Hyperbolic dynamics, Markov partitions and Symbolic Categories, Chapters 1 and 2.

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Contents

Abstract

Different faces of what we call *hyperbolicity* appear in Riemannian geometry, in the theory of holomorphic and of quasiconformal maps, in the combinatorial group theory and the theory of smooth, topological and symbolic dynamical systems.[1]

We discuss essential results in these domains with an emphasis on the (sometimes conjectural) links between different fields. Also we formulate a variety of open problems.

Our presentation is, for the most part, expository and aimed at non-experts: everything is defined in full and is accompanied by examples; notations and definitions are reminded to the reader everywhere in the text; the proofs are furnished in detail and garnished with informal explanations.

1 Hyperbolicity: Its Roots, Ubiquitousness, Functoriality.

Is there a unified theory of hyperbolicity?

I collected evidence in favour of a positive answer and formulated a few questions in a 1980-paper [49].

Here is a revision of [49] from a category theoretic perspective with an emphasis on functoriality of arrangements of hyperbolic patterns which underlies a combinatorial (symbolic) representation of a priori continuous objets and constructions.

This functoriality is most apparent in the following, almost half a century old, theorems by Michael Shub (1969) [104] and John Franks (1970) [40].

1.1 Topological Universality of Toral Endomorphisms.

Contracting, Expanding, Hyperbolic. A linear transformation T of a normed space Y, e.g. of $\mathbb{R}^n$, is (eventually) *contracting* if a power of T has norm < 1:

$$||T^k(y)|| \leq C||y||$$

for some positive integer k, some $C < 1$ and all non-zero $y \in Y$, where we use the standard abbreviation $T^k = T^{\circ k} = \underbrace{T \circ T \circ \cdots \circ T}_{k} : Y \to Y$.

[1]We do not know if *hyperbolic partial differential equations* need to be brought to this list.

A T is called (eventually) *expanding*, if it is invertible and T^{-1} is (eventually) contracting[2].

T is called *hyperbolic* if there exist *T-invariant splittings*,

$$Y = Y_{contr} \oplus Y_{exp} \text{ and } T = T_{contr} \oplus T_{exp},$$

where $T_{contr} : Y_{contr} \to Y_{contr}$ is a contracting map and $Y_{exp} \to Y_{exp}$ is an expanding one.

(If Y finite dimensional, hyperbolicity is equivalent to all eigenvalues of T having the absolute values $\neq 1$)

Notation: Abel's Maps etc. Let V be a compact locally contractible space, e.g. a compact manifold and let $A = A(V)$ denote the homological *Abel-Jacobi torus* of V,

$$A = H_1(V;\mathbb{R})/(H_1(V;\mathbb{Z})/torsion),$$

that is (non-canonically) isomorphic to the n-torus $\mathbb{T}^n$ for n equal the first Betti number $rank_{\mathbb{R}} H_1(V)$ of V.

Let

$$Ab : H_1(V;\mathbb{R}) \to H_1(A;\mathbb{R})$$

denote the tautological *Abel's* isomorphism and observe that there exists a unique homotopy class of continuous *Abel's maps* $q : V \to A$ for which the induced homology homomorphisms $q_* : H_1(V;\mathbb{R}) \to H_1(A;\mathbb{R})$ equal Ab.

Let $f : V \to V$ be a continuous map, let $f_* : H_1(V;\mathbb{R}) \to H_1(V;\mathbb{R})$ and $\underline{f}_* : A \to A$ be induced by f, where, observe, the linear map f_* and the torus endomorphism $\underline{f}_*$ mutually determine each other via the covering map $H_1(V;\mathbb{R}) \to A$.

[$\vec{\blacksquare}$] Topological Universality Theorem. Let either

[$\uparrow$] the homology endomorphism f_* is *expanding*, (Shub, 1969)

or

[$\uparrow\downarrow$] f is a *homeomorphism* and f_* is *hyperbolic* (Franks, 1970).

Then there exists a continuous map

$$\alpha : V \to A,$$

such that

(1) *the induced homomorphism* $\alpha_* : H_1(V;\mathbb{R}) \to H_1(A;\mathbb{R})$ *equals* Ab, that is α is in *the Abel's homotopy class* of maps $q : V \to A$.

(2) The map α is *a morphism* in the category of *spaces with $\mathbb{Z}_+$-actions*;[3] we call such an α *an f-morphism* for

$$(\alpha \circ f)(v) = (\underline{f}_* \circ \alpha)(v), v \in V$$

that expresses commutativity of the diagram

[2] Sometimes we say *strictly* expanding and/or contracting to distinguish λ-expansion with $\lambda > 1$ from mere $\lambda \geq 1$

[3] Non-isomorphic morphisms $X \to X$ (and $X \to Y$) in the category of $\mathbb{Z}_+$-spaces disrespectfully called *semicoqjugaces* in the dynamical systems community.

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ \downarrow \alpha & & \downarrow \alpha \\ A & \xrightarrow{f_*} & A \end{array}$$

Moreover,

(3) *α is unique up to group translations of (the Abelian group) A that preserve the fixed point set of α.* (There are at most finitely many of these translations.)[4]

These (1), (2), (3) feel highly unlikely – no sound minded person would ever dream that anything so sharp and precise might hold for *all* continuous maps (or homeomorphisms) f in certain homotopy classes.

But the proof due to Schub and Franks (see next section), which streamlined the original topological stability arguments by Smale and Anosov,[5] turns out as clean, short and simple as the statement of the theorem.

1.2 Shadowing, Quasimorphisms and Quasigeodesics.

The core ingredient of the proof of the Shub-Franks theorem is the following

SHADOWING LEMMA. Let X and Y be $\mathbb{Z}$-spaces where the actions of $\mathbb{Z}$ on X and Y is generated by maps that are denoted $S : X \to X$ and $T : Y \to Y$.

If Y is a metric, (e.g. linear normed) space, a map $Q : X \to Y$ is called a *$\mathbb{Z}$-quasimorphism* if the function

$$x \mapsto dist_Y(Q \circ S(x), T \circ Q(x))$$

is bounded on the S-orbits $\{...S^{-2}(x), S^{-1}(x), x, S(x), S^2(x)...\} \subset X$ for all $x \in X$.

[❒] *If Y is a Banach space and T is hyperbolic, than every $\mathbb{Z}$-quasimorphism Q_0 is shadowed by a unique morphism $M : X \to Y$, that is*

$$[\square] \qquad M \circ S = T \circ M$$

and where "shadowed" signifies that the the function

$$x \to dist_Y(Q_0(x), M(x)) = ||Q_0(x) - M(x)||$$

is bounded on all orbits in X.

Proof. Rewrite $[\square]$ as the fixed point relation $M = \mathcal{T}(M)$ for the transformation

$$\mathcal{T} : Q \mapsto T^{-1} \circ Q \circ S,$$

that applies to the space $\mathcal{Y}_0$ of maps $Q : X \to Y$ for which the function $dist_Y(Q(x), Q_0(x))$ is bounded on the S-orbits.

Since $T = T_{exp} \oplus T_{contr}$, the existence and uniqueness of a fixed point for $\mathcal{T}$ reduces to the two special cases where T is either *expanding* or *contracting.*

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[4]There is an enticing similarity between α and the Abel-Albanese map which we tried to understand in [58].

[5]Smale outlined the structural stability theory in his talks in Kiev (1961), [106] and in Stockholm (1962) [108] The proofs were sketched by Anosov in 1962 [7] and written down in details in 1967 [8].

If T is *expanding* then T^{-1} is contracting; hence, $\mathcal{T}$ is also (obviously) contracting for the space of maps with the *sup*-norm[6] and the proof follows from the *Banach attractive fixed point theorem.*

If T is *contracting*, then the Banach theorem applies to

$$Q \mapsto Q \mapsto T \circ Q \circ S^{-1}$$

that concludes the proof of [❐].

(In the contracting case, unlike the expanding one, the map $S : X \to X$ needs to be be invertible.)

Proof of [$\vec{\blacksquare}$]. Let $\tilde{V} \to V$ be the (Abel-Galois) covering with the Galois (deck transfrmation group) $\Gamma = H_1(V)/torsion$ and let $\tilde{f} : \tilde{V} \to \tilde{V}$ and $Q_0 : \tilde{V} \to H_1(V;\mathbb{R})$ be Γ-equivariant lifts of $f : V \to V$ and of $q_0 : V \to A$ to $\tilde{V}$.

Compactness of $V = \tilde{V}/\Gamma$ and Γ-equivariance of $\tilde{f}$ and of Q_0, imply that Q_0 is an $\tilde{f}$*-quasimorphism*, that is the (Γ-invariant!) function

$$||Q_0 \circ \tilde{f}(\tilde{v}) - f_* \circ Q_0(\tilde{v})||$$

is bounded on $\tilde{V}$ and by [❐] (with $\tilde{f}$ for S and f_* for T), Q_0 is shadowed by a unique $\tilde{f}$-mophism

$$Q : (\tilde{V}, \tilde{f}) \to (H^1(V,\mathbb{R}), f_*).$$

Since Q is unique, it is necessarily Γ-equivariant; hence, it descends to the required f-morphism

$$\alpha : (V = \tilde{V}/\Gamma, f) \to (A = H_1(V;\mathbb{R}))/\Gamma, \underline{f}_*)$$

for $\Gamma = H_1(V)/torsion$. QED.

The above proof of [❐] (due to Franks, 1970) that is (notationally) five times simpler than the one by Anosov of his original local version of shadowing,[7] also applies to $\mathbb{R}$-actions; yet, when it comes to the main class of intended examples – *geodesic flows* on manifolds with *negaitive sectional curvatures*[8] – this delivers only quantitative improvement of Anosov's local shadowing theorem.

But the correct global shadowing property for *quasigeodesics* in *hyperbolic* (*Lobachevsly*)*spaces* $\mathbf{H}^n$ was, in fact, proven by Marston Morse in 1924 for $n = 2$ [83] and by Efremovich and Tichomirova for all n in 1963, [37]

Namely,

[❐❐] *the curves in* $\mathbf{H}^n$ *that have smaller distortions than horocycles in* $\mathbf{H}^2$ *are shadowed by geodesics.*

DISTORTION, HOROCYCLES, QUASIGEODESICS, SHADOWS.

Given a subset Y in a Riemannian manifold X, the *induced path metric* $dist_Y$ in Y is defined by the infima of the lengths of curves in Y between pairs of points in Y, where this length is measured with respect to the Riemannian metric in $X \supset Y$. This $dist_Y$ is greater than the distance function $dist_X$ restricted to Y

[6]The sup-norms and the sup-metics in general are category theoretically preferable as they pass from metric spaces Y to spaces of maps $X \to Y$ for all *sets* X.

[7]In dynamics, shadowing is usually stated in an equivalently form for *quasiorbits* rather than quasimorphisms, i.e. for $X = \mathbb{Z}$ and $S : x \mapsto x + 1$.

[8]See section 2.3 for basics on negative curvature for Riemannian and non-Riemannian spaces.

(the distance $dist_Y$ may be easily infinite) and the ratio of the two is called *the distortion* of Y in X,

$$distor(y_1, y_2) = \frac{dist_Y(y_1, y_2)}{dist_X(y_1, y_2)}, \ y_1, y_2 \in Y.$$

Thus, we regard a $Y \subset X$ *undistorted* if $distor(y_1, y_2) = 1$ (with the agreement $distor(y, y) = 1$). For instance distance minimizing geodesics as well as geodesically convex subsets in X are undistorted.

The *l-distortion*, denoted $distor_l = distor_l(y_1, y_2)$, is the restriction of the distortion, that is a function on $Y \times Y$, to the subset of the pairs (y_1, y_2), where $dist_Y(y_1, y_2) = l$.

Horocycles in $\mathbf{H}^2$ are curves with curvatures one.[9] They are *equidistorted* in $\mathbf{H}^2$: the distortion function $distor_l$ of a horocycle depends only on l. We denote it by $dishor(l)$ and observe that it is asymptotic to $l/2\log l$ for $l \to \infty$,

$$\frac{dishor(l) \cdot 2\log l}{l} \to 1 \text{ for } l \to \infty.$$

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A curve C in a Riemannian manifold is called a (continuous) *quasigeodesic* if it has bounded distortion,

$$\sup_{c_1, c_2 \in C} distor(c_1, c_2) < \infty.$$

The argument by Morse (and of Efremovich-Tichomirova for this matter) applies to all *complete simply connected manifolds X with sectional curvatures bounded from above by a negative constant, say by -1 to save the notation,* and yields the following.

[□□□] *If a curve C in X that is parametrized by a length preserving map $\mathbb{R} \to X$ is strictly less distorted on some scale $l_0 > 0$ than horocycles in $\mathbf{H}^2$, namely, if*

$$\sup_{l_0 \le l \le 2l_0} (distor_l(c_1, c_2) - dishor(l)) \le -\varepsilon < 0,$$

then C is shadowed by a unique geodesic $G \subset X$ which means that there is a bijection $G \leftrightarrow C(=\mathbb{R})$, such that the corresponding points $g \leftrightarrow c$ satisfy

$$dist_X(g, c) \le D < \infty.$$

Moreover the geodesic shadow $G = G(C)$ is continuous in C.

(The same is true for $l_0 = 0$ with the curvature in the place of distortion.)

Proof. Let $I_L \subset C$ be a segment of length $L > 10l_0$, let $G_L \subset X$ be the geodesic segment between the ends of I_L and let

$$d(c) = dist(c, G_L), \ c \in I_L$$

[9]This definition applies only to $\mathbf{H}^2$ with curvature -1, while in the case of general metric spaces X, one defines *horospheres* in X as the boundaries of *horoballs* that are increasing unions of (non-concentric!) R-balls with radii $R \to \infty$ in X, see 2.2 and 2.13. For instance, hyperplanes in $\mathbb{R}^n$ (but not in $\mathbf{H}^n$) are horospheres.

If $D = \max_{c \in I_L} d(c) > 10 l_0$, then, by continuity of the function $d(c)$, there exists a subsegment $I_l \subset I_L$ of length $l \in [l_0, 2l_0]$, such that the distance function $d(c)$ on I_l satisfies:

• $d(c) \geq D/2$, $c \in I_l$,

• $d(c)$ assumes its minimum, say $D_0 \geq D/2$ at the two ends of I_l;

thus, $d(c) \geq D_0 \geq D/2$, $c \in I_l$.

(One can not guaranty the existence of such I_l for *individual* l, or even to replace 2 by $2 - \varepsilon$, unless $d(c)$ has a *unique* maximum point in I_L; however, [□□□] may reman valid with $distor_l$ instead of $\sup_{l_0 \leq l \leq 2l_0} distor_l$.)

Since the curvature of X is ≤ 0, the normal projection[10] of I_l to the hypersurface $H_{D_0} \subset X$, where $dist(x, G_L) = D_0$ is *convex* and the normal projection from I_L to H_{D_0} is distance decreasing, while the condition curvature ≤ -1 implies that

> *the l-distortions of the hypersurfaces $H_{D_0} \subset X$ are bounded from below, up to an error $\varepsilon \underset{D_0 \to \infty}{\to} 0$, by $dishor(l)$, that is the l-distortion of horocycles in the hyperbolic plane $\mathbf{H}^2$ with curvature -1.*

Therefore, the inequality ◖ implies a bound on the maximal distance D from $c \in I_L$ to G_L independently of $I_L \subset C$.

Now the geodesic G that shadows the curve C comes with $L \to \infty$,

$$G = G(C) = \lim_{L \to \infty} G_L,$$

where the proof of the existence and uniqueness of this (obviously defined) limit and of continuous dependence of the resulting G on C is straightforward.[11]

1.3 Negative Curvature, Geodesic flows and Geodesic Universality.

Given a Riemannian manifold, let $G(V)$ denote the space of naturally parametrized geodesics in V that are locally isometric maps $g : \mathbb{R} \to V$ and let $p_V : G(V) \to V$ be the evaluation at 0 map, $g \mapsto g(0) \in V$. The natural action of $\mathbb{R}$ on $G(V)$ is called the *geodesic flow* with the orbits being geodecis in V. (If V is C^2-smooth, then $G(V)$ equals the unit tangent bundle $UT(V) \to V$ and the geodesic flow is generated by the obvious vector field in $UT(V)$.)

Denote by $G_\sharp(V) \subset G(V)$ the (sub)set of the maps $g : \mathbb{R} \to V$ the lifts of which to the universal covering $\tilde{V}$ are *isometric* (also called *distance minimizing*) *geodesic.*

Notice that if V is a complete manifold with *non-positive curvatures,* e.g. a hyperbolic manifold $V = \mathbf{H}^n/\Gamma$ for a discrete isometry group Γ ($= \pi_1(V)$) of the hyperbolic $\mathbf{H}^n$, then $G_\sharp(V) = G(V)$.

The following theorem is (essentially) proved (but formulated differently) by Efremovich and Tichomirova in their 1963 paper [37]

✯→ E.T. Geodesic Universality Theorem. Let V, W be compact Riemannian manifolds where V has negative sectional curvatures and le $\phi : \pi_1(W, w_0) \to \pi_1(V, v_0)$ be an isomorphism between their fundamental groups.

[10] The *normal projection* to $Y \subset X$ sends $x \in X$ to the *nearest* point $y \in Y$.

[11] This argument extends to all geodesic metric spaces with *uniformly strictly convex* distance functions, see 2.5.

Then there exists a continuous map $F^{\to}: G_{\sharp}(W) \to G(V)$ with the following two properties.

(1) *The composed maps* $F^{\to} \circ g : \mathbb{R} \to V$ *send* $\mathbb{R}$ *to geodesics in* W *where the lifts of these maps to the universal covering* $\tilde{V}$ *are homeomorphisms from* $\mathbb{R}$ *onto (hyperbolic) geodesics in* $\tilde{V}$

(2) *There is a (unique up to homotopy) continuous map* $F: W \to V$, $F(w_0) = v_0$, *that induces the isomorphism* ϕ *and such that the composed maps* $F \circ p_W : G_{\sharp}(W) \to V$ *and* $p_V \circ F^{\to} : G_{\sharp}(W) \to V$ *are homotopic.*

Moreover,

the map $F^{\to}$ *is unique up to isotopies of the unite tangent bundle* $UT(V) = G(V)$ *that preserve geodesics (orbits) in* $UT(V)$.

Proof. Since V has negative curvatures, its universal covering $\tilde{V}$ is *contractible*; therefore, there exists a continuous Γ-equivariant map $\tilde{F}_0 : \tilde{W} \to \tilde{V}$, $\Gamma = p_1(V) = \pi_1(W)$, between the universal coverings of our manifolds, where, due to compactness of $W = \tilde{W}/\Gamma$, this map sends isometric geodesic from $\tilde{W}$ to quasigeodesics C in $\tilde{V}$.[12]

Compose $\tilde{F}_0$ on isometric geodesics $G_{\sharp} \subset \tilde{W}$ with the normal projections of the corresponding quasigeodesics C in $\tilde{V}$ to their geodesic shadows $G(C) \subset \tilde{W}$, and denote by

$$\tilde{F}_0^{\to} : G_{\sharp}(\tilde{W}) \to G(\tilde{V}),$$

the resulting Γ-equivarinat map that is a collection of maps, say $\{f_0^{\to}\}$, that send isometric geodesics $G_{\sharp}$ from $\tilde{V}$, to geodesics G in $\tilde{W}$.

These $f_0^{\to}$, albeit non-injective, are *Lipschitz* and *strictly monotone increasing on the large scale*:

$$C^{-1}(g_1 - g_2) \le f_0^{\to}(g_1) - f_0^{\to}(g_2) \le C(g_1 - g_2),\ \ g_1, g_2 \in G_{\sharp},$$

for some constant $C = C(F_0) \ge 1$ and all $r_1, r_2 \in G_{\sharp}$, such that $|r_1 - r_2| \ge D$ for some constant $D = D(F_0) \ge 0$ and where the formula is understood with the length parametrizations of geodesics $G^{\sharp} \subset \tilde{W}$ and $G \subset \tilde{V}$ (g denotes this length parameter in $G_{\sharp} \subset \tilde{V}$) that identifies them with $\mathbb{R}$.[13]

There are several simple ways to *canonically*, hence *coherently*, modify such maps $f_0^{\to} : \mathbb{R} \to \mathbb{R}$ in order to make them injective.

For instance, the injectivity is achieved with the operator $f_0^{\to} \mapsto f^{\to}$ that is the *convolution* of $f_0^{\to}$ with the *uniform probability measure on the segment* $[0, \Delta = 10CD]$; this operator, applied to all maps $f_0^{\to}$, transforms $\tilde{F}_0^{\to}$ to an *equivariant* map $\tilde{F}^{\to} : G(V) \to G(W)$ that descends to the required $F^{\to} : G(V) \to G(W)$.

Geodesic Rigidity Corollary.[14] Let V and W be compact Riemannian manifolds with negative sectional curvatures.

★$^{\leftrightarrow}$ *If* V *and* W *have isomophic fundamental groups, then there is a homeomorphism between their unit tangent bundles that sends the geodesics from* V *– that are the orbits of the geodesic flow in* $G(V) = UT(V)$ *– to geodesics in* W.

This rigidity, and even "mere" *topological stability* of the geodesic flow of V, i.e. where $W = V$ and the Riemannian W-metric is a *small* C^2-perturbation

[12]Strictly speaking, F_0 must be chosen smooth, or at least Lipschitz, for this purpose.

[13]This identification is unique *only* up to translations in $\mathbb{R}$ but this causes no problem.

[14]This was suggested to the author by William Veech in 1976 as a counterpart to the *Mostow rigidity theorem* for manifolds with *variable* negative curvatures.

of the V-metric, is as astounding as this property of hyperbolic toral automorphisms. Probabaly, no-one (except René Thom?) believed until 1960 that stability can be compatible with ergodicity. But when Smale conjectured structural stability of hyperbolic systems in 1961, this was proved by Anosov next year.

It is hard to say what is more beautiful: delightful implausibility of the statement or staggering simplicity of the proof.

1.4 Symmetric Spaces and Arithmetic Groups.

The most significant and intriguing aspect of negative curvature is the *existence* of *compact* manifolds and also of singular spaces V with curvatures < 0 of dimension n, for *all* n, where the central issue is the structure of the fundamental groups of these V.

At the first glance, on may expect a multitude of such manifolds, say with *constant sectional curvatures* that are quotients of the hyperbolic spaces, $V = \mathbf{H}^n/\Gamma$. Indeed, the hyperbolic spaces $\mathbf{H}^n$ are, along with the Euclidean $\mathbb{R}^n$, distinguished, besides by containing *infinite lines*, (isometric copies of $\mathbb{R} \subset \mathbf{H}^n$) by being *highly symmetric*, that is satisfying the following (almost) Euclidean

Congruence Axiom: Full Metric Homogeneity.[15] *Isometries*, i. e. bijective distance preserving maps, between (finite if you wish) subsets in $\mathbf{H}^n$, extends to isometries of $\mathbf{H}^n$,

$$\mathbf{H}^n \supset A_1 \underset{isom}{\leftrightarrow} A_2 \subset \mathbf{H}^n \text{ extend to } \mathbf{H}^n \underset{isom}{\leftrightarrow} \mathbf{H}^n.$$

But it is not at all obvious[16] that these $\mathbf{H}^n$ admit *free discrete* isometry groups Γ with *bounded* fundamental domains. There is no available *geometric* source of such groups for large n – all known (unknown?) constructions, however simple, rely on *arithmetics of number fields.*

It has not been realised by most (all?) differential geometers until mid-1960s that they have had no inkling of an idea if their beloved compact n-dimensional manifolds with negative curvatures existed at all for large n – even the existence of these for $n = 3$ was not a common knowledge.

The light came in 1963 from the study of arithmetic group by Armand Borel [19] who proved the following

❂ Compact Form Theorem. *All simply connected*[17] *Riemannan symmetric spaces X, e.g. $X = \mathbf{H}^n$, admit compact forms, that are compact manifolds V locally isometric to X; hence having their universal coverings $\tilde{V}$ isometric to X.*

$\sqrt{2}$-Example. The group Γ of Q-*orthogonal* (i.e. preserving Q) linear transformations of $\mathbb{R}^{n+1}$ for the quadratic form $Q = \sum_1^n x_i^2 - \sqrt{2}x_{n+1}^2$ defined

[15] If you want to allow $\mathbf{H}^\infty$ on the one hand and/or to rule out *Uryson-Fraïssé universal spaces* on the other hand you need to modify this condition

[16] The very existence of these (barely) non-Euclidean spaces $\mathbf{H}^n$ themselves is not "obvious" either: geometers have been struggling for 2000 years trying to prove the "obvious" non-existence of these spaces, where the stumbling block was confusion in formulating the existence/non-existence alternative in the absence of a mathematical concept of "*space*".

[17] Real projective spaces and flat tori are instances of non-simply connected symmetric spaces, but the existence of compact forms is trivial for them.

by $(n+1)\times(n+1)$ matrices with entries $\{a+b\sqrt{2}\}$, where a and b are *integers*, is *cocompact* in the group $O_{\mathbb{R}}(Q)(=O(n,1))$ of Q-orthogonal transformations with *real* coefficients, that is the quotient space $O_{\mathbb{R}}(Q)/\Gamma$ is *compact.* Thus, the isometric action of Γ on $\mathbf{H}^n$ represented by a connected component of the -1-sphere $\{x\}_{Q(x)=-1}\subset\mathbb{R}^{n+1}$ admits a *bounded* fundamental domain.

This Γ, by a theorem of Selberg, admits a finite index subgroup Γ' with *no torsion*,[18] the action of which on $\mathbf{H}^n$, besides being discrete, is *free.* Hence,

$V=\mathbf{H}^n/\Gamma'$ *is a compact manifold with constant negative curvature.*[19]

About the Proof. A relatively simple geometric argument shows – this goes back to a 1937 paper by B. A. Venkov [115] – that

if Q *is an indefinite quadratic form with coefficients in a totally real number field* K*, then the group* $\Gamma=\Gamma(Q)$ *of integer points from* K *of* Q*-orthogonal transformations is cocompact in* $O_{\mathbb{R}}(Q)$ *if and only if* Q *admits a non-trivial zero in* K.

Now arithmetic enters.

❆ Since $\sqrt{2}\leftrightarrow-\sqrt{2}$ extends to a *Galois automorphism* of the field $K=\mathbb{Q}(\sqrt{2})$, the absence of non-trivial zeros in K for the *positive* form $\sum_1^n x_i^2+\sqrt{2}x_{n+1}^2$ implies that for our $Q=\sum_1^n x_i^2-\sqrt{2}x_{n+1}^2$.

(Albeit the field $\mathbb{Q}(\sqrt{2})$ is contained $\mathbb{R}$, the Galois action can not be seen in the $\mathbb{R}$-geometry since it is discontinuous on its domain of definition that is $\mathbb{Q}(\sqrt{2})\subset\mathbb{R}$.)

Despite the explicit description, the algebraic structure of the groups $\Gamma(Q)$ clouded in mystery. For instance, one has a poor idea of what are (co)homologies of these groups and of their subgroups Γ' of finite index e.g. of the *congruence subgroups.*

What one knows, for instance, is that the rough asymptotics of $rank_{\mathbb{F}}H^i(\Gamma';\mathbb{F})$ as $index(\Gamma')\to\infty$ for coefficient fields $\mathbb{F}$ of zero characteristic.

(If Γ' has no torsion, than $H^i(\Gamma';\mathbb{F})=H^i(\mathbf{H}^n/\Gamma';\mathbb{F})$ for all fields $\mathbb{F}$ and the Euler characteristic $\chi(\Gamma')=C_n vol(\mathbf{H}^n/\Gamma')$ where $C_n\neq0$ for even n.)

Namely Atiyah's 1976 L_2*-index theorem* [9] in conjunction with *Kazhdans approximation of* L_2*-forms* on infinite coverings (1971) show that

$$\frac{rank_{\mathbb{F}}H^i(\Gamma';\mathbb{F})}{index(\Gamma')}\to 0 \text{ for } index(\Gamma')\to\infty$$

unless n is even and $i=n/2$, while this limit for $i=n/2$ and n even, equals to the absolute value (virtual) of the (non-zero!) Euler characteristic of Γ.

Questions. (a) Do similar asymptotics hold for *finite fields* $\mathbb{F}$? (See [11], [103] for some results in this direction.)

(b) What part of the homology of $\mathbf{H}^n/\Gamma'$ is representable by *totally geodesic submanifolds in* $\mathbf{H}^n/\Gamma'$?

(c) How much of the cohomology of the manifolds $\mathbf{H}^n/\Gamma'$, $m\leq n/2$, is generated by the *cup-products of the one dimensional classes*, in particular of those that are *Poincaré dual to totally geodesic hypersurfaces*?

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[18] One uses for this purpose the matrices with the entries divisible by a few first primes in the ring $\{a+b\sqrt{2}\}$.

[19] I do not know if this example was known prior to Borel's 1963 paper.

(d) What are possible homomorphisms $G : \Gamma_1 \to \Gamma_2$? What kind of homomorphism can such a G induce between the (co)homologies of Γ_1 and Γ_2?

For instance, how common (if these exisist at all) maps of *non-zero degrees* $V_1 \to V_2$, for $V_{1,2} = \mathbf{H}^n/\Gamma_{1,2}$, where $n \geq 4$, that are not homotopic to covering maps?

(e) What are finitely presented subgroups $\Pi \subset \Gamma$?

For instance, which compact manifolds W with (non-constant!) negative curvatures have fundamental groups isomorphic to these Π. (These W *can not be Kähler* by Siu's theorem stated below.)

Two Point Homogeneous Spaces

Besides $\mathbf{H}^n$, call it here *real hyperbolic space* $\mathbf{H}^n_{\mathbb{R}}$, there are three other families of remarkable Riemannian symmetric spaces with negative curvatures that are characterised as metric space by the following property (+ something obvious):

the groups $iso_x(X)$ *of isometries fixing* $x \in X$ *are transitive on the* R*-spheres* $S_x(R) \subset X$ *for all* $x \in X$ *and* $R \geq 0$.

I. *Complex Hyperbolic Spaces* $\mathbf{H}^n_{\mathbb{C}}$. These have topological dimensions $2n$ and they come with a natural complex analytic structures making them biholomorphic to the unit balls $B^{2n} \subset \mathbb{C}^n$. The Riemannian metrics in $\mathbf{H}^n_{\mathbb{C}}$ equal the unique (up to scaling) Riemannian metrics in B^{2n} that are invariant under biholomorphic transformations of B^{2n}.

The compact quotient manifolds $V = \mathbf{H}^n_{\mathbb{C}}/\Gamma (= B^{2n}/\Gamma)$, for discrete (biholomorphic) isometry groups Γ of $\mathbf{H}^n_{\mathbb{C}}$, which are plentiful by Borel's theorem, are all *projective algebraic* by *Kodaira's theorem* and they all enjoy the following holomorphic version of geodesic universality.

★ Holomorphic Universality of $V = \mathbf{H}^n_{\mathbb{C}}/\Gamma$, (Siu 1980) [107]. Let $f_0 : W \to V$ be continuous map where W is a compact *Kähler manifold*, e.g. a projective complex algebraic one.

Then either f_0 *is homotopic to a unique holomorphic map* $W \to V$ *or, to a map that factors through a map from* W *to a Riemann surface,* $W \to S \to V$.

The main applications of this theorem include the following.

- *Non-existence* of Kähler manifolds W with certain fundamental groups, e.g. isomorphic to subgroups $\Pi \subset \Gamma = \pi_1(B^n/\Gamma)$ that have *odd* homological dimensions$< dim_{\mathbb{R}}(W)$.[20]
- *Upper bounds* on the numbers of *deformations* of complex structures of certain Kähler manifolds associated to $V = B^n/\Gamma$, e.g the rigidity of the complex structure of V itself for $n \geq 2$.

On the other hand, there are no(?) visible examples of Kähler manifolds W and of continuous maps $f_0 : W \to V$ that are not just deformations of holomorphic maps used in the constructions of such W to start with.[21]

II. *Quaternionic Hyperbolic Spaces*, $\mathbf{H}^n_{\mathbb{H}}$ of dimensions $4n$.

These, except for $\mathbf{H}^1_{\mathbb{H}} = \mathbf{H}^4_{\mathbb{R}}$, are similar to symmetric spaces of $\mathbb{R}$-rank≥ 2 in their "rigidity". For instance, *cocompact* discrete isometry groups Γ of $\mathbf{H}^n_{\mathbb{H}}$ for

[20] I am not certain what are examples of such Π apart of fundametal groups of real hyperbolic manifolds and their subgroups.

[21] See [58] for a discussion of Siu theorem from the hyperbolic perspective similar to the present one.

$n \geq 2$ ($vol(\mathbf{H}^n_{\mathbb{H}}) < \infty$ suffices) have *Kazhdan's property* T (Kostant, 1969):

✦ *every isometric (allowing affine rather than only linear) action of such a* Γ *on the Hilbert space* $\mathbb{R}^\infty$ (*as well as on* $\mathbf{H}^\infty_{\mathbb{R}}$ *and on* $\mathbf{H}^\infty_{\mathbb{C}}$) *has a fixed point.*

Question. Do cocompact isometry groups Γ of $\mathbf{H}^n_{\mathbb{H}}$ for $n \geq 2$ admit "non obvious" factor groups Π? [22]

For instance, if such a Π serves as the fundamental group of a compact *aspherical manifold*[23] W, (e.g of a W that admits a Riemannian metric with non-positive sectional curvatures) is then the implied *epimorphism* $\Gamma \to \Pi$ necessarily an *isomorphism*?

III. *the Cayley (octonic) plane* $\mathbf{H}^2_{\mathbb{O}}$ of dimension 16.

The *known* geometric properties of $\mathbf{H}^2_{\mathbb{O}}$ are similar to these of $\mathbf{H}^n_{\mathbb{H}}$, but we do not know what the unknown ones are.

1.5 Small Cancellation

Arithmetic groups fall down from the sky but you can see something less spectacular but still interesting right under your feet.

Let W be a set of words in a formal language, let $\mathcal{T}$ be a semigroup of transformations acting on W and let $T \subset \mathcal{T}$ be a generating subset of what we call *elementary transformations* of words w in this language.

It is, in general, a fruitless task to describe the orbits of this action, e.g. to decide if a given word w_0 can be transformed to w_1 by consecutively applying transformations $\tau \in T$ since, typically, a T*-path* $w_0 \rightsquigarrow w_1$ needs to cross insurmountably high Gödel-Turing mountains composed of words that are incomparably longer than w_0 and w_1. But a satisfactory answer to the following question may lead to something interesting.

QUESTION. *What are* $(W, \mathcal{T})$*, where there is no such mountains and where there always exists a downhill path* $w_0 \rightsquigarrow w_1$*, assuming* w_1 *can be reached from* w_0 *at all?*

For instance let W be the set of words in a free groups, say in F_2 generated by a and b where the words are written in a, b, a^{-1}, b^{-1} and let $R \subset W$ be a subset of words called *relations*.

Let $\mathcal{T}$ be the group generated by a set T that consists of:

• elementary conjugations: $w \mapsto cwc^{-1}$, $c \in \{a, b, a^1, b^{-1}\}$;

• elementary cancellations: $ucc^{-1}v \mapsto uv$, $c \in \{a, b, a^1, b^{-1}\}$;

• multiplication from the left and from the right the relation words: $w \mapsto rw$ and $w \mapsto wr$, $r \in R$.

T-Paths between these words are faithfully represented by *homotopies* between *free loops*,[24] that represent words in the standard 2D-cell-complex K with the fundamental group $F_2/\mathcal{T}$:

K is obtained by attaching 2-cells D^2_r, $r \in R$, to the directed (a, b)-labeled graph $K^1 = a\infty b$, by continuous maps denoted $r^{\rightarrow}$ from the (circular) boundary of the disck D^2 to K^1,

$$r^{\rightarrow} : S^1 \to K^1, S^1 = \partial D^2 \subset D^2,$$

[22] These are called *underlattices* in section 7.A.IV of [51].

[23] A topological space is called *aspherical* if its universal covering is *contractible.*

[24] "Freedom" signifies that we "forget" the base points in the loops.

where r denote the maps corresponding to the words $r \in R$.

More generally, let K^1 be an arbitrary graph with its edges being asigned with the length structures and let $R^{\rightarrow}$ be a collection of closed curves in K^1 that are continuous maps $r^{\rightarrow} : S^1 \to K^1$.

Assume that all these maps are non-contractible and *length minimizing*, that means in the present case they are *locally injective*, where, observe, every curve is homotopic to a length minimizing one (that is constant in the contractible case).

Call a map from a real segment $I \subset \mathbb{R}$ to K^1, say $p^{\rightarrow} : I \to K^1$, *a piece of an* $r^{\rightarrow} : S^1 \to K^1$ if $p^{\rightarrow}$ factors through an imbedding $[x, y] \hookrightarrow S^1$,

$$I \hookrightarrow S^1 \overset{r^{\rightarrow}}{\to} K^1, \text{ where } r^{\rightarrow} \circ \hookrightarrow = p^{\rightarrow}.$$

$\frac{1}{k}$-CANCELLATION.

Say that $R^{\rightarrow}$ *satisfies* $[\leq \frac{1}{k}]$-*condition* if the lengths of the common pieces $p^{\rightarrow}$ of curves $r_1^{\rightarrow}, r_2^{\rightarrow} \in R^{\rightarrow}$ satisfy

$$length(p^{\rightarrow}) \leq \frac{1}{k} \min(length(r_1^{\rightarrow}), length(r_2^{\rightarrow}))$$

for all $r_1^{\rightarrow}, r_2^{\rightarrow} \in R^{\rightarrow}$, where the lengths of the curves $r^{\rightarrow}$ and $p^{\rightarrow}$ are those inherited from the graph K^1.

CRITICALITY OF $k = 6$.

Given a graph K^1 and a collection of continuous maps, $R^{\rightarrow} = \{r^{\rightarrow} : S^1 \to K^1\}$, let $K^2 = K^2(R^{\rightarrow})$ be the cell complex with the 2-cells attached to K^1 via the maps from $R^{\rightarrow}$,

$$D^2 \supset \partial D^2 = S^1 \overset{r^{\rightarrow}}{\to} K^1, \; r^{\rightarrow} \in R^{\rightarrow}.$$

It is (more or less) obvious that if $k < 6$, then the $[\leq \frac{1}{k}]$-condition *does not impose* any constraint on the fundamental group of K^2: every finitely presented group Γ can be realised as $\pi_1(K^2(R^{\rightarrow})$ for some $R^{\rightarrow} = R^{\rightarrow}(\Gamma)$.

What is more amusing, albeit (almost) equally simple – this was shown (in different terms) by Tartakovskii (1947 - 1949) [111] who introduced the concept of small cancellation – is that there is a natural metric in K^2 that extends the lenght metric in $K^1 \subset K^2$ (we shall describe it in Chapter 3), such that if $k > 6$ then

every curve $w_0^{\rightarrow} : S^1 \to K^1$ *that is contractible in* $K^2 \supset K^1$ *admits a homotopy*

$$w_t^{\rightarrow} : S^1 \to K^2 \supset K^1,$$

such that

$$w_1^{\rightarrow} \text{ maps } S^1 \text{ to } K^1 \subset K^2,$$

$$length(w_t^{\rightarrow}) \leq length(w_0^{\rightarrow}), \; 0 \leq t \leq 1,$$

and

$$length(w_1^{\rightarrow}) < length(w_0^{\rightarrow}).$$

Originally, the small cancellation theory was concerned with the solutions of the *word and the conjugacy problems* that was achieved by interpreting (combinatorial) curve shortenings as *algorithms* that solve these problems in groups. [25]

In the mid 1960s, the focus of small cancellation techniques shifted toward constructions of groups with interesting and sometimes unexpected properties, e.g. of *infinite torsion groups* by Novikov-Adian (1968).

Currently, small cancellations make a part of the hyperbolic (group) theory.

1.6 Locally Split Anosov-Smale Hyperbolic Systems.

Following Smale and Anosov, hyperbolicity of diffeomorphisms $f : X \to X$ is defined in terms of the corresponding properties of the differentials $Df : T(X) \to T(X)$ as follows.

Let $L \to X$ be a vector bundle with the (topological $\mathbb{R}$-linear spaces) fibers denoted $L_x \in L$, $x \in X$, and let $F : L \to L$ be a continuous fiberwise linear map. Denote the background self map of X by $\underline{F} : X \to X$ and write, accordingly, $F = \{F_x : L_x \to L_{\underline{F}(x)}\}_{x \in X}$.

Hyperbolicity. Let the linear spaces L_x be endowed with norms, and say F is (uniformly) *hyperbolic* if there is an $\varepsilon > 0$ and a splitting

$$L = L_{contr} \oplus L_{exp},$$

where L_{contr} and L_{exp} are *F-invarinat subbundles* in L, such that

$$||F^i(l)|| \le const(1-\varepsilon)^i ||l|| \text{ for } l \in L_{contr}$$

and

$$||F^i(l)|| \ge const'(1+\varepsilon)^i ||l|| \text{ for } l \in L_{exp}$$

for positive constants *cont* and $const'$ and all $i = 1, 2, 3, ...$

Observe that hyperbolicity implies *uniform expansiveness* of F that is the existence of an integer N such that

$$\max(||F^N(l)||, ||F^{-N}(l)||) \ge 2||l||, \text{ for all vectors } l \in L.$$

Also observe that L_{contr} and L_{contr} can be effectively defined as subbundles of L in two complementary ways.

A. *Asymptotic Exponential Contraction:*

$$l \in L_{contr} \Leftrightarrow \limsup_{n\to\infty} \frac{1}{i} log||F^i(l)|| \le \nu < 0$$

and

$$l \in L_{exp} \Leftrightarrow \limsup_{n\to\infty} \frac{1}{i} log||F^{-i}(l)|| \le \nu < 0$$

(where, in the case where F is non invertible, "$F^{-i}(l)$" reads as *all l' in the pullback $F^{-i}(l) \subset L$.)* with hyperbolicity coming along with the condition

$$L_{contr} + L_{exp} = L.$$

[25] An algoritmic solution of, the, say word, problem in the fundamental group of a compact length space K – that was already obvious to Dehn – needs a very modest "curve shortening" property: *every closed curve in K of length l must admit a strict shortening $w_0^{\to} \rightsquigarrow w_1^{\to}$ by a homotopy of curves of length $\le L(l)$ for a recursive function $L(l)$.*

B. *Cocontraction.* The subbundles $L_{contr}, L_{exp} \subset L$, seen as sections of the Grassmann bundle $G = G(L) \to X$ of linear subspaces in the fibers L_x, appear as *fixed points* of the obvious action $F_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ induced by $F : L \to L$ on the space $\mathcal{G}$ of sections $X \to G$.

This is a tautology. What is significant is that the fixed point of $F_{\mathcal{G}}$ in $\mathcal{G}$ that represents the subbundle L_{exp} is an *attractive* one.[26]

This trivially follows from the *exponential discrepancy* between the growths of the norms $||F^i(l)||$ for vectors l in L_{contr} and L_{exp}, that is the relation

$$\frac{||F^i(l_1)||}{||F^i(l_2)||} \geq const(1+\varepsilon)^i, \; i = 1, 2, 3, ...$$

for all unit vectors $l_1 \in L_{exp}$, $l_2 \in L_{contr}$.

Similarly, L_{contr} is an *attractive* fixed point for $F_{\mathcal{G}}^{-1}$ in $\mathcal{G}$ (understood in an obvious sense if F is non-invertible).

It follows from **A** that, say if X is compact and the fibers L_x are finite dimensional, then

hyperbolicity does not depend on the choice of norms in L_x;

while **B** implies that

small perturbations $F_\varepsilon : L \to L$ *of hyperbolic* $F : L \to L$ *remain hyperbolic.*

(All of this can be transparently expressed in purely linear terms, since F *is hyperbolic if and only if some power* $\mathcal{F}^N$ *of the associated linear map* $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ *on the Banach space* $\mathcal{L}$ *of sections* $X \to L$ *with the sup-norm is hyperbolic in the sens of 1.1* with the latter obviously accommodated for non-invertible F.)

Anosov Actions and Foliations A C^1-smooth action of the group $\mathbb{Z}$ on a smooth Riemannian manifold X is called *Anosov* if the differential $Df_{\mathbf{1}} : T(X) \to T(X)$ of the the diffeomorthism $f_{\mathbf{1}} : X \to X$ that corresponds to $1 \in \mathbb{Z}$ is hyperbolic, where the corresponding subbundles that split the tangent bundle $T(X)$ are denoted $T_{contr} \subset T(X)$ and $T_{exp} \subset T(X)$

A C^1-smooth action of $\mathbb{R}$ on X with no fixed point is called *Anosov* if there is a codimension one subbundle $T_{hyp} \subset T(X)$ (which is necessarily transversal to the $\mathbb{R}$-orbits of the action) such that the action of $Df_{\mathbf{1}}$, $1 \in \mathbb{R}$, on T_{hyp} is hyperbolic .

Dynamical Expansiveness. Uniform expansiveness[27] of $Df_{\mathbf{1}} : T(X) \to T(X)$ implies *expansivness* of Anosov $\mathbb{Z}$-actions, where a family (e.g. a group) of transformations $f_i : X \to X$, $i \in I$, of a metric space is called *(metrically) expansive* if

$$\sup_{i \in I} dist(f_i(x), f_i(y)) \geq \varepsilon$$

for all x and $y \neq x$ in X and some positive function $\varepsilon = \varepsilon(x) > 0$.

If X is *compact* and $f_i = f^i$, $i \in I = \mathbb{Z}$, are powers of a *homeomorphism* $f : X \to X$, then expansiveness of $\{f_i\}$ (obviously) equivalent to *exponential expansiveness*:

$$\max\left(dist(f^i(x), f^i(y)), dist(f^i(x), f^i(y))\right) \geq \min(\varepsilon, C(dist(x,y))(1+\varepsilon')^i)$$

[26]A fixed point of a transformation f of a metric space is called *attractive* if a power f^N of f is *strictly contractive* in some neighbourhood of this point.

[27]If X is compact uniformity is automatic.

for all $x, y \in X$, some $\varepsilon, \varepsilon' > 0$ and some positive monotone increasing function $C(d) > 0$. In fact, uniform expansiveness of the differential $Df : T(X) \to T(X)$ (obviously) implies *uniform* exponetial expansiveness of f that is the inequality

$$\max\left(dist(f^i(x), f^i(y)), dist(f^i(x), f^i(y))\right) \geq \min(\varepsilon, C(1+\varepsilon')^i)$$

with $C > 0$ independent of x and y.

Similarly, Anosov $\mathbb{R}$-actions are expansive "transversally to their orbits":

$$\max\left(dist(f^i(x), f^i(y)), dist(f^i(x), f^i(y))\right) \geq \varepsilon > 0,$$

for all x and y which *do not lie on the same* $\mathbb{R}$ *orbit*, where, as in the $\mathbb{Z}$-action case, Anosov $\mathbb{R}$-actions satisfy the stronger (and equally obvious) uniform exponential version of this inequality.[28]

The Anosov sub-bundles $T_{contr} \subset T(X)$ *and* $T_{exp} \subset T(X)$ *are integrable*: they serve as tangent bundle of foliations call them S_{contr} and S_{exp} that are partitions of X into C^1-smooth submanifolds, called *contracting* (*stable*) and *expanding* (*unstable*) *leaves* of dimensions equal the ranks of the bundles T_{contr} and T_{exp}.

These foliations can be dynamically (and obviously) reconstructed in two ways.

A$^\square$. The contracting leaves equal the equivalence classes of the relation

$$x \sim y \Leftrightarrow \lim_{i\to\infty} dist(f^i_{\mathbf{1}}(x), f^i_{\mathbf{1}}(y)) \to 0$$

and the expanding ones equal such classes for the relation

$$x \sim y \Leftrightarrow \lim_{i\to\infty} dist(f^i_{-\mathbf{1}}(x), f^i_{-\mathbf{1}}(y)) \to 0.$$

B$^\square$. Given a selfmap f of a metric space X, define the following map $f_{[\delta]}$ on families of subsets

$$\{A_x \subset X\}_{x\in X},$$

where the subsets A_x are contained in the δ-balls $B_x(\delta) \subset X$ of radius δ around the points x and

$$f_{[\delta]}(A_x) = f(A_x) \cap B_{f(x)}(\delta).$$

If $f = f_{\mathbf{1}}$, $1 \in \mathbb{Z}$, is an Anosov diffeomorphism on a compact manifold X, then, for all small $\delta > 0$ and $0 < \varepsilon \leq \delta$,

the families of subsets in X,

$$\{f^i_{[\delta]}(B_x(\varepsilon)) \subset X\}_{x\in X}$$

converge for $i \to \infty$ *to* (the connected components of) *the expanding leaves through the points* $x \in X$ *within the balls* $B_x(\delta)$.

And if $f = f_{\mathbf{1}}$, $\mathbf{1} \in \mathbb{R}$, is the time one diffeomorphism of an Anosov *flow* f_t (no contraction or expansion in the $\mathbb{R}$-orbits directions), then

[28]Diffeomorphisms with uniformly expansive differentials are called *quasi-Anosov*. Quasi-Anosov actions on compact manifolds X where *the union of minimal invarint subsets is dense* in X are Anosov. On the other hand there are quasi-Anosov diffeomorphisms, e.g. of the connect sum on two 3-tori, which are not Anosov (see [102] for more information and references).

the expanding leaves in the balls $B_x(\delta)$ emerge as the double limits

$$\lim_{\varepsilon\to 0}\lim_{i\to\infty}\{f^i_{[\delta]}(B_x(\varepsilon))\}_{x\in X}.$$

The basic examples of Anosov diffeomorphism are

(a) *hyperbolic automorphisms f of the n-tori $X=\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$* where Anosov' foliations descend from linear splittings $L_{contr}\oplus L_{exp}$ for the corresponding linear self-maps of $L=\mathbb{R}^n$,

(b) *geodesic flows in the unit tangent bundles $UT(V)$ of complete (e.g. compact) manifolds V with sectional curvatures* $\le\kappa<0$, where the contracting and expanding leaves are *horospheres* are lifted from the universal covering $\tilde{V}$ ov V to the unit tangent bundle $UT(\tilde{V})\to\tilde{V}$ of $\tilde{V}$ and then projected to $UT(V)$).[29]

But it remains unclear how representative these examples are since the influence of hyperbolic dynamics on topology of the underlying spaces remains mainly unknown. For instance,

?1. All *presetly known* Anosov diffeomorphisms $f:X\to X$ are topologically equivalent to hyperbolic endomorphisms f of *infranilmanifolds*; thus, every such f descends from an automorphism $\tilde{f}:\tilde{X}\to\tilde{X}$ where $\tilde{X}$ is a connected nilpotent Lie group.

However, one can not even rule out (confirm?) the existence of Anosov diffeomorphisms on compact *simply connected* manifolds, such as $S^3\times S^3$ for instance.

On the other hand, Margulis proved that the fundamental groups of 3-manifolds which support Anosov's $\mathbb{R}$-actions have *exponential growth*;[30] this was extended in 1972 by Plante and Thurston, [95] to the flows where the contracting (or expanding) foliations are 1-*dimensional*.

LOCAL SHADOWING.

Let X and Y be $\mathbb{Z}$-spaces (i.e. continuously acted upon by the group $\mathbb{Z}$) where Y is endowed with a metric. Call a map $Q:X\to Y$ *an ε-quasi-morphism* if the generators of the $\mathbb{Z}$-actions in X and Y, denoted by $f:X\to X$ and $g:Y\to Y$, satisfy

$$dist_Y(Q\circ f(x),g\circ Q(x))\le\varepsilon \text{ for all } x\in X.$$

Say that f and the $\mathbb{Z}$-action generated by f amit a *the unique local shadowing* if there exists a positive $\varepsilon_0=\varepsilon_0(Y,f)$ and a function $\delta=\delta(\varepsilon)\underset{\varepsilon\to 0}{\to}0$, such that

every ε-quasi-morphism for $\varepsilon\le\varepsilon_0$ is δ-close to a unique $\mathbb{Z}$-morphism that is an $M:X\to Y$, such that $M\circ f(x)=g\circ M(x)$.

Observe that if Y is compact, then this shadowing property is independent of the metric in Y.

---

[29] *The horoshere* in the universal covering $\tilde{V}$ that passes through a unit vector $\tau=\tau_{\tilde{v}}\in UT_{\tilde{v}}(\tilde{V})$ equals the limit of the R-spheres, $R\to\infty$, with the centers at the points $\tilde{v}(\tau,R)\in\tilde{V}$ that lie on the geodesic ray in $\tilde{V}$ that issues from $\tilde{v}\in\tilde{V}$ in the direction τ and such that $dist(\tilde{v}(\tau,R),\tilde{v})=R$.

[30] Margulis's proof is written down in an appendix to a 1967 survey article by Anosov and Sinai [?].

If Y is a complete metric space acted by a homeomorphism g that admits an invariant (Cartesian) splitting,

$$Y = Y_{contr} \times Y_{exp} \text{ and } g = g_{contr} \times g_{exp},$$

as in 1.1 without even assuming any linearity,[31] then by the argument from 1.2 the transformation T satisfies the (now global) shadowing property for *all*, arbitrarily large(!), $\varepsilon > 0$.[32] And if *all small neighbourhoods* $U \subset Y$ admits (coherent in an obvious way) invariant *contracting* × *expanding* splittings, then the same argument, or rather its obvious translation to the ε-δ-language, delivers the local shadowing property.

Thus one arrives at

ANOSOV SHADOWING LEMMA. *Anosov $\mathbb{Z}$-actions admit unique local shadowings.*

Indeed, small neighbourhoods $U \subset Y$ split into products of contracting and expanding leaves within these U. [33]

Similarly, Anosov proves the local shadowing for his (locally split hyperbolic) $\mathbb{R}$-*actions* $g_t : Y \to Y$, $t \in \mathbb{R}$, where *the ε-quasi-morphism property of a* Q : $(X, f_t) \to (Y, g_t)$ means that

$$dist_Y(Q \circ f_t(x), g_t \circ Q(x)) \le \varepsilon \text{ for all } x \in X \text{ and } t \in [-1, 1],$$

while an $M : X \to Y$ that shadows Q, by no means unique anymore, is required to be an ε'-quasi-morhism that *homeomorphically* sends the f_t-orbits from X onto the g_t-orbits in Y.

Smoothness and Continuity. Historically, this lemma was used by Anosov to prove the following

structural stability of smooth locally split hyperbolic systems.

Theorem. Let Y be a compact C^1-smooth manifold, $g : Y \to Y$ be an Anosov C^1-diffeomorphism and $f : Y \to Y$ a C^1-diffeomorphism. (Here $X = Y$).

If f is sufficiently close to g in the C^1-topology, then there exists a homeomorphism $M : Y \to Y$, that is C^0-close to g *such that* $f \circ M = M \circ g$.

In fact, take the $\mathbb{Z}$-morphism M delivered by the shadowing lemma (we already know this M is unique) for M.

Since the map M is C^0-close to the identity map, it can not bring together two points unless they are close one to another and since the underlying space Y is a closed manifold the map $M : Y \to Y$, being continuous and homotopic to identity, is necessarily *surjective.*

Since the differential of g is uniformly expansive the same is (obviously) true for our f that is (sufficiently) C^1-close to g and since M equivariantly sends the orbits of f to orbits of g, no two points, even if they are close one to another, can be brought together by such an M either. This means M is injective and the proof of the structural stability of g is concluded. QED.

[31] This is why the direct sum sign "⊕" is replaced by that of the Cartesian product "×".

[32] To make this argument work one needs an *uniform* (e.g. bi-Lipschitz) *equivalence*, between the original metric in Y and the "sum" of metrics in Y_{contr} and Y_{exp}, that was automatic in the linear case.

[33] Here and below in this section, Anosiov transformations are denoted by g and manifolds supporting these actions by Y.

WARNING. No regularity condition on f and g can guaranty C^1-smoothness of M.

Isolation and Expansiveness. Expansiveness of a family $G = \{g\}$ of transformations $g : X \to X$ can be expressed in topological (metric free) terms that leads to a more general concept than the metric expansiveness for *non-compact* spaces X.

Namely G is called *(topologically) expansive* if the diagonal $X = X_\triangle \subset X \times X$ is *(dynamically) isolated* with respect to the family of the diagonal action $G_\triangle = \{g, g\}$ of the diagonal actions $(x, y) \mapsto (g(x)g(y))$, where, in general,

a closed G invariant subset $Y \subset X$ is called *(dynamically) isolated* or *locally maximal*, if there is a neighbourhood $U \subset X$ of Y such that

$$\bigcap_{g \in G} g^{-1}(U) = Y.$$

Isolation is a well behaved concept:

pullbacks of isolated subsets $Y \subset X$ under morphims, say $\phi : X' \to X$, of G-spaces are (obviously) isolated.

(If the spaces X and X' are *compact* and ϕ is onto, and then a subset $Y \subset X$ is isolated *if* (as well as only if) its pullback $\phi^{-1}(Y) \subset X'$ is isolated.)

There is an abundance of results on smooth hyperbolic systems obtained in the 1960s - 1970s as well as of questions open since then. What we sketched above hardly gives a glimpse of these.

But looking from a different perspective, smooth manifolds, smooth transformations and Riemannian metrics mainly serve illustrative purposes, while the core hyperbolicity is located within *topological* dynamics.

1.7 Symbolic Dynamics, Periodic Points and ζ-Functions.

Given directed graphs G and H, possibly with loops and multiple edges, let G^H denote the set of *combinatorial maps* $H \to G$, where "combinatorial" means that vertices from H are sent to vertices in G and directed edges are sent to directed edges, such that these maps are compatible with the adjunctions between edges and vertices in the graphs.[34]

This *exponentiation* of graphs, $(G, H) \rightsquigarrow G^H$ is, obviously, *(bi)functorial.* In particular,

• the automorphisms groups and, more generally, endomorphisms semigroups of G and H naturally act on G^H where these action commute;

• there are natural restriction maps $R_I : G^H \to G^I$ for all subgraphs $I \subset H$.

Besides, G^H satisfies the following

⋆ *Localisation Property.* Let H be covered by two *edge saturated*[35] subgraphs $I_1, I_2 \subset H$ and let $I_1 \cap I_2 \subset H$ denote the edge saturated subgraph the vertex set of wich is the intersection of the vertex sets of I_1 and I_2 in H.

Take a points $p \in G^{I_1 \cap I_2}$, that is a combinatorial map $p : I_1 \cap I_2 \to G$, and let

$$P_{I_1}(p) \subset H^{I_1},\ P_{I_2}(p) \subset H^{I_1},\ \text{and}\ P_H(p) \subset G^H$$

[34] More precisely, our maps are morphisms in the category of (1-dimensional) semisimplicial complexes that are non-degenerate on the 1-simplices (edges).

[35] A subgraph I in a graph H is called *edge saturated* if it contains all edges of H the end vertices of which are in I.

be the pullbacks of p under the restriction maps

$$G^{I_1} \to G^{I_1 \cap I_2},\ G^{I_2} \to G^{I_1 \cap I_2} \text{ and } G^H \to G^{I_1 \cap I_2}$$

correspondingly.

If I_1 and I_2 are *edge disjoint in* H *away from* $I_1 \cap I_2$, that is no edge in H may have one vertex in the complement $I_1 \smallsetminus I_1 \cap I_2$ and another one in $I_2 \smallsetminus I_1 \cap I_2$, then $P_H(p)$ *naturally decomposes into the Cartesian product*

$$P_H(p) = P_{I_1}(p) \times P_{I_2}(p),$$

where "natural" means that the coordinate projections $P_H(p) = P_{I_1}(p) \times P_{I_2}(p) \to P_{I_1}(p)$ and $P_H(p) = P_{I_1}(p) \times P_{I_2}(p) \to P_{I_2}(p)$ are equal to the restrictions of the restriction maps $R_{I_1} : G^H \to G^{I_1} \supset P_{I_1}(p)$ and $R_{I_2} : G^H \to G^{I_2} \supset P_{I_2}(p)$ to the subset $P_H(p) \subset G^H$.

The set G^H possesses a rich combinatorial structure, but we are concerned at this stage with the topology that is defined as follows.

(i) If H is *finite* then G^H is endowed with the *discrete* topology.

(ii) In general, the topology in G^H is defined via *a basis*, namely the one that is comprised of the *pullbacks* of all subsets from G^I under restriction maps $G^H \to H^I$ for all *finite subfraphs* $I \subset H$.

MARKOV SHIFTS.

Commonly used graphs for the role of H are *the Cayley graphs* of the additive semigroup $\mathbb{N} = \{1, 2, 3, ...,\}$ generated by $1 \in \mathbb{N}$ and of the group $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3, ...,\}$ similarly generated by $1 \in \mathbb{Z}$. These are depicted as

$$\underset{\to}{\mathbb{N}} = \bullet\!\to\!\ldots,$$

$$\underset{\to}{\mathbb{Z}} = \ldots \to\!\bullet\!\to\!\ldots,$$

where the corresponding spaces of maps, denoted $G^{\rightsquigarrow} = G^{\underset{\to}{\mathbb{N}}}$ and $G^{\leftrightsquigarrow} = G^{\underset{\to}{\mathbb{Z}}}$, are visualized as the spaces of *marked edge-paths* in G which are *one sided future directed* in the case of $G^{\rightsquigarrow}$ and *two sided double infinite* for $G^{\leftrightsquigarrow}$ and where $G^{\rightsquigarrow}$ is naturally acted by $\mathbb{N}$ and $G^{\leftrightsquigarrow}$ is acted by $\mathbb{Z}$.

In symbolic dynamics, one usually assumes that the graph G is *finite* and, consequently, the spaces of paths $G^{\rightsquigarrow}$ and $G^{\leftrightsquigarrow}$ are *compact*. In this case the actions of $\mathbb{N}$ on $G^{\rightsquigarrow}$ and of $\mathbb{Z}$ on $G^{\leftrightsquigarrow}$ are called *Markov (sub)shifts* or *subshifts of finite type*.

They are called "*sub*shifts" since they are naturally realised as closed invariant subsets of the *(full) Bernoulli shifts* which act on the spaces of the maps from $\mathbb{N}$ and from $\mathbb{Z}$ respectively to finite set F, that are $F^{\mathbb{N}}$ and $F^{\mathbb{Z}}$; in the present case, one may take to the sets of edges in G for F:

$$G^{\rightsquigarrow} \subset edges(G)^{\mathbb{N}} \text{ and } G^{\leftrightsquigarrow} \subset edges(G)^{\mathbb{Z}}$$

"Finiteness" of these subshifts refers to the finiteness of the number of conditions (that are encoded in our picture by the combinatorics of the corresponding *finite* graphs G) which together with the invariance define these subshifts.

Instead of graphs, subshifts of finite type can be defined with a use of *basic open* (as well as closed) sets $U \subset F^{\mathbb{Z}}$: A subset $Y \subset F^{\mathbb{Z}}$ is *a subshift of finite type* according to this definition if it equals to the intersection

$$Y = \bigcap_{g \in \mathbb{Z}} g(U)$$

for some basic $U \subset \mathbb{Z}$, which is, recall, is the pullback of a subset $S \subset F^I$ under the restriction map $F^{\mathbb{Z}} \to F^I$ for some *finite subset* $I \subset \mathbb{Z}$. (The definition of $U = U_S \subset F^{\mathbb{Z}}$ does not depend on any structure in $\mathbb{Z}$.)

Since the basic subsets $U \subset F^{\mathbb{Z}}$ are closed as well as open, these Y are *dynamically isolated,* and since all closed subsets in $F^{\mathbb{Z}}$ equal to intersections of basic sets, the converse is also true.

Thus, subshifts of finite type can be defined as *dynamically isolated closed invariant subsets in* $F^{\mathbb{Z}}$. (The same applies to F^{Γ} for all groups and semigroups Γ including $\Gamma = \mathbb{N}$.)

Example: Expansivness. The diagonals in $G^{\rightsquigarrow} \times G^{\rightsquigarrow}$ and in $G^{\leftrightsquigarrow} \times G^{\leftrightsquigarrow}$ are, obviously, of finite type; hence, *Markov shifts are expansive.*

Annoying non-Example. The restriction of the obvious (kind of hyperbolic) north-pole→south-pole push diffeomorphism of the sphere S^n to an invariant Cantor set $X \subset S^n$ may be a subshift but *never of finite type.*

Shadowing and Symbolic Approximation.

Let X be a topological space, $f : X \to X$ a continuous map and $\mathcal{U} = \{U\}_{U \subset Y}$ be a covering of Y, where these U will to assumed open (sometimes closed) later on. Let $G = G(\mathcal{U}, f)$ be the graph on the vertex set $\mathcal{U}$ where U_1 is joint with U_2 by an edge whenever $f(U_1)$ intersect U_2.

Observe that the points of $G(\mathcal{U}, f)^{\rightsquigarrow}$ as well as the orbits of the action of $\mathbb{N}$ on $G(\mathcal{U}, f)^{\rightsquigarrow}$ are represented by certain $\mathbb{N}$-*families* of subsets $U \in \mathcal{U}$ that are sequences $\{U_i \in \mathcal{U}\}_{i \in N}$.[36]

Similarly, the orbits of the action of $\mathbb{Z}$ on $G(\mathcal{U}, f)^{\leftrightsquigarrow}$ are $\mathbb{Z}$-*families* $\{U_i \in \mathcal{U}\}_{i \in \mathbb{Z}}$, such that *the intersections* $U_i \cap f^{-1}(U_{i+1}) \subset X$ *are non-empty* for all $i \in \mathbb{Z}$. (Topology of X and continuity of f are not needed for all this so far.)

The local shadowing property of f, say if Y is compact and f is a homeomorphism which generates a $\mathbb{Z}$-action on Y, can be reformulated as follow.

Given an *open* cover $\mathcal{V}$ of X there exists a *finite open* cover $\mathcal{U}$ of X (which is finer than $\mathcal{V}$) and a *surjective* $\mathbb{Z}$*-morphism*

$$\phi : G(\mathcal{U}, f)^{\leftrightsquigarrow} \twoheadrightarrow (X, f),$$

such that the orbits $\{U_i\}$ of the $\mathbb{Z}$-action on $G(\mathcal{U}, f)^{\leftrightsquigarrow}$ are $\mathcal{V}$-*close* to the corresponding orbits $x_i = f^i(x_0) \in X$ of our $\mathbb{Z}$-action on X, where this "closeness" means the following.

The subset $U_i \subset X$ *and the point* $x_i \in Y$ *are contained together in a certain subset* $V \in \mathcal{V}$ *(depending on* $\{U_i\}$ *and* $i \in \mathbb{Z}$*) for all orbits* $\{U_i\}$ *and all* $i \in \mathbb{Z}$.

And *local uniqueness* now says that if the covering $\mathcal{V}$ is *sufficiently fine*, then there is *at most one* morphism $G(\mathcal{U}, f)^{\leftrightsquigarrow} \to X$ with the $\mathcal{V}$-closeness property.

❖❖ *Combinatorial Shadowing in Markov Shifts* The unique local shadowing for Markov shifts translates to the the following obvious combinatorial fact.

Let,

[36]We tacitly use the bijective correspondence between points of a G-space X and G-orbits for $x \leftrightarrow \{g(x)\}_{g \in G}$.

$$\phi_i : \{i, i+1, ...i+k-1, i+k\} \to G,$$
$$\{i, i+1, ...i+k-1, i+k\} \subset \mathbb{Z}, \quad i = \cdots -2, -1, 0, 1, 2, \ldots,$$

be a (double infinite) sequence of combinatorial maps that are paths in G of length k for some k, say $k = 10$, such that the restriction of

$$\phi_i \text{ to } \{i+2, ...i+k-1\} \subset \{i, i+1, ...i+k-1, i+k\}$$

is equal to the restriction of

$$\phi_{i+1} \text{ to } \{i+2, ...i+k-1\} \subset \{i+1, ...i+k, i+k+1\}$$

for all $i = \cdots -2, -1, 0, 1, 2, \ldots$.

Then there is *a unique combinatorial map (path) $\Phi : \mathbb{Z} \to G$, such that the restrictions of Φ to the subsets $\{i+3, ...i+k-2\} \subset \mathbb{Z}$ are equal to the restriction of ϕ_i on these subsets for all $i = \cdots -2, -1, 0, 1, 2, \ldots$.*

What is more interesting is the local shadowing we met earlier which ilmplies the following.

❖ ↠ ❐ *Compact manifolds with Anosov $\mathbb{Z}$-actions, as well as compact spaces with locally split hyperbolic – what Ruelle calls "Smale" – homeomorphisms,*[37] are representable as quotients of Markov shifts.

Moreover – this follows from the existence of *Markov partitions* as we shall see below,

❖• • • ↠ •❐ Compact spaces X with (locally split hyperbolic) Ruelle-Smale homeomorhsms $f : X \to X$ admit arbitrary finite open coveres $\mathcal{U}$ such that the surjective morhisms $\phi : G(\mathcal{U}, f)^{\leftrightsquigarrow} \twoheadrightarrow (X, f)$ are *finite-to one.*

ζ-Function Corollary (Manning 1971 [77]).[38] *The sequences Per_i of i-periodic points of Ruelle-Smale's diffeomorphisms f, that are the fixed points of the powers $f^i : X \to X$, $i = 1, 2, ...$, are finite linear combinations of geometric progressions,*

$$Per_i = \sum_{j=1,2,\ldots,k} c_j \lambda_j^i$$

Consequently, the generating functions

$$\sum_{i=0}^{\infty} Per_i z^i \quad and \quad \zeta(z) = \exp \sum_{i=1}^{\infty} \frac{1}{i} Per_i z^i$$

are rational functions in the variable z.

Proof. ([?]). If $X = G^{\rightsquigarrow}$ for a finite graph G and f is the Markov shift, then its i-periodic points are nothing but directed cycles in G of the length i and $Per_i = card(Fix(f^i))$ equals the trace of the i-th power of the matrix (g_{vw}) on the vertex set V of G, where g_{vw} equals the number of edges going from v to w in V.

Thus, the number Per_i equals the sum of the i-th powers of the eigenvalues λ_j, $j = 1, 2, ..., card(V)$, (taken in several copies according to their multiplicities) of this matrix and the decomposition $Per_i = \sum_j c_j \lambda_j^i$. follows.[39]

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[37]This means that every point in X admits a split neighbourhood $U = U_{contr} \times U_{exp}$ such that the following ineqialities are satisfied with respect to some metric on X.

$dist(f^n(x_1), f^n(x_2)) \leq const(1-\varepsilon)^n$ for $n = 1, 2, 3, ...$ and all x_1, x_2 in the *contr*-slices of U, i.e. for $x_1, x_2 \in U_{contr} \times u$, $u \in U_{exp}$, and $dist(f^{-n}(x_1), f^{-n}(x_2)) \leq const(1-\varepsilon)^n$ for x_1, x_2 in the *exp*-slices.

[38]Manning states in his paper that he followed a suggestion by R. Bowen.

[39]This is usually attributed to a 1970 paper by Bowen and Lanford but it is hard to believe this has not been known prior to 1970.

Now, let a $\mathbb{Z}$ action on a space Y generated by a homeomorphism $g: Y \to Y$ be a finite-to-one quotient of a Markov shift $f: X \to X$.

Then the numbers of fixed points $y \in Y$ of g^i can be expressed by those for Markov shifts by the following implementation of the

Moivre-Sylvester inclusion-exclusion counting argument. Divide the fixed points of f^i into several classes according to the types of the actions of f^i on the *finite* sets $\phi^{-1}(y) \subset X$, where these "types" are determined by the periods of the action of f^i and the number of these types t is bounded by a constant independent of i.

Thus, counting periodic points of g is reduced to an evaluation of the the numbers $Per_{i,t}$ of fixed points of f^i for all types t, and Manning's ζ-theorem follows from the "rational numerology" for the numbers $Per_{i,t}$ for all types t, where the latter, in turns, is obtained by by the above graph-theoretic arguments that works here to due to Markov (finite type) presentability of ϕ:

the subset of pairs $\{x_1, x_2\}_\sim \subset X \times X$ that satisfy the equivalence relation in X corresponding to ϕ,

$$x_1 \sim x_2 \Leftrightarrow \phi(x_1) = \phi(x_2)$$

is a subshift of finite type in $X \times X \subset (E \times E)^{\mathbb{Z}}$, where E denotes the set of edges of the graph that defines X.

(The Markov property of $\{x_1, x_2\}_\sim$, follows from expansiveness of our g, since $\{x_1 \sim x_2\}_\sim$, being the $\phi \times \phi$-pullback of the dynamically isolated diagonal in $Y \times Y$, is also dynamically isolated.)

1.8 Markov Partitions.

Brief History. Such "partitions" were used in 1967 by Adler and Weiss for showing that hyperbolic automorphisms of the 2-torus with equal entropies are isomorphic in the category of measure spaces with $\mathbb{Z}$-actions. [1].

In 1968, Sinai introduced the concept of a Markov partition, proved their existence for the Anosov systems, and established, albeit not in full generality at that time, the *measure theoretic* equivalence between Anosov systems and Markov shifts [105].

In 1970, Bowen constructed Markov partitions for topological Ruelle-Smale locally split hyperbolic systems [20].

What is called *partition* in this context means a *covering* of a topological space X by *closed* subsets V_i which are *fat* in the sense that the interior $int(V_i) = V_i \setminus \partial V_i$ are dense in V_i for all i and where, one requires that *the interiors* $U_i = int(V_i)$ (rather than V_i themselves) *do not pairwise intersect.*

Equivalently, such a partition can be defined as a collection of *mutually non-intersecting open* subsets $U_i \subset X$ – that correspond to the interiors $int(V_i)$, such that the union $\bigcup_i U_i$ is *dense* in X.

A partition $\{V'_j\}_{j \in J}$, is said to *refine* $\{V_i\}_{i \in I}$, if there exists a map $\iota: J \to I$, such that V'_j is contained in $V_{\iota(j)}$ for all $j \in J$.

The simplest kinds of *Markov* partitions are associate with *locally (including globally) expanding maps*[40] of metric spaces, $f: X \to X$, where the Markov prop-

[40]This means there exists a cover of X by open subsets $U_i \subset X$, such that the restrictions of f to U_i are homeomorphisms on their images, $U_i \underset{f}{\to} f(U_i)$ and such that the distances between the pairs points in U_i increase under these maps for all i.

erty of a partition $\{V_i\}$ of X in this case requires that the *f-pullback partition* $\{f^{-1}(V_i)\}$ *refines* $\{V_i\}$.

For instance, if $X = \mathbb{R}^n$ and $f : x \mapsto 2x$ then the standard partition of X into the integer unit n-cubes is Markov.

Also – this is a kindergarten exercise – an arbitrary strictly expanding linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ admits a Markov partition into *convex polyhedral* domains V_i of diameters $diam(V_i) \leq 1$ and with inradii (that are radii of maximal balls in them) bounded from below, $inrad(V_i) \geq \varepsilon > 0$.

On the other hand, Markov partitions $\{V_i\}$ for strictly expanding endomorphisms f of the n-tori $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are rarely so simple – piecewise linear or piecewise smooth. Their construction (due to Sinai) described below delivers *fractal* subsets V_i and, apparently, this fractality is unavoidable for majority of f.

Construction of Markov partitions for locally expanding maps $f : X \to X$.

Let $\{U_i\}_{i\in I}$, be a collection of open subsets in a topological space X and let $U' \subset X$ be yet another open subset.

Endow the index set I with a *well-order* structure[41] and let $i_{min} = i_{min}(U') \in I$ be the *minimal* i for which U_i *intersects* U'.

Define a new collection $\{U_i^{new} \subset X\}$ by attaching U' to $U_{i_{min}}$ at the expense of other U_i. Namely,

$$\textit{let } U_i^{new} = U_i \bigcup U' \textit{ if } i = i_{min}$$

and

$$U_i^{new} = int(U_i \smallsetminus U') = U_i \smallsetminus clos(U') \textit{ if } i \neq i_{min}.$$

Apply this to *all subsets* U'_j from a given *collection* (rather than a single U') $\{U_j \subset X\}$ and thus obtain $\{U_i^{new}\}$, which we now denote

$$\{U_i^{new}\} = \{U_i\} \swarrow \{U'_j\},$$

where

U_i^{new} is obtained from U_i by attaching to it those U'_j for which $i = i_{min}(U'_j)$ and by subtracting from U_i the closures of other U'_j.

Denote by $\mathcal{V}_{I\prec}$ the set of partitions $\{V_i\}_{i\in I}$ of X, where the index set I is endowed with a *well-order* structure and, given a continuous map $f : X \to X$, let

$$\Upsilon_f : \mathcal{V}_{I\prec} \to \mathcal{V}_{I\prec}$$

be defined in trems of the interiors $U_i = int(V_i)$ as follows

$$\Upsilon_f : \{U_i\} \mapsto \{U_i\} \swarrow \{f^{-1}(U_i)\}.$$

(We must allow *empty* subsets among V_i and/or U_i at this point. Alternatively, we could supress I and deal we the space $\mathcal{V}_\prec$ of well-ordered families of non-empty subsets in X.)

Now the property

the pullback partition $\{f^{-1}(V_i)\}$ refines $\{V_i\}$

reads:

$\{V_i\} \in \mathcal{V}_{I\prec}$ is a fixed point of the map $\Upsilon_f : \mathcal{V}_{I\prec} \to \mathcal{V}_{I\prec}$.

[41]In current examples, there are finitely many U_i and I can be represented by $\{1, 2, ..., i, ..., k\}$.

If $f : X \to X$ is a locally strictly expanding map, and X is compact (this is not truly needed) then the map Υ_f is eventually contracting with respect to the *Hausdorff metric* in $\mathcal{V}_I$[42] and the existence of a fixed point follows by the Banach fixed point theorem, where discontinuity of Υ_f is compensated by the *monotonicity* of the orbits of Υ_f in the space $\mathcal{V}_{I<}$,

$$\{V_i\} < \Upsilon_f(\{V_i\}) < \Upsilon_f \circ \Upsilon_f(\{V_i\}) < \Upsilon_f \circ \Upsilon_f \circ \Upsilon_f(\{V_i\}) < \ldots,$$

where "<" is the partial order in $\mathcal{V}_{I<}$ obtained by the I-lexicographic arrangement of I-families of sets of subsets $V \subset X$ with the partial orders defined by inclusions between subsets.

And, besides the mere existence, one sees in this picture that the *forward periodic points*, that are the fixed points of the powers $\Upsilon_f^i : \mathcal{V}_{I<} \to \mathcal{V}_{I<}$, $i \in \mathbb{N}$, are *dense* in $\mathcal{V}_{I<}$.

Markov Partitions for Split Hyperbolic Actions. Let $f : X \to X$ be a Ruelle-Smale, e.g. Anosov, locally split hyperbolic homeomorphism, and let us define the Markov property of a partition $\{V_i\}$ of X by two conditions.

(1) Each subset $V_i \subset X$ from $\{V_i\}$ Cartesian splits into an expanding $\times$ contracting factors,

$$V_i = V_i^{exp} \times V_i^{contr} \subset X$$

where f^{-1} strictly contracts $V_i^{exp} \times v^{contr} \subset X$ for all $v^{contr} \in Vi^{contr}$ while f itself similarly contracts all $v^{exp} \times V_i^{contr}$

(2) If $L = L^{exp} \subset X$ is an *expanding* (i.e. strictly contracted by f^{-1}) leaf, then the partition $\{L \cap V_i\}$ of this leaf by its intersections with V_i is *refined* by the pullback partition $\{f_{|L}^{-1}(V_i)\}$, where $f_{|L} : L \to X$ denotes the restriction of f to L); and, symmetrically, the intersections of $\{V_i\}$ with *contracting* leaves L are refined by the *images* of $\{f^{-1}(L) \cap V_i\}$ under $f_{|f^{-1}(L)}$ (that are pullbacks under $(f^{-1})_{|L}$).[43]

Route of the Proof. Close your eyes and proceed as follows. Take the space $\mathcal{V}_{I<}^{\times}$ of *split partitions*, i.e. of those which satisfy (1), show that (this is a trivial "soft" staff) that this $\mathcal{V}_{I<}^{\times}$ is non-empty and "split-hyperbolically adjust" the contracting fixed point argument. You end up with Markov partitions in your hands as well as with the density of these in $\mathcal{V}_{I<}^{\times}$ for the powers of $f^i : X \to X$.

Return to the ζ-Functions.

Recall the graphs associated with coverings, in particular, with what we call partitions $\{V_i\}_{i\in I}$ of spaces X which are acted by $\mathbb{Z}$ generated by homeomorphisms $f : X \to X$: in the present terms, such a graph $G = G(\{V_i\}, f)$ is based on the vertex set I with the edges that correspond to the pairs $i, j \in I$, such that

$$f(V_i) \bigcap V_{i+1} \neq \varnothing.$$

[42] This is defined for partitions via the ordinary Hausdorff distance (see 2.6) between the constituent subsets: $dist_{hau}(\{V_i\}, \{V'_i\}) = \sup_{i\in I} dist_{hau}(V_i, V'_i)$. This metric for partitions is finer than for individual sets but the map Υ_f is not truly contracting due to discontinuity of $V_j \mapsto i_{min}(V_j) \in I$.

[43] Strictly speaking, $\{L \cap V_i\}$ and $\{f_{|L}^{-1}(V_i)\}$ are not partitions according to our definition, since the intersections $V_i \cap L$ are not necessarily *fat* subsets in L and the unions of interiors of these intersections are not dense in L for certain L. In truth, the above conditions need to hold only for those L where these intersections are fat.

If the partition $\{V_i\}$ is *sufficiently fine*, then Anosov's shadowing lemma delivers a a $\mathbb{Z}$-morphism from the space of double infinite paths in $G(\{V_i\}, f)$ to (X, f),

$$\phi : G(\{V_i\}, f)^{\leftrightarrow} \to X,$$

where a straightforward check up shows that the *Markov property* of a partition $\{V_i\}$ makes the shadowing map ϕ *finite-to-one*.

Then, as it was explained earlier, the rationality of the ζ-function that counts the periodic points of f follows from such rationality of the ζ-functions for Markov shifts by the Manning-Moivre-Sylvester inclusion-exclusion counting argument.

Markov Presentation and Small Cancellation. There is a two (more?) way interplay between Markov partitions of $\mathbb{Z}$-actions and small cancellation of groups:

There are (several kinds of) Markov style structures associated to small cancellation and to some more general groups, while Markovian representations of (hyperbolic and similar) dynamical systems admit a combinatorial description similar to that of small cancellation groups. In particular, action of certain *relatively hyperbolic groups* on their ideal boundaries admit Markovian presentations (see [30] and references therein) and a similar theory for generalised *Grigorchiuk groups*[44] was developed by V. Nekrashevych [85], [86], [87].

We shall discuss this at length in chapters 3 and 4.

1.9 Holomorphic and Conformal Hyperbolicity.

Hyperbolicity of a complex manifold X may be measured by how large the spaces of holomorphic maps from Riemann surfaces to X are:

the smaller these spaces are the higher the "hypebolicity degree" of X is.

Exemplary hyperbolic manifolds X are *bounded domains* $B \subset \mathbb{C}^n$, e.g. the open unit ball $B^{2n} \subset \mathbb{C}^n$; also quotients of these by discrete, especially cocompact, groups of holomorphic transformatons, such as compact Riemann surfaces of genera ≥ 2

These X receive *no non-constant holomorphic maps* from $\mathbb{C}$ by *Liouville's theorem.*[45]

Another famous hyperbolicity phenomenon is

PICARD'S THEOREM.(1879) *The universal covering of the complex plane minus two points is biholomotphic to the unit disc $B^2 \subset \mathbb{C}$; hence $\mathbb{C} \setminus \{two\ points\}$ receives no non-constant holomorphic map from $\mathbb{C}$.*

In other words

[44]An essential feature of Grigorchuk's groups Γ from our perspective is the existence of expanding monomorphisms $\Gamma \times \Gamma \to \Gamma$ with finite index images.

[45]The unboundness of non-constant holomorphic maps $\mathbb{C} \to \mathbb{C}$ must have been known to Cauchy who apparently proved (?) that the oscilations, i.e. differences between maxima and minima, of holomorphic functions f in concentric disks of radii R and $r < R$ satisfy what is now-a-days called *Cauchy-Gårding elliptic inequality*:

$$osc f_{|B(r)} \leq \frac{r}{R} osc f_{|B(R)},$$

while Liouville stated (reproved?) I have not read the relevant 1847?,1879? papers by Liouville.) this for elliptic functions. (I have not read the relevant 1847?,1879? papers by Liouville and essentially rely on [116].)

a non-constant complex analytic map $\mathbb{C} \to \mathbb{C}$ *must assume all values, except, possibly, a single one* (as the function $z \mapsto \exp z$ does).

More generally, let f be a non-constant holomorphic map from a Riemann surface X to $\mathbb{C} \setminus \{two\ points\}$.

If X *admits a complete Riemannian metric of finite area* (as, for instance, $\mathbb{C}$ does) *and if it has finitely many ends* (this can be significantly relaxed), *then* X *is biholomorphic to a compact Riemann surface with* $k \geq 3$ *punctures and our* $f : X \to \mathbb{C} \setminus \{two\ points\}$ *is a finite-to-one ramified covering map with at most finitely many ramification points.*

There are two main avenues of generalisation of this theorem which have been especially thoroughly (yet, non-fully) explored.

1. Holomorphic maps from Riemann surfaces to complex manifolds of dimensions >1.

The first(?) result here, due to Borel's (1887), concerns maps from $\mathbb{C}$ to the complex projective space $\mathbb{C}P^n$:

If a holomorphic map $f : \mathbb{C} \to \mathbb{C}P^n$ *misses* $n+2$ *hyperplanes in general position then the image* $f(\mathbb{C}) \subset \mathbb{C}P^n$ *is contained in a hyperplane* $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

It easly follows (Montel 1927?) that

if f *misses* $2n+1$ *hyperplanes in general position then* f *is constant.* (If $n = 1$ this is just Picard's theorem.)

In 1928 -1933, Henry Cartan developed a Nevanlinna-type value distribution type theory for maps $f : \mathbb{C} \to \mathbb{C}P^n$ that was refined by Ahlfors in 1941 with some contributions by Herman and Joachim Weyl (1939, 1941).[46]

In 1967, Kobayashi [70] defined his *intrinsic metric*, that is the maximal metic on a complex manifold (and/or a complex space) X, such that every holomorphic map $\mathbf{H}^1_{\mathbb{C}} \to X$, is (non-strictly) distance decreasing, where $\mathbf{H}^1_{\mathbb{C}} = \mathbf{H}^2_{\mathbb{R}}$ stands for the unit disk $B^2 \subset \mathbb{C}$ with Poincaré's metric, that is the hyperbolic plane with the metric of constant curvature -1.

Obviously (but significantly) holomorphic maps $f : X \to Y$ are *distance decreasing* for *the Kobayashi metrics* in X and in Y while ĉovering maps f are *locally isometric*; this implies in particular, that the Kobayashi metric in $\mathbb{C}^n$ (as well as in any complex space with a selfsimilarity) is *everywhere zero.*[47]

Thus, Kobayashi metric in X *vanishes on the images of holomorphic maps* $\mathbb{C} \to X$; therefore,

> *manifolds* X *with non-vanishing Kobayashi metrics receive no nonconstant holomorphic maps* $\mathbb{C} \to X$.

Kobayashi versus Liouville.

Liouville's property is seductively simple and general:

[46] I follow here the expositions in [38] and [**?**].

The work by Cartan, this is pointed out by Eremenko in [38], remained in the dark for about half a century: several "difficult problems", I recall, that preoccupied the experts in 1970s turned out to be among lemmas and theorems proved in Cartan's papers.

Once, Serge Lang at his lecture on this subject matter asked Cartan, who was in the audience, why he had never mentioned his results to anybody. Cartan (born in 1904) responded that he was embarrassed by the mathematics he was doing in his youth. (I was present at this lecture but I am not certain that "embarrassed" was exactly the word Cartan used, but he definitely projected his feeling of embarrassment.

[47] Our convention for *metric* allows 0 and ∞.

an ordered pair of object (Y, X) in a category enjoys Liouville's property if every morphism $Y \to X$ *factors through a morphism from* Y *to a terminal object.* And this concept gain a particular significance when one fixes one of the variables X and/or Y, where the standard choice for X in complex analysis is that of the unit disk $B^2 = \mathbf{H}^1_{\mathbb{C}}$ while the distinguished X is $X = \mathbb{C}$.[48]

Accordingly, a natural, term for Y where (Y, B^2) is Liouville would be *Liouville-out* , while X in the Liouville pairs $(\mathbb{C}, X)$ shuld be called *Liouville-in.*

But, customary, Liouville-out's Y are called just *Liouville* [49] while Liouville-in's spaces X – which receive no non-constant holomophic maps from $\mathbb{C}$ – are, starting from the late 1970s, associated with the name of *Brody* for the reason we shall explain below.

"*Liouville per se*"– a bare non-existence of something – carries no significant message concerning the structures of the spaces involved. Non-Surprisingly, classical analysts – Nevanlinna, Landau, Bloch, Ahlfors,..., departing from Liouville and Picard theorems, shifted the focus to the study of more structurally rich properties, such as Nevanlinna-Ahlfors *value distribution theory* that was extended to higher dimensions by Cartan and later by Ahlfors.

A few decades later, holomorphic maps were approached from a geometrical and algebra-geometrical angle. Thus, in 1967, Kobayashi [70] introduced his metric as an essential *structurally significant* ingredient of hyperbolicity, now-a-days called *Kobayashi hyperbolicity*, responsible for arresting holomorphic maps $\mathbb{C} \to X$.

Notice, that that in a the case where X is *compact* Kobayashi hyperbolicity is equivalent to

compactness of the spaces of holomorphic maps $Y \to X$ *for all complex spaces* Y.

In 1978, Robert Brody [23] applied the *Bloch (Landau-Robinson-Zalcman) principle* to holomorphic maps into general complex manifolds and concluded that

if the Kobayashi metric in a compact complex space X *(or in a compact quasi-complex manifold for this matter)* X*, somewhere vanishes, that is if the above compactness fails, then there exists a non-constant (quasi)holomorphic map* $\mathbb{C} \to X$.

Thus, if X is compact, then

Liouville-in $\Leftrightarrow$ *Kobayashi hyperbolic.*[50]

This *Bloch-Lanadau-Zalcman-Brody* map $\mathbb{C} \to X$ is obtained as a limit of maps f_i from suitably rescaled hyperbolic planes, called Y_i, to X, where these Y_i themselves metrically converge to $\mathbb{C}$.

Namely, denote by Y the unit disk with the Poincaré metric, that is the hyperbolic plane with the metric of curvature -1, and also choose a metric in X.

---

[48] Is there a purely category theoretic definition of such "Liouville distinguished" objects?

[49] Also Riemannian manifolds Y as well as spaces wiith random walks defined on them are called *Liouville* if all *harmonic maps* $Y \to [01] \subset \mathbb{R}$ are constant.

[50] There is no reference to Bloch(1926) Landau (1929), Robinson (1973) or Zalcman(1975) in Brody's paper and it is unclear what was the level of generality of the Bloch principle, known to classical analists who worked it out, according to Robinson, [99] between 1915 and 1935. (Some references can be found in [117] and [2].)

Also Grothendiec's blow-up construction (≈1960) of rational curves employs a limit argument similar to that by Landau-Zalcmanin-Brody.

The vanishing of the Kobayashi metric in X says, in effect, that there exists a sequence of holomorphic maps $f_i : Y \to X$ and a sequence of balls $B_i \subset Y$, such that

$$\lim_{i\to\infty} diam_Y(B_i) = 0 \text{ while } diam_X(f(B_i)) \geq \delta_0 > 0 \text{ for all i.}$$

Assume $diam_Y(B_i) \leq 1$ for all i and let $Y_i' \subset Y$ denote the concemtric balls $B_i(1) \supset B_i$ with diameters 1 endowed with the Poincaré metrics.

Since these Poincaré metrics, call them P_i, *blow up at the boundaries* of the balls $B_i(1) \subset Y$, the resulting $Y_i' = (B_i(1), P_i)$ (which are isometric to the hyperbolic plane) contain balls $B_i^{\min} \subset Y_i'$, such that $diam_X(f(B_i^{\min})) = \delta_0$ and

the Y_i'-diameters ε_i of these $B_i^{\min}$ themselves, denoted ε_i, are minimal among the balls in Y_i' the f-images of which in X have diameters δ_0.

Rescale the spaces Y_i' by the factors ε_i^{-1}, denote the resulting spaces by

$$Y_i = \varepsilon_i^{-1} Y_i' = (B_i(1), \varepsilon_i^{-1} P_i)$$

and observe that the correspondingly rescaled balls

$$B_i^{\sharp} = \varepsilon_i^{-1} B_i^{\min} \subset Y_i = \varepsilon_i^{-1} Y_i'$$

have diametes 1 in Y_i.

Since the diameters of the f-images of *all* balls with diameters ≤ 1 in *all* Y_i are bounded by $\delta_0 < \infty$ by the minimality condition on $B_i^{\min}$, the family of maps $f_i : Y_i \to X$ is *uniformly continuous* according to the Cauchy inequality.

Since $diam_Y(B_i) \to 0$, the diameters ε_i of the balls $B_i' \subset Y_i'$ also tend to zero; therefore, Y_i (that are hyperbolic planes with the curvatures $-\varepsilon_i^2$) metrically converge to $\mathbb{C}$ and since X is compact, some subsequence of $f_i : Y_i \to X$ converges to a holomorphic map $f_\infty : C \to X$.

Finally, since $\delta_0 > 0$, this map is non-constant. QED.

Brody Curves. The Broody(Bloch-Lanadau-Zalcman) map f_∞ is, obviously, *Lipschitz*; in general Lipschitz holomorphic maps $\mathbb{C} \to X$ are called *Brody curves.*

The construction of f_∞ equally applies to maps $Y_i \to X_i$ for convergent sequences $X_i \to X$; thus, this was pointed out by Brody,

Kobayashi hyperbolicity is stable under perturbations of the complex structure in X.

(This is similar to the stability of Smale-Anosov hyperbolicity under C^1-perturbations.)

First Hyperbolic Examples.

The normalizing example which lies at the heart of Kobayshi's definition

is that of $X = \mathbf{H}^1_{\mathbb{C}}$ where the *Kobayashi metric* equals the original Riemannian metric with curvature -1 by *Schwarz Lemma*:

holomorphic maps $\mathbf{H}^1_{\mathbb{C}} \to \mathbf{H}^1_{\mathbb{C}}$ are distance decreasing.

(However simple, this, unlike all of the above, can not be proved by mere hand waving.)

More generally, if X is a bounded symmetric complex domain, e.g. $X = B^{2n} \subset \mathbb{C}^n$, then the *Kobayashi metric* equals the Riemannian (in fact Kählerian) metric invariant under the full group of holomorphic transformations of X. (The ball B^n with the Kobayashi metric is isometric to the complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ we met in section 1.4.)

By the same token, the Kobayashi metrics in *Kähler manifolds* X with *strictly negative* holomorphic curvatures is bounded from below by this Kähler metric times a positive constant.[51]

For instance, if X is a subvariety in the complex torus $\mathbb{C}^n/\Gamma$ for some lattice Γ isomorphic to $\mathbb{Z}^{2n}$, then it has strictly negative holomorphic curvature, unless it contains a complex subtorus of positive dimension; hence, the Kobayashi metric in such an X nowhere vanishes.

This in conjunction with the *Abel-Jacobi-Albanese theorem*, implies that

if the fundamental cohomology class $[X]^* \in H^{2n}(X;\mathbb{Q})$ *of a complex projective manifold* X *of complex dimension* n *decomposes into the* $\smile$*-product of 1-dimensional classes, and if* X *contains no Abelian subvariety of positive dimension, then* X *is Kobayashi hyperbolic.* This is attributed to Bloch but I did not try to trace the corresponding paper.

Besides fully hyperbolic spaces, there many *almost hyperbolic* X where holomorphic maps $f:\mathbb{C}\to X$, if not necessarily
constant, are, quite special, e.g. factors through maps from $\mathbb{C}$ to particular varieties or satisfy some differential equations [13], [44].

Also *higher order degree of hyperbolicity* of a complex manifold X can be measured by the maximum of ranks (of differentials) of holomorphic maps $f:\mathbb{C}^n\to X$, where a particular problem that have been studied since a 1971 article by Griffiths [45] concerns the bound $rank_{\mho} < n$ for maps to n-dimensional algebraic manifolds, of "general type".

Is there a holomorphic counterpart to Markov partitions?

One may search for the answer in terms the natural action of $\mathbb{C}$ in spaces of *Broody maps* in the spirit of [53], [78] with an eye to the *Weierstrass product theorem* which suggests the direction one should follow.

2. Quasiconformality.

The second well studied class of maps that are subjects to "hyperbolic constraints" is that of *quasiconformal maps* between n-dimensional manifolds, $n\geq 2$, defined by the following properties.

○ the complement of the domain $U\subset X$ on which the map f is locally homeomorphic has the Hausdorff dimension $dim_{haus}(X^n\smallsetminus U) < n-1$.

● there exist a positive functions $r(u)>0$ and a bounded positive function $K(u)\leq K_0<\infty$, such that the f-images of the r-balls $B_u(r)\subset X$ with $r\leq r(u)$ for all $u\in U$ contain balls in Y of radii $\geq K^{-1}(u)\cdot diam(f(B_u(r)))$;

(I am not certain what should be the most general/natural definition of these maps.)

Notice that such maps are often called *quasiregular* while *quasiconformal* is restricted to *locally homeomorphic* maps $X\to Y$, i.e. where $U=X$ in our notation.

[51] *Holomorphic curvature* is the function on the complex projective bundle $Pro_{\mathbb{C}}(X)$ of the complex lines in the tangent bundle $T(X)$ which is obtained by restriction of the Riemannian sectional curvature function from the Grassmannian bundle $Gr_2(X)\supset Pro_{\mathbb{C}}(X)$ of the real 2-planes in $T(X)$.

If $n = 2$ qusiconformal maps are quite similar to holomorphic ones. For instance, the proof of non-existence of non-constant quasi-comformal maps $\mathbb{R}^2 \to \mathbb{R}^2 \smallsetminus \{two\ points\}$ is as easy as that for conformal maps.

In fact (almost) all Nevanlinna's value distribution theory was extended by Ahlfors to the quasiconformal case.

If $n \geq 3$, quasiconformal maps, at least locally homeomorphic ones, become more constrained. For instance, global topological possibilities of *locally homeomorphic* quasiconformal maps in dimensions $n \geq 3$ are rather limited: according to a 1967 Zorich's solution of Lavrentiev's problem of 1938, [119]

a locally homeomorphic quasiconformal map from $\mathbb{R}^n$, $n \geq 3$, to the n-sphere S^n is a homeomorphism onto $S^n \smallsetminus \{point\}$.

(Recall that $S^n \smallsetminus \{point\}$ is conformally isomorphic to $\mathbb{R}^n$, where this isomorphism, i.e. a conformal diffeomorphism, is established by the stereographic projection $S^n \smallsetminus \{point\} \to \mathbb{R}^n$.

Also notice that there are lots of locally homeomorphic conformal maps $\mathbb{C} = \mathbb{R}^2$ to S^2, e.g. the integrals of $\exp f(z)$ for holomorphic functions $f : \mathbb{C} \to \mathbb{C}$.)

The situation with non-locally homeomorphic quasiconformal maps is less clear:

what are possible homotopy types (topologies?) of closed n-manifolds X that admit non-constant quasiconformal maps $f : \mathbb{R}^n \to X$ without assuming that these f are locally homeomorphic?

If such a quasiconformal map $f : \mathbb{R}^n \to X$ is *locally homeomorphic*, then – this follows by Zorich's argument – either X is homeomorphic to a manifold of *constant positive* curvature, hence having *S^n for the universal cover*, or X is homeomorphic to a manifold of *zero curvature*, hence, admitting *a finite cover homeomorphic to the torus $\mathbb{T}^n$*.

A definite result in this direction was obtained in 1980 by Rickman [100] (who calls his maps *quasiregular*):

A quasiconformal map $\mathbb{R}^n \to S^n$ may omit at most finitely many values.

But there is no bound on the number of the omitted points for $n \geq 3$ [33] and, annoyingly, there is

no known topological obstruction for the existence of non-constant quasiconformal maps $\mathbb{R}^n \to X$ for closed simply connected n-manifolds (compare [101] [16], [91].

Negative Codimensional Quasiconformality. The quasiconformality defined by ⬬ makes sense for maps $X \to Y$ where $dim(Y) < dim X$. In fact, the (quasi)conformal structure itself can defined on X by a sheaf of this kind of maps from $U \subset X$ to $\mathbb{R}^2$. But the following test question remains open.

For which m and $n > m$ does $\mathbb{R}^n$ admit a smooth ⬬-quasiconformal submersion to $\mathbb{R}^m$ with bounded image?

Quasiconformality in Positive Codimensions. There is no geometric obstructions for immersions $f : Y^m \to X^n$ with a prescribed infinitesimal geometry for $m < n$[52] and in order to have a nontrivial constrain on f, one need to augment quasiconformality by another geometric condition.

[52]It may be sometimes useful, e.g. in the study of the ideal boundaries of hyperbolic spaces (see section 2.13) to define a (quasi)conformal structure in X via (quasi)conformal maps of surfaces to X.

A known condition of this kind is *quasiminimality* of (the image of) f that says that for all domains $V \subset Y$ the volumes of all m-dimensional subvarieties $W \subset X$ with the boundaries ∂V equal to $f(\partial(\Omega))$ satisfy

$$vol_m(W) \geq c \cdot vol_m(f(V)) \text{ for some } c = c(f) > 0.$$

A pleasant *local* criterion which quarantines quasiminimality is *taming* (quasicalinbration) of f by an exact differential m-form $\omega = d\lambda$, where λ is (preferably) a bounded form, $\sup_{x \in X} ||\lambda_x|| < \infty$, where "taming" means that the induced form $f^*(\omega)$ is bounded from below in terms of the volume form R_f for the Riemannian metric on Y induced by f,

$$|f^*(\omega)| \geq c|R_f|, \ c > 0.$$

The most popular instances of such forms ω are powers of the *symplectic Kähler forms* ω on complex manifols X which calibrate (hence, tame) holomorhic maps to X. Such forms are not, in general, differentials of *bounded forms* (or of any forms at all), but if ω is such a differential, then, by *Ahlfors lemma*, (see $dist_{Kob}$-non-degeneracy proposition in section 2.12) X is Kobayashi hyperbolic (compare [56]).

Also the graphs $\Gamma_f : Y \to Y \times X$ of equidimensional locally orientation preserving quasiconformal maps between oriented Riemannian manifolds, $Y \to X$, are tamed by the sum of the oriented volume forms on X and on Y. In fact quaisconformality is equivalent to this taming property if X is orientable.

Question. Probabaly, taming is the only class of functional *local* criteria that would visibly enhance the quasiconformality condition for maps $f : Y \to X$ with positive codimension. But are there miningful *global* criteria besides quasiminimaliy, such as lower bounds on the *filling radii* of the f-images of spheres from Y ?

1.10 Old Problems and New Perspectives.

In the above sections, we summarised several aspects of hyperbolicity in

FOUR DOMAINS:

dynamics, Riemannian geometry, group theory,
geometry of mappings.[53]

These "aspects" had acquired their present shapes by the late 1970s, but the basic issues had remaining unresolved.

I. IS THERE A GENERAL THEORY SUCH THAT THE HYPERBOLIC PATTERNS SEEN IN THESE FOUR DOMAINS WOULD APPEAR AS SPECIAL CASES OF CONSTRUCTIONS AND PHENOMENA IN THIS THEORY?

II. WHAT ARE THE MOST GENERAL/NATURAL (CATEGORY THEORETICAL?) CONCEPTS BEHIND CONSTRUCTIONS AND PROPERTIES OF HYPERBOLIC OBJECTS INDIVIDUALLY IN EACH OF THESE DOMAINS? WHAT ARE CLASSES OF MATHEMATICAL OBJECTS (CATEGORIES?) WHERE IT MAKE SENSE TO SPEAK OF HYPERBOLICITY?

[53] One also encounters hyperbolicity in *algebraic geometry* where it is manifested by the rate of *growth of genera* of curves of degrees $d \to \infty$ and in *arithmetic* where it is associated with bounds on the *numbers of K-rational points* of algebraic varieties. Since I feel incompetent in these matters, I refer to [81] where one can learn about it.

III. What are sources of hyperbolicty? What are the essential/potential pools of examples of hyperbolic objects?

IV. Which classes of hyperbolic objets are classifiable?

V. What is relative hyperbolicity? What are hyperbolic morphisms. What are (functorial?) hyperbolic constructions?

VI. What are invariants which portray hyperbolic features of hyperbolic as well as of non-hyperbolic objects?

VII. What is the boundary of the realm of hyperbolicity and what happens at this boundary?

In what follows, we describe a few concepts and results in hyperbolicity, mainly developed since the 1970s. We try to do this in the most general (regrertfully, not in purely category theoretical) terms, which, we think, may bring us a step closer to approaching the above seven questions.

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2 Negative Curvature and Hyperbolic Spaces.

Perceiving the "true nature" of a class $\mathcal{C}$ of mathematical objects, such as *hyperbolic spaces* or a larger class of *metric spaces*, depends on simultaneous resolution of the two complementary problems.

■ *Choosing a language which would allow a simple logically/conceptually perfect definition of $\mathcal{C}$, possibly more general that the commonly used one.*

❄, ❆, ❄... *Uncovering pools of the examples of mathematically/aesthetically attractive objects from $\mathcal{C}$, often with unexpected properties, e.g. with an abnormal symmetry.*

2.1 Induced Metrics, Lipschitz Maps, Pythagorean Products, Amalgamation, Localization and Scaling of Metrics, Minimal Paths and Geodesic Spaces.

We do not exclude *metrics* which assume zero and infinite values at some pairs of distinct points, but when we say *metric space* we do make non-zero and finiteness assumption.

Induced Metrics $f^*(dist)$ Isometric Maps. Admitting zero values allow us to speak of *metrics induced* by (not necessarily injective) maps $f : X \to (Y, dist_Y)$ where the induces metric on X is

$$f^*(dist_Y)(x_1, x_2) = dist_Y(f(x_1), f(x_2)).$$

If X is also a metric space, f is called *isometric*, if $f^*(dist_Y) = dist_X$, where the word *isometry* is reserved for bijective isometric maps, or, better to say to *isomorphisms in the category metric spaces and isometric maps.*

Metric Lipschitz Categories $\mathcal{ML}_{\{\lambda\}}$ and $\mathcal{ML}_1$. A map between metric spaces, $f : X \to Y$, is called *λ-Lipschitz*, $\lambda \geq 0$, if the induced metric satisfies:

$$f^*(dist_Y) \leq \lambda \cdot dist_X.$$

A λ-Lipschitz map f is called *λ-bi-Lipschitz*, if it is *one-to-one* and the inverse map from the image of to X is also λ-Lipschitz.

Spaces X and Y are called *λ-bi-Lipschitz (isomorphic)* if there exists a *bi-Lipschitz bijection* between them.

Since λ's multiply under composition of maps, **m**etric spaces and λ-**L**ipschitz maps make a $\mathbb{R}_+^\times$*-graded category*, call it $\mathcal{ML}_{\{\lambda\}}$, where *1-Lipschitz*, i.e. (non-strictly) *distance decreasing* maps constitute a subcategory denoted $\mathcal{ML}_1$.

Much (but not all) of metric geometry can be expressed in terms of this category. For instance, *isometries* between metric spaces can be defined as *isomorphisms in* $\mathcal{ML}_1$.

Geodesics. Another useful $\mathcal{ML}_1$-expressible concept is that of a *minimizing geodesic* in X defined as a 1-Lipschitz map of a segment $[a,b] \to X$, such that

$$dist_X(f(a), f(b)) = |a-b|.$$

(Such maps, of course, are necessarily, isometric.)

Thus, *geodesic metric spaces* X, where all pairs of points are joint by a minimizing geodesic are those where

the 1-Lipschitz maps from subsets $A \subset \mathbb{R}$ *admit* 1*-Lipschitz extension to* $\mathbb{R}$.

Graphs and Trees. The basic examples of geodesic spaces are *metric graphs*, with assigned length values to the edges and where the distances are defined as the lengths of the shortest paths between pairs of points. The simplest, yet by no means trivial among these are *trees* X, where all pairs of points serve as the ends of *unique*; hence, *lengh minimizing*, topological segments in X.

As we shall see in section 2.3 the essential properties of graphs and trees extend to high dimensional spaces with generalised curvatures $\kappa \le 0$.

Notice in passing – this is an easy exercise – that metric trees X enjoy the *universal Lipschitz extension property*:

the 1*-Lipschitz maps* $Y \supset A \to X$ *extends to* 1*-Lipschitz maps* $Y \to X$ *for all metric spaces* Y *and all* $A \subset Y$.

Scaling and Parametric Scaling. Multiplication of the distance function of a metric space $X = (X, dist)$ by a consant $\lambda > 0$ is a functorial operation in $\mathcal{ML}_{\{\lambda\}}$ and in $\mathcal{ML}_1$, denoted

$$\times_\lambda : X \mapsto \lambda X = (X, \lambda \cdot dist).$$

More generally, given a positive *function* $\lambda = \lambda(d) > 0$, $d \ge 0$, let the correspondingly scaled space $\lambda(d)X$ carry the metric

$$\lambda(dist(x_1,x_2))dist(x_1,x_2),$$

where this $\lambda(dist)\cdot dist$ is a true metric, i.e it satisfies the *triangle inequality*, provided the function $\lambda(d)$ is *monotone decreasing*.

Product of Metric Spaces. The only metric in the Cartesian product of metric spaces $X = X_1 \times X_2$ that makes X a product in the category $\mathcal{ML}_1$ (as well as in the isometric maps category) is the *sup-metric*.

$$dist_X((x_1,y_1),(x_2,y_2)) = \sup(dist_{X_1}(x_1,y_1), didst_{X_2}(x_2,y_2)),$$

where we say "sup" rather than "max" to allow infinite products $\times_i X_i$.

In general, given a positive function in two variables, say $l(d_1,d_2)$, $d_1, d_2 \ge 0$, the corresponding $dist_X = l(dist_{X_1}, dist_{X_2})$ may be taken for a metric in X, if l satisfies a certain *sublinearity condition* that would guarantees the triangle inequality for $dist_X$.

An instance of such an l is the l_p-function $(d_1^p + d_2^p)^{\frac{1}{p}}$, $p \ge 1$, where the most beautiful one is the *Pythagorean* $\sqrt{d_1^2 + d_2^2}$.

Observe – this is obvious but significant – that the full set of the inequalities on l expressing the above mentioned sublinearity follows from the triangle inequalities for $l(dist_{X_1}, dist_{X_2})$ for all *3-point* metric spaces X_1 and X_2.

Sup-Amalagamation, Localization and Path Induction of Metrics. Given metrics $dist_i$ on subsets $U_i \subset X$, define the *sup-amalgamated metric*

$$\coprod_{i \in I} dist_i$$

on X as the supremum of the metrics which are majorized by $dist_i$ on the subsets U_i.

Given a metric space $X = (X, dist)$, let $dist_{loc}$ be the metric on X that is the supremum of the sub-amalgamations$\coprod_{i\in I} dist_i$, where $dist_i$ are the restrictions of $dist$ to subsets $U_I \subset X$ and where and where the supremum is taken over all open covers $\{U_i\}$ of X.

Example: Localization versus Parametric Scaling. The metric obtained from $dist$ by parametric scaling obviously have their liocalizations equal to that of $dist$ itself.

The localization of the induced metric on a Y mapped, e.g, embedded, to a metric space X is called the *path induced* metric on X.

If, for instance, f is a *smooth* map between smooth manifolds and $dist$ on X is Riemannian, e.g, if $X = \mathbb{R}^n$, then the path induced metric on Y is everywhere $< \infty$. But, these metrics are everywhere infinite for generic continuous maps.

Path Metric Spaces. The metrics $dist$ on X for which $dist_{loc} = dist$ are called *localized* and if X is path connected these are called *path metrics*, since the distances between points in such a metric are equal to the infima of the lengths of the paths between these points.

Accordingly, path connected spaces X with localized metrics are called *path metric spaces* or *length spaces.*

If these infima are assumed by some paths that serve as *minimizing geodesics* between the corresponding points and then

Thus, *locally compact complete* path metric spaces X are, in fact, *geodesic spaces.* but this not true for complete metric spaces in general. However, the *path* property is as good as the *geodesic* one for most practical purposes.

If X is a *connected localized* metric space then *locally λ-Lipschitz* maps $X \to Y$ (obviously) are λ-*Lipschitz* for all metric space Y. And if Y is a geodesic space, e.g. $Y = [0,1]$, then the the converse is also true (and obvious).

In fact, a *connected* metric space is *localized* if and only if the 1-Lipschitz functions $X \to [0,1]$ constitute a *subsheaf* (rather than a sub-pre-sheaf) in the sheaf of all functions on X.

Coproducts/Amalgamation. Given isometric maps $Y \to X_1$ and $Y \to X_2$ with the images $Y_i \subset X_i$, $i = 1,2$, define the *coproduct* alsso called *amalgamation* of X_i over (under?) Y

$$X = X_1 \coprod_Y X_2 = X_1 \coprod_{Y_1 \leftrightarrow Y_2} X_2$$

by gluing X_1 with X_2 along Y_i, $i = 1,2$, via the isometry $Y_1 \leftrightarrow Y_2$ which the composition $Y_1 \leftrightarrow Y \leftrightarrow Y_2$ and by endowing the resulting set X with the sup-amalgamation of the metrics $dist_{X_i}$.

This, indeed, is a coproduct in the *category of path metric spaces and 1-Lipschitz maps.*

Multiple Amalgamations. Let X_i, $i \in I$, be an arbitrary set of metric spaces, and let

$$X_i \supset Y_i \underset{f_{ij}}{\leftrightarrow} Y_j \subset X_j$$

be a set of isometries between certain subsets $Y_i \in X_i$. If the graph G on the vertex set I and with the edges corresponding to the arrows f_{ij} is *acyclic*, then the set

$X = \coprod_G X_i$ obtained by gluing all f_{ij}-corrspomnding points, $x_i \underset{f_{ij}}{\leftrightarrow} x_j$ is covered by copies of X_i and one equip the resulting set X – amalgamation of X_i – with the sup-amalgamation $dist_X$ of the metrics $dist_{X_i}$.

And if G is a *tree*, i.e. it is connected as well acyclic, then $dist_X < \infty$ and $(X, dist_X)$ is a bona fide metric space.

g **[U]** *Uryson's Amalgamation.* Albeit the above $X_1 \coprod_Y X_2$ is not a true coproduct in the category of metric spaces and *isometric maps*, it serves for construction of universal (category theoretically *injective*) objects $\mathcal{U}$ in this category as follows.

Let I be a set of metric spaces X_i and F be a set of isometric maps between them (which do not necessarily constitute a category). Let G be the directed graph on the vertex set I with the arrow-edges corresponding to $f_{ij} \in F$.

Let $\tilde{G}$ be the universal covering of this graph and let $\tilde{I}$ and $\tilde{F}$ be the corresponding sets of spaces and maps.

Since this graph is acyclic, the amalgamation

$$\mathcal{U} = \coprod_{\tilde{G}} X_{\tilde{i}},$$

where the directions of the arrows $\tilde{f}_{ij}$ are forgotten, is correctly defined and and, endowed with the sup-amalgamation $dist_{\mathcal{U}}$ of $dist_{\tilde{X}}.$, the space $\mathcal{U}$ is called the *Uryson amalgamation* of $\{X_i, f_{ij}\}$.

For example, let $\{X_i\}$ be a countable set of finite metric spaces, which contains isometric copies of all finite spaces with rational distances, and let F be the set of all isometric maps between these X_i.

Then the metric completion $\overline{\mathcal{U}}$ of Uryson's amalgamation of these X_i is a complete separable[54]) metric space, called *Uryson universal metric space*, which is characterised by the universal extension property of isometric maps:

the isometric maps $X \supset Y \to \overline{\mathcal{U}}$ extends to isometric maps $X \to \overline{\mathcal{U}}$ for all compact metric spaces X and all $Y \subset X$.

In categorical terms , the isometric arrows $Y \to \overline{\mathcal{U}}$ and $X \leftarrow Y$ can be completeed to commutative diagrams of isometric maps.

$$\begin{array}{ccc} X & \longleftarrow & Y \\ & \searrow & \downarrow \\ & & \overline{\mathcal{U}} \end{array}$$

To get a feeling for the geometry of Uryson's $\mathcal{U}$, look at a simple example where X_i are integer real segments $[m,n] \subset \mathbb{R}$, $m,n \in \mathbb{Z}$, and F is the set of integer translations $[m_1,n_1] \to [m_2,n_2]$ and of their compositions with the reflection $x \mapsto -x$.

In this case $\mathcal{U}$ is a tree with countably many branches at every vertex and with all edges of unit length.

The universal extension property now concerns isometric maps of finite trees A with unit edges to $\mathcal{U}$ which send vertices to vertices: these extend to such maps $X \to \mathcal{U}$ of all finite ambient trees $X \supset A$ with unit edges.

Metric Spaces as set_F-Functors. Many properties of metric spaces are expressible in terms of finite subsets in them that suggests a view on general metric spaces as functors F from a small category M of finite metric spaces to the category of sets. For instance, such a use of a category M that contains *all* isometry classes of finite metric spaces may provide an adequate language for the Hausdorff limits and ultra limits of metric spaces.

But one needs much smaller category M to define the (generalised?) metric structure, for instance the category $M_{2,\mathbb{R}}$, where the pairs of real numbrs $\{a,b\}$ are taken for the objects, suffices.

What makes a functorial representation of metric spaces via $M_{2,\mathbb{R}}$ pleasantly attractive is that the triangle inequality is functorially elevated to a map between set-values of the functors F from $M_{2,\mathbb{R}}$ to sets:

$$set_F\{a,b\} \times set_F\{b,c\} \to set_F\{a,c\}.$$

(This makes one think of these pairs $\{a,b\}$ as of morphisms $a \to b$.)

One can only wonder if there are interesting geometric representations of such functors besides metric spaces and if there are categories similar to $M_{2,\mathbb{R}}$ leading to worthwhile generalizations of the category of metric spaces.

[54] *Separable* means admitting countable dense subset.

2.2 Standard Hyperbolic Spaces and their Small Brothers.

The general hypebolic theory departs from classical hyperbolic spaces $\mathbf{H}^n_{\mathbb{R}}$, $n = 0, 1, 2, \dots$ which are members of a one parameter family of *standard n-dimensional spaces of constant curvatures* $\kappa \in (-\infty, +\infty)$ that are divided into three classes according to the sign of κ.

[$\kappa > 0$**].** *The standard n-spaces of positive curvatures* κ are *n-spheres*, where the usual Euclidean realization of such a sphere in $\mathbb{R}^{n+1}$ has radius $\frac{1}{\sqrt{\kappa}}$.

(Circles of lengths $2\pi/\sqrt{\kappa}$, $\kappa > 0$, are regarded in this contex as standard 1-dimensional spaces of curvature κ.)

[$\kappa = 0$**].** *The standard spaces with zero curvatures* are the Euclidean $\mathbb{R}^n$.

[$\kappa < 0$**].** *The standard spaces with negative curvatures* κ *are the hyperbolic spaces.*

The standard spaces X carry natural *geodesic distance functions* and much (all?) of what we care about these X and relations between them can be expressed in the language of *metric spaces* as it is briefly indicated below.

Scaling and Curvature. *The* λ*-Scaling* $X \mapsto \lambda X$, $\lambda > 0$, *transforms standard spaces* X *with curvatures* κ *to standard spaces with curvatures* κ/λ^2, where λX is isometric (better to say *isomorphic*) to X if and only if $\kappa = 0$.[55]

Isometric Maps, Planes and Lines. If X_1 and X_2 are standard spaces of the same curvature κ then X_1 admits an *isometric*, i.e. distance preserving, map to X_2 if and only if $dim(X_1) = m \leq n = dim(X_2)$; if $1 < m < n$, the images of these maps are called *m-planes*; if $m = 1$ they are called *lines* and if $(n-1)$-planes are called *hyperplanes.*

These planes satisfy the familiar Euclidean property:

Every $(m+1)$*-tuple of points in a standard n-dimensional space is contained in an* m*-plane, which is unique unless these points are contained in an* $(m-1)$*-plane.*

Equidistorting Maps and Equidistorted Hypersurfaces. There is no isometric maps between standard spaces with different curvatures[56] but there are *equidistorting* maps $f : Y \to X$, which *parametrically scale* our metrics:

$$f^*(dist_X) = \lambda(d) dist_Y \text{ for some function } \lambda(d).$$

Such an f is called *normalized* if $\lambda(0) = 1$, which is equivalent to f being *path isometric*, i.e. having the path induced metric equal the original $dist_Y$.

If $dim(Y) + m < n = dim(X)$ a normalized *equidistorting* map $f : Y \to X$ exists if and only if the curvatures κ' of Y and κ of X satisfy $\kappa' \geq \kappa$, where the equality $\kappa' = \kappa$ implies that such an f is *isometric.*

Every equidistorting f can be normalized by rescaling $Y \mapsto \lambda(0)Y$ (or by $X \mapsto \lambda(0)^{-1}X$) that does not change the image of f.

The images of *codimension one*, $m = n-1$, equidistorted maps called *equidistorted hypersurfaces*, play a special role in the geometry of Y.

◗ These hypersurfaces, call them $Y' \subset X$ are *complete convex* which means they serve as *boundaries* of closed (possibly unbounded) convex subset $V \subset X$, where the latter convexity means that the path induced metrics in these V is equal to the induced ones.

In fact, the hypersurfaces Y' are *strictly* convex unless the curvatures κ' of the path induced metrics in them are equal to the curvature κ of X in which case Y' are (flat undistorted) hyperplanes.

[55] Albeit 1-dimensional hyperbolic spaces $H^1(\kappa)$ are all isometric to the real line $\mathbb{R}^1$, they neither isomorphic to $\mathbb{R}^1$ nor they are mutually isomorphic for different $\kappa < 0$ in the present context.

[56] We disregard such maps for $m = 1$ for the reason indicated in the previous footnote.

There are three kinds of equidistorted hypersurfaces in n-dimensional hyperbolic spaces X.

(1) $(n-1)$-*Spheres* which are the boundaries of the balls.

(2) *Horospheres*, which, with the path induced metrics in them, are isometric to $\mathbb{R}^{n-1}$.

(3) *Equidistants to hyperplanes* $Y_0 \subset X$, where such an equidistant $Y' \subset X$ has the distance function $dist(y', Y_0)$ *constant* in $y' \in Y'$.[57]

In general, both δ-*equidistants* $Y'_{\pm\delta} \subset X$ of any equidistorted hypersurface $Y' \subset X$, i.e. which have $dist_X(x, Y') = \delta$ and which lie on the two sides from Y', say $Y'_{-\delta}$ is inside the convex set V bounded by Y' and $Y'_{+\delta}$ outside,[58] are also *equidistorted.*

For instance, the δ-equidistants of spheres of radii r are the concentric spheres of radii $r \pm \delta$ (the interior ones dissaper for $\delta > r$) and the $\pm\delta$-equidistants of horospheres (these can be thought of as spheres of infinite radii) are again horospheres for all $\delta \geq 0$.

Exponential Expansion.

◖◗. Given two δ-equidistant equidistorted hypersurfaces Y_1 and Y_2 in a standard space X, there is a unique bijective map

$$Y_1 \underset{\nu=\nu_\delta}{\longleftrightarrow} Y_2, \text{ where } y_1 \underset{\nu=\nu_\delta}{\longleftrightarrow} y_2 \text{ if and only if } dist_X(y_1, y_2) = \delta.$$

This ν, called *normal* map, *scales the metrics* in Y's, namely, the metrics $dist_{Y_1}$ and $dist_{Y_2}$ brought by ν to the same Y satisfy

$$dist_{Y_1} = \lambda_{12} dist_{Y_2} \text{ where } \lambda_{12} \text{ is a positive constant.}$$

In general, $\lambda_{12} \neq 1$, i.e. f is *not isometric.* In fact, it is isometric if and only if $Y_1 = Y_{+\delta}$ and $Y_2 = Y_{-\delta}$ for a hyperplane $Y \subset X$.

If the ambient space X is *not a sphere*, i.e. it has curvature $\kappa \leq 0$, if Y_1 is *strictly convex* and Y_2 is the *exterior* δ-equidistant of Y_1 (i.e. outside V with $\partial V = Y$), then $\lambda_{12} > 1$ for all $\delta > 0$.

And if $\kappa < 0$, then the path induced metrics of the exterior equidistants satisfy

$$dist_{Y_{+\delta}} \geq (1 - \varepsilon(\delta))(\exp \delta\kappa^2) \cdot dist_Y \text{ where } \varepsilon(\delta) \to 0 \text{ for } \delta \to \infty,$$

for all equidistorted hypersurfaces $Y \subset X$.

(The factor $1 - \varepsilon(\delta) < 1$, which is essential for *flat* Y, is unneeded for spheres and horospheres: equidistants to horospheres $Y \subset \mathbf{X}$ satisfy the *equality* $dist_{Y_{+\delta}} = (\exp \delta\kappa^2) \cdot dist_Y$ and spheres enjoy the *strict inequality* $dist_{Y_{+\delta}} > (\exp \delta\kappa^2) \cdot dist_Y$.)

This *exponential expansion inequality* is our main guide in working out a general concept of hyperbolicity.

2.3 Negative Curvature According to Alexandrov, Busemann, Pedersen and Bruhat-Tits.

In late 1940's-1950's, Alexandrov, Busemann and Pedersen introduced three different classes of geodesic metric spaces X with *generalized curvatures* ≤ 0.

$[\kappa \leq 0]_1$, Pedersen's Tube Convexity. [93]: *ε-neighbourhoods of geodesic segments G in X are geodesically convex.*

That is if two points in X have distances$\leq \varepsilon$ from a G, then all geodesic segments between these points also lie within distances$\leq \varepsilon$ from G;

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[57]Unlike the Euclidean case, the δ-equidistants to hyperplanes in hyperbolic spaces are *strictly* convex for all $\delta > 0$.

[58]This V is non-ambiguous for *strictly* convex Y' but *flat* Y', i.e. hyperplanes, divide X into two *half-spaces*, say $V_\pm$, which are both (non-strictly) convex and have $\partial V_+ = \partial V_- = Y'$.

These $V_\pm$ are *indistinguishable*: there exists a *unique involutive isometry* of X which interchanges $V_+ \leftrightarrow V_-$ while keeping Y' fixed.

$[\kappa \le 0]_{\mathbf{2}}$, Busemann's Convex Spaces [?]: *The distance function is geodesically convex.*

This means that

the restrictions of the distance function $dist_X : X \times X \to \mathbb{R}$ to the products of the pairs of the geodesic segments $G_1 = [a_1, b_1]$ and $G_2 = [a_2, b_2]$ in X are convex (meaning $\smile$) functions on these products $G_1 \times G_2 \subset X \times X$ which are isometrically identified with the rectangulars $[a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$.

$[\kappa \le 0]_{\mathbf{3}}$, Alexandrov's $CAT(0)$-Spaces [6] *The distance functions $d_y : X \to \mathbb{R}$ for $d_y : x \mapsto dist_X(x, y)$ restricted to the geodesic segments $G \subset X$ are more convex, for all $y \in Y$, than the corresponding distance function in the plane $\mathbb{R}^2$.*

This means that if the values of $d = d_y$ at the ends a and b of a segment $G = [a, b]$ are equal to these values of an $\mathbb{R}^2$-distance function d', then $d \le d'$ on $[a, b]$.

Observe that Alexanfrov's $[\kappa \le 0]_{\mathbf{3}}$ says, in effect, that certain 4-*point* metric spaces *admit no isometric maps* to X and Bisemann's $[\kappa \le 0]_{\mathbf{2}}$ is of the similar prohibitive nature for certain 6-point spaces.[59] But Pederson's $[\kappa \le 0]_{\mathbf{1}}$, at least in its present form, admits no such finitary description, besides, the definition of the distance to a geodesic segment depends on an *existential quantifier.*[60] Yet, it is obvious that $[\kappa \le 0]_{\mathbf{1}} \Leftarrow [\kappa \le 0]_{\mathbf{2}}$.

On the other hand the implication $[\kappa \le 0]_{\mathbf{2}} \Leftarrow [\kappa \le 0]_{\mathbf{3}}$, which amounts to an upgrading the convexity of $dist(x, y)$ in x and y separately to the convexity in two variables, is more subtle and it, probably, fails for the ordinary geodesic convexity. (I did not try making counterexamples.)

But Alexandrov's *strong* convexity in x, which, by definition, stronger than that of the distance in the Euclidean plane, does imply such strong convexity in two variable by a simple argument, which, however, relies on the *existence* of minimizing geodesics [4].

From Local to Global: Cartan-Hadamard Theorem.

All three definitions of $\kappa \le 0$ have their local counterparts, where each point in X admits a neighbourhood $U \subset X$, such that the $\kappa \le 0$ convexity condition holds for points and geodesic segments contained in U and

local $[\kappa \le 0]_{\mathbf{i}}$ integrates to $[\kappa \le 0]_{\mathbf{i}}$ one for the simply connected spaces X.

Namely, the (generalized) Cartan-Hadamard theorem, proven under somewhat different sets of assumptions in [26], [93], [6] and [3], says that

if a geodesic metric space X is locally $[\kappa \le 0]_{\mathbf{i}}$, $1, 2, 3$, then the universal covering $\tilde{X}$

is $[\kappa \le 0]_{\mathbf{i}}$.

On the surface of things, there are by far more Busemann's convex $[\kappa \le 0]_{\mathbf{2}}$-spaces than Alexandrov's $[\kappa \le 0]_{\mathbf{3}}$ that are $CAT(0)$. For instance, *all* strictly convex Banach spaces[61] are Busmann's $[\kappa \le 0]_{\mathbf{2}}$ but only the Hilbertian one among them are Alexandrov's $CAT(0)$.

And some naturally occurring Pedersen's $[\kappa \le 0]_{\mathbf{1}}$-spaces, notably convex Euclidean domains B with the projective Hilbertian metrics, are not $[\kappa \le 0]_{\mathbf{2}}$, unless B is an ellipsoid [93].

But the predominant majority of spaces with $\kappa \le 0$ which one encounters in geometry are Alexandrov's $CAT(0) = [\kappa \le 0]_{\mathbf{3}}$.[62] Apparently, all (known?) locally

[59] The original Busemann's definition rules out certain 5-tuples of points.

[60] The geodesity condition also uses an *existential quantifier*, but, at least in Alexandrov's case, this can shoveled under the rug, see [4].

[61] One can slightly generalize Buseman's definition of convexity to include all, possibly non-strictly convex, normed spaces, i.e. those where the spheres contain straight segments.

[62] One may disagree with this, especially when it comes to non-locally compact infinite

$[\kappa \le 0]_{\mathbf{1}}$-manifolds and polyhedra can be rendered $[\kappa \le 0]_{\mathbf{3}}$ by a simple modification of their constructions. For instance, the classes of groups that serve as fundamental groups of, say topologically polyhedral, locally $[\kappa \le 0]_{\mathbf{1}}$ spaces, probably(?), doesn't depend on $i = 1, 2, 3,$.[63]

The most prominent $CAT(0)$, besides the Euclidean/Hilbertian spaces $\mathbb{R}^n$, $n = 1, 2, 3, ...\infty$, are

✳ *metric trees* X, where the path metric on such an X is defined by assigning the length values to its segments, and

● *the spaces* $\bigcirc_N = SL_N(\mathbb{R})/SO(N)$, $N = 2, 3, \dots$, *of ellipsoids of volume one in* $\mathbb{R}^N$ *centerd at* 0, *which serve as the unit balls of the Euclidean metrics which are the positive definite quadratic forms on* $\mathbb{R}^N$ *with discriminants one.*

These $\bigcirc_N$ are endowed with Riemannian metrics invariant under the action of the special linear groups $SL_N(\mathbb{R})$, where such a metric on $\bigcirc_N$ is (obviously) unique up to scaling.

Next come other

irreducible symmetric space X of noncompact types,[64]

such as the hyperbolic spaces $\mathbf{H}^n_{\mathbb{R}}$, $\mathbf{H}^n_{\mathbb{C}}$ and $\mathbf{H}^n_{\mathbb{H}}$ which we met earlier (where the hyperbolic plane $\mathbf{H}^2_{\mathbb{R}}$ is isometric to $\bigcirc_2$.)

All these X admit *isometric embeddings* to $\bigcirc_N$ with large $N = N(X)$. Conversely, the complete Riemannian manifolds which admit isometric embeddings to $\bigcirc_N$ are, possibly reducible, symmetric spaces[65] with $\kappa \le 0$.

$[\kappa \le 0]$-Operations: Products and Coproducts.

The classes $[\kappa \le 0]_{\mathbf{1}}$ and $[\kappa \le 0]_{\mathbf{2}}$ as well as their local versions of spaces X are (this is obvious) stable under l_p-*products*:

the metric $(dist^p_{X_1} + dist^p_{X_2})^{\frac{1}{p}}$ *in the Cartesian product* $X = X_1 \times X_2$ *of* $[\kappa \le 0]_{\mathbf{i}}$-*spaces is* $[\kappa \le 0]_{\mathbf{i}}$ *if* $i = 1, 2$ *for all* $p > 1$.

But the (local as well as of global) class of (local as well as of global) $[\kappa \le 0]_{\mathbf{3}}$, i.e. $CAT(0)$, is closed only under (finite and infinite) *Pythagorean products*, i.e. where

$$dist_X = \sqrt{dist^2_{X_1} + dist^2_{X_2}}.$$

This seemingly innocuous, this product operation significantly enriches the geometry of spaces as it is already seen in the case of the Euclidean spaces which are Pythagorean products of lines. A less familiar example is that of the (finite and infinite) *Pythagorean products of metric trees* which have a beautifully elaborated, still not fully understood, $CAT(0)$-geometry.

Instance of a Question. What kinds of countable, in particular finitely presented, groups Γ *can isometrically, (discretely and non-discretely) act on products of trees?*

Convex Amalgamations. Recall that coproducts/amalgamations are obtained by gluing metric spaces, say X_1 with X_2, by isometries between subsets in them,

$$X_1 \supset Y_1 \overset{iso}{\longleftrightarrow} Y_2 \subset X_2,$$

where the corresponding coproduct

$$X = X_1 \coprod_{Y_1 \leftrightarrow Y_2} X_2$$

---

dimensional X, such as Banach spaces, for instance.

[63] It my be instructive to check this for *projectively flat* manifolds.

[64] *Noncompact type* in this context is synonimous to $\kappa \le 0$.

[65] This *reducible* means *decomposable* into a Cartesian/Pythagorian product, such as $\mathbb{R}^n$, for $n \ge 2$, for instance.

is endowed with maximal metric that is equal to $dist_{X_{1,2}}$ on $X_{1,2} \subset X$, $i = 1, 2$ (see section 2.1).

If the spaces X_1 and X_2 are $[\kappa \le 0]_{\mathbf{i}}$-for an $i = 1, 2, 3$, and the subspaces $Y_{1,2}$ are geodesically convex, which is equivalent to be images of isometric embeddings from a path isometric (necessarily $[\kappa \le 0]_{\mathbf{i}}$) space Y to X_1 and X_2, then one expects the coproduct,

$$X = X_1 \coprod_{Y_1 \leftrightarrow Y_2} X_2 = X_1 \coprod_{Y} X_2 = X_1 \coprod_{Y \rightrightarrows X_{1,2}} X_2$$

to be also $[\kappa \le 0]_{\mathbf{i}}$, since reflection of geodesics off convex subsets makes them diverge only faster.

I am not certain if this has been verified for Pedersen $[\kappa \le 0]_{\mathbf{i}}$, but the validity of this cases $[\kappa \le 0]_{\mathbf{2}}$ and $[\kappa \le 0]_{\mathbf{3}}$ are easily confirmed in the local and global cases, where this is shown by observing that convexity of function $d_1(x_1, y)$ and $d_2(x_2, y)$ implies that for

$$d(x_1, x_2) = \inf_{y \in Y} (d_1(x_1, y) + d_2(x_2, y))$$

where the supremum is taken over a *convex* domain Y. This is immediate for the ordinary geodesic convexity (Buseman's $[\kappa \le 0]_{\mathbf{i}}$) where it follows from the *convexity of projections* of convex sets, and it is slightly harder for the stronger $[\kappa \le 0]_{\mathbf{3}}$-convexity of Alexandrovs $CAT(0)$ spaces, where the proof was given by Reshetnyak in 1960, see [4]).

Alexandrov's $CAT(\kappa)$-Spaces for $\kappa \ne 0$. The concept of a metric space X with curvature bounded from above by 0 generalizes to an arbitrary number κ by replacing $\mathbb{R}^2$ in the above $[\kappa \le 0]_{\mathbf{3}}$-definition by the standard plane with curvature κ, i.e. by requiring the distance function in X to be more convex than the distance in the two sphere or radius $1/\sqrt{\kappa}$ for $\kappa > 0$ and/or in the hyperbolic plane with curvature $\kappa < 0$.

The basic examples of these are the following.

Graphs A non-simply connected metric graph X is $CAT(\kappa)$ for a $\kappa > 0$ if and only if all cycles in X have lengths $\ge 2\pi/\sqrt{\kappa}$, while the trees are $CAT(-\infty)$, i.e. $CAT(\kappa)$ for all κ.

Alexandrov's Cones. *Normal cones*[66] *over $CAT(1)$-spaces are $CAT(0)$.*

If a geodesic metric space X is locally $CAT(0)$ in the complement to a point $x_0 \in X$ and if X is *conical* at this point then

X is locally $CAT(\kappa)$ if and only if the base L of the cone is $CAT(1)$.

This generalizes to $\kappa \ne 0$ as follows.

Let X be a geodesic metric space X which is locally $CAT(\kappa)$ in the complement to a point $x_0 \in X$ and which is asymptotically conical at x_0: the scaled spaces λX *Hausdorff (ultra) converges* (see 2.7) for $\lambda \to \infty$ to a normal cone over some metric space L.

Then X is $CAT(\kappa)$ if and only if L is $CAT(1)$. (see [?] [4])

Question. What are the geometries of the bases of conical $[\kappa \le 0]_{\mathbf{i}}$-spaces for $i = 1, 2$?

$\frac{1}{\ge 6}$-Complexes.[67] Let X be a 2-dimensional simplicial polyhedron (complex) where each 2-simplex Δ is endowed with the metric of the regular unit triangle, say

[66]To get a fair picture of normal cones C over L, assume that L is path isometrically imbedded to the unit sphere $S^N \subset \mathbb{R}^{N+1}$ and take the ordinary cone with the apex $0 \in \mathbb{R}^{N+1}$ made of the straight segments $[0, l] \subset \mathbb{R}^{n+1}$, $l \in L$. If L is a smooth closed curve, such a cone with the induces path metric is flat Euclidean away from the apex. It is $CAT(0)$ (as well as $[\kappa \le 0]_{\mathbf{i=1,2}}$) if and only if this curve has length $\ge 2\pi$.

[67]This is, *morally,* the same $\frac{1}{6}$ as the one appearing in the small cancellation theory (see 1.5), but the *precise* meaning of this "morally" remains unclear.

of size ε, in the standard plane of curvature $\kappa \le 0$, i.e. in $\mathbb{R}^2$ for $\kappa = 0$ and in the hyperbolic plane of curvature $\kappa < 0$.

Let the links with the angular metrics at the vertices of X be $CAT(1)$, i.e. they have lengths $\ge 2\pi$.

(If $\kappa = 0$ this is equivalent to all cycles in these links to contain at *least* 6-*edges* and if $\kappa < 0$ and ε is small, say $\varepsilon \le 0.1\kappa^2$, the links are $CAT(1)$ if these cycles contain at *least* 7-*edges.*)

Since X is locally conical for the polyhedral metric (that is the sup-amalgamation of the standards metrics in $\Delta \subset X$), the $CAT(1)$-condition for the links implies that X, is locally $CAT(\kappa)$ and the universal covering $\tilde{X}$ of X is $CAT(0)$ by Cartan-Hadamard theorem.[68]

Uryson's CATs and Bruhat-Tits Buildings. Let A be a standard space of constant curvature κ, let Γ be a discrete isometry group of A and let Δ be a convex polyhedral fundamental domain of Γ. Let $\{X_i\}$ be the set of convex unions of Γ-translates of Δ and F be the set of maps $f_{ij} : X_i \to X_j$, $X_i, X_i \subset A$, which are isometries from Γ restricted to X_i and which move X_i to X_j.

The resulting Uryson amalgamations $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_{\Gamma,\Delta}$, defined via the universal coverings $\tilde{G}$ of the graphs G of f_{ij}-arrows (see 2.1) are $CAT(\kappa)$, by Reshetnyak's amalgamation theorem. (This theorem is valid for all κ, [4].)

For instance, if $A = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$ an $\Delta = [0,1]^n \subset \mathbb{R}^n$, then the resulting $\tilde{\mathcal{U}}$ are isometric to products of trees.

Among these $\tilde{\mathcal{U}}$ one distinguishes those which can be called *Uryson's* Γ-*buildings*, where $\Delta \subset A$ is a convex *simplex* and Γ is generated by *reflections* of Δ in the faces.

What we call here *buildings* are a spaces B which are obtained by convex amalgamation of copies of A, which admit isometric embeddings into some $\tilde{\mathcal{U}}$ and such that

• $B \subset \tilde{\mathcal{U}}$ are union of copies of Δ, which are called *chambers* in this context.

• every two chambers in a B are contained in an *apartment* – a copy of A in this parlance.

Besides, one often imposes some symmetry conditions B that, in particular, imply that the isometry group of B is transitive on the pairs (A, Δ), where A is an apartment and $\Delta \subset A$ is a chamber in it, [**?**] [**?**].

Two Perspectives on Singular Curvature. The apparent motivation of Alexandrov in introducing his curvature inequalities $\le \kappa$[69] was axiomatic scrutiny of the concept of curvature and recapturing basic properties of curvature in the synthetic terms of angular inequalities of geodesic triangles mimicking these for geodesic triangles in surfaces, while Busemann's idea of curvature≤ 0 was, probably, inspired by the Banach-Minkowski geometries.

Extension of Distance Decreasing Maps to $CAT(\kappa)$-Spaces.

A difference between $CAT(0)$ and alternative concepts of negative curvature is demonstratively visible in the following, category theoretically attractive, property of Alexandrov's spaces.

K.L.S. Lipschitz Extension Theorem. Let X_+ and X_- be complete geodesic metric spaces where

X_+ is Alexandrov' s space with curvature $\ge \kappa_+$; [70]

[68] I am not cerain on what happens for $\kappa > 0$.

[69] Alexandrov also introduced the mirror siblings of his $CAT(\kappa)$spaces where the distance functions are *more concave* than these in the standard spaces of curvature κ.

[70] You do not loose much if you assume that X_+ is a *Riemannian manifold* with the sectional curvatures *bounded from below by* κ_+. In fact, the theorem is significant already where X_+ is the standard space with constant curvature $\kappa \ge \kappa_+$, e.g. $X_+ = \mathbb{R}^n$ in the case $\kappa_+ = 0$.

X_- is Alexandrov's $CAT(\kappa_-)$, i.e. with curvature $\leq \kappa_-$, where in the case of $\kappa_- > 0$ we additionally assume that all geodesics in X_- are distance minimizing.[71] (This is always so for $\kappa_- \leq 0$.)

If $\kappa_+ \geq \kappa_-$, then all 1-Lipschitz maps from subsets $Y \subset X_+$ to X_- extend to 1-Lipschitz maps $X_+ \to X_-$.

This was proven by M.D. Kirszbraun[72] in 1934 for Euclidean spaces, that is for $\kappa_+ = \kappa_- = 0$, and extended by Lang and Schroeder in 1997 to all CAT-spaces [74],[4].

At the heart of the proofs in the Euclidean and non-Euclidean cases, lies charming *Kirszbraun's rigidity lemma* a leading case of which is formulated below in a dual form.

Let $\Delta, \Delta' \subset \mathbb{R}^n$ be convex n-simplices with (flat) (n-1)-faces denoted $F_i \subset \Delta$ and $F'_i \subset \Delta'$, $i = 1, 2, ..., n+1$, such that the dihedral angles between these faces satisfy

$$\angle(F_i, F_j) \geq \angle(F'_i, F'_j),\ \ i, j = 1, 2, ..., n+1.$$

Then, in fact, $\angle(F_i, F_j) = \angle(F'_i, F'_j)$.

Side Questions. What are convex polyhedra Q besides simplices with a similar *dihedral angles rigidity* property?

Which Q admit and which do not admit deformations Q' where the flatness of the deformed faces $F'_i \subset Q'$ is relaxed to *positivity of their mean curvatures* and such that the dihedral angles between the (tangent spaces of the) pairs (F'_i, F'_j) of adjacent faces are *smaller than the angles between the corresponding faces F_i and F_j in Q?* (See [**?**] for partial answers.)

Alexandrov and Busemann felt uncomfortable with the singularities allowed by their own definitions and tried to compensate by adding regularity conditions, such as *lower bounds* on the curvature (Alexandrov's school) and by the *uniqie continuation* of geodesics (Buseman's G-spaces).

Besides, it seems that Busemann, Alexandrov and their disciples were not interested in (aware of?) possibilities of particular geometries of concrete spaces, such as symmetric spaces and polyhedra with curvatures$\leq \kappa$: trees, $\frac{1}{6}$-complexes – none of these are ever mentioned in their papers.[73]

In 1972, two decades after Busemann (1948) and Alexandrov(1951), Bruhat and Tits have arrived at curvature≤ 0 from an opposite direction: they discovered/constructed a new class of spaces – *affine buildings* – comparable in their beauty to symmetric spaces. And albeit Bruhat and Tits established basic $\kappa \leq 0$-properties of their buildings, these were not recognised at the time as Alexandrov's $CAT(0)$. And a bit later, Tits introduced spherical buildings which, a posteriori, were classified as $CAT(1)$.

Thus, Busemann (generalized by Pederson, 1952) and Alexandrov contributed to the development of $[\kappa \leq 0]$-theory by introducing the basic concepts and proving essential prperties of axiomatically defined spaces with negative curvatures while the worthwhile classes of such spaces were furnished by $2d$-polyhedra underlying small cancellation groups of Tartakovskii (1947) and by Bruhat-Tits buildings.

2.4 $CAT(\kappa)$-Orbispaces and $CAT(\kappa)$-Constructions.

Ramified Coverings. The leading example of what we call *a ramified coverings* between topological spaces is the quotient map $\tilde{X} \to X = \tilde{X}/\Gamma$ where Γ is a group that

[71] This condition cannot be dropped, but, possibly, can be relaxed.

[72] *Über die zusammenziehende und Lipschitzsche Transformationen.* Fund. Math. 22: 77-108.

[73] Singular piecewise flat metric *on surfaces* have been studied in Alexanfrov's school where the $\frac{1}{6}$-condition was well appreciated. However, the singular metrics on surfaces only confirm what has been already known in the regular case – new phenomena start coming along with advent of *topologically singular* spaces, such, as 2-polyhedra of the small cancelation theory. But these were not approached in Alexandrov's school.

acts on X, where the action is discrete but may have fixed points.

We also allow on the roles of ramified coverings[74] the maps $f : \tilde{X} \to X$, for which there exist

a space $\tilde{\tilde{X}}$ with an action of a group Γ, such that $X = \tilde{\tilde{X}}/\Gamma$, and such that

$\tilde{X} = \tilde{\tilde{X}}/\tilde{\Gamma}$, for a subgroup $\tilde{\Gamma} \subset \Gamma$, and where f is equal to the natural map

$$\tilde{X} = \tilde{\tilde{X}}/\tilde{\Gamma} \overset{f}{\longrightarrow} \tilde{\tilde{X}}/\Gamma = X.$$

For instance, if $f : \tilde{X} \to X$ is an ordinary covering map, then the above works with the universal covering of X taken for $\tilde{\tilde{X}}$.

In general, f *ramifies* at the points in X over which f fails to be locally 1-to-1, but what we call *the ramification locus* $R_f \subset X$ may be greater than that: it is

the union of the images of the fixed points sets of subgroups of Γ *acting on* $\tilde{\tilde{X}}$ *under the quotient map* $\tilde{\tilde{X}} \to X = \tilde{\tilde{X}}/\Gamma$. [75]

In our examples, X and $\tilde{X}$ are path metric spaces, the actions of groups are discrete isometric with finite or infinite isotropy groups at the fixed points[76] and the maps f are path isometric.

It follows from Reshetnyak's amalgamation theorem, see [**?**] that

if X *is locally* $CAT(\kappa)$, *if the ramification locus* $R_f \subset X$ *is locally convex and if the pullback* $f^{-1}(R_f)$ *is locally* $CAT(\kappa)$, *then* $\tilde{X}$ *is also locally* $CAT(\kappa)$.

For instance, locally symmetric spaces X of non-compact type that contain locally convex (i.e. totally geodesic) submanifolds $Y \subset X$ often (always?) admit ramified coverings $f : \tilde{X} \to X$ with $R_f = Y$ which provide a significant source of (singular) $CAT(0)$-manifolds.

Orbistructures. An orbistructure on a topological space X is a representation of all, sufficiently small neighbourhoods $U \subset X$ as quotient spaces, $U = \tilde{U}/\Gamma_U$ for discrete actions of groups Γ_U on $\tilde{U}$, where these representations must agree in an obvious sense on the intersections of these neighbourhoods, [**?**].

An orbi-covering of an orbispace X is a ramified covering of X where the corresponding quotient representation of X as $\tilde{\tilde{X}}/\Gamma$ locally agrees with the representations $U = \tilde{U}/\Gamma_U$.

An orbicovering $f : \tilde{X} \to X$ is called *universal* if $\tilde{U}$ is connected and such that every connected orbicovering $f' : \tilde{X}' \to X$ admits an orbicovering $f'' : \tilde{X} \to \tilde{X}'$, such that $f = f' \circ f''$,

$$\tilde{X} \to \tilde{X}' \to X \quad (\tilde{X} \to X)$$

Universal orbi-coverings do not always exist: a connected locally contractible orbispace may admit no single orbicovering, but this rarely (never?) happens in the $CAT(0)$-category.

Namely, define a *local* $CAT(\kappa)$*-orbispace* as a geodesic metric space X with an orbistructure where all $\tilde{U}$ are endowed with local $CAT(\kappa)$ metrics such that the actions of Γ_U on $\tilde{U}$ are isometric and where the quotient maps $\tilde{U} \to U = \tilde{U}/\Gamma_U$ are path isometric.

All known examples of local (non-pathological?) $CAT(\kappa)$-orbispaces, at least for $\kappa < 0$, admit $CAT(\kappa)$ universal coverings, where the simplest instance of this is where the groups Γ_U are trivial, $U = \tilde{U}$ and where the universal $CAT(\kappa)$ covering is delivered by elementary topology with $CAT(\kappa)$-condition confirmed by the Cartan-Hadamard theorem.

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[74] This is not the most general definition.

[75] This definition allows the restriction of f to $f^{-1}(R) \subset \tilde{X}$ to be locally one-to-one.

[76] An instructive example of a ramified covering between $CAT(0)$-spaces, where such an isotropy is *infinite* is the obvious map from the metric completion of the universal covering of the punctures plane $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}^2$.

More amuzingly, every n-dimensional Brihat-Tits building X is the universal orbicovering of a flat n-simplex Δ with a suitable orbistructure on it with all grops Γ_U being finite.

$CAT(\kappa)$**-Orbicoverings.** It is known in general [**?**] that

◆ *a* $CAT(\kappa)$*-orbispace* X*,* $\kappa \le 0$*, admits the universal orbi-covering* $\tilde{X}$*, if the actions of the groups* Γ_U *on* $\tilde{U}$ *are rigid:* if a $\gamma \in \Gamma_U$ fixes a non-empty open subset in $\tilde{U}$ than it fixes $\tilde{U}$.[77]

The proof of this follows in the steps of that of the (generalized) Cartan-Hadamard theorem, but the $CAT(\kappa)$-role of ◆ is different: Cartan-Hadamard serves *to verify* the $CAT(0)$-condition for spaces known to be locally $CAT(0)$ while ◆ delivers *new* local $CAT(0)$-spaces.

For instance, let X be an arbitrary 2-dimensional simplicial polyhedron. Then choosing finite coverings $\tilde{L}_x$ of the links of the vertices $x \in X$ defines an orbistructure in X in an obvious way.

If these $\tilde{L}_x$ have no cycles with< 6 edges, then this structure is local $CAT(0)$ for the natuaral *p.l.* metric in X, and then the universal orbi-covering of X is $CAT(0)$. (See [**?**] for more general examples.)

$CAT(-1)$*-Uniformization Conjecture.* Every topological manifold admits a $CAT(-1)$ orbifold structure with finite groups Γ_U.

Cubical $CAT(0)$-Spaces.

We conclude this section by describing yet another attractive class of $CAT(0)$-spaces X, called *cubical* $CAT(0)$*-spaces/complexes*, where the $[\kappa \le 0]$-curvature condition is easily verifiable and which support many interesting isometry groups acting on them.

Such an X, say of dimension n, is covered by isometric copies of the cube $[0,1]^n$,[78] such that every two cubes in X intersect (if at all) across a common face and such that the links L of all faces with their natural piecewise spherical geometries are $CAT(1)$.

The pleasant feature of this class of spaces is a simple combinatorial criterion for recognition of the $CAT(1)$-property:

if L and all its links contain no empty triangles,[79] *then L is locally* $CAT(1)$.[80]

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2.5 Strict Convexity, Geodesics and Hyperbolicity Problem.

The $[\kappa \le 0]_{\mathbf{1,2,3}}$-spaces which allow zero curvatures lie on *the boundary of the hyperbolic domain*; to enter this domain the strict inequality $\kappa < 0$ is required.

If $i = 3$, hypebolicity is furnished by Alexandrov's $CAT(\kappa < 0)$-spaces but these have no(?) *canonical* $[\kappa < 0]_{\mathbf{1}}$ and $[\kappa < 0]_{\mathbf{2}}$ counterparts.

However, albeit non-canonical, *strict versions of the convexity inequalities* $[\kappa \le 0]_{\mathbf{1,2}}$ achieve hyperbolicity even in more visible manner than $CAT(\kappa < 0)$, where *Pedersen's strict tube convexity* defined below is most illuminating.[81]

[77] This ◆ holds true for rigid $[\kappa \le 0]_{\mathbf{3}}$-orbifolds as well.

[78] One may allow cubes of different. possibly unbounded. dimensions.

[79] An empty triangle in a simplicial complex is a triple of vertices pairwise joined by edges but not filled by a 2-simplex. For instance, empty triangles in graphs are cycles with 3 edges.

[80] A triangulated surface with $CAT(1)$-links is $CAT(1)$, even if it contains empty triangles.

[81] The charm of Alexandrov spaces mainly comes from the richness of their local/infinitesimal geometries while the large scale picture of hyperbolic spaces is better seen in the lights of $[\kappa < 0]_{\mathbf{1,2}}$.

Definition of $[\kappa < 0]_1$. Let X be a geodesic metric space and denote by $d(x) = d_Y(x)$ the distance functions from $x \in X$ to a geodesic segment Y in X.

Say that X *is strictly (tube) convex*, if the value of d_Y at *the mid-point* between $x_1, x_2 \in X$, such that $d_1 = d(x_1) = d(x_2) = d_2$ satisfy the following inequality for all segments $Y \subset X$ and all $x_1, x_2 \in X$:

❂$_\varepsilon$ $$d(x_{mid}) \leq d_1 - \varepsilon(d_1, d_{12}),$$

where $d_{12} = dist(x_1, x_2)$, where x_{mid} is defined by the equalities

$$dist(x_{mid}, x_1) = dist(x_{mid}, x_2) = d_{12}/2.$$

and where $\varepsilon = \varepsilon(d_1, d_{12})$, $d_1, d_{12} > 0$, is a function which is required to be *strictly positive* in a neighbourhood of $(0,0)$, say, for $0 < d_1, d_{12} < \epsilon_0 > 0$.

Since the inequality ❂$_\varepsilon$ for distance functions to the geodesic segments in X implies *the same inequality* for $d(x) = dist(x, Y)$ for *all convex subsets* $Y \subset X$, this ❂$_\varepsilon$ with any $\varepsilon > 0$ implies *uniform convexity*, hence exponential growth, of the external equidistants of convex hypersurfaces $H = \partial Y$ in X (e.g. of concentric spheres, which we already know for $\mathbf{H}^n$, see 2.2) which yields all familiar properties of manifolds with negative curvatures.

This growth of these equidistants, say H_D, which are the sets of points *in*X with $dist_X(x, Y) = D$ is accompanied with the *lower bounds* on their l-distortions (defined as in 1.2) which can be formulated as follows.

The diameters of the curves $C \subset H_D$ of length l are bounded by a function $\Delta_X(D, l)$such that:

◖ $\Delta_X(D, l) < l$ *for all* $L, D > 0$.

This is, essentially, (but not quite) a reformulation of ❂$_\varepsilon$.

◖ $\Delta_X(D, l) \leq a \cdot \log(l + 1) + b$ *for some constants* $a, b > 0$, *all* $l > 0$ *and* $D > l$. This follows from the exponential growth of (the lengths of curves in) H_D for $D \to \infty$.. and instantaneously implies *shadowing of quasigeodesics* $C \subset X$ *by geodesics.*

Moreover, (compare [❒❒❒] in 1.2) there is a function $D_X(l)$, such that

[❒'] *if a double infinite curve* $C \subset X$ *has the diameters of all its segments* $I_l \subset C$ *of length* l *for* $l_0 \leq l \leq 2l_0$, $l_0 > 0$, *bounded from below by* $diam(I_l) \geq \Delta_X(D, l)$ *for all* $D \geq D_X(l)$, *then* C *lies within finite Hausdorff distance (see 2.6) from a geodesic in* X.

Horospheres and Geodesic Flows.

Horospheres in complete geodesic spaces X with (generalized) curvatures $\kappa \leq 0$ are associated with *geodesic rays* $R \subset X$, also denoted $[x_0, x_\infty) \subset X$, which are subsets isometric to $[0, \infty)$ where $x_0 \in R \subset X$ corresponds to $0 \in [0, \infty)$.

The horoball associated to R, is the union of the r-balls with the centers $x \in R$ and with $dist(x, x_0) = r$; the corresponding *horosphere* is the boundary of this ball (compare 2.13).

It is easy to see that if X is strictly tube convex then the distortion of all convex equidistance H_D is asymptotic for $D \to \infty$ to that of horospheres. It follows that

the shadowing conclusion in [❒'] (similarly to [❒❒❒] from 1.2) *only needs the distortion of* C *to be strictly smaller than that of the horospheres.*

Horospheres also serve as "building blocks" of the expanding and/or contracting leaves of the (not quite) foliations associated with the *geodesic flow* on the space $G(X)$ of isometric maps $g : \mathbb{R} \to X$, which is defined by the obvious action of $\mathbb{R}$ on $G(X)$.

What we call "leaves" and pass through all points $g \in G(X)$, denote them $L_{exp}(g)$ and $L_{contr}(g.)$ where their projections to X are the horospheres for the two rays in the geodesic $g(\mathbb{R}) \subset X$ starting at $x_0 = g(0)$ which are $g((-\infty, 0]) \subset g(\mathbb{R}) = g((-\infty, +\infty)) \subset X$ and $g([0, +\infty)) \subset g(\mathbb{R})$.

The projections $L_{exp}(g), L_{contr}(g) \to X$ are far from 1-to-1 for singular spaces X and formally speaking the geodesic flow applies to the leaves $L_{exp}(g)$ and $L_{contr}(g)$ in $G(X)$ but not their projections yo X.

Yet, there is a *half-flow* that acts on the space $G_+(X)$ of isometric maps $[0,\infty) \to X$ which sends horospheres H to their internal equidistants H_{-t} by the normal projections $H \to H_{-t}$ which are contracting but not necessarily injective or surjective maps.

(To appreciate the picture, look at $G(X)$ ad $G_+(X)$ for 2-dimensional polyhedra X build of convex simplices with constant curvatures $\kappa < 0$ (the case of $\kappa = 0$ is also instructive) and with all links being $CAT(1)$.)

If you think dynamically, the horosphere $H = H(R) \subset X$ associated with a ray $R_0 \subset X$ should be defined as follows.

A point $x_1 \in X$ is in H if and only if there exists a ray $R_1 \subset X$ issuing from x_1, such that

$$dist_X(x_0(r), x_1(r)) \to 0 \text{ for } r \to \infty,$$

where $x_i(r) \in R_i$, $i = 1, 2,$, are the points within distances r from the end points of the rays.

The essence of $\kappa < 0$ can be now expressed in terms of geometry of such "horospheres". In fact, to be consistent, one should forfeit X and work directly in the the space of rays where hyperbolicity of the $\mathbb{R}_+$-action plays the role of $k < 0$.

From Strict Convexity to Hyperbolicity.

The shadowing property [❐], as we know, implies the geodesic universality and the geodesic rigidity (see ✯$^{\to}$ and ✯$^{\leftrightarrow}$ in 1.3) which say, in effect, that

If X is compact locally strictly convex manifold, then the space $G(X)$ of geodesic in X and the geodesic flow are uniquely determined, up to reparametrization of geodesics. by the fundamental groups $\Gamma = \pi_1(X)$.

This begs several questions:

1. *Can one describe/reconstruct $G(X)$ in terms of Γ?*

2. *Which properties of Γ make it suitable for the role of the fundamental group of a locally strictly convex manifold?*

3. *Is there a natural class of what may be called "hyperbolic spaces" that would enjoy the shadowing and the geodesic universality?*

This class must be defined in terms applicable to manifolds as well as to discrete spaces such as finitely generated groups and where discontinuous maps are allowed.

In the next section we shall introduce terminology needed for responding to these questions.

2.6 Almost Isometries, Controlled Maps, Nets, Blow-up Graphs, Quasiisometries, Quasigeodesic Spaces, Word Metrics, etc.

Geometry of metric spaces X on a given scale $D > 0$ concerns pairs of points with $dist_X(x_1, x_2) \geq D$, where the essential large scale concepts are as follows.

Almost Isometries. An almost isometry or, if you want to be specific, *a D-almost isometry*, between metric spaces, say between Y and Y', is *a correspondence* $Y \leftrightarrow Y'$ such that the pairs of corresponding points (y, y') satisfy

$$|dist(y_1, y_2) - dist(y_1', y_2')| \leq D,$$

where our "correspondences", are subsets in $Y \times Y'$ of pairs of *corresponding points* $y \leftrightarrow y'$,

$$\{y, y'\} \subset Y \times Y',$$

and where all $y \in Y$ and $y' \in Y'$ participate in such a correspondence: the projections of $\{y, y'\}$ to Y and to Y' are *onto.*

Notice that a metric space B is bounded, i.e. $diam(B) < \infty$, if and only if it is almost isometric to a single point; thus, passing from isometries to almost isometries is like taking metric spaces modulo bounded ones.

Question. Can one formally define almost isometry via *factorization* of the category of metric spaces by the subcategory of bounded ($diameter < \infty$) spaces?

Are there other factorizations of the category of metric spaces by classes of "small spaces"?

Lipschitz on the Large Scale. A (possibly discontinuous) map between metric spaces is called *λ-Lipschitz on the scale* $D > 0$ (better to say $\geq D$) if

$$dist_Y(f(x_1), f(x_2)) \leq \lambda dist_X(x_1, x_2)$$

for the pairs of points in X, for which $dist_X(x_1, x_2) \geq D$.

Displacements and Nets. A subset $X' \subset X$ is *a D-net* if the D-balls with the centers in X' cover X; equivalently, if there exists a (possibly discontinuous) map $\Phi' : X \to X'$ with *displacement bounded by* D, which means that

$$dist_X(\phi(x), x) \leq D \text{ for all } x \in X.$$

Often we do not specify D and say just "net" meaning "D-net for some D".

Hausdorff Distances. The directed Hausdorff distance $dist_{Hau}(X_1 \to X_2)$ between subsets $X_1, X_2 \subset X$ is the infimum of the numbers $D \geq 0$ for which there exists a (possibly discontinuous) map $X_1 \to X_2$ with displacement $\leq D$.

Then the ordinary symmetric Hausdorff distance is

$$\max\left(dist_{Hau}(X_1 \to X_2), dist_{Hau}(X_2 \to X_1)\right).$$

Quasiisometries between metric space X and Y were defined by Margulis in 1970 as *bi-Lipschistz* (i.e. λ-Lipschitz in both directions for some $\lambda < \infty$) maps between *nets* X' in X and Y' in Y,

$$X \supset X' \overset{f}{\leftrightarrow} Y' \subset Y.$$

Observe that the all X, being nets in $G_{\leq D}(X)$ are quasiisometric to $G_{\leq D}(X)$ for all $D > 0$.

Also notice that pairs of nets $X_1', X_2' \subset X$ contain *δ-separated*[82] subnets $X_i'' \subset X_i$, $i = 1, 2$, which are related by bijections $X_1'' \leftrightarrows X_2''$ with *bounded displacements* and because of the δ-separation these bijections are necessarily bi-Lipschitz.[83] Hence, all nets in X are mutually quasiisometric. It follows that quasiisometry is an *equivalence relation* between metric spaces.

Quasiisometries can be also defined as compositions of almost isometries and bi-Lipschitz isomorphisms or else, as pairs of large scale Lipschitz maps $X \leftrightarrows Y$ (for some λ and D), such that both composed maps $X \to X$ and $Y \to Y$ have bounded displacements.

Exercise: Quasiisometries with Trees. Show that metric spaces quasiisometric to $\mathbb{R}$ are almost isometric to $\mathbb{R}$.

More generally if X is quasiisometric to metric tree with a geodesic metric then it is almost isometric to such a tree.

Periodic Metrics and Burago's Theorem. Quasiisometries are far from being almost isometries for higher dimensional spaces. For instance, the balls $B(R, \kappa)$ of radii R

[82]A subset $X' \subset X$ is called *δ-separated* if $dist(x_1', x_2') \geq \delta > 0$ for $x_1' \neq x_2'$.

[83]δ-Separated nets themselves, e.g. such nets in the plane $\mathbb{R}^2$, *are not* always mutually bi-Lipschitz equivalent.[24],[80].

in the standard spaces of constant curvatures κ with $\kappa^2 = R$ are λ-bi-Lipschitz to the Euclidean balls, say with $\lambda = 10$ for all R, but there is no D-almost isometries between these balls for $R \to \infty$. In fact, that would imply *isometry* between $B(1,\kappa)$ and an Euclidean ball, since the rescaled balls $\frac{1}{R}B(R, R^2\kappa)$ are isometric to $B(1,\kappa)$.

However, according to a theorem by D.Burago (1992) [?], this is true for *almost homogeneous spaces:*

a metric space X is called *almost homogeneous* if the action of the isometry group G on X is *cobounded* , i.e. the G-transforms of a *bounded* subset in X *cover* X.

◆ *If an almost homogeneous geodesic metric space X is quasiisometric to $\mathbb{R}^n$, then X is almost isometric to $\mathbb{R}^n$ with some,* (typically non-Euclidean) *Banach-Minkovsky metric, i.e. a geodesic metric invariant under translations of $\mathbb{R}^n$.*

An essential step in Burago's proof is a construction of an *almost selfsimilarity* of X, namely of an almost isometry between X and $2X = (X, 2dist_X)$, which is obtained with a use of the following rendition of *the Borsuk-Ulam theorem.*

Let μ_i, $i = 1, 2, ..., n$, be summable (not necessarily positive), measures on the unit segment $[0,1]$ ("summable" means $|\mu_i|[0,1] < \infty$) *with continuous densities. Then there exists a partition of the $[0,1]$ into two subsets – finite unions of subintervals – say $R_\pm \subset [0,1]$, which have equal μ_i-masses,*

$$\mu_i(R_+) = \mu_i(R_-),\ i = 1, ..., n,$$

and such that the common boundary set of these subsets, $Z = \partial R_+ = \partial R_- \subset [0,1]$, consists of at most n-points.

(The space of ordered pairs $R_\pm$, $|\partial R_\pm| \leq n$, with the topology borrowed from the space of 1-dimensional $\mathbb{Z}_2$-chains, is homeomorphic to the sphere S^n with the antipodal map $S^n \leftrightarrow S^n$ corresponding to the ±-involution, where the customary linear Borsuk-Ulam can be recognized by looking at the space of polynomials of degrees $\leq n$, the zero sets of which, albeit non-uniquely, represent our Z.)

Questions. (a) Which almost selfsimilar spaces, in particular almost homogeneous ones, are almost isometric to selfsimilar spaces? (This question is resolved in [72] for some nilpotent groups, while the extent of deviation of general nilpotent groups from being selfsimilar is evaluated in [21])

(b) Can one relax the coboundness of the isometry group action in ◆ by the existence of a *transitive set* of D-almost isometries of X for some D? (Should we call such X *almost-almost homogeneous*?)

(c) Let the group $\mathbb{R}^n$ freely acts on a compact Riemannian manifold. Under what conditions and to what extent (in the spirit of [21]) are the orbits with the induced Riemannian metrics almost selfsimilar?

Blow-up Graph $G_{\leq D}(X)$. This is the graph on the vertex set X where the pairs of points with $dist(x_1, x_2) \leq D$ are joined by the edges and where this graph is endowed with the supremum of the metrics where all edges are isometric to the segment $[0, D]$ and where the distances between all pairs of vertices $x_1, x_2 \in X \subset G_{\leq D}(X)$ are bounded by $\max(D, dist_X(x_1, x_2))$.

It is clear that

- $G_{\leq D}(X)$ with this metric is *almost isometric* to X,
- if X is a *localized, e.g. path isometric*, space then $G_{\leq D}(X)$ is a *geodesic space,*
- *large scale Lipschitz* maps $X \to Y$ correspond to *Lipschitz* maps $G_{\leq D}(X) \to G_{\leq D}(Y)$ for large D.

However *quasiisometries* $X \leftrightarrow Y$ do not always yield bi-Lipschitz isomorphisms $G_{\leq D}(X) \leftrightarrow G_{\leq D}(Y)$, essentially, because the spaces X and Y may have *different cardinalities.*

(One can artificially "thicken" the spaces by taking their products with some bounded space Z of huge cardinality, but this only shows that quasiisometries are inherently *equivalences* – they are vaguely similar to *Morita equivalence*– rather than isomorphisms.)

$\lambda(d)$*-Control* The concept of large scale Lipschitz maps admits the following quantitative (essentially cosmetic) refinement.

A map between metric spaces, $f_{12} : X_1 \to X_2$, is called *controlled* by a (positive monotone increasing) function $\lambda(d)$, if the induced distance function $f_{12}^*(dist_{X_2})$ on X_1 satisfies

$$f_{12}^*(dist_{X_2}) \leq \lambda(dist_{X_1}).$$

For instance, λ-Lipschitz maps are controlled by the linear function $d \mapsto \lambda \cdot d$ and being *λ-Lipschitz on the scale D*, is essentially the same as being controlled by the affine function $\lambda d + D$.

The $\lambda(d)$-version of "large scale" has a (little) bonus:

the composed maps maps $f_{13} = f_{12} \circ f_{23}$, where

$$X_1 \overset{f_{12}}{\to} X_2 \overset{f_{23}}{\to} X_3,$$

are controlled by the compositions $\lambda_{23} = \lambda_{12} \circ \lambda_{23}$ of the corresponding functions

$$\mathbb{R}_+ \overset{\lambda_{12}}{\to} \mathbb{R}_+ \overset{\lambda_{23}}{\to} \mathbb{R}_+.$$

This can be interpreted as *a grading of the category* of maps between metric spaces by the semigroup of maps $\mathbb{R} \to \mathbb{R}$, where, in the large scale Lipschitz case, this semigroup lies in the group $\mathrm{aff}_+(\mathbb{R})$ of orientation preserving affine maps of the line.[84]

Quasigeodesic Spaces. A metric space X is called *quasigeodesic* if it is quaisiisometric to a geodesic space. In fact, this is only needed for the canonical embedding $X \hookrightarrow G_{\leq D}D(X)$ where the graph $G_{\leq D}(X)$ defines above is *now* given the *geodesic* metric with the edges of length D. (This is the localization of the metric defined earlier.)

One can also define quasigeodesicity of spaces in an intrinsic way by requiring that every two points $x_1, x_2 \in X$ are joined by (i.e. contained in) a *quasigeodesic segment* or a *quasisegment* $E \subset X$ that is the image of a D-almost isometric isometric map of a real segment $[0, d]$ to X where $D >$ depend on X but not on (x_1, x_2).

Exercise. Show that *quasi*geodesic spaces are *almost* isometric to geodesic ones.

An essential instance of the concept of quasiisometry used by Margulis (and by everybody else since 1970) is as follows.

☛*the universal coverings of compact Riemannian manifolds with isomorphic fundamental groups are quasiisometric.*[85]

Albeit ostentatiously "soft" and childishly simple (see the proof below), this allowed Margulis to conceptualise, simplify and generalize Mostow's proof of his

[84]The (connected Lie) group $\mathrm{aff}_+(\mathbb{R})$, when endowed with a (left)-invariant Riemannian metric g, becomes isometric to the hyperbolic plane $\mathbf{H}^2$ with curvature $\kappa = \kappa(g) < 0$. Similarly, $\mathbf{H}^{n+1}$ is isometric to the group ho of homotheties $x \to \lambda x + y$ of $\mathbb{R}^n$ with a (left)invariant Riemannian metric. Amazingly, the symmetries of these metrics, besides $(n+1)$-parameters corresponding to the group translations, harbour extra $\frac{n(n+1)}{2}$-parameters which correspond to rotations of $\mathbf{H}^{n+1}$ around a point.

[85]Margulis (privately) attributes the idea to Efremovich (1953) [36] and Ratcliff (see p. 572 in [97]) directs to a 1912 paper by Dehn.)

Hyperbolic Rigidity Theorem: *Compact Riemannain manifolds with constant curvatures* -1 *of dimensions* $n \geq 3$ *with isomorphic fundamental groups are isometric.*[86]

The link between algebra and geometry furnished by ☛ is demystified by an introduction of the concept of a *word metric* in a finitely generated group Γ.[87] Below is the definition.

Cayley Graphs and Word Metrics. Given a group Γ and a (usually generating) subset $\Delta \subset \Gamma$, let G_Δ be the graphs with the vertex set Γ and where edges correspond to the pairs $(\gamma, \gamma\delta)$ for all $\gamma \in \Gamma$ and $\delta \in \Delta^{\pm 1}$, i.e. either δ or δ^{-1} is in Δ.

The quasigeodesic (in fact, $\mathbb{Z}$-*geodesic*) metric on $\Gamma \subset G_\Delta$ induced from the geodesic metric on this graph where all edges are assigned unit lengths is called the Δ-*word metric in* Γ.

Equivalently, this can be defines as

an integer valued metric where the i*-balls* $B_\gamma(i) \subset \Gamma$, $\gamma \in \Gamma$, *are the left*[88] γ*-translates of the set of the group products of the* i*-tuples of elements from* Δ,

$$B_\gamma(i) = \gamma \cdot \underbrace{\Delta^{\pm 1} \cdot \Delta^{\pm 1} \cdot \ldots \cdot \Delta^{\pm 1}}_{i} \subset \Gamma.$$

It is also obvious (but significant) that if

$$\Delta_2 \subset \underbrace{\Delta_1^{\pm 1} \cdot \Delta_1^{\pm 1} \cdot \ldots \cdot \Delta_1^{\pm 1}}_{i} \text{ and } \Delta_1 \subset \underbrace{\Delta_2^{\pm 1} \cdot \Delta_2^{\pm 1} \cdot \ldots \cdot \Delta_2^{\pm 1}}_{i} \text{ for some } i,$$

then the Δ_1- and Δ_2-metrics are (bi-Lipschitz) equivalent:

$$0 < c \leq \frac{dist_1}{dist_2} \leq C < \infty.$$

Observe that

- *the left translations* $\Gamma \to \Gamma$, *where* $\gamma \mapsto \gamma_0\gamma$, *are isometries* for these metrics,
- *the right translations* $\gamma \mapsto \gamma\gamma_0$ *have bounded displacements;*
- *the* Δ*-metrics associated with finite generating subsets* Δ *in* Γ *are mutually bi-Lipschitz equivalent*; they collectively called *word metrics.*[89]

Now, let Γ be the *fundamental group* of a compact Riemannian manifold X, or of any *compact geodesic metric space* for this matter. Then the word metrics in Γ are

bi-Lipschitz equivalent to the metrics induced on the orbits of the Galois (deck transformation) action of $\Gamma = \pi_1(X)$ *on the universal covering* $\tilde{X} \to X$.

Proof. The Galois action of Γ on $\tilde{X}$ is free and discrete with the orbits, which are the pullbacks of points $x \in X$ under the covering map $\tilde{X} \to X$, being D-*nets* in $\tilde{X}$ for $D = diam(X)$.

Since the Galois action is *isometric*, the orbit maps $O_{\tilde{x}} : \Gamma \to \tilde{X}$ for $\gamma \mapsto \gamma(\tilde{x})$, are *Lipschitz* for all $\tilde{x} \in X$ and all Δ-word metrics in Γ with *finite* $\Delta \subset \Gamma$.

Since X is compact, Γ is finitely generated and, on the other hand, all R-balls $B_{\tilde{x}}(R) \subset \tilde{X}$ intersect these orbits over *finite* subsets. Hence, these intersections, when regarded as the pullbacks

$$\Delta_R = O_{\tilde{x}}^{-1}(B_{\tilde{x}}(R)) \subset \Gamma,$$

[86] In his original paper Mostow assumes the manifolds to be diffeomorphic.

[87] I do not know where the word metric was explicitely defined for the first time. (Was it in A.S. Svarc', *Volume invariants of coverings* of 1955?

[88] You can not tell this "left" from"right" because there is a golden reason why a mathematician's mind cringes away from the concepts brewed in the pot of human culture.

[89] There are interesting metrics with *infinite* Δ, e.g. where Δ is invariant under conjugations in Γ, such as the commutator $[\Gamma, \Gamma]$.

are *finite* subsets which *generate* Γ for all *sufficiently large* R (in fact, for $R \geq 2diam(X)$).

Finally, since the subsets $\Delta_R \subset \Gamma$ *are finite* and the orbit map $O_{\tilde{x}}$ is *injective*, the inverse maps $O_{\tilde{x}}^{-1} : \Delta_R \to B_{\tilde{x}}(R)$ are *Lipschitz*, and since the induced metric in the orbits are *quasigeodsic* by ✐, the full inverse map $O_{\tilde{x}}^{-1}$ is *Lipschitz on the* Γ*-orbit* of $\tilde{x}$. QED.

Thus, if X is *compact*, then

✍ the orbit maps $O_{\tilde{x}} : \Gamma \to \tilde{X}$ of the Galois action of $\Gamma = \pi_1(X)$ on $\tilde{X}$, are *quasiisometries* for *all word metrics* in Γ and all Γ*-invariant* path metrics in $\tilde{X}$ (which are the same as the path metrics path induced from path metrics in X).

2.7 Logic and Language of Metric Spaces: Errors, Points, Maps, Limits, Asymptotic Cones and Isoperimetric Inequalities.

The definitions of almost isometries quasiisometries, and of large scale Lipschitz maps $f : X \to Y$ in general, present us with the following problems.

Specification/Quantification of Error Parameters. The quasification of metric concepts depends numbers, such as D and λ which specify by how much the properties of "quaisiobjects" deviate from these of the true ones: almost isometries and quasiisometries from isometries, quasigeodesics from geodesics and –this will come in 2.9 – classes/concepts of hyperbolic spaces from these of metric trees, etc.

On the one hand, we do not care about specific values of these D and λ – on the other hand, we must keep tracks of them in the course of arguments.

Question. Is there a concise way of writing down such arguments without an explicit use of numerical constants; yet, keeping the exposition rigorous and non-ambiguous?

This question hardly can be answered without

(1) a modification/generalization of the concept of *metric space*

and of

(2) a logical analysis of manipulations with numbers in arguments concerning "quasipropertes" of properties of spaces and maps specified by parameters.

As far as (1) is concerned, we want to be able to operate with "spaces" and morphisms between them, *without*, at least notationally, appealing to points in them, where our motivation is as follows.

∗ A property of a map $f : X \to Y$ to be a an almost isometry or a quasiisometry is rather insensitive to what happens at particular points $x \in X$ – removing a point does not change the relevant property of f.

This is partly compensated by allowing *multivalued maps* $X \to Y$ that are *correspondences* between X and subsets $Y' \subset Y$, but this is not a long term solution.

∗ Albeit the distances $dist(x_1, x_2) \leq D > 0$ are not present in the definition of quasiisometry, you can't exclude them from metric spaces X, since $dist(x, x) = 0$.

Formally, "self-interacting" points can be excluded by introducing what can be called

Pre-metric Spaces. These are defined (similar to a set_F-functor from 2.1) as sets P with the following structures:

• a partially defined composition, $P \times P \supset R \to P$, denoted

$$(p_1, p_2) \to p_1 \triangleright p_2;$$

• a map $P \to \mathbb{R}_+$, denoted $p \mapsto ||p|| = ||p||_P$

The principal examples of such $P = (P, \triangleright, ||...||)$ are associated with metric spaces X:

$P = P(X) = X \times X$, where $R \subset X \times X$ consists of the pairs $((x_1, x_2), (x_3, x_4))$, such that $x_2 = x_3$, where

$$(x_1, x_2) \triangleright (x_2, x_4) = (x_1, x_4)$$

and where the triangle inequality translates to $||p_1 \triangleright p_2|| \leq ||p_1|| + ||p_2||$.

Two other mutually logically similar classes of examples are *normed spaces* and groups Γ with (left) invariant metrics, where $||\gamma|| = dist(\gamma, id)$.

The arrow $X \to P(X)$ for metric space X can be reversed by observing that x in $p = (x, y) \in P(X)$ is determined by the subset $P' \subset P$, such that $(p', p) \in R$ for all $p' \in P$.

This suggests a translation the language of metric spaces to the P-languge, where, in particular, the D-scale definition is conveniently expressed in terms of

$$P_{\geq D}(X) = \{p\}_{||p|| \geq D} \subset P(X).$$

However, this P, does not give a new conceptualization of quasiisometries, such as their representation by isomorphisms (which is impossible anyway).

☀ Probably, the true solution, IF at all desired,[90] must involve a more significant extension of the category of metric spaces.

The first step could be passing from individual quasiobjects to families of these over domains $\mathcal{D}$ of real parameters, where dimensions of these domains may grow in the course of introduction of new quasiobjects.

For instance the full family of blow-up graphs $G_{\leq D}(X)$, where D is running over all of $\mathbb{R}_+$, must be regarded as a single object.

Geometric properties of metric spaces X on large scales $D \to \infty$ can be (partly) expressed in terms of the asymptotic cones

$$Con_\infty(X) = \lim_{D \to \infty} D^{-1} X.$$

Recall that $D^{-1}X = (X, D^{-1}dist_X)$, and that "lim" must be understood as the *ultralimit* of the sequence $X_i = i^{-1}X$, over a *non-principal ultrafilter* μ which is an additive $\{0, 1\}$-valued measure on the set $\{1, 2, ...\}$, where "non-principal" means that it takes value *zero* at some *infinite* subsets in $\{1, 2, ...\}$.

Granted such a μ, the limit cone $Con_\infty(X) = lim_\mu i^{-1}X$ is defined as the set of μ-*asymptotic classes* of bounded sequences $x_i \in i^{-1}X$, i.e. such that $i^{-1}dist_X(x_i, o)$ remains bounded for $i \to \infty$ and a fixed reference point $o \in X$, and where "μ-asymptotically equivalent" $x_i \sim_\mu y_i$ signifies that $x_i = y_i$ on a subset of $i \in \{1, 2, ...\}$ of μ-measure 1.

A pleasant feature of such limits in general, $\lim_\mu X_i$, is that they define a *functor* from the category of sequences $\{(X_i, o_i)\}$ of metric spaces with reference points and D_i-almost isometries for $D_i \to 0$, to the category of metric spaces.

Also these can be seen as set theoretic models of limit points of the *first order theory over* $\mathbb{R}$ [91] of finite subsets Y in metric spaces X with induced metric on Y, where the basic operation is what corresponds to the union $Y_1 \cup Y_2$ *in the language of this theory.*

This can be also expressed in terms of the first order $\mathbb{R}$-theory (with limits) of contravariant functors F from the category $\mathcal{M}$ of finite metric spaces to the category of sets (where $F_X(M)$ is the set of isometric maps $M \to X$.)

[90]The purpose of such a generalization is not so much application to metric geometry but rather an extension of "metric ideas" to other categories.

[91]I am not certain this is the standard terminology.

The first order language of finite tuples of "points" $\{x_i\}$ decorated with numbers $r_{ij} \geq 0$ is used in definitions of hyperbolic spaces in section 2.9, where we augment this language with "segments" E and extend it further by allowing "quasisegments". And in section 2.11 we introduce surfaces and their areas, which is harder to reconcile with the idea of "first order language".

Below is a rare and quite amusing instance of possible utility of logical scrutiny of a class of geometric arguments.

Sharp Isoperimetric Problem in $CAT(0)$-Spaces X. Do all $(n-1)$-cycles in X with $(n-1)$-volumes A bound n-chains with volumes $V \leq V_{\text{☻}}(A)$, where $V_{\text{☻}}(A) = V_{\text{☻},n}(A)$ denotes the the volume of the Euclidean n-ball with the boundary of $(n-1)$-volume A?

There are several proofs of this in the classical case where X is the Euclidean space $\mathbb{R}^n$ and where especially logically simple proofs are known to exist in the dimensions $n = 2, 4$.

At the core of these proofs – Santalo's (1953?) for $n = 2$ and Croke's (1980) for $n = 4$ – lie "elementary" integral-geometric "formulas" that relate the volumes of Euclidean balls to the volumes of the boundary spheres.

The essential feature of these "elementary formulas" is that they make sense for all Riemannian manifolds, where in the case of $\kappa \leq 0$ they have the correct error terms and yield sharp isoperimetric inequalities. (Santalo and Croke adapted their arguments to Romanian manifolds but they apply to all $CAT(0)$ spaces.)

Besides $n = 2, 4$, the sharp equidimensional ($dim(X) = n$) isoperimetric inequality $V \leq V_{\text{☻}}(A)$ is known for 3-*dimensional Riemannain manifolds* with negative curvatures where it was proven by Kleiner (1992) (see [69] where further references can be found) but his (variational Almgren's style) proof is significantly "more transcendental" than those by Santalo and Croke. (In principle Klener's proof may work in general 3-dimensional $CAT(0)$-spaces but it seems technically more difficult.)

Historically, all of the above was preceded by technically simple but logically rather transcendental *symmetrization arguments* (Steiner 1838, Schwarz 1884) which show that

> *sharp equidimensional isoperimetric inequalities hold in the standard spaces of constant curvatures κ and yield our $V \leq V_{\text{☻}}(A)$ for $\kappa \leq 0$.*
> *Morover the inequality $V \leq V_{\text{☻}}(A)$ is inherited by Pythagorean products of spaces, $X = \times_i X_i$.*

Non-equidimensional Isoperimetry. If $dim(X) > n$, then the principal issue is not *evaluation* of volumes of *given* domains (chains) in X in terms of their boundaries S, but rather *construction* of small chains that *fill-in* given S.

This is relatively easy for $dim(S) = 1$, since, for instance, the geodesic cones over circles in the $CAT(0)$ spaces $Cone_{s_0}(S) \subset X$, $s_0 \in S$, have their induced path metrics of curvatures ≤ 0. [92] Thus, the general inequality $V \leq V_{\text{☻}}(A)$ for $n = 2$ follows from the equidimensional case.

And starting from a 1912 article by Dehn, combinatorial renditions of such (non-sharp) inequalities – we shall meet them in section 2.11 – have been playing key roles in the study of the small cancellation and of hyperbolic groups.

The sharp isoperimetric, or rather *filling inequality* $V \leq V_{\text{☻}}(A)$ in the Euclidean spaces $\mathbb{R}^N$, for all n and N was proven my Almgren in 1986, where the argument is both logically and mathematically quite complicated and seems hard to extend even to the hyperbolic spaces $\mathbf{H}^N$.[93]

[92] This follows from the Gauss formula in the Riemannian case and the proof in singular $CAT(0)$-spaces relies on an elementary/synthetic construction of "inexpanding mappings" by Reshetnyak (1968).

[93] Non-sharp filling inequality in $\mathbb{R}^N$ with a constant depending on N was proven by Federer

Question. What would be a rigorous formulation of *impossibility to proof* the sharp inequality $V_{\mathbb{R}^n}(A)$ in $\mathbb{R}^n$ for all n by a same "level of transcendence" argument as was used for $n = 2$ and 4?

2.8 Cylindrical Towers of Spaces and Geometric Asymptotics of Group Actions.

There is a variety of large scale properties of spaces with $\kappa \le \kappa_0 < 0$ which can be taken for definition of *hyperbolicity*, where the essential requirements for such a definition are:

quasiisometry invariance and *quasi-locality* for *geodesic* metric spaces X,

where the latter means that hyperbolicity of the R-balls in X for some $R < \infty$ and *simply connectedness* of X must yield hyperbolicity of X.

Besides, this definition must accommodate the following

MODEL EXAMPLES OF HYPERBOLIC SPACES.

The logic of hyperbolicity is a *deformation of the logic of metric trees* with geodesic metrics (the meaning of this will become clarified later on), where, recall, a topological space X is a tree if every two points $x_1, x_2 \in X$ serve as the end points of a *unique* topological segment (homeomorphic to $[0,1]$, unless $x_1 = x_2$) in X.

A metric tree T with a vertex $\mathbf{t}_0 \in T$ taken for the root, can be described in terms of the *r-spheres* $S_{\mathbf{t}_0}(r) \subset T$ of points with distances r from $\mathbf{t}_0$ or in the language of *shortest paths* in T issuing from the root.

Accordingly there are two constructions of such trees.

Trees as $[\substack{\downarrow\\ \downarrow}]$-*Cylindrical Towers.* Think of the set $\mathbb{R}_+$ of real numbers $r \ge 0$ as a category, where the inequalities $r_2 \le r_1$ are taken for morphisms $r_2 \to r_1$ and let $\tau : r \mapsto S(r)$ be a covariant functor from this category to the category of sets, such that $\tau(0)$ is a one point set. (This will serve as the root of the tree.)

Thus, τ defines an $\mathbb{R}_+$-family of sets $S(r)$ and maps $P_{r_1 r_2} : S(r_2) \to S(r_1)$ which satisfy the composability relation for

$$S(r_3) \to S(r_2) \to S(r_1) \text{ for } r_3 \ge r_2 \ge r_1.$$

Let $\tilde{T}$ be the graph with the disjoint union of $\coprod_{r \in \mathbb{R}} S(r)$ taken for the set of vertices and where the arrows

$$S(r_2) \ni s_2 \overset{P_{r_1 r_2}}{\mapsto} s_1 \in S(r_1)$$

are taken for the edges with the lengths $r_2 - r_1$ assigned to them.

Denote by $\tilde{\mathbf{t}}_\mathbf{0} \in \tilde{T}$ the vertex corresponding to $\tau(0)$ and define the tree $T = T_\tau$ as the *quotient space* of $\tilde{T}$ where pairs of points $\tilde{t}_1, \tilde{t}_2 \in \tilde{T}$ are identified if they have *equal distances* to the (future) root,

$$dist_{\tilde{T}}(\tilde{t}_1, \tilde{\mathbf{t}}_\mathbf{0}) = dist_{\tilde{T}}(\tilde{t}_2, \tilde{\mathbf{t}}_\mathbf{0})$$

and if there exists a vertex $s \in \tilde{T}$ and *minimal paths*, say $[\tilde{\mathbf{t}}_\mathbf{0}, s]_1, [\tilde{\mathbf{t}}_\mathbf{0}, s]_2 \subset \tilde{T}$ which join $\tilde{\mathbf{t}}_\mathbf{0}$ with s and such that $t_i \in [\tilde{\mathbf{t}}_\mathbf{0}, s]_i$, $i = 1, 2$.

$[\curlyvee]$ *Amalgamations of Segments.* Let $\underline{T}$ be a set of isometric copies of segments $[0, r(\underline{t})] \subset \mathbb{R}_+$, $\underline{t} \in \underline{T}$, let R be the equivalence relation on $\underline{T}$ which

(a) identifies all zero points in these segments (which makes the root of the tree);

(b) glues pairs of segments by isometries along subsegments which, because of (a) both contain zeros in them.

Then the quotient space $\underline{T}/R$ is a tree which we take for our T.

Hyperbolic Riemannian Cylinders. Let $g_r = g_r(s)$, $r \in \mathbb{R}_+$, be Riemannian metrics on a smooth manifold S and let $g = g_r + dr^2$ be the Riemannain metric on $X = S \times \mathbb{R}_+$.

---

and Fleming in 1960. Another proof with a constant depending on n was found by Michael and Simon (1973) whose argument easily extends to all $CAT(0)$-spaces.

Observe that $S(r) = S \times r \subset X$ are *mutually equidistant hypersurfaces* in X, where

$$dist_X(S(r_1), S(r_2)) = |r_1 - r_2|$$

and where

$$dist_X((s_1,r_1),(s_2,r_2)) = |r_1 - r_2| \textit{ if and only if } s_1 = s_2.$$

It follows that the curves $s \times \mathbb{R}_+$ are geodesic rays in X which are *distance minimizing* on all subsegments $[a,b] \subset \mathbb{R}_+ = [0,\infty)$ and that *every point* $x = (s,r) \in X$ admits a *unique nearest point* s_x in $S(r_0)$, for all r_0, namely, $s_x = (s, r_0)$.

Thus, the *normal projections* $X \to S(r_0)$ are *non-ambiguously defined* for all $r_0 \in \mathbb{R}_+$ and are given by $(s,r) = x \mapsto s_x = (s, r_0)$

The essential property of the family g_r which leads to hyperbolicity (yet to be defined) of X is the *exponential growth* of the metrics g_t, which means that

$\exp_k\uparrow$ $\qquad \cdot g_{r+r'} \geq (\exp kr')g_r$ for some $k > 0$ and all $r, r' \geq 0$.

In fact, if S *is compact* then $\exp_k\uparrow$ does imply hyperbolicity of X the definition of which is given later in this section.

About Curvature. The inequality $\exp_k\uparrow$ is equivalent to the *lower bound* on the *principal curvatures* of the (convex!) hypersurfaces $S(r) \subset X$ by our k.

The leading example of this is provided by manifolds with sectional curvatures $\kappa \leq -k_2$ where, not only the principal curvatures $K(r)$ of $S(t)$ but also the derivatives $K'(r)$ are bounded from below.[94] This bound implies that the X-*distances* $d_{12}(r) = dist_X((s_1,r),(s_2,r))$ grow exponentially, albeit with a rate $k' \lessapprox k$, in-so-far as d_{12} remains below $\leq 1/k$. But I did not check whether $\exp_k\uparrow$ suffices for this.

Exponential contraction. The inequality $\exp_k\uparrow$ can be turned upside down, and then it implies *exponential contraction of normal projections*

$$S \times [r + r', \infty) = X_{\geq r+r'} \overset{P_r}{\to} X_{\leq r} = S \times [0, r].$$

(Recall that $P_r(x) \in S(r_0)$ is, by the definition of normal projection, *the nearest point to* x *in* $S(r_0)$.)

Namely, if a curve $Y \subset X$ lies r'-far from the subset $X_{\leq r} \subset X$, i.e. $dist_X(y, X_{\leq r}) \geq r'$ for all $y \in Y$, then

$\exp_k\downarrow$ $\qquad length(P_r(Y) \leq (exp - k(r'))length(Y).$

This parallels expansion $\leftrightarrow$ contraction correspondence (e.g. for foliations) under time reversal of dynamical (e.g. Anosov's) systems.

$[\curlyvee]$-*Perspective: Hyperbolic Divergence of Rays in* X. If X is a hyperbolic geodesic metric space then (we shall see it later) the pairs of minimizing geodesic rays in X, that are isometric maps $\mathbb{R}_+ = [0,\infty) \to X$, denoted $x_1(r), x_2(r) \in X$, $r \in [0,\infty)$, satisfy the following condition.

There exist constants $D_\curlyvee = D_\curlyvee(X)$ and $C_\curlyvee = C_\curlyvee(X)$, such that if

$$dist(x_1(r_1), x_2(r_2)) \geq D_\curlyvee$$

for all $r_1, r_2 \geq 0$, then

$[\curlyvee]$ $\qquad dist(x_1(r), x_2(r)) \geq 2r - dist(x_1(0), x_2(0) - C_\curlyvee$ for all $r \geq 0$.

(Notice that $dist(x_1(r), x_2(r)) \leq 2r + dist(x_1(0), x_2(0))$ by the triangle inequality.)

Moreover, hyperbolicity implies the following.

[94]The sectional curvatures of X, are, essentially by definition, [48], equal to the derivatives of the *shape operators* of equidistant deformations of hypersurfaces in X.

[ⵜ] *Quasi Convexity of Geodesic Pairs.* This property says, in particular, that the unions $R_1 \cup R_2$ of pairs of minimizing geodsic rays $R_1, R_2 \subset X$ are *geodesically quasiconvex*, which means that

distance minimizing segments with the end-points in $R_1 \cup R_2$ lie within distance $\leq D$ from $R_1 \cup R_2$, where $D \leq dist(R_1, R_2) + C_{\text{ⵜ}}$ and where $C_{\text{ⵜ}}$ depends only on X.

Notice that [$\curlyvee$] and [ⵜ] are (obviously) *satisfied by trees with $D_{\curlyvee} = C_{\curlyvee} = C_{\text{ⵜ}} = 0$.* (Both properties sharply contrast with how it is in the Euclidean spaces.)

Also, since the above manifolds $X = S \times \mathbb{R}_+$ with the exponential grows of the metrics in $S(r)$ are hyperbolic, the rays $R_s = s \times \mathbb{R}_+$ in them satisfiy [$\curlyvee$] and [ⵜ]. This make X look rather similar to rooted trees which are obtained by amalgamations of copies of $\mathbb{R}_+$. (The branches in such trees are are infinite and there are no leaves.)

Remarks. (a) Dispite a rather explicit description, there is no (?) apparent direct proof of [$\curlyvee$] and [ⵜ] for the rays R_s in these X. (The proof we indicate below, albeit short and simple, depends on a general hyperbolic setting.)

(2) It will be become clear later on (this is easy) that if two such (hyperbolic) manifolds $X_1 = S_1 \times \mathbb{R}_+$ and $X_2 = S_2 \times \mathbb{R}_+$ are *quasiisomtric*, then the underlying manifolds S_1 and S_2 are *homeomorphic.*

Conversely, if S_1 and S_2 are *diffeomorphic* (*quasi-conformal* homeomorphism will do), and both families g_{r1} and g_{r2} grow *conformally* and *moderately exponentially*,

$$g_{ri}(s) = \phi_i(r,s) g_{0i}(s), \;\; i = 1,2, \;\; \text{and } \exp k_i' r_1 \geq \phi_i(r + r_1, s) \geq \exp k_i r_1,$$

then the spaces X_1 and X_2 are *quasiisometric.*

(Spheres in manifolds with negative sectional curvatures *bounded from below* grow moderately, while conformal growth of the spheres $S_x(r) \subset X$ around all points $x \in X$, with $\phi_i(r,s)$ *constant in $s \in S_x(r)$* is *characteristic for manifolds X with constant curvatures.*)

Hyperbolic Horospherical Cylinders. Let us replace $\mathbb{R}_+$ by the full line $\mathbb{R} = (-\infty, +\infty)$, and, given a family g_r of metrics on a manifold S, now with $r \in \mathbb{R}$, endow X with the metric $g = g_r + dr^2$ as earlier.

We shall see with definition of hyperbolicity given in the next section that

if the metrics g_r are complete and

$$g_{r+r_1} \geq (\exp k r_1) g_{r_1} \textit{ for some } k > 0, \textit{ all } r \in \mathbb{R} \textit{ and } r_1 \geq 0,$$

then (X, g) is hyperbolic.

A simple, yet instructive, instance of this is where S is a Banach space, e.g. the Euclidean $\mathbb{R}^n$, with a strictly contracting linear operator $A : S \to S$ and $g_r = (A^r)^* g$ for the metric g associated with $||...||_S$.

Another example is where X is a complete simply connected Riemannian manifold with strictly negative curvature and $\{S_r\} \subset X$ is the family of mutually equidistant horospheres associated with a geodesic ray $[x_0, x_\infty) \subset X$, where, recall, S_r is the limit of the $(r + R)$-spheres with the centers $x_R \in [x_0, x_\infty)$ and with $dist(x_R, x_0) = R$ for $R \to \infty$.

On Expansion and Contraction. If we want the geometry of the $\mathbb{R}$-warped $X - S \times \mathbb{R}$ to be *locally bounded*, e.g. having sectional curvatures bounded from above and from below, we need a uniform bound on the geometries, e.g. on the curvatures, of the metrics g_r on S. Such bounds come for free for expanding families. For instance, scaling g_0 by (arbitrary) large constants may only simplify local geometry.

But not all manifold S admit *complete arbitrarily small* metrics with, say, *bounded curvatures.*

It is known, for example, that among compact manifolds (without boundaries) only *infranil manifolds* may have *almost flat* metrics of small diameters and similar non-compact S are also exceptional.

(It is apparent that most metrics g_0 on S admit no families $g_t \leq \varepsilon(t)g_0$ with uniformly bounded curvatures for $\varepsilon(t) \to 0$ and topology of manifolds S which support such families also is a subject to many obvious and non-obvious constraints, but a comprehensive picture is yet to be developed.

Expanding $[^{\downarrow}_{\downarrow}]$-*Cylinders.* Recall, that *the cylinder* Cyl_P *of a map* $P : A \to B$ is a subset $Y \subset A \times B \times [0,1]$ such that

$$(a,b,t) \in Y \Leftrightarrow P(a) = b.$$

In other words, Y is the union of the unit segments $[a,b]$, where $b = P(a)$ and where $A \subset Y$ is the top and $B \subset Y$ is the bottom of this cylinder.

If A and B are metric spaces, this Y is equipped with the metric $dist_Y$ which is induced from the, say Pythagorean, product metric in $Y \subset A \times B \times [0,1]$.

Now, let $S(i)$, $i = 0,1,2,...$, be disjoint unions of geodesic metric spaces, called *components of* $S(i)$, where we let $S(0)$ be a single point.

Let $P_i : S(i) \to S(i-1)$, $i = 1,2,3,...$, be *uniformly contracting maps.* This means, P_i sends each component of $S(i)$ to a component of $S(i-1)$ by a λ-Lipschitz map for some $\lambda < 1$, e.g. with $\lambda = 1/2$ for all i and all components in $S(i)$.

Let X be the union of *the cylinders* $Y_i = Cyl_{P_i}$ of the maps P_i with the obvious identifications of the tops of Y_i, that are $S(i)$, with the bottoms of Y_{i+1}, which are also qual to $S(i)$, and endow X with the supremum of the metrics which are equal to $dist_{Y_i}$ for (all components of) all $Y_i \in X$, $i = 1,2,....$

Two Examples. (1) If $S(i)$ are disjoint unions of points – no edges between vertices, then this construction can be seen as a discretization of the above $[^{\downarrow}_{\downarrow}]$-representation of trees, where the resulting trees *now* have the edges in them, which are the vertical segments $[s_i, P(s_i)]$, all of the unit lengths.

(2) The $[^{\downarrow}_{\downarrow}]$-construction also provides discretization of Riemannian cylinders. Thus, for instance, if $S(i)$, $i = 0,1,2,3,4....$, are (isometric to) the real segments of lengths $0,1,2,4,8...$ and P_i are the obvious surjective $\frac{1}{2}$-Lipschitz maps between them, then the resulting $[^{\downarrow}_{\downarrow}]$-cylinder X is bi-Lipschitz equivalent, to the hyperbolic half-plane.

We shall see in 2.11 that

✵ *a geodesic metric space* X *is quasiisometric to an expanding* $[^{\downarrow}_{\downarrow}]$-*cylinder if and only if it is hyperbolic as it is defined in 2.9.*

And below in this section, hyperbolicity of metric spaces is understood as being quasiisometric to expanding $[^{\downarrow}_{\downarrow}]$-cylinders.

Dynamical Expansion and Spatial Hyperbolicity.

Let a metric space $S = (S, dist_0)$ be continuously acted upon by a group Γ. Given a (preferably finite) subset $\Delta \subset \Gamma$, let $dist_\Delta$ be the supremum of the induced metrics $\gamma^*(dist_0)$, $\gamma \in \Delta$ and let $dist_\Delta{\uparrow}^\varepsilon$ be the supremum of the metrics which are majorized by $dist_\Delta$ on the ε-balls for the metric $dist_0$ in S.[95]

The asymptotics of metric invariants of the spaces

$$S_{\Delta\uparrow^\varepsilon} = (S, dist_\Delta{\uparrow}^\varepsilon) \text{ for } \Delta \to \Gamma \text{ and } \varepsilon \to 0$$

carry much informations about the topological dynamics of (S,Γ). For instance, the *topological entropy* for $\mathbb{Z}$-actions is customary defined with

$$\Delta_i = \{-i, -i+1, ..., i-1, i\} \subset \mathbb{Z}$$

in terms of the asymptotics of the minimal numbers $N_{i\varepsilon}$ of ε-balls needed to cover $S_{\Delta_i\uparrow^\varepsilon}$ as

$$ent_{top}(S,\mathbb{Z}) = \lim_{\varepsilon\to 0}\lim_{i\to\infty}\frac{1}{i}\log N_{i\varepsilon}.$$

[95]Think of $\Delta \mapsto dist_\Delta$ as kind of a distance valued measure on Γ.

The spaces $S_{\Delta\uparrow^{\varepsilon}}$ can be brought together by making *double cylinders*, using the identity maps arrows $S_{\Delta'\uparrow^{\varepsilon'}} \to S_{\Delta\uparrow^{\varepsilon}}$ for $\Delta' \supset \Delta$ and $\varepsilon' \le \varepsilon$, e.g. as follows

Let Δ be a finite generating set in Γ and let

$$\Delta^i = \underbrace{\Delta^{\pm 1} \cdot \Delta^{\pm 1} \cdot \ldots \cdot \Delta^{\pm 1}} \subset \Gamma$$

and let X_ε be the $[^{\downarrow}_{\downarrow}]$-cylinder associated with the chains of maps

$$\ldots \to S_{\Delta^i\uparrow^{\varepsilon}} \to S_{\Delta^{i-1}\uparrow^{\varepsilon}} \to \ldots \to S_{\uparrow^{\varepsilon}} \to \{\cdot\}$$

. Then let X be the $[^{\downarrow}_{\downarrow}]$-cylinder for $\{\ldots \to X_{\varepsilon_i} \to X_{\varepsilon_{i-1}} \to \ldots\}$, where, say, $\varepsilon_i = 2^{-i}\varepsilon_0$.

Thus, the asymptotics of geometries of the dynamically generated $S_{\Delta\uparrow^{\varepsilon}}$ is encoded in a single metric space X, which opens a possibility of using invariants of metric spaces, for characterising dynamics of group actions.

Questions. (a) It it is not difficult to describe geometric invariants of these cylinders (and double cylinders), such as the *L_p-cohomology* and the *Novikov Shubin invariants* directly in terms of the metrics $dist_\Delta$ on S.

But what is the dynamical meaning of these invariants?

(b) Topologically, the $[^{\downarrow}_{\downarrow}]$-cylinder for a $\mathbb{Z}$-action is kind of "folded" cyclic covering of the *mapping torus* of the generator of this action, where this torus is fibered over the circle S^1 – *the classifing space* of the group $\mathbb{Z}$. This points toward a general constructions grounded on the multidimensional classifing spaces of groups Γ.

But apparently, what is closer to the $[^{\downarrow}_{\downarrow}]$-cylinders is suggested by the assignments $\Delta \mapsto dist_\Delta$, $\Delta \subset \Gamma$, which define a kind of Γ-equivariant measures on Γ with value in the space of metrics on S. Here, similarly to the $[^{\downarrow}_{\downarrow}]$-cylinders and unlike how it is in the constructions of the classifying spaces, the arrows are the identity maps, while the role of Γ is shifted from the action on S to the induced action on space of metrics on S.

What is the correct (multiparmetric?) generalization, of the (one-parametric) $[^{\downarrow}_{\downarrow}]$-cylinders for actions of non-cyclic groups Γ?

For instance, if $(S, \Gamma) = (S_1, \Gamma_1) \times (S_2, \Gamma)$, we want to be able to reconstruct the invariants of the actions of Γ_i on S_i, $i = 1, 2$.

Such invariants may serve for evaluation of the *relative rates* of growth of actions of Γ on two metric spaces S and T, where such rates can be associated with *homotopy classes* of continuous maps $S \to T$, as follows.

Given a class h, characterize such a "rate" by $\lambda(\Delta, h)$, $\Delta \subset \Gamma$, which is the minimal number such that h contains a $\lambda(\Delta)$-Lipschitz representative for the metrics $dist_{S,\Delta}$ and $dist_{T,\Delta}$; similarly define $\lambda(\Delta, h, \varepsilon)$ with $dist_\Delta\uparrow^{\varepsilon}$ instead of $dist_\Delta$.

Another quantity one my like to measure is the range of $(\lambda, D)(\Delta)$ for which the spaces $(S, dist_{S,\Delta})$ and $(T, dist_{T,\Delta})$ are λ-bi-Lipschitz on the scale D.

****

Say that the action of Γ is *uniformly expansive* (compare 1.6) if there exist an $\varepsilon_0 > 0$ and a finite subset $\Delta_0 \subset \Gamma$ such that

$$dist_{\Delta_0}(s_1, s_2) \ge 2 dist_0(s_1, s_2) \text{ whenever } dist_0(s_1, s_2) \le \varepsilon_0.$$

It is straightforward to check that if $\Delta' \supset \Delta \cdot \Delta_0$, then the maps $S_{\Delta'\uparrow^{\varepsilon'}} \to S_{\Delta\uparrow^{\varepsilon}}$, are 1/2-Lipschitz for all $\Delta \subset \Gamma$ and all $\varepsilon \le \varepsilon' \le \varepsilon_0$. Cosequently,

if the action of Γ on S is uniformly expansive, then the above $[^{\downarrow}_{\downarrow}]$-cylinders X_ε are hyperbolic for all $\varepsilon \le \varepsilon_0$.

In fact, albeit these cylinders are not exactly what we called "expanding" – the metrics $dist_\Delta\uparrow^{\varepsilon}$ are not necessarily geodesic – X_ε can be easily fit into our expanding category, e.g. by passing to the blow-up graphs $G_{\le D}(S_{\Delta\uparrow^{\varepsilon}})$.

2.9 Tree-like Arrangements of Points and Lines, Slim Triangles and Rips Collapsibility Theorem.

Informally, a metric space X is *hyperbolic* if it looks on the large scale as a space M with negative curvatures $\kappa \leq \kappa_0 < 0$. There are many ways to make it precise by picking up a particular feature common to these M and taking it for a definition of hyperbolicity.

Eventually, it became clear that almost all (all?) general properties of trees that are shared by the classical hyperbolic spaces $\mathbf{H}^n$ with $\kappa = -1$ but not by the Euclidean $\mathbb{R}^n$ lead to equivalent definitions of hyperbolicity.

Below is a strongest such definition with the explanation of terminology to follow.

$_{\forall N}$. A quasigeodesic, e.g. geodesic, metris space X is called *hyperbolic* if the finite unions of *quasisegments* in it,

$$Y = \bigcup_{i=1,\dots N} E_i \subset X,$$

are *almost isometric*[96] to a subset in a tree,

$$Y \leftrightarrow Y' \subset T$$

.

Segments, Quasisegments, Rays and Lines A subset E in a metric space is called a *segment* with end points $,x_1 x_2 \in X$, denoted $[x_1, x_2] \subset X$ (even if such a segment is not unique), if it is isometric to a real segment $I = [a_1, a_2]$ $subset(-\infty, \infty)$, where we also allow infinite $I = [a, \infty)$ and $I = (-\infty, \infty)$, where the corresponding $E \subset X$ are called (geodesic) *rays* and *lines*.

Recall that an $E \subset X$ a *quasisegment* if it is quasiisometric to a real segment I, where, recall a quasiisometry is a λ-bi-Lipschitz bijection between D-nets in the two spaces:

$$E \subset E' \leftrightarrow I' \subset I.$$

We agree that the net I' in I contains the end points of I and the corresponding points in E are regarded as the end points of E.

Segments versus Quasigeodesics. Quasigeodesics in section 2 defined as curves $Y \subset X$ parametrised by the arc length such that $dist_X(y_1, y_2) \geq \lambda^{-1}|y_1 - y_2|$ are instances of λ-quasisegments. And general quasisegments E in *geodesic spaces* X can be approximated by quasigeodesics E' composed of segments $[x_i, x_{i+1}]$, $i = 1, \dots n$, where $e_i \in E$ and where the lengths of these segments are between λ and 10λ.

Also recall that *an almost isometry* between metric spaces Y and Y', is *a correspondence* $Y \leftrightarrow Y'$ such that

$$y \leftrightarrow y' \Rightarrow |dist((y_1, y_2) - dist(y'_1, y'_2)| \leq D < \infty,$$

where we say D-*almost isometry* if we want to specify the error parameter D (compare 2.7).

Specification/Quantification of Error Parameters. The above definitions contain numbers, such as λ and D which specify by how much the properties of our "quaisiobjects", deviate from the "ideal" case: quasisegments from segments, almost isometries and quasiisometries from isometries, D-nets in X from all of X. (compare 2.7)

[96] It may seem more natural to require Y to be *quasiisometric* rather than *almost isometric* to subsets in trees T. But, due to somewhat peculiar topology of trees, "quasiisometric" can be upgraded to "almost isometric" in the key case where subsets Y consist of quasisegments joining *all pairs* of points in finite subsets in X (e.g. for quasigeodesic triangles $\triangle \subset X$) by modifying the geodesic metric in T.

And the hyperbolic spaces, which appear as *quasitrees* in this setting, are also burdened with error parameters, often denoted δ, that must be specified in individual cases.

Now we make $\oslash_{\forall N}$ precise by quantifying it as follows.

X is hyperbolic if the finite unions of quasisegments,

$$Y = \bigcup_{i=1,\dots N} E_i \subset X,$$

admit almost isometries $Y \overset{f}{\leftrightarrow} Y' \subset T$ the error parameters of which (D in this case) depend on N and Err_{E_i} (which are pairs (λ_i, D_i)), written as

$$Err_f \leq \delta_N(Err_{E_i})$$

for some functions δ_N, where the collection $\{\delta_N\}_{N=1,2,\dots,}$ serves as the error parameter $Err(X)$.

Let us see what almost isometries f tell us for particular $Y = \bigcup E_i$.

Geodesic Shadowing. The condition $\oslash_{\forall N}$ for $N = 2$ implies *the shadowing property* for geodesic hyperbolic spaces X, which says that

$\big($ *all quasisegments $E_1 \subset X$ are shadowed by segments $E_2 \subset X$.*

To properly quantify this, let E_2 have the same endpoints as E_1 and let shadowings be implemented by correspondences $E_1 \overset{f}{\leftrightarrow} E_2$ with displacement $\leq D$, (i.e. $dist(e_1, e_2) \leq D$ for $e_1 \overset{f}{\leftrightarrow} e_2$), where this D – the error parameter of f – depends only on $Err(E_1)$ which specifies "quasi" of E_1.

It is obvious that the existence of such f is equivalent to the existence of almost isometries $E_1 \cup E_2 \overset{F}{\leftrightarrow} Y' \subset T$, such that $1/4 \leq Err_F/Err_f \leq 4$.

Thus, if X is hyperbolic, such f exist with the error parameters depending on those of E_1 and X,

$$Err_f = Err_f(Err(E_1), \delta_2(X)).$$

The converse implication $\big(\Rightarrow \oslash_{\forall N}$ – this parallels the definition of *dynamical hyperbolicity* via (local) shadowing (see chapter 4) – is also true. The proof of it, due to Bonk [14], is rather elaborate.

The simplest class of subsets $Y \subset X$ for which the existence of almost isometries $Y \overset{F}{\leftrightarrow} Y' \subset T$ is significant is where Y are *four point sets*[97] and where the hyperbolicity condition $\oslash_{\forall N}$ reduces to the following.

[•⋰•] *All quadruples of points in X are almost isometric to quadruples in trees.*[98]

Somewhat surprisingly, albeit this is not hard to prove, [•⋰•] $\Rightarrow \oslash_{\forall N}$. In fact, the following special case of [•⋰•] can be taken for the definition of hyperbolicity [?].

.

[•)(•]-*Condition*: the diameters of the intersections of balls $B_1, B_2 \subset X$ are bounded by the radii R_1, R_2 of the balls and the distances d_{12} between their centers according to the following inequality,

$)(_{\delta}$
$$Diam(B_1 \cap B_2) \leq 2(R_1 + R_2 - d_{12} + \delta).$$

[97] The existence of quasiisometric embeddings of *finite sets* to trees, unlike, for instance, this property for geodesic triangles, does not tell much: every finite metric space Y with n points admits a λ-bi-Lipschitz map to $\mathbb{R}$ with $\lambda < n$.

[98] Here and below we don't write down the error specification since misinterpretation is improbable.

Observe that [•)(•] makes sense for *non-(quasi)geodesic* spaces and that despite the appearance, it concerns (rather special) *quadruples* of points in X and, at the same time, it expresses the idea of *uniform quasiconvexity of balls* for $d_{12} \to \infty$ with $R_1 + R_2 - d_{12}$ kept bounded.

And albeit neither [•)(•] nor [•˙.˙] are quasiisometry invariant in general, they do enjoy this property for *quasigeodesic* metric spaces. In fact, most of what comes below refers to geodesic spaces.

Also notice that Alexandrov's $CAT(\kappa)$ spaces are characterised by the inequalities $Diam(B_1 \cap B_2) \leq Diam(B'_1 \cap B'_2)$, where B'_1, B'_2 are balls in the standard spaces with curvatures κ, and where the raddii and the distances between the centers of B'_1, B'_2 are equal to these for B_1 and B_2 in X, [4]

A geometrically most transparent case of $\varnothing_{\forall N}$ is where $N = 3$ and Y is a *geodesic triangle* $\triangle \subset X$ – the union of three segments between the vertices of $\triangle$.

The existence of almost isometry between $Y = \triangle \subset X$ and a subset in a tree, says in effect that $\triangle$ is Y-*slim*, or simply *slim*, if it is D-*slim* for some $D > 0$ according to the following definition.

$\triangle \subset X$ is called D-*slim* if there a tree Y embedded to X which is the union of three segments $[x_0, x_i] \subset X$, $i = 1, 2, 3,$, where x_i are the vertices of $\triangle$, and such that

$\triangle \subset X$ *lies* D-*close to* $\mathsf{Y} \subset X$, i.e $\triangle$ admits a map to Y with displacement $\leq D$.

the tree geometry in Y, that is the *induced path metric* in it, satisfies

$$dist_{\mathsf{Y}} \leq dist_X + D,$$

(it is always $dist_{\mathsf{Y}} \geq dist_X$),

This motivates the following.

$[\triangle \rightsquigarrow \mathsf{Y}]$-*Condition: the geodesic triangles in X are slim.*

Exercises. (a) Show that slimness of the triangles having a given point $x_0 \in X$ for a vertex implies slimness of all triangles in X.

(b) Show that the unions Y of n-tuples of segments in hyperbolic spaces are D_n-almost isometric to $Y' \subset Tree$, where

APPR$_{\log}$ $\quad D_n \leq \delta \log n$ for some (possibly large) $\delta = \delta(X) > 0$.

(c)$_{\mathsf{Y}}$ Show that if X is hyperbolic and subsets in $Y_i \subset X$, $i = 1, 2, ..., n$, are almost isometric to trees then their union $Y = \cup_i Y_i$ is almost isometric to a subset in a tree, where the error parameter of the latter "almost" depends (only) on those of X, of Y_i and on n.

Recollection. Slimness of triangles in (the Cayley graphs of) hyperbolic groups was emphasised by Ilya Rips, who was concerned with construction of *combinatorially slim* triangles ▲ in 2-dimensional combinatorial (cellular) spaces X, where combinatorial sliminess of ▲ and/or of $\triangle = \partial$▲, means, according to Rips, that the set of interior points of ▲ can be covered by at most $n = n(X)$ combinatorial units – cells in X.

He conjectured that if X is the universal covering of a polyhedral space $\underline{X}$ with finitely many cells and with *hyperbolic fundamental group* $\Gamma = \pi_1(\underline{X})$, then all geodesic triangles in X with their vertices in the 0-skeleton of X can be simultaneously Γ-equivariantly deformed to combinatorially slim ones. (This may be regarded a counterpart to Markov partitions in hyperbolic dynamics.)

Rips told me about this in ≈1980 along with his other unpublished results and ideas about hyperbolic groups, e.g. his *collapsibility argument* (see below) of *Vietoris' complexes* of hyperbolic groups. But, for all I know, the existence of combinatorially slim triangles was confirmed by him only for small cancellation groups, the (probably quite complicated) proof of which still remains unpublished(?).

From $[\triangle \rightsquigarrow \mathsf{Y}]$ to $\varnothing_{\forall N}$.

The slimness $[\triangle \rightsquigarrow Y]$-condition, similarly to $[\bullet\!\!\!)\!\!\!(\ \bullet]$ is, a priori, weaker than $\varnothing_{\forall N}$ and it is not quasiisometry invariant. But we shall see in (2.11) (compare [47] [28]) that *hyperbolic universality of* $[^{\downarrow}_{\downarrow}]$- cylinders *yields the implication*

$$[\triangle \rightsquigarrow Y] \Rightarrow \varnothing_{\forall N}.$$

Exercise (d). Discretise the proof of [⊓⊓⊓] in 1.2 or the arguments concerning $[^{\downarrow}_{\downarrow}]$- cylinders in 2.11 and show, without a use of $[^{\downarrow}_{\downarrow}]$- cylinders and area estimates, that $[\triangle \rightsquigarrow Y]$ yields shadowing of quasigeodesics by geodesics, which trivially implies the above implication

Dehn Contraction and Rips Collapsibility. Let Y be a bounded subset in a geodesic δ-hyperbolic space X. Then, for every $C > 0$,

✥ *there exist points* $y_\wedge \in Y, x_\downarrow \in X$*, such that*

$$[\wedge] \qquad dist(y, x_\downarrow) \le dist(y, y_\wedge) - C + D$$

for all $y \in Y$ *for which* $dist(y, y_\wedge) \ge 2C$ *and all sufficiently large* $D \ge D_0(\delta)$.

Proof. Assume without loss of generality that X is unbounded and that $C > D$. Let $x_0 \in X$ satisfy

$$\sup_{y \in Y} dist(y, x_0) > 2C.$$

Let $y_\wedge \in Y$ be the *the farthest point from* x_0 – one may assume such exists since everything is estimated up to a positive error anyway – and let $x_\downarrow$ be the point in a segment $[x_0, y_\wedge] \subset X$ within distance $2C$ from $y_\wedge \in Y$. Then $[\wedge]$ is seen by either looking at the (slim!) triangles $\{y, y_\wedge, x_0\}$ or by comparing with how it is trees.

Vietoris Blow-up Complex. Recall the blow-up graph $G^1 = G_{\le D}(X)$ on the vertex set X with the edges (x_1, x_2) where $dist_X \le D$, and let $G^* = G^*_{\le D}(X)$ be the maximal simplicial complex with the 1-skeleton G^1. Thus, the simplices in G^* corresponds to *cliques* in G^1, which are tuples of points in X with mutual distances $\le D$.

Rips Lemma. *If X is hyperbolic then, for sufficiently large D, every finite subcomplex in $G \subset G^*$ is contained in a collapsable*[99] *subcomplex $G_+ \subset G^*$. Consequently G^* is contractible.*

Proof. Let $Y \subset X$ be a finite subset, join all $y \in Y$ by segments with a point $x_0 \in X$ and let $Y_+ \supset Y$ be a finite set such that the intersections of Y_+ with these segments are ε-nets in these segments for some (relatively) small $\varepsilon > 0$, say $\varepsilon = 1$.

Then ✥, say with with $D \ge 2$ and $C = 2D$, shows that removing $y_\wedge$ from Y_+ effectuates an elementary collapse from $G^*_{\le D'}(Y_+)$ to $G^*_{\le D'}(Y_+ \setminus \{y_\wedge\})$ for $D' \ge 2C$ and the proof follows.

However simple this lemma may appears, it has significant corollaries for *word hyperbolic groups* Γ, which are finitely generated groups with *hyperbolic* word metrics.

In this case, the complexes $G_{\le D}(\Gamma, dist_\Delta)$ are locally finite and the actions of Γ on them are cocompact, i.e. admit bounded fundamental domains.

Besides these action are free on vertex sets of $G_{\le D}(\Gamma, dist_\Delta)$ since these are are equal to Γ.

At this point one applies Rips lemma to $G_{\le D}(\Gamma, dist_\Delta)$ with large D and, observe that contractibility of $G_{\le D}(\Gamma, dist_\Delta)$ implies, in particular,

Rips Theorem. *Finitely generated word hyperbolic groups are finitely presented.*

[99] An elementary collapse of a simplicial complex G is defined by a decomposition $G = G_{-1} \cup \Delta$ where G_{-1} is a subcomplex and Δ is a simplex in G, such that the intersection $\Delta \cap G_{-1}$ is a proper face in Δ. A collapse in general is a chain of elementary collapses terminating at a single point,

$$G \supset G_{-1} \supset G_{-2} \ldots \supset G_{-i} \supset \ldots \supset \{\cdot\}.$$

2.10 Rough Convexity, Slim Triangles and Limit Trees.

So far, we have been only munching definitions; now let us bring forth a simple argument that is needed for showing that $[^{\downarrow}_{\downarrow}]$-*cylinders* from 2.8 are hyperbolic.

$[Ct+D]$-*Convexity of Real Functions and of Metric Spaces.* A positive function $f(r)$ on an interval in $\mathbb{R}$ is called $[Ct+D]$-*(quasi)convex*, where $C \geq 1$ and $D \geq 0$, if the inequality

$$r_2 - r_1 \geq C(f(r_1) + f(r_2)) + D$$

is satisfied for given $r_1 < r_2$ in the domain of f, then there exists a point r between r_i, i.e. $r_1 \leq r \leq r_2$, such that

$$f(r) \leq D/2.$$

Remark. In applications, such an r can be taken relatively close to the center of the segment $[r_1, r_2]$, e.g. such that $r_1 + \varepsilon(r_2 - r_1) \leq r \leq r_2 - \varepsilon(r_2 - r_1)$.

A geodesic metric space X is called $[Ct+D]$-*convex* if the distance functions to *quasisegments* $E_1 \subset X$ are $[Ct+D]$-convex on all *segments* $E_2 \subset X$, where C and D depend on the error parameters of E_1 but not on E_1 and/or E_2.

For instance, *trees are* (obviously) $[Ct+D]$-*convex.* Consequently,

Hyperbolic spaces X are $[t+D]$-convex with $D = D(X)$.

In fact, D depends on the error in the approximation of X by trees in the sense of $\oslash_{\forall N}$.

$[Ct+D]$-*Lemma. Let $f(r)$ be $[Ct+D]$-convex 1-Lipschitz function on $[a,b]$. Then there exists a subinterval $[a', b'] \subset [a,b]$, such that:*
- $f(r) \leq (C+2)D$ *for all* $r \in [a', b']$;
- $f(r) \geq \frac{1}{3C} dist(r, [a', b'])$ *for r in the complement to $[a', b']$, i.e. for $a \leq r \leq a'$ an $b' \leq r \leq b$.*

Proof. (a) The lenght of an interval $[\alpha, \beta]$, that supports a $[Ct+D]$-convex function $f(r)$ such that $f(\alpha) = f(\beta) \leq D$ and $f(r) > D/2$ for $\alpha < r < \beta$ is at most $2CD + D$. Hence, if f is 1-Lipschitz, then

$f(r) \leq D + CD + D/2 \leq (C+2)D$ for all $r \in [\alpha, \beta]$.

(b) If a 1-Lipschitz $[Ct+D]$-convex function $f(r)$ on $[0, \beta]$, satisfies $f(0) = D$ and $f(r) > D$ for $r > 0$, then $f(r) \geq \frac{r}{3C}$ for all r.

Indeed, since f is 1-Lipschitz, the inequality $f(r_0) < \frac{r_0}{3C}$, implies that $r_0 \geq \max(3DC, 3Cf(r_0))$ and, by the $[Ct+D]$-convexity, there is a point $0 < r < r_0$ where $f(r) \leq D/2 \leq D$.

Now, if a' is equal to the smallest r where $f(r) \leq D$ and b' is the largest such r, then the segment $[a'b']$ satisfy the conditions of the lemma.. QED.

From $[Ct+D]$-Convexity to $\oslash_{\forall N}$-Hyperbolicity.

Hyperbolicity of $[Ct+D]$-convex spaces X follows in two steps.

(A) *Shadowing.* If the above segment E_2 has the same end points as E_1 then, it is $(C+2)D$-close to E_1 by (a).

(B) *Linear Divergence of Segments and Rays.* Let $[x_0, x_1]$ and $[x_0, x_2]$ be two segments in X, where $[x_0, x_2]$ is longer than $[x_0, x_1]$.

Let $[x_0, x_1'] \subset [x_0, x_1]$ be the maximal subsegment for which

$$dist(x, [x_0, x_2]) \leq D \text{ for all } x \in [x_0, x_1']$$

and let $x_2' \in [x_0, x_2]$ be the nearest point to x_1'. Then the union

$$E = [x_1, x_1'] \cup [x_2', x_2] \subset X$$

is a *quasisegment* in X, since the distance between $x \in [x_1', x_1]$ and $y \in [x_2', x_2]$ grows linearaly in $r = dist(x, x_1')$ according to (b).

(One could take the broken geodesic $E' = [x_1, x'_1] \cup [x'_1, x'_2] \cup [x'_2, x_2] \subset X$ instead of E, but this does not change anything, since $dist(x'_1, x'_2) \leq D$.)

(C) *Slimness of Geodesic Triangles.* Since E is a quasisegment, it is shadowed by the segment $[x_1, x_2]$, which means that the geodesic triangle with the vertices x_0, x_1, x_2 is slim.

(D) *Hyperbolicity.* Since quasisegments are shadowed by segments, quasigeodesic triangles are also slim, and, as we explained earlier, this (trivially) implies $\varnothing_{\forall N}$.

About $C = 1$. Isn't it amusing that albeit $[Ct + D]$-convexity of *functions does not* imply $[t + D_+]$-convexity, this *is so* for *geodesic metric spaces*?.

Pedersen's Style $[Ct + D]$*-Convexity.* The above definition of $[Ct + D]$-Convex spaces has a purely geodesic counterpart where $[Ct + D]$-Convexity is required only of distance functions to segments (rather than to quasisegments) in X, which is in the spirit of Pedersen's $[\kappa < 0]_1$ (see 2.3, 2.5).

It is, probably, obvious to anybody how geodesic $[Ct+D]$-convexity yields *slimness of geodesic triangles* – it is like a high school exercise in geometry of triangles..., except for the annoying task of keeping track of the "error paramenters" .

The purpose of what is written below is to challenge the reader to find a better language which would reduce the following (non fully detailed) argument to five(ten?) lines of effortless and rigorous reasoning.

Start by observing that, by the above (B), the "purely geodesic" $[Ct+D]$-convexity implies the following.

Let $\{x_0, x_1, x_2\} \subset X$ be (the vertex set of) a geodesic triangle $\triangle$ in X and let $x'_i \in [x_0, x_i]$, $i = 1, 2$ be points with equal distances from x_0.

$[\star_\lambda]$ *If*

$$dist(x'_1, [x_1, x_2]) \geq \lambda dist(x_1, x_2)$$

for some $\lambda \leq \lambda_0(C, D) < \infty$*, then*

$$dist(x'_1, x'_2) \leq const = const(C < D)$$

(where $const \leq 10CD$*).*

Let us upgrade $\star$ as follows. Subdivide the segment $[x_1, x_2]$ by some pints $x_i \in [x_1, x_2]$ into $n \leq 10\lambda + 1$ subsegments of lengths$\leq 0.1\lambda$, apply $[\star_\lambda]$, to the n (narrow) triangles $\{x_0, x_i, x_{i+1}\}$ add the resulting bounds on distances between the points $x_i \in [x_0, x_i]$.

By the triangle inequality, this yields the following qualitative self-improvement of $[\star_\lambda]$. where the gain in the multiplicative constant is paid for by an increase of the additive error. ("Additive" is insignificant on the large scale: $At + b$ beats $at + B$ for large t.)

$[\bigstar_{0.1}]$ *If*

$$dist(x'_1, [x_1, x_2]) \geq 0.1 dist(x_1, x_2),$$

then

$$dist(x'_1, x'_2) \leq const = const(C, D)$$

(where $const(C, D) \leq 10nCD$).

It follows, that

$\big[\ast\big]$ *every geodesic triangle* $\triangle = \{x_0, x_1, x_2\} \subset X$ *contains a vertex, namely the one which is opposite to the shortest edge, let it be* x_0*, such that the points* $x_{0i}(r) \in [x_0, x_i]$*,* $i = 1, 2$*, with distances* r *from* x_0 *satisfy*

$$dist(x_{01}(r), x_{02}(r)) \leq const(C, D) \text{ for } r \leq \tfrac{1}{2} diam\{x_0, x_1, x_2\}$$

Given a (large) geodesic triangle $\triangle = \{x_1, x_2, x_3\}$, let

$$(x_{ij} \in [x_i, x_j], x_{ij'} \in [x_i, x_{j'}])$$

where $i, j = 0, 1, 2$, $j \neq i$, $j' \neq j$, be three similar pairs of points on the pairs of edges of $\triangle$ at the vertices x_i, with distances

$$dist(x_{ij'}, x_i) = dist(x_{ij}, x_i) = r_i, \ \ i = 1, 2, 3,$$

such that

r_i are the largest numbers for which the distances beween x_{ij} and $x_{ij'}$ are $\leq const(C, D)$.

According to the above (B), the edges of the (eight smaller) triangles $\triangle = \{x_{0j_0}, x_{1j_1}, x_{2j_2}\}$ with the vertices on the three edges of $\triangle$ (one needs only one such triangle, say $\{x_{01}, x_{12}, x_{20}\}$) keep within a controlled bounded distance from the corresponding edges of $\triangle = \{x_0, x_1, x_2\}$; hence, the edges in $\triangle$ diverge at the vertices x_{ij} as much, up to a bounded error, as the corresponding edges of $\triangle$ do beyond the points $x_{i,j}$ on their edges where this divergence begins by the definition of these points.

It follows that $diam(\triangle) \leq const'(C, D)$, otherwise $\triangle$ would violate $[*]$. Hence, the (large) triangle $\triangle$ is slim. QED

Exercises.(a) Let the subsets Y in a geodesic metric space X with diameters D and cardinalities $\leq N = N(D)$ be $\varepsilon_0 D$*-almost isometric* to subsets in trees. (This means that there are correspondences $Y \leftrightarrow Y' \subset Tree$, such that

$$|dist_X(y_1, y_2) - dist_{Tree}(y_1', y_2')| \leq \varepsilon_0 \cdot D \text{ for } (y_1, y_2) \leftrightarrow (y_1', y_2').)$$

Show that if $\varepsilon_0 > 0$ is sufficiently small, e.g. $\varepsilon_0 < 0.1$, and $N(D) \to \infty$ for $D \to \infty$, then X is hyperbolic. (According to APPR_{log}-exercise (that is (b) in 2.9) such an approximation in hyperbolic spaces X is possible for $N = \epsilon D$, $\epsilon = \epsilon(X) > 0$.)

(b) Show with (a) that

$*_{lim} \Leftrightarrow \oslash_{\forall N}$: *a geodesic metric space X is hyperbolic if and only if all ultra limits of the spaces $\lambda X = (X, \lambda dist(X))$ for $\lambda \to 0$ are trees.*

2.11 Lengths-Areas Inequalities and Hyperbolicity of [↓↓]-Cylinders.

Let $X = \bigcup_i Cyl_{P_i}$ be a [↓↓]-*Cylinder* that is the (amalgamated) union of the the cylinders of maps $P_i : S_i \to S_{i-1}$ with the distinguished point, $x_0 = S(0) \in X$, where, by the construction of the metric in X, each $x \in X$ is joined with x_0 by a unique segment $[x, x_0] \subset X$ (see 2.8).

Let $Y \subset X$ be a, say piecewise geodesic, curve an observe that the induced path metric in the the geodesic cone Z over Y,

$$Z = Cone_{x_0}(Y) = \bigcup_{y \in Y} [x + 0, y] \subset X,$$

is a Riemannian one away from x_0, provided we use the *Pythagorean* product metrics in the constituent cylinders $Cyl_{P_i} \subset S(i) \times S(i-1) \times [0, 1]$ (see 2.1).

Let the maps P_i in the definition of X be *uniformly contracting*, say $e^{-\lambda}$-Lipschitz for some $\lambda > 0$. Then the the lengths of the equidistant curves Y_r obtained by moving Y by distance r toward x_0 satisfy

$$length(Y_r) \leq (\exp -\lambda r) length(Y)$$

and

$$area(Z) = \int_0^\infty length(Y_r)dr \le length(Y)\int_0^\infty \exp -\lambda r \le \lambda^{-1} length(Y).$$

Thus, X satisfies

$\bullet_\lambda$ Linear 2D-Filling Inequality. *all closed curves $Y \subset X$ bound discs $Z \subset X$ with areas linearaly bounded by the lenghts of Y*

$$area(Z) \le \lambda^{-1} length(Y).$$

This serves for estimating distances in hyperbolic space via the following

■ *Besicovitch-Loewner Square Inequality.* Let Z be a disk with a Riemannian metric, and let the circular boundary $Y = \partial Z$ is partitioned, similarly to □, into four arcs A, B, A', B'.

Then the Riemannian distances between these arcs (measured by the minimal lengths of curves between points) satisfy"

$$dist(A, A') \cdot dist(B, B') \le area(Z).$$

Proof. Let $A_{+r} \subset Z$ denotes the r-equidistant to A, that is the set of points $z \in Z$ with $dist(z, A) = r$. If $r \le d = dist(A, A')$ then, by an elementary toplogy, there is a connected component in A_{+r} that meats B and B'; hence, the length of such an A_{+r} (understood as the 1-dimensional Hausdorff measure, if you wish) is bounded from below by

$$lengthA_{+r} \ge dist(B, B').$$

On the other hand, since

$$area(Z) = \int_0^d length(A_{+r})dr$$

by the *coarea formula* (rather obvious in the present case), the proof follows.

Derivation of $[Ct + D]$-Convexity from $\bullet$&■. Let $E_1 \subset X$ be a Δ_1-quasisegment that we may assume being a piecewise geodesic simple curve in X, where every subsegment $E' \subset E_1$ with end points e_1', e_2' satisfies

$$length(E') \le \Delta_1 dist(e_1', e_2')$$

where $\Delta_1 \ge 1$ is the distortion of E_1, which is equal to one for undistorted E_1 i.e. (geodesic) segments in X.

Let $E_2 = [x_1, x_2] \subset X$ be a (geodesic) segment, let e_1', e_2' be the points in E_1 nearest to x_1 and to x_2 and let Y be the closed curve composed of the segments $[x_1, x_2]$, $[x_1, e_1']$ $[x_2, e_2']$ and the subsegment $E' \subset E_1$ between $e_1' \in E_1$ and $e_2' \in E_1$.

If

$$L = length(E_2) = dist(x_1, x_2) \ge C(dist(x_1, e_1') + dist(x_2, e_2')) + D, \ \ C \ge 1, D \ge 0$$

then

$$dist([x_1, e_1'], [x_2, e_2']) \ge L - (L - D)/C \ge \frac{C-1}{C} L$$

by the triangle inequality. Then we estimate the length of Y by comparing its piaces to L and conclude:

$$length(Y) \le 2L + 2\Delta_1 dist(e_1', e_2') \le 4\Delta_1 L.$$

Invoke $\bullet_\lambda$ and Span Y by a disc of area

$$A \le 4\lambda^{-1}\Delta_1 L.$$

Apply ■ to this disc and evaluate the distance from E_2 to E_1 by

$$dist(E_2, E_1) \leq dist(E_2, E') \leq \frac{A}{dist([x_1, e'_1], [x_2, e'_2])} \leq \frac{4\lambda^{-1}\Delta_1 C}{C-1}.$$

Thus, the $[Ct+D]$-inequality holds in X for all $C > 1$ and

$$D \geq \frac{8\lambda^{-1}\Delta_1 C}{C-1}.$$

Hyperbolic Universality of $[^{\downarrow}_{\downarrow}]$*-Cylinders.* Let us complete the circle of hyperbolic implications by showing that all geodesic hyperbolic spaces X are almost isometric to $[^{\downarrow}_{\downarrow}]$-*cylinders.*

To do this, fixe a point $x_0 \in X$, let $[x_0, x_1], [x_0, x_2]$ be geodesic segments and attach to X the geodesic triangle $\blacktriangle' = \mathbf{H}^2_\kappa$, with vertices $x'_i \in \mathbf{H}^2_\kappa$, where

• $\mathbf{H}^2_\kappa$ is the hyperbolic plane with constant curvature κ

• $$dist_{\mathbf{H}^2_\kappa}(x'_i, x'_j) = dist_X(x_i, x_j) \; i, j = 0, 1, 2,$$

• the triangles $\blacktriangle'$ are attached to X along the edges on their boundaries, namely at the pairs issuing from x'_0, by isometries $[x'_0, x_i] \to [x_0, x_i]$, $i = 1, 2$.

Perform this attachment for all pairs of edges $[x_0, x_1], [x_0, x_2] \in X$, where $dist_X(x_1, x_2) \leq D$ for some $D > 0$ and, as we did it many times earlier, we take the supremum of the metrics d on the union of the so attached triangles, call it $X_\kappa = X \coprod \{\blacktriangle'\}$, which are majorized by the $\mathbf{H}^2_\kappa$-metrics on these triangles, where we identify points where the resulting metric, call it $dist_\kappa$, vanishes.

(All such point lie in $X \subset X_\kappa$. For instance, $dist_\kappa(x_1(r), x_2(r)) = 0$ if $x_1(R) = x_2(R)$ for some $R \geq r$ or if the segments $[x_0, x_1(r), [x_0, x_2(r)]$ extend to two rays with the distance $\leq D/2$ between them.)

Such an X_κ carries a natural $[^{\downarrow}_{\downarrow}]$- cylindrical structure as the normal projections of the $(r+1)$-spheres in X_κ around x_0 to the concentric r-spheres are uniformly contracting maps. (They contract as much as such projections in $\mathbf{H}^2_\kappa$.)

It is also clear that all of X_κ lies within bounded distance from X, or rather from what remains of X after identification of points in it with $dist_\kappa$ zero. In fact, X may collapse to a single ray for $\kappa << -1$, which happens, for instance, to $\mathbf{H}^2_{\kappa'}$ if $\kappa' > \kappa$.

But if $\kappa > 0$ is sufficiently close to zero, such that the asymptotics of the growth of X far away from x_0 dominates the growth of $\mathbf{H}^2_{\kappa'}$, then the collapse is insignificant:

the difference $dist_\kappa - dist_X$ *stays bounded on* X*.* QED.

$[\triangle \rightsquigarrow \mathsf{Y}] \Rightarrow \varnothing_{\forall N}$-COROLLARY. "Insignificance of collapse" follows from the $[\mathsf{Y}]$-slimmnes of the triangles $\triangle$ in X which have x_0 for one of their vertices. Thus, $[\triangle \rightsquigarrow \mathsf{Y}]$ does imply the hyperbolicity of X which was defined in 2.9, by $\varnothing_{\forall N}$ via approximation of n-tuples of *quasi*segments in X by trees.

Remark+Open Problem. The space X_κ is locally $CAT(\kappa)$ at all points except for x_0. One can make such a space everywhere locally $CAT(\kappa)$ by moving x_0 to infinity, but then the resulting space X_κ loses simple connectivity.

it is probable, however, that all hyperbolic spaces X are quasiisometric (almost isometric?) to Pedersen's strict tube convex $[\kappa < 0]_1$-spaces (see 2.3) but it is unlikely with $CAT(\kappa)$ instead of $[\kappa < 0]_1$. The obvious candidates for counterexamples are $[^{\downarrow}_{\downarrow}]$-spaces $S \times \mathbb{R}$ with the metric $e^r ds^2 + dr^2$ where S is an infinite dimensional non-Riemannian space, such as $L_\infty[0,1]$. .

What is known in this regard is that *locally bounded* geodesic hyperbolic spaces X, i.e. where the balls of radius R can be covered by at most $\exp R$ of unit balls, are *almost isometric to* $CAT(\kappa)$*-spaces, namely to convex subsets in* $\mathbf{H}^N_\kappa$, according to a theorem by Bonk and Shramm (2000) [18].

Their construction also allows similar embeddings of a class of, possibly infinite dimensional, hyperbolic spaces with *uniform exponential divergent rate of geodesics* to Pedersen's strict tube convex spaces like the above $(S\times\mathbb{R}, e^r ds^2 + dr^2)$, but we are far from answering the following.

QUESTION. *What are hyperbolic spaces with discrete isometry groups* Γ *acting on them which are* Γ***-equivariantly*** *quasiisometric to Pedersen's negatively curved* Γ*-spaces.*

2.12 Quasiminimal Surfaces, Conformal Kobayashi Metrics and Spaces of Curves.

The linear filling inequality $\bullet_\lambda$, as it was explained above, is equivalent to hyperbolicity of path metric spaces X, provided one has a reasonable concept of *area* of maps from surfaces Y to X, e.g. if X is a polyhedral space with a piecewise Riemannian metric. For instance – this suffices for our present purposes[100] – one can define the area of a map $f:Y\to X$ as

the infimum of the areas of the Riemannian metrics g *on* Y *for which the map* f *is distance decreasing.*

Granted this, one defines

filling area $ar_{fl}(S)$ of a closed (possibly disconnected) curve $S\subset X$, that is the infimum of areas of surfaces Y mapped to X with boundaries $\partial Y = S$ with an obvious convention for the meaning of this equality.

In these terms, the corresponding linear filling inequality reads

☻ $ar_{fl}(S)\le \lambda length(S)$ for all closed curves $S\subset X$ and some constant $\lambda=\lambda(X)$.

(This ☻ is weaker than $\bullet_\lambda$ since it allows surfaces Y which do not have to be disks. However, it implies hyperbolicity similarly to $\bullet$ because the Besicovitch-Loewner square inequality ■ is impervious to the topology of Y. Thus, the validity of ☻ for *all* closed curves in a *simply connected* X implies $\bullet_\lambda$.)

Since we are mainly concerned with the large scale geometry the existence of minimal surfaces with the areas equal to $ar_{fl}(S)$ is non-essential: *almost minimal surfaces* with areas $\le ar_{fl}(S) + const$ serve equally well. Another class of suitable surfaces Y in X are *quasiminimal ones* where the areas of the domains $Y'\subset Y$ with boundaries S' satisfy

$$area_X(Y')\le cost\cdot ar_{fl}(S').$$

On the other hand, there is no problem with minimality in 2-*dimensional* cell spaces G^2 with all 2-cells having *equal areas*: the closed curves in the 1-skeleta $G^1\subset G^2$ do bound minimal surfaces. And since the geodesic spaces X are quaisiisometric to the 2-skeleta $G^2_{\le D}(X)$ of the Vietoris blow-up complexes $G^*_{\le D}(X)$, one may discard of the existence problem in the study of large scale properties of X.

Conformal Surfaces in X. A Lipschitz map from a Riemann surface, i.e. a surface with a Riemannian metric, to a metric space, $f:Y\to X$ is *metaconformal* if it becomes path isometric after a conformal change $g_0\rightsquigarrow g_1$ of the original Riemannian metric g_i on Y, which means that $g_1=\phi g_0$ for a positive locally square summable measurable function $\phi=\phi(y)$.

In general, such maps are far from conformality, after all Riemann surfaces admits path-isometric maps to the plane $\mathbb{R}^2$, but if such a surface is *minimal*, then *metaconformal* is what one could regard as *conformal*, at least at the points $y\in Y$ where the

[100] *Sharp* geometric inequalities need a finer notion of area, such, e.g. as *Hilbert 2-volume* introduced in [57].

map $Y \to X$ is locally one-to-one.[101]

Kobayashi Construction. The *conformal Kobayashi* (possibly degenerate) metric on a metric space X is the maximal metric, such that all minimal (meta)conformal maps from the hyperbolic plane $\mathbf{H}^2$ (with curvature -1) to X are 1-Lipschitz.

This definition and the preceding discussion are justified by the following

❖ $dist_{Kob}$-Nondegeneracy Proposition (*Ahlfors Lemma*). *If the space X satisfies ☻, and if closed curves in X of length $l \le 1$ bounds discs of areas $\le C_2 \cdot l^2$, then*

$$dist_{Kob}(x_1, x_2) \ne 0 \text{ for } x_1 \ne x_2.$$

Proof.[102] To prove the non-vanishing of $dist_{Kob}$ it is sufficient to bound the diameters of the images of the unit balls under conformal minimal maps $\mathbf{H}^2 \to X$.

We start with observing that the linear filling inequality ☻ for curves of lengths $l \ge 1$ and the quadratic inequality for $l \le 1$ (trivially) imply, when taken together, that the curves S of all lengths l satisfy

$$ar(s) \le C_\alpha l^\alpha \text{ for all } 0 < \alpha < 2,$$

where the constant $C = C_\alpha$ also depends on the above C_2 and the constant in ☻.

Then the proof reduces to the following

Lemma. Let $\phi = \phi(y)$ be a positive measurable function on $\mathbf{H}^2$, such that the integrals of ϕ^2 over the balls (discs) $B \subset \mathbf{H}^2$ are bounded in terms of the integrals of ϕ over the boundaries $S = \partial B$ as follows.

$$\bigcirc_\alpha, \qquad \int_B \phi(y)^2 dy \le C_\alpha \left(\int_S \phi(s) \right)^\alpha ds,$$

where α is a constant in the interval $1 < \alpha < 1\frac{1}{3}$.

Then every pair of points in $\mathbf{H}^2$ with the distance ≤ 1 between them can be joined by a curve $\Sigma \subset \mathbf{H}^2$, such that

$$\int_\Sigma \phi(\sigma) d\sigma \le const_\star,$$

where $const_\star$ depends only on C_α.

Proof of the Lemma. It suffices to show that, for every point $y \in \mathbf{H}^2$, there exist

[$\bullet_E$] a segment $E = E_y$, say of of length 0.11, issuing from y, such that $\int_E \phi(e) de \le const_1$

and

[$\bullet_S$] a circle $S = S_y$ centered at y of radius $r = r(y)$, where $0.1 \le r \le 0.11$ and such that $\int_S \phi(s) ds \le const_2$.

Let $\overline{\phi}_0 = \overline{\phi}_0(y)$ be obtained by *averaging ϕ over the rotation group* around $y_0 \in \mathbf{H}^2$ and observe that the existence of E_{y_0} and S_{y_0} for ϕ follows from that for $\overline{\phi}_0$ by the *convexity* of the function(al) $\phi \mapsto \int \phi^2$.[103]

To conclude the proof with a minimal computation, let us pass from the the punctured hyperbolic plane $\mathbf{H}^2 \setminus \{y_0\} \subset \mathbf{H}^2$ to the conformally equivalent to it cylinder $(-\infty, 0) \times S^1$ with $length(S^1) = 1$.

The (S^1-symmetric!) function $\overline{\phi}_0$ becomes here a function in $r \in (-\infty, 0)$, call it $\psi(r)$, $r < 0$, where the inequality $\bigcirc_\alpha$ says that

$$\Psi(r) = \int_{-\infty}^r \psi(r')^2 dr' \le C_\alpha \psi(r)^\alpha,$$

[101] "Conformal" is hardly applicable to the points in Y small neighbourhoods of which have 1-*dimensional* images in X.

[102] This, probably, goes back to Ahlfors.

[103] This is the moment where linearization built into the lemma becomes essential.

or

$$\psi(r)^2 = \frac{d\Psi(r)}{dr} \geq C_\alpha^{-\frac{\alpha}{2}} \Psi(r)^{\frac{2}{\alpha}} = c_\beta \Psi(r)^\beta, \text{ where } 3/2 > \beta > 1.$$

Since $\beta > 1$, the inequality $\frac{d\Psi(r)}{dr} \geq c_\beta \Psi(r)^\beta$ implies the bound

$$\Psi(r) \leq \overline{C}(r), \ r < 0$$

for some function

$$\overline{C}(r) \sim -r^{-\frac{1}{\beta-1}}, \ r < 0,$$

since

$$\frac{d(-r^{-\frac{1}{\beta-1}})}{dr} = a_\beta \cdot \left(-r^{-\frac{1}{\beta-1}-1}\right) = a_\beta \cdot \left(-r^{-\frac{1}{\beta-1}}\right)^\beta \text{ where } a_\beta = (\beta-1)^{-1}.$$

And since $\frac{1}{\beta-1} = 2+\varepsilon$, $\varepsilon > 0$, for $\beta < 3/2$, the integrals $\int_{-\infty}^{-1} \psi(r)dr$ are bounded in terms of

$$I_i = \int_{-(i+1)}^{-i} \psi(r)^2 dr = \Psi(i) - \Psi(i+1) \leq C_\varepsilon \frac{1}{i^{2+\varepsilon}}, \ i = 1, 2..., \ \varepsilon > 0.$$

as follows

$$\int_{-\infty}^{-1} \psi(r)dr \leq \sum_1^\infty I_i^{\frac{1}{2}} \leq const_\varepsilon < \infty.$$

The bound $\int_{-\infty}^{-1} \psi(r)dr \leq const$ directly implies the above $[\bullet_E]$ and, together heter with the inequality

$$\min_{0.101 r_0 \leq r \leq 0.1 r_0} \psi(r) \leq 1000 \int_{-\infty}^{-1} \psi(r)dr \text{ for all } r_0 \leq -10,$$

it yields $[\bullet_S]$. This concludes the proof of the lemma and

non-vanishing of the metric $dist_{Kob}$ in hyperbolic metric spaces follows. QED.

It is also not hard to show that the converse is true:

non-vanishing of $dist_{Kob}$ on the blow-up 2-complex $G^2_{\leq D}(X)$ implies hyperbolicity of X, compare section 6.8 in [?].

In fact, there is a stronger result due to Bruce Kleiner (unpublished) which proivides a direct derivation of the linear isoperimetric inequality from a bound on dilations of conformal maps.

Exercise: Ahlfors-Picard Theorem. Prove that the discs Y immersed into the triply punctured sphere, $S^2 \setminus \{\cdots\}$, satisfy $area(Y) \leq const \cdot lenght(\partial(Y))$ and show that that there is no (non-constant) quaisconfomal map $\mathbb{R}^2 \to S^2 \setminus \{\cdots\}$.

What is "Space"?

There is something wrong with the above proof of ❖:

why two pages instead of two lines?[104]

Apparently, the concepts of *metric space* and *large scale geometry* are poorly adapted to the 2-dimensionality of surfaces, areas and conformal maps.

A more expressive language (motivated by the ideas of *the conformal field theories*) would be that of spaces (categories?) $\mathcal{S}^*$ with the properties imitating these of the spaces $\mathcal{S}^*(X)$ of closed oriented(?) curves S in metric spaces X (with any number $i = 0, 1, 2, ...$ of components in S) with the structure(s) imposed by the geometries of (quasi)minimal (and harmonic?) surfaces bounded by these curves.

This $\mathcal{S}^*$, seen as the space of 1-dimensional $\mathbb{Z}$-cycles, has a structure of an Abelian group, where the genera and areas of (minimal) surfaces Y bounded by $s \in \mathcal{S}^*$ serve

[104] This proof is not even, strictly speaking, complete: a few technicalities, let them be trivial ones, were swept under the carpet.

as "norms" of these s. And finer parameters are encoded in the (possible) conformal structures of Y.

Questions. What would be the counterpart(s) of the large scale geometries for such $\mathcal{S}^*$?

How does this "geometry" look, for example, if $\mathcal{S}^* = \mathcal{S}^*(X)$ where X is a compact negatively curved manifold?

Which $\mathcal{S}^*$ can be regarded as "hyperbolic"?

2.13 Ideal Boundary, Conformal Geometries and Asymptotic Invariants of Metric Spaces.

The asymptotic cones $Con_\infty(X) = \lim_{D\to\infty} D^{-1}X$ give only a rough idea of the geometry of metric spaces X at infinity. For instance, these cones are mutually isometric for most (all?) interesting $CAT(-1)$-spaces, including all manifolds with negative curvatures≤ -1. (Trees with finitely many branches, such as $\mathbb{R}$ and $\mathbb{R}_+$, are among the exceptions.)

An incomparably richer spectrum of images unveils in front of your eyes if instead of large balls $B(R) \subset X$ you look at large spheres $S(R) = \partial B(R)$ with *localized induced metrics*, possibly, scaled by D^{-1}, say, with $D = \exp -\beta R$, $0 < \beta < \infty$. (Thinking quasi-isometrically, it is better, instead of spheres, to use annuli between concentric spheres $S(R_1)$ and $S(R_2)$ with arbitrarily large ratios R_2/R_1.)

Then, traditionally, the sphere at infinity, denoted $\partial_{ray}(X)$, is defined as the set of rays $R \subset X$ – images of isometric embeddings $[0,\infty) \to X$ – modulo the equivalence relation $R_1 \sim_{Hau} R_2$ which signifies the boundness of the Hausdorff distance (see 2.6) between rays: $dist_{Hau}(R_1, R_2) < \infty$.

This serves well for $CAT(0)$ spaces and for expanding $[\downarrow\uparrow]$-cylinders: the boundary $\partial_{ray}(X)$ for such an X is equal to the *projective (inverse) limit* of the concentric R-spheres $S(R) \subset X$, around a reference point $x_0 \in X$, where these $S(R)$ map to $S(r < R)$ by radial (which are also normal) projections.

In other words, $\partial_{ray}(X)$ is equal to the space of geodesic rays issuing from $x_0 \in X$, which makes the pictured most transparent for manifolds with negative curvatures where the space of rays issuing from x_0 identifies with the *unit sphere in the tangent* space $T_{x_0}(X)$.

This $\partial_{ray}(X)$ may be not so nice for such spaces as $\mathbb{R}^n$ with the sup-norm $\|(x_1,...,x_i,...,x_n)\| = max|x_i|$, where there are two many rays. For instance, the map $r \mapsto (x_1(r), x_2(r), ..., x_n(r))$, $r \in [0,\infty)$, is isometric (hence, it defines a ray) for *all monotone increasing 1-Lipschitz* functions $x_i(r)$ if one of them, say, $x_1(r)$ is equal to r.

This problem does not arise in the alternative definition, where X is embedded to the space F of real functions on X with the sup-norm by $x \mapsto f(y) = dist(x,y)$ and then F is factorized by a subspace F_{small} of "small" functions, e.g, of constants or of bounded functions. Then $\partial_\infty(X)$ is defined as the set of the *limit points*[105] of the image of X in F/F_{small}.

This boundary, if it is defined with *small = bounded*, is (obviously) invariant under almost isometries. What is more interesting is that

if X is geodesic hyperbolic then $\partial_\infty(X)$ is a quasiisometry invariant. Moreover, quasisometric embeddings $Y \to X$ induce topological embeddings $\partial_\infty(Y) \to \partial_\infty(X)$.

In fact, it (easily) follows from the shadowing property that

$$\partial_\infty(X) = \partial_{geo}(G_{\le D}(X)), D > 0.$$

Why $G_{\le D}(X)$? The blow-up graph $G_{\le D}(X)$ is needed to take care of (artificial) cases where almost rays are not shadowed by rays; one could avoid using $G_{\le D}$ by

[105] *A limit point* of a subset $A \subset B$ is a point in the closure of A which is not in A itself.

redefining $\partial_{geo}(X)$ with *almost rays* – images of *almost isometric* maps $[0,\infty)\to X$ – instead of rays.

In what follows below, we assume for the sake of brevity that almost rays *are* are shadowed by rays and also *almost lines shadowed are by lines*; thus we dispose of $G_{\leq D}$.

Then we have no problem, for instance, with the existence of

lines joining pairs of points at infinity,

denoted $]p_1,p_2[\subset X$ for all $p_1,p_2\in\partial_\infty(X)$, $p_1\neq p_2$.

Quasiconformal Structure on ∂_∞. If X is equal to the standard hyperbolic space $\mathbf{H}^n$, then the radial projection of spheres, $S_x(R)\to S_x(r)$, $x\in X$, scale the metrics in the spheres. This is not so for the radial projection $S_x(R)\to S_y(r)$, but this projection is *asymptotically conformal* for $R\to\infty$, since the geodesic rays issuing from y are *asymptotically normal* to the spheres $S_x(R)$ for $R\to\infty$.

Thus,

the conformal structure in $\partial\infty(X)$ induced by the radial projection to $S_x(r)$ does not depend either on r (which is obvious) *or on x* (which is more interesting).

Consequently,

The isometries of $X=\mathbf{H}^n$ extend to *conformal* homeomorphisms of the sphere $S^{n-1}=\partial_\infty(\mathbf{H}^n)$.

Less obviously,

❂ *quasiisometries* of H^n extends to *quaisiconformal* homeomorphisms of the sphere $S^{n-1}=\partial_\infty(\mathbf{H}^n)$.

Historical Remark. Generalizing/refining earlier analytic results by Mori (1957, $n=2$,) and Gering (1963, $n=3$) Mostow (1968) [**?**] proved (on 30 pages in [84] that

(A) *quasi conformal maps* of the unit n-ball, $n\geq 3$, quasiconformally extend to the boundary of the ball.

Since H^n is conformally equivalent to the (open) ball, (A) implies that

(B) *bi-Lipschitz homeomorphisms* of H^n extends to *quasiconformal* homeomorphisms of the sphere $S^{n-1}=\partial_\infty(\mathbf{H}^n)$.

Then Margulis (1970) gave a half a page proof of ❂following in steps of the Morse-Efremovich-Tichomirova shadowing argument (see below.)

Notice also that (A) follow from ❂, since

quasiconformal (not necessarily locally homeomorphic) maps $f:\mathbf{H}^n\to\mathbf{H}^n$, $n\geq 2$, are Lipschitz on the large scale.

This follows from Ahlfors Lemma which we proved in section 2.12 for $n=2$, but the argument automatically extends to all n.

(Our argument needs f to be almost everywhere differentiable but this is not difficult to take care of.)

Margulis-Sullivan Quasiconformal Structure on $\partial_\infty X$ and the proof of ❂.

Start with the case at hand where $X=\mathbf{H}^n$ and describe annuli between $(n-2)$-spheres in $S^{n-1}=\partial\mathbf{H}^n$ in terms of geodesic rays issuing from a point $x_0\in\mathbf{H}^n$ as follows.

Given a ray R_0 in $\mathbf{H}^n$ from x_0 to a point $s_0\in\partial\mathbf{H}^n$ let $U_{L,d}(R)\subset\partial\mathbf{H}^n$ be the subset represented by rays R issuing from x_0, such that the initial segments R of length L lie within distance $\leq d$ from R_0.

These subsets $U_{L,d}=U_{L,d}(R_0)$ are balls in $S^{n-1}=\partial\mathbf{H}^n$ around s_0 for the metric in S^{n-1} coming from the unit sphere $S_x^{n-1}(1)\subset\mathbf{H}^n$, where the radii r of these balls are, roughly $r\approx d\cdot e^{-L}$ for $d<<L$ and where the essential point is that

the ratios of the radii of the balls U_{L_1,d_1} and U_{L_2,d_2} inS^{n-1}, where $L_2\geq L_1$ and $d_1\geq d_2$, are bounded by

$$\frac{r_1}{r_2}\leq const\frac{d_1}{d_2}\exp(L_2-L_1).$$

Now let $f : \mathbf{H}^n \to \mathbf{H}^n$ be a quasisometry that sends $x_0 \mapsto x'_0$ and let R'_0 be the ray issuing from x'_0 which shadows $f(R_0) \subset \mathbf{H}^n$. The segments in R'_0 which correspond to the initial L-segments in R have lengths $L' \approx a \cdot L^{\pm 1}$; thus the extension of f to $S^{n-1} = \partial \mathbf{H}^n$, still called f, is not usually Lipschitz but Hölder C^a.

And since the boundness of the differences $L_2 - L_1$ and of the ratios d_1/r_d is preserved under quasiisometries,

the f-images of the r-balls in S^{n-1} are pinched between balls the radii of which r'_- and r'_+ have the ratios bounded by $r'_+/r'_- \leq const_f$. (These radii themselves may be $\approx r^a$.)

Hence, the map f is quasiconformal on S^{n-1}. QED.

The description of annuli via the sets of (quasi)rays make sense for all hyperbolic spaces X, where it serves as *the definition* of the quasiconformal structure $\partial_\infty X$ (I recall Sullivan making this remark many years ago), where this structure was analysed earlier in different terms by Mostow (1973) in his proof of the (strong) rigidity of the locally symmetric spaces with negative (not necessarily constant) sectional curvatures.

Conformal Invariants of Flows. The growth rate characteristics of $\mathbb{R}$-actions on spaces S, such as the topological entropy, are not invariant under the time reparametrization, but certain ratios of these "characteristics" are invariant.[106] This may be seen in terms of the $[{}^{\downarrow}_{\downarrow}]$-cylinders X_ε (see section 2.8) associated with these actions, where the quasiisometry classes of X_ε are invariant under time reparametrization and, at least in the hyperbolic case, the conformal invariants of $\mathbb{R}$-actions are seen in S which is identified for this purpose with $\partial_\infty(X_\varepsilon)$.

Clarification. The definition of the metrics $dist_\Delta$ and $dist_\Delta\uparrow^\varepsilon$ on S and of the corresponding $[{}^{\downarrow}_{\downarrow}]$-cylinders X_ε given in 2.8 for actions of discrete groups, applies to $\mathbb{R}$, with $\Delta_i = [-i, i] \subset \mathbb{R}$, but this action can not be uniformly expansive, since it does not expand the $\mathbb{R}$-orbits. This can be (artificially) compensated by scaling the metric $dist_{\Delta_i}$ along the orbits with the factor $(1+\varepsilon)^i$. Alternatively, one may restrict $dist_{\Delta_i}$ to a transversal slice to the orbits.

Higher Dimensional Groups and Foliations. The above applies to (generously understood) foliations. e.g. to orbits of continuous groups actions, where $dist_{\Delta_i}$ measures the Hausdorff distances (see 2.6) between i-balls in the leaves with the localized induced metrics in them.

2.14 Semihyperbolic Spaces.

We search for a class (classes?) of spaces X which could be taken for a "boundary" of the class of the hyperbolic spaces, where the failure of hyperbolicity in these X should be localized on (quasi)flat (or similarly simple) subspaces.

Below are a few (potential) examples of subclasses of such spaces.

[**B**]. This is an extreme generalization of Bruhat-Tits buildings: the minimal class of geodesic spaces which contains convex subsets in Banach spaces and which is closed under amalgamation over convex subsets. (Also one may insist that [**B**] is closed under inductive limits and/or ultra limits.)

Example: Gersten's spaces. Let $\mathcal{H}$ be the set of the affine hyperplanes $H = H_{ij} \subset \mathbb{R}^n$ given by the equations $x_i = j$, $i = 1, 2, ..., n$, $j = ... - k, ..., 0, ..., k....$

The Gersten space X_{Ger} is obtained from copies of $\mathbb{R}^n$ by amalgamation along hyperplanes $H \in \mathcal{H}$, such that there are two copies of $\mathbb{R}^n$ attached to every copy of each H in X.

[106]We discuss specific invariants of this kind, such as *Pansu's conformal dimension* [92], [51] and *conformal entropy spectrum* [52] later in this paper.

For instance if $n = 1$, this $\mathcal{H}$ is the set of the integer points in $\mathbb{R}$ and X_{Ger} is isometric to the regular tree with four edges at every vertex in it.

Exercises. (a) Show that the R-spheres $S_x(R) \subset X$ have $diam(S_x(R)) \sim R^2$, where these diameters are measured with respect to the path metrics in the complements to the open balls $B_x(R) \subset X$, i.e. $dist(s_1, s_2)$ is defined as the length of the shortest path in $X \setminus B_x(R)$ between these points. (See [43], [10], [76], [112] for more about it.)

(b) Describe the isometry groups of the spaces X_{Ger} for $n \geq 2$.

$[\mathbf{B}_\delta]$ This class of geodesic metric spaces X is defined by the following property:
if subsets $Y_i \subset X$, $i = 1, 2, ..., k$, are almost isometric to spaces from $[\mathbf{B}]$
*then the union $Y = \cup_i Y_i$ is almost isometric to a subset in a space from [**B**],*
where the almost isometry parameter D of Y depends, besides the parameter δ which characterises X itself, only on these parameters D_i of Y_i and on k. (compare exercise (c)$_{\mathsf{Y}}$ in 2.9)

Another rather general class, call it $[\updownarrow]_\lambda$, is comprised of $[\updownarrow]$-cylinders, where the maps $S(i) \to S(i-1)$ are λ_i-Lipschitz, e.g. for $\lambda_i \leq \frac{i}{i-1}$.

Among these "cylinders" with $\lambda_i \leq \frac{i}{i-1}$ one finds
Pederesens $[\kappa \leq 0]_{\mathbf{1}}$*-spaces*, as well some (undesirable?) nilpotent Lie groups with *Carnot-Caratheodory metrics.*

Two other examples – these are motivated by the group theory – are the following.

[⬡] 2-dimensional polyhedral spaces where every closed 2-cell Δ has the following property: *if an open topological disc in $D^2 \subset X$ contains Δ then D^2 intersects the interiors of at least* 6 *2-cells in X besides Δ.* (The universal coverings of the complexes associated with $[\leq \frac{1}{6}]$-presentations of groups are instances of such spaces.)

[✿] Complete polyhedral, e.g. Riemannian, spaces X where all complete simply connected quasi-minimal surfaces are quasiisometric to simply connected surfaces with non-positive curvatures.

The above list neither perfect nor complete. Other directions of generalizations of $\kappa \leq 0$, are suggested by the concepts of *combing* and *bicombing* and and/or of *divergence rates of geodesics* in X. But the main questions remain open (compare [**?**]).

What are classes of spaces deserving the name "SEMIHYPERBOLIC"?
What are relations between different classes of semihyperbolic spaces X, especially between classes of almost homogeneous X?

3 Hyperbolic Symmetries and Hyperbolic Groups.

In the same vein, as the class of hyperbolic spaces is a geometric/logical perturbation of the class of trees, the class of *hyperbolic group* is an *algebraic/logical perturbation* of the class of *free groups.*

4 Symbolic Coding and Markov Partitions.

Similarly to how it is with hyperbolic groups, the class of hyperbolic dynamical systems is a combinatorial/logical perturbation of the class of the Bernoulli actions.

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