Plateau–Stein Manifolds.

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Abstract

We study/construct (proper and non-proper) Morse functions f on complete Riemannian manifolds X such that the hypersurfaces $f(x) = t$ for all $-\infty < t < +\infty$ have positive mean curvatures at all non-critical points $x \in X$ of f. We show, for instance, that if an X admits no such (not necessarily proper) function, then it contains a (possibly, singular) complete (possibly, compact) minimal hypersurface of finite volume.

1 Introduction.

1.1 Mean Curvature Convexity.

Let X be a smooth Riemannian manifold. Given a smooth function f on X, define the *mean curvatures $mn.curv_x(f)$* at *non-critical* points $x \in X$, i.e. where $df(x) \neq 0$, as the mean curvatures of the level hypersurfaces $Y = Y_r = f^{-1}(r)$, for $r = f(x)$,

$$mn.curv_x(f) =_{def} mn.curv_x(Y_r)$$

where the mean curvatures of the hypersurfaces Y are defined by evaluation of their second fundamental forms on the normalized *downstream* gradient field $-grad(f)/\|grad(f)\|$. Call a function f *strictly mean curvature convex* (sometimes we say say "$(n-1)$-*mean curvature convex*" instead of just "mean curvature convex" for $n - 1 = dim(Y) = dim(X) - 1$.) if

$mn.curv_x(f) \geq \varepsilon(x) > 0$ for all f-*non-critical* points $x \in X$, i.e. where $df(x) \neq 0$,

for a positive continuous function ε on X.

Remarks. (a) We are especially concerned with strictly mean curvature convex *Morse* function, i.e. where the critical points of f are *non-degenerate*. Even though our "convexity" definition *formally* makes sense for all smooth functions f, one has, in reality, to impose some, possibly weaker than Morse, constrains on the critical points of f — we do not want to accept, for example, constant functions.

(b) In what follows, most our manifolds X are non-compact where $\varepsilon(x)$ may tend to 0 for $x \to \infty$. This happens, for instance, to the squared distance function in the Euclidean space $\mathbb{R}^n$ from the origin, that is an archetypical example of a strictly mean curvature convex Morse function.

(c) If we compose $f : X \to \mathbb{R}$ with a smooth nowhere locally constant function $\psi : \mathbb{R} \to \mathbb{R}$, then $\psi \circ f : X \to \mathbb{R}$ has, at least locally, the same levels as f. The sign of the mean curvatures of the levels is preserved if the derivative derivative of ψ is positive, $\psi' > 0$; however, it changes where $\psi' < 0$.

Conclude by observing that Morse properties of a function are influenced by the sign of the mean curvature of the levels via the following obvious inequality.

[**ind$\leq$ n $-$ 2**] *the critical points of strictly mean curvature convex Morse functions have their Morse indices $\leq n - 2$ for $n = dim(X)$.*

1.2   Non-Proper and Proper Plateau–Stein Manifolds.

[**n-n-proPS**]$_\smile$. Call a possibly non-complete, Riemannian manifold X of dimension $n \geq 2$ without boundary *Plateau–Stein* if it admits a strictly mean curvature convex Morse function $f : X \to \mathbb{R}$.

Sometimes, to emphasize that f is *not* assumed proper, we call these *non-proper* Plateau–Stein or [n-n-proPS]$_\smile$, where "$\smile$" stands for "convex" with "$\frown$" in the next section for"concave" and where

non-proper must be always understood as *not necessarily proper.*

Three other similar conditions on X are as follows.[1]

[PS](1) Given a compact subset $B \subset X$ and $\varepsilon > 0$, there exist a *strictly $(n-1)$-volume contracting* continuous map $\Psi_\varepsilon : B \to X$, that is

$$vol_{n-1}(\Psi_\varepsilon(H)) < vol_{n-1}(H)$$

for all smooth hypersurfaces $H \subset X$, and such that

$$dist_X(\Psi_\varepsilon(x), x) \leq \varepsilon \text{ for all } x \in B.$$

(This condition does not truly need any smooth structure in X.)

[PS](2) X admits a C^1-smooth *strictly $(n-1)$-volume contracting* vector field V, $n = dim(X)$, i.e., for every compact subset $B \subset X$, there exists $\varepsilon > 0$, such that V integrates on B to a flow up to the time $t = \varepsilon$ where the flow maps, say $V_t : B \to X$, are strictly $(n-1)$-volume contracting on B for $0 < t < \varepsilon$.

[PS](3) X admits a *strictly $(n-1)$-mean convex function* f i.e. such that the *gradient field* of $-f$ is strictly $(n-1)$-volume contracting. (See section 3.3 for an alternative definition.)

Clearly, [PS](3)$\Rightarrow$[PS](2)$\Rightarrow$[PS](1) and also [PS](3) implies Plateau–Stein, since every strictly $(n-1)$-mean convex function f admits an arbitrarily small perturbation that makes it strictly $(n-1)$-mean curvature convex.

What seems non-obvious — I do not see a direct proof of this — is the following corollary to the *inverse maximum principle* stated in the next section.

Contraction Corollary for Covering. *Let X be an infinite covering of a compact manifold. If X is [PS](1) i.e. if it admits strictly $(n-1)$-volume contracting continuous maps $\Psi_\varepsilon : B \to X$, for all compact subsets $B \subset X$ and all $\varepsilon > 0$, with $dist_X(\Psi_\varepsilon(x), x) \leq \varepsilon$, then X is Plateau–Stein.*

Remark/Question. Probably, if f is strictly mean convex, then one can arrange a smooth function a on X with a large derivative along the gradient field $grad(f)$, such that the field $-e^{a(x)}grad(f)$ would be strictly mean contracting.

But it is less clear what should be *exactly* the class of (non-covering) manifolds X where the existence of a strictly $(n-1)$-volume contracting field implies Plateau–Stein.

[**proPS**]$_\smile$. Say that X is *proper Plateau–Stein*, if it admits a *proper positive* strictly mean convex Morse function $f : X \to \mathbb{R}_+$ where proper for (a not necessarily positive) function f on X means that $x \to \infty \Rightarrow |f(x)| \to \infty$, where "$\to \infty$" means "eventually leaves every compact subset".

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[1]Bringing forth these properties was motivated by our conversations with Bruce Kleiner.

Sometimes we say that *the Riemannian metric on X is proper/non-proper Plateau–Stein.*

Proper Plateau–Stein manifolds are reminiscent of *complex Stein manifolds* X that, by definition, support proper positive *strictly $\mathbb{C}$-convex,* traditionally called *plurisubharmonic,* functions. An obvious *necessary* condition for the existence of such a (not necessarily proper) function is the absence of *compact complex submanifolds* of positive dimensions in X.

A theorem by Grauert says that this condition is also *sufficient,* if X can be exhausted by compact domain with strictly $\mathbb{C}$-convex (pseudoconvex) boundaries.

We shall prove in this paper a Riemannian counterpart to Grauert's theorem with "suitable compact minimal hypersurfaces" ("hypersurface" always means a *codimension 1 subvariety,* possibly with singularities) instead of "compact complex submanifolds".

The possible topologies of Plateau–Stein manifolds are rather transparent.

Plateau–Stein manifolds X are *non-compact* and, if *proper,* they have *zero homology* $H_{n-1}(X; \mathbb{Z}_2)$ for $n = dim(X)$. In particular, they are connected at infinity. Moreover, *proper* Plateau–Stein manifolds X are diffeomorphic to regular neighborhoods of *codimension two* subpolyhedra in X.

This follows from [**ind$\leq$ n – 2**]. For example, proper Plateau–Stein surfaces are homeomorphic to the 2-plane $\mathbb{R}^2$ and proper Plateau–Stein 3-folds are topological handle bodies, while non-proper Plateau–Stein allows a complete (warped product) metric on the topological cylinder $X^n = X_0^{n-1} \times \mathbb{R}$ for all $(n-1)$-manifold X_0^{n-1} as a simple argument shows.

In fact one can show (we leave this to the reader) the following.

Let $f : X \to \mathbb{R}$ be a proper, not necessarily positive, Morse function $f : X \to \mathbb{R}$, where all critical points have indices $\leq n - 2$. Then there exists a complete Riemannian metric on X, (which is proper Plateau–Stein according to our definition if f is positive) for which this function is strictly mean curvature convex.

Unlike the proper Plateau–Stein the non-proper Plateau–Stein condition is not topologically restrictive for open manifolds.

every open manifold X admits a (possibly non-complete) non-proper Plateau–Stein Riemannian metric.

In fact, a simple argument shows that

given a smooth function f without critical points on a smooth manifold X, there obviously exists a (possibly non-complete) Riemannian metric on X, such that the level hypersurfaces of f are convex with respect to this function.

Probably, every open manifold X of dimension $n \geq 3$ admits a *complete* non-proper Plateau–Stein Riemannian metric.

(Complete Plateau–Stein surfaces X are homeomorphic either to $\mathbb{R}^2$ or to the cylinder $S^1 \times \mathbb{R}$, since other *non-compact complete connected* surfaces necessarily contain (non-contractible) closed geodesics that is incompatible with being (complete or not) Plateau–Stein in dimension $n = 2$.)

The geometry of Plateau–Stein manifolds, unlike their topology, is not as apparent as their topology.

Examples and Questions. (a) Let $X = (X, g_0)$ be a complete simply connected n-manifold, $n \geq 2$, of non-positive sectional curvature $\kappa(g_0) \leq 0$. Since the spheres $S_{x_0}(R) \subset X$ around a given point $x_0 \in X$ are strictly convex, such an X is proper Plateau–Stein and all open subsets $U \subset X$ are Plateau–Stein.

Even though the inequality $\kappa(g_0) \leq 0$ is unstable under smooth perturbations of g_0 the Plateau–Stein may be stable.

For example, let the Ricci curvature of X be bounded from below by $-\delta g_0$, $\delta > 0$, e.g. X is a symmetric space of non-compact type with no Euclidean factor. Then $mn.curv(S(R)) \geq \delta > 0$ for all $R > 0$. Since this inequality is stable under uniformly C^1-small perturbations g_ε of the original metric g_0 on X, the function $x \to dist_{g_0}^2(x, x_0)$ remains mean curvature convex with respect to g_ε; hence, these g_ε are proper Plateau–Stein .

If a *non-flat* symmetric space (X, g_0) of non-compact type does have a Euclidean factor, then the perturbed metrics g_ε are, obviously, non-proper Plateau–Stein. Probably, they are proper Plateau–Stein.

On the other hand, the Euclidean metric g_0 on $\mathbb{R}^n$, $n \geq 2$, admits arbitrarily C^∞-small perturbation g_ε that are *not* Plateau–Stein.

Indeed, one can arrange g_ε, such that $g_0 - g_\varepsilon$ is supported in an annulus A pinched between two large spheres, say $S^{n-1}(R)$ and $S^{n-1}(R+1)$ in $\mathbb{R}^n$), $R \gg 1/\varepsilon$, and such that (A, g_ε) is isometric to the Riemannian product $S^{n-1}(R) \times [0, 1]$.

It is clear such a g_ε is not Plateau–Stein, since the mean curvature of a smooth function f on $X = (\mathbb{R}^n, g_\varepsilon)$ is, obviously, non-positive at the maximum point of f on $S^{n-1}(R)$.

(b) Let G_ε be the space of ε-small C^∞ perturbations g_ε of g_0 that are *invariant under the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$*, i.e. these g_ε correspond to perturbations of the flat metric on the torus $\mathbb{R}^n/\mathbb{Z}^n$. Divide the space G_ε into three classes:

[1] Proper Plateau–Stein;

[2] Plateau–Stein but not proper Plateau–Stein;

[3] not even non-proper Plateau–Stein.

What is the topological structure of this partition? Are all three subsets $[1], [2], [3] \subset G_\varepsilon$ dense in G_ε for small ε? Is any of these $[1], [2], [3] \subset G_\varepsilon$ a meager set? (I am uncertain of what happens even for $n = 2$ where the answer might be known, albeit in different terms.)

(c) Let $p : X_1 \to X$ be a *Riemannian submersion* between Riemannian manifolds, i.e. the differential $dp : T(X_1) \to T(X)$ everywhere has $rank(dp) = n = dim(X)$ and it is isometric on the *horizontal tangent (sub)bundle* $T(X_1) \ominus ker(dp) \subset T(X_1)$ (normal the fibers $p^{-1}(x) \subset X_1$). The simplest instance of this is the projection of a Riemannian product $X_1 = X \times X'$ onto the X factor.

Let the action of the normal holonomy (by the parallel transport along the horizontal bundle) on the fibers be *volume preserving*, e.g. $p = X \times X' \to X$. Then the p-pullback of hypersurfaces from X to X_1 preserves their mean curvatures. Therefore, if a function $f : X \to \mathbb{R}$ is Morse strictly mean convex, then a *generic* smooth perturbation of $p \circ f : X_1 \to \mathbb{R}$ is also Morse as well strictly mean convex.

Consequently:

(c$_1$) *If X is a Plateau–Stein then so is X_1.*

(c$_2$) *If X is a proper Plateau–Stein and the fibers $p^{-1}(x) \subset X_1$ are closed manifolds for all $x \in X$, then X_1 is also proper Plateau Stein.*

(c_3) *if X and the fibers $p^{-1}(x)$ are proper Plateau Stein, if the action of normal holonomy is isometric on the fibers, and if the normal holonomy group is compact, then X_1 is proper Plateau Stein.*

(c_4) It follows from (a) and (c_2) that

non-compact semisimple groups with finite centers are proper Plateau–Stein,

while (c_1) implies that

unimodular solvable, e.g. nilpotent, groups are (not necessarily proper) Plateau– Stein.

Probably, all non-compact Lie group X with left invariant metrics, except for *compact* $\times \mathbb{R}$, are Plateau–Stein but it is less clear which Lie groups, and Riemannian homogeneous spaces in general, are *proper* Plateau–Stein.

(d) The Riemannian cylinders that are product $\mathbb{R} \times X'$ are Plateau–Stein for many (all?) *open* Riemannian X', e.g. for the interiors X' of compact manifolds with boundaries (this is obvious) and for complete connected manifolds of with infinite volume. (This is not hard.)

(e) What are non-compact, e.g. complete, Riemannian manifolds X', such that the Riemannian products $X \times X'$ are proper Plateau–Stein for all proper Plateau–Stein manifolds X?

Conclude with the following questions where topology and geometry are intertwined.

Let V be a closed Riemanian manifold.

(I) *When does V admit a Riemannian metric such that the universal covering X of V with this metric is proper, or at least non-proper, Plateau–Stein?*

(II) *When is the universal covering of V bi-Lipschitz equivalent to a (proper) Plateau–Stein manifold?*

Probably, the answers are invariant under the codimension two surgeries of V, and even possibly, depend only on the fundamental group $\Gamma = \pi_1(V)$.

Anyway, the fundamental groups Γ of such manifolds V (where the answers are positive) may be called (I) or (II) "Plateau–Stein" [proPS]$_\smile$ and [n-n-proPS]$_\smile$.

The best candidates for "Plateau–Stein" Γ are non-amenable groups with one end. On the other hand, there may exist some "tricky" (forget about virtually cyclic) amenable groups that are not even [n-n-proPS]$_\smile$.

Question (II) makes sense for all complete manifolds X, not only coverings of compact ones:

When is such an X is bi-Lipschitz equivalent to a Plateau–Stein manifold?

("Bi-Lipschitz" seems too restrictive in this context; one needs something half-way from bi-Lipschitz to *quasiisometric* in the spirit of the *directed Lipschitz metric* [3].)

1.3 Inverse Maximum Principle.

[n-n-proPS]$_{\min}$ Say that a Riemannian manifold X is [n-n-proPS]$_{\min}$ if it contains *no compact minimal hypersurface.*

One can not exclude such a hypersurface being singular. Below is smooth version of this condition with minimal replaced by "almost concave"

[n-n-proPS]$_\frown$. Say that a Riemannian manifold X is [n-n-proPS]$_\frown$ if it admits a continuos positive function $\varepsilon(x) > 0$, such that every compact smooth domain, i.e. a relatively compact open compact subset $U \subset X$ with smooth boundary in X, has a point $x \in \partial U$ where

$$mn.curv_x(\partial U) \geq \varepsilon(x).$$

Another way to put it is that X contains *no* bounded domain with ε-mean-concave boundary.

[proPS]$_{\min}$. Say that X is [proPS]$_{\min}$ if it is connected at infinity and if, for every compact subset $B \subset X$, there is a (larger) compact subset $C = C(B) \subset X$, such that

all compact minimal hypersurfaces $H_i \subset X$ with boundaries contained in $B \subset X$ are themselves contained in C.

A smooth almost concave version of this condition is as follows.

[proPS]$_\frown$. Say that X is [proPS]$_\frown$ if it is connected at infinity and there are continuos positive function $\varepsilon(x) > 0$ and a proper continuous function $\phi : X \to \mathbb{R}_+$, such that:

given a compact subset $V \subset X$ and a smooth domain $U \subset X$ where

$$\sup_{x \in U} \phi(x) > \sup_{x \in V} \phi(x),$$

there exists a point $x \in \partial U \smallsetminus V$, where

$$mn.curv_x(\partial U) \geq \varepsilon(x).$$

One immediately sees by looking at the *maxima points* of strictly mean curvature convex functions $f(x)$ on the boundaries ∂U that proper/non-proper Plateau–Stein manifolds satisfy the corresponding [...]$_{\min}$-conditions. Namely:

$$[\text{n-n-proPS}]_\smile \Rightarrow [\text{n-n-proPS}]_\frown$$

and

$$[\text{proPS}]_\smile \Rightarrow [\text{proPS}]_\frown.$$

Also it is not hard to see by a simple approximation argument (see Step 2 in the next section and sections 3.4, 5.6, 5.7) that

$$[\text{n-n-proPS}]_\frown \Rightarrow [\text{n-n-proPS}]_{\min}$$

and

$$[\text{proPS}]_\frown \Rightarrow [\text{proPS}]_{\min}.$$

IMP for Thick Manifolds. The main purpose of the present paper is proving inverse implications for Riemannian manifolds X that are *thick at infinity* (see section 2.1). Examples of these include:

$\bullet_{conv}$ complete manifolds X where the balls of radii$\leq \varepsilon$ are *convex* for some $\varepsilon > 0$;

$\bullet_{Ricc}$ complete manifolds X where $Ricci(X) \geq -const \cdot Riem.metric(X)$ and, at the same time, the volumes of the unit balls $B_x(1) \subset X$ for all $x \in X$ are bounded
from below by an $\varepsilon > 0$;

$\bullet_{Lip}$ complete manifolds X where the ε-balls $B = B_x(\varepsilon) \subset X$ for some $\varepsilon > 0$ admit λ-*bi-Lipschitz* embeddings $B \to \mathbb{R}^n$ for $n = dim(X)$ and for some constant λ independent of B.

Notice that coverings of compact manifolds are thick at infinity by either of these conditions.

Main Theorem: Inverse Maximum Principle. *Let X be a complete C^2-smooth Riemannian n-manifold, $n \geq 2$. If X is thick at infinity, then*

IMP[non-proper] $[\text{n-n-proPS}]_{\min} \Rightarrow [\text{n-n-proPS}]_{\smile}$;

IMP[proper] $[\text{proPS}]_{\min} \Rightarrow [\text{proPS}]_{\smile}$.

Convex/Minimal Existence Alternative. Observe that IMP[non-proper] says, in effect, that either X can be "filled" by *strictly mean convex* hypersurfaces (that are the levels of a strictly mean curvature convex Morse function $f(x)$) or, alternatively, X contains a *compact minimal* hypersurface Y and IMP[proper] encodes a similar alternative.

1.4 ϕ-Bubbles, Plan of the Proof of IMP and Trichotomy Theorem.

Let μ be a Borel measure μ in X and define the μ-*area* of a *domain $U \subset X$* with boundary $Y = \partial U$ as

$$area_{-\mu}(U) =_{def} vol_{n-1}(Y) - \mu(U),$$

where "domain" means either a closed subset $U \subset X$ with the interior $int(U) \subset U$ being *dense* in U or an open subset that *equals the interior* of its closure.

Call U a μ-*bubble* if it *locally minimizes* the function $U \mapsto area_{-\mu}(U)$ where "local" may be understood at this point relative to the Hausdorff metric in space of pairs $(U, Y = \partial U)$. (We return to this in section 2.)
.

For instance, if $\mu = 0$ then μ-bubbles are domains bounded by stable minimal hypersurfaces in X.

If μ is given by a measurable density function $\phi(x)$, $x \in X$, i.e. $\mu = \phi dx$ for the Riemannian n-volume (measure) dx, then we speak of ϕ-*bubbles* and observe that if ϕ is a continuous function, then

the mean curvatures of the boundaries $Y = \partial U$ of ϕ-bubbles satisfy

$mn.curv_x(Y) = \phi(y)$ *for all regular points x of Y.*

In particular, if $\phi \geq 0$, then ϕ-bubbles are *mean convex* at all regular points of their boundaries, i.e. $mn.curv_x(Y) \geq 0$ at all regular $x \in Y$ and *strictly mean convex* at such points if $\phi > 0$.

In sequel, if ϕ is not specified, these are called just *(strictly) mean convex bubbles* .

(By definition, Y is *regular* at a point $x \in Y$ if Y is a C^2-*smooth* hypersurface in a neighbourhood of this point.)

Bubbles with Obstacles. If $\phi_0 = \infty_B$ equals $+\infty$ on some, say compact subset $B \subset X$ and zero outside B, then the boundaries of ϕ-bubbles $U \supset B$ are, almost by definition, minimal hypersurfaces Y in the closure of the complement $X \smallsetminus B$ that solve the Plateau problem with the *obstacle B.*

We shall often use positive continuous functions $\phi = \phi_\varepsilon > 0$ that approximate such an ϕ_0, being large, say $1/\varepsilon$, on B and $\varepsilon > 0$-away from B. Then the corresponding ϕ_ε-bubbles $Y = Y_\varepsilon$ lie close to ϕ_0-bubbles for small $\varepsilon \to 0$ that helps to understand their overall geometry, while the continuity of ϕ makes the local (quasi)regularity of these Y_ε similar to that of minimal varieties.

We divide the proof of IMP into five steps.

Step 0: *Excluding "Narrow Ends".* The representative case of the theorem is where X is *one-ended*, i.e. *connected at infinity*, and where this end has *infinite area*. This means that the boundaries $Y_j = \partial V_j$ of an arbitrary exhaustion of X by bounded domains $V_j \in X$ satisfy

$$vol_{n-1}(Y_j) \underset{j \to \infty}{\to} \infty \text{ for } n = dim(X).$$

(In fact, one needs a slightly more general version of this condition as we shall explain in sections 2.2, 2.3.)

Step 1: *Mean Convex Exhaustion.* Let X be one-ended complete with infinite area at infinity. Then

there exist strictly mean convex compact bubbles $U_j \subset X$, $j = 1, 2, ...$, that exhaust X,

$$U_1 \subset U_2, \subset ... \subset U_j \subset ... \text{ and } \cup_j U_j = X.$$

Notice that the thickness at infinity condition is not needed beyond this point. On the other hand it is essential for the existence of the bubbles U_j.

In fact, such an ϕ_j-bubble U_j is obtained for $\phi_j > 0$ that is large on the ball $B = B_{x_0}(j) \subset X$ of radius j around a fixed point $x_0 \in X$ and small away from B.

The existence of U_j is proven in sections 2.2, 2.3 by the standard minimization argument of the *geometric measure theory*, that works on our (non-comact) X because of the thickness condition that is designed exactly for this purpose (see section 2.1) in order to prevent *partial escape* of minimizing sequences of (boundaries of) bounded domains U in X to infinity. (Such an escape can be imagined as a "long narrow tentacle" protruding from the "main body" of U).

Step 2: *Mean Convex Regularization.* One can not guaranty at this point that the boundary hypersurfaces $Y_j = \partial U_j$ are smooth for $n = dim(X) \geq 7$. Yet, they do have positive mean curvatures in a generalized sense. Moreover, (this proves the implication [n-n-proPS]$_\frown \Rightarrow$ [n-n-proPS]$_{\min}$) these Y_j can be

approximated by C^2-smooth hypersurfaces $Y_j' \subset U_j$
with positive mean curvatures.

In fact, since (the boundaries of) ϕ-bubbles U are *quasiregular* (as defined in section 3.2) for all continuous functions ϕ by the *Almgren-Allard regularity theorem*, the minus distance function $d(x) = -dist(x, Y = \partial U)$ can be *regularized* almost without loss of the lower mean curvature bound near the boundary of U. Namely, we shall see in section 3.4, and, from a different angle, in section 5.6, that

every strictly mean convex bubble $U \subset X$ admits a continuous function
$d' : U \to (-\infty, 0]$, such that

• *$d'(x) = 0$ for $x \in Y = \partial U$,*
• *$d'(x)$ is smooth strictly negative in the interior $int(U) \subset U$*
and all critical points of d' in $int(U) \subset U$ are non-degenerate,
• *there exists a continuous function δ in U that vanishes on*
the boundary $Y = \partial U$, such that $mn.curv_x(d') \geq \phi(x) - \delta(x)$
for all d'-non-critical points $x \in int(U)$.

Remarks and Questions. (a) The above regularization is non-essential at this stage of the proof; yet, it will becomes relevant later on.

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(b) This regularization, along with the simple but "non-elementary" minimization argument at Step 1 in the framework of the geometric measure theory provides an exhaustion of X by compact domains $U'_j \subset X$ with smooth strictly mean convex boundaries Y'_j.

Is there an "elementary" proof of this?

(c) Our regularization procedures (see sections 3.4, 5.6), however simple, needs C^2-smoothness of the Riemannian metric in X and it does not work for C^1-manifolds with the sectional curvatures bounded from above and from below.

But the inverse maxima principles, if properly formulated, may hold for C^1-smooth manifolds and for some singular spaces, e.g. for *Alexandrov spaces* with curvatures bounded from below.

Step 3: *Inverse Maximum Principle for Manifolds with Boundary.* Let V be a smooth compact Riemannian manifold with boundary and ϕ be a continuous function on V such that the boundary of V is *strictly mean ϕ-convex*, i.e. $mn.curv_v(\partial V) > \phi(x)$ for all $v \in \partial V$. Then, assuming $\phi > 0$,

- *either the interior of V contains a ϕ-bubble,*
- • *or V admits a Morse function $f : X \to \mathbb{R}_- = (-\infty, 0]$ that vanishes on the boundary of X and that is strictly mean ϕ-convex i.e. $mn.curv_v(f) > \phi(v)$ for all f-non-critical points $v \in V$.*

The proof of this is divided into two half-steps.

Half-Step 3A: *Shrinking Bubbles.*

Take a (eventually small), positive $\rho > 0$, a (large) $C > 1$ and a monotone decreasing sequence $\varepsilon_i > 0$ $i = 0, 1, 2, ...$, where $\varepsilon_1 = \varepsilon_1(V) > 0$ is (very) small and where $\varepsilon_i \to 0$ for $i \to \infty$.

Construct step by step a sequence

$$U_0 = V, \phi_1, U_1, \phi_2, U_2, \phi_3, ..., U_i, \phi_{i+1}, U_{i+1}, ...,$$

where ϕ_{i+1}, $i = 0, 1, 2, ...$, is a continuous function on U_i and where $U_{i+1} \subset U_i$ is a ϕ_{i+1}-bubble where the following three conditions must be satisfied.

$(*)_\varepsilon$ $\phi_i \geq \phi + \varepsilon_i$ *for all* $i = 1, 2, ...;$

$(*)_{\varepsilon,\rho}$ $\phi_i(u) = \phi(u) + \varepsilon_i$ *for all u in the ρ-neughbourhood of $\partial U_{i-1} \subset U_{i-1}$.*

$(*)_{C\rho}$ ∂U_{i+1} *is contained in the $C\rho$-neighbourhood of $\partial U_i \subset U_i$ for all* $i = 0, 1, 2, ...$

This $(*)_{C\rho}$ is the only non-trivial requirement on our sequence, where the existence of a ϕ_{i+1}-bubble $U_{i+1} \subset U_i$ with the boundary ∂U_{i+1} lying close to ∂U_i needs a suitably chosen ϕ_{i+1} that must be large away from the ρ-neigbourhood of $\partial U_i \subset U_i$ (see sections 2.4, 41, 4.2).

Half-Step 3B: *Regularization.* On can show (see section 2.) that if V contain no ϕ-bubble, the sets U_i become empty for large i.

Then, if $\rho > 0$ is sufficiently small, one can construct the required f by "splicing and regularizing" minus distance functions $u \mapsto -dist(u, \partial U_i)$ on the subsets U_i (see section 4)

Remarks. (a) "Shrinking bubbles" can be seen as discretization of a some "gradient flow" for a non-Hilbertian norm in the (tangent space to the) space of subvarieties in X, where the Hilbertian norm leads to the mean curvature flow.

(b) The condition $\phi > 0$ can be dropped with a slightly more general notion of "ϕ-bubble", that would allow, for instance, the central geodesic in the (hyperbolic) Möbius band for the role of a ($\phi = 0$)-bubble.

Step 4. *Limits by Exhaustion.* We shall use in section 4.2 a simple compactness property (of sets of the boundaries of) our mean convex bubbles $U_{i,j} \subset U_j$ construct mean convex functions f on X as limits of such functions f_j in bounded mean convex domains $U_j \subset X$ that exhaust X.

Looking closer (see section 4) one obtains with the above argument the following.

Trichotomy Theorem. Let X be a complete Riemannian C^2-smooth manifold (not assumed thick at infinity). Then (at least) one of the three conditions is satisfied.

(1) *X admits a proper (positive, if X is connected at infinity) strictly mean curvature convex Morse function;*

(2) *X contains a complete (possibly compact) minimal hypersurface H of finite volume;*

(3) *X admits a non-proper strictly mean curvature convex Morse function and such that*

   either there is a non-compact minimal hypersurface H with finite volume that is closed in X as a subset and that has compact boundary,

   or there is a sequence of compact minimal hypersurfaces $H_i \subset X$ with no uniform bound on their diameters, such that the boundaries ∂H_i are contained in a fixed compact subset in X.

Remarks. (a) A complete Riemannian manifold X with two ends that admits a proper strictly mean curvature convex Morse function $f : X \to \mathbb{R}$, may contain, however, arbitrarily large compact minimal hypersurfaces with boundaries in a given compact subset in X.

For instance, the 2D *hyperbolic cusp* X_0 (the hyperbolic plane divided by a parabolic isometry) has this property and the Cartesian products $X = X_0 \times V$ for compact V furnish example of all dimensions.

What are other examples of minimal hypersurfaces protruding toward "concave ends" in complete manifolds?

Are there such examples with thick ends, e.g. for manifolds with bounded geometries?

(b) The above theorem (and, in particular, its special case stated in the abstract to our paper) shows that the inverse maximum principle does not truly need thickness at infinity, but the direct maximum principle, probably, does. Quite likely, there exist complete Plateau–Stein n-manifolds for $n \geq 3$ that contain complete minimal hypersurfaces of finite $(n-1)$-volume.

1.5 Miscellaneous Remarks, Questions and Corollaries.

(A) The most essential ingredient of our proof — the *Almgren-Allard regularity theorem* for "soap bubble" is trivial for $n = dim(X) = 2$: curves with continuous curvature in surfaces are, obviously, smooth.

Consequently, our argument is quite elementary for $n = 2$. In fact, both IMP hold with (almost) no restrictions on X, where IMP[non-proper] reduces to the following, most likely known, proposition.

IMP[dim=2]. *Let X be a surface with a complete C^2-smooth* (probably, C^1 will do in this case) *Riemannian metric. Then one of the following three properties holds.*

(1) X contains a simple closed geodesic,

(2) X supports a smooth function $f : X \to \mathbb{R}$ which has no critical points and such that the sublevels $f^{-1}(-\infty, r] \subset X$ are strictly geodesically convex for all $r \in \mathbb{R}$. (Such an X, obviously, is homeomorphic to the plane $\mathbb{R}^2$ or to the cylinder $S^1 \times \mathbb{R}$.)

(3) X is homeomorphic to the sphere with three point removed.

(Originally I overlooked (3); it was pointed to me by Yura Burago that one can not always ensure a *simple* closed geodesic with this topology where non-simpe geodesics are abundant for all metrics. But it should be noticed that the essence of this IMP resides in the surfaces that are homeomorphic to $\mathbb{R}^2$ and to $S^1 \times \mathbb{R}$.)

(B) An instance of a corollary to, say, IMP[non-proper], is the validity of the counterparts of the stability of Plateau–Stein under Riemannian submersions (see (c) from the previous section) for the corresponding $[...]_{\min}$.

For example, let X be a complete and thick at infinity.

If X is [n-n-proPS], then so does the Riemannian product $X \times X'$, for all closed Riemannian manifolds X'.

This, however, looks almost as obvious as the original Plateau–Stein case and, moreover, "thick" seems unnecessary. Indeed,

if a compact smooth domain $U_1 \subset X \times X'$ is *mean concave*, i.e. its boundary satisfies

$$mean.curv_x(\partial U_1) \leq 0 \text{ for all } x \in \partial U_1$$

then the boundary of the projection $U \subset X$ of U_1 to X is also mean concave at all *regular* points $x \in \partial U$, while singular points have generalized mean curvatures = $-\infty$. This allows a approximation/regularization of ∂U with $mean.curv(\partial U) \leq \varepsilon$ everywhere for all $\varepsilon > 0$.

On the other hand, the $[...]_{\min}$ counterpart of the above IMP[non-proper] is not fully trivial.

If $X \times X'$ contains a closed minimal (possibly singular) hypersurface then so do X and X', provided X and X' are complete and thick at infinity.

But the direct proof of this by the geometric measure theory is very simple.

Notice that minimal hypersurfaces in split Riemannian manifolds $X_1 = X \times X'$ do not always split, e.g. in flat 3-tori. Probably,(?) there are *non-split* compact domains U_1 *with minimal boundaries* in certain Riemannian products $X_1 = X \times X'$ for open manifolds X and closed X', where "split" means $U_1 = U \times X'$.

But it seems unclear, for example, if such non-split U_1 with minimal boundaries exist in the products $X_1 = X \times S^1$ of *a hyperbolic surfaces X of finite areas* by circles and if there are compact domains with minimal boundaries in the products $X_1 = X \times X'$ of complete hyperbolic surfaces X and X' of finite areas. (These X_1 are *not* uniformly locally contractible but some IMP may hold.)

(C) The existence of an exhaustion of a Riemannian manifold X by compact mean convex domain is an interesting property in its own right, call it *strict*

mean convexity at infinity.

For instance, a Galois covering X of a closed Riemaniinan manifold $\underline{X}$ is strictly mean convex at infinity unless the Galois group Γ of the covering is virtually cyclic and if, moreover, Γ is non-amenable, then X can be exhausted by domains with mean curvatures $\geq \varepsilon > 0$. (A representative counterexample for cyclic Γ is provided by manifolds $\underline{X}$ that admit fibrations over the circle, $X \to S^1$, such that the fibers are *minimal* hypersurfaces.)

Mean convexity at infinity is visibly "cheaper" than Plateau–Stein; yet, there are non-proper Plateau-Stein manifolds that are *not* mean convex at infinity. For instance, let X be a topological cylinder, i.e. homeomorphic to $X = X_0 \times \mathbb{R}$ where X_0 is a compact manifold and let g_0 be a Riemannian metric on X_0. Let $\phi = \phi(t)$, $t \in \mathbb{R}$, be a positive function and observe that the metric $\phi(t)g_0 + dt^2$ on X is *concave* at the $t \to -\infty$ end of X rather than mean convex. But such an X, obviously is Plateau–Stein since $f(x_0, t) = \phi(t)$ is a strictly mean convex function.

Question Under what conditions does Plateu-Stein imply strict mean convexity at infinity?

(An easy instance of such a condition is *thickness at infinity+ connectedness at infinity.*)

(D) Smoothness conditions we impose on functions and on hypersurfaces in the definitions of Plateau–Stein manifolds and of their $[...]_{\smallsmile}$-counterparts allows a glib formulation of our results with no need for concept of "minimal hypersurface". But insistance on this smoothness look facetious in view of the geometric measure theory techniques that underly the essential part of the argument while "regularization of bubbles" that excludes $[...]_{\min}$ may strike one as a waste of effort.

In fact, an expected generalization of the IMP-implications must apply to non-smooth objects in *singular* spaces X. On the other hand, the regularization process we employ delivers — this is implicit in the arguments in section 4 — a simple but non-trivial information on geometry of singular minimal varieties. (This "information" is by no means new).

(E) Here another obvious consequence of the inverse maximal principle implications

$$[...]_{\min} \Rightarrow [...]_{\smallsmile},$$

where minimal varieties and their singularities do not appear.

$(n-1)$-**Contraction Corollary.** Let X be a C^2-smooth complete Riemannian manifold that is thick at infinity.

If X admits a strictly $(n-1)$-volume contacting vector field V then X is Plateau–Stein.

If, moreover, X is connected at infinity and if there are vector fields V_i, such that the supports of V_i exhaust X and such that V_i are strictly $(n-1)$-volume contacting in the complement of their supports, then X is proper Plateau–Stein.

(Recall, that this theorem was stated for coverings of compact manifolds in section 1.2 and that a vector field V is *strictly $(n-1)$-volume contacting* if the V-derivatives of the volumes of all smooth hypersurfaces in X are negative.)

(F) Let X, not necessarily thick at infinity, contain no *compact* minimal hypersurface. Then does it admit a strictly $(n-1)$-volume contracting vector

field?

(This question is motivated by such a result for 1-volume (i.e. length) contracting fields that was pointed out to me by Bruce Kleiner. Possibly, there is something like that for all *k-volume contracting* vector fields.)

2 Construction of ϕ-Bubbles and of Minimal Hypersurfaces.

We shall describe in this section a few standard ϕ-area (including (n-1)-volume) minimization constructions that deliver minimal hypersurfaces, such as ϕ-bubbles, under the thickness condition.

2.1 Thickness at Infinity.

An n-dimensional Riemannian manifold X is called *thick at infinity* if it contain *no non-compact minimal* hypersurface with *compact* boundary and with *finite* $(n-1)$-volume.

Such a hypersurface $Y \subset X$ must, by definition, be closed in X as a subset and be *ε-locally vol_{n-1}-minimizing in X at infinity.* This means that

there exists an $\varepsilon > 0$ ($\varepsilon = 1$ is good for us) and a compact subset $A = A(Y) \subset X$ (that contains the boundary of Y) such that the intersection $Y \cap B$ with every ε-ball $B = B_x(\varepsilon) \subset X$ for $x \in X \smallsetminus A$ is vol_{n-1}-minimizing in X in the class of hypersurfaces (integral currents) with the boundary equal $Y \cap \partial B$.

Examples. The Paul Lévy (Buyalo-Heintze-Karcher) tube volume bound shows that the condition $\bullet_{Ricc}$ from section 1.3 implies this thickness, while the conditions $\bullet conv$ and $\bullet_{Lip}$ are taken care by the following corollary to the implication

$$[cone\ inequality] \Rightarrow [filling\ inequality]$$

and the lower bound on volumes of minimal varieties by the filling constant [11].

$\bullet_{fill}$ *If every closed integral k-chain S in X, $k = 1, 2, ...,$ of diameter ≤ 1 bounds a $(k+1)$-chain T such that*

$$vol_{k+1}(T) \leq const \cdot diam(S) \cdot vol_k(S)$$

for some $const = const(X)$, then X is thick at infinity.

Remark. This thickness concept obviously generalizes to all dimensions $2 \leq m \leq n-1$ with minimal subvarieties $Y \subset X$ of dimension m, where the above criterion remans valid, while $\bullet_{Ricc}$ should be replaced by $\bullet_{sect}$ with the lower sectional curvature bound instead of such a bound on Ricci.

2.2 Convex and Concave Ends.

Let X be a Riemannain manifold possibly with compact boundary ∂X and a single end, such that X is *thick at infinity.* Then one of the following three possibilities is realized.

[1⌣] *X can be exhausted by compact strictly mean convex bubbles;*
[2⌢] *X can be exhausted by compact strictly mean concave bubbles;*

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[$3_\equiv$] *There exists a continuous positive proper function* $h : X \to \mathbb{R}_+$,
such that the levels $Y_t = h^{-1}(t) \subset X$ *are minimal hypersurfaces,*
that are the boundaries of 0-bubbles, for all $t \geq t_0 = t_0(X) \geq 0$.

Proof. Start by observing that 1 and 2 are *not* mutually exclusive and if there are these two kinds of exhaustions then there also (obviously) exists an exhaustion by 0-bubbles. But 3 is incompatible with 1 and with 2 by the maximum principle.

Let 3 do not hold and, moreover, assume that X *can not be exhausted* by compact 0-bubbles. Then minimization of $vol_{n-1}(Y)$ for $Y = \partial U$ where $U \subset X$ is a compact sufficiently large domain, either move Y to infinity or brings to a compact region $X_0 \subset X$.

In the former case, let $-\phi$ be a negative function on X, where ϕ is very large at infinity and very small in the vicinity of some X_0 that contains the boundary of X. Then minimization of ϕ-area brings U to a strictly mean concave $-\phi$-bubble in X.

Similarly, if volume minimization brings Y to a fixed compact $X_1 \subset$ we use a positive ϕ that is very large on some compact $X_1' \supset X_1$ and very small at infinity. Thus we obtain a strictly mean convex bubble.

And, keep obviously modifying ϕ by moving the change of its size to infinity, we exhaust X either by strictly mean convex or by strictly mean concave bubbles.

Now, let X be exhausted by by compact 0-bubbles. Then X contain infinitely many "empty bands", say W between the boundaries of these bubbles say U_1 and $U_2 \supset U_1$, i.e. $W = U_2 \smallsetminus U_1$, where such a W is bounded by the minimal hypersurfaces $Y_1 = \partial U_1$ and $Y_2 = \partial U_2$ with no 0-bubble between the two. Then the obvious adjustment of the above argument delivers both a strictly mean convex and a strictly mean concave bubble pinched between U_1 and U_2.

Besides, *Almgren's min-max argument* delivers a non-stable minimal hypersurface in W that separates Y_1 from Y_2.

2.3 Minimal Separation of Ends.

Here X has several ends and no boundary, where the set of ends is given its natural topology.

Notice that every *isolated* end E can be represented/isolated by an equidimensional submanifold (domain) $X_E \subset X$ with compact boundary and a single end; we say "exhaustion of E" instead of "exhaustion of X_E for some $X_E \subset X$".

If the set of ends of X contains at least two limit points, (e.g. if it has no isolated ends), then X contains a compact two-sided smooth hypersurface $H \subset X$ such that there are infinitely many ends of X on either side of H. Then, clearly,

if X is complete and thick at infinity, then the homology class of H contains a minimizing hypersurface.

Now let X has *at least two isolated* ends, say E_1 and E_2.

If none of these ends admits a strictly mean convex exhaustion, then H contains a compact (non-stable) minimal variety $M \subset X$ where this M may be a varifold.

Proof. Let $h : x \mapsto t = h(x) \in (-\infty, +\infty)$ be a smooth proper Morse function $X \to \mathbb{R}$ such that $h(x) \to -\infty$ for $x \to E_1$ and $h(x) \to +\infty$ for $x \to E_2$.

The vol_{n-1}-minimimzation process starting from the levels $h^{-1}(t)$ moves some connected component of $h^{-1}(t)$ for small negative t (approaching $-\infty$) to E_1, while some component for positive large t goes to E_2. Since the manifold X is thick at infinity, Almgen's min-max theorem applies and the proof follows.

By combining the above with $[1_\smile]/[2_\frown]/[3_\equiv]$ in the previous section, we conclude to the following.

Let X be a complete Riemannian manifold that is thick at infinity. Then

$[\smile/-]$ *Either an isolated end of X admits a strictly mean convex exhaustion, or X contains a compact minimal subvariety.*

2.4   Shrinking Mean Convex Ends.

Let X be a complete connected Riemannian manifold of dimension n with non-empty compact boundary $Y_\smile = \partial X$ with $mn.curv(Y_\smile) > \varepsilon_0 > 0$, let $0 < \varepsilon_i < \varepsilon_0$, $i = 1, 2, ...$ be a sequence of positive numbers that converges to 0 and let $\rho(x) > 0$ be a continuous function on X. Then either

$\star_A$ *X contains a minimal hypersurface $H \subset X$ of finite $(n-1)$-volume that is closed in X as a subset and that does not meet $Y_\smile$,*

or

$\star_B$ *X can be exhausted by an increasing sequence of compact strictly concave bubbles U_i in X that contain $Y_\smile$,*

$$Y_\smile \subset U_1 \subset U_2 \subset ... \subset U_i \subset ... \subset X, \quad \cup_i U_i = X,$$

such that

- *U_i is contained in the ρ_{i-1}-neighborhood of U_{i-1}*

    for $\rho_{i-1} = \inf_{x \in U_{i-1}} \rho(x)$ and all $i = 1, 2, ...$, where $U_0 =_{def} Y_\smile$;

,
- *the mean curvatures of the topological boundaries $Y_i = \partial U_i \subset X$ satisfy $mn.curv_x(Y_i) = -\varepsilon_i$ at all those regular $x \in Y_i$ where $dist(x, U_{i-1}) \leq \rho/2$.*

Remarks. If X is thick at infinity, then, as we know, the above minimal hypersurface H, if it exists at all, must be compact.

If X is compact and no minimal H exists, then the sequence U_i stabilizes and the boundaries Y_i of the bubbles U_i become empty for large i. On the other hand if H does exist and $\rho \leq \rho_0 = \rho_0(H) > 0$, then *no U_i intersects H.* (If ρ is large then U_i *may* exhaust X even in a presence of H.)

Proof of the $\star_A/\star_B$-Alterantive. Proceed as at the half-step 3A from 1.4. Namely, granted U_{i-1} for some i, let ϕ_i be a positive continuous functions ϕ_i on X, such that

$\phi_i = \varepsilon_i$ in the $\rho_i/2$-neighbourhood of U_{i-1},

$\phi_i(x)$ is very large for $dist(x, U_{i-1}) \geq \frac{2}{3}\rho_i$.

Then, clearly, there exists a compact $-\phi_i$-bubble $U_i \supset U_{i-1}$ that satisfies all of the above properties.

Since $\varepsilon_i \to 0$ the boundary of the union $\cup_i U_i \subset X$ must be a *minimal* hypersurface H in X with $vol_{n-1}(H) < vol_{n-1}(Y_0)$; if no such hypersurface exists, then this boundary must be empty and $\cup_i U_i = X$. QED.

If $\star_B$ holds for all $\rho(x) > 0$ then, by the maximum principle, X contains no minimal hypersurface H of *any volume.* This leads to the following

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Almgren's min/max Theorem for Non-Compact Manifolds. *If a complete Riemannian manifold X with non-empty compact strictly mean convex boundary contains a complete minimal hypersurface then it also contains a complete minimal hypersurface of finite volume.*

("Complete" means being closed in X as a subset, not intersecting the boundary of X and having no boundary of its own.)

2.5 Shrinking to Concave Boundary.

The above admits a relative version where X has a concave component $Y_\frown$ in its boundary (or several such components) that serves as an obstacle for shrinking bubbles and where the minimal hypersurface H that (if it exists at all) obstructs shrinking of bubbles may have non-empty boundary that is contained in $Y_\frown$.

Namely, let again X be a complete connected Riemannian manifold of dimension n with compact boundary that is now decomposed into two disjoint parts $\partial X = Y_\smile \cup Y_\frown$ (these $Y_\smile$ and $Y_\frown$ are unions of connected components of ∂X) where $Y_\frown$ (possibly, empty as in the previous section) is strictly mean concave and where $Y_\smile$ is non-empty and has $mn.curv(Y_\smile) > \varepsilon_0 > 0$.

Let $\rho(x) > 0$ be a continuous function on X. Then either

$\frown \star_A$ *X contains a minimal hypersurface $H \subset X$ of finite $(n-1)$-volume that is closed in X as a subset, that does not meet $Y_\smile$, and that may have boundary contained in $Y_\frown$,*

   or

$\frown \star_B$ *the complement $X \smallsetminus Y_\frown$ can be exhausted by an increasing sequence of* compact strictly concave bubbles U_i in X that contain $Y_\smile$,

$$Y_\smile \subset U_1 \subset U_2 \subset ... \subset U_i \subset ... \subset X, \ \cup_i U_i = X \smallsetminus Y_\frown,$$

such that
   • *U_i is contained in the ρ_{i-1}-neighborhood of U_{i-1}*

$$\text{for } \rho_{i-1} = \inf_{x \in U_{i-1}} \rho(x) \text{ and all } i = 1, 2, ..., \text{ where } U_0 =_{def} Y_\smile.$$

Remarks. We could also impose here additional constrains on the mean curvatures of the boundaries $Y_i = \partial U_i \subset X$ as we did in the absence of $Y_\frown$ but we postpone doing this until section 4 where this becomes relevant.

An essential case of the above is where X is compact with two boundary components, $Y_\smile$ and $Y_\frown$ and these shrinking bubbles are used (see section ....) for construction of a strictly mean curvature convex Morse function f on X that equals 1 on $Y_\smile$ and 0 on $Y_\frown$, where
   such an f exists if and only if there is no minimal hypersurface
   H in X with $\partial H \subset Y_\frown$.
Here (and everywhere in this kind of context) "only if" follows by the maximum principle, while "if" is what we call the inverse maximum principle.

Proof of $\frown \star_A / \frown \star_B$.   Proceed as earlier and keep pushing boundaries of bubbles closer and closer to $Y_\frown$. Then, say in the compact case, we arrive at a *maximal* compact bubble $U_{max} \subset X$ the boundary of which can not be moved closer to $Y_\frown$ anymore. Then either $U_{max} = X$ or the topological boundary of U_{max} is non-empty. Then this boundary, call it H makes our *minimal* hypersurface in the interior of X: if not "minimal" it could be moved closer to $Y_\frown$.

Notice that this H is *tangent* (rather than transversal) to Y, where the two hypersurfaces meet.

3 Distance Functions, Equividistant Hypersurfaces and k-Mean Convexity.

We fix in this section our terminology/notation and state a few standard facts on distance functions in Riemannian manifolds X.

3.1 Signed Distance Function and Equidistant Hypersurfaces.

Interior Domains $U_{-\rho}^<$, $U_{-\rho} = U_{-\rho}^\leq$ and Equidistant Hypersurfaces $Y_{-\rho}$. Let U be a (closed or open) domain or an open subset (possibly dense) in a Riemannian manifold X (or in any metric space for this matter) and denote by $x \mapsto d(x) = d_U(x) = dist_\pm(x, \partial U)$ the *signed distance function* to the topological boundary $Y = \partial U$, i.e.

$d_U(x)$ equals the distance from x to Y outside U,

$d_U(x)$ equals *minus* the distance from x to Y in U.

In writing,

$$d_U(x) = dist(x, Y) =_{def} \inf dist_{y \in Y}(x, y) \text{ for all } x \in X \smallsetminus U,$$

$$d_U(x) = -dist(x, Y) = -dist(x, X \smallsetminus U) \text{ for all } x \in U,$$

where

$$dist(x, Y = \partial U) = dist(x, U), \text{ for } x \in X \smallsetminus U,$$

since the Riemannian distance is a *length metric* being defined via the lengths of curves between pairs of points;

Let $\rho \geq 0$ and denote by $U_{-\rho} = U_{-\rho}^\leq \subset U$ and $U_{-\rho}^< \subset U_{-\rho}$, $\rho \geq 0$, the closed/open $(-\rho)$-sublevels of d_U, that are

$$U_{-\rho}^\leq = d_U^{-1}(-\infty, -\rho] \text{ and } U_{-\rho}^< = d_U^{-1}(-\infty, -\rho),$$

where, clearly, $U_0 = U_{-0} = U_0^\leq$ equals the topological closure of U and $U_{-\rho}^\leq$ are closed subsets in U for $\rho > 0$ with $U_{-\rho}^<$ being equal the interior of $U_{-\rho}^\leq$ for $\rho > 0$.

Let

$$Y_{-\rho} = d_U^{-1}(-\rho) \subset X, \rho \geq 0,$$

be the *(interior) ρ-equidistant hypersurface to Y*, that is the subset of points u in U, where $dist_X(u, Y = \partial U) = \rho$ and that equals the topological boundary $\partial U_{-\rho}$ since Riemaniinan manifolds X are *length metric spaces*.

Similarly define $U_{+\rho} = U_{+\rho}^\leq$ and $U_{+\rho}^< \subset U_{+\rho}^\leq$ or $\rho \geq 0$ as

$$U_{+\rho}^\leq = d_U^{-1}(-\infty, \rho] \text{ and } U_{+\rho}^< = d_U^{-1}(-\infty, \rho).$$

Thus, $U_{+\rho} = U_{+\rho}^\leq$ equals the *closed ρ-neighbourhood of U* in X and $U_{+\rho}^<$ is the *open ρ-neighbourhood*.

On Hausdorff (Dis)Continuity. Clearly, the boundaries of the open sublevels of d_U satisfy
$$\partial U_{-\rho}^< \subset Y_{-\rho} = \partial U_{-\rho}^\leq,$$
where the *local minima* of the d_U on U make the difference set $Y_{-\rho} \smallsetminus \partial U_{-\rho}^<$.

The set valued function $\rho \mapsto U_{-\rho} \subset X$, $\rho \in \mathbb{R}_+$, is continuous for the Hausdorff metric in the space of subsets in X at those ρ where $\partial U_{-\rho}^< = Y_{-\rho}$, or, equivalently, where the closure of the interior of $U_{-\rho}$ equals $U_{-\rho}$. Since $\rho \mapsto U_{-\rho}$ is a *monotone decreasing* function in ρ for the inclusion order on subsets, it has *at most countably many* discontinuity points ρ.

Also observe that the function $\rho \mapsto Y_{-\rho} = d_U^{-1}(-\rho) = \partial U_{-\rho}$ is Hausdorff continuous at the Hausdorff continuity points of the function $\rho \mapsto U_{-\rho}$ and the word "hypersurface" is justifiably applicable to $Y_{-\rho}$ at these continuity points ρ.

Exercise. Let $Z \subset X$ be a compact subset that is contained in a smooth hypersurface in X. Then, for all sufficiently small $\rho > 0$, there exists an open subset $U \supset Z$ in X with smooth boundary Y, such that $Z = Y_{-\rho} = U_{-\rho}$, i.e. Z serves in U as the set of the minima of the (minus distance to Y) function $d_U : U \to (-\infty, 0)$.

Example: $U_{-\rho}$ as the Intersection of Translates of U. If $X = \mathbb{R}^n$ then, obviously, $U_{-\rho}$ equals the intersection of the parallel r-translates $U + r \subset \mathbb{R}^n$ for all $r \in \mathbb{R}^n$ with $\|r\| \leq \rho$ and $Y_{-\rho}$ equals the topological boundary of this intersection.
$$U_{-\rho} = \bigcap_{\|r\| \leq \rho} U + r \text{ and } Y_{-\rho} = \partial (\bigcap_{\|r\| \leq \rho} U + r).$$

Thus,

the transformation $U \mapsto U_{-\rho}$ preserves all classes of Euclidean domains

(e.g. the class of convex domains) that are closed under intersections.

It is also clear that
$$Y_{-\rho} \subset \partial (\bigcap_{\|r\| = \rho} U + r)$$
and if the boundary of U is *connected*, then
$$U_{-\rho} = \bigcap_{\|r\| = \rho} U + r \text{ and } Y_{-\rho} = \partial (\bigcap_{\|r\| = \rho} U + r).$$

More generally, let $iso_{\leq \rho}$ denote the set of isometries $r : X \to X$ such that $dist(x, r(x)) \leq \rho$ for all $x \in X$. Then, obviously,
$$U_{-\rho} \subset \bigcap_{r \in iso_{\leq \rho}} r(U).$$

Furthermore, if X is a *compact two-point homogeneous space*, i.e. the isometry group of X is transitive on the unit tangent bundle of X, then, as in the Euclidean case,
$$U_{-\rho} = \bigcap_{r \in iso_{\leq \rho}} r(U) \text{ and } Y_{-\rho} \subset \partial (\bigcap_{r \in iso_{=\rho}} r(U)) \text{ for } iso_{=\rho} = \partial(iso_{\leq \rho}).$$

3.2 Accessibility and Quasi-regularity.

A point x in the boundary $Y = \partial U$, is called ρ-*accessible* (from U),

$$x \in (U_\rho)_{+\rho} \subset U \cup Y.$$

In other word, x is contained in some Riemannian ρ-ball in X that is contained in the closure of U. (The referee pointed out to me that this is usually called "with reach ρ", with a possible origin of the concept due to Federer.)

Say that an open subset U in X is C^2-*quasiregular* (at its boundary) if, loosely speaking, the singular locus $sing_Y \subset Y = \partial U$ is non-accessible from U. More precisely, the following two conditions must be satisfied.

(1) The subsets

$$Acc_{>\rho_0}(Y) = A_{>\rho_0}(Y) = \bigcup_{\rho > \rho_0} A_\rho(Y) \subset Y \subset X$$

are *open* in Y for all $\rho_0 \geq 0$.

Notice that this condition implies that, besides being open, the subsets $A_{>0}(Y_{-\rho}) \subset Y_{-\rho}$, $\rho > 0$, are $C^{1,1}$ in $C^{1,1}$-smooth Riemannian submanifolds in X.

(2) The subset $A_{>0}(Y) \subset X$ is a C^2-*smooth* hypersurface in X, that is a C^2-smooth $(n-1)$-submanifold without boundary that, topologically, is a locally closed subset in X.

If $U \subset X$ is a *closed* domain then its quasiregularity means that of the *interior* $int(U) \subset X$.

On the other hand, "quasiregularity of a hypersurface" $H \subset X$ is understood as quasiregularity if its complement $X \smallsetminus H \subset X$.

ALMGREN-ALLARD QUASIREGULARIRTY THEOREM. *Let X be a C^2-smooth Riemannian manifold. Then ϕ-Bubbles $U \subset X$ are C^2-quasiregular for all continuous functions $\phi(x)$. Also, all kinds of minimal hypersurfaces $H \subset X$ are quasiregular.* (See [13] for a simple prove of this.)

Here "minimal hypersurface" is understood as a *minimal varifold* that does not, necessarily, bound any domain in X.

The following two instances of quaisiregularity are, unlike Allmgren-Allard theorem, are fully obvious.

(A) *Locally finite intersections of C^2-smooth domains with transversally intersecting boundaries are C^2-quasiregular.*
And
(B) *If U a C^2-quasiregular domain in a C^2-smooth Riemannian manifold, e.g. the boundary Y is C^2-smooth to start with, then the sub-domains $U^<_{-\rho}$ are also C^2-quasiregular.*

3.3 Smooth and non-Smooth k-Mean Convex Functions and Hypersurfaces.

Let X be a C^2-smooth Riemannian manifold and let $Gr_k(X)$ be the Grassmann space of the tangent k-planes τ in X. Define the k-*Laplacian* Δ_k from C^2-functions f on X to functions on $Gr_k(X)$ by taking the traces of the Hessian

of f on all $\tau \in Gr_k(X)$

$$\Delta_k(f)(\tau) = trace_\tau Hess(f).$$

For example $\Delta_n = \Delta$ is the ordinary Laplacian for $n = dim(X)$ and $\Delta_1(f)(\tau)$ equals the second derivative of f on the geodesic in X tangent to the tangent line τ.

Say that f is *k-mean φ-convex* for a given continuous function $\varphi = \varphi(\tau)$ on $Gr_k(X)$ if $\Delta_k(f)(\tau) \geq$, where, as usual strictly corresponds to $> \varphi$ and plain "convex " stands for 0-convex.

Observe that the k-mean convexity says, in effect that the gradient of $-f$ is strictly k-volume contracting. Also notice that

$$k\text{-}mean\ convex \Rightarrow l\text{-}mean\ convex\ \text{for}\ l \geq k,$$

$$1\text{-}mean\ convex = convex, \quad \text{n-}mean\ convex = subharmonic$$

and that
$$(n-1)\text{-}mean\ convex \Rightarrow\ mean\ curvature\ convex,$$

while the converse implication is not, in general true.

On the other hand, a C^2-smooth strictly mean convex co-oriented hypersurface $Y \subset X$ (e.g. if $Y = \partial U$) can be realized as
the zero set of a C^2-smooth strictly $(n-1)$-convex function $f(x)$ defined in some neighbourhood of Y.

Bending $d(x)$ to a Mean Convex $f(x)$. (Compare [5], [10].) A strictly mean convex function f can be obtained, for example, by "bending" the signed distance function $d(x) = \pm dist(x, Y)$ (that is $d(x) = d_U(x)$ for $Y = \partial U$), i.e. where bending is achieved by means of a smooth strictly monotone increasing function $\beta(d)$, $-\infty < d < +\infty$, that vanishes at $d = 0$, that has the first derivative $d'(0) = 1$ and positive second derivative $d''(x) > 0$.

If Y is compact, then the function

$$f(x) = \beta \circ d(x) = \beta(d(x))$$

is strictly $(n-1)$-mean convex in some neighborhood of Y, provided the second derivative $d''(0)$ is sufficiently large (compared to the absolute values of the negative principal curvatures of Y).

If Y is non-compact one needs to modify this f by making its second derivative normal to Y to be large as a *function* on Y.

Remark. The above remains true (and equally obvious) for *k-mean convex* hypersurfaces $Y \subset X$, $k = 1, 2, ..., n-1$, where the traces of the second fundamental forms are positive on the k-planes tangent to Y.

The notion of k-mean convexity extends from C^2-functions to all continuous ones via *linearity* of the operator Δ_k by declaring a *continuous* function $f(x)$ being k-mean φ-convex if it is contained in the *localized weak convex hull* of the space of *smooth* k-mean φ-convex functions.

In other words $f(x)$ is k-mean φ-convex if

$\Delta_k(f)(\tau) - \varphi(\tau)$, *understood as a distribution, is representable by a positive measure on $Gr_k(X)$.*

Then one defines the set of *strictly k-mean φ-convex function* as the intersection of the sets of $(\phi + \varepsilon)$-convex ones, where the intersection is taken over all positive functions $\varepsilon = \varepsilon(\tau)$ on $Gr_k(X)$.

For example, a continuous function f is strictly k-mean convex if there exists a continuous function $\varepsilon(x) > 0$ such that the restriction of f to every (local) k-dimensional submanifold $Y \subset X$ with principal curvatures $\kappa_y(Y)$ bounded by $|\kappa_y(Y)| \leq \varepsilon(y)$ is a *subharmonic* function on Y.

Also one easily sees that

if f_i, $i \in I$, are strictly k-mean ϕ-convex functions, then
$f(x) = \max_i f_i(x)$ is also strictly k-mean ϕ-convex.

Linearized Definition of k-Mean Curvature Convexity. A cooriented hypersurface Y is called *strictly k-mean φ-convex* for $\varphi = \varphi(\tau)$ defined on a neighborhood of the pullback of Y under the tautological map $Gr_k(X) \to X$, if iY is representable as the zero set of a continuous strictly k-mean φ-convex function $f(x)$ defined in some neighbourhood of Y.

Here, "cooriented hypersurface" means that there is a neighbourhood X_0 of Y where Y serves as the boundary of a closed domain $U \subset X_0$. Then our f must be positive inside U and positive outside. We say in this situation that U itself *strict k-mean (curvature) ϕ-convex (at the boundary).*

Mean Curvature Convexity of Functions Revisited. (Compare [5] [10]) A continuous function $f(x)$ is called *strictly k-mean φ-curvature convex* if, for every point $x \in X$, there exists a convex C^2-function $\beta : \mathbb{R} \to \mathbb{R}$ with strictly positive derivative $\beta' > 0$, such that the composed function $x \mapsto \beta \circ f(x) = \beta(f(x)))$ is strictly k-mean φ-convex in some neighbourhood of $x \in X$.

Notice that the so defined strict k-mean curvature φ-convexity is
stable under small C^2-perturbations of functions.

Also,

maxima of families of strictly k-mean curvature φ-convex functions
are strictly k-mean curvature φ-convex,

since

$$\beta \circ \max_i f_i = \max_i \beta \circ f_i \text{ for monotone increasing } \beta$$

and since strict k-mean φ-convexity is stable under taking maxima.

Remark Probably, little (essentially nothing?) changes if one allows *non-smooth* convex monotone increasing β in this definition.

k-Convexity Lemma. Let X be a C^2-smooth Riemannian manifold and $U \subset X$ a quasiregular domain with boundary $Y = \partial U$ which is k-mean φ-convex on the regular locus $reg_Y = Y \setminus sing_Y$ for a positive continuous function $\varphi = \varphi(\tau) > 0$. Then

the minus distance function $d_U(x) = -dist(x, Y)$ is k-mean curvature $(\varphi - \varepsilon)$-convex in the interior of U for some continuous function $\varepsilon = \varepsilon(\tau)$ that vanishes on the pullback of Y in $Gr_k(X)$.

Moreover, $\varepsilon(x)$ is bounded in terms of $|\inf_x Ricci_x(X)|$ for x running over the ball $B_x(R) \subset X$ for $r = dist(x, Y)$ (For instance, $\varepsilon(x) = 0$ if the Ricci curvature is non-negative in this ball.)

The proof is quite simple and, I guess, is well known in some quarters. Yet, for the completeness sake, we spell it down in section 5.6. where our argument

is essentially the same as that in [5] and [10].

Question. Is there a meaningful characterization of Plateau–Stein n-manifolds that admit (proper) strictly $(n-1)$-mean convex functions?

3.4 Smoothing and Approximation.

Continuous strictly k—mean convex function f can be approximated by *smooth* strictly k-mean convex ones, by convolving with the following

Standard ε-Smoothing Kernel. Such a kernel is a function in two variables, $K_\varepsilon(x_1, x_2)$, $\varepsilon > 0$, on a Riemannian manifold X that is defined with some Ψ by

$$K_\varepsilon(x_1, x_2) = \lambda(x_1)\Psi(\varepsilon^{-1}dist(x_1, x_2)), \ \varepsilon > 0,$$

for

$$\lambda(x_1) = \left(\int_X \Psi(\varepsilon^{-1}dist(x_1, x_2))dx_2 \right)^{-1}.$$

A standard ε-smoothing of functions on X is

$$f(x) \mapsto f_\varepsilon(x) = \int_X f(x_2) \cdot K_\varepsilon(x, x_2)dx_2.$$

It is obvious that if f is a *continuous* strictly k-mean ϕ-convex function and $V \subset X$ is a compact subset, then f_ε is strictly k-mean ϕ-convex on V for all sufficiently small $\varepsilon > 0$.

It follows that f can be uniformly, and even in the fine C^0-topology, approximated by C^2-*smooth* strictly k-mean ϕ-convex functions, where, moreover such approximating functions can be chosen equal f on a closed subset $X_0 \subset X$ if f itself is smooth in a neighborhood of X_0 in X.

Recall that C^0-*fine approximation* means that the difference between f and an approximating function can be made less than a given strictly positive continuous function on X.

Curvature Smoothing Corollary. Let $f(x)$ be a continuous strictly k-mean curvature ϕ-convex function on a C^2-smooth Riemannian manifold X. Then

f can be C^0-finely approximated by smooth strictly mean curvature ϕ-convex functions with non-degenerate critical points.

Proof. Locally, in a neighborhood $U_x \subset U$ of a given point x, such an approximation obtained by finely approximating $f \circ \beta(x) = f(\beta(x))$, for a suitable β, by a smooth k-mean ϕ-convex function on U_x, call such an approximation by $(f \circ \beta)_{appr}$ and then applying the inverse $-\beta$-function, thus approximating f by $f_{appr} = \beta^{-1} \circ (f \circ \beta)_{appr}$.

Then the global C^2-smooth approximation of f is obtained, by a usual argument with a covering of X by open subsets U_i, $i = 1, 2, ..., n + 1 = dim(X) + 1$ where each of U_i them equals the disjoint union of arbitrarily small subsets.

Finally, "C^2-smooth" is upgraded to "generic C^∞" by an arbitrarily C^2-small perturbation. QED.

4 Splicing, Smoothing and Extending Distance Functions.

We shall prove in this section the inverse maximum principles stated in section

Smoothing Bubbles and Minimal Hypersurfaces. This smoothing applies, in

4.1   Staircase of Distance Functions.

Let X be a Riemannian manifold let $U_1 \supset U_2 \supset U_3 \supset ... \subset U_i \supset ... \subset X$ be closed domains with boundaries $Y_i = \partial U_i$, let $d_i(x)$, $x \in U_i$, denote the minus distance function from x to the boundary $Y_j = \partial U_j$ and let some numbers $\delta_i > 0$ satisfy

$$\delta_i > \sup_{x \in Y_{i+1}} dist(x, Y_i).$$

Then there exists a negative proper continuous function $h : X \to \mathbb{R}_-$, such that locally, in a neighbourhood of every point $x \in X$, this h equals the maximum of the functions $\beta_i \circ d_i$, for $k_x \le i \le l_x$, where
- *l_x is the maximal l such that $x \in U_l$;*
- *k_x is the minimal k such that $dist(x, Y_k) \le \delta_k$;*
- *β_j are smooth monotone increasing functions, $\beta_k : \mathbb{R} \to \mathbb{R}$, with strictly positive derivatives, $\beta'_j > 0$.*

Proof. The required max-function h is determined by its sublevels, call them $Y_{-\rho} = h^{-1}(-\infty, -\rho) \subset X$, that come as *intersections* of certain sublevels of the functions d_i that are $(U_i)_{-\rho_i} \subset U_i = d_i^{-1}(\infty, \rho_i]$ for some ρ_i that must be continuous strictly increasing functions in ρ.

The essential point is to choose these ρ_i, such that
if the boundary of some $(U_i)_{-\rho_i}$ passes through a point x contained in the boundary of the intersection $\cap_i (U_i)_{-\rho_i}$, then $dist(x, Y_i) \le \delta_i$.

Since $\delta_i > \sup_{x \in Y_{i+1}} dist(x, Y_i)$, this inequality can be obviously satisfied with some ρ_i and the proof follows. .

Corollary. Non-Smoothed Inverse Maximal Principle for compact manifolds. Let X be a compact Riemannian C^2-smooth manifold with strictly mean convex boundary.

Then either X contains a compact minimal hypersurface in its interior or it admits a continuous negative strictly mean convex function that vanishes on the boundary of X.

Proof. Shrinking the mean convex "ends" of X (see 2.4) provides a finite descending sequences of ϕ-convex bubbles U_i with a *fixed* (albeit very small) *strictly positive ϕ* and with arbitrarily small $\sup_{x \in Y_{i+1}} dist(x, Y_i) > 0$. Then the above h is strictly mean convex being local maximum of distance functions that are strictly convex by the k-convexity lemma in section 3.3.

Remark.   Bruce Kleiner explained to me how a version of this follows by an application of the mean curvature flow, but this does not seem to be simpler than our more pedestrian argument.

4.2   Proofs of Inverse Maximum Principles.

What remans is to justify step 4 in the proof of IMP in section 1.4.

Let, for instance X be complete C^2-smooth Riemannian manifold that is connected and thick at infinity.

We already know (see section 2.4) that if X contains no minimal hypersurface then it can be exhausted by compact strictly mean convex bubbles U_j.

We also know that each U_j can be shrunk via smaller bubbles $U_{ji} \subset U_j$,

$$U_{j0} = U_j \supset U_{j1} \supset U_{j2} \supset ... \supset U_{ji} \supset ...,$$

where the minus distance functions $d_{ij}(x) = -dist(x, \partial U_{ji})$, $x \in U_{ji}$, can be "spliced" to continuous mean curvature convex functions h_j on U_j.

If the (positive!) mean curvatures of the boundary hypersurfaces $Y_{ji} = \partial U_{ji}$ are bounded from above at all points $x \in Y_{ji}$ by $\alpha(x)$, where $\alpha(x)$ is a (possibly very fast growing) continuous function on X, then, by the usual compactness principle of the geometric measure theory, some subsequence of h_j converges on all compact subsets in X to the required h.

A transparent way to achieve the control over $\sup mn.curv(\partial U')$ of a bubble U' inside a given bubble U is to see the construction of U' in terms of an obstacle (see section 1.4) that is a subdomain $V \subset U$ that must be engulfed by U'. If the mean curvatures of the boundary of V at all boundary points are bounded by $\alpha(x)$, then the same bound will hold for $\partial(U')$.

If, for instance, X has Ricci curvature bounded from below, one may take $V = (U_{-\rho})_{+\rho/2}$ where this V (pinched between $U_{-\rho/2}$ and $U_{-\rho}$) has its mean curvatures bounded by above roughly by $\rho^{-(n-1)}$.

In general, one modifies this by replacing constant ρ by a positive function $\rho(x)$ on X, that must decay, roughly, as $(1 + |R(x)|^{(n-1)})^{-1}$, for a negative function $R(x)$ that serves as a lower bound for the Ricci curvature of X.

The curvature of the boundary of such a V, that is obtained by pushing U inward by $\rho(x)$ and then outward by $\rho(x)/2$, can be easily bounded by some (very fast growing) $\alpha(x)$.

This argument, that extends to multi-ended manifolds with the preparations made in sections 2.2-2.4, yields the following non-regular IMP stated in section 1.4.

Trichotomy Theorem. Let X be a complete Riemannian C^2-smooth n-manifold. Then (at least) one of the the following three conditions is satisfied.

(1) *X admits a proper strictly mean curvature convex function*
$h : X \to \mathbb{R}_+$.
(2) *X contains a minimal hypersurface H that is closed in X as a subset*
and such that $vol_{n-1}(H) < \infty$.
(3) *X admits a non-proper continuous strictly mean curvature*
convex function $h : X \to \mathbb{R}$ and also there is a sequence of minimal
hypersurfaces $H_i \subset X$ with boundaries ∂H_i contained in a fixed compact
subset $X_0 \subset X$, where these H_i are closed in X as subsets and such that

$$vol_{n-1}(H_i) < \infty, \quad diam(H_i) \to \infty \ for \ i \to \infty.$$

Proof of the Regularized Maximum Principles. The above functions h_i can be approximated by C^2-Morse functions and minimal H approximated by slightly concave hypersurfaces according to the smoothing lemma.(see 3.4). This accomplishes the proof of the IMP's stated in section 1.3.

Remark. It seems, I did not check the details, the above theorem remans true with "convex" replaced everywhere by ϕ-convex for a given continuous (not even necessarily positive) function $\phi(x)$, where the minimality condition on H

must be replaced by $mn.curv_x(H) = \phi(x)$, $x \in H$, and where the finiteness requirement for the $n-1$-volume of H must be replaced by a suitable finiteness condition for some ϕ-area.

5　Generalized Convexity.

We look at the mean convexity from a broader prospective in this section and we prove the k-mean convexity lemma from section 3.4. All of what we say is known but dispersed in the literature.

5.1　Smooth and Non-Smooth Convexity Classes.

A coorintation of a germ of hypersurface Y at a point x in a manifold X is expressed by calling the closure of one of the two "halves" in the complement $B_x \smallsetminus Y$, for a small ball at x, being *inside Y* and the closure of the other half *outside Y*.

Thus, cooriented germs at $x \in X$ are partially ordered. We agree, thinking of Y_2 being *more* convex than Y_1, that

$$Y_2 \geq_x Y_1 \text{ stands for } Y_2 \text{ is } inside\ Y_1.$$

Formally, being inside a cooriented Y_1 does not need any coorientation of Y_2. In fact, Y_2 is *inside Y_1* implies that Y_1 is *outside Y_2 only for one* of the two coorientations of Y_2. So the above "Y_2 is *inside Y_1*" tacitly assumes that this does imply "Y_1 is *outside Y_2*"; moreover, if, geometrically, without coorientations, $Y_1 = Y_2$, then "inside" means that their coorientations are equal as well.

Assume X is smooth, let $T(X)$ denote the tangent bundle of X and $\mathcal{H}$ be the space of tangent cooriented hyperplanes $H = H_x \subset T_x(X)$, $x \in X$, that are the tangent spaces to germs of smooth cooriented hypersurfaces in X. Observe that the relation $Y_2 \geq_x Y_1$ between C^1-*smooth* cooriented hypersurfaces implies that they have equal *oriented* tangent spaces (hyperplanes) at x. Accordingly, we may write $Y_2 \geq_H Y_1$ instead of $Y_2 \geq_x Y_1$ for their common cooriented tangent hyperplane $H \subset T_x(X)$.

Given a cooriented hyperplane $H \subset T_x(X)$, denote by $\mathcal{Q}_H = \mathcal{Q}_H(X)$ the space of quadratic functions (forms) $H \to T_x(X)/H = \mathbb{R}$ and by $\mathcal{Q} = \mathcal{Q}(X)$ the space of $\mathcal{Q}_H$ over all $H \subset T_x(X)$, $x \in X$.

The affine space $aff(\mathcal{Q}_H)$ naturally represent the space of 2-jets $J_x^2(Y)$ of germs of cooriented hypersurfaces Y at x that are tangent to H and one may speak of the difference

$$J_x^2(Y_1) - J_x^2(Y_2) \in \mathcal{Q}_H.$$

Obviously,

$$Y_2 \geq_H Y_1 \Rightarrow J_x^2(Y_1) - J_x^2(Y_2) \geq 0$$

where we refer to the usual partial order on the space $\mathcal{Q}_H$ regarded as a space of $\mathbb{R}$-valued functions on H. where this implication is reversible for

Strict Order. The above implication is not, in general, reversible but it is reversible in the strict form:

$$Y_2 >_H Y_1 \Leftrightarrow J_x^2(Y_1) - J_x^2(Y_2) > 0,$$

where the *strict inequality between germs* signifies that not only $Y_2 \geq_H Y_1$, but also that this non-strict inequality is *stable* under small C^2-perturbations of the germs that remain tangent to H.

If X is endowed with an affine connection, then one may identify $aff(\mathcal{Q}_H)$ with $\mathcal{Q}_H$; thus, every germ Y is assigned the quadratic form Q on $H = T_x(Y) \subset T_x(X)$ with values in $T_x(X)/H$. If, moreover, X is a Riemannian manifold, then $T_x(X)/H$ is *canonically* isomorphic to $\mathbb{R}$, and Q equals the *second fundamental form of Y at x.*

Denote by $\mathcal{Q}_{aff}(X)$ the space of 2-jets of cooriented hypersurfaces $Y \subset X$ and call a subset $\mathcal{R} \subset \mathcal{Q}_{aff}(X)$ a *convexity relation* (of second order) if

$$J_x^2(Y_1) \in \mathcal{R} \Rightarrow J_x^2(Y_2) \in \mathcal{R} \text{ for all germs } Y_2 \geq Y_1,$$

where $Y_2 \geq Y_1$ signifies that both Y have the *same* underlying cooriented tangent space (hyperplane) $H \subset T_x(X)$ where this inequality makes sense.

We say that a cooriented C^2-smooth hypersurface $Y \subset X$ *satisfies* $\mathcal{R}$, or it is $\mathcal{R}$-*convex*, if the 2-jets of Y are contained in $\mathcal{R}$ at all points $y \in Y$.

If X is a Riemannian manifold, then $\mathcal{Q}_{aff}(X) = \mathcal{Q}(X)$ and such a relation is expressed in terms of the second fundamental forms of hypersurfaces.

5.2 k-Convexity and $(n-k)$-Mean Convexity.

Let X be an n-manifold with an affine, e.g. Riemannian, connection and say that a cooriented C^2-hypersurface $Y \subset X$ is $([k_\geq] + [k_>])$-*convex* if the second fundamental form of Y with values in $T(X|Y)/T(Y)$, when diagonalized, has at least $k_\geq$ nonnegative terms and $k_>$ positive terms. If only one of the two terms in the sum $k = k_\geq + k_>$ is present, one speaks of k-*convexity* for $k = k_\geq$ and of *strict k-convexity* for $k = k_>$.

Accordingly, a domain $V \subset X$ is called $([k_\geq] + [k_>])$-*convex* if its boundary is $([k_\geq] + [k_>])$- convex.

For instance, a small ε-neighbourhood of compact smooth submanifold P^{n-k-1} of codimension $k+1$ in a Riemannian X is strictly k-convex and it is easy to show that every curve-linear subpolyhedron in X of codimension $k + 1$ also admits an arbitrarily small strictly k-convex neighbourhood.

If $X = \mathbb{R}^n$, these $[k_\geq] + [k_>]$ are *the only* convexity relations that are invariant under *affine* transformations of $\mathbb{R}^n$, where $k = n - 1$ corresponds to the ordinary local convexity, while 1-convex hypersurfaces are nowhere concave.

The distinction between "$\leq$" and "$<$" is nonessential for *compact* $Y \subset \mathbb{R}^n$, since, (almost) obviously (see §1/2 in [2])

every smooth compact (possibly with a boundary and with a self-intersection) k-convex hypersurface Y, in $\mathbb{R}^n$ can be C^2-approximated by *strictly* k-convex hypersurfaces Y' that may be positioned, depending on what you wish, inside or outside Y.

Remark/Question. If Y is non-compact, then a "strict" approximation of Y by Y' may be possible in one topology, e.g. for Y' being obtained from Y, by a map f with $dist(y, f(y)) \leq \varepsilon$ but not in a finer topology where $\varepsilon = \varepsilon(y) \to 0$ for $y \to \infty$. Besides, an approximation of $([k_\geq] + [k_>])$-convex hypersurfaces by $([k'_\geq - l] + [k'_> + l])$-convex ones may depend on l and on your positioning Y' inside/outside Y.

Is there a comprehensive description of what may happen?

Since a generic linear function f on a k-convex domain $V \subset \mathbb{R}^n$ bounded by k-convex hypersurface adds no l-handles to sublevels of f at the critical points of f on Y,

a compact k-convex domain $V \subset \mathbb{R}^n$ is diffeotopic to a regular neighbourhood of $(n-k-1)$-dimensional subpolyhedron $P^{n-k-1} \subset \mathbb{R}^n$.

Questions. Does there exist such a diffeotopy $f_t : V \to \mathbb{R}^n$, (that eventually "shrinks" V to P^{n-k-1}) where all intermediate domains $f_t(V)$, $t > 0$ (for $f_0(V) = V$) are k-convex?

What are topological possibilities of k-convex domains in the Euclidean n-sphere?

Observe that the complement to a disjoint union of ε-neighbourhoods of two or more equatorial spheres of dimension k, $k < n/2$, is strictly k-convex; it is contractible to some P^{n-1} but not to any P^{n-k-1}.

(A more traditional problem concerns k-convex domains $V \subset S^n$, such that, moreover, the complementary domains $S^n \smallsetminus V$ are $(n-1-k)$-convex.)

Recall that a C^2-smooth coorineted hypersurface $Y \subset X$ is called $(n-k)$-*mean convex* if the traces of the second fundamental form of Y restricted to the tangent $(n-k)$-planes $H^{n-k} \subset T(Y)$ are non-negative. In other words,

the principal curvatures of $Y \subset X$, say $\kappa_1 \leq \kappa_2 \leq ... \leq \kappa_{n-1}$ satisfy

$$\kappa_1 + \kappa_2 + ... + \kappa_{n-k} \geq 0 \text{ at all points } y \in Y.$$

(If $k = 1$ this means that the Y is convex and if $k = n-1$ this says that the mean curvature of Y is non-negative.)

Accordingly, *strict $(n-k)$-mean convexity* requires this inequality to be strict, i.e. all traces to be positive.

Question. Can every closed $(n-k)$-mean convex hypersurface in $\mathbb{R}^n$ be approximated by *strictly $(n-k)$-mean convex* ones?

(This is easy for $k = 1$ and $k = n-1$, but I see no immediate proof it for other k. Am I missing something obvious?[2]

Clearly, (strictly) $(n-k)$-mean convex hypersurfaces are (strictly) k-convex, and

every embedded closed k-convex hypersurface in the Euclidean space $\mathbb{R}^n$ is isotopic to a strictly $(n-k)$-mean convex one (since it can be brought to a neighbourhood of k-subpolyhedron $P^k\mathbb{R}^n$).

But this if far from being true, even on the homotopy level, in non-Eulidean spaces.

Mean Convex Surgery. Let $V \subset X$ be a smooth $(n-k)$-mean convex domain and let $B^l \subset X$ be a smooth disk that all lies outside V except for its boundary sphere $S^{l-1} = \partial B^l \subset Y = \partial V$, where we assume (just for the civility sake) that B^l meets Y *normally* i.e. under the angle$= \pi/2$ along $S^{l-1} = B^l \cap Y$.

Let us slightly thicken B^l by taking its ε-neighbourhood, denoted $_\varepsilon B^l \subset X$, and observe, assuming $\varepsilon > 0$ is sufficiently small, that

• the union $V \cup _\varepsilon B^l$ has smooth boundary except for a $\sim \pi/2$ corner along the boundary of a small tubular neighbourhood of $S^{l-1} \subset Y$;

---

[2]Bruce Kleiner pointed out to me that such approximation is possible with the mean curvature flow.

- the new smooth part of the boundary of $V \cup_\varepsilon B^l$, that is

$$\partial(V \cup_\varepsilon B^l) \smallsetminus \partial V = (\partial(_\varepsilon B^l) \cap (X \smallsetminus V)$$

is $(l+1)$-mean convex.

If $l \leq n-k-1$, then the corner in $V \cup_\varepsilon B^l$ can be $(n-k)$-mean convexly smoothed.

About the Proof. The boundary of the union $V \cup_\varepsilon B^l$ is concave along the corner and the obvious smoothing of $V \cup_\varepsilon B^l$ does not give us a $(n-k)$-mean convex domain. However the $(n-k)$-mean curvature of the boundary of the ε-tube around B^l for $l \leq n-k-1$ tends to $+\infty$ for $\varepsilon \to 0$. This "infinite excess of positivity" allows one to to construct strictly $(n-k)$-mean smoothing similarly but easier than how it was done in [4] for scalar curvature.

5.3 Convergence Stability.

The limit behavior of *embedded $\mathcal{R}$-convex hypersurfaces* is opposite to what is demanded by the C^0-*dense h-principle*: the spaces of such hypersurfaces are *closed* rather than dense in the C^0-topology for closed subsets $\mathcal{R} \subset \mathcal{Q}$.

Moreover, let $\mathcal{R} \subset Q(X)$ be a closed convexity relation and let $Y \subset X$ be a C^2-smooth cooriented hypersurface that is closed in X as a subset. Let $U_i \supset Y$, $i = 1, 2...$, be a sequence of neighbourhoods such that $\bigcap_i U_i = Y$ and let $Y_i \subset U_i$ be smooth cooriented hypersurfaces closed in U_i as subsets, the closures of which do not intersect the boundaries of U_i and that separate the components of the boundaries ∂U_i in the same way Y does. In other words, Y_i are homologous to Y in U_i (in the sense of homology with infinite supports if Y is non-compact.)

If all Y_i satisfy a closed convexity relation $\mathcal{R}$ then Y also satisfies $\mathcal{R}$.

Proof. In fact let Q_0 be the jet of Y at some point Y_0 and $\Omega_0 \subset \mathcal{Q}$ be an arbitrarily small neighbourhood of Q_0. Then, by the *weak* (and obvious) *maximal principle*, every Y_i for all $i \geq i_0 = i_0(\Omega_0)$ contains a point y_i^- such that the 2-jet $J^2_{y_i^+}(Y_i) \in \mathcal{Q}$ (or a germ at this point, if you wish) satisfies

$$J^2_{y_i^-}(Y_i) \leq \omega_i \text{ for some } \omega_i \in \Omega_0$$

(as well as a point y_i^+, where

$$J^2_{y_i^+}(Y_i) \geq \omega_i' \text{ for some } \omega_i' \in \Omega_0.)$$

Q.E.D

(Counter)examples. (a) Every curve Y in the plane can be (obviously) C^0-approximated by locally convex *immersed* curves Y_i. By the above, these Y_i must have lots of double points as Y_i come close to the region where Y is concave.

(b) Similarly, according to Lawson and Michelson [6] every co-orientable immersion f of an $(n-1)$-manifold Y to a Riemannian n-manifold X, can be C^0-approximated by immersions $f_i : Y \to X$ with positive mean curvatures.

(The building blocks of f_i are finite coverings maps onto the boundaries of ε-neighbourhoods of $(n-2)$-submanifolds in X, where, observe, such boundaries have mean curvatures $\sim \varepsilon^{-1}$ for $\varepsilon \to 0$.)

(c) In contrast with the above, if $k > n/2$, then

every closed cooriented strictly $(n-k)$-mean convex hypersurface Y in a complete Riemannian manifolds X with non-negative sectional curvatures bounds a compact Riemannian manifold U, i.e. $\partial U = Y$, such that the immersion $Y \to X$ extends to an isometric immersion $U \to X$.

(This U is contractible to its k-skeleton, since the minus distance function $u \mapsto -dist_U(u, \partial U = Y)$ admits an approximation by an $(n - k)$-mean convex Morse function on U that provides an isotopy of U to a regular neighbourhood of a k-dimensional subpolyhedron $P^k \subset U$ [10].)

On the other hand, k-convex hypersurfaces for $k < n - 1$ in general *non-flat* n-manifolds X with non-negative curvatures *do not necessarily bound* immersed n-manifolds in X. But this is true in the presence of many "movable" totally geodesic submanifolds in X by the the Euclidean argument from §1/2 in [2], where the standard examples of such manifolds are Riemannian cylinders $X = X_0 \times \mathbf{R}$ and complete simply connected n-spaces X of *constant negative* curvature.

Questions. What are possible topologies of (embedded and immersed) k-convex hypersurfaces in the Euclidean n-sphere?

Are there any constrains on immersed $(n - k)$-mean convex hypersurfaces in the Euclidean n-space for $n \geq 2k$?

The convergence stability suggests that that the notion of $\mathcal{R}$-convexity can be extended to non-smooth subsets. The cheapest way to produce non-smooth examples starting with the class $\mathcal{U}$ of smooth $\mathcal{R}$-*convex domains* $U \subset X$, i.e. having smooth $\mathcal{R}$-convex boundaries, is enlarging/completing $\mathcal{U}$ by some/all of the following three operations over subsets.

$[\pitchfork]_{<\infty}$ Locally finite intersections of smooth domains $U_i \in \mathcal{U}$ with mutually transversal boundaries.

$[\cap \downarrow]_\infty$ Intersections of infinite *decreasing* families of subsets.

$[\cap]_{\leq\infty}$ Finite and infinite intersections of smooth domains, that is essentially the same as $[\pitchfork]_{<\infty} + [\cap \downarrow]_\infty$.

$[\cup \uparrow]_{-\infty}$ Union of infinite *increasing* families.

Question. Given, say an open, convexity relation $\mathcal{R} \subset \mathcal{Q}(X)$, let $\mathcal{C}(\mathcal{R})$ denotes the class of all compact subsets in X obtained from compact smooth $\mathcal{R}$-convex domains $U \subset X$ by some of the above operations, e.g. by $[\cap]_{\leq\infty}$, i.e. by taking infinite intersections of compact smooth $\mathcal{R}$-convex domains U.

Is there any, not necessarily exhaustive, characterization of subsets in $\mathcal{C}(\mathcal{R})$ in terms of $\mathcal{R}$?

Is every $\mathcal{R}$ *uniquely* determined by $\mathcal{C}(\mathcal{R})$?

For instance, which *Cantor sets* $C \subset \mathbb{R}^n$ are representable as infinite intersections of disjoint finite unions of compact *convex* subsets?

Clearly, this is possible if the Hausdorff dimension of C satisfies $dim_{Hau}(C) < 1$, but "generic" subsets C with $dim_{Hau}(C) > 1$ admit no such representation.

In fact, the geometry of a Cantor set $C \subset \mathbb{R}^n$ at a point $x \in C$ may be characterized by the minimal possible "oscillatory complexity", $osc_\varepsilon = osc_\varepsilon(C, x)$, $\varepsilon > 0$, e.g the *total curvature curv$_\varepsilon$* (that is the $n - 1$-volume of the tangential Gauss map counted with multiplicity) of the boundaries of smooth neighborhoods $U_\varepsilon \subset \mathbb{R}^n$ of x such that $diam(U_\varepsilon) \leq \varepsilon$ and where the boundaries ∂U_ε do not intersect C.

It seem "most" Cantor sets in $\mathbb{R}^n$, $n \geq 2$, (I checked this only for a few particular classes of sets) have $osc_\varepsilon \to \infty$, e.g. $curv_\varepsilon \to \infty$ for $\varepsilon \to 0$) and they do not belong to any convexity class $\mathcal{R} \subset \mathcal{Q}$, unless $\mathcal{R}$ equals $\mathcal{Q}$ minus a "very thin" subset).

Convergence Stability for k-Mean Convexity for Functions. Since this convexity is defined by *linear* inequalities on the (second) derivatives of functions f it is stable under all kinds of weak limits and it non ambiguously extends to continuous functions as we saw in section 3.3.

5.4   Riemannian Curvature Digression.

The above is a baby version of the following Riemannian problems.

Given two 2-jets, or germs g_1 and g_2 of Riemannian metrics at a point x in a smooth manifold X, write $g_1 \preccurlyeq g_2$, if the two have equal 1-jets and their sectional curvatures satisfy

$$curv_\tau(g_1) \geq curv_\tau(g_2) \text{ for all tangent 2-planes } \tau \subset T_x(X).$$

For example, metrics with "large amount" of positive curvature are regarded as small.

A *lower curvature relation/bound* $\mathcal{B}$ is a subset of 2-jets g of Riemannian metrics at the origin in $\mathbb{R}^n$, such that

- 1-jets of g equal the 1-jet of the Euclidean metric;
- if $g_2 \in \mathcal{B}$ and $g_1 \preccurlyeq g_2$ then $g_1 \in \mathcal{B}$;
- the subset $\mathcal{B}$ in the space of jets is invariant under orthogonal transformations of $\mathbb{R}^n$.

The latter condition allows one to invariantly speak of $\mathcal{B}$-positive metrics on *all* smooth n-manifolds X that are, in other words, *Riemannian n manifolds that satisfy $\mathcal{B}$* (compare with §7 in [2]).

The fundamental questions are as follows.

[A] Given $\mathcal{B}$ what is the weakest topology/convergence $\mathcal{T} = \mathcal{T}(\mathcal{B})$ in the space of Riemannian manifolds, such that the limits of $\mathcal{B}$-positive manifolds are $\mathcal{B}$-positive?

[B] What are singular $\mathcal{B}$-positive metrics spaces?

[C] What are $\mathcal{B}$ for which the above two questions have satisfactory answers?

If $\mathcal{B}$ consists of the metrics with a given bound *all* sectional curvatures, then the (best known) answer to [A] is the *Hausdorff convergence* of metric spaces and [B] is essentially resolved by the theory of Alexandrov spaces.

The starting point of the theory for spaces with a lower bound on the *Ricci* curvature is the (almost obvious) stability of the inequality $Ricci(g) \geq const \cdot g$ under C^0-limits of Riemannian metrics on a given underlying (and unchangeable) smooth manifold X while the general theory, albeit not fully established, is well underway, see [1], [7], [9] and references therein.

The most tantalizing relation $\mathcal{B}$ is expressed with the *scalar curvature* by $scal(g) \geq cost$ where even the C^0-limit stability is not fully established and where some possibilities are suggested by the *intrinsic flat distance* [12].

Nothing seems to be known about other $\mathcal{B}$, e.g. those encoding some positivity of the curvature operator, e.g. positivity of the *complexified sectional curvature*, see [8], and §7 in [2].

5.5 Cut Locus, Focality and Curvature Blow-up.

Let us see what happens to convexity under equidistant deformations of a hypersurface $Y \subset X$, where an attention must be paid to singularities on the *cut locus* that may be aggravated by the presence of *focal points*.

Recall that the cut locus $cut(U) \subset X$ of an open subset $U \subset X$ (or of a closed domain U) with respect to $Y = \partial U$ is defined as the closure of the set of points $u \in U$ that have more than one *ancestor* in Y, where a point x in the closure of U is called a *d-ancestor*, for $d = dist(x, u)$, or just "ancestor" of a point $u \in U$, with u being called a *d-descender*, or "descender" of x, if $dist(x, u) = dist(u, Y) - dist(x, Y)$.

Assume that X is a complete C^2-smooth Riemannian manifold and recall a few obvious properties of $cut(U)$.

If $Y = \partial U$ is a C^2-hypersurface, than the cut locus of U does not intersect Y and the ρ-equidistant hypersurfaces denoted $Y_{-\rho} \subset U$ are C^2-smooth away from $cut(U)$, i.e. the complements $Y_{-\rho} \smallsetminus cut(U)$ are C^2-smooth (locally closed) hypersurfaces in U.

If the boundary $Y = \partial U$ is C^1-*smooth*, then an $x \in Y$ is ρ-accessible from U if and only if the geodesic segment of length ρ issuing from x normally to Y inward U either *does not* intersect $cut(U)$, or, if it meets $cut(U)$, then only at its terminal in U.

All open $U \subset X$ satisfy (by a simple *Čech homology* argument).

$\rho \le dist(y, cut(U)) \Rightarrow y$ *is ρ-accessible from U for all $y \in Y = \partial U$.*

Consequently, if U is C^2-quasi-regular, then $Y \smallsetminus cut(U)$ is C^2-smooth.

Focal Points. Let $y_0 \in Y$ be an ancestor of $u_0 \in U$, i.e. a *(global) minimum* point of the function $y \mapsto dist(y, u_0)$ on Y. Assume X is complete and let $\gamma = \gamma(s)$ in X be a geodesic ray issuing from x_0 inward U, such that

$$\gamma(s_0) = u_0 \text{ for } s_0 = dist(u_0, y_0),$$

where $s \ge 0$ denotes the geodesic length parameter. (If Y is C^1-*smooth hypersurface* at y_0 then γ is unique being normal to Y.)

The point u_0 is called *non-focal* for y_0 along γ if y_0 remains a *local* minimum of the function $y \mapsto dist(y, u)$ on Y as we slightly move along γ inward, i.e. for $u = \gamma(s_0 + \varepsilon)$ and all sufficiently small $\varepsilon > 0$.

In other words, the $(s_0 + \varepsilon)$-ball in X around $u_\varepsilon \in U$, say

$$B_{u_\varepsilon}(s_0 + \varepsilon) \supset B_{u_0}(s_0) \subset U,$$

is "contained in U at y_0", i.e. the intersection of $B_{u_\varepsilon}(s_0 + \varepsilon)$ with a small neighbourhood of y_0 in X is contained in U.

Notice that focal/non-focal for $y_0 \in Y$ depends only on the geometry of Y in a small neighbourhood of Y_0 plus on how one defines "inward". Thus, one

can extend the above definition by taking an arbitrarily small neighbourhood $B_0 \subset X$ of y_0, (e.g. a small ε-ball around y_0), letting

$$U_0 = X \smallsetminus \left(B_0 \bigcap (X \smallsetminus U)\right) \supset U$$

and defining focal/non-focal along geodesic segments in U_0 that starts at y_0 and may go beyond U.

If Y is a C^1-smooth hypersurface and y is an ancestor of u with $dist(u,y) = dist(u,Y) = \rho$ then the ρ-sphere around u, say $S_u(\rho) = \partial B_u(\rho) \subset U$, that contains y is C^2-smooth at y, provided our Riemannian metric is C^2-smooth. If, moreover, Y is a C^2-smooth hypersurface, then the second fundamental form Q_Y of Y at y is minorized by the form Q_S at y, i.e. $Q_Y - Q_S$ is *negative semidefinite* since $B_u(\rho) \subset U$. (Our sign convention for $Q's$ is the one for which the boundaries of convex subsets $U \subset X$ have *positive* definite second fundamental forms Q.)

Obviously, $u \in U$ is *non-focal* for $Y = \partial U$, (along the minimal geodesic segment between the two points) if and only if the quadratic form $Q_Y - Q_S$ is *negative definite*.

Denote by $foc(U) \subset U \cap cut(U)$ the subset of *the focal points* where u is called focal if it is focal for some ancestor of u in $Y = \partial U$ and observe that

If Y is C^2-quasi-regular, e.g. C^2-smooth, then the subset $foc(U) \subset U$, is closed in U. (This is not, in general, true for C^1-hypersurfaces Y.)

The appearance of focal points can be seen in terms of the hypersurfaces $Y_{-\rho} \subset U$ equidistant to $Y = \partial U$ as follows. Join a point $u_0 \in Y_{-\rho}$ with one of its ancestors, say $y_0 \in Y$ by a minimal geodesic segment γ in the closure of U, where $length(\gamma) = \rho$, and observe that the hypersurfaces $Y_{-\rho+\varepsilon}$, $0 < \varepsilon \le \rho$, are C^2-smooth at the points $u_{+\varepsilon} = Y_{-\rho+\varepsilon} \cap \gamma$, provided Y is C^2-smooth at y_0. Then,

the second fundamental forms Q_ε of $Y_{-\rho+\varepsilon}$ at the points $u_{+\varepsilon}$ are uniformly bounded from below.

If u_0 is a *non-focal* for y_0 then the forms Q_ε are also bounded from *above*; moreover, the hypersurfaces $Y_{-\rho+\varepsilon}$ can be locally, around γ, be included into a C^2-smooth family of local equidistant hypersurfaces to a small neighbourhood of $y_0 \in Y$.

But if u_0 is *focal for y_0* then these forms "blow up" for $\varepsilon \to 0$ as follows.

The $((n-1)$-dimensional) spaces $T(\varepsilon)$ normal to γ at the points $u_{+\varepsilon}$, that serve as tangent spaces to Y for $\varepsilon > 0$, admit orthogonal splittings $T(\varepsilon) = T_0(\varepsilon) \oplus T_1(\varepsilon)$, where these $T_0(\varepsilon)$ and $T_1(\varepsilon)$ continuously depend on $\varepsilon \in [0,\rho]$ and such that

• the subspace $T_0(\varepsilon = \rho) \subset T_{y_0}(Y)$ equals the kernel of the above difference form $Q_Y - Q_S$ at y_0;

• the forms Q_ε restricted to $T_1(\varepsilon)$ are continuous for all $0 < \varepsilon \le \rho$ and they continuously extend to the space $T_1(\varepsilon = 0)$.

• The forms Q_ε on the subspaces $T_0(\varepsilon)$ tend to $+\infty$ for $\varepsilon \to 0$. In fact, the values of Q_ε on the unit vectors in $T_0(\varepsilon)$ is of order $1/\varepsilon$.

5.6 C^2-Approximation with Corners.

We show here how equidistant hypersurfaces to a quasiregular Y can be approximated by piecewise smooth hypersurfaces with one sided controls on their

curvatures.

Let $U \subset X$ be a C^2-quasiregilar open subset (domain) with boundary $Y = \partial U$ in a complete Riemannian manifold X, let $Y_{-\rho} = \partial U_{-\rho} = U_{-\rho}^{\leq} \subset U$, $\rho > 0$, be the equidistant hypersurface, where, as earlier, $U_{-\rho}^{\leq}$ denotes the set of $u \in U$ where $dist(u, Y) \leq \rho$.

$\smallsmile$*Approximation Lemma.* Given $\varepsilon > 0$ and $0 < \rho' < \rho$, there exists a domain $U' = U_{\rho'}^{\smallsmile \varepsilon}$ in X such that

$$U_{-\rho} \supset U' \supset U_{-(\rho+\varepsilon)},$$

and such that

      the boundary $Y' = \partial U'$ is piecewise C^2-smooth.
In fact, there are C^2-diffeomorphisms $D_i : X \to X$ such that
$D_i(U_{-\rho})$ *do not intersect the singular locus of $Y_{-\rho'}$ and*
Y' *equals the union of the D_i-pullbacks of $Y_{-\rho'}$*

$$Y' = \cup_i D_i^{-1}(Y_{-\rho'}).$$

Moreover, if $\rho - \rho'$ is small, then these D_i are C^2-close to the identity map $X \to X$; consequently,

      the curvatures of the smooth pieces $D_i^{-1}(Y_{-\rho'})$ are close
      to the curvatures of their D_i images in $Y_{-\rho'}$.

Proof. Let $\gamma \subset X$ be a minimal geodesic segment. Then, obviously, there exists smooth vector field $V_\gamma(x)$ on X that is tangent to γ where it equals the unit field directed from x_0 to x_1 and such that the norm of V_γ satisfies

$$\|V_\gamma(x)\| < 1 \text{ for } x \notin \gamma \text{ and } \limsup_{x \to \infty} \|V_\gamma(x)\| < 1.$$

Integrate V_γ for the flow time equal $length(\gamma)$; thus, obtain a C^2-diffeomorphism $D_\gamma : X \to X$ such that
    D_γ *sends one end of γ, say x_0, to the other one, called $x_1 = D_\gamma(x_0)$,*
where this diffeomorphism is *sharp at γ* in the sense that
$$dist(x, D_\gamma(x)) < length(\gamma) \text{ for all } x \notin \gamma,$$
and where one can achieve a map $\gamma \mapsto D_\gamma$ to be *continuous for the C^2-topology* in the space of diffeomorphisms.

Remark. It is easy to arrange the maps $D_\gamma : X \to X$ such that their differentials $T_{x_0}(X) \to T_{x_1}(X)$ are *isometries* for all γ. Moreover, if X has *positive* sectional curvatures, one can make D_γ *second order isometries* at these points, i.e. such that every geodesic through x_0 goes to a curve with zero curvature at x_1. However, this is impossible for manifolds of *negative* curvature.

Now, let $\delta > 0$ be very small (depending, in particular, on ε), take all minimal segments γ between the points $y \in Y_{-\rho}$ and its $(\rho' + \delta)$-anscestors in U and let

$$U' = U_{-\rho} \smallsetminus \cup_\gamma D_\gamma^{-1}(X \smallsetminus U_{-\rho'}).$$

Finally, take a sufficiently dense locally finite set of geodesic segments, say $\{\gamma_i\}$, and take D_{γ_i} for the required diffeomorphisms D_i.

This $\smallsmile$-approximation implies in particular that the distance function d to the boundary Y of U can be approximated by the maximum of *smooth* distance functions with their second partial derivatives close to those of h at nearby

points. It follows that all k-convexity bounds extends from smooth to non-smooth points of d. In particular the k-convexity lemma from section follows from this $\smile$-approximation since one, obviously, has a uniform bound on the "bending" β in this case.

On External Approximation. The above piecewise smooth hypersurfaces $Y_{-\rho}^{\delta\smile\Delta}$ that approximate the boundary $\partial U_{-\rho}^{<}$ are positioned *inside* $U_{-\rho}^{<}$. Probably, there is no similar approximation by hypersurfaces lying *outside* but this is obviously possible if $Y_{-\rho}$ is compact: just apply the inside approximation to $Y_{-\rho'}$ for $\rho' < \rho$ and let $\rho' \to \rho$.

5.7 Cornered Domains and Smoothing the Corners.

Let us indicate here a geometric alternative to the smoothing operators we used in section 3.4.

A cornered domain of class C^k in a C^k-smooth n-manifold X is a closed subset V, such that every *boundary* point v in V admits a neighbourhood $U(v)$ in V that is C^k-diffeomorphic to the intersection of $k \le n$ mutually orthogonal halfspaces in $\mathbb{R}^n$.

The regular part of the boundary of V, denoted $reg_{\partial V} \subset \partial V$, consists of those v, where $U(v) \subset V$ is diffeomorphic to a half space, i.e. $k = 1$.

The $(n-1)$-faces W_i, $i \in I$, of V are the closures of the connected components of $reg_{\partial V} \subset V$ where, obviously,

$$\bigcup_{i \in I} W_i = \partial V.$$

(Sometimes, one takes finite unions of disjoint connected components for faces.)

Corners or $(n-2)$-faces of V are, by definition, non-empty pairwise intersections of $(n-1)$-faces,

$$W_{i_1 i_2} = W_{i_1} \cap W_{i_2},$$

(Since we assume the corner structure being "simple", there is no non-empty intersection $W_{i_1} \cap W_{i_2}$ of dimension $< n-2$. On the other hand, corners may be disconnected.)

It is easy to see that a cornered V equals an intersection

$$V = \bigcap_{j \in J} V_j,$$

where

(1) $V_j \subset X$ are n-submanifolds with *smooth* boundaries ∂V_j,

(2) all intersections between (k-tuples of) ∂V_j are *transversal*,

(3) there are at most *finitely many* boundaries that intersect a given *compact* subset in X.

These (1)-(3) imply that the intersections of ∂V_j with V equal finite union of disjoint $(n-1)$-faces of V where these $V \cap \partial V_j \subset \partial V_j$ are cornered domains in ∂V_i. If one wishes, one may let $J = I$ and choose V_i such that $V \cap \partial V_i = W_i$.

If X is a Riemannian manifold than one may speak of the dihedral angles between pairs of $(n-1)$-faces along $(n-2)$-faces. Clearly all these angles

$\angle(W_{i_1}, W_{i_2})$, that are *continuous functions* on $W_{i_1 i_2} = W_{i_1} \cap W_{i_2}$, are bounded by

$$\angle(W_{i_1}, W_{i_2}) < \pi.$$

Essential Example. A generic C^∞-perturbation of the smooth pieces of U' that approximate $U_{-\rho}$ in the previous section turn U' into a cornered domain.

Since the corners of V are convex for the dihedral angles $< \pi$ one expects that cornered domains V admit approximations by *smooth* domains that are, up to an arbitrary small error, "as convex" as the faces W_i of V. Indeed, this is possible for quite a few, classes of *convexity relations* (see next section) including strict k-mean convexity where the picture is most transparent for the mean curvature convexity.

Corner Smoothing Lemma. Let X be a C^2-smooth Riemannian n-manifold, $\phi : X \to \mathbb{R}$ a continuous function and V a cornered domain of class C^2 such that the mean curvatures of the regular part of the boundary $\partial V \subset V \subset X$ satisfy,

$$mn.curv_x(\partial V) > \phi(x) \text{ for all } x \in reg_{\partial V} \subset \partial V.$$

Then, for an arbitrary neighbourhood $\Delta \subset V$ of the boundary $\partial V \subset V$, there exists a domain $V' \subset V$ with C^2-smooth boundary, such that

$$\partial V' \subset \Delta \text{ and } mn.curv_x(\partial V') > \phi(x) \text{ for all } x \in \partial V';$$

moreover, the normal projection $\partial V \to \partial V'$ is a C^2-diffeomorphism on every $(n-1)$-face of V.

Proof. Let ∂V be compact and let $\partial_{+\delta} V \subset X$ be the boundary of the δ-neighbourhood of V for a small $\delta > 0$. Clearly, $\partial_{+\delta} V$ is a C^1-smooth hypersurface, that is, moreover, piecewise C^2.

This is seen with the normal projection $\partial_{+\delta} V \to \partial V$ that sends every C^2-piece of $\partial_{+\delta} V$ onto an m-face of V, for some $m = 1, 2, ..., n-1$. If $m = n-1$, then the mean curvature of this piece is δ-close to that of the corresponding $(n-1)$-face and if $m < n-1$ then the mean curvature is $\sim \delta^{-1}$.

Thus, the mean curvature of $\partial_{+\delta} V$ is a piecewise continuous function on $\partial_{+\delta} V$ that satisfies

$$mn.curv_x(\partial_{+\delta} V) > \phi(x) \text{ for all sufficiently small } \delta > 0 \text{ and all } x \in \partial_{+\delta} V.$$

Now, observe that $\partial_{+\delta} V$ equals the δ-level of the distance function $d(x) = dist(x, V)$ and let $d_\varepsilon(x)$ be the average of $d(x)$ over the ε-ball $B_x(\varepsilon) \subset X$ for a small $\varepsilon > 0$.

Since the second differential of $d(x)$ is a bounded measurable function and $\|grad(d)\| = 1$ the δ-level say $\partial_{+\delta,\varepsilon} V \subset X$ of d_ε is a C^2-smooth hypersurface that C^1-converges to $\partial_{+\delta}$ for $\varepsilon \to 0$. Since the mean curvature of a level of a function is *linear* in the second derivatives of the function the mean curvatures of $\partial_{+\delta,\varepsilon}$ are, up to an ε-error, equal the $B_x(\varepsilon)$-averages of these of $\partial_{+\delta}$; hence, the mean curvatures of $\partial V' = \partial_{+\delta,\varepsilon}$ are $> \phi(x)$ for sufficiently small $\varepsilon > 0$.

Finally, in order to have $\partial V'$ inside rather than outside V, we apply the above to an interior equidistant hypersurface $\partial V_{-\delta}$ instead of $\partial V = \partial V_0$, where a minor readjustment of this argument is needed non-compact ∂V.

This, together with "essential example" allows an alternative proof of

Smoothing of Quasiregular Hypersurfaces . Let U be an open domain in X with quasiregular boundary and let the mean curvatures at all regular points of ∂U are strictly minorized by a continous function ψ on X, i.e.

$$mn.curv_x(\partial U) > \psi(x) \text{ for all regular ponts } x \in \partial U.$$

Then U can be exhausted by closed subsets $U_i \subset U$ with smooth boundaries $Y_i = \partial U_i$, where the mean curvatures of these are strictly minorized by $\psi(x)$ at all $x \in \partial U_i$ and all U_i.

The two basic examples where this smooth approximation is used in the present paper are

(1) *Strictly Mean Convex Bubbles $U \subset X$ with compact boundaries Y.*

(2) *Minimal Hypersurfces H.*

In both cases the Almgren-Allard quasiregularity theorem applies and, in the case (1), allows a *smooth strictly mean convex approximation of Y* while in the case (2) one approximates the boundary Y_ε of the ε-neighbourhood $U\varepsilon(H) \subset X$ of H by a *smooth c_ε-concave hypersurface*, i.e. with $mn.curv(Y_\varepsilon) \leq c_\varepsilon$ where $c_\varepsilon \to 0$ for $\varepsilon \to 0$.

Remarks.(a) When we discussed smoothing minimal hypersurfaces H with Joachim Lohkamp a few years ago he, on one hand, said he was well aware of possibility of such smoothing, but, on the other hand, he expressed a concern about singularities at the focal points.

Focal points are invisible in the argument with bending and standard linear smoothing (see section 3.4) but the above makes it clear why singularities at these points cause no additional complication.

(b) The corner smoothing lemma. remain valid for the mean $(n-k)$-convexity for all k but it fails, in general, for k-convexity, probably for all $k \neq 1, n-1$.

To see this for even $n - 1 \geq 4$ and $k = (n-1)/2 \geq 2$, let $V \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$ be a compact domain with smooth boundary. Then there obviously exist C^∞-small perturbations V', V_+ and V_- of V in $\mathbb{R}^n$, such that V_+ and V_- transversally meet along the boundary $\partial V'$ and bound together a domain $U' \subset \mathbb{R}^n$ that is k-convex away from the corner along $\partial V'$.

This $U' \supset V'$ can be seen as a small thickening of V' that is homeomorphic to $V' \times [0,1] = V \times [0,1]$. Therefore, if the homology group $H_{n-2}(V) \neq 0$, then $H_{n-2}(U) \neq 0$ as well; hence, U can not be approximated by smooth k-convex domains if $k < n - 2$.

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