\Psi = X^{\{1,...,N\}}$, such as our (ad hoc) packing energy for a metric space X,

$$E\{x_1, ..., x_i, ..., x_N\} = \sup_{i \neq j = 1,...,N} dist^{-1}(x_i, x_j)$$

that is invariant under the symmetric group Sym_N. It is often profitable, as we shall see later on, to exploit the symmetry under certain *subgroups* $G \subset Sym_N$.

Besides the group Sym_N, energies E on $X^{\{1,...,N\}}$ are often invariant under some groups H acting on X, such as the isometry group $Is(X)$ in the case of packings.

If such a group H is compact, than its role is less significant than that of Sym_N, especially for large $N \to \infty$; yet, if H properly acts on a *non-compact* space X, such as $X = \mathbb{R}^m$ that is acted upon by its isometry group, then H and its action become essential.

MIN-Symmetrized Energy. An arbitrary function E on a G-space Ψ can be rendered G-invariant by taking a symmetric function of the numbers $e_g = E(g(\psi)) \in \mathbb{R}$, $g \in G$. Since we are mostly concerned with the order structure in $\mathbb{R}$, our preferred symmetrisation is

$$E(\psi) \mapsto \inf_{g \in G} E(g(\psi)).$$

Minimization with Partitions. This inf-symmetrization does not fully depends on the action of G but rather on the partition of Ψ into orbits of G. In fact, given an arbitrary partition α of Ψ into subsets that we call *α-slices*, one defines the function

$$E_{inf_\alpha} = inf_\alpha E : \Psi \to \mathbb{R}$$

where $E_{inf_\alpha}(\psi)$ equals the infimum of E on the α-slice that contains ψ for all $\psi \in \Psi$. Similarly, one defines $E_{sup_\alpha} = sup_\alpha E$ with E_{min_α} and E_{max_α} understood accordingly.

Example: Energies on Cartesian Powers. The energy E on Ψ induces N energies on the space $\Psi^{\{1,...,N\}}$ of N-tuples $\{\psi_1, ..., a_i, ..., a_N\}$, that are

$$E_i : \{\psi_1, ..., a_i, ..., a_N\} \mapsto E(a_i).$$

It is natural, both from a geometric as well as from a physical prospective, to symmetrize by taking *the total energy* $E_{total} = \sum_i E_i$. But in what follows we

in this regard. On the other hand, general locally isometric maps $\phi : \Psi_1 \to \Psi_2$ between symmetric spaces as well as continuous maps $S \to \Psi$ of positive degrees, where S and Ψ are equidimensional manifolds with only Ψ being locally symmetric, are also interesting.

shall resort to $E_{min} = \min_i E_i = \min_i E(a_i)$ and use it for bounding the total energy from below by

$$E_{total} \geq N \cdot E_{min}.$$

For instance, we shall do it for families of N-tuples of balls U_i in a Riemannian manifold V, thus bounding the k-volumes of k-cycles c in the unions $\cup_i U_i$, where, observe,

$$vol_k(c) = \sum_i vol_k(c \cap U_i)$$

if the balls U_i *do not intersect*.

$$E(c) = \sum_i c \cap U_{x_i}.$$

This, albeit obvious, leads, as we shall see later on, to non-vacous relations between

homotopy/homology spectrum of the vol_k-energy on the space $C_k(X;\Pi)$
and
equivariant homotopy/homology of the spaces of packings of X by ε-balls.

13 Equivariant Homotopies of Infinite Dimensional Spaces.

If we want to understand homotopy spectra of spaces of "natural energies" on spaces of infinitely many particles in non-compact manifolds, e.g. in Euclidean spaces, we need to extend the concept of the homotopy and homology spectra to infinite dimensional spaces Ψ, where infinite dimensionality is compensated by an additional structure, e.g. by an action of an infinite group Υ on Ψ.

The simplest instance of this is where Υ is a countable group that we prefer to call Γ and $\Psi = B^\Gamma$ be the space of maps $\Gamma \to B$ with the (obvious) *shift action* of Γ on this Ψ, motivates the following definition (compare [7]). Let H^* be a graded algebra (over some field) acted upon by a countable amenable group Γ. Exhaust Γ by finite *Følner subsets* $\Delta_i \subset \Gamma$, $i = 1, 2, ...$, and, given a finite dimensional graded subalgebra $K = K^* \subset H^*$, let $P_{i,K}(t)$ denote the Poincare polynomial of the graded subalgebra in H^* generated by the γ-transforms $\gamma^{-1}(H^*_K) \subset H^*$ for all $\gamma \in \Delta_i$.

Define *polynomial entropy* of the action of Γ on H^* as follows.

$$Poly.ent(H^* : \Gamma) = \sup_K \lim_{i \to \infty} \frac{1}{card(\Delta_i)} \log P_{i,K}(t).$$

Something of this kind could be applied to subalgebras $H^* \subset H^*(\Psi; \mathbb{F})$, such as images and/or kernels of the restriction cohomology homomorphisms for (the energies sublevel) subsets $U \subset \Psi$, IF the following issues are settled.

1. In our example of moving balls or particles in $\mathbb{R}^m$ the relevant groups Υ, such as the group of the orientation preserving Euclidean isometries are connected and act *trivially* on the cohomologies of our spaces Ψ.

For instance, let $\Gamma \subset \Upsilon$ be a discrete subgroups and Ψ equal the *dynamic Υ-suspension of B^Γ*, that is $B^\Gamma \times \Upsilon$ divided by the diagonal action of Γ.

$$\Psi = \left(B^\Gamma \times \Upsilon\right)/\Gamma.$$

The (ordinary) cohomology of this space Ψ are bounded by those of B tensored by the cohomology of Υ/Γ that would give *zero* polynomial entropy for finitely generated cohomology algebras $H^*(B)$.

In order to have something more interesting, e.g. the *mean Poincaré polynomial* equal that of B^Γ, which is the ordinary Poincaré$(H^*(B))$, one needs a definition of some *mean (logarithm) of the Poincaré polynomial* that might be *far from zero* even if the ordinary cohomology of Ψ vanish.

There are several candidates for such *mean Poincaré polynomials*, e.g the one is suggested in section1.15 of my article *Topological Invariants of Dynamical Systems and Spaces of Holomorphic Maps.*

Another possibility that is applicable to the above $\Psi = \left(B^\Gamma \times \Upsilon\right)/\Gamma$ with *residually finite* groups Γ is using finite i-sheeted covering $\tilde{\Psi}_i$ corresponding to subgroups $\Gamma_i \subset \Gamma$ of order i and taking the limit of

$$\lim_{i\to\infty} \frac{1}{i} \log \text{Poincaré}(H^*(B)).$$

,

(Algebraically, in terms of actions of groups Γ on abstract graded algebras H^*, this corresponds to taking the normalised limit of logarithms of Γ_i*-invariant* sub-algebras in H^*; this brings to one's mind a possibility of a generalisation of the above polynomial entropies to *sofic groups* (compare [5]).

2. The above numerical definitions of the polynomial entropy and of the mean Poincaré polynomials beg to be rendered in categorical terms similarly to the ordinary entropy (see [20]).

3. The spaces $\Psi_\infty(X)$ of (discrete) infinite countable subsets $\psi \subset X$ that are meant to represent infinite ensembles of particles in non-compact manifolds X, such as $X = \mathbb{R}^m$, are more complicated than $\Psi = B^\Gamma$, $\Psi = \left(B^\Gamma \times \Upsilon\right)/\Gamma$ and other "product like" spaces studied eralier.

These $\Psi_\infty(X)$ may be seen as as limits of finite spaces $\Psi_N(X_N)$ for $N \to \infty$ of N-tuples of points in compact manifolds X_N where one has to chose suitable approximating sequences X_N.

For instance, if $X = \mathbb{R}^m$ acted upon by some isometry group Υ of $\mathbb{R}^m$ one may use either the balls $B^m(R_N) \subset \mathbb{R}^m$ of radii $R_N = const \cdot R^{N/\beta}$ in $\mathbb{R}^m$, $\beta > 0$ for X_N or the tori $\mathbb{R}^m/\Gamma_N$ with the lattices $\Gamma_N = const \cdot M \cdot \mathbb{Z}^m$ with $M = M_N \approx N^{\frac{1}{\beta}}$ for some $\beta > 0$.[42]

Defining such limits and working out functional definitions of relevant structures the limit spaces, collectively callused $\Psi_\infty(X)$ are the problems we need to solve where, in particular, we need to

• incorporate actions of the group Υ coherently with (some subgroup) of the infinite permutations group acting on subsets $\psi \subset X$ of particles in X that represent points in $\Psi_\infty(X)$

and

• define (stochastic?) homotopies and (co)homologies in the spaces $\Psi_\infty(X)$, where these may be associated to limits of families of n-tuples $\psi_{P_N} \subset X_N$ parametrised by some P_N where $dim(P_N)$ may tend to infinity for $N \to \infty$.[43]

[42] The natural value is $\beta = m$ that make the volumes of X_N proportional to N but smaller values, that correspond to ensembles of points in $\mathbb{R}^m$ of *zero densities*, also make sense as we shall see later on.

[43] We shall meet families of dimensions $dim(P_N) \sim N^{\frac{1}{\gamma}}$ where $\gamma + \beta = m$ for the above β.

4. Most natural energies E on infinite particle spaces $\Psi_\infty(X)$ are everywhere infinite[44] and defining "sublevels" of such E needs attention.

14 Symmetries, Families and Operations Acting on Cohomotopy Measures.

Cohomotopy "Measures". Let T be a space with a distinguished *marking point* $t_0 \in T$, let $H^\circ(\Psi;T)$ denote the set of homotopy classes of maps $\Psi \to T$ and define the "T-measure" of an open subset $U \subset \Psi$,

$$\mu^T(U) \subset H^\circ(\Psi;T),$$

as the set of homotopy classes of maps $\Psi \to T$ that send the complement $\Psi \setminus U$ to t_0.

For instance, if T is the Cartesian product of Eilenberg-MacLane spaces $K(\Pi;n)$, $n = 0,1,2,...,$, then $H^\circ(\Psi;T) = H^*(\Psi;\Pi)$ and μ^T identifies with the (graded cohomological) ideal valued "measure" $U \mapsto \mu^*(U;\Pi) \subset H^*(\Psi;\Pi)$ from section 1.5.

Next, given a category $\mathcal{T}$ of marked spaces T and homotopy classes of maps between them, denote by $\mu^{\mathcal{T}}(U)$ the totality of the sets $\mu^T(U)$, $T \in \mathcal{T}$, where the category $\mathcal{T}$ acts on $\mu^{\mathcal{T}}(U)$ via composition $\Psi \overset{m}{\to} T_1 \overset{\tau}{\to} T_2$ for all $m \in \mu^T(U)$ and $\tau \in \mathcal{T}$.

For instance, if $\mathcal{T}$ is a category of Eilenberg-MacLane spaces $K(\Pi;n)$, this amounts to the natural action of the (unary) cohomology operations (such as Steenrod squares Sq^i in the case $\Pi = \mathbb{Z}_2$) on ideal valued measures.

The above definition can be adjusted for spaces Ψ endowed with additional structures.

For example, if Ψ represents a *family of spaces* by being endowed with a partition β into closed subsets – call them *β-slices* or *fibers* – then one restricts to the space of maps $\Psi \to T$ *constant on these slices* (if T is also partitioned, it would be logical to deal with maps sending slices to slices) defines $H^\circ_\beta(\Psi;T)$ as the set of the homotopy classes of these slice-preserving maps and accordingly defines $\mu^{\mathcal{T}}_\beta(U) \subset H^\circ_\beta(\Psi;T)$.

Another kind of a relevant structure is an action of a group G on Ψ. Then one may (or may not) work with categories $\mathcal{T}$ of G-spaces T (i.e. acted upon by G) and perform homotopy, including (co)homology, constructions equivariantly. Thus, one defines equivariant T-measures $\mu^T_G(U)$ for G-invariant subsets $U \subset \Psi$.

(A group action on a space , defines a partition of this space into orbits, but this is a weaker structure than that of the the action itself.)

Guth' Vanishing Lemma. The supermultiplicativity property of the cohomology measures with arbitrary coefficients Π (see 1.5) for spaces Ψ acted upon by finite groups G implies that

$$\mu^*\left(\bigcap_{g\in G} g(U;\Pi)\right) \supset \smile_{g\in G} \mu^*(g(U;\Pi))$$

for all open subset $U \subset \Psi$.

[44]In the optical astronomy, this is called *Olbers' dark night sky paradox.*

This, in the case $\Pi = \mathbb{Z}_2$ was generalised by Larry Guth for families of spaces parametrised by spheres S^j as follows.

Given a space Ψ endowed with a partition α, we say that a subset in Ψ is *α-saturated* if it equals the union of some α-slices in Ψ and define two operations on subsets $U \subset \Psi$,

$$U \mapsto \cap_\alpha(U) \subset U \text{ and } U \mapsto \cup_\alpha(U) \supset U,$$

where

$\cap_\alpha(U)$ is the *maximal α-saturated subset that is contained in U*

and

$\cup_\alpha(U)$ is the *minimal α-saturated subset that contains U.*

Let, as in the case considered by Guth, $\Psi = \Psi_0 \times S^j$ where $S^j \subset \mathbb{R}^{j+1}$ is the j-dimensional sphere, let α be the partition into the orbits of $\mathbb{Z}_2$-action on Ψ by $(\psi_0, s) \mapsto (\psi_0, -s)$ (thus, "α-saturated" means "$\mathbb{Z}_2$-invariant") and let β be the partition into the fibres of the projection $\Psi \to \Psi_0$ (and "β-saturated" means "equal the pullback of a subset in Ψ_0").

Following Guth, define

$$Sq_j : H^{*\geq j/2}(\Psi; \mathbb{Z}_2) \to H^*(\Psi; \mathbb{Z}_2) \text{ by } Sq^j : H^p \to H^{2p-j}$$

and formulate his "Vanishing Lemma" in μ_β-terms as follows,[45]

$$[\cup\cap] \qquad \mu^*_\beta(\cup_\beta(\cap_\alpha(U)); \mathbb{Z}_2) \supset Sq_j(\mu^*_\beta(U; \mathbb{Z}_2)) \subset H^*_\beta(\Psi; \mathbb{Z}_2),$$

where, according to our notation, $H^*_\beta(\Psi; \mathbb{Z}_2) \subset H^*(\Psi; \mathbb{Z}_2)$ equals the image of $H^*(\Psi_0; \mathbb{Z}_2)$ under the cohomology homomorphism induced by the projection $\Psi \to \Psi_0$.

If $E : \Psi \to \mathbb{R}$ is an energy function, this lemma yields the lower bound on the *maxmin*-energy[46]

$$E_{max_\beta min_\alpha} = max_\beta min_\alpha E$$

evaluated at the cohomology class $St_j(h)$, $h \in H^*_\beta(\Psi; \mathbb{Z}_2)$:

$$[maxmin] \qquad E^*_{max_\beta min_\alpha}(St_j(h)) \geq E^*(h).$$

.

Question. What are generalisations of $[\cup\cap]$ and $[maxmin]$ to other cohomology and cohomotopy measures on spaces with partitions α, β, γ,...?

15 Pairing Inequality for Cohomotopy Spectra.

Let Ψ_1, Ψ_2 and Θ be topological spaces and let

$$\Psi_1 \times \Psi_2 \overset{\circledast}{\to} \Theta$$

be a continuous map where we write

$$\theta = \psi_1 \circledast \psi_2 \text{ for } b = \circledast(\psi_1, \psi_2).$$

[45]Guth formulates his lemma in terms of the complementary set $V = \Psi \setminus U$:
if a cohomology class $h \in H^*_\beta(\Psi; \mathbb{Z}_2)$ vanished on V, then $St_j(h)$ vanishes on $\cap_\beta(\cup_\alpha(V))$.

[46]Recall that $min_\alpha E(\psi)$, $\psi \in \Psi$, denotes the minimum of E on the α-slice containing ψ and max_β stands for similar maximisation with β (see 1.12).

For instance, composition $\psi_1 \circ \psi_2 : X \to Z$ of morphisms $X \overset{\psi_1}{\to} Y \overset{\psi_2}{\to} Z$ in a topological category defines such a map between sets of morphisms,

$$mor(X \to Y) \times mor(Y \to Z) \overset{\circledast}{\to} mor(X \to Z).$$

A more relevant example for us is the following

CYCLES × PACKINGS.

Here, Ψ_1 is a space of locally diffeomorphic maps $U \to X$ between manifolds U and X,

Ψ_2 is the space of cycles in X with some coefficients Π,

Θ is the space of cycles in U with the same coefficients,

$\circledast$ stands for *"pullback"*

$$\theta = \psi_1 \circledast \psi_2 =_{def} \psi_1^{-1}(\psi_2) \in \Theta.$$

This U may equal the disjoint unions of N manifolds U_i that, in the spherical packing problems, would go to balls $B_x(r) \subset X$; since we want these balls *not to intersect*, we take the space of *injective* maps $U \to X$ for Ψ_1.

If the manifold X is *parallelizable* and the balls $B_{x_i}(r) \subset X$ all have some radius radius r smaller then the injectivity radius of X, then corresponding $U_i = B_{x_i}(r)$ can be identified (via the exponential maps) with the r-ball $B^n(r)$ in the Euclidean space $\mathbb{R}^n$, $n = dim(X)$. Therefore, the space of k-cycles in $U = \bigsqcup_i U_i = B^n(r)$ equals in this case the Cartesian N-th power of this space for the r-ball:

$$\Theta = \mathcal{C}_k(U;\Pi) = (\mathcal{C}_k(B^n(r);\Pi))^N.$$

Explanatory Remarks. (a) Our "*cycles*" are defined as *subsets* in relevant manifolds X and/or U with Π-*valued functions* on these subsets.

(b) In the case of *open* manifolds, we speak of cycles with *infinite supports*, that, in the case of compact manifolds with boundaries or of open subsets $U \subset X$, are, essentially, *cycles modulo the boundaries* ∂X.

(c) *"Pullbacks of cycles"* that preserve their codimensions are defined, following Poincaré for a wide class of smooth *generic* (not necessarily equividimensional) maps $U \to X$ (see [29]).

(d) It is easier to work with *cocycles* (rather than with cycles) where contravariant functoriality needs no extra assumptions on spaces and maps in question (see [21]).

Let h^T be a (preferably non-zero) cohomotopy class in Θ, that is a homotopy class of non-contractible maps $\Theta \to T$ for some space T, (where "*cohomotopy*" reads "*cohomology*" if T is an Eilenberg-MacLane space) and let

$$h^{\circledast} = \circledast \circ h^T : [\Psi_1 \times \Psi_2 \to T]$$

be the induced class on $\Psi_1 \times \Psi_2$, that is the homotopy class of the composition of the maps $\Psi_1 \times \Psi_2 \overset{\circledast}{\to} \Theta \overset{h^T}{\to} T$.

(Here and below, we do not always notationally distinguish *maps* and *homotopy classes* of maps.)

Let h_1 and h_2 be homotopy classes of maps $S_1 \to \Psi_1$ and $S_2 \to \Psi_2$ for some spaces S_i, $i = 1, 2$,

(In the case where h^T is a *cohomology* class, these h_i may be replaced by *homology* – rather than homotopy – classes represented by these maps.)

Compose the three maps,

$$S_1 \times S_2 \overset{h_1 \times h_2}{\to} \Psi_1 \times \Psi_2 \overset{\circledast}{\to} \Theta \overset{h^T}{\to} T,$$

and denote the homotopy class of the resulting map $S_1 \times S_2 \to T$ by

$$[h_1 \circledast h_2]_{h^T} = h^{\circledast} \circ (h_1 \times h_2) : [S_1 \times S_2 \to T]$$

Let $\chi = \chi(e_1, e_2)$ be a function in two real variables that is monotone unceasing in each variable. Let $E_i : \Psi_i \to \mathbb{R}$, $i = 1, 2$, and $F : \Theta \to \mathbb{R}$ be (energy) functions on the spaces Ψ_1, Ψ_2 and Θ, such that the $\circledast$-pullback of F to $\Psi \times \Theta$ denoted

$$F^{\circledast} = F \circ \circledast : \Psi_1 \times \Psi_2 \to \mathbb{R}$$

satisfies

$$F^{\circledast}(\psi_1, \psi_2) \le \chi(E(\psi_1), E(\psi_2)).$$

In other words, the $\circledast$-image of the product of the sublevels

$$(\Psi_1)_{e_1} = E_1^{-1}(-\infty, e_1) \subset \Psi_1 \text{ and } (\Psi_2)_{e_2} = E_2^{-1}(-\infty, e_2) \subset \Psi_2$$

is contained in the f-sublevel $B_f = F^{-1}(-\infty, f) \subset \Theta$ for $f = \chi(e_1, e_2)$,

$$\circledast((\Psi_1)_{e_1} \times (\Psi_2)_{e_2}) \subset \Theta_{f = \chi(e_1, e_2)}.$$

$\circledast$-Pairing Inequality.

Let $[h_1 \circledast h_2]_{h^T} \neq 0$, that is *the composed map*

$$S_1 \times S_2 \to \Psi_1 \times \Psi_2 \to \Theta \to T$$

is non-contractible. Then the values of E_1 and E_2 on the homotopy classes h_1 and h_2 are *bounded from below* in terms of a lower bound on $F^\circ[h^T]$ as follows.

$[_{\circ\circ}\ge^\circ]$ $$\chi(E_{1\circ}[h_1], E_{2\circ}[h_2]) \ge F^\circ[h^T].$$

In other words

$$\big(E_{1_\circ}[h_1] \le e_1\big) \& \big(E_{2\circ}[h_2] \le e_2\big) \Rightarrow \big(F^\circ[h^T] \le \chi(e_1, e_2)\big)$$

for all real numbers e_1 and e_2; thus,

upper *bound* $E_1^\circ[h_1] \le e_1$ $+$ ***lower*** *bound* $F^\circ[h^T] \ge \chi(e_1, e_2)$

yield

upper *bound* $E_2^\circ[h_2] \ge e_2$,

where, observe, E_1 and E_2 are interchangeable in this relation.

In fact, all one needs for verifying $[_{\circ\circ}\ge^\circ]$ is unfolding the definitions.

Also $[_{\circ\circ}\ge^\circ]$ can be visualised without an explicit use of χ by looking at the $h^{\circledast}$*-spectral line* in the (e_1, e_2)-plane

$$\Sigma_{h^{\circledast}} = \partial\Omega_{h^{\circledast}} \subset \mathbb{R}^2$$

(we met this Σ section 1.3) where $\Omega_{h^{\circledast}} \subset \mathbb{R}^2$ consists of the pairs $(e_1, e_2) \in \mathbb{R}^2$ such that the restriction of $h^{\circledast}$ to the Cartesian product of the sublevels $\Psi_{1e_1} = E_1^{-1}(-\infty, e_1) \subset \Psi_1$ and $\Psi_{2e_2} = E_2^{-1}(-\infty, e_2) \subset \Psi_2$ vanishes,

$$h^{\circledast}_{|\Psi_{1e_1} \times \Psi_{2e_2}} = 0.$$

16 Inequalities between Packing Radii, Waists and Volumes of Cycles in a Presence of Permutation Symmetries.

The most essential aspect of the homotopy/homology structure in the space of packings of a space X by $U_i \subset X$, $i \in I$, is associated with the permutation group $Sim_N = aut(I)$ that acts on these spaces.

A proper description of this needs a use of the concept of "homological substantionality for *variable* spaces" as in section 8. This is adapted to the present situation in the definitions below.

Let

- Ψ_1 be space of I-packing $\{U_i\}$ of X by disjoint open subsets $U_i \subset X$, $i \in I$, for some set I of cardinality N, e.g. by N balls of radii r, $N = card(I)$,
- $\mathcal{U}^\cup$ be the space of pairs $\big(\{U_i\}, u\big)_{i\in I}$, $i \in I$, where $\{U_i\} \in \Psi_1$ and $u \in \bigcup_{i\in I} U_i$,
- $\mathcal{U}^\times$ be the space of pairs $\big(\{U_i\}, \{u_i\}\big)_{i\in I}$, where $\{U_i\} \in \Psi_1$ and $u_i \in U_i$,
- $\mathcal{C}_k^\cup$ be the space of k-cycles with Π coefficient in the unions of U_i for all packing $\{U_i\} \in \Psi_1$ of X,

$$\mathcal{C}_k^\cup = \bigcup_{\{U_i\}\in\Psi_1} \mathcal{C}_k(\bigcup_{i\in I} U_i; \Pi),$$

- $\mathcal{C}_{Nk}^\times$ be the space of $N\cdot k$-cycles with the N-th tensorial power coefficients in the Cartesian products of U_i for all $\{U_i\} \in \Psi_1$,

$$\mathcal{C}_{Nk}^\times = \bigcup_{\{U_i\}\in\Psi_1} \mathcal{C}_{Nk}^\cup \left(\bigtimes_{i\in I} U_i; \Pi^{\otimes N} \right).$$

The four spaces $\mathcal{U}^\cup$, $\mathcal{U}^\times$, $\mathcal{C}_k^\cup$ and $\mathcal{C}_{NK}^\times$ tautologically "fiber" over Ψ_1 by the maps denoted $\varpi^\cup$, $\varpi^\times$, $\varpi_k^\cup$, and $\varpi_{Nk}^\times$; besides, the Cartesian products of cycles $C_i \subset U_i$ defines an embedding

$$\mathcal{C}_k^\cup \hookrightarrow \mathcal{C}_{Nk}^\times.$$

Observe that

▪ the fibers $\times_i U_i$ of the map $\varpi^\times : \mathcal{U}^\times \to \Psi_1$ for $(\big(\{U_i\}, \{u_i\}\big) \overset{\varpi^\times}{\mapsto} \{U_i\}$ are what we call "variable spaces" in section 8 where $\mathcal{U}^\times$ playing the role of $\mathcal{X}$ and Ψ_1 that of Q from section 8,

▪ the above four spaces are naturally/tautologically acted upon, along with Ψ_1, by the symmetric group Sym_N of permutations/automorphisms of the index set I and the above four maps from these spaces to Ψ_1 are *Sym_N-equivariant,*

▪ the G-factored map $\varpi^\times$, denoted

$$\varpi_{/G}^\times : \mathcal{U}^\times/G \to \Psi_1/G,$$

has the same fibres as $\varpi^\times$, namely, the products $\times_i U_i$. However, if $card(G) > 1$ and $dim(X) = n > 1$, then $\varpi_{/G}^\times$ is a *non-trivial* fibration, even if all U_i equal translates of a ball $U = B^n(r)$ in $X = \mathbb{R}^n$, where the $\mathcal{U}^\times = \Psi_1 \times U^I$ and where $\varpi^\times : \Psi_1 \times U^I \to \Psi_1$ equals the coordinate projection.

The "$\circledast$-pairing" described in the previous section via the intersection of cycles in X with $U_i \subset X$ for $\{U_i\}_{i\in I} \in \Psi_1$ followed by taking the Cartesian product of these intersections defines a pairing

$$\Psi_1 \times \Psi_2 \overset{\circledast^\times}{\to} \Theta^\times = \mathcal{C}^\times_{Nk} = \bigcup_{\{U_i\}\in\Psi_1} \mathcal{C}_{Nk}\left(\bigtimes_{i\in I} U_i; \Pi^{\otimes N}\right),$$

where, recall, Ψ_1 is the space of packings of X by $U_i \subset X$, $i \in I$, and $\Psi_2 = \mathcal{C}_k(X;\Pi)$ is the space of cycles in X.

And since this map $\circledast^\times$ is equivariant for the action of Sym_N on $\Theta^\times$ and on the first factor in $\Psi_1 \times \Psi_2$, it descends to

$$\circledast^\times_{/G} : \Psi_1/G \times \Psi_2 \to \Theta^\times/G$$

for all subgroups $G \subset Sym_N$.

Detection Of Non-Trivial Families of Cycles and of Packings.

A family $S_2 \subset \Psi_2$ of cycles in X is called *homologically G-detectable* by (a family of) packings of X, if there exists a family S_1 of cycles in $\Psi_1/G \times \Psi_2$ such that the corresponding family of product $N\cdot k$-cycles in "variable spaces" $\bigtimes_i U_i$, that is a map from $S_1 \times S_2$ to $\Theta^\times/G$ by $\circledast^\times_{/G}$ is *homologically substantial.*

Recall (see 8) that this substantiality is non-ambigously defined if $\circledast^\times_{/G}$ is a fibration with *contractible fibres*, which is the case in our examples where $U_i \subset X$ are topological n-balls and observe that *unavoidably* variable nature of $\bigtimes_i U_i$ is due to non-triviality of the fibration $\varpi^\times_{/G} : \mathcal{U}^\times/G \to \Psi_1/G$ for permutation groups $G \neq \{id\}$ acting on packings that is most essential for what we do.

In his "*Minimax-Steenrod*" paper Guth shows that all homology classes h_2 of the space $\Psi_2 = \mathcal{C}_k(B^n;\mathbb{Z}_2)$ of relative k-cycles in the n-ball are *G-detectable by families $S_1 = S_1(h_2)$ of packings* of B^n by sufficiently small δ-balls $U_i = B^n_{x_i}(\delta) \subset B^n = B^n(1)$, for some 2-subgroup $G = G(h_2) \subset Sym_N$.

This is established with the help of "Vanishing Lemma" stated in section 14; where, if understand it correctly, the detective power of such a family $S_1 \subset \Psi_1/G$ (of "moving packings" of X by N balls) is due to *non-vanishing* of some cohomology class in Ψ_1/G that comes from the classifying space $\mathcal{B}_{cla}(G)$ via the classifying map $\Psi_1/G \to \mathcal{B}_{cla}(G)$.

(Probably, a proper incorporation of the cohomology coming from X would imply similar "detectability" *all* manifolds X, where it is quite obvious for *parallelizable* X.)

Apparently Guth' "Vanishing Lemma" shows that, every $\mathbb{Z}_2$-homology class h_* in $K(\mathbb{Z}_2, m)$ equals the image of a homology class from the classifying space $\mathcal{B}_{cla}(G)$ of some finite 2-groups $G = G(h_*)$ under a map $\mathcal{B}_{cla}(G) \to K(\mathbb{Z}_2, m)$. Equivalently, this means that given *a non-trivial* $\mathbb{Z}_2$-cohomology operation *op* from degree m to n, (that is, necessarily, a polynomial combination of Steenrod squares) there exists a class $h \in H^m(\mathcal{B}_{cla}(G);\mathbb{Z}_2)$ for some finite 2-group G, such that $op(h) \neq 0$.

Possibly, this is also true(?) for other primes p (which, I guess, must be known to people working on cohomology of p-groups).

On the other hand it seems unlikely that the (co)homologies of spaces of I-packing for all finite sets I and/or of (the classifying spaces of) all p-groups are fully detectable by the (co)homologies of the space $\mathcal{C}_k(B^n;\mathbb{Z}_p)$ of cycles in the n-ball, where, recall, the space $\mathcal{C}_k(B^n;\mathbb{Z}_p)$ is homotopy equivalent to Eilenberg-MacLane's $K(\mathbb{Z}_p, n-k)$.

Pairing Inequalities Between k-Volumes and Packing Radii.

Let $S_1 \subset \Psi_1$ be a G-invariant family of I-packings of a Riemannian manifold X by open subsets $U_i = U_{i,s_1}$, $i \in I$, $s_1 \in S$, where G is a subgroup of the group $Sym_N = aut(I)$, and let S_2 be a family of k-cycles (or more general "virtually k-dimensionl entities") $Y = Y_{s_2}$ in X.

Let the coupled family that is a map from $S_1/G \times S_2$ to $\Theta^\times/G$ by $\circledast^\times_{/G}$ be *homologically substantial.* Then, by the definition of " the wast of a variable space " (see section 8) the supremum of the volumes of Y_{s_2} is bounded from below by

$$\sup_{s_2 \in S_2} vol_k(Y_{s_2}) \geq \sum_{i\in I} waist_k(U_{i,s_1}).$$

In particular, since small balls of radii r in X have k-*waists* $\sim r^k$ the above implies

$$\frac{\inf_{s_1\in S_1}\sum_{i\in I} inrad_k(U_{i,s_1})^k}{\sup_{s_2\in S_2} vol_k(Y_{s_2})} \leq const(X),$$

where $inrad(U)$, $U \subset X$ denote the radius of the largest ball contained in U.

This inequality, in the case where $U_i \subset B^n = B^n(1) \subset \mathbb{R}^n$ are Euclidean r-balls, is used by Guth in [30], (as it was mentioned earlier, for obtaining a (nearly sharp) *lower bound* on the k-volume spectrum of the n-ball that is on $\sup_{s_2\in S_2} vol_k(Y_{s_2})$ for families S_2 of k-cycles Y in $X = B^n$ such that a given cohomology class $h \in H^*(\mathcal{C}_k(X;\mathbb{Z}_2);\mathbb{Z}_2)$ does not vanish on this $S_2 \subset \mathcal{C}_k(X;\mathbb{Z}_2)$. This is achieved by constructing a G-invariant family $S_1 = S_1(h)$ of I-packings of X by r-balls for some 2-group $G \subset Sym_N$, $N = card(I)$, such that the coupled family

$$S_1/G \times S_2 \overset{\circledast^\times_{/G}}{\to} \Theta^\times/G$$

is homologically substantial und such that the radii r of the balls $U_i = B_{x_i}(r) \subset X$ are sufficiently large.

Conversely, and this is what we emphasise in this paper, one can *bound from above* the radii r of the balls in a G-invariant family S_1 of I-packing of X by r-balls in terms of a cohomology class h' in the space Ψ_1/G (that is of packings/G) such that h' does not vanish on S_1, where one can use for this purpose suitable families $S_2 = S_2(h')$ of k-cycles in X with possibly small volumes, e.g. those used by Guth in [30] for his *upper bound* on the vol_k-spectra.

Thus, in the spirit of Guth' paper, one gets such bounds for $N = card(I) \to \infty$ and some $h' \in H^*(\Psi_1/G;\mathbb{Z}_2)$ for certain 2-groups $G \subset Sym_N$, where $ord(G) \to \infty$ and $deg(h') \to \infty$ along with $N \to \infty$.

And even though the packing spaces Ψ_1 looks quite innocuous being the complements to the $2r$-neighbourhoods to the unions of the diagonals in X^I there is no(?) parent alternative method to obtain such bounds.[47]

Multidimensional Rendition of the Pairing Inequalities. Let Ψ be a space of pairs $\psi = (\{U_i\}_{i\in I}, Y)$ where $U_i \subset X$ are disjoint open subsets and $Y \subset X$ is a k-cycle, say with $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients.

[47]A natural candidate for such method would be the Morse theory for the distance function to the union of the diagonals in X^I, see [6], but this does not(?) seem to yield such bounds.

Let $inv(U)$ be some geometric invariant of open subsets $U \subset X$ that is monotone under inclusions between subsets, e.g. the n-volume, inradius, some kind of waist of U, etc.

Define $\mathcal{E} = (E_0, E_1, ..., E_N) : \Psi \to \mathbb{R}^{N+1}$ by

$$E_0(\{U_i\}_{i\in I}, Y) = vol_k(Y) \text{ and } E_i(\{U_i\}_{i\in I}, Y) = inv(U_i)^{-1}$$

assume that Ψ is invariant under the action of some subgroup $G \subset Sym_N$ that permutes U_i, and observe that all of the above can be seen in the kernels of the cohomology homomorphisms from $H^*(\Psi/G)$ to $H^*(\Psi_{e_0,e_1,...,e_N}/G)$ for the subsets $\Psi e_0, e_1, ..., e_N \subset \Psi$ defined by the inequalities $E_i(\psi) < e_i$, $\psi = (\{U_i\}, Y)$, $i = 0, ...N$, as in the definition of the multidimentional (co)homology spectra in sections 4,10,11.

For example, besides the pairing inequalities for packing by balls, this also allows an encoding of similar inequalities for *convex partitions* from [22].

17 Sup$_\vartheta$-Spectra, Symplectic Waists and Spaces of Symplectic Packings.

Let Θ be a set of metrics ϑ on a topological space X and define *sup_ϑ-invariants* of (X, Θ) as *the suprema* of the corresponding invariants (X, ϑ) over all $\vartheta \in \Theta$. (In many cases, this definition makes sense for more general classes Θ of metrics spaces that do not have to be homeomorphic to a fixed X.)

Problem. Find general criteria for finiteness of sup_ϑ-invariants.

Two Classical Examples: Systoles and Laplacians. (1) Let Θ be the space of Riemannian metrics on the 2-torus X with

$$sup_{\theta\in\Theta} area_\theta(X) \leq 1.$$

Then the sup$_\vartheta$*-systole*$_1$ *of* (X, ϑ) *is* $< \infty$.

In fact,

$$sup_\vartheta\text{-}systole_1(X, \Theta) = \sqrt{\frac{2}{\sqrt{3}}}$$

by *Lowener's torus inequality* of 1949.

This means that all toric surfaces (X, ϑ) of *unit areas* admit closed *non-contractible* curves of lengths $\leq \sqrt{\frac{2}{\sqrt{3}}}$, where, observe, the equality $systole_1 = \sqrt{\frac{2}{\sqrt{3}}}$ holds for $\mathbb{R}^2$ divided by the *hexagonal lattice.* (See Wikipedia article on systolic geometry and references therein for further information.)

(2) Let Θ be the space of Riemanian metrics ϑ on the 2-sphere X with $area_\vartheta(X) \geq 4\pi$ (that is the area if the unit sphere). Then

the first sup$_\vartheta$*-eigenvalue of the Laplace operator on* (X, Θ) *is* $< \infty$.

In fact sup_ϑ-$\lambda_1(X, \Theta) = 2$, that is the first eigenvalue of the Lapalce operator on the unit sphere, by *the Hersch inequality* of 1970.

Symplectic Area Spectra and Waists. Let X be a smooth manifold of even dimension $n = 2m$ and let $\omega = \omega(x)$ be a differential 2-form on X.

A *Riemannian metric* ϑ on X, is called *adapted to* or *compatible with* ω, if

$\bullet_{\geq\omega}$ $$area_\vartheta(Y) \geq |\int_Y \omega|$$

for all smooth oriented surfaces $Y \subset X$;

2. the n-volume $d_\vartheta x$ element, satisfies

$\bullet_{\leq\omega^m}$, $$d_\vartheta(x) \leq |\omega^m| \text{ at all ponts } x \in X.$$

that is $vol_\vartheta(U) \leq \int_U \omega^m$ for all open subsets $U \subset X$.[48]

Question. Which part of the (suitably factorized/coarsend) homotopy/homology area spectra of (X, ϑ) remains finite after taking suprema over $\vartheta \in \Theta(\omega)$?

Partial Answer Provided by the Symplectic Geometry. The form ω is called *symplectic* if it is *closed*, i.e $d\omega = 0$, and $\omega^m = \omega^m(x)$ does not vanish on X. In this case $X = (X, \omega)$ is called *a symplectic manifold.*

The symplectic k-waist of X may be defined as the *supremum of the k-waists*[49] of the Riemannian manifolds (X, ϑ) for all metrics ϑ *compatible* with ω.

It is easy to see that

the space $\Theta = \Theta(\omega)$ of metric ϑ compatible with ω is *contractible,*

and that

ω-compatible metrics are *extendable* from open subsets $X_0 \subset X$ to all of X with the usual precautions at the boundaries of X_0.

It follows, that the symplectic waists are monotone under equividimensional symplectic embeddings:

$$sympl\text{-}waist_k(X_0) \leq sympl\text{-}waist_k(X)$$

for all open subsets $X_0 \subset X$.

If $k = 2$, then upper bounds on symplectic waists are obtained by proving homological stability of certain families of *ϑ-psedoholomorphic curves* in X under deformation of compatible metrics ϑ, where "psedoholomorphic curves" are oriented surfaces $Y \subset X = (X, \omega, \vartheta)$, such that $area_\vartheta(Y) = \int_Y \omega$.

This, along with the symplectomorphism of the ball $B = B^{2m}(1) \subset \mathbb{C}^m$ onto the complement $\mathbb{C}P^m \setminus \mathbb{C}P^{m-1}$, where the symplectic form in $\mathbb{C}P^m$ is normalised in order to have the area of the projective line $\mathbb{C}P^1 \subset \mathbb{C}P^m$ equal $area(B^2(1)) = \pi$, implies that

$$sympl\text{-}waist_2(\mathbb{C}P^m; \mathbb{Z}) = sympl\text{-}waist_2(B; \mathbb{Z}) = waist_2(B; \mathbb{Z}) = \pi,$$

where $waist_2(...; \mathbb{Z})$ stands for the $\mathbb{Z}$-waist that is defined with homologically substantial families of $\mathbb{Z}$-cycles.

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[48]The *inequalities* $area_\vartheta(Y) \geq |\int_Y \omega|$ and $d_\vartheta(x) \leq |\omega^m|$ imply the *equality* $d_\vartheta x = |\omega^m|$ and if $\omega^m(x) \neq 0$, then there is a $\mathbb{R}$-linear isomorphism of the tangent space T_x to $\mathbb{C}^m$, such that $(\sqrt{-1}\omega, \vartheta)_x$ go to the imaginary and the real parts of the diagonal Hermitian form on $\mathbb{C}^m$. But $\bullet_{\leq\omega^m}$ is better adapted for generalisations than $\bullet_{=\omega^m}$.

[49]This may be any kind of a waist defined with families S of "virtually k-dimensional entities" of a particular kind and with a given type of homological substantiality required from S, where the most relevant for the symplectic geometry are $\mathbb{Z}$-waists $waist_k(X; \mathbb{Z})$ defined with families of $\mathbb{Z}$-cycles that are assumed as regular as one wishes.

On the other hand, it is probable that *sympl-waist*$_k(X;\mathbb{Z}) = \infty$ for all X, unless $k = 0, 2, n = dim(X)$ and this is also what one regretfully expects to happen to *the symplectic k-systoles* $syst_k(X,\omega) =_{def} \sup_\vartheta syst_k(X,\vartheta)$, for $k \neq 0, 2, n$.

For instance, the complex projective space $\mathbb{C}P^m$ may(?) carry Riemannian metrics ϑ_s for all $s > 0$ compatible with the standard symplectic form on $\mathbb{C}P^m$ such that the k-systoles, i.e the minimal k-volumes of all non-homologous zero k-cycles in $(\mathbb{C}P^m, \vartheta_s)$ of all dimensions k except for $0, 2, 2m$ are $> s$.

Yet, some geometric (topological?) invariants of the functions $\vartheta \mapsto syst_k(X,\vartheta)$ and $\vartheta \mapsto waist_k(X,\vartheta)$ on the space of metrics ϑ compatible with ω may shed some light on the symplectic geometry of (X,ω), where possible invariants of such a function $F(\vartheta)$ may be the asymptotic rate of some kind of "minimal complexity" of the Riemannian manifolds (X,ϑ) (e.g. some integral curvature or something like the minimal number of contractible metric balls needed to cover (X,ϑ)) for which $F(\vartheta) \geq s$, $s \to \infty$.

Let us generalise the above in the spirit of "multidimensional spectra" by introducing the space $\Psi = \Psi(X,I)$ of triples $\psi = (\{U_i\}_{i\in I}, Y, \vartheta)$, where U_i are disjoint open subsets, $Y \subset X$ is an integer 2-cycle and ϑ is a Riemannian metric compatible with ω.

Let $\mathcal{E} : \Psi \to \mathbb{R}^{N+1}$ be the map defined by

$$\mathcal{E}(\{U_i\}_{i\in I}, Y, \vartheta) = (sympl\text{-}waist_2(U_i), area_\vartheta(Y))_{i=1,2,\dots,N},$$

where, as an alternative to *sympl-waist*$_2(U)$, one may use $inrad_\omega(U)$, $U \subset (X,\omega)$, that is the supremum of the radii r of the balls $B^{2m}(r) \subset \mathbb{C}^m$ that admit symplectic embedding into U. Let $f : \mathbb{R}^{N+1} \to \mathbb{R}$ be a positive function that is symmetric and monotone decreasing in the first N-variables and monotone increasing in the remaining variable (corresponding to $area(Y)$), where the simplest instance of this is $-\sum_{i=1,\dots N} z_i + z_{N+1}$

Let $G \subset Sym_N$ be a permutation group that observe, naturally acts on Ψ, let S be a topological space with a free action of G and let $[\varphi/G]$ be a homotopy class of maps $\varphi : S/G \to \Psi/G$.

Let $E_f(\psi) = f(\mathcal{E}(\psi))$ and define *the* $\sup\vartheta$*-homotopy spectrum* of E_f (compare section 4)

$$E_f[\varphi/G]_{\sup_\vartheta} = \sup_\vartheta \inf_{\varphi\in[\varphi]} \sup_{s\in S} E_f(\varphi(s)).$$

Playing $\inf_\phi$ *Against* $\sup_\vartheta$. Our major concern here is the possibility of $E_f[\varphi/G]_{\sup_\vartheta} = \infty$ that can be outweighed by enlarging the homotopy class $[\phi]$ to a homology class or to a set of such classes. With this in mind, given a cohomology class h in $H^*(\Psi/G, \Pi)$ with some coefficients Π, one defines

$$E_f[h]_{\sup_\vartheta} = \sup_\vartheta \inf_{\varphi^*(h)\neq 0} \sup_{s\in S} E_f(\varphi(s)).$$

where the infimum is taken over all G-spaces S and all maps $\varphi : S/G \to \Psi/G$. such that $\varphi^*(h) \neq 0$.

Another possible measure against $E_f[\varphi/G]_{\sup_\vartheta} = \infty$ is taking $\mathbb{Z}_2$-cycles Y instead of $\mathbb{Z}$-cycles, but this is less likely to tip the balance in our favour.

On the other hand, one may enlarge/refine the outcome of minimisation over φ, yet, still keeping the final result finite, by restricting the topology of Y, e.g. by allowing only Y represented by surfaces of genera bounded by a given number. Also one may incorporate the integral $\int_Y \omega$ into E_f.

But usefulness of all these variations is limited by the means we have at our disposal for proving finiteness of the $\sup_\vartheta$-spectra that are limited to the homological (sometimes homotopical) stability of families of psedoholomorphic curves.

Packing Inequalities. If $X = (X, \omega)$ admits a nontrivial stable family of psedoholomorphis curves $Y \subset X$, then, there are non-trivial constrains on the topology of the space of packings of X by U_i with $inrad_\omega(U_i) \geq r$.

Namely there are connected G-invariant subsets $S_\circ$ in the space of I-tuples of disjoint topological balls in X for all finite sets $I \ni i$ of sufficiently large cardinalities N and some groups $G \subset Sym_N = aut(I)$, such that

every family S of I-tuples of disjoint subsets $U_{i,s} \subset X$, $s \in S$, that is G-equaivariantly homotopic, or just homologous, to $S_\circ$ satisfies,

$$\inf_{s \in S} \sum_{i \in I} sympl\text{-}waist_2(U_{i,s}) \leq \int_Y \omega.$$

Consequantly,

$$\inf_{s \in S} \sum_{i \in I} \leq \pi \cdot inrad_\omega(U_{i,s}) \leq \int_Y \omega.$$

And effective and rather precise unequalties of this kind estimates are possible for particular manifolds, say for the projective $\mathbb{C}P^m$ and domains in it where psedoholomorphic curves are abundant.

But if X is a closed symplectic manifold $X = (X, \omega)$ *with no psedoholomorphic curves in it* , e.g. the $2m$-torus, one does not know whether there are non-trivial constrains on the homotopy types of symplectic packing spaces.

Also it is unclear if such an X must have $syst_2(X, \omega) = \infty$ and/or $sympl\text{-}waist_2(X) = \infty$.

(See [4] [9] [10] [12] [13] [32] [16] [38] [39] [42] for what is known concerning *individual* symplectic packings of X and *spaces* of embeddings of *a single* ball into X.)

Conclude this section by observing that spaces of certain symplectic packings can be described entirely in terms of the set $\Theta = \Theta(\omega)$ of ω-compatible metrics ϑ on X with the following definition applicable to general classes Θ of metric spaces.

Θ-*Packings by Balls.* A packings of (X, Θ) by I-tuples of r-balls $B^n(r) \subset \mathbb{R}^n$ is a pair $(\vartheta, \{f_i\})$ where $\vartheta \in \Theta$ and an $\{f_i\}$, $i \in I$, is an I-tuple of *expanding maps* $f_i : B^n(r) \to (X, \vartheta)$ with disjoint images.

Problem. Find further (non-symplectic) classes Θ with "interesting" properties of the corresponding spaces of Θ-Packings.

For example, a Riemannain metric ϑ may be regarded as compatible with a *pseudo-Riemannian* (i.e. non-degenerate indefinite) h on a compact manifold X if the h-lengths of all curves are bounded in absolute values by their ϑ-length and if the ϑ-volume of X equals the h-volume.

Are there instances of (X, h) where some Θ-packings and/or sup_ϑ-invariants carry a non-trivial information about h?

Are there such examples of other *G-structures* on certain X for groups $G \subset GL(n)$, $n = dim(X)$, besides the symplectic and the orthogonal ones, where metrics ϑ serve for reductions of groups G to their maximal compact subgroups?

18 Packing Manifolds by k-Cycles and k-Volume Spectra of Spaces of Packings.

Define *an I-packing of the space* $\mathcal{C}_*(X;\Pi) = \bigoplus_{k=0,1,...}\mathcal{C}_k(X;\Pi)$ *of cycles with* Π-*coefficients* in a Riemannian manifold X as *an I-tuple* $\{V_i\}$ *of cycles in* X with a given *lower bound* on *some distances*, denoted "*dist*", between these cycles, say

$$"dist"(V_i, V_j) \geq d,$$

or, more generally,

$$"dist"(V_i, V_j) \geq d_{ij},\ i, i \in I.$$

Leading Example: $dist_X$-Packings of the Space $\mathcal{C}_*(X;\Pi)$. A significant instance of "distance between cycles in X" is

$$dist_X(V, V') =_{def} \inf_{v \in V, v' \in V'} dist_X(v, v'),$$

where we use here the same notation for cycles V and their supports in X, both denoted $V_i \subset X$.

Question 1. What are the homotopy/homology properties of the spaces

$$\Psi_{d_{ij}} = \mathcal{C}_*(X;\Pi)^I_{<d_{ij}} \subset \mathcal{C}_*(X;\Pi)^I$$

of k-tuples of cycles $V_i \subset X$ that satisfy the inequalities $dist_X(V_i, V_j) \geq d_{ij}$.?

(Here, as at the other similar packing occasions, one should think in Sym_N-equivaruinat terms that makes sense since the *totality* of the spaces $\mathcal{C}_*(X;\Pi)^I_{<d_{ij}}$ for all $d_{ij} > 0$ is Sym_N-invariant.)

Volume Sectra of $X^I_{>d_{ij}}$. One can approach this question from an opposite angle by looking at the k_N-volume spectra of (the spaces of cycles in the) Riemannian manifolds $X^I_{>d_{ij}}$ for various $k_N \leq N \cdot dim(X)$, where $X^I_{>d_{ij}} \subset X^I$ are defined by the inequalities $dist_X(x_i, x_j) > d_{ij}$, since mutual distances between cycles V_i in X can be seen in terms of locations of their Cartesian products in X^I, as follows:

$$dist_X(V_i, V_j) \geq d_{ij} \Leftrightarrow V^\times = \bigtimes_{i \in I} V_i \subset X^I_{>d_{ij}}.$$

Recall at this point that the spaces $X^I_{>d_{ij}=d}$ represent packings of X by balls of radii $d/2$. and observe that $V^\times \in \mathcal{C}_*(X^I; \Pi^{N\otimes})$ are rather special, namely split, cycles in X^I.

Now the above Question 1 comes with the following companion.

Question 2. What are the volume spectra of the spaces $X^I_{>d_{ij}}$ and how do they depend on d_{ij}?

Recall the following relation between volume of cycles W and W' in the Euclidean space and distances between them.

Gehring's Linking Volume Inequality. Let $W \subset \mathbb{R}^n$ be a k-dimensional sub(psedo)manifolds of dimension k.

Suppose,

W is non-homologous to zero in its open d-neighbourhood $U_d(W) \subset \mathbb{R}^n$,

or, equivalently, there exists an $(n-k-1)$-dimensional subpseodomanifold $W' \subset \mathbb{R}^n$ that has

a non-zero linking number with W and such that $dist_{\mathbb{R}^n}(W, W') \geq d$.

("Linking" is understood mod 2 if W is non-orientable.)

Then, according to the *Federer Fleming Inequality*, (see next section)

$$vol_k(W) \geq \varepsilon_n d^k, \ \varepsilon_n > 0,$$

where, moreover,

$$\varepsilon_n = \varepsilon_k = vol_k(S^k)$$

($S^k = S^k(1) \subset \mathbb{R}^{k+1}$ denotes the unit sphere.) by the Bombieri-Simon solution of Gehring's Linking problem (see next section).

Proof. Map the Cartesian product $W \times W'$ to the sphere $S^{n-1}(d) \subset \mathbb{R}^n$ of radius d by

$$f : (w, w') \mapsto \frac{d(w - w')}{dist_{\mathbb{R}^m}(w, w')}$$

and observe that

• the family of the f-images of the "slices" $W \times w' \subset W \times W'$, $w' \in W'$, in $S^{n-1}(d)$ is *homologically substantial* (in the sense of section 6) since the degree of the map $f : W \times W' \to S^{n-1}(d)$ equals the linking number between W and W';

• the map f is *distance decreasing, hence, k-volume decreasing* on the "slices" $W \times w'$ for all $w' \in W'$, since $dist(w, w') \geq d$ for all $w \in W$ and $w' \in W'$.

Therefore, by the definition of *waist* (see section 6)

$$vol_k(W) \geq wast_k(S^{n-1}(d))$$

where $wast_k(S^{n-1}(d)) = vol_k(S^k(d)) = d^k vol_k(S^k(1))$ by the sharp spherical waist inequality (see sections 6 and 19). QED.

Linking Waist Inequality. The above argument also shows that whenever a k-cycle $W \subset \mathbb{R}^n$ is *non-trivially homologically linked* with some $W' \subset \mathbb{R}^N$, and $dist(W, W') \geq d$, then

$$waist_l(W) \geq waist_l(S^n(d)) = d^l vol_l(S^l) \text{ for all } l \leq k.$$

This provides non-trivial constrains on the spaces of packings of $\mathbb{R}^n$ by l-cycles, since $(k-l)$-cycles in the space of l-cycles $Y \subset \mathbb{R}^n$ make l-cycles $W \subset \mathbb{R}^n$.

Another generalisation of Gehring inequality concerns several cycles, say Y_i linked to some k-cycle W. In this case the intersections of Y_i with any chain implemented by a subvariety $V = V^{k+1} \subset \mathbb{R}^n$ that fills-in W, i.e. has W as its boundary, $\partial V = W$, make packings of this V by 0-cycles.

This, applied to the *minimal* V filling-in (spanning) W, suggests a (sharp?) lower bound on distances between such Y_i in terms of what happens to the ordinary packings of the ball $B^{k+1}(r)$ which has $vol_k(\partial B^k) = vol_k(W)$.

Question 3. Let Y_i, $i \in I$, be m-cycles in $\mathbb{R}^{2m+1}$, such that

$$vol_m(Y_i) \leq c \text{ and } dist(Y_i, Y_j) \geq d.$$

What are, roughly, possibilities for the linking matrices $L_{ij} = \#_{link}(Y_i, Y_j)$ of such Y_i depending on c and d?

What are the homotopies/homologies of spaces of such I-tuples of cycles depending on L_{ij}?

19 Appendix: Volumes, Fillings, Linkings, Systoles and Waists.

Let us formulate certain mutually interrelated fillings, linkings and waists inequalities extending those presented above and earlier in section 6.

1. *Federer-Fleming Filling-by-Mapping (Isoperometric) Inequality.* ([18]) Let $Y \subset \mathbb{R}^n$ be a closed subset with finite k-dimensional Hausdorff measure for an integer $k \leq n$. Then there exisists a continuous map $f : Y \to \mathbb{R}^n$ with the following properties.

$\bullet_{k-1}$ The image $f(Y) \subset \mathbb{R}^n$ is *at most* $(k-1)$*-dimensional.* Moreover, $f(Y)$ is contained in *a piecewise linear subset* $\Sigma^{k-1} = \Sigma^{k-1}(X) \subset \mathbb{R}^n$ *of dimension* $k-1$.

$\bullet_{disp}$ *The displacement of* Y *by* f *is bounded in terms of the Hausdorff measure of* Y by

$$\sup_{y \in Y} dist_{\mathbb{R}^n}(f(y), y) \leq const_n Haumes_k(Y)^{\frac{1}{k}}.$$

$\bullet_{vol}$ The $(k+1)$-dimensional measure of *the cylinder* $C_f \subset \mathbb{R}^n$ *of the map* f that is the union of the straight segments $[y, f(y)] \subset \mathbb{R}^n, y \in Y$, satisfies

$$Haumes_{k+1}(C_f) \leq const'_n Haumes_k(Y)^{\frac{k+1}{k}}.$$

(A possible choice of Σ^{k-1} is the $(k-1)$-skeleton of a standard decomposition of the Euclidean n-space $\mathbb{R}^n$ into R-cubes for $R = C_n Haumes_k(Y)^{\frac{1}{k}}$ and a sufficiently large constant C_n, where the map $f : Y \to \Sigma^{k-1}$ is obtained by consecutive radial projections from Y intersected with the m-cubes $\square^m$ to the boundaries $\partial\square^m$ from certain points in $\square^m$ starting from $m = n$ and up-to $m = k$.)

It remains unknown if this holds with $const_k$ instead of $const_n$ but the following inequality with $const_k$ is available.

2. *Contraction Inequality.* Let the $Y \subset \mathbb{R}^n$ be a k-dimensional polyhedral subset. Then there exists a continuous map $f : Y \to \mathbb{R}^n$ with the following properties.

$\bullet_{k-1}$ The image $f(Y) \subset \mathbb{R}^n$ is contained in *a piecewise linear subset of dimension* $k-1$.

$\bullet_{dist}$ The image of f lies within a *controlled distance* from Y. Namely,

$$\sup_{y \in Y} dist_{\mathbb{R}^n}(f(y), Y) \leq const_k vol_k(Y)^{\frac{1}{k}}.$$

$\bullet_{hmt}$ There exists *a homotopy between the idenetity map and* f, say $F : Y \times [0,1] \to \mathbb{R}^n$ with $F_0 = id$ and $F_1 = f$, such that the image of F satisfies

$$vol_{k+1}(F(Y \times [0,1])) \leq const'_k vol_k(Y)^{\frac{k+1}{k}}.$$

This is proven appendix 2 in [24] for more general spaces X in the place of $\mathbb{R}^n$, including all Banach spaces X.

3. *Almgen's Sharp Filling (Isoperimetric) Inequality.* Almgen proved in 1986 [?] the following sharpening of the (non-mapping aspect of) Federer-Fleming inequity.

the volume minimising $(k+1)$*-chains* Z *in Euclidean spaces satisfies*:

$$\frac{vol_{k+1}(Z)}{vol_k(\partial Z)^{\frac{k+1}{k}}} \leq \frac{vol_{k+1}(B^{k+1}(1))}{vol_k(S^k(1))^{\frac{k+1}{k}}} = (k+1)^{-\frac{k+1}{k}} vol_{k+1}(B^{k+1}(1))^{-\frac{1}{k}},$$

where $B^{k+1}(1) \subset \mathbb{R}^{k+1}$ is the unit Euclidean ball and $S^k(1) = B^{k+1}(1)$ is the unit sphere.

In fact, *Almgren's local-to global variational principle* [?], [21] reduces filling bounds in Romanian manifolds X to lower bounds of the suprema of mean curvatures of subvarieties $Y \in X$ in terms of k-volumes of Y, where such a sharp(!) bound for $Y \subset \mathbb{R}^n$ is obtained by Almgren by reducing it to that to that for *the Gaussian curvature* of the boundary of the convex hull of Y.

4. *Divergence Inequality.* Recall that the *the* k*-divergence* of a vector field $\delta = \delta_x$ on a Riemannian manifold X is the function on the tangent k-planes in X that equals *the* δ*-derivative of the* k*-volumes* of these k-planes. Thus,

the δ*-derivative of the* k*-volume of each* k*-dimensional submanifold* $V \subset X$ *moving by the* δ*-flow equals* $\int_V div_k(\delta)(\tau_v)dv$*, for* τ_v *denoting the tangent* k*-plane to* $V \subset X$ *at* v*.*

For instance, the k-divergence of *the standard Euclidean radial field focused at zero* $\delta_x = \overrightarrow{x} = grad(\frac{1}{2}||x||^2)$ on $\mathbb{R}^n$, where $\overrightarrow{x}$ denotes this very x seen as the tangent vector parallely transported from $0 \in \mathbb{R}^n$ to x, equals the norm $||x||$. (The δ-flow here equals the homothety $x \mapsto e^t x$, $t \in \mathbb{R}$.)

Let Z be a compact *minimal/stationary* $(k+1)$-dimensional subvariety with boundary $Y = \partial Z$ and let $\nu = \nu_y$ be the unit vector field tangent to Z, normal to Y and facing outside Z.

Then

$$\star_= \qquad \int_Z div_k(\delta)(\tau_z)dz = \int_Y \langle \delta_y, \nu_y \rangle.$$

If Z and Y are non-singular, this equality, that follows from the definition of "stationary" and the Gauss-Stokes formula, goes back to 19th century, while the singular case is, probably, due to Federer-Fleming (Reifenberger? Almgren?).

5. *Isoperimetric Corollary.* If $div_k(\delta) > 0$, then every on subset $Z_0 \subset Z$ satisfies:

$$\star_{\partial\geq} \qquad vol_k(Y) \geq vol_{k+1}(Z_0) \cdot \frac{\inf_{z \in Z_0} div_{k+1}\delta}{\sup_{y \in Y} ||\delta_y||}.$$

This inequality may be applied to a *radial field* δ focused in $X \supset Z$ at some (say non-singular) point $z_0 \in Z$ (such a filed is tangent to the geodesics issuing from z_0) in conjunction with *the coarea inequality* for the intersections Y_r of Z with spheres in X around z_0 of radii r,

$$\int_0^d vol_k(Y_r)dr \geq vol_{k+1}(Z_d),$$

$Z_r \subset Z$ denoting the intersection of Z with the d-ball in X around in z_0.

For instance, if $X = \mathbb{R}^n$ and δ is the standard radial field focused at some point $z_0 \in Z$, then one arrives this way at the classical (known since 1960s, 50s, 40s?)

6. *Monotonicity (Isoperimetric) Inequality.*

$$\frac{vol_{k+1}(Z_r)}{vol_k(Y_r)^{\frac{k+1}{k}}} \leq \frac{vol_{k+1}(B^{k+1}(1))}{vol_k(S^k(1))^{\frac{k+1}{k}}}.$$

Remark. If Z is volume minimising, rather than being being only "stationary", this follows from the above Almgren sharp filling inequality.

Corollary. If the boundary ∂Z lies d-far from z_0, ten the volume of Z_d is bounded from below by that of the Euclidean ball $B^{k+1}(d) = B_{\mathbf{0}}(\mathbb{R}^{k+1}, d) \subset \mathbb{R}^{k+1}$ of radios d:

$$vol_{k+1}(Z_d) \geq vol_{k+1}(B^{k+1}(d)),$$

provided the boundary ∂Z lies d-far from z_0.

7. *Application to Linking.* The boundary $Y = \partial Z$ (as well as $Y_d = \partial Z_d$) satisfies

$\bullet_{\partial \geq}$
$$vol_k(Y) \geq vol_k(S^k(d)),$$

where $S^k(d) = \partial B^{k+1}(d)$ is the Euclidean sphere of radios d and where, we keep assuming that Y lies *outside the d-ball* in $\mathbb{R}^n$ around z_0.

Indeed, this follows by applying $\star_{\geq}$ to the radial field that is focused at z_0, that equals the standard one (i.e. $\overrightarrow{x - z_0}$) *inside the ball* $B^n_{z_0}(d) \subset \mathbb{R}^n$ and that has *norm (length) equal d everywhere outside* this ball.

Consequently, every (mildly regular) k-cycle $Y \subset \mathbb{R}^n$ *bounds a $(k+1)$-chain Z in its d-neigbourhood for d equal the radius of k-sphere with volume equal $vol_k(Y)$.*

Namely, the solution Z of *the Plateau problem* with boundary Y does the job. (This is how Bombieri and Simon solve the Gehring linking problem, see [11].)

8. *Systolic Inequality.* A simple adjustment of the above "monotonicity argument" yields the following short-cut in the proof of *the systolic unequalty*[50]

Let $K = K(\Pi, 1)$, $\Pi = \pi_1(K)$, be *an aspherical space* and let $\mathcal{X} = \mathcal{X}(h, R)$, where $h \in H_n(K)$, $R > 0$, be the class of all n-dimensional pseudo manifolds X with piecewise Riemannian metrics on them along with maps $f : X \to K$, such that

$\bullet_h$ *the fundamental homology class $[X]_n \in H_n(X)$ of X (defined mod 2 if X is non-orientable) goes to a given non-zero homology class $h \in H_n(K)$;*

$\bullet_R$ *the restrictions of the map f to the R-balls in X, that are maps $B_x(R) \to K$, are contractible for all $x \in X$.*

Then

$$vol(X) \geq \alpha_n R^n \text{ for } \alpha_n = \frac{(2\sigma_{n-1})^n}{n^n \sigma_n^{n-1}},$$

[50]This was proven in [24] by a reduction to the contraction (filling) inequality generalized to Banach spaces, see [25], [36].

where $\sigma_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ *is the volume of the unit sphere* S^n*; thus,* $\alpha_n \sim \frac{(2\sqrt{e})^n}{n^n}$*, that is* α_n *equals* $\frac{(2\sqrt{e})^n}{n^n}$ *plus a subexponential term.* (The expected value of the constant in the systolic inequality is $\sim \frac{c^n}{n^{\frac{n}{2}}}$.)

Proof. Assume that X is *volume minimising* in $\mathcal{X}$, i.e. under conditions $\bullet_h$ and $\bullet_R$.[51] Then volume of each r-ball $B_x(r) \subset X$ with $r < R$ is bounded by the volume of *the spherical* r*-cone* $B_{\circ r}(S)$ over the r-sphere $S = S_x(r) = \partial B_x(r) \in X$.[52]

In fact, if you cut $B_x(r)$ from X and attach $B_{\circ r}(S)$ to $X \setminus B_x(r)$ by the boundary $\partial B_{\circ r}(S) = S = \partial B_x(r)$, the resulting space X' will admit map $f' : X' \to K$ with the conditions $\bullet_h$ and $\bullet_R$ satisfied.

Therefore,

$$vol_{n-1}(S_x(r)) = \frac{d}{dr} vol(B_x(r)) \geq \beta_n vol(B_x(r))^{\frac{n-1}{n}} \text{ for } \beta_n = \frac{\sigma_{n-1}}{(\frac{1}{2}\sigma_n)^{\frac{n-1}{n}}},$$

which implies, by integration over $r \in [0, R]$, the bound $vol(B_x(R)) \geq \alpha_n R^n$. for $\alpha_n = (\beta_n/n)^n$. QED.

Remark. Earlier, Guth, [33] suggested a short proof of a somewhat improved systolic inequality, that says, in particular, that the R-ball $B_x(R) \subset X$ at some point $x \in X$ has $vol_n(B_x(R)) \geq (4n)^{-n} R^n$, provided the fundamental class $[X]_n \in H_n(X)$ equals the product of one dimensional classes.

His argument, based on minimal hypersurfaces and induction on dimension, generalises to minimal hypersurfaces with boundaries as in [27] and yields a similar systolic inequality for spaces X with "sufficiently large" fundamental groups $\Pi = \pi_1(X)$.

But the proof of the bound $vol_n(B_x(R)) \geq \varepsilon_n R^n$ for general groups Π, also due to Guth (see [34] where a more general inequality is proven), is rather complicated and gives smaller constant ε_n.[53]

9. *Negative Curvature and Infinite Dimensions.* The divergence inequality and its corollaries applies to many non-Euclidean spaces X, such as $CAT(\kappa)$-*spaces* with $\kappa \leq 0$, that are

complete simply connected, possibly infinite dimensional spaces, e.g. Riemanian/Hilbertian manifolds, with non-positive sectional curvatures $\leq \kappa$ *in the sense of Alexandrov*

Albeit vector fields are not, strictly speaking, defined in singular spaces, *radial semigroups of transformations* $X \to X$ *with controllably positive* k*-divergence* are available in CAT-spaces. This along with a solution of Plateau's problem, (let it be only approximate one) shows that

every (mildly regular) k*-cycle* Y *in a* $CAT(\kappa)$*-space* X *is homologous to zero in its* d*-neighbourhood* $U_d(Y) \subset X$ *for* d *equal the radius of the corresponding* k*-sphere in the hyperbolic space with curvature* κ*, that is* $S^k(\kappa, d) \subset H^{k+1}(\kappa)$*, such that* $vol_k(S^k(d)) = vol_k(Y)$.

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[51] This is assumption is justifiable according to [44], or, approximately, that is sufficient for the present purpose, by the formal (and trivial) argument from section 6 in [24].

[52] The spherical r-cone over a piecewise Riemannian manifold S can be seen by isometric imbedding from S to the equatorial sphere $S^{N-1}(r) \subset S^N(r) \subset \mathbb{R}^{N+1}$ and taking the geodesic cone over $S \subset S^{N-1}$ from a pole in $S^N \supset S^{N-1}$ for $B_{or}(S)$.

[53] Most (all?) *known* examples of fundamental groups of closed aspherical manifolds, e.g. those with non-postive curvatures, are "sufficiently large". But, conjecturally, there are many "non-large" examples.

Remark. If X is *finite dimensional as well as non-singular,* this also follows from the spherical waist inequality that, in fact, does not need k-volume contracting (or expanding) radial fields but rather (controllably) k-volume contracting maps from X to the unit tangent spheres $S_x^{n-1}(1) \subset T_x(X)$, $n = dim(X)$, at all $x \in X$, see Appendix 2 in [24]. But proper setting for such an inequality for infinite dimensional spheres and in a presence of singularities remains problematic.

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10. *Almgren's Inequality for Curvature* ≥ 0. Let X be a complete n-dimensioanl Riemannian manifold X *with non-negative sectional curvatures* and with *strictly positive volume density at infinity*

$$dens_\infty(X) =_{def} \limsup_{R\to\infty} \frac{vol_n(B_{x_0}(X,R))}{vol_n(B_{\mathbf{0}}(\mathbb{R}^n,R))} > 0.$$

Observe, that since $curv(X) \geq 0$, this X admits a *unique tangent cone* $T_\infty(X)$ *at infinity* and the volume density of X at infinity equals the volume of the init ball in this cone centered at the apex $o \in T_\infty(X)$, where, recall, the tangent cone $T_\infty(X)$, of a metric space X at infinity is *the pointed Hausdorff limit* of the metric spaces obtained by scaling X by $\varepsilon \to 0$:

$$T_\infty(X) = \lim_{\varepsilon\to 0}(\varepsilon X =_{def} (X, \varepsilon \cdot dist_X)).$$

Let $Y \subset X$ be a compact k-dimensional subvariety such that the distance minimizing segments $[x,y] \subset X$ almost all points $x \in X$ and Y have their Y-endpoints y contained in the $C^{1,Lipshitz}$-regular locus of Y where moreover they are normal to Y. (This is automatic for closed smooth submanifolds $Y \subset X$ but we need it for more general Y.)

If the norms of the mean curvatures of Y *at almost all of these points* $y \in Y$ *are bounded by a constant* M, *then*

$$dens_\infty(X) \leq \frac{vol_k(Y)}{vol_k(S^k(k/M))}$$

for the k-sphere of radius k/M *in* $\mathbb{R}^{k+1}$ *that has its mean curvature equal* M.

Proof. The volumes of the R-tubes $U_R(Y) \subset X$ around Y are bounded by those of $S^k(k/M) \subset \mathbb{R}^n$ by *the Hermann Weyl tube formula* extended as a (volume comparison) inequality to Riemannian manifolds X with $curv \geq \kappa$ by Bujalo and Heintze-Karcher.

Then it follows by Almgren's variational local-to-global principle mentioned in the above 3, that

the volume minimising $(k+1)$*-chains* Z *in* X *satisfy:*

$$[curv \geq 0]_{almg} \qquad \frac{vol_{k+1}(Z)}{vol_k(\partial Z)^{\frac{k+1}{k}}} \leq (k+1)^{-\frac{k+1}{k}}(dens_\infty(X)\cdot vol_{k+1}(B^{k+1}(1))^{-\frac{1}{k}}.$$

Shareness of $[curv \geq 0]_{almg}$. This inequality is *sharp*, besides $X = \mathbb{R}^n$, for certain conical (singular if $\delta < 1$) spaces X, in particular, for $X = (\mathbb{R}^{k+1}/\Gamma) \times \mathbb{R}^{n-k-1}$ for finite isometry group Γ acting on $\mathbb{R}^{k+1}$.

Linking Corollary. The inequality $[curv \geq 0]_{almg}$, combined with the above coaria inequality, yields the following generalisation of the Bombieri-Simon (Gehring) linking volume inequality.

The $(k+1)$-volume minimising chains Z in a complete Riemannian manifold X of non-negative curvature that fill-in a k-cycle Y in X are contained in the d-neighbourhood of $Y \subset X$ for d equal the radius of the ball $B_o(T_\infty(X), d) \subset T_\infty(X)$, such that $vol_k(\partial B_o(T_\infty(X), d)) = vol_k(Y)$.

11. *Convex Functions and Monotonocity Inequality for $curv \geq 0$.* Let X be a metric space, let $x_0 \in X$ be a preferred point in X and $\mu_\bullet$, also written as $d\mu_{x_\bullet}$, be a probability measure on X.

Let $h_{x_0}(x, x_\bullet)$, $x, x_\bullet \in X$, be defined as

$$h_{x_0}(x_\bullet, x) = \max(0, (-dist(x, x_\bullet) + dist(x_0, x_\bullet))$$

and

$$H_{x_0,\mu_\bullet}(x) = \int_X h^2_{x_0}(x, x_\bullet) d\mu_{x_\bullet}.$$

If X is a complete manifold with $curv(X) \geq 0$ and $\mu_{\bullet,i}$ is a sequence of measures with supports tending to infinity (thus weakly convergent to 0), then the limit $H(x) = H_{x_0}(x)$ of the functions $H_{x_0,\mu_{\bullet,i}}$ for $i \to \infty$, assuming this limit exists, is a *convex* function on X by the old Gromoll-Meyer lemma on convexity of *the Busemann functions.*

Moreover,

if $\mu_{\bullet,i} = \rho_i(x_\bullet)dx$ for *radial functions* ρ_i, i.e $\rho_i(x_\bullet) = \phi_i(dist(x_0, x_\bullet))$, for some real functions ϕ_i, and *if $dens_\infty(X) > 0$*, then the corresponding limit functions $H(x)$ are *strictly convex.*

In fact this strictness is controlled by the density $\delta = dens_\infty(X) > 0$ as follows.

The second derivatives of these H along all geodesics in X are bounded from below by $\varepsilon = \varepsilon_n(\delta) > 0$.[54]

Now, the strict convexity of the (smoothed if necessary) function $H(x)$ can be seen as a lower bound on the k-divergence of the gradient of H and, as in the above 6, we arrive at

the monotonicity inequality of intersections of stationary $(k+1)$-dimensional subvarieties $Z \subset X$ with the balls $B_{x_0}(r) \subset X$, $x_0 \in Z$,

$$\frac{vol_{k+1}(Z_r)}{vol_k(\partial Z_r)^{\frac{k+1}{k}}} \leq const = const_n(\delta) \text{ for } n = dim(X) \text{ and } \delta = dens_\infty(X).$$

Remark. A specific evaluation of this *const* depends on a lower bound on the scalar products $\langle s_0, s\rangle$ averaged over subsets $U \subset S^{n-1}$ with spherical measures $\geq \delta \cdot vol_{n-1}(S^{n-1})$ in the tangent spheres $S^{n-1} = S^{n-1}_x \subset \mathbb{R}^n_x = T_x(X)$. But even the best bound

$$\inf_{s_0 \in S^{n-1}} \frac{1}{vol(U)} \int_U \langle s_0, s\rangle ds \geq c(\delta)$$

[54]This, which is obvious once it has been stated, was pointed out to me in slightly different terms by Grisha Perelman along with the following similar observation that constitutes the geometric core of *the Grove-Petersen finiteness theorem.*

If $curv(X) \geq -1$ and if the volume of the unit ball $B_{x_0}(1) \subset X$ around an $x_0 \in X$ has volume $\geq \delta$, then x_0 admits a convex neighbourhood $U_\varepsilon \subset B_{x_0}(1)$ that contains the ball $B_{x_0}(\varepsilon)$ for $\varepsilon = \varepsilon_n(\delta) > 0$.

with sharp $c(\delta)$ does not seem(?) to deliver the sharp constant

$$const_{almgren} = (k+1)^{-\frac{k+1}{k}}(dens_\infty(X)\cdot vol_{k+1}(B^{k+1}(1))^{-\frac{1}{k}}.$$

11. *On Singularities and Infinite Dimensions with $curv \geq \kappa$.* Since singular Alexandrov spaces X with $curv \geq \kappa$ admit strictly contracting minus "gradient fields" of strictly convex functions allow *conical filling* of cycles in balls, $Y \subset B_{x_0}(R)$, such that $vol_{k+1}(Z) \leq cost \cdot R \cdot vol_k(Y)$ and, according to [24], [44] *the cone inequality* in a space X implies a non-sharp filling inequality in this X.

Probably, these singular spaces enjoy the filling and waist inequalities with similarly *sharp constants* as their non-singular counterparts, (This is easy for *equidimensional Hausdorff limits* of non-singular spaces, since waists and filling constants are Hausdorff continuous in the presence of lower volume bounds.)

In fact, one expects a full fledged contravariantly Hausdorff continuous (i.e. for collapsing $X_i \to X = X_\infty$) theory of volume minimizing as well as of *quasi-stationary* (similar to quasi-geodesics of Milka-Perelman-Petrunin) subvarieties in X.

Anther avenue of possible generalisations is that of infinite dimensional (singular if needed) spaces X with positive curvatures. Here one is encouraged by *the stability of $dens_\infty$*:

$$dens_\infty(X) = dens_\infty(X \times \mathbb{R}^N)$$

that suggests a class of infinite dimensional spaces X with "small positive" curvatures where the differentials of various exponential maps are isometric up-to small (trace class or smaller) errors. In this case, one may try to defined $dens_\infty$ that would allow one to formulate and prove an infinite dimensional counterpart of $[curv \geq 0]$.

12. *Almgren's Morse Theory for Regular Waists.* Recall (see section 6) that k-waists of Riemannian manifolds X are defined via classes $\mathcal{D}$ of diagrams $D_X = \{X \overset{\chi}{\leftarrow} \Sigma \overset{\varsigma}{\rightarrow} S\}$ where S and Σ are pseudomanifolds with $dim(\Sigma) - dim(S) = k$ that represent *homologically substantial* S-families of k-cycles Y_s in X, that are the χ-images of the pullbacks $\varsigma^{-1}(s), s \in S$ and where homological substantiality may be understood as non-vanishing of the image of the fundamental homology class $h \in H_n(Z)$ under the homomorphism $\chi_* : H_*(\Sigma) \to H_*(X)$. Namely $waist_k(X)$ is defined as

$$waist_k(X) = \inf_{D_X \in \mathcal{D}} \sup_s vol_k(Y_s) \text{ for } Y_s = \chi(\varsigma^{-1}(s)).$$

If a class $\mathcal{D}$ consists of diagrams $\mathcal{D}_X = \{X \overset{\chi}{\leftarrow} \Sigma \overset{\varsigma}{\rightarrow} S\}$ with *sufficiently regular* maps ς and χ, e.g. *piecewise real analytic* ones, then the resulting waists, call them *regular*, admit *rather rough* lower bounds in terms of *filling* and of *local contractibility* properties of X, where the latter referees to the range of pairs of numbers (r, R) such that every r-ball in X is contractible in the concentric R-ball.

On the other hand, *the sharp* lower bound on a regular k-waist of the unit n-sphere S^n, that (trivially) implies the equality

$$reg\text{-}waist_k(S^n) = vol_k(S^k)$$

can be derived from the Almgren-Morse theory in *spaces of rectifiable cycles with flat topologies*, [41], [31]. This theory implies that

$$reg\text{-}waist_k(X) \geq \inf_{M^k \in \mathcal{MIN}_k} vol_k(M^k)$$

for "inf" taken over all *minimal/stationary* k-subvarieties $M^k \subset X$ and the inequality

$$reg\text{-}waist_k(S^n) \geq vol_k(S^k)$$

follows from the lower volume bound for minimal/stationary subvarities in the spheres S^n:

$$[volmin_k(S^n)] \qquad vol_k(M_k) \geq vol_k(S^k).$$

Almgren's theory, that preceded the homological localisation method from [22], albeit limited to the regular case, has an advantage over the lower bounds on the $\mathbb{Z}_2$-waists indicated in section 6 of being applicable to the integer and to $\mathbb{Z}_p$-cycles, that allows minimisation over the maps $\chi : \Sigma \to X$ of non-zero *integer degree* in the case of oriented X, S and Σ.[55]

Besides, Almgren's theory plus Weyl-Buyalo-Hentze-Karcher volume tube bound yield the following

sharp waist inequality for closed Riemannian manifolds X with $curv(X) \geq 1$, (see section 3.5 in [21] and [40]).

$$[waist]_{curv\geq 1} \qquad waist_k(X) \geq vol_k(S^k)\frac{vol_n(X)}{vol_n(S^n)},\ n = dim(X).$$

If X is a manifold with a *convex boundary*, this inequality may be applied to the (smoothed) double of X. but the resulting low bound on $waist(X)$ is non-sharp, unlike those obtained by the convex partitioning argument from [22].

Nevertheless, Almgern's Morse theory seems better suitable for proving the counterpart of $[waist]_{curv\geq 1}$ for general singular Alexandrov spaces with curvatures ≥ 1 (where even a properly formulated volume tube bound is still unavailablele.)

(13) *Regularization Conjecture.* Probably, general homologically substantial families of virtually k-dimensional subsets $Y_s \subset X$, should admit an ε-approximation, for all $\varepsilon > 0$, by *regular* nomologically substantial families $Y_{s,\varepsilon} \subset X$ with

$$Haumes_k(Y_{s,\varepsilon}) \leq Haumes_k(Y_s) + \varepsilon.$$

This would imply that all kinds of k-waists of S^n are equal to $vol_k(S^k)$ with no regularity assumption, that is for all homologically substantial diagrams with continuous χ and ς and with k-volumes understood as Hausdorff measures. (This remains unknown except for $k+1$ and $k = n-1$.)

14. *Waists at Infinity.* Let X be a complete manifold with $curv(X) \geq 0$. Do the waists of the complements to the balls $B_{x_0}(R) \subset X$ satisfy

$$waist_k(X \smallsetminus B_{x_0}(R) \geq R^k dens_\infty(X) vol_k(S^k)?$$

Are there some non-trivial (concavity, monotonicity) inequalities between these waists for different values of R?

[55] I am not certain in all implications of Almgren's theory, since I have not studied the technicalities of this theory in detail.

Does the above lower bound hold for waists of the subsets $X \smallsetminus B_x(R)$ seen as variable ones (in the sense of section 8) parametrised by $X \ni x$?

If "yes" this would imply that every m-dimensional subvariety $W \subset X$ that is *not homologous to zero in its R-neighbourhood*, satisfies (compare section 18)

$$waist_k(W) \geq R^k dens_\infty(X) vol_k(S^k).$$

Another class of spaces X where evaluation of waists (and volume spectra in general) at infinity may be instructive is that of *symmetric spaces with non-positive curvatures.*

(One may define "waists at infinity" via integrals of positive radial functions $\phi(dist(x,x_0))$ over k-cycles in X. For instance, the function $\phi(d) = \exp \lambda d$, $\lambda < 0$, may serve better than $+\infty$ on the ball B_{x_0} and 1 outside this ball which depicts $X \smallsetminus B(R)$.)

(15) *Waists of Products and Fibrations.* Let X be a Riemannian product, $X = \underline{X} \times X_\varepsilon$, or more generally, let $X = X(\varepsilon)$ be fibered over $\underline{X}$.

If X_ε is sufficiently small compared to $\underline{X}$, e.g.

$$X_\varepsilon = \varepsilon \cdot X_0 =_{def} (X_0, dist_\varepsilon = \varepsilon \cdot dist_0) \text{ for a small } \varepsilon > 0,$$

then, conjecturally,

$$wast_k(X) = wast_k(\underline{X})$$

and something similar is expected for fibrations $X(\varepsilon) \to \underline{X}$ with *small* fibres $X_{\underline{x}} \subset X$ that also for this purpose must vary *slowly* as functions of $\underline{x}$.

Here, conjecturally, the Hausdorff k-waist of $X(\varepsilon)$ equals the maximum of the k-volumes of the fibres of the normal projection $X(\varepsilon) \supset \underline{X}$ plus a lower order term.[56]

This, *the regular case,* follows from the Almgren-Morse theory whenever "small" minimal subvarieties $M^k \subset X(\varepsilon)$ are "vertical", namely, if those with $vol_k(M^k) \leq wast_k(X(\varepsilon))$, go to points under $X(\varepsilon) \to \underline{X}$, where a sufficient condition to such "verticality" is a presence of *contracting* vector fields, in sufficiently large balls in $\underline{X}$ and where such filds often coms as gradients of convex functions. But it is unclear how to sharply bound from below the Hausdorff waists in the non-regular case, compare 7.4 in [22] and 1.5(B) in [28].

16. *Waists of Thin Convex Sets.* A particular case of interest is where $\underline{X}$ is a compact convex m-dimensional subset in an n-dimensional space, call it Z, of constant curvature, and $X(\varepsilon) \supset \underline{X}$ are n-dimensional convex subsets in Z that are ε-close to this $\underline{X}$.

These $X(\varepsilon)$ *normally* project to $\underline{X}$ with convex ε small k-dimensional fibres $X_{\underline{x}}$ over the interior points $\underline{x} \in int(\underline{X}) = \underline{X} \smallsetminus \partial \underline{X}$ for $k = n - m$[57] and, conjecturally,

$$\frac{wast_k(X(\varepsilon))}{\max_{\underline{x} \in int(\underline{X})} vol_k(X_{\underline{x}})} \to 1 \text{ for } \varepsilon \to 0.$$

This is known for δ-$Mink_k$-waists, and this implies, in conjunction with homological localisation, the sharp lower bounds on the Minkowski waists spheres,

[56] This would imply the sharp lower bound on the Hausdorff waist of spheres by the homological localisation argument in [22].

[57] The fibres over the boundary points $\underline{x} \in \partial \underline{X}$ have $dim(X_{\underline{x}}) = n$.

see [22]. But the case of Hausdorff waistes with *no regularity* assumption remains open.

17. *Waists of Solids.* Does the regular k-waist of the rectangular solid

$$[0, l_1] \times [0, l_2] \times ... \times [0, l_n], \ 0 < l_1 \leq l_2 \leq ... \leq l_i \leq ... \leq l_n,$$

equal $l_1 \cdot l_2 \cdot ... \cdot l_k$?

This is not hard to show for *fast growing* sequences $l_1 << l_2 << ... << l_i << ... << l_n$ but the case of roughly equal l_i, especially that of the cube $[0, l]^n$, remains problematic.

18. *Waist of the Infinite Dimensional Hilbertian Sphere.* "Homologically substantial" families of k-cycles in S^∞ may be defined with either Fredholm maps $S^\infty \mathbb{R}^\infty$ of index k, (i.e. with virtually k-dimensilnal fibres) or, with Fredholm maps $Y \times S^\infty \to S^\infty$ of Fredholm degree one, or by bringing the two diagrammatically together as in section 6.

But it is unclear if these waists are not equal to zero.

19. *Linking Inequalities with δ-$Mink_k$.* Such inequalities provide lower bounds on the volumes of δ-neighbourhoods $U_\delta(W) \subset X$ of k-dimensional subvarities $W \subset X$ that are not homologous to zero in their R-neighbourhoods for some $R > \delta$. For instantce,

If $X = \mathbb{R}^n$, then

$$vol_n(U_\delta(W)) \geq vol_n(U_\delta(S^k(R)))$$

for the R-sphere $S^k(R) \subset \mathbb{R}^{k+1} \subset \mathbb{R}^n$.

Proof The argument from section18 (also see section 8 in [24]) with the radial projections from W to the R-spheres with centers $x \in \mathbb{R}^n \setminus W$ applies here, since:

• the spherical waist inequality holds true with δ-$Mink_k$ defined with δ-neighbourhoods of subsets $Y \subset S^{n-1} \subset \mathbb{R}^n$ taken in $\mathbb{R}^n \supset S^{n-1}$ (see [22]);

• these projections $W \to S^{n-1}_x(R)$ are not only distance decreasing, but they are obtained from the identity map by a distance decreasing homotopy. Therefore it diminishes the volume of δ-neighbourhoods by *Csikós' theorem.* See [14] and [8] and references therein.)

20. *Geometric Linking Inequalities.* Let two closed subsets $W, W' \subset \mathbb{R}^n$ of dimensions k and $n-k-1$ *can not be unlinked*, i.e. moved apart without mutual intersection on the way by a certain class $\mathcal{M}$ of geometric motions.

Can one bound from below the Hausdorff measures of these subsets in terms of $dist(W, W')$?

For instance – this, according to Eremenko [17], was observed by I. Syutric in 1976 or in 1977 – if $\mathcal{M}$ consists of homotheties $h_t : x \mapsto x_t = x_0 + (1-t)(x - x_0)$, $t \in [0, 1]$, for some point $x_0 \in W$, then $Haumes_1(W) \geq \pi R$ for connected sets W and $hausmes_1(W) \geq 2\pi R$ for closed curves W.

Indeed, the image of this homothety, that is the unit cone over W from x_0, intersects W', say at $x' \in W' \cap h_{t'}(W)$ for some $t' \in [0, 1]$; thus the radial projection from W to the R-sphere $S^{n-1}_{x'}(R) \subset \mathbb{R}^n$, that is distance decreasing, contains two diametrically opposite points, namely the images of x_0 and $x_1 = h^{-1}_{t'}(x') \in W$. QED.

A bound on $vol(W \times W')$. If W and W' can not be unlinked by *parallel translations*, then, obviously, the map $W' \times W \to S_0^{n-1}(R)$ for $(w,w') \mapsto R\frac{w-w'}{dist(w,w')}$ is onto. Hence,

$$Hausmes_k(W) \cdot Hausmes_{n-k-1}(W) \geq const_n R^{n-1}.$$

Thus, either W or W' must have large Hausdorff measure, but one of them may arbitrarily small. This is seen by taking (large) W that ε-approximates the k-skeleton of a (large) sphere $S^{n-1}(R+r) \subset \mathbb{R}^n$ and where W' much finer, say with $\varepsilon' = 0.1\varepsilon$ approximates the $(n-k-1)$-skeleton of a (tiny) r'-ball $B_0^n(r') \subset B_0^n(R+r) \subset \mathbb{R}^n$ for $r' \leq r/2$.

If $r' \geq 2\varepsilon$ then W "cages" W' inside $S^{n-1}(R+r)$, provided $r' \geq 10\varepsilon$:

W' can not be moved outside of $S^{n-1}(R+r)$ by an isometric motion without meeting W on the way.

Non-accessible Articles. There is a dozen or so other papers on Gehring linking problem but, since they are not openly accessible, one can not tell what is written in there.

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