Progress in Mathematics, Vol. 144, © 1996 Birkhäuser Verlag Basel/Switzerland

Carnot-Carathéodory spaces seen from within

MIKHAEL GROMOV*

Contents

*Acknowledgment. I thank Richard Montgomery for reading the manuscript and locating a multitude of errors.

0. Basic definitions, examples and problems

Let V be a smooth manifold where we distinguish a subset $\mathcal{H}$ in the set of all piecewise smooth curves c in V. We assume that $\mathcal{H}$ is defined by a *local* condition on curves, i.e. if c is divided into segments $c_1, \dots, c_k$, then

$$c \in \mathcal{H} \Longleftrightarrow c_i \in \mathcal{H},\ i = 1, \dots, k.$$

Next we pick up some Riemannian metric g in V and define

$$\operatorname{dist}(v_1, v_2) = \operatorname{dist}_{\mathcal{H},g}(v_1, v_2),\ v_1, v_2 \in V,$$

as the infimum of the lengths of the *distinguished* curves joining v_1 and v_2 in V. This distance obviously satisfies the usual axioms of a metric, provided every two points in V can be joined by a (distinguished) curve $c \in \mathcal{H}$. Otherwise, dist becomes infinite at the pairs of points in V which admit no distinguished curve joining them.

Example. Let V be the Euclidean plane $\mathbb{R}^2$ and $\mathcal{H}$ consist of piecewise linear curves, where each segment is either vertical or horizontal. Then the corresponding distance between the points $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ equals $|x_1 - x_2| + |y_1 - y_2|$, where we use the Euclidean metric of $\mathbb{R}^2$ for g. Notice that this $\operatorname{dist}_{\mathcal{H}}$ is *equivalent* to the ordinary Euclidean distance $\operatorname{dist}_{\mathrm{Eu}}$ in the sense that the identity map $(\mathbb{R}^2, \operatorname{dist}_{\mathcal{H}}) \to (\mathbb{R}^2, \operatorname{dist}_{\mathrm{Eu}})$ is *bi-Lipschitz*, i.e. $C^{-1} \operatorname{dist}_{\mathrm{Eu}} \le \operatorname{dist}_{\mathcal{H}} \le C \operatorname{dist}_{\mathrm{Eu}}$ for some $C > 0$. (Here one may take $C = \sqrt{2}$). This equivalence makes $\mathcal{H}$ and $\operatorname{dist}_{\mathcal{H}}$ rather non-interesting from our present point of view.

0.1. Polarizations, horizontal curves and Carnot-Carathéodory metrics. A *polarization* of a manifold V is, by definition, a subbundle of the tangent bundle, say $H \subset T(V)$. One may think of H as a distinguished set of directions (tangent vectors) in V which are called in sequel *horizontal*. (This terminology is motivated by the picture where V is smoothly fibered over some manifold B and H is normal to the fibers.)

A piecewise smooth curve in V is called *horizontal* with respect to H if the tangent vectors to this curve are horizontal.

The metric defined with (the set $\mathcal{H}$ of) the horizontal curves in V is called the *Carnot-Carathéodory metric* associated to H and denoted

dist_H. Notice that the definition of dist_H also involves an auxiliary Riemannian metric g as

$$\mathrm{dist}_H(v_1, v_2) \underset{\text{def}}{=} \inf(g\text{-lengths of } H\text{-horizontal curves between } v_1 \text{ and } v_2),$$

but the effect of g on dist_H is non-essential from our point of view. Namely, two metrics defined with different g_1 and g_2 and some H are bi-Lipschitz equivalent (on each compact subset in V). On the other hand, the role of H is crucial as is seen in the following

0.2. Basic contact example. Let $V = \mathbb{R}^3$ and H be the standard *contact* subbundle, which is the kernel of the (contact) 1-form $\eta = dz + x dy$ on $\mathbb{R}^3$. This means that the tangent space (plane) $H_{v_0} \subset T_{v_0}(\mathbb{R}^3) = \mathbb{R}^3$ is given at each $v_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ by the equation $z + x_0 y = 0$. Notice, that H is generated by the following two independent vector fields, $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$. These fields do not commute. In fact, their commutator equals $-\frac{\partial}{\partial z}$ and so the three fields ∂_1, ∂_2, and the Lie bracket $[\partial_1, \partial_2]$ span the tangent bundle $T(\mathbb{R}^3)$ at each point $v \in \mathbb{R}^3$.

0.2.A. Connectivity theorem for the contact polarization H.

Theorem. *Every two points in $\mathbb{R}^3$ can be joined by a smooth H-horizontal curve.*

Proof. Take a curve $\underline{c} = \big(x(t), y(t)\big)$, $t \in [0,1]$, in the (x,y)-plane joining two given points (x_1, y_1) and (x_2, y_2) and such that the *formal area* "bounded" by $\underline{c}$, defined by the integral $\int_{\underline{c}} x\,dy = \int_0^1 x(t)y'(t)dt$, equals a given number a. (One easily finds such $\underline{c}$, say among curves of constant curvature). Then we take the horizontal lift of $\underline{c} = \big(x(t), y(t)\big)$ to $\mathbb{R}^3$ by letting $z(t) = z_1 - \int_0^t x(t)y'(t)dt$ for a given value z_1 of z. The lifted curve $c = \big(x(t), y(t), z(t)\big)$ is indeed horizontal as $dz(t) = z'(t)dt = -x(t)y'(t)dt = -x(t)dy(t)$ and it joins the given points (x_1, y_1, z_1) and $(x_2, y_2, z_2 = z_1 + a)$. ∎

Historical Remarks. This result (which seems obvious by the modern standards) appears (in a more general form) in the 1909-paper by Carathéodory on formalization of the classical thermodynamics where horizontal curves roughly correspond to *adiabatic* processes. In fact, the above proof may be performed in the language of Carnot (cycles) and for this reason the metrics dist_H were christened "Carnot-Carathéodory" in [G-L-P].

I suspect that some form of the connectivity theorem was known to Lagrange in the framework of the *nonholonomic mechanics*. (Compare [Ve-Fa] and [Ver-Ger$_{1;2}$]; an instance of a nonholonomic system is given by a billiard ball rolling on the plane, such that the velocity at the lowest point of the ball where it touches the plane must be zero. Here V equals the configuration space, that is $\mathbb{R}^2 \times SO(3)$, and the nonholonomic constraint (on the velocity) is represented by a subbundle $H \subset T(V)$ of rank 3. Now, a child can roll the ball from one given position to another thus proving the connectivity property for H.) Various forms of the connectivity theorem have been persistently appearing in the literature but I have not tried to keep track of them. By tradition, a general commonly used connectivity theorem (see 0.4.) is attributed to Chow (see [Cho], though his paper was neither the first (see, e.g. [Rash]) nor the last on that matter. Finally we notice that the phenomenon of H-connectivity is also seen in the theory of optimal control and in robotics where it is named "controllability" (see [Brock]).

0.2.A$'$. Contact C-C metric on $(\mathbf{V}, \mathbf{H})$. Now, let us look more closely at the C-C metric (C-C = Carnot-Carathéodory) associated to a contact subbundle $H \subset T(\mathbb{R}^3)$. We recall that the H-horizontal curves c in $\mathbb{R}^3$ are the lifts of curves in the (x, y)-plane, such that the z-coordinate of c equals the formal area of the (x, y)-projection $\underline{c}$ of c. If two points v_1 and v_2 in $\mathbb{R}^3$ lie on the same vertical line (or z-line), i.e. have equal (x, y)-coordinates, then the (x, y)-projections $\underline{c}$ of curves c joining these points are *closed* in the (x, y)-plane and so the (formal) area of these curves $\underline{c}$ is bounded by

$$\operatorname{area}\underline{c} \leq \operatorname{const}(\operatorname{length}\underline{c})^2 \leq \operatorname{const}(\operatorname{length} c)^2,$$

where $\operatorname{const} = (4\pi)^{-1}$. It follows that the C-C distance between v_1 and v_2 is bounded from below by the Euclidean distance as follows,

$$\mathrm{dist}_H \geq \operatorname{const}^{-\frac{1}{2}} (\mathrm{dist}_{\mathrm{Eu}})^{\frac{1}{2}}, \qquad (*)$$

since the Euclidean distance $\mathrm{dist}_{\mathrm{Eu}}$ between our points equals $z_1 - z_2 = \operatorname{area}\underline{c}$. One also has the upper bound

$$\mathrm{dist}_H \leq \alpha \operatorname{const}^{-\frac{1}{2}} \mathrm{dist}_{\mathrm{Eu}}^{\frac{1}{2}}$$

where α is a certain positive function depending on the Euclidean norms of the points v_1 and v_2. (One may take $\alpha = \left(1 + \|v_1\| + \|v_2\|\right)^2$). In fact, one can join v_1 and v_2 by a curve c in $\mathbb{R}^3$ which projects to a circle $\underline{c}$ in $\mathbb{R}^2$, such that

$$\mathrm{dist}_{\mathrm{Eu}} = \mathrm{area}\,\underline{c} = (4\pi)^{-1}(\mathrm{length}\,\underline{c})^2$$

while

$$\mathrm{dist}_H \leq \mathrm{length}\,c \leq \alpha\,\mathrm{length}\,\underline{c}.$$

($\sqrt{\ }$)-Conclusion. *On every vertical line in $\mathbb{R}^3$ the Carnot-Carathéodory metric is locally equivalent to $\sqrt{\text{Euclidean metric}}$.*

On the other hand, the C-C metric is locally equivalent to the Euclidean one on every horizontal curve, for example, on every (straight) x-line in $\mathbb{R}^3$. This makes the C-C metric highly non-isotropic. In fact, a small ε-ball in the C-C metric around each point $v \in \mathbb{R}^3$ looks roughly as a rectangular solid (box) with two ε-edges and one edge of length ε^2 (see 0.3.C). The $\varepsilon\times\varepsilon$-face of this solid is positioned in R^3 approximately tangent to H at v, while the ε^2-edge is approximately normal to H. (Our idea of "approximate" is such that the Euclidean ε-ball is roughly the same thing as an ε-cube). It follows, that the (Euclidean) volume of the ε-C-C ball is about ε^4 and consequently *the Hausdorff dimension of $\mathbb{R}^3$ with the* C-C *metric* dist_H *equals* 4.

Integrable non-example. If we take the form dz instead of $dz + x\,dy$, then the kernel subbundle H becomes integrable and the C-C metric degenerates: on each horizontal plane dist_H equals the Euclidean distance while every two points with non-equal z-coordinates have $\mathrm{dist}_H = \infty$. This can be remedied by introducing another metric which, by definition, equals $\sqrt{\text{Euclidean}}$ in the z-directions, see [N-S-W] and 1.5.

0.2.B. Internal versus external in C-C geometry. If we live inside a Carnot-Carathéodory metric space V we may know nothing whatsoever about the (external) infinitesimal structures (i.e. the smooth structure on V, the subbundle $H \subset T(V)$ and the metric g on H) which were involved in the construction of the C-C metric. For example, the relations between the C-C metric and the Euclidean one (such as the above equivalence C-C metric $\approx \sqrt{\text{Euclidean metric}}$ on the vertical lines in $\mathbb{R}^3$ and the ε^4-estimates for the *Euclidean* volume of C-C balls) remain invisible for a C-C insider. On the other hand the equality $\dim_{\mathrm{Hau}} V = 4 = \dim_{\mathrm{top}} V + 1$ is an internal feature stated in the intrinsic metric terms.

Main problems

(1) Develop a sufficiently rich and robust internal C-C language which would enable us to capture the essential external characteristics of our C-C spaces. For example, we would like to recognize rank H by a purely C-C metric consideration invariant under a sufficiently large class of transformations, such as bi-Lipschitz transformations and, even better, C^α-Hölder transformations, where $\alpha = 1-\varepsilon$ for a *positive* (albeit small) ε.

(2) Develop external (analytic) techniques for evaluation of internal invariants of V. For example, in order to determine $\dim_{\mathrm{Hau}} V$ we need the external information on the Euclidean volume of small C-C balls.

The purpose of the present paper is to expose what is known in these directions. The basic internal invariants we look at are concerned with subspaces $V' \subset V$ and here the best understood ones are curves ($\dim_{\mathrm{top}} V' = 1$) and hypersurfaces ($\mathrm{codim}_{\mathrm{top}} V' = 1$). On the other hand, the subspaces of intermediate dimensions offer more challenging geometric problems where we are still far from the final solution.

0.3. Heisenberg group view on the contact example. The geometry of the contact C-C metric becomes infinitely more transparent if we replace the Euclidean metric by another Riemannian metric as follows. (As we mentioned earlier, this change does not affect the essential C-C features.) In fact, instead of $\mathbb{R}^3$ we take the three-dimensional *Heisenberg group* G which can be defined as the only simply connected nilpotent non-Abelian Lie group. The Lie algebra $L = L(G)$ of G admits a basis x, y, z, such that $[x, z] = [y, z] = 0$ and $[x, y] = z$. (These relations uniquely define L and, hence, G.) We introduce a polarization $H \subset T(G)$ by taking the left translates of the (x, y)-subspace $H_0 \subset L = T_{\mathrm{id}}(G)$. (One knows, that there exists a diffeomorphism between G and $\mathbb{R}^3$ sending this H to the standard contact subbundle in $\mathbb{R}^3$ but this is not crucial at the present moment.) Next we take a left invariant metric g on G and let dist_H be the C-C metric defined with H and g. First, we must make sure that this is indeed a *metric* by checking the connectivity property for H. This can be done in (at least) two ways. One possibility is to look at the homomorphism (projection)

$$G \to \mathbb{R}^2 = G/\mathrm{center},$$

where the center of G (obviously) equals the 1-parameter subgroup obtained by the exponentiation of the (central) line spanned by z in $L(G)$. The (contact) geometry of this projection of $G = \mathbb{R}^3$ to $\mathbb{R}^2$ is identical to the (x, y)-projection of the basic contact example and the proof of the connectivity theorem 0.2.A applies.

Here is another approach.

0.3.A. Lie group theoretic proof of connectivity. Consider the one-parameter groups G_x and G_y of (right) translations of G corresponding to x and y in $L(G)$. The orbits of these subgroups are obviously tangent to H. On the other hand, G_x and G_y, viewed as subgroups in G, generate G since x and y Lie-generate $L(G)$. It follows, that every two points in G can be joined by a piecewise smooth curve whose every segment is a piece of an orbit of G_x or of G_y. Q.E.D.

0.3.A′. Connectivity theorem for general Lie groups. Let G be an arbitrary connected Lie group and H_0 a linear subspace in the Lie algebra of G. This H_0 defines a left invariant polarization $H \subset T(G)$ and then one defines the Carnot-Carathéodory metric dist_H on G. This is an honest metric (nowhere ∞) if and only if the subspace H_0 Lie-generates the Lie algebra $L(G)$. Furthermore, if the auxiliary Riemannian metric used in the definition of dist_H were left invariant, then dist_H is also left invariant on G.

0.3.B. Self-similarity. The C-C metric on the Heisenberg group has an additional remarkable feature which is absent for the left invariant Riemannian metrics on general G. Namely, besides being left invariant and thus admitting a transitive group of isometrics, (which is, of course, a property shared by all left invariant Riemannian metrics) the Carnot-Carathéodory metric on G admits non-trivial *self-similarities*. In fact *there exists a 1-parameter group of diffeomorphisms $A_t : G \to G$, $t \in \mathbb{R}_+^\times$, such that $A_t \,\mathrm{dist}_H = t \,\mathrm{dist}_H$ for all $t \in \mathbb{R}_+^\times$, which means*

$$\mathrm{dist}\big(A_t(v_1),\, A_t(v_2)\big) = t\,\mathrm{dist}(v_1, v_2)$$

for all v_1 and v_2 in G.

Proof. Define automorphisms a_t of the Lie algebra $L(G)$ by $(x, y, z) \mapsto (tx, ty, t^2 z)$ for the (x, y, z)-basis, where z is central and $[x, y] = z$. These are, clearly, automorphisms and they exponentiate to automorphisms of G, called $A_t : G \to G$. These A_t preserve H (which is generated by x and y) and they scale the Riemannian metric on H (but not on all of $T(G)$) by t. Therefore, the length of all horizontal (i.e. tangent H) curves is also scaled by t and then dist_H is scaled by t as well. ∎

General self-similar Lie groups. Let G be a simply connected nilpotent Lie group where the Lie algebra L admits a grading, $L = \bigoplus_{i=1}^{d} L_i$, such that $[L_i, L_j] \subset L_{i+j}$. Then the operator $a_t : L \to L,\ t \in \mathbb{R}_+^\times$, defined by $a_t \ell = t^i \ell$ for $\ell \in L_i$ are automorphisms of L. These a_t integrate to a 1-parameter group of automorphisms $A_t : G \to G$, which are similarities for the Carnot metric defined with the polarization corresponding to L_1. (Here as earlier we should assume that L_1 Lie generates L in order to have dist $< \infty$.)

Two-step example. Let G be a two-step nilpotent group. Then L can be graded with $L_2 = [L, L]$ and some subspace $L_1 \subset L$ complementary to L_2. This L_1 obviously Lie-generates L and thus gives us a self-similar C-C metric on G.

Remarks. If V is a homogeneous *Riemannian* manifold which admits a non-trivial self-similarity, then V, clearly, is isometric to $\mathbb{R}^n$ for some n. Analogous non-Riemannian examples are provided by Banach spaces of finite or infinite dimension but the self-similar nilpotent Carnot-Carathéodory manifolds do not spring in one's mind so readily. One knows now-a-days that there are no additional essential examples among *finite dimensional manifolds* (see [Be-Ve]) but there are interesting *infinite dimensional* and/or *disconnected* homogeneous self-similar metric spaces.

Infinite dimensional and totally disconnected examples

(a) Let $L = \bigoplus_{i=1}^{d} L_i$ be an infinite dimensional graded nilpotent Banach-Lie algebra which is (algebraically) Lie-generated by L_1. Then the corresponding nilpotent Banach-Lie group carries a natural homogeneous self-similar C-C metric. The simplest example is that of the infinite dimensional Heisenberg group for $L = L_1 \oplus L_2$, where L_1 is the Hilbert space and $L_2 = \text{center } L = \mathbb{R}$.

(b) Every field with a norm, e.g. the field of p-adic numbers, is homogeneous and self-similar. Furthermore, vector spaces and suitable nilpotent groups over such fields share this property.

(c) Let G be a topological group generated by a family of 1-parameter subgroups. We give a metric to each of these subgroups by choosing parameters and then we define the corresponding C-C metric in G as the *maximal* (or "supremal") left invariant metric for which the inclusions of all our 1-parameter subgroups into G are 1-Lipschitz, i.e. (non-strictly) distance decreasing maps. (This is similar to the definition of the word metric in G with a given generating subset). If G admits a (dilating) automorphism $A : G \to G$, which preserves the distinguished subset of subgroups and dilates the corresponding parameters with a fixed constant $\lambda > 1$, then this A also dilates the C-C metric by λ, which means A is a nontrivial self-similarity. An especially interesting class of such self-similar groups is associated to infinitely graded Lie algebras $L = \bigoplus_{i=1}^{\infty} L_i$ which are Lie-generated by L_1. (This was pointed out to me by I. Babenko.)

0.3.C. Uses of self-similarity: infinitesimal versus asymptotic. If a metric space V admits a non-trivial (i.e. non-isometric) self-similarity fixing a point $v_0 \in V$ then all of geometry of V can be seen in an arbitrary small neighbourhood of v_0. In particular, if V is a C-C manifold then the asymptotic geometry of V (at infinity) can be read in terms of infinitesimal data of the implied polarization at v_0 and vice versa.

Let us apply this to the Heisenberg group G with the dilations A_t (defined by $(x, y, z) \mapsto (tx, ty, t^2 z)$ on the Lie algebra $L = L(G)$) and determine an approximate shape of the Carnot-Carathéodory balls in G. Denote by $\tilde{B}'(\rho) \subset L$ the box defined by the inequalities

$$|x| \leq \rho \ , \ |y| \leq \rho \ , \ |z| \leq \rho^2$$

and let $B'(\rho) \subset G$ be the exponential image of $\tilde{B}'(\rho)$. Clearly, A_t transforms each box $B'(\rho)$ into $B'(t\rho)$ (because A_t commute with exp) and the actual C-C balls in G around id $\in G$ are transformed by A_t in a similar way, $B(\rho) \mapsto B(t\rho)$ (since A_t is a t-similarity for dist_H). It follows that the boxes $B'(\rho)$ approximate the balls $B(\rho)$ in the sense that

$$B'(C^{-1}\rho) \subset B(\rho) \subset B'(C\rho) \tag{$*$}$$

for all $\rho \geq 0$ and some constant $C > 0$.

The relation $(*)$ for $\rho \to 0$ justifies our earlier claims (see 0.2.A$'$) on the shape of the small C-C balls associated to the contact structure. On the other hand, the same $(*)$ for $\rho \to \infty$ provides an asymptotic information on the ordinary Riemannian left invariant metric in G. Namely the balls in this metric, call them $B^+(\rho) \subset G$, (obviously) have the same asymptotics for $\rho \to \infty$ as the balls $B(\rho)$ (since the two metric involved are both left invariant and determined by length of curves). In fact,

$$B(\rho) \subset B^+(\rho) \subset B(C^+\rho),$$

for all $\rho \geq 1$ and some constant C^+ (where the first constantless inclusion is due to the fact that the Riemannian metric we speak about is the one which underlies the C-C metric and so the corresponding Riemannian distance $\leq$ C-C distance).

It follows that the Riemannian balls $B^+(\rho)$ are asymptotically approximated by the (exponentiated) boxes $B'(\rho)$. In fact, this remains valid for all simply connected nilpotent Lie groups with graded Lie algebras by the above self-similarity argument. (An arbitrary simply connected nilpotent Lie group G is *asymptotic* to a group G_∞ which does admit a self-similarity by a theorem of Pansu cited in Remark (b) below. Thus the large balls in G are box-shaped as well as those in G_∞. Also see [Bass] and [Kari] on this matter.)

0.3.D. Self-similar spaces appearing as tangent cones of equiregular ones. Let $V = (V, \mathrm{dist})$ be a metric space with a reference point v_0. We set $tV = (V, t\,\mathrm{dist})$ for $t \in]0, \infty[$ and we want to go to some limits of tV for $t \to 0$ and for $t \to \infty$. An appropriate notion of a limit for our present purpose is the one associated to the *Hausdorff topology* on the "set" of isometry classes of pointed metric spaces (see [$\mathrm{Gro}_{\mathrm{GPG}}$] and [G-L-P]). If such a limit exists for $t \to \infty$ it can be thought of as the tangent space (cone) of V at v_0 while the limit of tV for $t \to 0$ looks like the asymptotic cone of V or the tangent space of V at infinity. Notice, that the very definition of the limit makes these tangent cones (spaces) self-similar whenever they exist. On the other hand, if V already admits t-self-similarities fixing v_0 for all $t > 0$ then (tV, v_0) is isometric to (V, v_0) for all $t > 0$ and so the tangent cones to V at v_0 and at infinity exist and are isometric to V.

Examples

(i) If V is a Riemannian manifold then the Hausdorff limit $\lim_{t\to\infty} tV$ does exist and it equals the ordinary tangent space $T_{v_0}(V)$.

(ii) Let G be the Heisenberg group with a left invariant Riemannian metric g and set $g_t = t^{-1}A_t^*(g)$ for the above A_t. It is obvious, that the distance function dist_t associated to g_t converges (in the usual sense) to the Carnot-Carathéodory metric dist_∞ on G for $t \to \infty$. On the other hand the space (G, dist_t) is isometric to $t^{-1}G \underset{\mathrm{def}}{=} (G, t^{-1}\,\mathrm{dist})$. It follows that the *Hausdorff* limit of the spaces tG for $t \to 0$ also exists and is isometric to $(G, \mathrm{dist}_\infty)$. Hence, *the asymptotic* (tangent) *cone of the Riemannian manifold* (G, dist) *equals the Carnot-Carathéodory space* $(G, \mathrm{dist}_\infty)$.

Thus the (infinitesimal data of the) C-C geometry can be recaptured from the asymptotic Riemannian geometry of G.

Remarks on C-C limits of discrete groups. The above example appears in [Gro$_{\mathrm{GPG}}$] in the surrounding of *discrete* groups G of *polynomial growth*. The polynomial growth ensures the existence of a Hausdorff *sublimit* of tG, $t \to 0$, *an* asymptotic cone, which is a nilpotent Lie group with a C-C metric where the degree of the growth of G translates to the Hausdorff dimension of the limiting C-C metric. In fact, the asymptotic geometry of discrete nilpotent groups provided the major source of inspiration for the initial study of Carnot-Carathéodory spaces.

0.3.D$'$. Pansu convergence theorem. The existence of an actual Hausdorff limit (not just a sublimit) $\lim_{t\to\infty} tG$ was proven by P. Pansu (see [Pan$_{\mathrm{CBN}}$]) for an arbitrary nilpotent Lie group with a left invariant Riemannian metric. (In fact, Pansu allows in [Pan$_{\mathrm{CBN}}$] a more general class of spaces G including discrete virtually nilpotent groups with word metrics.) Pansu shows that such a limit is isometric to a nilpotent Lie group G_∞ (which is, in general, not isomorphic to G) with a self-similar C-C metric. Thus the asymptotic geometry of every nilpotent group G reduces to the local C-C geometry of G_∞. In particular the asymptotics of the ρ-balls in G for $\rho \to \infty$ is encoded in the (infinitesimal) behavior of ρ-balls in G_∞ as $\rho \to 0$.

0.3.D″. Mitchell cone theorem for equiregular spaces. Consider a C-C metric in V defined with a polarization $H \subset T(V)$ which satisfies the following (genericity) assumption. Choose some tangent vector fields in H which span H and denote by $H_i(v) \subset T_v(V)$, $v \in V$, the subspace in $T_v(V)$ spanned by all commutators of the chosen fields of order $\leq i$. Clearly, this H_i does not depend on the choice of the spanning fields.

Equiregularity Assumption. The dimension $\dim H_i(v)$ is constant in $v \in V$ for each i. proves in [$\mathrm{Mit}_{1;2}$] under this assumption that the Carnot-Carathéodory metric in V (for an arbitrarily given underlying Riemannian metric) admits the tangent cone at each point $v_0 \in V$ (i.e. the Hausdorff limit $\lim\limits_{t\to\infty} tV$ exists) and this cone is isometric to some self-similar nilpotent Lie group (see 1.4.).

This shows that the local Carnot-Carathéodory geometry essentially reduces to (asymptotic) geometry of a nilpotent group with a dilation. (One can imagine by looking at Pansu and Mitchell theorems that there is an "inversion $t \longleftrightarrow t^{-1}$" which interchanges local C-C spaces with nilpotent Lie groups, such that the "fixed point set" of this "inversion" consists of the self-similar groups.)

0.4. Chow connectivity theorem. (see 1.1).

Theorem. *Let $X_1, \ldots, X_m$ be C^∞-smooth vector fields on a connected manifold V, such that successive commutators of these fields span each tangent space $T_v(V)$, $v \in V$. Then every two points in V can be joined by a piecewise smooth curve in V where each piece is a segment of an integral curve of one of the fields X_i.*

Remarks and corollaries

(a) *Lie group motivation.* This theorem is quite obvious on the formal level. In fact, let L be the Lie algebra generated by the fields X_i, $i = 1, \ldots, m$, and $G \subset \operatorname{Diff} V$ be the subgroup of diffeomorphisms generated by the one-parameter subgroups corresponding to X_i, $i = 1, \ldots, m$. The theorem claims that G is transitive on V provided L spans $T(V)$. This is immediate if L is *finite dimensional* as L can be identified with the Lie algebra of G (which makes G finite dimensional as well) and then the "L spans $T(V)$"-condition amounts to surjectivity of the differential of the orbit map $G \to V$ for $g \mapsto g(v_0)$

for every given point $v_0 \in V$. In fact, this argument applies in the infinite dimensional case as well (see 1.1).

(b) *Polarizations.* If the dimension of the span $H_v \subset T_v(V)$ of the fields X_i at v is independent of v, the span H of these fields is a subbundle in $T(V)$, i.e. a polarization of V in our sense where the orbits of X_i are horizontal. Thus Chow theorem implies the connectivity property for H-horizontal curves mentioned earlier.

(c) *Generic fields.* The assumption of the theorem is satisfied by *generic* C^∞-fields whenever $m \geq 2$ as successive commutators of *two* generic C^∞-fields on V span $T(V)$ as is easy to show. On the other hand, the conclusion of the theorem may be satisfied by some non-smooth continuous fields where the commutators are not even defined. In fact, the connectivity property is valid for a single (!) *generic continuous* field as everybody knows (compare II.5 in [Hart]). Probably (this seems obvious) a generic pair of C^k-fields have the connectivity property for every k.

(d) *Polarizations defined by* 1*-forms.* As we mentioned in (b) the Chow theorem applies to polarizations $H \subset T(V)$ *viewed as spans of systems of vector fields.* But one also can define H as the common zero set of a system of 1-form on V. This suggests a dual approach to (the proof of) the connectivity property of H which does not directly use orbits of vector fields tangent to H but rather appeals to leaves of 1-dimensional foliations obtained by intersecting H with submanifolds $W \subset V$ with $\operatorname{codim} W = \operatorname{rank} H - 1$. For example, if H is a contact subbundle on a 3-dimensional manifold V and $W_0 \subset V$ is a curve *transversal* to H, one takes 2-dimensional cylinders $W_\varepsilon \subset V$, $\varepsilon > 0$, around W_0 and $H \cap T(W_\varepsilon)$ give us (spiral) curves in W_ε tangent to H which closely follow W_0 for small ε, see Fig 1.

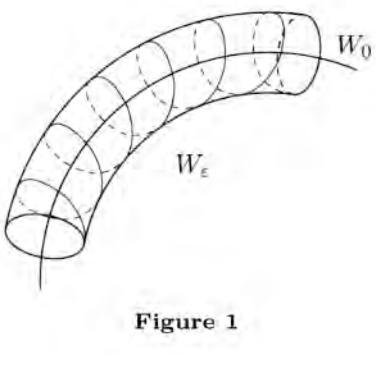


Figure 1

0.5. The shape of C-C balls: Mitchell-Gershkovich-Nagel-Stein-Wainger theorem. Let $H \subset T(V)$ be a smooth polarization (i.e. subbundle) spanned by some vector fields $X_1, \ldots, X_m$. (Every H of rank $H = n_1$ can be spanned by $m \leq n_1 + n$ fields for $n = \dim V$ and locally one needs only $m = n_1$ fields. In fact, our considerations are local and so $m = n_1$ suffices.) We denote successive commutators of our fields by X_i for suitable indices $i > m$ and we assign to each X_i the number $\deg X_i$ which is the degree of the corresponding commutator. Thus, for example, $\deg X_i = 1 \iff i \leq m$, and

$$\deg X_i = 2 \iff X_i = [X_\mu, X_\nu],\ 1 \leq \mu,\ \nu \leq m.$$

(It may happen that $X_i = X_j$ for $i \neq j$ and, moreover, $\deg(X_i) \neq \deg(X_j)$ for $X_i = X_j$, i.e. the degree is assigned to the name (i.e. the subindex) of a field rather than the field itself.)

Now, let X_i, $i = 1, \ldots, M$, be the successive commutators of $X_1, \ldots, X_m$ of degree $\leq d$ and let us assume that these X_i, $i = 1, \ldots, M$, span $T(V)$. We want to characterize the C-C metric in V corresponding to H in terms of the integral curves (1-parameter subgroups) of the fields X_i, $i = 1, \ldots, N$, and of the linear combinations of these fields with constant coefficients. First, for an arbitrary field X on V, we recall the notation $\exp_v X \in V$ which is the result of applying to v the 1-parameter group (flow) corresponding to X at the time $t = 1$. In other words, we obtain $v_1 = \exp_v(X)$ by taking the integral curve of X issuing from v and following it with speed X for time $t = 1$. (If V is non-compact, the field may be globally non-integrable but this is irrelevant at the moment.) Next, we define the exponential map $\exp_v : \mathbb{R}^M \to V$ by

$$(t_1, \ldots, t_M) \mapsto \exp_v(t_1 X_1 + \cdots + t_M X_M).$$

This is, clearly, a smooth map (as we assume X_1 are smooth) and the differential of $\exp_v$ is surjective at the origin $0 \in \mathbb{R}^M$.

Notice that if V is non-compact, the map $\exp_v : \mathbb{R}^M \to V$ need not be defined on all of $\mathbb{R}^M$ (as some field $X = \sum\limits_{i=1}^{M} t_i X_i$ may fail to be globally integrable) but it is always defined in a small neighbourhood of the origin of $\mathbb{R}^M$ and this is all we need for our purpose.

Consider the following box in $\mathbb{R}^M$,

$$\mathrm{Box}(\rho) = \{|t_i| \leq \rho^{\deg X_i},\ i = 1, \ldots, M\} \subset \mathbb{R}^M.$$

0.5.A. Ball-box-theorem. ([2])

Theorem. *The small* C-C *balls $B_v(\rho)$ in V around $v \in V$ are uniformly equivalent to the exponential images of the boxes.* This means, *there are strictly positive continuous functions $C = C(v)$ and $\rho_0 = \rho_0(v)$, such that*

$$\exp_v \mathrm{Box}(C^{-1}\rho) \subset B_v(\rho) \subset \exp_v \mathrm{Box}(C\rho)$$

for all $v \in V$ and $\rho \le \rho_0(v)$.

Remarks and corollaries

(a) *Hölder equivalence of* C-C *and Euclidean.* The first inclusion $B_v(\rho) \supset \exp_v \mathrm{Box}(C^{-1}\rho)$ gives us a quantitative version of Chow's theorem as it shows that every point v_1 in the exponential image of $\mathrm{Box}(C^{-1}\rho)$ can be joined with v by an H-horizontal curve of length $\le \rho$. Since the exponential map has surjective differential at the origin, the image of this box contains the Euclidean (or Riemannian) ball in V around v or radius $\approx \rho^d$ (where, recall, $d = \max \deg X_i$). It follows that the identity map

$$(V,\ \text{Riemannian metric}) \xrightarrow{\text{id}} (V,\ \text{C-C metric})$$

is C^α-*Hölder* with the exponent $\alpha = d^{-1}$. (Notice that the map

$$(V, \text{C-C metric}) \xrightarrow{\text{id}} (V,\ \text{Riemannian metric})$$

is, obviously, Lipschitz.)

(b) Let us generalize the above and determine the Hölder class of a smooth map of a Riemannian or, more generally, Carnot-Carathéodory manifold W into V. To simplify the matter we assume our polarization H is *equiregular* in the sense that the dimension of the subspace in $T_v(V)$ generated by the commutators of degree $j \le d$ does not depend on v (compare Mitchell cone theorem cited in 0.3.D). In this case the bundle $T(V)$ is filtered by smooth subbundles

$$H = H_1 \subset H_2 \subset \cdots \subset H_j \subset \cdots \subset H_d = T(V)$$

such that H_j is spanned by the j-th degree commutators of the fields in H. Then we take some polarization $H' \subset T(W)$ whose successive

[2] See [$\mathrm{Mit}_{1;2}$], [$\mathrm{Ger}_{\mathrm{TSE}}$], [$\mathrm{Ger}_{\mathrm{EN}}$], [N-S-W] and 1.1 – 1.3.

commutators span $T(W)$ and turn to the question of the Hölder exponent for a smooth map $f : W \to V$ with respect to the C-C metrics associated to H' and H. Here is the answer.

0.5.B. Hölder exponent evaluation. *The Hölder exponent of f equals j^{-1} for the minimal j, such that the differential $\mathcal{D}$ of f sends H' into H_j.* (This includes the case of a Riemannian W for $H' = T(W)$.)

Notice that this evaluation of the Hölder exponent contains two statements: the fact that f is C^α for $\alpha = j^{-1}$ and that it is *not* C^β for $\beta > j^{-1}$, where the latter is immediate with the inclusion C-C ball $\subset$ exp Box in V while the former uses the $\sqrt[j]{\text{Euclidean}}$ property of H_j-horizontal curves in V.

It is not at all clear what is the precise geometrical (infinitesimal) significance of f being C^α with respect to C-C metrics *without* assuming f is smooth. Yet one can think of such maps f as of *generalized solutions* of the P.D.E.-system expressing the inclusion $\mathcal{D}f(H') \subset H_j$ for $j = (\operatorname{ent} \alpha^{-1}) + 1$. In fact, one of the general questions concerning C-C manifolds is the following

0.5.C. Hölder mapping problem. Given two C-C spaces V and W and a real number $0 < \alpha \leq 1$, describe the space of C^α-maps $f : W \to V$. For example, when can each continuous map $W \to V$ be uniformly approximated by C^α-maps? When can W be C^α-*embedded* into W? When are V and W C^α-homeomorphic? etc.

These questions can be approached in the smooth case by the P.D.R.-techniques (see [Gro$_{\text{PDR}}$] and §4) and what little we can say about general (non-smooth Hölder) maps also uses some P.D.R. (see §4).

0.5.D. Hölder surfaces in contact 3-manifolds. Let V be a 3-dimensional contact C-C manifold (see 0.2) and $f : \mathbb{R}^2 \to V$ a C^α-*embedding*, where $\mathbb{R}^2$ is endowed with the Euclidean metric. Then one can show that $\alpha \leq \frac{2}{3}$ while the natural expectation is $\alpha \leq \frac{1}{2}$ (compare 0.6.C).

0.5.E. Homotopy count of Hölder maps. If V and W are compact Riemannian manifolds, then the number $Nm(\lambda)$, of the homotopy classes of Lipschitz maps $V \to W$ with the Lipschitz constant $L(f) \leq \lambda$, satisfies

$$Nm(\lambda) \leq C(1+\lambda)^r, \tag{$*$}$$

provided the fundamental group of W is finite and where the exponent r depends only on the homotopy types of V and W, while the constant C may depend on the metrics as well. This is proven in [$\mathrm{Gro}_{\mathrm{HED}}$] by appealing to Sullivan's minimal model of the de Rham algebra of differential forms.

Warning. *The argument in* [$\mathrm{Gro}_{\mathrm{HED}}$] *uses minimal models only for maps of* spheres *into W and then the general case is obtained by an induction on skeletons of V. This induction, however, contains a gap (pointed out to me by Pansu) which I am still unable to fill in. Yet, one can apply the techniques of minimal models directly to V and obtain by the argument in* [$\mathrm{Gro}_{\mathrm{HED}}$] *a polynomial bound on the number of mutually non-equivalent morphisms* Mod$V$ $\to$ Mod$W$ *in terms of $L(f)$. Then the rational homotopy theory yields such a bound (i.e. $(*)$) for maps $V \to W$ themselves. (This is one of the creeds of the rational homotopy theory, so I believe.)*

Next we observe that a Hölder C^α-map $f : V \to W$ can be approximated by a Lipschitz map f' such that $L(f') \approx \big(L_\alpha(f)\big)^{-\frac{1}{\alpha}}$. This is done by working on the ε-balls B_ε in V with $\varepsilon \approx \big(L_\alpha(f)\big)^{\frac{1}{\alpha}}$ where $\operatorname{Diam} f(B_\varepsilon) << 1$. Thus $(*)$ implies a similar inequality for C^α-maps $f : V \to W$ satisfying $L_\alpha(f) \leq \lambda$, namely

$$Nm_\alpha(\lambda) \leq C(1+\lambda)^{r_\alpha} \tag{$*$}_\alpha$$

for $r_\alpha \leq r_1^{\frac{1}{\alpha}}$ for the exponent $r = r_1$ in $(*)$.

Now we return to C-C manifolds V and W, invoke the Hölder equivalence between the Riemannian and the C-C metrics and conclude to the bound $(*)$ for compact C-C manifolds. Notice that the exponent r in $(*)$ depends in the C-C case on the polarizations of V and W as well as on their homotopy types.

Problem. Give a precise formula for r in $(*)$ for C-C manifolds.

Notice that this problem is open even in the Riemannian case for most homotopy types of V and W and in the C-C case one would be content to know the answer in simple cases where the homotopy theoretic structure is sufficiently transparent (compare 1.4.E$'$, 2.4.A).

Remark. The Riemannian inequality $(*)$ can be sharpened by replacing the Lipschitz constant $L(f)$ by more precise measures of dilation of f such as the L_p-norm of the differential $\mathcal{D}f$ on the k-th exterior power $\wedge^k T(V)$, for suitable k and p (see [Gro$_{\text{HED}}$] and note that the L_∞-norm of $\mathcal{D}f$ on $\wedge^1 T(V)$ equals $L(f)$). This will be partially extended to the C-C framework in 2.4, 2.5 and §4.

Example. Let V and W be closed connected orientable n-dimensional Riemannian manifolds where W is homeomorphic to the sphere. Then $Nm(\lambda) \sim C\lambda^n$ as the only homotopy invariant of a map $f : V \to W$ is the degree (see [Gro$_{\text{HED}}$] and [G-L-P]). Now let V be C-C associated to a generic subbundle $H \subset T(V)$ of codimension one (e.g. a contact structure). Then, by an easy argument, $Nm(\lambda) \sim C\lambda^{n+1}$ which agrees with the fact that the Hausdorff dimension of such a V is $n+1$ (see below as well as 1.4.E$'$ and 2.5). Notice, that this is sharper than the bound $Nm(\lambda) \lesssim \lambda^{2n}$ provided by the Riemannian estimate $Nm(\lambda) \lesssim \lambda^n$ via the Hölder $C^{\frac{1}{2}}$-equivalence between the contact and Riemannian structures.

0.6. The volume of C-C balls and the Hausdorff dimension. We assume here for simplicity's sake that the polarization $H \subset T(V)$ defining our C-C structure is equiregular and take a frame of vector fields $X_1, \ldots, X_n$, $n = \dim V$, which agrees with the commutators filtration $H = H_1 \subset \cdots \subset H_j \subset \cdots \subset H_d = T(V)$ defined above. (This means the first m_1 fields for $m_1 = \operatorname{rank} H_1$ belong to H_1, the following m_2 fields for $m_2 = \operatorname{rank} H_2/H_1$ belong to H_2, etc.) We denote by $\deg X_i (= \deg i)$ the minimal j such that X_i belongs to H_j. Thus $\deg X_i = 1$ for $i = 1, \ldots, m = \operatorname{rank} H_1$, $\deg X_i = 2$ for $i = m+1, \ldots, \operatorname{rank} H_2/H_1$, etc. The corresponding box $\operatorname{Box}(\rho) \subset \mathbb{R}^n$ given by the inequalities $|t_i| \leq \rho^{\deg X_i}$ (obviously) has the Euclidean volume equal $2^n \rho^D$ for $D = \sum_{i=1}^n \deg X_i = \sum_{j=1}^d j \operatorname{rank}(H_j/H_{j-1})$, where we assume $H_0 = 0 \subset H$, and by the ball-box theorem the Riemannian volumes of the C-C balls $B_v(\rho)$ in V are roughly the same. Namely, *there* *exist continuous strictly positive functions $C_1(v)$, $C_2(v)$ and $\rho_0(v)$, on V, such that*

$$C_1(v) \leq \big(\mathrm{Vol}\, B_v(\rho)\big)/\rho^D \leq C_2(v) \tag{+}$$

for all $\rho \leq \rho_0(v)$. Consequently, the Hausdorff dimension of V with respect to the C-C metric associated to H equals D. (See [Mit$_{1;2}$] and 1.3.A).

Notice that the inequality (+) partially extends to non-equiregular structures (see [N-S-W]) which allows an evaluation of $\dim_{\mathrm{Hau}} V$ for some non-equiregular H, e.g. for the real analytic ones (see 1.3.A).

0.6.A. On the intrinsic size of C-C balls. If ρ is small, the C-C ball $B_v(\rho)$ approximated by the box with the sides ρ^{d_i}, $i = 1, \ldots, n$, (extrinsically) appears much smaller than the Euclidean ball of radius ρ (or the ρ-cube which is roughly the same as the ρ-ball from our viewpoint) if ρ is small and at least one of the degrees $d_i = \deg X_i$ exceeds one (i.e. if $\operatorname{rank} H_1 < n$). Yet we know that both, the Euclidean and Carnot-Carathéodory balls, can be rescaled to the unit size where they look roughly the same and so for each ρ the C-C ball *cannot* be smaller than the Euclidean one of the same radius ρ. In fact, one may imagine the C-C balls as being incomparably larger since their Hausdorff dimensions are greater than n. The following observation justifies this view. *Every C-C ball $B_v(\rho)$, where v is a fixed point and $\rho \leq 1$, admits a surjective distance decreasing map f onto the Euclidean ball $B^{\mathrm{Eu}}(\rho')$, such that f has infinite average multiplicity and $\rho' \geq \delta\rho$ for a fixed (independent of ρ) $\delta > 0$.*

Recall that the average multiplicity, also called the *total volume* or the *variation* of a map f with the range in a Euclidean ball $B \subset \mathbb{R}^n$ is defined by the integral

$$\int_B \operatorname{card} f^{-1}(x)dx.$$

If the domain V of f is an n-dimensional Riemannian manifold and f is Lipschitz, this integral equals the total volume of the induced (singular) Riemannian metric on V and, hence, is finite for *compact* V. But if V is non-Riemannian of the Hausdorff dimension $N > n$, one can only claim the finiteness of the integral

$$\int_B \mathrm{mes}_{N-n}\left(f^{-1}(x)\right) dx,$$

as this, for Lipschitz maps f, does not exceed const $\mathrm{mes}_N V$ by the coarea inequality (where const $\leq C_n L(f)$ for the Lipschitz constant L of f and an universal constant C_n).

Now let us explain how to construct the above claimed map $f : B_v(\rho) \to B^{\mathrm{Eu}}(\rho')$ of infinite average multiplicity. We start with some smooth distance decreasing map f_0 of $B_v(\rho)$ onto $B^{\mathrm{Eu}}(\rho')$ (which is trivial to arrange) and then we use the fact that the C-C metric in $B_v(\rho)$ can be approximated by Riemannian metrics g_t for $t \to \infty$ (see 0.8). When t is large, the map f_0 becomes almost distance decreasing for g_t and it can be easily made honestly g_t-distance decreasing by a small perturbation of f_0. Then f_0 can be uniformly approximated by a g_t-*isometric* (sic!) map f_t, i.e. preserving the g_t-length of the smooth curves in $B_v(\rho)$ (see 2.4.11 in [Gro$_{\mathrm{PDR}}$]), which has $\int \mathrm{card}\, f_t^{-1}(x)dx = \mathrm{Vol}_{g_t} B_v(\rho) \to \infty$. Finally, one sees that f_t subconverge to a C-C-Lipschitz map $f = f_\infty$ with $\int \mathrm{card}\, f^{-1}(x)dx = \infty$.

Questions

- Can one find a Lipschitz $f : B_v(\rho) \to B^{\mathrm{Eu}}(\rho')$ such that $\mathrm{card}\, f^{-1}(x) = \infty$ for *all* $x \in B^{\mathrm{Eu}}(\rho')$?
- Can one have a Lipschitz f with the *positive* measure $\mathrm{mes}_{N-n} f^{-1}(x)$ for all x in $B^{\mathrm{Eu}}(\rho')$ or at least for x in a subset $X \subset B^{\mathrm{Eu}}(\rho')$ of positive measure? (Recently some such f were constructed in [Bat$_3$]).

0.6.B. Hausdorff dimension of submanifolds. Let V' be a smooth submanifold in V and let us evaluate the Hausdorff dimension of V' with respect to the Carnot-Carathéodory metric in $V \supset V'$. We intersect the tangent spaces $T_v(V') \subset T_v(V)$, $v \in V' \subset V$, with H_j and denote these intersections by $H'_j(v)$. These are linear subspaces which filter $T_v(V')$ and we denote by $m'_j(v)$, $v \in V'$, the ranks $\mathrm{rank}\big(H'_j(v)/H'_{j-1}(v)\big)$. Next we let

$$D'(v) = \sum_{j=1}^{d} j\, m'_j(v),\ v \in V',$$

and finally we define $D_H(V')$ by

$$D_H(V') = \max_{v \in V'} D'_H(v).$$

For example, if V' *horizontal*, i.e. is everywhere tangent to H then $D_H(V') = \dim V'$. In general, $D_H(V') \geq \dim V'$ and $D_H(V')$ is maximal for generic submanifolds V'. For instance if V' is a *generic* submanifold of dimension n', then the number $m_j'(v)$ is given for almost all $v \in V'$ by the following obvious formula depending on $\operatorname{codim} V' = n - n'$,

$$m_j'(v) = \begin{cases} m_j = \operatorname{rank} H_j/H_{j-1} & \text{if} \quad \operatorname{codim} V' \le \operatorname{rank} H_{j-1}, \\ m_j + n' - n & \text{if} \quad \operatorname{rank} H_{j-1} \le \operatorname{codim} V' \le \operatorname{rank} H_j, \\ 0 & \text{if} \quad \operatorname{codim} V' \ge \operatorname{rank} H_j. \end{cases}$$

It follows in the case $d = 2$ that

$$D_H(V') - \dim V' = \begin{cases} m_2 = \dim V - \operatorname{rank} H & \text{if} \quad \operatorname{codim} V' \le \operatorname{rank} H, \\ n' = \dim V' & \text{if} \quad \operatorname{codim} V' \ge \operatorname{rank} H. \end{cases}$$

Evaluation of $\mathbf{dim_{Hau} V'}$. *The Hausdorff dimension of an arbitrary smooth submanifold V' in a manifold V with a C-C metric associated to H equals the number $D_H(V')$.*

This is an easy corollary of the ball-box theorem for V (see 4.1.A).

Warning. The restricted C-C metric dist_H on V' by no means equals the C-C metric on V associated to the polarization $H' = T(V') \cap H$ on V. In fact, this restricted metric is of more general nature than Carnot-Carathéodory. (But it fits in many cases the definition in §1 of [N-S-W].) Yet there are important examples (e.g. hypersurfaces V' in contact manifolds of dimension ≥ 5) where the restricted C-C metrics $\operatorname{dist}_H |V'$ is Lipschitz equivalent to the C-C metric $\operatorname{dist}_{H'}$ on V' (see 2.4.B).

Let us look at the inequality $\dim_{\mathrm{Hau}} V' \le N'$ for a given $N' > 0$ as an equation imposed on $V' \subset V$. If V' is a smooth submanifold (and so $\dim_{\mathrm{Hau}} V' = D_H(V')$) this can be indeed represented by a system of partial differential equations on V' expressing some degree of tangency of V' to H_j. For example, the relation $\dim_{\mathrm{Hau}} V' \le \dim V'$ says, in the case where V' is smooth, that V' is everywhere tangent to H. Thus non-smooth subspaces $V' \subset V$ with $\dim_{\mathrm{Hau}} V' \le N'$ can be interpreted as generalized solutions of a certain system of P.D.E. (compare the earlier Hölder discussion for mappings). Here is the main question we want to address:

0.6.C. Hausdorff dimension problem for subspaces. Determine the (topological) structure of (the space of) the subsets $V' \subset V$ having $\dim_{\mathrm{Hau}} V' \leq N'$ with respect to a given C-C metric in V. For instance, find the infimum of $\dim_{\mathrm{Hau}} V'$ among all compact subsets $V' \subset V$ of a given topological dimension n'.

Contact example. Every smooth surface in the contact 3-space V has the C-C Hausdorff dimension 3 as follows from the equality $\dim_{\mathrm{Hau}}(V') = D_H(V')$. Then with a little extra effort one can show that *every subset* V' *of topological dimension* 2 *has* $\dim_{\mathrm{Hau}} V' \geq 3$ (see 2.1). Notice that this inequality immediately implies non-existence of C^α-embeddings $\mathbb{R}^2 \to V$ for $\alpha > \frac{2}{3}$ (as was claimed in 0.5.D earlier) since the Hausdorff dimension (obviously) agrees with C^α-Hölder maps $f : W \to V$ by the rule $\dim_{\mathrm{Hau}} f(W) \leq \alpha^{-1} \dim_{\mathrm{Hau}} W$.

Exercise. Show that the Euclidean ε-ball in the contact 3-space V can be covered by $\approx \varepsilon^{-1}$ C-C balls of radius ε while every C-C ε-ball needs $\approx \varepsilon^{-2}$ Euclidean ε^2-balls to cover it. Use this to compare the C-C and Euclidean Hausdorff dimensions of subsets in V. State and prove corresponding results for general C-C manifolds (V, H).

0.7. Isoperimetric filling problem. Let S be a k-dimensional cycle in V which is homologous to zero. We want to evaluate the minimal possible Hausdorff dimension of (the supports of) $(k+1)$-chains D in V *filling in* S, i.e. having $\partial D = S$. More specifically we want to bound this minimal $j = \dim_{\mathrm{Hau}} D$ in terms of $i = \dim_{\mathrm{Hau}} S$ and then, moreover, we look for a bound on the (minimal possible) j-dimensional Hausdorff measure of fillings D of S in terms of the i-dimensional Hausdorff measure of S.

0.7.A. Isoperimetric inequality for $\mathbf{k = n - 1}$. If S is a closed hypersurface in V then there is little choice for D : this is a domain in V bounded by S. Thus we exercise no control over the Hausdorff dimension of D but we still may try to bound its Hausdorff measure of an appropriate dimension in terms of the Hausdorff measure of S one dimension less.

A first result of this kind is due to Pansu (see [$\mathrm{Pan}_{\mathrm{The3}}$] and [$\mathrm{Pan}_{\mathrm{InIs}}$]) who proved the following isoperimetric inequality for the 3-dimensional Heisenberg group V with an equivariant C-C metric.

Pansu isoperimetric inequality. *The 3-dimensional Hausdorff measure of the boundary $S = \partial D$ of a domain $D \subset V$ restricts the 4-dimensional measure of D by*

$$\operatorname{mes}_4 D \leq \operatorname{const}(\operatorname{mes}_3 S)^{\frac{4}{3}}. \qquad (*)$$

Notice that mes_4 in this case equals the ordinary (3-dimensional) Haar measure on the group V.

Now we recall that every left invariant *Riemannian* metric on the group V has the same asymptotic geometry as the C-C metric. Thus $(*)$ is essentially equivalent to the following inequality for the ordinary volume and area with respect to a left-invariant Riemannian metric g in V,

$$\operatorname{Vol} D \leq \operatorname{const}_g(\operatorname{Area} \partial D)^{\frac{4}{3}}. \qquad (**)$$

(This is stronger for large domains D than the Euclidean inequality $\operatorname{Vol} D \leq \operatorname{const}\big(\operatorname{Area} \partial D)^{\frac{3}{2}}\big)$.

The inequality $(**)$ was extended by Varopoulos (see [Var-Sa-Co]) to an arbitrary simply connected nilpotent Lie group V,

$$\operatorname{Vol}_n D \leq \operatorname{const}(\operatorname{Vol}_{n-1} \partial D)^{\frac{N}{N-1}},$$

where $n = \dim V$ and N is the *asymptotic Hausdorff dimension* or the exponent of the volume growth of the balls $B(R) \subset V$, that is

$$N = \lim_{R \to \infty} \big(\log \operatorname{Vol} B(R)\big) / \log R.$$

The isoperimetric inequality for general C-C spaces V of the Hausdorff dimension N reads

$$\operatorname{mes}_N D \leq \operatorname{const}_V(\operatorname{mes}_{N-1} \partial D)^{\frac{N}{N-1}} \qquad (+)$$

and we shall prove this in 2.3 under suitable assumptions on V. (The inequality $(+)$ in the form of a Sobolev inequality for smooth functions on V is due to Varopoulos, see [Var].)

0.7.B. Filling in curves in V. Here $k = 1$ and we look for a "minimal" surface $D \subset V$ which fills in a given closed curve $S \subset V$. We assume for the moment that S is rectifiable (i.e. $\dim_{\mathrm{Hau}} = 1$) and, in fact, we do not loose much by assuming that S is smooth and horizontal (with respect to the polarization underlying the C-C-geometry). But even in this case the evaluation of the (minimal) Hausdorff dimension (and measure) of surfaces D filling in S is a rather subtle matter as is seen in the following

Contact example. Let V be the 3-dimensional Heisenberg group with a left invariant and self-similar C-C metric. We know already (see 0.6.C) that surfaces in V have Hausdorff dimension ≥ 3 and moreover, one can show (using $(*)$ for instance) that every filling D of a simple closed curve S has $\mathrm{mes}_3\, D \geq \varepsilon > 0$ for $\varepsilon = \varepsilon(S)$. On the other hand the Riemannian area of a smooth surface bounds mes_3 which implies the (filling) inequality

$$\mathrm{mes}_3\, D \leq C(\mathrm{mes}_1\, S)^3$$

for a suitable filling D of S.

The situation radically changes if we take the Heisenberg group V of dimension $n > 3$. (Recall that the Heisenberg group of dimension $n = 2m+1$ is characterized by its Lie algebra which admits a basis $x_i, y_i, z,\ i = 1, \ldots, m$, where z is central, $[x_i, x_j] = [y_i, y_j] = 0$ and the only non-zero commutators between x_i and y_j are $[x_i, y_i] = z,\ i = 1, \ldots, m$.) This V admits an essentially unique left invariant self-similar C-C metric where the underlying (contact) polarization H has codimension one (being spanned by x_i and y_i). Now, for $n > 3$, the filling of closed C-C-rectifiable curves X can be made more efficiently than for $n = 3$. Namely, S can be filled by surfaces D of Hausdorff dimension 2, which means, in effect, that these are horizontal, i.e. are everywhere tangent to H. Moreover, the area of these horizontal surfaces can be made as small as $\mathrm{const}(\mathrm{length}\, S)^2$. *Thus every S admits a filling D for which*

$$\mathrm{mes}_2\, D \leq \mathrm{const}(\mathrm{mes}_1\, S)^2 \tag{$\star$}$$

(see 3.5 and 4.8 where $(\star)$ generalizes to some non-contact C-C manifolds).

0.7.C. Filling for dim S > 1. Here the situation is rather unsatisfactory, as we lack a non-trivial upper bound on the (best) filling in most cases. For example, we have the following

Open question. Let V be the $2m+1$-dimensional Heisenberg group with the C-C metric. Does there exist, for every k-dimensional cycle S for $k < m$, a filling D of S satisfying

$$\mathrm{mes}_{k+1}\, D \leq \mathrm{const}(\mathrm{mes}_k\, S)^{\frac{k+1}{k}}\,?$$

This is motivated by the fact that the underlying (contact) polarization H admits plenty of $(k+1)$-dimensional submanifolds tangent to it for $k+1 \leq m$. In particular, if S is smooth horizontal, it can be filled in by a horizontal D and thus having $\operatorname{mes}_{k+1}(D) < \infty$. What is unclear at the present moment is how to bound $\operatorname{mes}_{k+1} D$ in terms of $\operatorname{mes}_k S$.

Above middle dimension. If $k = m$ then the "correct" (isoperimetric) filling inequality reads

$$\operatorname{mes}_{k+2} D \leq \operatorname{const}(\operatorname{mes}_k S)^{\frac{k+2}{k}}$$

(which means: "there exists D satisfying ...") and this can be easily derived from an ordinary (Riemannian) filling estimate.

Finally, the k-dimensional cycles S for $k > m$ are expected to admit fillings D satisfying

$$\operatorname{mes}_{k+2} D \leq \operatorname{const}\,(\operatorname{mes}_{k+1} S)^{\frac{k+2}{k+1}} \qquad (++)$$

as in the (only known) case $k = \dim V - 1$.

Remark

(a) Since the Heisenberg group V admits self-similarities (dilations) $A_t : V \to V$ which scale the r-dimensional Hausdorff measure by t^r for each r, every one of the above inequalities can be reduced to the case where the relevant measure of S equals one and the problem boils down to finding any bound on the measure of a suitable filling D of S.

(b) One knows that the constant in $(++)$ is definitely different from zero. In fact, for every closed smooth k-dimensional submanifold $X \subset V$ with $k \geq m$, there exists a constant $\varepsilon = \varepsilon(S) > 0$, such that *every* filling D of S has

$$\operatorname{mes}_{k+2} D \geq \varepsilon. \qquad (+-)$$

This follows by the argument in 3.1.A.

0.8. Carnot-Carathéodory metrics as limits of Riemannian ones. Consider a smooth manifold V where the tangent bundle is decomposed into the sum of two complementary subbundles, $T(V) = H \oplus H^{\perp}$, and let $A_t : T(V) \to T(V)$ be defined by $(h, h^{\perp}) \mapsto (h, t\,h^{\perp})$ for all $t > 0$. Then we pick up some Riemannian metric g on V and look at the family $g_t = A_t^*(ga)$ as t varies between 1 and ∞.

0.8.A. Riemannian homogeneous spaces and their limits. Let V be a Lie group and $H, H^\perp$ and g be left invariant. Then the metrics g_t are also left invariant and one may naively think that their geometry is fairly simple. But even in the first non-trivial (i.e. non-commutative) case of $V = SU(2)$ and $\operatorname{rank} H = 2$, one has insufficient understanding of the asymptotic behavior of the geometry of (V, g_t) for $t \to \infty$. For example, one still does not know if the path (ray) (V, g_t), $t \in [1, \infty[$ is (at least roughly) minimizing in the space $\mathcal{M}$ of Riemannian manifolds with a suitable metric on $\mathcal{M}$. (I have more respect for this problem now than ten years ago when I first faced it, see my note [$\mathrm{Gro}_{\mathrm{AGHS}}$] following a meeting in Torino in 1983 on homogeneous spaces.)

If the complementary bundle $H^\perp$ has rank > 0, then the family of the Riemannian metrics (quadratic forms) g_t *diverges* at each point $v \in V$. Yet the associated distance functions $\operatorname{dist}_t = \operatorname{dist}_{g_t}$ may converge for $t \to \infty$. In fact, if the subbundle H (polarization) Lie generates the tangent bundle (i.e. successive commutators of H-horizontal, fields span $T(V)$) then dist_t (obviously) converges to the Carnot-Carathéodory metric $\operatorname{dist}_\infty = \operatorname{dist}_{H,g}$ on V (compare 1.4.D).

0.8.B. Contact example in the spherical clothing. Let $V = S^3$ and $H \subset T(V)$ be the (2-dimensional horizontal) subbundle normal to the fibers of the Hopf fibration $S^3 \to S^2$. As we take the metrics g_t for t getting larger and larger, the Hopf fibers are becoming longer and longer but the diameter of S^3 with respect to the metric dist_t remains bounded for $t \to \infty$, as every two points in S^3 can be joined by an H-*horizontal* curve whose g_t-length is independent of t. (This drastically contrasts with what happens to the trivial fibration $V = S^2 \times S^1 \to S^2$, where $\operatorname{diam}(V, \operatorname{dist}_t) \to \infty$ for $t \to \infty$.)

Let us formulate two basic problems concerning asymptotic geometry of $V_t = (V, g_t)$ (for $t \to \infty$).

0.8.C. The asymptotic mapping problem. Find the asymptotics for $t \to \infty$ of the *Lipschitz constant* Lip_t of the *homotopy class* of the identity map $(V, g_1) \to (V, g_t)$, where the Lipschitz constant of a class Φ of maps φ is defined as the infimum over $\varphi \in \Phi$ of the Lipschitz constants $L(\varphi) = \sup\limits_{v \in V} \|\mathcal{D}\varphi(v)\|$. More specifically, one wants to know if the Lipschitz constant of the identity map can be significantly decreased (for large t) by homotopying this map. One also asks this question for other dilation characteristics such as the L_p-norms of $\mathcal{D}f$ on $\wedge^k T(V)$ (compare 0.5.E.).

0.8.D. The intermediate volume problem. Take a k-dimensional homology class $\alpha \in H_k(V)$ and evaluate the asymptotics of $\mathrm{Vol}_k(\alpha)$ with respect to g_t as $t \to \infty$, where the volume of a homology class in a Riemannian manifold is defined as the infimum of the volumes of the cycles in this class.

Notice that these two problems are vaguely parallel to the Hölder mapping problem and the Hausdorff dimension problem correspondingly (see 0.5.C and 0.6.B) and we see later clearer relations between this kind of problems.

0.8.E. Families of metrics associated to dynamical systems. Instead of moving g_t by automorphisms $A_t : T(V) \to T(V)$ one may use a family of diffeomorphisms $a_t : V \to V$ and define

$$\mathrm{dist}_t = \sup_{1 \le \tau \le t} a_\tau^*(\mathrm{dist})$$

for a fixed (Riemannian) distance function on V. Many dynamical characteristics of a_t, e.g. *the topological entropy*, can be expressed as asymptotic invariants of dist_t. On the other hand, for some dynamical systems (first of all for Anosov systems) one can renormalize dist_t such that the limit for $t \to \infty$ exists and is of Carnot-Carathéodory type. Here is a typical question where these ideas are useful (see 4.10).

0.8.F. The intermediate entropy problem. Recall that for every self-homeomorphism $a : V \to V$ and each compact subset K one can define the topological entropy $\mathrm{ent}(a; K)$ (by using suitable ε-covers of K for the metrics $\mathrm{dist}_i = \sup_{1 \le j \le i} (a^{(i)})^* \mathrm{dist})$. Then one defines $\mathrm{ent}_k(a)$ as the infimum of $\mathrm{ent}(a; K)$ over all compact subsets K of topological dimension k. The question is how to evaluate this ent_k for specific (e.g. Anosov) transformations a.

0.8.G. Intrinsic approximation of C-C spaces by Riemannian ones. The approximation of a C-C metric by the Riemannian metric g_t at the beginning of 0.8 makes an essential use of the polarization H underlying the C-C structure, and this is hard to see in purely metric (intrinsic) C-C terms. An alternative intrinsic approximation V_ε to V (where ε corresponds to t^{-1}) appeals to the nerve of a suitable ε-covering of V where each simplex of the nerve is given the metric of the standard

Euclidean ε-simplex. (These V_ε are piecewise Riemannian rather than Riemannian but this hardly matters.) We shall see later on that these V_ε rather closely approximate V for $\varepsilon \to 0$ in a suitable category (albeit V_ε are not necessarily homeomorphic to V) and many geometric invariants of V can be extracted from those of V_ε for $\varepsilon \to 0$ (see 1.4.D and [Ger$_{\mathrm{SBRM}}$]).

0.9. Conformal C-C geometry and hyperbolic geometry. The most profound geometric applications of the Carnot-Carathéodory structures are centered around the rigidity problems for non-compact symmetric spaces of rank one and are due to Mostow and Pansu (see [Most], [Pan$_{\mathrm{QIR1}}$]). Recall that every compact metric space V serves as the *ideal boundary* of certain *hyperbolic spaces* CV. Namely, we take $CV = V \times [0, \infty[$ with the maximal (or better to say supremal) metric satisfying the following two conditions.

(i) for each $v \in V$ the embedding $[0, \infty[\to CV$ for $t \mapsto (v, t)$ is (non-strictly) distance decreasing.

(ii) for each $t \in [0, \infty[$ the embedding $V \mapsto CV$ for $v \mapsto (v, t)$ is 2^t-Lipschitz.

Then one can show that the quasi-isometry type of CV determines (suitably defined) quasi-conformal types of V (compare [Mar], [Gro$_{\mathrm{HG}}$], [Pan$_{\mathrm{QCM}}$]).

Example

(a) Let $V = S^3$ with the C-C metric associated to the above (contact) polarization (normal to the Hopf fibers). Then the corresponding space CV is quasi-isometric to the complex hyperbolic plane and quasi-isometries of CV induce (possibly non-smooth) contact maps on the ideal boundary $\partial_\infty CV = V$ which are quasi-conformal for the C-C metric (notice that the quasi-conformality is automatic for *smooth* contact maps).

(b) Let $V = S^7$ and $H \subset T(V)$ be horizontal for the Hopf fibration $S^7 \to S^4$. Then the corresponding CV is quasi-isometric to the quaternionic hyperbolic plane. Quasi-isometries of CV induce C-C quasi-conformal transformations of V. But these are quite special (unlike the contact maps in the complex hyperbolic case) by the following

Pansu rigidity theorem. *The above transformations of V are, in fact, conformal and the corresponding quasi-isometrics of CV are equivalent (asymptotic) to actual isometrics of the quaternionic hyperbolic plane.*[3]

Remark. Usually one starts with the complex (or quaternionic) hyperbolic space (e.g. plane) X and reconstructs the C-C structure at the sphere at infinity $\partial_\infty X$ as the limit of normalized Riemannian metrics induced on the concentric spheres $V_t \subset X$ around a fixed point $x_0 \in X$. Namely, one takes g_t on V_t equal to (induced metric)/Diam_t, where Diam_t denotes the diameter of the induced Riemannian metric on V_t (which is about $\exp t$ for $t \to \infty$), and then the distance functions $\mathrm{dist}_t = \mathrm{dist}_{g_t}$ (on $\partial_\infty X$ identified with V_t) converge to the Carnot-Carathéodory metric on the corresponding sphere with the horizontal subbundle (polarization) associated to the Hopf fibration. (If one does all that to the real hyperbolic space of dimension n one gets just the usual round Riemannian metric on S^{n-1} and one does not know what happens for non-symmetric spaces of negative curvature, compare [$\mathrm{Gro}_{\mathrm{AI}}$]).

0.10. Dimension and growth in the asymptotic geometry. We have seen in 0.8. how certain problems for C-C manifolds appear in the limit for families of growing Riemannian metrics on a compact manifold. Then such families can be realized on growing concentric spheres in a single non-compact complete manifold X whose ideal boundary comes along with a C-C geometry which is determined by the asymptotic geometry of X. One can generalize further and take a quite general (not necessarily hyperbolic) complete manifold X and transplant our basic C-C problems to X. For example, one may look for lower bounds on the (volume) growth of subsets $Y \subset X$ in terms of a suitable *asymptotic dimension* of Y in the spirit of [$\mathrm{Gro}_{\mathrm{AI}}$].

1. Horizontal curves and small C-C balls

We analyze the behavior of short horizontal curves issuing from a given point $v \in V$ by a systematic (and somewhat boring) use of the Taylor remainder formula and thus prove several versions of the ball-box theorem (see [Bell] for a more elegant treatment of these problems). We apply this

[3] See the original paper [$\mathrm{Pan}_{\mathrm{QIR1}}$] for a more precise and general statement.

to the evaluation of the Hausdorff dimension of V and submanifolds V' in V and also to a count of homotopy classes of Lipschitz maps $V \to W$. Our first step is

1.1. Proof of the Chow connectivity theorem. (compare [Herm]). Recall that we are given vector fields $X_1, \dots, X_m$ on a connected manifold V which Lie-generate $T(V)$ and we want to join a given pair of points in V by a piecewise smooth curve where each piece is a segment of an integral curve of some field $X_i, i = 1, \dots, m$.

Since the problem is local, we may assume the fields X_i integrate to one-parameter groups of diffeomorphisms of V and we must join given points by piecewise orbit curves. In other words we must prove that the group G of diffeomorphisms of V generated by these subgroups is transitive on V. We start with the following

Trivial Lemma. *If G contains one-parameter subgroups, say $Y_1(t)$, $Y_2(t), \dots, Y_p(t)$, where the corresponding vector fields $Y_1, Y_2, \dots, Y_p$ span $T(V)$ (without taking commutators), then G is transitive on V.*

Proof. This follows from the implicit function theorem. Namely, for each $v \in V$ we consider the composed action map $E_v : \mathbb{R}^p \to V$ defined by

$$(t_1, \dots, t_p) \mapsto Y_1(t_1) \circ \cdots \circ Y_p(t_p)(v).$$

The differential of E_v at the origin $0 \in \mathbb{R}^p$ sends $\mathbb{R}^p$ onto the span of the fields Y_i in $T_v(V)$ and, hence, is surjective in our case. Thus the orbit $G(v)$ is open in V for each $v \in V$ by the implicit function theorem and, as V is connected, $G(v) = V$. ■

Now, let $X(t)$ be a one-parameter group (flow) on V, Y be a vector field and let us look at the transport of Y by $X(t)$, denoted $X_*(t)Y$. We observe that for small $t \to 0$

$$X_*(t)Y = Y + t[X, Y] + o(t)$$

(by the very definition of the Lie bracket) and conclude that, since the commutators of X_i span $T(V)$, there exist vector fields Y_j, $j = 1, \dots, p \geq m$, on V which span $T(V)$ and such that

(i) $Y_i = X_i$ for $i = 1, \dots, m$,

(ii) each field Y_j for $j > m$ equals $\big(X_i(t_j)\big)_* Y_{j'}$, i.e. the transport of some $Y_{j'}$ for $j' < j$ by the flow $X_i(t)$ at $t = t_j$, (where i also depends on j).

Finally we note that the one-parameter groups $Y_j(t)$ are contained in G because the transport of a field Y by $X(t)$ corresponds to the conjugation of the one parameter group $Y(\tau)$, i. e.

$$X_*(t)Y = X(t)Y(\tau)X^{-1}(t),$$

and the proof follows by the Trivial lemma.

1.1.A. Quantitative version of the Chow theorem for $\deg \mathbf{X_i} \le 2$ and a preliminary Hölder bound on the C-C metric. Let us try to estimate the length of the piecewise integral curve between given nearby points v and v' in V provided by the above argument, where the length of an orbit is given by the time parameter. (Thus, for example, the length of the orbit for the field $\frac{1}{t}\frac{\partial}{\partial t}$ in $\mathbb{R}$ between the points $t = 1$ and $t = \varepsilon \in]0,1[$ equals $|\log \varepsilon|$). We assume that $T(V)$ is spanned by the commutators of degree ≤ 2 of X_i (i.e. by X_i and $[X_i, X_j]$) and we want to show that *v and v' can be joined by a piecewise integral curve, such that*

$$\text{length of the curve} \lesssim (\text{Riemannian distance between } v \text{ and } v')^{\frac{1}{2}}.$$

Proof. Take vector fields $Y_1, \dots, Y_n$ on V such that

(i) The first m_0, for certain $m_0 < m$, among Y_i are some of the fields X_i say $X_1, \dots, X_{m_0}$, which are linearly independent at v.

(ii) The remaining fields $Y_{m_0+1}, \dots, Y_n$ are of the form $\big(X_i(\varepsilon)\big)_* X_j$ for some small ε (specified below), such that the corresponding commutators $[X_i, X_j]$ complete $X_1, \dots, X_{m_0}$ to a full n-frame at v.

Recall, that every field Y_k for $k > m_0$ is of the form $X_j + \varepsilon[X_i, X_j] + o(\varepsilon)$, therefore the vectors $Y_1, \dots, Y_n$ are linearly independent at v for small ε, and so the differential $\mathcal{D} : \mathbb{R}^n \to T_v(V)$ of the composed orbit map

$$E = E_\varepsilon : (t_1, \dots, t_n) \mapsto Y_1(t) \circ \cdots \circ Y_n(t)(v)$$

does not degenerate. Moreover, it is clear that the norm of the inverse operator is bounded by

$$\|\mathcal{D}^{-1}\| \le \operatorname{const} \varepsilon^{-1} \quad \text{for small } \varepsilon \to 0.$$

It follows that the E_ε-image of the ε-ball in $\mathbb{R}^n$ around the origin contains the Riemannian ball $B_v(\delta) \subset V$ for $\delta \geq \text{const}' \varepsilon^2$, provided $\varepsilon > 0$ is sufficiently small. This follows from the implicit function theorem since the second derivatives of the map E_ε are uniformly bounded in ε (as this is true for the fields Y_i of the form $\big(X_i(\varepsilon)\big)_* X_j$). ■

Reformulation in terms of the C-C metric. The length of piecewise integral curves defines a metric on V which is of a slightly more general type than Carnot-Carathéodory since the fields X_i are not supposed to be independent. But if these fields are independent this metric essentially majorizes the Carnot-Carathéodory one as our curves are (special) H-horizontal for $H = \text{Span}\{X_i\}$. In fact, the two metrics are equivalent as we shall see later and we call our present metric C-C anyway. Now we can reformulate the above proposition in terms of the following Hölder bound on this C-C metric by a Riemannian one,

$$\text{C-C dist} \lesssim (\text{Riem.dist})^{\frac{1}{2}}. \qquad (*)$$

1.1.A′. Upper box bound on C-C dist for deg $\mathbf{X_i} \leq 2$. Let us refine $(*)$ by taking into account the difference in the behavior of the C-C distance in different directions. Namely, we want to show that in the direction of H the C-C metric is equivalent to the Riemannian one. Namely,

let $c(t)$ be a smooth curve in V issuing from $v \in V$ parametrized by the length parameter t. If $c(t)$ is tangent to H at $t = 0$, then

$$\text{C-C dist}\big(v = c(o), c(t)\big) \lesssim t. \qquad (**)$$

Proof. The tangent vector of $c(t)$ at $v = c(o)$ can be written as a linear combination of X_i at v, say $c'(t) = \sum_{i=1}^m a_i X_i$. Then the value of the composed orbit map

$$E : (t_1, \ldots, t_m) \mapsto X_1(t) \circ \cdots \circ X_m(t)(v)$$

at the point $(ta_1, \ldots, ta_m)$ approximates $c(t)$ for small t as

$$E(ta_1, \ldots, ta_m) = c(t) + O(t^2),$$

by the Taylor remainder theorem (applied to $c(t)$ and to E with respect to some Euclidean structure in V near v). It follows, with $(*)$, that

$$\text{C-C dist}\big(c(t),\ E(ta_1, \ldots, ta_m)\big) \lesssim t$$

and the proof of $(**)$ is implied by the triangle inequality as

$$\text{C-C dist}\big(v, E(ta_1, \dots, ta_m)\big) \le t$$

by the very definition of our C-C distance with piecewise integral curves.

1.1.B. Lower box bound on C-C dist for $\deg \mathbf{X_i} \le \mathbf{2}$. The above shows that every small C-C ball of radius ε around v contains a box of size about ε in the directions tangent to H_v and about ε^2 in the transversal direction (where "about ε" signifies const ε for const > 0 independent of ε). Now we want to show that the ε-C-C ball is contained in such a box. Namely, we want to show that along a curve $c_0(t)$ *transversal* to H the C-C distance is $\gtrsim \sqrt{\text{Riemannian distance}}$. Here are two slightly different proofs.

First proof. Assume for the moment that the dimension of the span $\{X_i\}$ is constant at v and let α be a smooth 1-form defined near v, such that

$$\alpha(X_i) = 0,\ i = 1, \dots, m$$

and

$$\alpha\big(c_0'(t)\big) = 1, \quad \text{for our curve } c(t)$$

issuing from v in the direction $c'(t)$ transversal to H.

Let c be a piecewise smooth curve tangent to H joining v with $v_\varepsilon = c_0(t = \varepsilon)$ and let b denote the closed curve formed by $c_0[0, \varepsilon]$ and c, see Fig. 2 below.

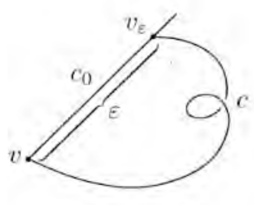


Figure 2

This b bounds a disk D of Riemannian area about ℓ^2 for

$$\ell = \text{length } b = \text{length } c + \varepsilon,$$

and by the Stokes formula

$$\text{Area } D \gtrsim \int_D d\alpha = \int_{c_0[0,\varepsilon]} \alpha \approx \varepsilon,$$

since α vanishes on c (which is tangent to H). It follows that $(\operatorname{length} c + \varepsilon)^2 \gtrsim \varepsilon$ and consequently

$$\operatorname{length} c \gtrsim \sqrt{\varepsilon}. \qquad \blacksquare$$

Now, even without assuming that the span of X_i has constant rank, we have a form α which annihilates all X_i at v and has $\alpha\big(c_0'(0)\big) = 1$. Then the integral $\int_D d\alpha = \int_b \alpha$ acquires an extra term, namely $\int_c \alpha$, which is of the order at most $(\operatorname{length} c)^2$ since $\alpha(X_i)$ is of order $\leq \varepsilon$ in the Riemannian ε-ball around v. This gives us the relation

$$(\operatorname{length} c_1 + \varepsilon)^2 \gtrsim \varepsilon \pm (\operatorname{length} c)^2,$$

which implies $\operatorname{length} c \geq \sqrt{\varepsilon}$ just the same.

Second Proof. Take a smooth hypersurface $V_0 \subset V$ passing through v and tangent to $H_v \subset T_v(V)$. Then every curve c issuing from v and tangent to H is also tangent to V_0 at v and by the Taylor remainder theorem the Riemannian distance from $c(t)$ to V_0 is bounded by $\approx t^2$. Hence, the Riemannian distance between $v = c(0)$ and $c(t)$ is bounded by t^2 on each curve c_0 transversal to V_0. $\blacksquare$

Remark. Notice that both proofs need the fields X_i to be C^1-smooth and fail for continuous fields where, in fact, the quadratic Hölder bound may be invalid.

1.1.C. Corollary: Ball-box theorem for deg $= 2$. The small ε-C-C ball in V around v is *equivalent* to the following $\varepsilon\times\varepsilon^2$-box. Take a smooth m_0-dimensional submanifold $V_1 \subset V$ through v with $T_v(V_1) = H_v$, take the Riemannian ε-ball in V_1 and then the Riemannian ε^2-neighbourhood of this ball in V. This is our $\varepsilon\times\varepsilon^2$-box. It is equivalent to the ε-C-C ball in the sense that

$$\operatorname{Box}(\delta\varepsilon\times\delta^2\varepsilon^2) \subset \varepsilon\text{-C-C ball} \subset \operatorname{Box}(\Delta\varepsilon\times\Delta^2\varepsilon^2),$$

where δ and Δ are positive constants independent of ε (which can be chosen continuously depending on v in a suitable sense).

1.2. A new proof of the Chow theorem and the Hölder bound on the C-C metric for arbitrary degree. (compare [Mit$_{1;2}$] and [N-S-W]). Our proof of the bound

$$\text{C-C dist} \lesssim (\text{Riem.dist})^{\frac{1}{d}} \qquad (+)$$

for the case where $T(V)$ is spanned by the commutators of the fields X_i of degree $d \leq 2$, does not easily extend for $d \geq 3$ but the following more straightforward approach proves out to be more forceful as it yields $(+)$ for all d. The key ingredient is the following (well known) approximate expression for the commutator of one-parameter groups $X_1(t)$ and $X_2(t)$ in Diff V in terms of the 1-parameter group corresponding to the Lie brackets of the fields X_1 and X_2,

$$[X_1(t), X_2(t)]^\circ \overset{\text{def}}{=} X_1(t) \circ X_2(t) \circ X_1^{-1}(t) \circ X_2^{-1}(t) = [X_1, X_2](t^2) + o(t^2), \qquad (*)$$

where the additive notation refers to some Euclidean structure in a relevant neighbourhood (and where one should note that $X_i^{-1}(t) = X_i(-t)$, $i = 1, 2$).

Proof of $(*)$. We need the following three elementary formulas

(1) $$(\tau Y)(t) = Y(t\tau)$$

(2) $(X+\tau Y)(t)=X(t)\circ Y(\tau t)+o(t\tau)=Y(\tau t)\circ X(t)+o(t\tau)$, for $t, \tau \to 0$.

(3) $$X_1(t)X_2(\tau)X_1^{-1}(t) = (X_2 + \tau[X_1 X_2])t + o(t\tau).$$

Notice, that (1) is obvious, (2) follows from the Taylor remainder theorem for the composition $X(t) \circ Y(\tau t)$, and (3) is implied by (1), (2) and the definition of the Lie bracket (compare 1.1).

Now we obtain $(*)$ in the form

$$X_1(t) \circ X_2(t) \circ X_1^{-1}(t) = [X_1, X_2](t^2) \circ X_2(t) + o(t^2)$$

by applying first (3) and then (2) and (1) to the left-hand side.

Next we observe that $(*)$ by induction implies

$$\big[X_1(t), [X_2(t), X_3(t)]^\circ\big]^\circ = \big[X_1, [X_2 X_3]\big](t^3) + o(t^3)$$

$$\cdots\cdots\cdots$$

$$\big[X_1(t), [\ldots, X_d(t)]^\circ \ldots\big]^\circ = \big[X_1, [\ldots X_d] \ldots\big](t^d) + o(t^d).$$

Proof of (+). We concentrate on the simplest case where $\dim V = 3$ and $T(V)$ is generated by $X_1,\ X_2$ and $Y = [X_1, X_2]$. We denote by $Y^\circ(t)$ the one-parameter family (not a subgroup) of diffeomorphisms defined by

$$Y^\circ(t) = \begin{cases} \left[X_1(|t|^{\frac{1}{2}}), X_2(|t|^{\frac{1}{2}})\right]^\circ & \text{for } t \geq 0 \\ \left[X_2(|t|^{\frac{1}{2}}), X_1(|t|^{\frac{1}{2}})\right]^\circ & \text{for } t \leq 0 \end{cases}$$

and we observe that the composed map

$$E^0 : (t_1, t_2, t_3) \mapsto X_1(t) \circ X_2(t) \circ Y^\circ(t)(v)$$

sends the box $B(\varepsilon) \subset \mathbb{R}^3$ defined by $|t_1| \leq \varepsilon,\ |t_2| \leq \varepsilon,\ |t_3| \leq \varepsilon^2$ into the ε'-C-C ball in V around v for $\varepsilon' \approx \varepsilon$ (in fact, for $\varepsilon' \leq 10\varepsilon$). What remains to show is that the image of this box contains a Riemannian δ-ball around v for $\delta \gtrsim \varepsilon^2$. To see that we compare E^0 with the composed map

$$E : (t_1, t_2, t_3) \mapsto X_1(t) \circ X_2(t) \circ Y(t)(v),$$

for which the image of the ε-box is δ-large by the implicit function theorem. In fact, the E-image of the ε^2cube defined by $|t_1| \leq \varepsilon^2,\ i = 1, 2, 3$, contains the required δ-ball. Then we observe with $(*)$ that $E^0 = E + o(\varepsilon^2)$ in the ε^2-cube. It follows by elementary topology (see below) that the E^0-image of the ε^2-cube is essentially as large as the E-image. ∎

Elementary topology lemma. *Let E and E^0 be continuous maps of a compact manifold with boundary into a Riemannian manifold, say $B \to V$ such that*

(i) $\operatorname{dist}_V(E, E^0) \underset{\text{def}}{=} \sup_{b \in B} \operatorname{dist}\big(E(b), E^0(b)\big) \leq \delta_0$ for some $\delta_0 > 0$.

(ii) *Every two points in V within distance $\delta \leq \delta_0$ can be joined by a unique geodesic segment of length δ.*

(iii) *The map E is a homeomorphism of B onto its image $E(B)$ in V.*

Then the image $E^0(B)$ contains every point $v \in E(B)$ for which the δ-ball in V around v is contained in $E(B)$.

The (standard) proof of this is left to the reader.

Finally we observe that with the provisions we have made the above proof of (+) extends to the general case (of any number of fields and arbitrary $d = 1, 2, 3, \ldots$) by just adjusting the notations.

1.2.A. Upper box bound on the C-C distance for arbitrary degree. The above argument does not provide decent box-shaped domains inside the C-C balls but the following simple modification of this argument does just that. To see this we concentrate again on the case of three fields X_1, X_2 and $Y = [X_1, X_2]$ spanning $T(V)$ and we observe the following

1.2.A′. C^1-Lemma. *The above family $Y^\circ(t)$ is C^1-smooth.*

Proof. What we know about $Y^\circ(t)$ is the relation

$$Y^\circ(t^2) = [X_1, X_2](t^2) + o(t^2) \qquad (*)'$$

where $Y^\circ(t^2)$ and $[X_1, X_2](t^2)$ are smooth (as we *assume* our fields X_i are C^∞-smooth). Hence $(*)'$ implies that

$$Y^\circ(t^2) = [X_1, X_2](t^2) + t^3\varphi(t)$$

where $\varphi(t)$ is smooth, and so

$$Y^\circ(t) = [X_1, X_2](t) + t^{\frac{3}{2}}\varphi(\sqrt{t})$$

is C^1-smooth. ∎

We observe that the differential of $Y^\circ(t)$ at $t = 0$ equals that of $Y(t)$ and so $E(t)$ and $E^0(t)$ also have equal differentials at $t = 0$ (where the C^1-smoothness of E^0 is ensured by that of $Y^0(t)$), and so the E^0-image of the box

$$B(\varepsilon) = \{|t_1| \le \varepsilon,\ |t_2| \le \varepsilon, |t_3| \le \varepsilon^2\}$$

is sent by E^0 onto a box-shaped domain in V by the implicit function theorem (which we now can apply to E^0). Thus the proof of the upper box bound on dist_H is concluded.

1.2.B. Making "smooth" instead of "piecewise smooth" in the Chow connectivity theorem for polarizations $\mathbf{H} \subset \mathbf{T(V)}$. In the original Chow theorem curves joining given points must necessarily consist of pieces of orbits of different fields and so they cannot be made smooth. But if we have a smooth polarization H, where the commutators of H-horizontal fields span $T(V)$, we may slightly improve the result by showing that

Every two points in V can be joined by a smooth H-horizontal curve in V, i.e. by a smooth immersion $f : [0,1] \to V$ with $f(0) = (v_0)$ and $f(1) = (v)$ and $f'(t) \in H$ for $t \in [0,1]$.

Proof. Either of our two proofs of the Chow theorem provides a smooth family Φ of piecewise smooth curves issuing from v_0, say $\varphi(t) \in \Phi$, $t \in [0,1]$, such that the map $\Phi \to V$ defined by $\varphi \mapsto \varphi(1)$ contains a given point v in its image, i.e. $\varphi(1) = v$ for a certain φ, and this map $\Phi \to V$ is a submersion near φ. The curve φ consists of segments of orbits of certain H-horizontal vector fields $Y_1, \dots, Y_k$ and when these fields are fixed, then φ is uniquely determined by the lengths of the segments. In fact, these lengths serve as coordinates in Φ and so the curves in Φ close to φ are obtained by slightly varying these lengths, called $\ell_i = \ell_i(\varphi)$, $i = 1, \dots, k$. Next, let us smoothly interpolate between Y_i and Y_{i+1} for all $i = 1, \dots, h$. Namely, we introduce a smooth family of fields $Y_t = Y_t(\ell_1, \dots, \ell_k)$, $t \in [0, L_k]$ for $L_k = \sum_{i=1}^{k} \ell_i$, such that

(i) $Y_t = Y_{i+1}$ for $t \in [L_i + \varepsilon, L_{i+1} - \varepsilon]$, for $L_i = \ell_1 + \dots + \ell_i$ and small $\varepsilon > 0$.

(ii) $\|Y_t\| \le \text{const}$ for some $\text{const} \ge 0$ independent of ε.

Now we define $f(t)$ as the integral curve of the field Y_t issuing from v_0, i.e. $f(O) = v_0$ and $f'(t) = Y_t$ at $v = f(t)$, and observe that $f \to \varphi$ for $\varepsilon \to 0$. It easily follows (e.g. with Elementary topology lemma) that the map $f \mapsto f(1)$ contains v in its image for a sufficiently small ε. ∎

Acknowledgment. The smoothing problem in Chow's theorem was brought to my attention by Lucas Hsu.

1.3. Lower box bound on the C-C distance for deg ≥ 2. We want to show that on each smooth curve $c_0(t)$ in V issuing from v in the direction $c_0'(t) \in T_v(V)$ transversal to the span of the commutators of given smooth fields X_i, $i = 1, \dots, m$, of degrees $\le s$, the C-C distance satisfies for small $t > 0$,

$$\text{C-C dist}\big(c(o) = v, c(t)\big) \gtrsim t^{\frac{1}{s+1}}.$$

We shall do it by adopting the second proof from 1.1.B for which we need the following

Definition of tangency. A smooth function f on V is called *constant of order s at $v \in V$* with respect to given fields X_i, if for every differential operator of order r obtained by composing $r \leq s$ of our fields, denoted

$$X_I = X_{i_1} X_{i_2} \dots X_{i_r} \text{ for } I = (i_1, \dots, i_r),$$

the function $X_I(f)$ vanishes at v, where vector fields are thought of as differential operators of first order acting on functions. Next, a submanifold $V_0 \subset V$ passing through v is called *s-order tangent to X_i*, if every function on V vanishing on V_0 is constant of order s at v.

Examples

(a) If $s = 1$ then this tangency means that the tangent space $T_v(V_0) \subset T_v(V)$ contains the span of the fields X_i at v.

(b) If the system of fields X_i is integrable in the sense that it gives a tangent frame to a foliation of V then each leaf is tangent to X_i of infinite order and the same is true for every submanifold containing a leaf.

The following obvious lemma relates the above definition with our problem.

Lemma

(a) *Let f be constant of order s with respect to X_i at v and $c(t)$ be a smooth curve issuing from v and tangent to X_i in the sense that $c'(t) = \sum_{i=1}^{m} a_i(t) X_i$, for some smooth functions $a_i(t)$. (If the span H of X_i has dimension independent of v, then this tangency amounts to the inclusion $c'(t) \in H$ for all t.) Then $f\big(c(t)\big) \lesssim t^{s+1}$.*

(b) *Let V_0 be tangent to X_i at v with order s then the Riemannian distance from $c(t)$ to V_0 is bounded by*

$$\text{Riem.dist}\big(c(t), V_0\big) \lesssim t^{s+1}.$$

In other words, the C-C *ε-neighbourhood of V near v_0 is contained in the Riemannian ε^{s+1}-neighbourhood. In particular, the* C-C *distance on each curve $c_0(t)$ transversal to V_0 at v is $\gtrsim$* (Riemannian distance)$^{\frac{1}{s+1}}$.[4]

[4] Notice, that the only assumption on the fields X_i is C^∞-smoothness.

In order to use this lemma and exhibit actual boxes in small C-C balls, we need sufficiently many submanifolds tangent to X_i with prescribed order. These are provided by the following

Infinitesimal lemma. *Let $T_0 \subset T_v(V)$ be a linear subspace containing the commutators of X_i of orders $\leq s$ at v. Then there exists a submanifold $V_0 \subset V$ passing through v, having $T_v(V_0) = T_0$ and tangent to X_i at v with order s.*

One may prove this by a straightforward linear algebra in the space of d-th order jets of submanifolds (and functions) at $v \in V$. But one can also make a short-cut with the following

Exponential lemma. *Let $Y_1, \dots, Y_{m_1}, \dots, Y_{m_2}, \dots, Y_{m_r}, \dots, Y_{m_s}$ be linearly independent vector fields on V, where for each $r = 1, \dots, s$ the fields $Y_{m_{r-1}+1}, Y_{m_{r-1}+2}, \dots, Y_{m_r}$ are taken among commutators of X_i of degree r, such that the fields $Y_1, \dots, Y_{m_r}$ have the same span at v as the commutators of X_i of degree $\leq r$.* (Obviously, such Y_j exist.) *Then the image V_0 of the exponential map $\exp_v : \mathbb{R}^{m_s} \to V$ corresponding to Y_j is tangent to X_i at v with order s.*

Proof. To make it simple we start with the case $s = 2$ and assume for the moment that the fields X_i are linearly independent and so $m_1 = m$ and $Y_j = X_j$ for $j = 1, \dots, m$. Let f be a smooth function vanishing on V_0 (or rather on a germ of V_0 at v) and observe that

(1) $[X_i, X_j]f(v) = 0,\ i, j = 1, \dots, m$, since $T_v(V_0)$ contains the second degree commutators of X_i.

(2) For every field $X = \sum_{i=1}^m a_i X_i$, the operator X^2 satisfies $X^2 f(v) = 0$. In fact $X^p f(v) = 0$ for all $p = 1, 2, 3, \dots$, as f vanishes on the orbit $X(t)(v)$ (which is contained in the exponential image of $\mathbb{R}^m$ corresponding to X_i).

It follows that

$$(X_i X_j + X_j X_i) f(v) = \big((X_i + X_j)^2 - X_i^2 - X_j^2\big) f(v) = 0$$

which implies with (1) that $(X_i X_j) f(v) = 0$ as well. This gives us the required tangency of V_0 to X_i in the case where X_i are independent and the dependent case is taken care of as follows. Every field among X_i, say X_{i_0}, can be written as a sum, $X_{i_0} = Y'_{i_0} + X^0_{i_0}$, where Y'_{i_0} is some combination of Y_j, $j = 1, \ldots, m_0$, with constant coefficients, such that Y'_{i_0} equals X_{i_0} at v and so $X^0_{i_0} = X_{i_0} - Y'_{i_0}$ vanishes at v. The relations $[X_i, X_j]f(v) = 0$ imply that $[X^0_{i_0}, X_i]f(v) = 0$ since $X^0_{i_0}$ is a combination of X_i. Therefore the (obvious) vanishing $X^0_{i_0}X_i f(v) = 0$ implies $X_i X^0_{i_0} f(v) = 0$, and consequently

$$X_i X_j f(v) = (Y'_i + X^0_i)(Y'_j + X^0_j)f(v) = \\ Y'_i Y'_j f(v) + Y'_i X^0_j f(v) + X^0_i Y'_j f(v) + X^0_i X^0_j f(v) = 0,$$

where the vanishing of the first term is ensured by (the argument in) the independent case.

Case s = 3. Now we start with the relations $\big[X_i, [X_j, X_k]\big]f(v) = 0$ and we use $Y^p f(v) = 0$ for every combination Y of Y_j with constant coefficients. Thus we get $Y_i Y_j f(v) = 0$ for $i = 1, \ldots, m_1$ and $j = m_1 + 1, \ldots, m_2$ as well as for $i = m_1 + 1, \ldots, m_2$ and $j = 1, \ldots, m_1$, by using $[Y_i, Y_j]f(v) = 0$ and $Y^2 f(v) = 0$ for $Y = Y_i, Y_j$ and $Y_i + Y_j$. This implies that the (second order) operators $X_i[X_j, X_k]$ and $[X_j, X_k]X_i$ also vanish at $f(v)$ where dependencies among X_i and $[X_j, X_k]$ are taken care of as above for $s = 2$.

Next we show that the symmetrization of $Y_i Y_j Y_k f(v)$, over all (six) permutations of the three indices i, j and k, vanishes for $i, j, k = 1, \ldots, m_0$. This is done with the identity $(aY_i + bY_j + cY_k)^3 f(v) = 0$ for $a, b, c = 0, \pm 1$. This implies with the above that $Y_i Y_j Y_k f(v) = 0$ for $i, j, k = 1, \ldots, m_0$. What remains is to pass from Y_i to X_i, where X_i may be dependent, but these dependencies are easily taken care of as earlier. ■

Case s ≥ 4. The proof of this is clear by now.

We conclude by explicitly describing a box-shaped domain in V containing the C-C ball around v provided by the above lemmas.

Step 1. For every $r = 1, 2, \ldots$ we take a submanifold $V_r \subset V$ passing through v such that $T_v(V_r)$ equals the span of the commutators of X_i of degrees $1, \ldots, r$. A specific such V_r is provided by the exponential map corresponding to $Y_1, \ldots, Y_{m_r}$ from the exponential lemma.

Step 2. We take the Riemannian ε^{r+1}-neighbourhood of V_r for each r and intersect these over $r = 0, 1, \ldots$ (with the convention $V_0 = \{v\}$). This intersection is, indeed, box-shaped if we take $V_0 \subset V_1 \subset \cdots \subset V_2 \subset \cdots$, (which is possible with the exponential lemma) and it contains the ε'-C-C ball around $v \in V$ for $\varepsilon' \approx \varepsilon$.

Matching upper and lower box bounds. Our (families of) boxes inside (see 1.2.A) and outside (see above) the C-C balls are slightly different but this does not bring any confusion into the geometric picture as these families are equivalent in an obvious sense as a simple argument shows. (See [N-S-W] for further information.)

1.3.A. Doubling and covering properties for balls; equisingularity and the Hausdorff dimension. The ball-box theorem (as stated in 0.5.A) immediately implies the following universal bound on the Riemannian volume of concentric C-C balls in a compact manifold V,

$$\operatorname{Vol} B_v(2\rho) \leq C \operatorname{Vol} B_v(\rho) \tag{$*$}$$

for all $v \in V$ and real ρ and some constant $C = C(V)$. (This is one of the major applications of the ball-box theorem indicated in [N-S-W].) Consequently we obtain, as an obvious corollary, the following (purely internal) metric property of V.

Every ball $B_v(2\rho)$ can be covered by at most k balls of radii ρ for some k depending only on V (but not on v or ρ).

Now, suppose H is equiregular. Then, clearly, the ball-box theorem shows that small ρ-balls have volumes $\approx \rho^D$ (see (+) in 0.6) and thus $\dim_{\mathrm{Hau}} V = D$ with $0 < \operatorname{mes}_D < \infty$, where D is computed in terms of the commutator filtration $0 \subset H = H_1 \subset H_2 \subset \cdots \subset H_d = T(V)$ by $D = \sum_{i=1}^{d} i \operatorname{rank}(H_i/H_{i-1})$. Furthermore, let $V' \subset V$ be a submanifold in V such that the intersections $H'_i = H_i \cap T(V')$ have constant ranks for all $i = 1, \ldots, d$. Then in the (proof of the) ball-box theorem one may use frames *adapted* to V': if some vector from the frame is tangent to V' at a given point v', then the corresponding vector field is tangent to V' near v'. Thus we extend the ball-box theorem to the restricted metric $\operatorname{dist}_H |V'$, and see that a small ρ-ball in this metric is approximated by a box in V' having k'_i (among his $\sum_{i=1}^{d} k'_i$) sides of length $\approx \rho^i$ for $i = 1, \ldots, d$, where k'_i denotes $\operatorname{rank} H'_i/H'_{i-1}$. Now we see as earlier that the ρ-balls in V' have volumes $\approx \rho^{D'}$ for $D' = \sum_{i=1}^{d} ik'_i$ and so satisfy the same properties as those in V. In particular we see that $\dim_{\mathrm{Hau}}(V', \operatorname{dist}_H) = D'$.

Non-equiregular fields. First, let us try to understand how generic those frames of fields $X_1, \ldots, X_m$ are which Lie-generate $T(V)$. As the Lie generation condition involves jets of arbitrary large orders, one might think that for $m \geq 2$ the (jets of) non-generating frames have infinite codimension. In fact this is true modulo the following trivial

Observation. If C^∞-fields X_i, $i = 1, \ldots, m$ on V vanish at some point $v \in V$, then so do their commutators of all orders. Thus the jets of a non Lie-generating frame at a fixed point $v \in V$ have codim $\leq mn$ for $n = \dim V$.

On the other hand let X be a non-vanishing field on V. Then for generic fields Y the successive commutators

$$Y_1 = Y,\ Y_2 = [X, Y_1],\ Y_3 = [X, Y_2] \ldots,$$

generate $T(V)$, where *the jets of exceptional* Y*'s have infinite codimension.* In particular, *generic polarization of* rank ≥ 2 *Lie-generate* $T(V)$.

Proof. Let $X_1 = X$, $X_2 \ldots, X_n$ be a frame of commuting fields at a point $v \in V$ and $Y = \sum_{i=1}^n a_i X_i$. Then generically, up to infinite codimension, all but finitely many iterated Lie derivatives $L_X L_X \ldots L_X a_i$ do not vanish at a given point and so the fields $Y_1, Y_2, \ldots$ are not contained in a given hyperplane $H' \subset T_v(V)$. This hyperplane, for the above frame, is given by the span of $X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_n$ for some i and as we are free to change our frame it may be arbitrary. This implies our assertion.

Now we see with the above two facts that the space of jets of m-frames of fields for $m \geq 2$ contains two strata Σ_0 and Σ_1 where $\operatorname{codim} \Sigma_0 = mn$ and $\operatorname{codim} \Sigma_1 = \infty$ such that a frame Lie-generates $T(V)$ at v if and only if its jet at v misses $\Sigma = \Sigma_0 \cup \Sigma_1$.

Questions. Let $\Sigma^r \supset \Sigma$ correspond to the frames whose commutators of order $\leq r$ do not generate $T_v(V)$. What is the structure of Σ^r and $\Sigma^r - \Sigma^{r-1}$? What is the possible geometry of the subset $V^{(r)} \subset V$ where the r-th order commutators (of some fields) fail to generate $T_v(V)$ for $v \in V^{(r)}$ while the following commutators do generate $T(V)$?

Example. Let $n = \dim V = 2$ and one of the fields in the frame, say X, does not vanish at v. Then the intersection of $V^{(r)}$ with every orbit of X is discrete (as all commutators vanish at an accumulation point) and so $\Sigma^{(r)}$ is locally contained in a smooth curve.

Our interest in the above questions stems from the problem of evaluation of the Hausdorff dimension of non-equiregular (V, H) where we should know the (discontinuity) structure of the functions $n_i(v) = \operatorname{rank} H_i(v)$ on V.

Let us look what happens in the above plane example where we have two fields $X = \frac{\partial}{\partial x}$ and $Y = a(x,y)\frac{\partial}{\partial y}$ on the (x,y)-plane $\mathbb{R}^2$ such that the function a and the derivatives $\frac{\partial a}{\partial x}, \frac{\partial^2 a}{\partial^2 x}, \dots, \frac{\partial^s a}{\partial^s x}$ vanish on some closed subset A in the line $\{x = 0\}$ and $a \neq 0$ outside A. We give $\mathbb{R}^2$ the maximal (or supremal) metric for which all X- and Y-orbits $\mathbb{R} \to \mathbb{R}^2$ are (non-strictly) distance decreasing (where one should use partial orbits if the fields are not integrable) and determine $\dim_{\mathrm{Hau}} \mathbb{R}^2$ with this metric $\mathrm{dist}_{X,Y}$ as follows. This metric, outside A, is equivalent to the Euclidean metric dist and so

$$\dim_{\mathrm{Hau}}(\mathbb{R}^2 - A,\ \mathrm{dist}_{X,Y}) = 2.$$

On the other hand, $\mathrm{dist}_{X,Y}$ on A is equivalent to $(\mathrm{dist})^{\frac{1}{s+1}}$ and thus

$$\dim_{\mathrm{Hau}}(A, \mathrm{dist}_{X,Y}) = \big(\dim_{\mathrm{Hau}}(A, \mathrm{dist})\big)^{s+1}.$$

Consequently

$$\dim_{\mathrm{Hau}}(\mathbb{R}^2, \mathrm{dist}_{X,Y}) = \max\Big(2, \big(\dim_{\mathrm{Hau}}(A, \mathrm{dist})\big)^{s+1}\Big),$$

and we see that *any* real number ≥ 2 may appear as $\dim_{\mathrm{Hau}}(\mathbb{R}^2, \mathrm{dist}_{X,Y})$.

A similar example can be arranged in $\mathbb{R}^3$ with a *polarization* spanned by two *independent* fields, e.g. $X = \frac{\partial}{\partial x}$ and $Z = a(x,y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, where the function a is as above, and the commutator $Y = [X, Z] = \frac{\partial a}{\partial x}\frac{\partial}{\partial y}$ is of the same kind as earlier.

Thus, in the general case, one cannot say much more about $\dim_{\mathrm{Hau}}(V, \mathrm{dist}_H)$ than

(1) $\dim_{\mathrm{Hau}} V \leq D = \max\limits_{v \in V} \sum_{i=1}^{d} ik_i$, for $k_i = \mathrm{rank}\big(H_i(v)/H_{i-1}(v)\big)$.

(2) There is an open dense set $U \subset V$ where $D \leq \frac{n(n-1)}{2} + 1$ and so $\dim_{\mathrm{Hau}} U$ is also bounded by $\frac{n(n-1)}{2} + 1$.

But the picture appears more regular for sufficiently generic H admitting so-called *equisingular stratifications*. Such a stratification is a partition of V into locally closed submanifolds (strata) V_ν, such that

(i) the ranks $n_i(v)$ of $H_i(v) \subset T_v(V)$ are constant on each V_ν,

(ii) the numbers $n_{i,\nu} = \mathrm{rank}\big(H_i(v) \cap T(V_\nu)\big)$ are also constant on each V_ν.

If this is the case we see as earlier that

$$\dim_{\mathrm{Hau}} V_\nu = \sum_{i=1}^{d} i(n_{i,\nu} - n_{i-1,\nu})$$

and $\dim_{\mathrm{Hau}} V = \max_\nu \dim_{\mathrm{Hau}} V_\nu$.

It is clear that real analytic H admits equisingular stratifications and the same seems to be true for the generic C^∞-polarizations H by virtue of (E_1) on p. 34 in [Gro$_{\mathrm{PDR}}$]. Similarly one defines equisingularity of $V' \subset (V,H)$ and obtain, in particular, the integrality of $\dim_{\mathrm{Hau}}(V', \mathrm{dist}_H)$ in the real analytic and C^∞-generic cases.

1.4. Canonical coordinates, almost Lie groups, nilpotent tangent cones and a sharp version of the ball-box theorem for equiregular polarizations. Suppose we are given a frame of smooth vector fields $Y_1, \ldots, Y_n$ on V, $n = \dim V$, where each Y_i is assigned an integer $\deg Y_i = \deg i \geq 1$, such that commuting fields at most add degrees, i.e.

$$[Y_i, Y_j] = \sum_k c_{ijk}(v) Y_k, \tag{$*$}$$

where $c_{ijk} = 0$ for $\deg k > \deg i + \deg j$ (compare 0.5). Then for each $v \in V$ one defines the following Lie algebra L_v with a preferred basis, denoted Y_i^v where the Lie brackets are given by the formulae

$$[Y_i^v, Y_j^v] = \sum_k \delta_{ijk} c_{ijk}(v) Y_k^v, \tag{$*^{v}$

where $\delta_{ijk} = 1$ for $\deg k = \deg i + \deg j$ and $\delta_{ijk} = 0$ otherwise. To comprehend the meaning of $(*)^v$ we ε-scale the fields Y_i according to their degrees, denote ${}^\varepsilon Y_i = \varepsilon^{\deg Y_i} Y_i$, and express the multiplication table $(*)$ in terms of ${}^\varepsilon Y_i$. This gives

$$[{}^\varepsilon Y_i, {}^\varepsilon Y_j] = \sum_k \varepsilon^{d_{ijk}} c_{ijk} {}^\varepsilon Y_k, \tag{${}^\varepsilon(*)$}$$

for $d_{ijk} = \deg i + \deg j - \deg k$. Now we see that $(*)^v$ equals the limit of ${}^\varepsilon(*)$ at v for $\varepsilon \to 0$ which shows that L_v is indeed a Lie algebra which is nilpotent of degree at most $\max_i \deg Y_i$.

Definition. (compare [Good], [Bell]). The simply connected (nilpotent) Lie group N_v corresponding to L_v with distinguished left invariant fields corresponding to Y_i^v is called the *nilpotent tangent* cone of the frame $\{Y_i\}$ at v.

We still denote the distinguished fields on N_v by Y_i^v and we want to show that the formal limit relation ${}^\varepsilon(*) \to (*)^v$ for $\varepsilon \to 0$ implies an actual convergence ${}^\varepsilon Y_i \to Y_i^v$ in suitable local coordinates in V and N_v. In fact, one can use for this purpose any system of coordinates in V near v which is made up in a canonical way out of the fields ${}^\varepsilon Y_i$ but we shall stick to the coordinates t_i defined with the composition of the one parameter groups $Y_i(t)$, namely, with the map

$$E_v : (t_1, \dots, t_n) \mapsto Y_1(t) \circ Y_2(t) \circ \dots \circ Y_n(t)(v),$$

defined on a certain cube

$$B(\rho) = \{|t_i| \le \rho\} \subset \mathbb{R}^n.$$

We identify the Lie algebra $L_v = L(N_v)$ (which comes along with the basis Y_i^v) with $\mathbb{R}^n$ and we denote by $E_0 : L_v = \mathbb{R}^n \to N_v$ the map defined by composing the one-parameter subgroups $Y_i^v(t)$ corresponding to the fields Y_i^v on N_v. Now we assume the "radius" $\rho > 0$ is so small that the maps E_v and E_0 are diffeomorphisms of $B(\rho)$ onto their respective images in V and N_v and we transport the (left invariant) fields Y_i^v from N_v to V (or rather to the image $E_v\big(B(\rho)\big) \subset V$) by (the differential of) the map

$$E_v \circ E_0^{-1} : E_0\big(B(\rho)\big) \to E_v\big(B(\rho)\big).$$

(Notice that the map E_0^{-1} is defined for all ρ as E_0 is, in fact, a diffeomorphism of L_v onto N_v, but this is irrelevant for our local discussion.) We denote the transported fields by $\bar{Y}_i^v$ and we want to compare them with the fields Y_i in smaller and smaller "boxes" around v in V obtained by the following ε-scaling of a fixed (cubical) box $\bar{B}(\rho) = E_v\big(B(\rho)\big) \subset V$. We denote by $a_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ the linear operator sending the Euclidean basis $\{e_i\}$ to $\{\varepsilon^{\deg i} e_i\}$ and we apply this notation to our systems of vector fields Y_i, Y_i^v, etc. For example, we write $a_\varepsilon(Y_i)$ for ${}^\varepsilon Y_i = \varepsilon^{\deg i} Y_i$. Then the ε-scaled box $\bar{B}_\varepsilon(\rho) \subset V$ is defined as the E_v-image of $a_\varepsilon B(\rho) \subset \mathbb{R}^n$. Equivalently, $\bar{B}_\varepsilon(\rho)$ can be defined as the image of $B(\rho)$ under the E_v-map corresponding to the fields $a_\varepsilon Y_i = {}^\varepsilon Y_i$.

Next we observe that the operators a_ε on $L_v = \mathbb{R}^n$ are automorphisms of the Lie algebra L_v and we denote by $A_\varepsilon : N_v \to N_v$ the corresponding automorphisms (self-similarities) of N_v. Then we transport these for $\varepsilon \leq 1$ to V via $E_v E_0^{-1}$ and denote the transported maps by

$$\bar{A}_\varepsilon : \bar{B}(\rho) \to \bar{B}_\varepsilon(\rho) \subset \bar{B}(\rho),$$

where the "transport" is defined by

$$\bar{A}_\varepsilon = (E_v E_0^{-1}) A_\varepsilon \left(E_v E_0^{-1}\right)^{-1}.$$

Let us summarize what we have obtained so far. We have chosen a small "curved cube" $\bar{B}(\rho) \subset V$ around v, that is the E_v-image of an actual ρ-cube in $\mathbb{R}^n$. We have on this cube two systems of fields, Y_i and $\bar{Y}_i^v$ which are related as follows.

(1) The two systems of fields coincide at v,

$$\bar{Y}_i^v(v) = Y_i(v), \ i = 1, \ldots, n.$$

(2) The multiplication table for $\bar{Y}_i^v$ has constant coefficients,

$$\left[\bar{Y}_i^v, \bar{Y}_j^v\right] = \sum_k \bar{c}_{ijk} \bar{Y}_k^v \qquad (\bar{*})^v$$

where $\bar{c}_{ijk}$ are the structure constants of the Lie algebra L_v with the distinguished basis Y_i^v.

(3) The multiplication table $(\bar{*})^v$ is obtained from that for the fields ${}^\varepsilon Y_i = a_\varepsilon(Y_i)$ (see ${}^\varepsilon(*)$) by sending $\varepsilon \to 0$ and by evaluating the limit at v.

(4) We have diffeomorphisms $\bar{A}_\varepsilon$ of $\bar{B}(\rho)$ onto smaller box-like domains $\bar{B}\varepsilon(\rho) \subset \bar{B}(\rho)$, such that (the differentials of) $\bar{A}_\varepsilon$ on $\bar{Y}_i^v$ commute with a_ε, i.e. $\bar{A}_\varepsilon(\bar{Y}_i^v) = a_\varepsilon(\bar{Y}_i^v)$, or equivalently, $\bar{A}_\varepsilon^{-1}\left(a_\varepsilon(\bar{Y}_i^v)\right) = \bar{Y}_i^v$.

Now we want to understand what happens to the fields $\bar{Y}_i^\varepsilon = \bar{A}_\varepsilon^{-1}\left(a_\varepsilon(Y_i)\right)$ on $\bar{B}(\rho)$ (where the corresponding fields Y_i need be defined on $\bar{B}_\varepsilon(\rho)$) for $\varepsilon \to 0$.

1.4.A. Convergence Proposition. *If the fields Y_i are C^1-smooth (which we assume all along) then the fields $\bar{Y}_i^\varepsilon$ uniformly converge to $\bar{Y}_i^v$ on $\bar{B}(\rho)$.*

Proof. The multiplication table for the fields $\bar{Y}_i^\varepsilon$ on $\bar{B}(\rho)$ is obtained from that of ${}^\varepsilon Y_i$ on $\bar{B}_\varepsilon(\rho)$ (see ${}^\varepsilon(*)$ above) with the map $\bar{A}_\varepsilon : \bar{B}(\rho) \to \bar{B}_\varepsilon(\rho)$. Namely,

$$[\bar{Y}_i^\varepsilon, \bar{Y}_j^\varepsilon] = \sum_k \bar{c}_{ijk}^\varepsilon \bar{Y}_k^\varepsilon, \qquad (\bar{*})^\varepsilon$$

where

$$\bar{c}_{ijk}^\varepsilon(v') = \varepsilon^{d_{ijk}} c_{ijk}\big(\bar{A}_\varepsilon(v')\big),\ v' \in \bar{B}(\rho),$$

for $d_{ijk} = \deg i + \deg j - \deg k$.

It follows, by the continuity of c_{ijk} at our point v (in the "center" of $\bar{B}(\rho)$), that the oscillations of c_{ijk} on $\bar{B}_\varepsilon(\rho)$ go to zero for $\varepsilon \to 0$ and so the same is true for the oscillation of $\bar{c}_{ijk}^\varepsilon$ on $\bar{B}(\rho)$. Thus we have uniform convergence on $\bar{B}(\rho)$,

$$\bar{c}_{ijk}^\varepsilon \to \bar{c}_{ijk} \quad \text{for } \varepsilon \to 0.$$

Now we are going to prove the required convergence $\bar{Y}_i^\varepsilon \to \bar{Y}_i^v$ by observing that $\bar{Y}_i^\varepsilon$ and $\bar{Y}_i^v$ satisfy similar systems of ordinary differential equations (in the coordinates $t_1, \dots, t_n$) with the coefficients $\bar{c}_{ijk}^\varepsilon$ and $\bar{c}_{ijk}$ playing identical roles. To do this we lift all our objects to the cube $B(\rho) \subset \mathbb{R}^n = L_v$ where we use the coordinates $t_1, \dots, t_n$ and we introduce the following notations $\tilde{c}_{ijk}^\varepsilon$, the lifts of the functions $\bar{c}_{ijk}^\varepsilon$ to $B(\rho)$ via the map $E_v : B(\rho) \to \bar{B}(\rho)$.

$\tilde{Y}_i^\varepsilon$, the lifts of the fields $\bar{Y}_i^\varepsilon$ to $B(\rho)$ by $(E_v)^{-1}$. (This makes sense since E_v^ε is a diffeomorphism of $B(\rho)$ onto $\bar{B}(\rho)$ by our assumption.)

$\tilde{Y}_i^v$, the lifts of the fields Y_i^v to $B(\rho) \subset \mathbb{R}^n = L_v$ via the map $E_0 : L_v \to N_v$. Notice that the map $E_v : B(\rho) \to \bar{B}(\rho)$ sends $\tilde{Y}_i^v$ to $\bar{Y}_i^v$.

First observation. The fields $\tilde{Y}_i^\varepsilon$ are related to the fields $\partial_i = \frac{\partial}{\partial t_i}$ on $B(\rho) \subset \mathbb{R}^n$ by the following identities,

$$\left.\begin{aligned}
\tilde{Y}_1^\varepsilon &= \partial_1 \text{ on } B(\rho)\\
\tilde{Y}_2^\varepsilon &= \partial_2 \text{ on the subspace } \{t_1 = 0\} \subset B(\rho),\\
\tilde{Y}_3^\varepsilon &= \partial_3 \text{ on} \{t_1 = 0, t_2 = 0\} \subset B(\rho)\\
&- - - - - - - - - - -\\
\tilde{Y}_n^\varepsilon &= \partial_n \text{ on the } t_n\text{-line } \{t_i = 0,\ i = 1, \dots, n-1\} \subset B(\rho).
\end{aligned}\right\} \text{(A)}$$

Furthermore, the fields Y_i^v satisfy the same system of relation on $B(\rho)$.

To see that, we first observe that the lifts $\tilde{Y}_i$ to $B(\rho)$ of the original fields $Y_i = \bar{Y}_i^1$ satisfy (A) as immediately follows from the definition of the map $E_v : \mathbb{R}^n \to V$ via the composition of the one-parameter subgroups $Y_i(t)$. Similarly, the fields $\tilde{Y}_i^v$ satisfy (A) since these are the lifts of Y_i^v from N_v to $L_v = \mathbb{R}^n$ via the map $E_0 : L_v \to N_v$ obtained by composing $Y_i^v(t)$. What remains to show is that the passage from $Y_i = \bar{Y}_i^1$ to $\bar{Y}_i^\varepsilon = \bar{A}_{\varepsilon^{-1}}\big(a_\varepsilon(Y_i)\big)$ does not change the fields $\tilde{Y}_i^\varepsilon$ on those parts of $B(\rho)$ where the relations (A) apply. Namely

$$\begin{aligned} \tilde{Y}_1^\varepsilon &= \tilde{Y}_1 = \partial_1 \ (= \tilde{Y}_1^v) \text{ on all of } B(\rho) \\ \tilde{Y}_2^\varepsilon &= \tilde{Y}_2 = \partial_2 \ (= \tilde{Y}_2^v) \text{ on the subspace } \{t_1 = 0\} \subset B(\rho) \end{aligned}$$

and so on. To see this we shall bring $\bar{A}_\varepsilon$ from V to $\mathbb{R}^n$ by taking

$$\tilde{A}_\varepsilon = E_v^{-1} \bar{A}_\varepsilon E_v : B(\rho) \to B_\varepsilon(\rho) = a_\varepsilon\big(B(\rho)\big).$$

Then $\tilde{Y}_i^\varepsilon = \tilde{A}_{\varepsilon^{-1}}\big(a_\varepsilon(\tilde{Y}_i)\big)$ and (A) for $\tilde{Y}_i^\varepsilon$ follows from the relations (A) for $\tilde{Y}_i^v$, which are

$$\begin{aligned} &\tilde{Y}_1^v = \partial_1 \text{ on } B(\rho), \\ &\tilde{Y}_2^v = \partial_2 \text{ on } \{t_1 = 0\} \subset B(\rho), \end{aligned}$$

etc., and the commutation between $\tilde{A}_\varepsilon$ and a_ε on the fields $\tilde{Y}_i^v$, i.e. $\tilde{A}_{\varepsilon^{-1}}\big(a_\varepsilon(\tilde{Y}_i^v)\big) = \tilde{Y}_i^v$, which follows from the corresponding relation for $\bar{A}_\varepsilon$ (and eventually for A_ε). Q.E.D. (This extra argument for $\tilde{Y}_i^\varepsilon$ was needed as the fields $\tilde{Y}_i^\varepsilon$ were defined with E_v rather than with the map corresponding to $\bar{Y}_i^\varepsilon(t)$.)

Second observation. The fields $\tilde{Y}_i^\varepsilon$ satisfy the following linear differential equations,

$$\left.\begin{aligned} \partial_1 \tilde{Y}_i^\varepsilon &= \sum_k \tilde{c}_{1,i,k}^\varepsilon \tilde{Y}_k, \ i = 2, 3, \ldots, n, \text{ on } B(\rho) \\ \partial_2 \tilde{Y}_i^\varepsilon &= \sum_k \tilde{c}_{2,i,k}^\varepsilon \tilde{Y}_k, \ i = 3, \ldots, n \text{ on } \{t_1 = 0\} \subset B(\rho) \\ &- - - - - - - - - - - - \\ \partial_{n-1} \tilde{Y}_n^\varepsilon &= \sum_k \tilde{c}_{n-1,n,k}^\varepsilon \tilde{Y}_k \text{ on the } (t_{n-1}, t_n)\text{-plane in } B(\rho). \end{aligned}\right\} \text{(B)}$$

Furthermore, the fields $\tilde{Y}_i^v$ satisfy an identical system with (the constants) $\bar{c}_{ijk}$ instead of $\tilde{c}_{ijk}^2$.

The equations (B) follow from the commutation relations for $\tilde{Y}_i^\varepsilon$, that are

$$[\tilde{Y}_i^\varepsilon, \tilde{Y}_j^\varepsilon] = \sum_k \tilde{c}_{\varepsilon_{ijk}} \tilde{Y}_k^\varepsilon, \qquad (\tilde{*})^\varepsilon$$

the identities (B) and the obvious formula $[\partial_i, Y] = \partial_i Y \underset{\text{def}}{=} \frac{\partial Y}{\partial t_i}$ for all fields Y on $\mathbb{R}^n$. Similarly, we see the validity of these relations for $\tilde{Y}_i^v$ with $\bar{c}_{ijk}$ in place of $\tilde{c}_{ijk}^\varepsilon$.

To understand the meaning of (A) and (B) let us read these equations from bottom to top. The last identity in (A), i.e. $\tilde{Y}_n^\varepsilon = \partial_n$, should be thought of as an initial value datum for the (last among (B)) equation $\partial_{n-1}\tilde{Y}_n^\varepsilon = \sum_{k=1}^n \tilde{c}_{n,n-1,k}^\varepsilon \tilde{Y}_k$ on the (t_{n-1}, t_n)-plane (given by the equations $t_i = 0$, $i = 1, \dots, n-2$). Notice that this equation also involves the fields $\tilde{Y}_k$ for $k \leq n-1$ on the (t_{n-1}, t_n)-plane but these are equal to ∂_k on this plane according to (A). Next, we take $\tilde{Y}_n^\varepsilon$ on the (t_{n-1}, t_n)-plane obtained by solving our initial value problem and we also take $\tilde{Y}_{n-1}^\varepsilon = \partial_{n-1}$ on this plane as given by the second from the bottom relation (A). Then the pair $(\tilde{Y}_n^\varepsilon, \tilde{Y}_{n-1}^\varepsilon)$ serves for initial values for the second from bottom equations in (B) on the (t_{n-2}, t_{n-1}, t_n)-space which are

$$\partial_{n-2}\tilde{Y}_{n-1}^\varepsilon = \sum_k \tilde{c}_{n-2,n-1,k}^\varepsilon \tilde{Y}_k^\varepsilon, \ \partial_{n-2}\tilde{Y}_n^\varepsilon = \sum_k \tilde{c}_{n-2,n,k}^\varepsilon \tilde{Y}_k^\varepsilon,$$

where the field $\tilde{Y}_k^\varepsilon$ on the right hand side for $k \leq n-2$ equals ∂_k according to (A).

Conclusion. The fields $\tilde{Y}_i^\varepsilon$ on $B(\rho)$ are uniquely determined by their values at v and by the functions $\tilde{c}_{ijk}^\varepsilon$ via the equation (A) and (B). It follows, by an elementary theorem on dependence of solutions of linear O.D.E. upon initial conditions and coefficients, that the fields $\tilde{Y}_i^\varepsilon$ are continuous in $\tilde{c}_{ijk}^\varepsilon$. In particular, $\tilde{Y}_i^\varepsilon \to \tilde{Y}_{ijk}^v$ for $\varepsilon \to 0$ as $\tilde{c}_{ijk}^\varepsilon \to \bar{c}_{ijk}$. Consequently, the fields $\bar{Y}_i^\varepsilon$, which are images of $\tilde{Y}_i^\varepsilon$ under E_v, converge to $\bar{Y}_i^v$. Q.E.D.

1.4.A$'$. Uniformity of the convergence. Recall that the objects appearing in 1.4.A are constructed with the use of a distinguished point $v \in V$. These are the "curved cube" $\bar{B}(\rho) = \bar{B}(v, \rho)$, the fields $\bar{Y}_i^v$ coming from N_v and the fields $\bar{Y}_i^\varepsilon$ obtained by some rescaling and "homotopies" $\bar{A}_{\varepsilon^{-1}} = \bar{A}^v_{\varepsilon^{-1}}$ of Y_i. Now, we claim that the norm $\|Y_i^v - \bar{Y}_i^\varepsilon\|$ can be bounded independently of v. More precisely, we have the following

Uniform version of 1.4.A. *If the fields Y_i are C^1-smooth, then, for each compact subset $V_0 \subset V$, there exists a positive number ρ and a function $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$, such that for each $v \in V_0$ the fields $\bar{Y}_i^v$ and $\bar{Y}_i^\varepsilon$ are well defined on $\bar{B}(\rho) = \bar{B}(v, \rho)$ and $\|\bar{Y}_i^v - \bar{Y}_i^\varepsilon\| \leq \delta(\varepsilon)$, where the norm refers to the Riemannian metric on V which makes the frame Y_i orthonormal.*

This is immediate by observing that the proof of 1.4.A is "uniform in v".

1.4.A$''$. On C^r-convergence. *If the fields Y_i are C^{r+1}-smooth then $\bar{Y}_i^\varepsilon$ converge to $\bar{Y}_i^v$ in the C^r-topology and this convergence is uniform in v in the above sense.*

Proof. If Y_i are C^{r+1} then the coefficients c_{ijk} are C^r and so for $r \geq 1$ the Lie derivatives of the functions $\bar{c}^\varepsilon_{ijk}$ with respect to the fields $\bar{Y}_\mu^\varepsilon$ go to zero as fast as ε, i.e.

$$|\bar{Y}_\mu^\varepsilon \bar{c}^\varepsilon_{ijk}| \leq \operatorname{const} \varepsilon.$$

This estimate lifts to $B(\rho)$ where it reads

$$|\tilde{Y}_\mu^\varepsilon \tilde{c}^\varepsilon_{ijk}| \leq \operatorname{const} \varepsilon,$$

and since the fields $\tilde{Y}_\mu^\varepsilon$ are close to a fixed frame (namely $\tilde{Y}_\mu^v$) this implies

$$|\partial_\mu c_{ijk}| \leq \widetilde{\operatorname{const}}\, \varepsilon.$$

Then by going through (A) and (B) one obtains a C^1-bound on $\tilde{Y}_i^\varepsilon$, namely $|\partial_\mu \tilde{Y}_i^\varepsilon| \leq \operatorname{const}' \varepsilon$. With this, if $r \geq 2$, one can pass from the bound on the Lie derivatives $\bar{Y}_\mu^\varepsilon \bar{Y}_\nu^\varepsilon \bar{c}^\varepsilon_{ijk}$ to the bound on $\partial_\mu \partial_\nu \tilde{c}^\varepsilon_{ijk}$ which yields, in turn, a bound on $\partial_\mu \partial_\nu \tilde{Y}^\varepsilon_{ijk}$ and so on. ∎

1.4.B. Approximation of equiregular Carnot-Carathéodory spaces by self-similar nilpotent groups. Let $H \subset T(V)$ be an equiregular polarization on V which means that the subsets of tangent vectors $H = H_1 \subset H_2 \subset \cdots \subset H_i \subset \cdots$ spanned by the commutators of H-horizontal fields of degrees $\leq i$ are actual subbundles in $T(V)$, i.e. their fibers $(H_i)_v \subset T_v(V)$ have dimensions constant in $v \in V$ (compare 0.3.D). We assume, moreover, that the commutators of order $\leq d$ span all of $T(V)$, i.e. $H_d = T(V)$, and then we take a frame of vector fields $Y_1, \dots, Y_n$ adapted to H_i, i.e. $Y_1, \dots, Y_{n_1}$, for $n_1 = \operatorname{rank} H_1$, belong to H_1, then $Y_{n_1+1}, \dots, Y_{n_2}$, $n_2 = \operatorname{rank} H_2$, belong to H_2 etc. This frame comes along with a deg-function, where the first n_1 vectors Y_i have deg $= 1$, the following $n_2 - n_1$ have deg $= 2$ etc. We invoke the nilpotent tangent cone N_v associated to Y_i and we recall our diffeomorphism $E_v E_0^{-1}$ which sends a small neighbourhood of the identity element in N_v onto some neighbourhood of a given point v in V. We fix some Riemannian metric in V (in order to have the C-C metric) and we endow N_v with a corresponding left invariant metric for which the differential of $E_v E_0^{-1}$ at $\mathrm{id} \in N_v$ is isometric. Now both, V and N_v, have C-C metrics, say dist in V and dist* in N_v and the diffeomorphism $E_v E_0^{-1}$ brings dist* to V (or rather to our small neighbourhood in V around v where the action takes place). This transported metric is denoted dist_v^* on V. Denote by $B^*(v, \varepsilon) \subset V$ the ε-ball in V around v with respect to dist* and observe that 1.4.A implies the following

Local approximation theorem. [5] *If the subbundle $H \subset T(V)$ is sufficiently smooth then the difference between the metrics* dist *and* dist_v^* *on $B^*(v, \varepsilon)$ is $o(\varepsilon)$, i.e.* $\varepsilon^{-1}\big(\operatorname{dist}(v_1, v_2) - \operatorname{dist}_v^*(v_1, v_2)\big) \underset{\varepsilon \to 0}{\longrightarrow} 0$, *for all pairs of points $v_1, v_2 \in B^*(v, \varepsilon)$.*

Proof. To see the picture clearly for $\varepsilon \to 0$, we rescale our metrics and neighbourhoods by ε^{-1} using the (expanding) diffeomorphism $\bar{A}_{\varepsilon^{-1}}$ acting in V near v (see 1.4.A). To simplify the matter, we assume for the moment that our Riemannian metric in V comes from the left invariant metric in N_v via $E_v E_0^{-1}$. (This assumption, in fact, does not restrict the generality as our old metric agrees with the new one at v.) Then we observe that $\bar{A}_\varepsilon$ scales dist_v^* by ε, i.e. $\bar{A}_\varepsilon^* \operatorname{dist}_v^* = \varepsilon \operatorname{dist}_v^*$, wherever $\bar{A}_\varepsilon^*$ and dist_v^* are defined. In particular, $\bar{A}_{\varepsilon^{-1}}^*$ transform $B^*(v, \varepsilon)$ to $B^*(v, 1)$, provided E_v^{-1},

[5] Compare [$\mathrm{Mit}_{1,2}$].

and hence $\bar{A}_\varepsilon^*$ are defined on the ball $B^*(v,1)$. In fact, this can be always achieved by multiplying the underlying Riemannian metric by a fixed large constant and we assume from now on that the unit ball $B^*(v,1)$ is small enough for our game.

Now we compare $\varepsilon^{-1}\,\mathrm{dist}$ and $\varepsilon^{-1}\,\mathrm{dist}^*$ in $B^*(v,\varepsilon)$ by bringing them to $B^*(v,1)$ by $A_{\varepsilon^{-1}}$. We observe that $A_{\varepsilon^{-1}}(\varepsilon^{-1}\,\mathrm{dist}^*) = \mathrm{dist}^*$ and so we must prove the uniform convergence on the unit ball,

$$\left(A_{\varepsilon^{-1}}(\varepsilon^{-1}\,\mathrm{dist}) - \mathrm{dist}^*\right) \to 0 \ \text{ for } \ \varepsilon \to 0.$$

To prove this we use 1.4.A which shows that the polarization $A_{\varepsilon^{-1}}(H)$ on $B^*(v,1)$ converges to the polarization H^* corresponding to dist^* and the Riemannian metric (i.e. quadratic form) on this polarization transported from the original metric on H converges to the metric in H^*, since the vectors $Y_1,\dots,Y_{n_1}$ spanning H satisfy, according to 1.4.A,

$$\bar{A}_{\varepsilon^{-1}}(\varepsilon Y_i) \underset{\varepsilon\to 0}{\to} \bar{Y}_i^v,\ i=1,\dots,n_1,$$

where $\bar{Y}_i^v$ are certain vector fields spanning H^*.

To conclude the proof we would need the following continuity of the Carnot-Carathéodory metrics defined by (H,g) where $H \subset T(V)$ is a polarization and g is a Riemannian metric on H,

if $(H_\varepsilon, g_\varepsilon)$ converges to (H^*,g^*), then the C-C metrics also converge, i.e. $\mathrm{dist}_\varepsilon \to \mathrm{dist}^*$.

This is indeed so if the convergence $H_\varepsilon \to H^*$ is understood in the C^{d-1}-topology (where d is the bound on the degrees of the commutators of the fields in H^* spanning $T(V)$) due to a uniform bound on the metrics $\mathrm{dist}_\varepsilon$ (see below) but, in general, e.g. for C^0-convergence $H_\varepsilon \to H^*$, the functions $\mathrm{dist}_\varepsilon$ for arbitrarily small $\varepsilon > 0$ may be, a priori, infinite on certain pairs of points in V and so one cannot speak of the ordinary (uniform) convergence $\mathrm{dist}_\varepsilon \to \mathrm{dist}^*$. However we do have the following weak convergence defined with the Hausdorff distance between subsets in V with respect to dist^*.

Weak convergence lemma. *If $(H_\varepsilon, g_\varepsilon)$ uniformly converge to (H^*, g^*) then every* $\operatorname{dist}_\varepsilon$*-ball $B^\varepsilon(v,\rho)$ Hausdorff-converges to the corresponding* dist^**-ball $B^*(v,\rho)$ and this convergence is uniform on compact subsets in $V\times\mathbb{R}_+$, i.e.*

$$\operatorname{dist}^*_{\mathrm{Hau}}\big(B^\varepsilon(v,\rho), B^*(v,\rho)\big) \le \delta(\varepsilon)$$

for $\delta(\varepsilon)\to 0$ for $\varepsilon\to 0$, where one may use a fixed function $\delta(\varepsilon)$ for each compact subset of points (v,ρ).

Proof. Let H_1 and H_2 be mutually close polarizations with close quadratic forms. Then, at least locally, there exist mutually close frames of orthonormal vector fields spanning H_1 and H_2 and with these fields one establishes a correspondence between H_1- and H_2-horizontal curves as follows. The curves $c_1(t)$ and $c_2(t)$ parametrized by arc length (coming from the quadratic forms in H_1 and H_2 respectively) *correspond* to each other if at each moment t their derivatives,

$$c_1'(t)\in (H_1)_{v_1=c_1(t)} \text{ and } c_2'(t)\in (H_2)_{v_2=c_2(t)}$$

have identical decompositions with respect to the frames in H_1 and H_2. Clearly, corresponding curves issuing from nearby points remain close for a certain time which implies the required closeness of the C-C balls. (We suggest the reader would check that this "close" talk can be made rigorous and uniform.)

Corollary. *If the metrics* $\operatorname{dist}_\varepsilon$ *are uniformly bounded,* i.e. *if each* $\operatorname{dist}_\varepsilon$*-ball of radius ρ around v contains the* dist^**-ball around v of radius ρ^* for some strictly positive function $\rho^*(\rho)$, $\rho>0$, then $|\operatorname{dist}_\varepsilon - \operatorname{dist}^*|\to 0$ uniformly on compact subsets in V.*

This is obvious by the triangle inequality.

Conclusion of the proof of the approximation theorem for C^{2d-2}-smooth polarizations $H\subset T(V)$. If H is C^r-smooth, then the corresponding frame $Y_1,\dots,Y_n$ obtained by taking commutators of degree $\le d$ are C^{r-d+1}-smooth and then H_ε converges to H^* in C^{r+d}-topology according to 1.4.A$''$. In particular, the convergence $H_\varepsilon\to H^*$ is C^{r+d}. Now we use the fact that $T(V)$ is generated by commutators of H^*-horizontal fields of degree $\le d$ which, according to the Chow connectivity theorem, makes $\operatorname{dist}^* < \infty$. Since the Chow theorem appeals to the derivatives of H^* of order $\le d-1$, it is stable under small C^{d-1}-perturbations of H^* which implies a uniform bound on $\operatorname{dist}_\varepsilon$ whenever H_ε is sufficiently C^{d-1}-close to H^*. ∎

The case of H being C^d-smooth. We start by giving a bound on a metric in pure "Hausdorff terms". To grasp the idea, imagine we have two metrics on V, say dist and dist*, such that *every* dist-ball $B(v,\rho)$ is sufficiently dist*-Hausdorff close to the corresponding dist*-ball $B^*(v,\rho)$. Say,

$$\operatorname{dist}^*_{\mathrm{Hau}}\big(B(v,\rho), B^*(v,\rho)\big) \leq \rho/10.$$

Then the triangle inequality implies that the metrics are close as functions on $V\times V$. In particular (and most importantly)

$$B(v,\rho) \subset B^*(v,\alpha\rho), \text{ for a fixed } \alpha > 0,$$

(where one can take $\alpha = \frac{3}{5}$).

Now we turn to the proof of the approximation theorem and observe that the weak convergence lemma makes every small dist-ball around v quite close to the corresponding dist^*_v-ball. But here, unlike the above discussion, the metric dist^*_v depends on v. To remedy that we exclude v by setting $\operatorname{dis}^*(v,v') = \operatorname{dis}^*_v(v,v')$. Of course, dis* is not, in general, a metric, but the above argument also works for *quasi-metrics*, that are positive functions on $V\times V$, vanishing exactly on the diagonal and satisfying the following *approximate triangle inequality*,

$$\operatorname{dis}(v,v'') \leq \operatorname{const}\big(\operatorname{dis}(v,v') + \operatorname{dis}(v',v'')\big), \tag{+}$$

which must be uniform on compact subsets in $V\times V\times V$ (i.e. for each $K \subset V\times V\times V$ there exists const, so that (+) holds for all $v,v',v'' \in K$).

In order to prove (+) for dis* we recall that the function dist* comes from the nilpotent group N_v with self-similarities $A_\varepsilon : N_v \to N_v$ defined with $a_\varepsilon : L_v \to L_v$ on the Lie algebra $L_v = L(N_v)$. Since A_ε and a_ε commute with the (composed orbit) map $E_0 : L_v \to N_v$, (i.e. $E_0 a_\varepsilon = A_\varepsilon E_0$) the ε-balls in N_v around id $\in N_v$ are equivalent to the E_0-images of the ε-boxes B_ε in $L_v = \mathbb{R}^n$ defined by $|t_i| \leq \varepsilon^{\deg i}$. (We have already seen this picture for the exponential map in 0.3.C). Therefore, the inequality (+) for dis* reduces to the corresponding property of the E_v-images of the boxes $B_\varepsilon \subset \mathbb{R}^n = T_v(V)$. Namely, we need to show, that if two such images, say $E_v(B_\varepsilon)$ and $E_{v'}(B_\delta)$ in V intersect, then $E_{v'}(B_\delta)$ is contained in $E_v(B_{\varepsilon'})$ for $\varepsilon' \leq \operatorname{const}(\varepsilon+\delta)$. (Warning: the commutation relations $\deg[Y_i,Y_j] \leq \deg i + \deg j$ is crucial for this property.) To see this we recall that $E_v(B_\varepsilon)$ consists of the second ends of piecewise smooth curves issuing from v which are built of n segments $c_1,\ldots,c_n$, where c_1

is a piece of orbit of Y_1 of length $\leq \varepsilon^{\deg 1}(=\varepsilon)$, c_2 such a piece for Y_2 of length $\leq \varepsilon^{\deg 2}$ and so on. Thus the problem reduces to showing that if we add to such curve a new piece c_{n+1} of the orbit of Y_j of length $(\delta)^{\deg j}$, then there exists a curve of n pieces $c'_1, \ldots, c'_i, \ldots, c'_n$ with lengths $\leq (\varepsilon')^{\deg i}$ for $\varepsilon' \leq \mathrm{const}(\varepsilon + \delta)$, such that the second end of the new curve equals the free end v'' of c_{n+1}, see Fig. 3 below.

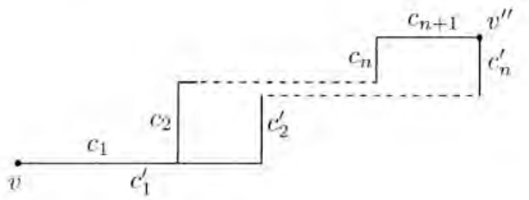


Figure 3

In other words, we must compensate for changing the order of the orbits and this can be achieved with the relation $(*)$ in 1.2. (compare [N-S-W]). This is straightforward and we leave it to the reader.

Finally, the weak convergence lemma shows that the "Hausdorff distance" between dist and dis* on $B^*(v,\varepsilon) \subset V$ is $O(\varepsilon)$ uniformly in v (because of 1.4.A′). This yields a bound on the metric dist by the above argument (but now, of course, with a constant different from 3/5) which concludes the proof of the approximation theorem for C^d-smooth H.

Remarks and corollaries

(a) It seems, the conclusion of the theorem should stand for H being C^{d-1}.

(b) If H is C^{d+1}, it is easy to show that $o(\varepsilon)$ in the approximation theorem can be replaced by $O(\varepsilon^2)$.

(c) The ball-box theorem is an immediate corollary of the approximation theorem as the balls in N_v *are* (obviously) box-like as we have mentioned several times.

(d) The Mitchell theorem concerning the tangent cones of C-C manifolds (see (iii) in 0.3.D) immediately follows from the approximation theorem. In fact, one only needs here the (weak) dis*-Hausdorff convergence of dist_ε rather than the final result on the uniform convergence (see [$\text{Mit}_{1;2}$]).

(e) The ball-box theorem implies (at least in the equiregular case) that small balls in V are "essentially contractible", i.e. *each small ε-ball is contractible within the concentric ball of radius $C\varepsilon$ for a fixed $C \geq 1$.* In fact, one can squeeze a *topological* ball (box) between the ε and the $C\varepsilon$-balls. Furthermore, the balls in nilpotent groups with self-similarities are honestly (and obviously) contractible (i.e. the above C equals one). It follows by the approximation theorem that for all V one has $C \to 1$ for $\varepsilon \to 0$ and it is likely (for sufficiently smooth C-C data) that $C = 1$ for small ε, i.e. small balls are probably contractible.

1.4.C. Pinching and related problems for C-C metrics. The approximation of a frame of fields Y_i with almost constant coefficients in the multiplication table $[Y_i, Y_j] = \sum_k c_{ijk} Y_k$ by a frame where the corresponding coefficients are truly constant (see 1.4.A) represents a simplest instance of the *stability phenomenon* for (homogeneous) geometric structures. In general, one looks for a weakest possible local or infinitesimal criterion for a given geometric structure to be homogeneous or almost homogeneous in a suitable sense. More specifically, when dealing with a Riemannian metric, one makes up such a criterion in terms of curvature (and covariant derivatives of the curvature) sometimes by *pinching* the curvature between two constants. (This explains the "pinching" terminology.) In our case the structure was given by a frame of vector fields which form *an almost Lie group* in the terminology of [Ru] and there are several results due to Min-Oo and Ruh allowing an approximation of an almost Lie group by an actual Lie group which goes infinitely deeper than our proposition 1.4.A. In fact, our argument with choosing O.D. equations (A) and (B) parallels the initial phase of the proof of Rauch's comparison and pinching theorems for Riemannian manifolds. (See [Gro_{SAP}] for an exposition of these techniques and ideas.)

Now we notice that the study of a general geometric structure, say σ on V, can be often reduced to that of an auxiliary Riemannian metric $\bar{g}$. For example, every frame σ of vectors $Y_1, \ldots, Y_n$ on V defines a unique metric $\bar{g} = \bar{g}(\sigma)$ on V for which this frame becomes orthonormal and so the almost Lie group problem can be, in principle, viewed as a special case of the Riemannian pinching problem.

The essential feature of σ which allows one to make the step $\sigma \mapsto \bar{g}$ is the *compactness* of the (isotropy) groups $\mathrm{Aut}(V, v, \sigma)$ for all $v \in V$. So we may expect that the C-C metrics given by pairs (H, g), where g is a positive quadratic form on the polarization H, should give rise to Riemannian metrics $\bar{g}$ naturally (or at least canonically) associated to (H, g). Here is an example where everything is perfectly nice.

Contact* C-C *manifolds. Let $H \subset T(V)$ be contact, i.e. H is (locally) given by a 1-form η on V such that the differential $d\eta$ is non-singular on H. Notice that in this case $n - 1 = \operatorname{rank} H$ is even, say $2m$. Now, using g on H we can specify η by requiring the $2m$-form $(d\eta)^m$ on H to be equal up to $\pm$ sign to the volume element of g on H which defines η up to $\pm$ sign and consequently $d\eta$ on V. This $d\eta$ defines a 1-dimensional subbundle ℓ transversal to H, namely $\ell = \operatorname{Ker} d\eta$, and we have a metric g' on ℓ defined by the condition $\|\eta\|_{g'} = 1$. Finally, the pair (g, g') defines the required metric $\bar{g} = g \oplus g'$ on V.

Let us indicate a general (and rather ugly) construction which provides $\bar{g}$ for a C-C structure (H, g) on V which is everywhere infinitesimally close to a fixed (model) homogeneous structure (H_0, g_0) on V_0. We fix some Riemannian metric $\bar{g}_0$ on V_0 invariant under automorphisms of (H_0, g_0) (which is possible since the isotropy group $\mathrm{Aut}(V_0, v_0, H_0, g_0)$ is compact as we assume our C-C is a *metric* on V_0) and then for a large i and each point $v \in V$ we consider the i-jets of maps $(V, v) \to (V_0, v_0)$ which are "ε-isometric" for a fixed small $\varepsilon > 0$, i.e. for which the image of the jet of (H, g) at v is sent ε-close to the jet of (H_0, g_0) at v_0. Using these jets we induce $\bar{g}_0$ at each point $v \in V$ and thus obtain a family $\bar{g}_\varepsilon$ of metrics on V. Finally, one extracts a single metric $\bar{g}$ out of $\bar{g}_\varepsilon$ by some averaging or envelope construction, e.g. by taking $\bar{g}$ which has the maximal volume element among all metrics smaller than all g_ε. (The reason why we first bring in some ε and then smooth it out is due to the fact that the infinitesimal symmetries of (H_0, g_0) are usually destroyed by small perturbations. Yet we want our $\bar{g}$ on V to remember these symmetries.)

The above indicates an approach to an infinitesimal stability (pinching) problem for C-C structures but we are attracted by more interesting (and more difficult) purely metric stability and/or homogeneity problems. Here are examples of these.

Metric criteria for homogeneity. Let (V, dist) be a Carnot-Carathéodory manifold. Suppose that the small ε-balls in V are mutually *δ-isometric* in an appropriate sense where $\delta = \delta(\varepsilon)$ fast goes to zero for $\varepsilon \to 0$. For example, let the Hausdorff distance between these balls, thought of as abstract metric spaces, satisfy

$$\mathrm{dist}_{\mathrm{Hau}}\big(B(v_1, \varepsilon), B(v_2, \varepsilon)\big) \leq \delta = \mathrm{const}\, \varepsilon^p \tag{$*$}$$

for a sufficiently large p. Does this imply that (V, dist) is locally homogeneous?

Example. If (V, dist) is a C^p-smooth Riemannian manifold then $(*)$ implies that (V, dist) is infinitesimally homogeneous of order $p-1$ (see below) and then the positive answer is provided by a theorem of Singer (see [Sin], p. 165 in [$\mathrm{Gro}_{\mathrm{PDR}}$] and [D-G]). The key step here is the implication $(*) \Rightarrow$ infinitesimal homogeneity which says, in effect, that the curvature of V and its covariant derivatives can be read of the distance (properties) on finite subsets in small balls in V. This is done by observing that $(*)$ implies the existence of a (possibly discontinuous) δ-isometric map $\varphi : B(v_1, \varepsilon) \to B(v_2, \varepsilon)$ which can be smoothed e.g. using the Riemannian center of mass construction (see [Kar]). Thus we obtain a smooth δ'-isometric map φ' where $\delta' \approx \delta$ and where we hold control over the derivatives of the map so that the metrics dist on $B(v_2, \varepsilon)$ and $\varphi'(\mathrm{dist})$ brought from $B(v_1, \varepsilon)$ become close with many (about p) derivatives. (We leave the details to the reader. Notice that even in the Riemannian framework one does not know how to get rid of the smoothness assumption on dist.)

A somewhat easier version of the problem appears if we are already given a homogeneous (model) C-C space (V_0, dist_0) and replace $(*)$ by the corresponding distance inequality between the balls $B(v_0, \varepsilon) \subset V_0$ and $B(v, \varepsilon) \subset V$. (Here, for example, the low smoothness of V does not cause serious problems.)

Almost homogeneity and pinching problems. Now we require the inequality $(*)$ for a *fixed* small ε, or for ε in a fixed interval $[\varepsilon_1, \varepsilon_2]$ where $\varepsilon_1 > 0$. The expected conclusion is the existence of a locally homogeneous C-C metric on V which is close to the original C-C metric. Of course, one needs here extra topological and local geometric assumptions on V (see below).

One arrives at a more traditional pinching problem if one replaces $(*)$ (now for a fixed ε) by the corresponding relation between the balls in V and in a homogeneous model space V_0. The desired conclusion is the existence of an almost isometric covering map $V_0 \to V$.

Necessary restriction on V. Usually, one insists while "pinching" V that it should be compact or at least metrically complete in order to avoid irrelevant complications. (One should separately consider the case where V_0 is incomplete and, moreover, admits no complete homogeneous manifold locally isometric to V_0). Another group of extra conditions should take care of the possibility that the "injectivity radius" of V becomes small of order ε. One can rule this out by insisting on "essential contractibility" of the ε-balls (compare (e) preceding 1.4.C). Yet one may wish to allow Inj.Rad. $\to 0$. Then one should either replace the comparison between balls by that between appropriate coverings of the balls (e.g. universal coverings of $C\varepsilon$-balls restricted to ε-balls for some $C > 1$), or to work out a more subtle comparison between the balls themselves that would allow $\varepsilon \to 0$. (Notice that the traditional pinching condition on the curvature, i.e. $K \in [\kappa_1, \kappa_2]$, can be detected by looking metrically at arbitrarily small balls but conditions imposed on the derivatives of the curvature become invisible on very small balls around $v \in V$ unless K vanishes at v.)

There are two basic approaches to the above problems. The first uses special curves (e.g. geodesics) in V satisfying certain O.D.E.'s similar to the system (B) we met in 1.4.A. Some results in this direction, allowing one to recapture the smooth structure of V out of the metric via the geodesics, appear in [Ham]. (Notice that O.D.E.'s may govern not only curves but also some functions on V, such as $v \mapsto \operatorname{dist}(v_0, v)$, and these functions may be used for embeddings $V \to \mathbb{R}^q$ (with large q) which sometimes serve almost as good as canonical coordinates on V.) The second approach uses some P.D.E.'s. In the Riemannian case the best results were obtained with non-linear P.D.E.'s (see [Ru]) but for C-C manifolds these are not yet available. Yet, we do have the (linear) Hörmander-Laplace operator Δ on V and the corresponding diffusion which can be reconstructed in many (all?) cases out of dist as follows. Define the *energy-density* $ef(v)$ of a Lipschitz function $f : V \to \mathbb{R}$ at $v \in V$ by

$$\lim_{\varepsilon \to 0} \varepsilon^{-N} \int_{B(v,\varepsilon)} |f(v) - f(v')|^2 dv',$$

where N denotes the Hausdorff dimension of (V, dist) and dv' refers to the Hausdorff measure. Then the energy of f on V is

$$E(f) = \int_V ef(v)dv.$$

Equivalently we can define $E(f)$ as the limit of the integrals $\int_{V\times V} |f(v) - f(v')|^2 \psi_\varepsilon dv\, dv'$ for $\varepsilon \to 0$ with appropriate weights $\psi_\varepsilon(v, v')$ which localize at the diagonal for $\varepsilon \to 0$. Then the extremal functions f for $E = E(f)$ satisfy the Hörmander-Laplace equation $\Delta f = 0$ and therefore are smooth (see [Hör]). These can be used to build up the smooth structure on V and one may try with this structure to reduce the metric problems (of pinching and homogeneity) to the corresponding infinitesimal problems. (Notice that the diffusion can be also constructed purely geometrically as the limit of convolutions of some kernels $\Phi_\varepsilon = \Phi_\varepsilon(v, v')$, namely

$$\lim_{i\to\infty} \underbrace{\Phi_{\varepsilon_i} * \Phi_{\varepsilon_i} * \cdots * \Phi_{\varepsilon_i}}_{i}$$

where $\Phi_\varepsilon(v, v')$ are suitably chosen functions of the form $\Phi_\varepsilon = \varphi_\varepsilon(\text{dist}(v, v'))$ which localize at the diagonal in $V\times V$ for $\varepsilon \to 0$ and where ε_i goes to zero at an appropriate rate for $i \to \infty$.)

1.4.D. Riemannian and piecewise Riemannian approximation of C-C metrics. Let V be an equiregular C-C manifold as in 1.4.B with the polarization $H \subset T(V)$ and the subbundles

$$H = H_1 \subset H_2 \subset \cdots \subset H_i \subset \cdots \subset H_d = T(V)$$

generated by the commutators of orders $1, 2, \dots, i, \dots, d$. We assume V is compact and we want to approximate the C-C metric dist on V by suitably controlled Riemannian metrics dist_ε on V (compare 0.8).

Approximation theorem. *There exist Riemannian metrics* dist_ε *on* V *for positive* $\varepsilon \to 0$, *such that*

(1) $|\,\text{dist} - \text{dist}_\varepsilon\,| = O(\varepsilon)$ uniformly on $V\times V$.

(2) The curvatures of (the Riemannian metric tensors underlying) dist_ε are $O(\varepsilon^{-2})$ and the injectivity radii of dist_ε are $O(\varepsilon^{-1})$.

Proof. Fix a Riemannian metric g on V, let $H_i^\perp = H_i \ominus_g H_{i-1}$, and let A_ε be the linear operator $A_\varepsilon : T(V) \to T(V)$ fixing H and acting by $\tau \mapsto \varepsilon^{-(i-1)}$ on the vectors $\tau \in H_i^\perp$ for $i = 2, \dots, d$. Then the distance functions $\mathrm{dist}_\varepsilon$ of the Riemannian tensors (metrics) $A_\varepsilon^*(g)$ on $T(V)$ satisfy the requirement of the theorem. This is immediate if V is a self-similar Lie group and the general case follows from 1.4.A - 1.4.A$''$. (The details are left to the reader.)

Remarks

(a) The metrics $\mathrm{dist}_\varepsilon$ we constructed increase as $\varepsilon \to 0$ and their total volumes grow as ε^{-m} for $m = \dim_{\mathrm{Hau}} V - \dim_{\mathrm{top}} V$.

(b) The weaker bound $O\big(\varepsilon^{-2(d-1)}\big)$ on the curvature of g_ε is obvious and needs no special (commutator) relations between H_i.

(c) The conditions (2) of the theorem says that $\mathrm{dist}_\varepsilon = \varepsilon\,\mathrm{dist}_\varepsilon^*$ where the metrics $\mathrm{dist}_\varepsilon^*$ have uniformly bounded (local) geometries as $\varepsilon \to 0$.

(d) It is unclear if one can improve (1) to $|\,\mathrm{dist} - \mathrm{dist}_\varepsilon| = o(\varepsilon)$ (for a suitable $\mathrm{dist}_\varepsilon$ satisfying (2)).

Uniqueness of the approximation. Let $\mathrm{dist}_\varepsilon$ and $\mathrm{dist}'_\varepsilon$ be two families of Riemannian metrics on V approximating dist with the above conditions (1) and (2). Then, ideally, one would like to have a bi-Lipschitz map $f_\varepsilon : (V, \mathrm{dist}_\varepsilon) \to (V, \mathrm{dist}_\varepsilon)$, such that

(1) $\sup\limits_{v \in V} \mathrm{dist}\big(v, f_\varepsilon(v)\big) \le \varepsilon$ for the original C-C metric dist in V.

(2) The Lipschitz constants $L(f_\varepsilon)$ and $L(f_\varepsilon^{-1})$ are bounded independently of ε.

This is likely to be true but may be hard to prove and also appears too refined at the present crude state of the study of C-C geometries. On the other hand the following more natural equivalence between $\mathrm{dist}_\varepsilon$ and $\mathrm{dist}'_\varepsilon$ comes quite effortlessly.

Equivalence Proposition. *There exist Lipschitz maps $f_\varepsilon : (V, \mathrm{dist}_\varepsilon) \to (V, \mathrm{dist}'_\varepsilon)$ and $f'_\varepsilon : (V, \mathrm{dist}'_\varepsilon) \to (V, \mathrm{dist}_\varepsilon)$ with the implied Lipschitz constants independent of ε such that the composed maps $f'_\varepsilon \circ f_\varepsilon$ and $f_\varepsilon \circ f'_\varepsilon$ are ε-close to the identity with respect to the metrics $\mathrm{dist}_\varepsilon$ and $\mathrm{dist}'_\varepsilon$ correspondingly. Moreover, these composed maps can be joined with the identity by the Lipschitz homotopies $V \times [0, \varepsilon] \to V$ where again the Lipschitz constants do not depend on ε.*

Proof. Since $(V, \mathrm{dist}'_\varepsilon)$ has ε^{-1}-bounded geometry one can smooth the identity map $(V, \mathrm{dist}_\varepsilon) \to (V, \mathrm{dist}'_\varepsilon)$ by using the center of mass construction in $(V, \mathrm{dist}'_\varepsilon)$ for the ε-balls coming from $(V, \mathrm{dist}_\varepsilon)$. Alternatively, one may triangulate $(V, \mathrm{dist}_\varepsilon)$ into standard (fat geodesic) ε-simplices and then approximate the identity map by a piecewise "linear" map. In either way one arrives at a Lipschitz map $f_\varepsilon : (V, \mathrm{dist}_\varepsilon) \to (V, \mathrm{dist}'_\varepsilon)$ which is ε-close to Id and f'_ε is constructed in the same way. Then the required ε-homotopies are constructed in a similar fashion with the additional ingredient of the local contractibility of C-C balls proved in 1.4.B. The details are left to the reader.

Exercise. Generalize the above to approximations $\mathrm{dist}_\varepsilon$ and $\mathrm{dist}'_{\varepsilon'}$ where ε' may be non-equal to ε.

Open problem. Characterize C-C spaces in terms of their (controlled) approximation by Riemannian manifolds with respect to the Hausdorff distance between metric spaces. Study other metric spaces admitting sufficiently nice approximations by Riemannian ones.

1.4.D$'$. Approximation by nerves. Every compact metric space can be ε-approximated by the nerve of a suitable ε-covering and in the C-C case this approximation enjoys some particularly nice features due to the following two facts.

I. *The doubling property for* C-C *balls:* $\mathrm{mes}\, B(2\varepsilon) \leq \mathrm{const}\, \mathrm{mes}\, B(\varepsilon)$.

II. *Contractibility of* $B(\varepsilon)$ *within* $B(2\varepsilon)$.

We shall use the following standard construction of a "good" covering by ε-balls. Take a maximal δ-separated net $\Sigma \subset V$ and cover V by the ε-balls with the centers in Σ for $\varepsilon = 3\delta$. Denote by V_ε the nerve of this cover which we equip with the piecewise Riemannian metric $\mathrm{dist}_\varepsilon$ where each simplex is isometric to the regular Euclidean ε-simplex of the corresponding dimension. One sees with I that $\dim V_\varepsilon$ is bounded by a constant independent of ε, and, in fact, bounded, for small ε, by const_n, $n = \dim V$. Moreover, the local geometry of V_ε is bounded by ε^{-1} in the following strong sense: every ε'-ball in $(V_\varepsilon, \mathrm{dist}_\varepsilon)$ for $\varepsilon \leq \varepsilon' \leq 1$, contains at most $M \leq \mathrm{const}_n (\varepsilon'/\varepsilon)^N$ simplices for $N = \dim_{\mathrm{Hau}} V$.

Next, we introduce a partition of unity of V inscribed into our cover where all functions can be made λ-Lipschitz with $\lambda \leq \mathrm{const}_n\, \varepsilon^{-1}$. Then the corresponding continuous map $f_\varepsilon : V \to V_\varepsilon$ is Lipschitz with $L(f_\varepsilon) \leq \mathrm{const}_n$. Finally, using II, we construct a continuous map $\overline{f}_\varepsilon : V_\varepsilon \to V$ which sends each 0-simplex to the center of the corresponding ball and such that the image of each simplex has diameter $\leq \mathrm{const}_n$. The composed maps $f_\varepsilon \circ \overline{f}_\varepsilon$ and $\overline{f}_\varepsilon \circ f_\varepsilon$ are both $C\varepsilon$-close to the identities for $C \leq \mathrm{const}_n$ and by II $\overline{f}_\varepsilon \circ f_\varepsilon : V \to V$ is homotopic to Id by a homotopy moving all points by at most $C\varepsilon$. This is almost true for $f_\varepsilon \circ \overline{f}_\varepsilon : V_\varepsilon \to V_\varepsilon$. Namely we invoke simplicial maps $M^\varepsilon_{\varepsilon'} : V_\varepsilon \to V_{\varepsilon'}$ for all $\varepsilon' > 5\varepsilon$ where each 1-simplex of V_ε goes to a 1-simplex of $V_{\varepsilon'}$ such that the corresponding ball $B(\varepsilon) \subset V$ is contained in $B(\varepsilon')$. Then we (obviously with I and II) have a homotopy of $M^\varepsilon_{\varepsilon'} \circ f_\varepsilon \circ \overline{f}_\varepsilon$ to Id in $V_{\varepsilon'}$ whenever ε'/ε is sufficiently large, i.e. $\geq \mathrm{const}_n$. Moreover, this homotopy can be realized by a Lipschitz map $V \times [0, \varepsilon] \to V_{\varepsilon'}$ with the implied Lipschitz constant independent of ε.

Now one can see that the above properties of V_ε make this approximation equivalent to the one considered earlier where the equivalence used in the above Equivalence Proposition should be argumented by stabilizing maps $M^\varepsilon_{\varepsilon'}$.

Remark. One can avoid $M^\varepsilon_{\varepsilon'}$ by slightly modifying V_ε to ensure their sufficient local contractibility (and thus the homotopy equivalence to V). But this is somewhat artificial and one should not be worried by $M^\varepsilon_{\varepsilon'}$, albeit they complicate the notations. In what follows we just pretend that $M^\varepsilon_{\varepsilon'}$ are not there.

Example. Pretend V_ε is homotopy equivalent to V and represent the cohomology classes in $H^*(V;\mathbb{R})$ by simplicial cocycles in V_ε. Then for each $h \in H^*(V;\mathbb{R})$ we minimize the ℓ_p-norm of these cocycles, denote this minimum by $\|h\|^\varepsilon_{\ell_p}$, and observe that the asymptotic rate of growth of $|h\|^\varepsilon_{\ell_p}$ for $\varepsilon \to 0$ gives us an interesting geometric invariant of $(V, \mathrm{dist}; h)$ similar to the norm on the Alexander-Spanier cochains considered in 3. Furthermore, the ε-asymptotics of the spectrum of the combinatorial Laplace operator in V_ε (as well as the ℓ_{pq}-spectra) gives us further invariants of V.

Combinatorial Hyperbolization. Take $\varepsilon_i = 5^{-i}$, $i = 1, 2, \ldots$ and let W be the joint infinite mapping cylinder of the maps $M^{\varepsilon_i}_{\varepsilon_{i-1}} : V_{\varepsilon_i} \to V_{\varepsilon_{i-1}}$. This W carries a natural hyperbolic metric where the (formerly ε) simplices are given the unit size, such that $V_i \subset W$ appears as a sphere of radius i (if we add to W the mapping cone of $V_1 \to \{\text{point}\}$) and V serves as the ideal hyperbolic boundary of W. Then W harbours many geometric invariants of V. In fact, all quasi-isometry invariants of W are quasi-conformal invariants of V. Most prominent of these are ℓ_p and ℓ_{pq}-cohomology of W (where W must be slightly regularized, i.e. made contractible and having all R-balls contractible within concentric λR-balls for a fixed large λ).

Finally, observe that W is quasi-isometric to the hyperbolic "cone" CV in 0.9 and one can think of W as a $p.\ell.$ regularization of CV which is essentially equivalent to the Rips complex of CV or the nerve of a covering of CV by unit balls.

1.4.D″. Anisotropic blow-up of $(\mathbf{V}, \mathbf{H})$. Let us return to the Riemannian metric $\mathrm{dist}_\varepsilon$ constructed in the proof of approximation theorem and look more closely at the rescaled (blown up) metric $\mathrm{dist}^*_\varepsilon = \varepsilon^{-1}\,\mathrm{dist}_\varepsilon$ (compare Remark (c)). This $\mathrm{dist}^*_\varepsilon$ (or, rather, the corresponding g^*_ε) is obtained from a given Riemannian metric g on V by scaling (blowing up) by ε^{-1} in the direction of $H = H_1$, by ε^{-2} in the direction of $H_2 \ominus_g H_1$, and so on up to ε^{-d} in $H_d \ominus_g H_{d-1}$. The metric $\mathrm{dist}^*_\varepsilon$, as we have seen, has uniformly bounded (local) geometry for $\varepsilon \to 0$ and if $Y_1, \ldots, Y_{n_1}, Y_{n_1+1}, \ldots, Y_{n_2}, \cdots, Y_{n_d = n}$, where $n_i = \operatorname{rank} H_i$, is an orthonormal frame of fields on V near some $v \in V$ which agree with the splitting $T(V) = \bigoplus_{i=1}^{d} (H_i \underset{g}{\ominus} H_{i-1})$, then the frame

$$a_\varepsilon\{Y_i\} = \varepsilon Y_1, \ldots, \varepsilon Y_{n_1}, \varepsilon^2 Y_{n_1+1}, \ldots, \varepsilon^2 Y_{n_2}, \ldots, \varepsilon^d Y_{n_{d-1}}, \ldots, \varepsilon^d Y_n,$$

becomes unitary (orthonormal) in the rescaled Riemannian metric g^*_ε. Now, as $\varepsilon \to 0$, the family $V_{\varepsilon^{-1}} = (V, v, \mathrm{dist}^*_\varepsilon, a_\varepsilon\{Y_i\})$ converges (in the pointed Lipschitz topology, see [G-L-P]) to a Riemannian manifold, say V_∞, with a distinguished orthonormal frame of fields, say Y_i^∞, whose commutators obviously satisfy

$$[Y_i^\infty, Y_j^\infty] = \sum_k c_{ijk} Y_k^\infty$$

where c_{ijk} are defined at the beginning of 1.4. Now, one clearly sees that this V_∞ can be identified with the (tangent) nilpotent Lie group N_v (see the beginning of 1.4.) which appears this time as a geometric limit of anisotropically rescaled copies of V rather than an abstract Lie group with a given Lie algebra.

Question. (Compare 4.11). What is the behavior of the standard analytic objects (fields), such as harmonic forms and spinors on $V_{\varepsilon^{-1}}$, for small $\varepsilon > 0$? What happens in the (anisotropic adiabatic) limit as $\varepsilon \to 0$. (These kinds of questions are often asked for fibrations over $V_{\varepsilon^{-1}}$ where the geometry of the fibers does not change as $\varepsilon \to 0$.)

1.4.E. Smoothing Lipschitz functions on C-C manifolds. Let (V, H) be a compact equiregular C-C manifold as in 1.4.D. with the commutator subbundles $H_1 = H \subset H_2 \subset \cdots \subset H_i \subset \cdots \subset H_d = T(V)$ and let $f : V \to \mathbb{R}$ be a Lipschitz function with the implied Lipschitz constant bounded by some number $\lambda > 0$.

Smooth approximation theorem. *There exists a family of smooth functions $f_\delta : V \to \mathbb{R}$ for small positive δ, say for $\delta \in]0, 1]$, satisfying the following two inequalities.*

(1) $|f - f_\delta| \le \delta$ *everywhere on* V,

(2) $\|\mathcal{D} f_\delta | H_i\| \le C \left|\lambda \delta^{-1}\right|^i$, $i = 1, \ldots d$, *everywhere on* V,

where $\mathcal{D} f_\delta | H_i$ denotes the differential of f_δ restricted to H_i and where C is a positive constant depending on (the geometry of) V but not on f.

Proof. Let V_ε be the Riemannian approximation to V with $\varepsilon = \delta/\lambda$ and let $K(v, v')$ be a standard Riemannian smoothing kernel on V on the scale slightly smaller than ε. Namely $K(v, v')$ is supported in the $C_1^{-1}\varepsilon$-neighbourhood of the diagonal in $V \times V$ for a fixed large constant C_1 and the Riemannian covariant derivatives of K are bounded by $C_2 \varepsilon^{-1}$. (For example, for each v one takes the Riemannian ball $B_v \subset V$ of radius $C_1^{-1}\varepsilon$ and defines $K(v, v')$ to be $\mu_v \operatorname{dist}(v', \partial B_v)$ for $v' \in B$ and $K(v, v') = 0$ for $v' \in V - B$, where the constant μ_v is chosen so that $\int_{B_v} K(v, v') dv' = 1$.)

Then the Riemannian smoothing of f with K, i.e.

$$f_\varepsilon(v) = \int f(v') K(v, v') dv',$$

is the desired approximation as an obvious argument shows.

Corollary. *Let W be a compact Riemannian manifold or, more generally, a complete manifold with bounded local geometry. Then every λ-Lipschitz map $f : V \to W$ is homotopic to a smooth map $f_1 : V \to W$, such that $\|\mathcal{D}f_1|H_i\| \leq C\,\lambda^i$, $i = 1, \dots, d$.*

1.4.E′. On the homotopy count of Lipschitz maps. Let W be the sphere S^n for $n = \dim V$ and let us estimate the degree of a λ-Lipschitz map $V \to W$ f in terms of λ. We replace f by the above f_1 and use the bound of $\deg f_1$ by the norm of the differential on the n forms. Thus we get

$$\deg f = \deg f_1 \leq \text{const} \,\| \wedge^n \mathcal{D}f\| \leq \text{const}_1 \prod_{i=1}^{d} \lambda^{im_i}$$

for $m_i = \operatorname{rank} H_i/H_{i-1}$. This agrees with the formula for $N = \dim_{\text{Hau}} V$, i.e.

$$\deg f \leq \text{const}_1\,\lambda^N, \text{ for the Lipschitz constant } \lambda = \lambda(f).$$

Notice that this bound is sharp as V contains $\approx \lambda^N$ disjoint balls of radii λ^{-1} where each can be λ-Lipschitz mapped onto S^n with degree 1.

The above argument also allows an improvement of the estimate in 0.5.E of the number $Nm(\lambda)$ of the homotopy classes of maps $f : V \to W$ in terms of $\lambda = L(f)$, where W is an arbitrary compact simply connected Riemannian manifold. Let us spell it out for maps of the contact C-C sphere S^3 to S^2. Here the number $Nm(\lambda)$ is controlled by the Hopf invariant $h(f)$ which is defined as follows (compare 2.5). Fix the normalized area form ω on S^2, pull it back to S^3 by f, i.e. take the 2-form $f^*(\omega)$ on S^3. This "integrates" to some 1-form α on S^3, where "integrates" means $d\alpha = \omega$. Then

$$h(f) = \int_{S^3} \alpha \wedge f^*(\omega).$$

If S^3 were Riemannian, this would imply $h(f) \leq \text{const}\,\lambda^4_{\text{Ri}}$ since $\|f^*(\omega)\| \leq \text{const}_1 \,\| \wedge^2 \mathcal{D}f\| \leq \text{const}_1\,\lambda^2_{\text{Ri}}$ and since (this is a key point) one could find α with $\|\alpha\| \leq \text{const}_2\,\|f^*(\omega)\|$. Since the contact C-C metric is $C^{\frac{1}{2}}$-equivalent to the Riemannian one this implies the bound $h(f) \leq \text{const}\,\lambda^8$ for C-C Lipschitz constant λ as was indicated in 0.5.E. But now the above corollary yields the bound $\|f_1^*(\omega)\| \leq \text{const}\,\lambda^3_{\text{Ri}}$ for the Riemannian norm of a suitable smooth f_1 approximating f which leads to the improved bound

$$h(f) = h(f_1) \leq \text{const}_3\,\lambda^6.$$

Question. What is the exponent r in the true bound of $h(f)$ by $C\lambda^r$ for $\lambda \to \infty$? (The above shows that $r \leq 6$ and it is easy to see that $r \geq 4$). A closely related question concerns an estimate of $h(f)$ of an f which is λ-Lipschitz with respect to the Riemannian ($SU(2)$-invariant) metric g_t approximating the C-C metric on S^3 (see 0.8.B), where the asymptotics of h now depend on $t \to \infty$ as well as on $\lambda \to \infty$ (as $d = 2$, this g_t equals the above g_ε^* for $\varepsilon = t^{-1}$).

1.5. Anisotropic metrics beyond Carnot-Carathéodory. Let us indicate a class of metrics on V where the ball-box theorem is built-in into the definition. We consider a Euclidean vector bundle X over V endowed, besides the projection $X \to V$, with a smooth ("exponential") map $E : X \to V$ which fixes the zero section and such that the differential $\mathcal{D}E$ maps the tangent space of each fiber X_v at the origin, say $T_0(X_v)$, *onto* $T_v(V)$. Then we equip X with a fiber preserving and fiberwise linear automorphism $A : X \to X$ with norm ≤ 1 (i.e. distance decreasing) on each fiber X_v, and we say that a metric dist on V *agrees* with (X, E, A) if for some $\rho = \rho_i$ and all $i = 1, 2, \ldots$ the balls $B_v(\rho) \subset V$ are equivalent to the E-images of the ellipsoids $A^i(B_0(1)) \subset X_v$ for the unit (Euclidean) ball $B_0(1)$ in the fiber X_v, for all $v \in V$, where the equivalence means, as earlier, the inclusion

$$E\,A^{i+c}\big(B_0(1)\big) \subset B_v(\rho) \subset E\,A^{i-c}\big(B_0(\rho)\big).$$

Then a metric on V is called (pleasantly) *anisotropic* if it agrees with some (X, E, A). This generalizes C-C metrics as well as those defined in [N-S-W]. In particular, the restriction of C-C metrics to submanifolds are anisotropic in our sense.

It is unclear if there are anisotropic metrics significantly different from those defined in [N-S-W]. To figure this out one should be able to decide for which (X, E, A) there exists a metric (or quasi-metric) on V agreeable with these data.

Let us indicate another generalization by first recalling that the ball-box theorem applies to the metrics associated to systems of vector fields on V. Such fields, say $X_1, \ldots, X_m$, viewed as linear functions on the cotangent bundle $T^*(V)$, define a semipositive quadratic form, namely $h = \sum_{i=1}^{m} (X_m)^2$ on $T^*(V)$. Then one observes that every suitably generic smooth semidefinite form h on $T^*(V)$, which does not have to be a sum of squares of *smooth* linear functions (vector fields), also defines a metric on V, where the geometry of the small balls is somewhat similar to that for $h = \Sigma(X_i)^2$, as was proven by Fefferman and Phong (see [Fe-Ph] and [Je-Sa]), but the argument is significantly harder than that for $h = \Sigma(X_i)^2$. (All that was explained to me by N. Varopoulos.) We conclude by suggesting a study of more general infinitesimally defined classes of horizontal curves and associated metrics, e.g. corresponding to semipositive forms on $T^*(V)$ of (even) degree > 2. (A geometric aspect of the problems is related to the structure of the convex hull of the spaces of maps $V \to S \subset \mathbb{R}^n$ for a smooth submanifold $S \subset \mathbb{R}^n$, where one of the questions reads

$$\text{conv.hull}(\text{maps}(V \to S)) \stackrel{?}{=} \text{maps}\,(V \to \text{conv.hull}\,(S)).$$

(Compare pp. 170, 205 and 206 in [Gro_{PDR}].)

2. Hypersurfaces in C-C spaces

Much of the classical analysis in $\mathbb{R}^n$ (and in Riemannian manifolds) deals (explicitly) with functions and (often implicitly) with hypersurfaces which may serve as levels of functions. A typical example is provided by the Sobolev inequalities reflecting the isoperimetry of hypersurfaces in $\mathbb{R}^n$. We shall see in this section that many classical Euclidean results extend to C-C spaces where the main technical tools are provided by H-horizontal curves, often appearing as orbits of H-horizontal fields, and by certain closed $(n-1)$-forms vanishing on H. Our major applications concern Sobolev-type spaces of maps $V \to W$ associated to H, where we establish C-C counterparts of corresponding Riemannian results due to Karen Uhlenbeck.

2.1. A lower bound on the Hausdorff dimension of a hypersurface $V' \subset V$. Let V' be a compact subset in an equiregular C-C manifold V of topological dimension $\dim V' = n-1$ for $n = \dim V$. We show below that the Hausdorff dimension of V' with respect to the induced C-C metric satisfies

$$\dim_{\text{Hau}} V' \geqslant \dim_{\text{Hau}} V - 1, \tag{$*$}$$

as was stated in § 0.6.

The first result of this kind was obtained by P. Pansu who observed that such lower bound on $\dim_{\mathrm{Hau}}$ of hypersurfaces follows from his (and Varopoulos) isoperimetric inequality (see [Cor] and p.187 in [$\mathrm{Gro}_{\mathrm{AI}}$]). In fact, the lower bound $(*)$ on $\dim_{\mathrm{Hau}} V'$ is a more basic and elementary fact than the isoperimetric inequality and we shall establish it by a simple direct argument. Then we shall elaborate this argument in order to obtain a lower bound on the Hausdorff measure of V', needed for the isoperimetric inequality.

Proof of $(*)$ for equiregular polarizations H. We shall use the following

Characteristic property of $\dim = n-1$. *If V' is a compact subset of topological dimension $n-1$ in an n-dimensional manifold V, then there exists an $\varepsilon > 0$ and a simple curve c in V i.e. an embedding $c : [0,1] \to V$, such that every continuous curve $f : [0,1] \to V$ which is ε-close to c (in the sense of the inequality*

$$\mathrm{dist}_V(c(t), f(t)) \leqslant \varepsilon\,, \quad t \in [0,1],$$

for a fixed distance in V) meets V', i.e. $f([0,1]) \cap V' \neq \varnothing$.

This follows from the elementary homological dimension theory (see [Nag] and 4.5; a non-fastidious reader may just use the above intersection property as a definition of the dimension).

We use an approximation of c by piecewise horizontal curves and then make them smooth as in 1.2.B. In fact the argument in 1.2.B provides a smooth family of smooth horizontal curves close to c, say $f(t,x)$ for $t \in [0,1]$ and x running over a (small) ball $B \subset \mathbb{R}^{n-1}$, such that the global map $f : [0,1] \times B \to V$ is smooth generic with a (possible) folding along some smooth hypersurface $W \subset [0,1] \times B$ as the only singularity (where f is not an immersion). If V' contains an "essential piece" of W, i.e. if $f^{-1}(V') \cap W$ contains a non-empty open subset $W' \subset W$, then $f(W') \subset V'$, being a *smooth* hypersurface in V must be almost everywhere *transversal* to H (by the connectivity property for H-horizontal curves) which makes

$$\dim_{\mathrm{Hau}} f(W') \geqslant \dim_{\mathrm{Hau}} V - 1$$

by the discussion in 0.6.A.

Now we turn to the essential case where W has nothing to do with V' and then we may assume that f is an immersion, and moreover, a smooth embedding $[0,1]\times B\subset V$, where all curves $[0,1]\times b\subset V$ are H-horizontal and intersect V'. Thus the projection $p:[0,1]\times B\to B$ is surjective on $V_0'=V'\cap[0,1]\times B$ and the levels of p are horizontal. Now we recall that every small ε-ball in $[0,1]\times B\subset V$ looks as a box with the edges $\varepsilon^{\deg i}$, $i=1,\dots,n$ where $\deg i=1$ for $i=1,\dots,n_1=$ rank H, $\deg i\geqslant 2$ for $i=n_1+1,\dots,n$, where the first edge of the box may be assumed equal to a segment of some level of p (see 1.3.A). It follows that the p-image of this box is a box with the edges $\approx\varepsilon^{\deg i}$, for $i=2,\dots,n$. In particular, the volume of this image satisfies

$$(n-1)\text{-volume }(p(\text{box}))\lesssim\varepsilon^{-1}(n\text{-volume (box)}).$$

Now, if a system of ε_j-boxes B_j, $j=1,2,\dots,$ cover X_0, then their projections cover B and so $\Sigma_j\ \varepsilon_j^{-1}\operatorname{Vol}B_j\geqslant$ const. This implies the desired inequality

$$\dim_{\text{Hau}}V'\geqslant\dim_{\text{Hau}}V_0'\geqslant\dim_{\text{Hau}}V-1,$$

by the discussion in 0.6.

2.1.A. A systolic bound on $\operatorname{mes}_{N-1}V'$ of $(*)$. Suppose V' is $(n-1)$-dimensionally *essential* in V, i.e. it can not be homotoped to a subset of dimension $\leqslant n-2$. This is the case, for example, where V' supports a (Čech) cycle of dimension $n-1$ non-homologous to zero in V. Then, if V is compact, *the $(N-1)$-dimensional Hausdorff measure of V' for $N=\dim_{\text{Hau}}V$ is bounded from below by a constant $s=s(V,$ C-C dist$)$*, (called the $(n-1)$-*systole* of V, see [Gro$_{\text{SISI}}$] for a survey of systoles). This is proven with an obvious modification of the above argument.

Exercise. Show that every closed curve in V can be approximated by a smoothly immersed closed horizontal curve. (This is somewhat more than needed for the proof of our claim.)

Remark. If V' supports an essential $(n-1)$-cycle in V modulo some subset $V_0\subset V$, then again we have a bound on the measure of V' by some $s=s(V,V_0,\text{dist})$ and such a bound was, in fact, instrumental in our proof of $(*)$.

Exercise. Assume V admits an equisingular stratification by V_ν (see 1.3.A), such that $\dim(V'\cap V_\nu)=\dim V_\nu-1$ for certain ν and bound from below the Hausdorff dimension of $V'\cap V_\nu$.

2.2. Lower systolic bounds for families of metrics on compact manifolds V. Let $A : T(V) \to T(V)$ be a smooth automorphism (fixing V) and let us look at the Riemannian metrics $A^t(g)$, $t = 1, 2, 3, \ldots$, for a fixed g as $t \to \infty$ (compare 0.8). To simplify the matter we assume the absolute values of the eigenvalues λ_i of A and their multiplicities are constant on V. Then the volume growth of (V, g_t) for $g_t = (A^t)^* g$ can be expressed in terms of the eigenvalues of A by

$$\lim_{t\to\infty} (\mathrm{Vol}(V, g_t))^{\frac{1}{t}} = \prod_{i=1}^{n} |\lambda_i|.$$

Now, in order to measure the volume growth of submanifolds $X \subset V$ of positive codimension we recall the following

Definitions. The *volume of a k-dimensional* (integral) *homology class* is the infimum of the volumes of the cycles realizing this class. Then the ($\mathbb{Z}$-homological) *systole* $\mathrm{sys}_k(V, g)$ is defined as the infimum of these volumes over all non-zero homology classes. Finally, the *absolute* (homotopy) *systole* absys_k is defined as the infimum of the volumes of k-dimensional subsets in V which can not be contracted to $(k-1)$-dimensional subsets in V. (Here "subset" means a piecewise smooth sub-polyhedron in V.) It is clear that $\mathrm{absyst}_k \leqslant \mathrm{syst}_k$.

Let $m \leqslant n$ be the first integer such that $\lambda_{m+1} > \lambda_m$ and the (eigen) subbundle H_m corresponding to $\lambda_1, \ldots, \lambda_m$ Lie-generates $T(V)$.

Theorem. *The absolute systole of codimension one of (V, g_t) for $t \to \infty$ is bounded from below according to the inequality*

$$\lim_{k\to\infty} \infty \, (\mathrm{absys}_{n-1}(V, g_t))^{\frac{1}{t}} \geqslant |\lambda_m^{-1}| \prod_{i=1}^{n} |\lambda_i|. \qquad (\star)$$

The proof of $(\star)$ is the same as that of the above inequalities $(*)$ and $(+)$.

Remark. The inequality $(\star)$ is by no means sharp. For example if V is the 2-torus and $H_1 \subset T(V)$ is the standard 1-foliation with slope α, then the asymptotics of $\mathrm{sys}_1(V, g_t)$ depends on the continuous fraction expansion of α.

Problem. Find the actual asymptotics of $\operatorname{sys}_{n-1}(V, g_t)$ for *generic* A.

2.2.A. A lower systolic bound via closed horizontal $(n-1)$-forms. Our lower bounds on hypersurfaces in V used families of H-horizontal curves (for a given $H \subset T(V)$ which Lie spans $T(V)$) or, equivalently smooth maps $p : V \to B^{n-1}$ with H-horizontal fibers. Notice that such a p pulls back (necessarily closed) $(n-1)$-forms on B^{n-1} to closed $(n-1)$-forms ω on V which *vanish* on H, i.e. vanish on every (local) hyperplane field containing H, and such forms, called *H-horizontal*, can be then used to study hypersurfaces in $X \subset V$. In fact, one can produce sufficiently many such forms by a purely algebraic argument. Namely we have the following

Linear Lemma. *If the subbundle $H \subset T(V)$ Lie generates $T(V)$ then every $(n-1)$-dimensional de Rham cohomology class in V can be represented by a closed horizontal $(n-1)$-form ω.*

First (not quite linear) proof. We may assume (by passing to the double cover of V if necessary) that V is oriented and so the cohomology $H^{n-1}(V;\mathbb{R})$ is dual to $H_1(V;\mathbb{R})$. Then every integral class in $H_1(V;\mathbb{R})$ can be realized by a closed horizontal curve c (see the above exercise) which gives us a closed $(n-1)$-current, called c^*, representing the class $[c]^* \in H^{n-1}(V;\mathbb{R})$ where the latter "$*$" denotes the Poincaré duality. Now, in order to pass from currents to forms one needs some smoothing or diffusion of currents preserving H-horizontality. This is easy if V admits a transitive action of a connected group G preserving H as one can diffuse the current c^* by taking $\int_G c^* d\mu$, where $d\mu$ is a smooth measure with a compact support on G (localized near $\mathrm{id} \in G$). For example, this diffusion is available if our polarization H is a constant structure. In the general case, the diffusion is achieved with a smooth family of horizontal curves, say $c_b \subset V$, $b \in B$, such that the corresponding map $S' \times B \to V$ (for c_b parametrized by the circle S') is a submersion. The existence of such family is proven in the same manner as of an individual c (the details are left to the reader).

Second (purely linear) proof. We still assume V is oriented and take a non-vanishing oriented volume form Ω on V. Then the *interior product* with Ω establishes an isomorphism between vector fields X and exterior $n-1$ forms, i.e.

$$X \leftrightarrow X \cdot \Omega,$$

and similarly bivectors correspond to $(n-2)$-forms,

$$X \wedge Y \leftrightarrow (X \wedge Y)\cdot\Omega.$$

Closed $(n-1)$-forms correspond in this picture to *divergence free* vector fields, where the divergence δX of X is the function defined by the equality

$$L_X\,\Omega = (\delta X)\Omega,$$

where L_X denotes the Lie derivative. We recall the formula

$$d((X \wedge Y)\cdot\Omega) = [X,Y]\cdot\Omega + Y\cdot L_X\,\Omega - X\cdot L_Y\,\Omega$$

which implies that the field

$$[X,Y] + \delta(X)Y - \delta(Y)X$$

has zero divergence and, moreover, corresponds to an *exact* $(n-1)$-form. It follows, that for all functions a, the field

$$a[X,Y] + (Xa + a\delta(X))Y - \delta(aY)X$$

corresponds to an exact $(n-1)$-form, or in other words $a[X,Y]$ equals $(Xa + a\ \delta(X))Y - \delta(aY)X$ modulo (the fields corresponding to) exact forms. (The latter expression is antisymmetric in X and Y, as it should be, since $Xa + a\ \delta(X) = \delta(aX)$.)

Next, we look at $H \subset T(V)$ and observe that $(n-1)$-forms vanishing on H correspond to vector fields sitting in H. Thus, to prove the lemma, we must find a divergence free H-horizontal field in a given cohomology class. We pick up some fields $X_1, \ldots, X_s$ spanning H (here s may be greater than rank H) add to these X_i their successive commutators, say X_j, $j = s+1, \ldots, r$, which span $T(V)$ and observe that every cohomology class in $H^{n-1}(V;\mathbb{R})$ can be represented by a divergence free field of the form $\Sigma_{i=1}^r\ a_i\ X_i$. But the above formulae allow us to replace every (commutator) term in this sum with $i > s$ by (cohomologically) equivalent lower terms and thus we obtain a desired divergence free representative of the form $\Sigma_{i=1}^s\ a_i'\ X_i$. ∎

Remarks

(a) The above algebraic discussion can be neatly expressed with the differential operator d^H which is obtained by composing the exterior differential $d : \Lambda^{n-2}(V) \to \Lambda^{n-1}(V)$ with the quotient homomorphisms $\Lambda^{n-1}(V) \to \Lambda^{n-1}(V)/H^\perp$ where $H^\perp$ denotes the (sheaf of sections of the) subbundle of $(n-1)$-forms vanishing on H. Namely, we have shown that this d^H is a *surjective* operator. Moreover, our argument provides another differential operator, say $\delta^H : \Lambda^{n-1}(V)/H^\perp \to \Lambda^{n-2}(V)$ (which order equals the degrees of commutators needed to span $T(V)$) such that $d^H \delta^H = \mathrm{Id}$. Furthermore, the "coefficients" of δ^H are expressed by some universal rational functions in the components of some jet of H represented by a section $V \to \mathrm{Gr}_{n_1}(T(V))$ for $n_1 = \mathrm{rank}\, H$. (This is not surprizing as the P.D.E. system $d^H\varphi = \omega$ is underdetermined in our case and the theorem proven in 2.3.8 (E) in [Gro$_{\mathrm{PDR}}$] predicts the existence of δ for generic H.)

Exercise. Show that the existence of a differential operator δ^H making $d^H\delta^H = \mathrm{Id}$ is not only implied by the commutator generation property of H but also implies this property of H.

(b) Observe that our second proof is purely algebraic (modulo de Rham theorem), technically trivial and self-contained unlike the first proof based on an elaborated version of Chow connectivity theorem (paradoxally, the first proof is geometrically obvious while the second one makes no lasting impression on a geometrically moulded mind).

(c) The abundance of closed $(n-1)$-forms vanishing on H is dual to what happens to 1-forms: every closed 1-forms vanishing on H (obviously) equals zero. In fact, every closed 1-current φ satisfying $X\varphi = 0$ for all H-horizontal fields X necessarily vanishes. Furthermore, there exists a differential operator δ'_H acting from $\Lambda^2(V)$ to the subspace $H_\perp \subset \Lambda^1(V)$ of 1-forms vanishing on H, such that

$$(\delta'_H\ \omega) \wedge \lambda = \omega \wedge (\delta_H\ \lambda) \mod d\Lambda^{n-1}(V),$$

where ω is an arbitrary 2-form and λ is an $(n-1)$-form modulo $H^\perp$. In fact, this formula (essentially) defines δ'_H as a formal adjoint of δ^H which makes $\delta'_H\ d = \mathrm{id}$ since the differential d on 1-forms is a formal adjoint of d on $(n-2)$-forms and $d^H\delta^H = \mathrm{id}$.

(d) The operators d_H and d^H make sense for forms of all degrees, see 3.3 and 4.1.E.

Now let us return to the metrics $g_t = (A^t)^* g$ on V and notice that the g_t-norms of the forms ω vanishing on H satisfy for the same m as in $(\star)$,

$$\|\omega\|_{g_t} \leqslant \mathrm{const}\, |\lambda_m|^t \Big/ \prod_{i=1}^{n} |\lambda_i|^t. \qquad (-)$$

(sheaf of sections of the) subbundle of $(n-1)$-forms vanishing on H. Namely, we have shown that this d^H is a *surjective* operator. Moreover, our argument provides another differential operator, say $\delta^H : \Lambda^{n-1}(V)/H^\perp \to \Lambda^{n-2}(V)$ (which order equals the degrees of commutators needed to span $T(V)$) such that $d^H\delta^H = \text{ Id}$. Furthermore, the "coefficients" of δ^H are expressed by some universal rational functions in the components of some jet of H represented by a section $V \to \text{ Gr}_{n_1}(T(V))$ for $n_1 = \text{rank } H$. (This is not surprizing as the P.D.E. system $d^H\varphi = \omega$ is underdetermined in our case and the theorem proven in 2.3.8 (E) in [Gro$_{\text{PDR}}$] predicts the existence of δ for generic H.)

Exercise. Show that the existence of a differential operator δ^H making $d^H\delta^H = \text{ Id}$ is not only implied by the commutator generation property of H but also implies this property of H.

(b) Observe that our second proof is purely algebraic (modulo de Rham theorem), technically trivial and self-contained unlike the first proof based on an elaborated version of Chow connectivity theorem (paradoxally, the first proof is geometrically obvious while the second one makes no lasting impression on a geometrically moulded mind).

(c) The abundance of closed $(n-1)$-forms vanishing on H is dual to what happens to 1-forms: every closed 1-forms vanishing on H (obviously) equals zero. In fact, every closed 1-current φ satisfying $X\varphi = 0$ for all H-horizontal fields X necessarily vanishes. Furthermore, there exists a differential operator δ'_H acting from $\Lambda^2(V)$ to the subspace $H_\perp \subset \Lambda^1(V)$ of 1-forms vanishing on H, such that

$$(\delta'_H\ \omega) \wedge \lambda = \omega \wedge (\delta_H\ \lambda) \mod d\Lambda^{n-1}(V),$$

where ω is an arbitrary 2-form and λ is an $(n-1)$-form modulo $H^\perp$. In fact, this formula (essentially) defines δ'_H as a formal adjoint of δ^H which makes $\delta'_H\ d = \text{ id}$ since the differential d on 1-forms is a formal adjoint of d on $(n-2)$-forms and $d^H\delta^H = \text{ id}$.

(d) The operators d_H and d^H make sense for forms of all degrees, see 3.3 and 4.1.E.

Now let us return to the metrics $g_t = (A^t)^*g$ on V and notice that the g_t-norms of the forms ω vanishing on H satisfy for the same m as in $(\star)$,

$$\|\omega\|_{g_t} \leqslant \text{const}\,|\lambda_m|^t \Big/ \prod_{i=1}^{n} |\lambda_i|^t. \qquad (-)$$

If ω is closed then the volume of every homology class $C \in H_{n-1}(V;\mathbb{R})$ is bounded from below by

$$(\mathrm{Vol}_{g_t} C)\ \|\omega\|_{g_t} \geqslant \mathrm{const}' \left|\int_C \omega\right|.$$

Since for each C there is a closed ω vanishing on H_m with $\int_C \omega \neq 0$ by the linear lemma, we obtain once more the bound

$$\mathrm{Vol}_{g_t} C \gtrsim \left(\prod_{i=1}^{n} |\lambda_i|\right)^t \Big/ |\lambda_m|^t.$$

2.3. Isoperimetric inequality. We show in this section that the $(N-1)$-dimensional measure of a closed hypersurface S is a C-C manifold V of Hausdorff dimension N is minorized under certain restrictions on V, by the N-dimensional measure of the domain $D \subset V$ bounded by S as follows

$$\mathrm{mes}_{N-1} S \geqslant \mathrm{const}_V (\mathrm{mes}_N D)^{\frac{N-1}{N}}. \qquad (*)$$

For example, we shall see that $(*)$ is valid for all simply connected nilpotent Lie groups with left-invariant C-C metrics as well as for all compact equiregular manifolds V.

This result for the Heisenberg group is due to Pansu (see [$\mathrm{Pan}_{\mathrm{InIs}}$]) and in the general case to Varopoulos (see [Var]) who uses a rather elaborate random walk argument and proves $(*)$ in the (equivalent) form of a Sobolev inequality for smooth functions on V. Our argument presented below is more direct, using only the ball-box theorem and the issuing Vitali covering lemma, and it does not presupposes any a priori regularity of S. The basis for a relation between mes_N and mes_{N-1} is established by the following lemma which is implied by the ball-box theorem.

2.3.A. Flow tube estimate for small C-C balls. *Let X be a smooth H-horizontal vector field on an equiregular C-C manifold (with the underlying polarization $H \subset T(V)$), take a C-C ball $B = B(v,\varepsilon) \subset V$ and consider the $X(t)$-orbit $T = T(v,\varepsilon,\ell)$ of B under the flow $X(t)$ for $t \in [0,\ell]$, i.e. $T = \{X(t)(v), t \in [0,\ell], v \in B\} \subset V$. Then the top-dimensional Hausdorff measure of T satisfies in the limit for the radius of B going to zero the following inequality*

$$\lim_{\varepsilon\to 0}\ (\mathrm{mes}\, T \,/\, \varepsilon^{-1}\, \mathrm{mes}\, B) \leqslant \ell\, \mathrm{const}_X\ (v,\ell),$$

where the function $\mathrm{const}_X(v,\ell)$ *is uniformly bounded on compact subsets in* $V\times\mathbb{R}_+$.

Proof. The ball-box theorem implies that the Hausdorff measure is equivalent to the Riemannian (Lebesgue) measure and that the transversal section to the tube T at v has $(n-1)$-dimensional Riemannian measure $O(\varepsilon^{-1}\,\mathrm{mes}\,B)$. ∎

2.3.A′. Corollary: Flow tube estimate for hypersurfaces. *Let* X *be as above and let* S *be a hypersurface contained in a compact subset* $V_0\subset V$. *Then the* $X(t)$*-orbit* $T(S,\ell)$ *of* S *for* $t\in[0,\ell]$, *satisfies*

$$\mathrm{mes}\,T(S,\ell)\leqslant \ell\,\mathrm{const}_{X,V_0}\,(\ell)\,\mathrm{mes}_{N-1}\,S,$$

for $N=\dim_{\mathrm{Hau}}V$, *where* $\mathrm{const}(\ell)$ *is bounded on each segment* $[0,\ell_0]\subset\mathbb{R}$.

Proof. Use the definition of the Hausdorff measure of S recall the relation $\mathrm{mes}\,B(\varepsilon)\approx\varepsilon^N$ and apply 2.3.A.

Using 2.3.A′ one may bound from below the measure $\mathrm{mes}_{N-1}\,S$ of a closed hypersurface S in V in terms of the volume (or, equivalently of mes_N) of the domain $D\subset V_0$ bounded by S as follows. Suppose that every orbit of $X(t)$ starting from a point $v\in V_0$ for $t=0$ leaves V_0 at some moment $t_0=t_0(v)\leqslant\ell_0=\ell_0(V,X)$. Then, D is contained in the flow tube $T(S,\ell_0)$ and so

$$\mathrm{mes}_N\,D\leqslant\mathrm{const}\,\ell_0\,\mathrm{mes}_{N-1}\,S$$

for $\mathrm{const}=\mathrm{const}(V_0,X)$.

We want a similar bound in the relative case where S may have a boundary which is contained in the boundary of V_0. Here one should take proper care in choosing X so that sufficiently many orbits of X intersect S which now constitutes only a part of the boundary of D. For example if D is a flow tube for X starting and terminating on ∂V_0, then X is useless for our purpose, but we shall see below how to find finitely many fields X_i, $i=1,\dots,k$ such that at least one of them will serve our purpose. We denote by $T(U,\ell,\{X_i\})$ the *iterated orbit* of a given subset $U\subset V$ defined as the union of piecewise smooth curves in V which issue from U and which consist of k segments where the i-th segment equals a piece of the orbit of X_i of length $\leqslant\ell$. It is clear (by Chow connectivity theorem) that there exist smooth H-horizontal fields X_i, $i = 1, \dots, k$, such that the iterated orbit of every point v in our compact subset V_0 is sufficiently large, i.e. for each positive $\ell \leqslant \ell_0 = \ell_0(V_0) > 0$ the "orbit" $T(v, \ell, \{X_i\})$ contains the C-C ball B in V around v of radius ℓ. (Notice that $T(v, \ell, \{X_i\})$ is contained in the ball $B = B(v, R_+)$ for $R_+ = \mathrm{const}\,\ell$.)

Now take some point $v \in V_0$ and suppose that some R-ball $B = B(v, R)$ is contained in V_0. Assume, moreover, that the "orbit" $T(B, \ell, \{X_i\})$ is also contained in V_0 for $\ell = 2R$ (for which it suffices to assume that the concentric ball of radius $R' = \mathrm{const}\, R$ is contained in V_0). We claim that for every sufficiently small subset $D \subset V_0$ a definite percentage of the measure of $D \cap B$ can be moved away from D by some field X_i in time $\equiv \ell = 2R$. Namely, we have the following trivial

2.3.B. Measure moving lemma. *There exist positive constants ℓ_0 and α (depending on V_0 via X_i), such that for every $R \leqslant \ell_0/2$ and every measurable subset $D \subset V_0$ with $\mathrm{mes}\, D \leqslant \frac{1}{2} \mathrm{mes}\, B$ for $B = B(v, R)$, there exist some field among X_i, say X_{i_0} and a measurable subset $D_0 \subset D$, such that*

(1) $\mathrm{mes}\, D_0 \geqslant \alpha\, \mathrm{mes}\, D \cap B$

(2) every X_{i_0}-orbit of length $\ell = 2R$ issuing from D_0 is contained in V_0 but is not contained in D.

Proof. If otherwise, the "orbit" $T(D \cap B, \ell, \{X_i\})$ would be "almost contained" in D which is impossible as this orbit contains $B(v, R)$ which has significantly greater volume than D.

2.3.B′. Corollary: Local isoperimetric inequality. *The part S of the boundary of D strictly inside V_0, i.e. $S = \partial D \cap \mathrm{Int}\, V_0$, is bounded from below by*

$$\mathrm{mes}_N(D \cap B) \leqslant \beta\ R\ \mathrm{mes}_{N-1} S, \qquad (\star)$$

for some constant $\beta = \beta(V_0)$.

Proof. Apply 2.3.A′ to X_{i_0}.

Important remark. Notice that all action takes place in the concentric ball $B' \supset B$ of radius $R' \leqslant \mathrm{const}_{V_0} R$ and so the inequality $(\star)$ may be strengthened to

$$\mathrm{mes}_N(D \cap B) \leqslant \beta\ R\ \mathrm{mes}_{N-1}(S \cap B'). \qquad (\star)'$$

Now we want to prove the global scale invariant inequality $\mathrm{mes}_N D \lesssim (\mathrm{mes}_{N-1} S)^{\frac{N}{N-1}}$ by covering a substantial part of D by balls of various size to which $(\star)'$ applies. This is achieved with the following version of Vitali covering lemma which is quite standard in this framework and follows from the doubling property for the concentric balls (this was clarified to me by N. Varopoulos),

$$\mathrm{mes}\, B(v, 2R) \leqslant \mathrm{const}_{V_0}\, \mathrm{mes}\, B(v, R)$$

for all $v \in V_0$ and $R \leqslant \mathrm{diam}\, V_0$, where this property is immediate with the ball-box theorem.

2.3.C. Vitali covering lemma. *For each $\lambda > 1$ there exist positive numbers $\mu > 0$ and $\delta > 0$, such that for every measurable subset $D \subset V_0$ of measures μ there exist balls $B_i = B(v_i, R_i) \subset V$, $i = 1, \ldots, m$ around some points $v_i \in V_0$ satisfying the following properties.*

(1) The balls B_i are mutually disjoint; moreover, the concentric balls $B(v_i, \lambda R_i)$ are also mutually disjoint.

(2) The balls B_i contain at least δ-part of the total measure of D, i.e.

$$\sum_{i=1}^{m} \mathrm{mes}(B_i \cap D) \geqslant \delta\, \mathrm{mes}\, D.$$

(3) The intersection $B_i \cap D$ is δ-substantial in each ball, i.e.

$$\mathrm{mes}(B_i \cap D) \geqslant \delta\, \mathrm{mes}\, B_i\ , \ i = 1, \ldots, m.$$

(4) The intersections of D with the (larger) λR_i-balls $B(v_i, \lambda R_i)$ are somewhat smaller than B_i, i.e.

$$\mathrm{mes}(D \cap B(v_i, \lambda R_i)) \leqslant \frac{1}{2}\, \mathrm{mes}\, B(v, R_i) \quad \text{for} \quad i = 1, \ldots, \tau.$$

Proof. Consider concentric balls $B(v, R_j)$ for $v \in D$ of radii $R_j = 2^{-j} R_0$ for $R_0 = \mathrm{Diam} V_0$ and $j = 1, 2, \ldots$. If v is a density point of D, then $\mathrm{mes}(D \cap B(v, R_j)) \geqslant \frac{1}{2} \mathrm{mes}\, B(v, R_j)$ for large j. If $\delta > 0$ is small and $\mu < \delta \,\mathrm{mes}\, B(m, R_1)$, then there exists first j, say j_0, such that $\mathrm{mes}(B(v, R_{j_0}) \cap D) \geqslant \delta \,\mathrm{mes}\, B(v, R_{j_0})$. Furthermore, by making μ and δ smaller, we arrive at the situation where $\lambda R_{j_0} < R_1$ and the intersection of D with $B(v, \lambda R_{j_0})$ is somewhat smaller than B_j in the sense of (4). Thus for each density point $v \in D$ we constructed a ball $B(v, R = R(v))$ satisfying the above (3) and (4) and now we select the required B_i among them. We start with the ball $B_1 = B(v_1, R(v_1))$ for the point v_1 where the function $R(v)$ assume its maximum on D (notice that $R(v)$ takes finitely many values). Then we take the point $v_2 \in D$ outside $B(v_1, 2\lambda R(v_1))$ where again $R(v)$ is maximal on $D - B(v_1\ 2\lambda R(v_1))$. Clearly the ball $\lambda B_2 = B(v_2, \lambda R(v_2))$ does not intersect $\lambda B_1 = B(v_1, \lambda R(v_1))$. Then we take the maximal ball outside $2\lambda B_1 \cup 2\lambda B_2$ for B_3 and so on. The resulting balls B_i satisfy (1), (3) and (4). Furthermore, the concentric balls $2\lambda B_i$ cover D, (this is obvious) and so by the doubling property these B_i contain definite part of D, i.e. satisfy (2) with some $\delta' > 0$ which may be somewhat smaller than the one used above.

To conclude the proof we need to ensure that all (or almost all) points $v \in D$ are density points with respect to the C-C distance. In fact, the proof of that follows by the above covering argument presented in a proper light. On the other hand, the subset D we work with is open (or at least contains an open dense set of full measure) and so the density problem becomes irrelevant.

Remark. The above proof does not need equiregularity of V if "mes" is understood in the Riemannian (Lebesgue) sense.

2.3.D. Isoperimetric inequality in compact regions $V_0 \subset V$. *There exist constants $\mu > 0$ and $C \geqslant 0$ (depending on V_0) such that every domain D inside V_0 with $\mathrm{mes}\, D \leqslant \mu$ bounded by a closed hypersurface S satisfies*

$$\mathrm{mes}\, D \leqslant C(\mathrm{mes}_{N-1} S)^{\frac{N}{N-1}}, \qquad (**)$$

where $N = \dim_{\mathrm{Hau}} V$ and $\mathrm{mes} = \mathrm{mes}_N$.

Proof. Take B_i from the covering lemma with a sufficiently large λ and apply $(\star)'$ to each intersection $D \cap B_i$. Then we observe that each (small) ball $B = B(R)$ has $\operatorname{mes} B \gtrsim R^N$ and since the measure of $D \cap B$ is compatible with $\operatorname{mes} B$ the inequality $(\star)'$ gives us the bound $\operatorname{mes}_{N-1}(S \cap B') \gtrsim R^{N-1}$, which implies, in turn, that

$$R \operatorname{mes}_{N-1}(S \cap B') \lesssim (\operatorname{mes}_{N-1}(S \cap B'))^{\frac{N}{N-1}}.$$

Thus we can free $(\star)'$ of R and obtain

$$\operatorname{mes}_N(D \cap B_i) \leqslant C_{V_0} \operatorname{mes}_{N-1}(S \cap \lambda B_i) \qquad (\star)_i$$

for all i. These $(\star)_i$ add up to $(**)$ became the balls B_i exhaust an essential part of D (property (2)) while the balls λB_i do not intersect according to (1).

Remarks

(a) If V is a compact manifold to start with we do not need any V_0 but we still have to restrict D in size. For example, if V is a closed connected manifold, then $(**)$ holds true with $C = C(V)$ for all $D \subset V$ with $\operatorname{mes} D \leqslant \frac{1}{2} \operatorname{mes} V$. But in general, removing V_0 from the picture is possible only under special favourable circumstances as indicated in 2.3.E below.

(b) The inequality $(**)$ and its proof can be transplanted to the asymptotic framework of 2.2 where this can be used for evaluating the Sobolev constant and the first eigenvalue of (V, g_t) for $t \to \infty$.

(c) The inequality $(**)$ implies a similar inequality (with the same constant C) for *multiple* domains D over V which are *immersions* of D to V ore more generally (ramified) maps $D \to V$ without foldings. This is proven by applying $(**)$ to each subset $D_i \subset V$ consisting of the points covered by D at least i times. Furthermore, this works for continuous families of domains $D_t \subset V$ represented by levels of positive functions $f : V \to \mathbb{R}_+$ (for $D_t = f^{-1}[0, t]$) and gives one a Sobolev inequality for f. (This argument is due to Mazia.)

2.3.D′. Excluding V_0 from the game. If V is a nilpotent Lie group with a self-similarity, then every bounded domain can be scaled into a fixed compact subset $V_0 \subset V$ and so the constant C can be assumed independent of V_0. In particular, $(**)$ implies the ordinary isoperimetric inequality in $\mathbb{R}^n$ (with $N = n$ and non-sharp constant). In fact, since an arbitrary simply connected nilpotent Lie group is asymptotic to a group V with a self-similarity, the inequality $(**)$ holds true in such V with $C = C(V)$. In general, one looks for inequalities like $(**)$ and $(\star)$ with constants controlled by some easily computable invariants of V. In the Riemannian category one likes inequalities with the constants depending on the curvature. Such inequalities are available in two somewhat opposite cases, namely $K \leqslant 0$ and Ricci $\geqslant -\kappa$, see [B-Z] and one wishes to distinguish appropriate C-C manifolds in a similar manner.

Examples of C-C manifolds with "$K < 0$". Let V_0 be nilpotent Lie group with a polarization H_0 a metric g_0 on H_0 and with self-similarities $A_t : V_0 \to V_0$. Take $V = V_0 \times \mathbb{R}$, with the polarization $H \subset T(V)$ obtained as the pull-back of H_0 under the projection $V \to V_0$ and with the metric g on H given by $g = A_t\ g_0 + dt^2$. (This correspond to the horospherical coordinates in the ordinary hyperbolic space $H^n = \mathbb{R}^{n-1} \times \mathbb{R}$ with $g = e^t g_0 + dt^2$ for $g_0 = \Sigma_{i=1}^{n-1} dx_i^2$.) One can show (we leave it to the reader) that this V satisfies $(**)$ with some $C = C(V)$. Another class of negatively curved C-C manifolds appears in [$\mathrm{Ge}_{\mathrm{GG3d}}$] who studies horizontal contact structures in S^1-bundles over surfaces of genus $\geqslant 2$ where the condition $K < 0$ refers to *Webster* curvature. Probably, some of Ge's results may be extended to (S^1 and more general) bundles over Riemannian (e.g. Kählerian) manifolds of negative curvature.

Examples of C-C manifolds with Ricci $\geqslant$ 0. The round sphere S^{2n-1} with the standard contact structure and metric looks as a good candidate for "$K > 0$" (compare [$\mathrm{Hsu}_{\mathrm{SRCT}}$]). The same can be said about S^{4n-1} (with the polarization normal to the Hopf fibers) and S^{15}. Here one expects sharp isoperimetric inequalities in the spirit of Paul Levy (see [$\mathrm{Gro}_{\mathrm{PLI}}$]). Further examples of positive (or at least bounded from below) curvature (may) appear in the constructions parallel to those in the Riemannian geometry. These are

(i) *Cartesian products and some non-trivial fiber bundles.*

(ii) *Factors by compact isometry groups* (which may act non-freely if one allows spaces with singularities).

(iii) *Special "cone" and "joint" constructions.*

(iv) *Convex hypersurfaces.* We do not know what corresponds to these in C-C manifolds but the idea of convexity suggests looking at strictly convex (and pseudoconvex) hypersurfaces in $\mathbb{C}^n$ with the natural C-C structures on them.

Examples of $|K| \leqslant$ const. The standard examples are of the form $V = \widetilde{V}/\Gamma$ where $\widetilde{V}$ is a nilpotent Lie group and Γ a discrete free isometry group of $\widetilde{V}$. One probably can show with our argument that every $D \subset V$ satisfies

$$\operatorname{mes}_N D \leqslant C(\operatorname{Diam} D) \operatorname{mes}_{N-1} \partial D$$

with C depending on (the C-C metric on) $\widetilde{V}$ but not on Γ.

The above $\widetilde{V}/\Gamma$ correspond to flat manifolds in the Riemannian geometry. The "small variable curvature" can be expressed in term of systems of (locally defined) vector fields on V compatible with H and g (in an appropriate sense) and having "multiplication table" for the Lie bracket with small coefficients. (The difficulty which emerges here is the same as in the pinching discussion in § 1, that is the absence of canonical coordinates in V and/or of a parallel transport.)

2.3.D″. Isoperimetric inequality in non-equiregular spaces. The key role of equiregularity appears in the equivalence of the N-dimensional Hausdorff measure with the Riemannian (Lebesgue) measure on V. In the general case, the Hausdorff measure can be supported on a proper subset $A \subset V$, such as a Cantor sets in the example in, and then, of course, the inequality $(**)$ fails to be true. On the other hand if V admits an equisingular stratification, one, probably, has a meaningful isoperimetric inequality on each stratum V_ν of dimension $\geqslant 2$. Observe that the metric on such a stratum is not, in general, Carnot-Carathéodory and the corresponding inequality for N_ν-dimensional measure for $D_\nu \subset V_\nu$ may involve the N'_ν-dimensional measure of ∂D_ν for $N'_\nu < N_\nu - 1$. For example, V_ν may have such metric as $(\operatorname{dist}_{\text{Riem}})^{\frac{1}{d}}$ and then the isoperimetric inequality bounds $\operatorname{mes}_{N_\nu} D$ by $\operatorname{mes}_{N_\nu - d} \partial D$. It seems, inequality of this kind are satisfied by all equiregular submanifold $V' \subset V$ (not only on $V' = V_\nu$).

On the other hand, there is a meaningful version of the isoperimetric inequality $(**)$ which needs no equiregularity assumption. To formulate this we denote by mes the *Riemannian* measure in V and then define the *Minkowski volume* $\operatorname{mes}' S$ by

$$\operatorname{mes}' S = \liminf_{\varepsilon \to 0} f\varepsilon^{-1} \operatorname{mes} U_\varepsilon(S)$$

where $U_\varepsilon(S)$ denotes the Carnot-Carathéodory ε-neighbourhoods of S.

Theorem. *If D, C and S satisfy the assumptions of* 2.3.D *then*

$$\operatorname{mes} D \leqslant C(\operatorname{mes}' S)^{\frac{N}{N-1}} \tag{$**$}'$$

where N is defined in terms of the (variable) $n_i(v) = \operatorname{rank} H_i(v)$ by

$$N = \max_{v \in V} \sum_{i=1}^{d} i(n_i(v) - n_{i-1}(v))$$

(and H is not assumed equiregular anymore).

The proof is essentially identical to that of $(**)$.

Notice that this theorem follows from a result by Varopoulos in [Var] where the isoperimetric inequality is directly linked to the volume behaviour of (small) balls in V. Also see $[\text{F-G-W}]_{1,2}$, [F-L-W], $[\text{Haj}_{\text{SSAM}}]$, [Ha-Ko] and [Co-SC].

2.3.E. Green forms, pencils of curves and an integral geometric proof of the isoperimetric inequality. We start with a nilpotent group V with a self-homotopy $A : V \to V$ and we call a closed $(n-1)$-form ω_0 on $V - \{0\}$, where 0 stands for the identity element, a *Green form* if it is

(i) H-horizontal.

(ii) A-invariant.

(iii) closed and non-exact.

Notice that the A-invariance implies that

(iv) $\|\omega_0\|_v \leqslant \operatorname{const} \operatorname{dist}^{-(N-1)}(0, v)$, $v \in V$.

Also observe that the non-exactness of ω_0 makes

$$\int_S \|\omega_0\|_s \, ds \geqslant \left| \int_S \omega_0 \right| = c_0 \neq 0$$

for a fixed c_0 and all smooth closed hypersurfaces S around the origin, where ds refers to the $(N-1)$-dimensional Hausdorff measure on S.

Example. If $V = \mathbb{R}^n$ such an ω_0 may be obtained as the radial pull-back of the volume form on $S^{n-1} \subset \mathbb{R}^n$.

Lemma. *Every V admits a Green form.*

Proof. Divide $V - \{0\}$ by the (infinite cyclic) group $\{A^i\}$ generated by A, take some H-horizontal closed non-exact $(n-1)$-form $\overline{\omega}$ on the quotient space $(V-0)/\{A^i\}$ (which exists according to Linear Lemma in 2.2.A) and pull $\overline{\omega}$ back to V for the quotient map

$$V - \{0\} \to V - \{0\}/\{A^i\}.$$

Remarks

(a) A slight readjustment of the proof of 2.2.A yields a Green form invariant under the one parameter group $\{A^t\}$, $t \in \mathbb{R}$.

(b) A standard ω_0 comes as the dual of the horizontal gradient of the fundamental solution of the Hörmander-Laplace operator on V.

Santalo-type proof of the isoperimetric inequality for a compact domains $D \subset V$ with a smooth boundary S. If D contains 0 we have the following lower bound on the integral of $\mathrm{dist}^{-(N-1)}$ with respect to the Hausdorff measure on S,

$$\int_S \mathrm{dist}^{-(N-1)}(0,s)ds \geqslant \varepsilon > 0, \tag{+}$$

since $\int_S \omega_0 = c_0 \neq 0$. On the other hand

$$\int_D \mathrm{dist}^{-(N-1)}(0,v)dv \leqslant \mathrm{const}_0(\mathrm{mes}\, D)^{\frac{1}{N}}, \tag{++}$$

as the left hand side integral is obviously bounded by $r^{N-1}\,\mathrm{mes}\, B$ for the ball $B = B_0(r) \subset V$, where $r = r(D)$ is chosen so that $\mathrm{mes}\, B = \mathrm{mes}\, D$. We apply (+) to $\mathrm{dist}(v,s)$, for all $v \in D$ and integrate over D. Thus we get

$$I = \int_D dv \int_S \mathrm{dist}^{-(N-1)}(v,s)ds \geqslant \varepsilon\,\mathrm{mes}\, D.$$

Then we change the order of integration and obtain with (++),

$$I = \int_S ds \int_D \mathrm{dist}^{-(N-1)}(s,v)dv \leqslant \mathrm{const}_0\,\mathrm{mes}_{N-1} S(\mathrm{mes}\, D)^{\frac{1}{N}}.$$

Thus

$$(\mathrm{mes}\, D)^{\frac{N-1}{N}} \leqslant \varepsilon^{-1}\,\mathrm{const}_0\,\mathrm{mes}_{N-1} S.$$

Remarks. The original argument by Santalo proceeds slightly differently and leads to the *sharp* isoperimetric inequality on $\mathbb{R}^2$ as well as on S^2 and H^2. Also this argument provides the basic (non-sharp) isoperimetric inequality on all complete simply connected Riemannian manifolds with $K \leqslant 0$ which suggests a similar extension for C-C manifolds.

Green forms for equiregular V. Now a Green form ω_v is defined in a punctured neighbourhood of some point $v \in V$ and it must satisfy the above (i), (iii) and (iv). Such an ω_v is constructed by using the approximation of V at v by a nilpotent group $\overline{V}$ (called N_v in 1.4). Namely, according to 1.4.A$''$ our polarization on each sufficiently small annulus $\mathrm{An}_\rho = B_v(3\rho) - B(\rho)$ can be smoothly approximated by that on the corresponding annulus in a certain nilpotent Lie group $\overline{V} = \overline{V}(V, v)$, such that when An_ρ and $\overline{\mathrm{An}}_\rho$ are scaled to the unit size and brought together, the polarization H becomes ε-close to $\overline{H}$ in C^n-topology (as we assume H as smooth as we need) with $\varepsilon = \varepsilon(\rho) \to 0$ for $\rho \to 0$. Then we cover $B_v(\rho) - \{v\}$ by slightly overlapping annuli, say

$$\mathrm{An}_i = B(\lambda 2^i) - B(2^{i-1}/\lambda) \quad \text{for } \lambda = \sqrt{\frac{3}{2}}$$

and large $i = i_0, i_0 + 1, \ldots$, and we construct a closed horizontal form ω_i on each An_i representing the generator of $H^{n-1}(A_i; \mathbb{Z}) = \mathbb{Z}$ (where we ignore possible (?) minor nuisance at $\partial \mathrm{An}_i$) such that on the rescaled unit annulus this form become ε_i-close to a fixed form $\overline{\omega}_1$ on $\overline{\mathrm{An}}_1$. This is done by slightly adjusting the proof of Linear Lemma which also allows us to match ω_i with ω_{i-1} and ω_{i+1} on the overlaps of the annuli. Everything is quite simple and left to the reader. We also trust the reader to reprove, using ω_v, the inequality $(**)'$.

Remark. The new proof can be made "softer" and thus brought closer to our first proof by observing that the form ω_v does not need to be closed but rather having the differential $d\omega_v$ growing no faster near v than $(\mathrm{dist}(v, \cdot))^{-(N-1)}$. This makes $\int_D d\omega_v$ small for small D and does not affect our estimates. Then a suitable ω_v can be constructed by just prescribing several first terms of its (Laurent type) asymptotic expansion at v. (Our $\omega_v = \omega_v(v')$ can be found in the form $\omega_v = d^{-(N-1)}(v, v')\, \omega'_v(v')$, where d is a properly smoothed distance function and $\omega'_v(v')$ is a smooth form for all v' near v. Here the relevant properties of ω_v are ensured by first Taylor terms of $\omega'_v(v')$ at v.)

Pencils of curves emanating from v. We saw in 2.2.A how a closed horizontal $(n-1)$-form comes along with (a measure on) a family of closed horizontal curves in V. In our present situation we need families (pencils) of non-closed horizontal curves issuing from a fixed point v. (For example, the standard form ω_0 on $\mathbb{R}^n$ corresponds to the family of the straight rays coming from the origin with the obvious spherical measure on these rays.) Every such family provides certain information on hypersurfaces S around v since the measure on S is recorded by the curves in the pencil as they intersect S leaving the domain $D \ni v$ bounded by S. The best pencils are those where the corresponding ω_v is a true form (not just a current) which grows no faster than $(\mathrm{dist})^{-(N-1)}$ at v. Such pencils can be constructed as follows. Take sufficiently many horizontal fields $X_0, \dots, X_k$ on V in general position and let

$$X_\mu = \sum_{i=1}^{k} \mu_0 \, X_i$$

for $\mu = (\mu_0, \dots, \mu_k)$ being the points in the standard k-simplex Δ. Then for each $v \in V$ we have the family of integral curves $c_{v,\mu}$, $\mu \in \Delta$, issuing from v with the natural measure $d\mu$ on this family. We suggest the reader would follow the integral geometric proof with ω_v associated to this $c_{v,\mu}$ and observe how close this comes to our first proof. In fact much of the standard analysis on (V, H) can be performed with suitable measures on the space of horizontal curves and we shall encounter later on similar measures on higher dimensional horizontal submanifolds.

2.4. Singular integrals and Sobolev inequalities. As we mentioned earlier the isoperimetric inequality yields, by Mazia argument, the following Sobolev inequality for the L_p-norm, $p = \frac{N}{N-1}$, of a function f on V in terms of the L_1-norm of the *differential of f restricted to H*,

$$\int_V |f(v)|^{N/N-1} dv \leqslant \mathrm{const} \left(\int_V \|df(v) \mid H\| \, dv \right)^{N/N-1} . \qquad (*)$$

This implies the bound

$$\|f\|_{L_p} \leqslant \mathrm{const}_q \, \|df \mid H\|_{L_q} \qquad (*)_q$$

for all q in the interval $1 \leqslant q < N$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{N}$ as follows. Apply $(*)$ to $|f|^a$ for $a = \frac{p(N-1)}{N}$ and get

$$\|f\|^a_{L_p} \lesssim \||f|^{a-1} df \mid H\|_{L_1}$$

where "$\lesssim$" means "$\leqslant \mathrm{const}_q$". Then use the Hölder inequality

$$\||f|^{a-1}df \mid H\|_{L_1} \lesssim \||f|^{a-1}\|_{L_b} \cdot \|df \mid H\|_{L_q}$$

for $b = \left(1 - \frac{1}{q}\right)^{-1}$, and observe that $\||f|^{a-1}\|_{L_b} = \|f\|^c_{L_p}$ for $c = pb^{-1}$. Thus $\|f\|^{a-c}_{L_p} \lesssim \|df \mid H\|_{L_q}$, which yields $(*)_q$ since $a - c = 1$ with our choice of a, b and c.

The inequality $(*)$ for $q > 1$ can be also derived from the following estimates for convolution integrals. Let V be a nilpotent group and $K(v)$ be a function (convolution kernel), such that

$$|K(v)| \leqslant (\mathrm{dist}(0, v))^{-(N-1)}\ ,\ v \in V.$$

Then

$$\|K * f\|_{L_p} \leqslant \mathrm{const}_q \|f\|_{L_q}, \qquad (**)_q$$

for all q in the interval $1 < q < N$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{N}$.

This is classical for $V = \mathbb{R}^N$ and the Euclidean proof (see, e.g. Ch.V in [Ste]) extends to the general V (see [Fol]).

Finally, we recall the Green form $\omega_0(v)$ of 2.3.E and take the associated divergence free vector fields $X(v)$ which clearly has

$$\|X(v)\| \leqslant \mathrm{const}(\mathrm{dist}(0, v))^{-(N-1)}.$$

We observe that every function f on V decaying at ∞ can be reconstructed from $df \mid H$ by "convolution" with $X(v)$, as $f(0) = \int_V df(X(v))dv$, and so

$$(**)_q \Rightarrow (*)_q \quad \text{for } q > 1.$$

2.4.A. Dilation and homotopy. The above inequality $(*)_q$ fails to be true for $q = N$ and $p = \infty$ (where $\|f\|_{L_\infty}$ refers to $\sup_{v \in V} |f(v)|$) but the proof of $(*)_q$ implies a bound on $\|f\|_{L_\infty}$ by $\|df \mid H\|_{L_q}$ for all $q > N$. In fact, one may reconstruct $f(v_1) - f(v_2)$ from the differential df on H for given v_1 and v_2 using a divergence free vector field X_{v_1,v_2} on $V - \{v_1, v_2\}$ which vanishes away from a ball containing v_1 and v_2 and has non-zero fluxes through small spheres around v_1 and v_2. One sees instantaneously with such a field (using the Hölder inequality) that the norm $\|df \mid H\|_{L_q}$ for $q = N + \alpha$ bounds the Hölder constant $L_\beta(f)$ for the exponent $\beta = \alpha / N + \alpha$ for all $\alpha > 0$ (compare 3.6.B$_1$) which implies a similar bound on the Hölder constant of a map f of V into a Riemannian manifold by the L_q-norm of the differential $\mathcal{D}f$ restricted to H.

Corollary. *Let V be a compact equiregular* C-C *manifold and W be a compact Riemannian one. Then, for every $\alpha > 0$ and $C > 0$, the space of maps $f : V \to W$ with $\|\mathcal{D}f \mid H\|_{L_{N+\alpha}} \leqslant C$ is compact in the uniform topology and, in particular, contains at most finitely many homotopy classes of maps. Furthermore, if C is sufficiently small, then f is null-homotopic.* (Here, as earlier, $N = \dim_{\mathrm{Hau}} V$.)

Remarks

(a) The fields X_{v_1,v_2} in the above argument can be replaced by a pencil of curves c_μ between v_1 and v_2 (e.g. the orbits of X_{v_1,v_2}). In fact, the difference $f(v_1) - f(v_2)$ is given by the integral of df on c_μ over c_μ for each μ. On the other hand the μ-average of the latter integrals can be bounded in term of $\|df|H\|_{L_q}$ for sufficiently "rich" (or "thick") pencils (compare 3.6.B$'$, 3.6.D and 5.4.B).

(b) The above corollary (and its proof) remains valid for non-equiregular V with $N = \max\limits_{v\in V}\ \Sigma_{i=1}^{a}\ i(n_i(v) - n_{i-1}(v))$ as in 2.3.D$''$.

2.4.B. High levels of functions f with $\|df \mid H\|_{L_N} < \infty$ on (codim 1)-stable C-C manifolds of (formal) Hausdorff dimension N. The above compactness property fails to be true for $\alpha = 0$ as a suitable positive function $\varphi = \varphi(\rho)$ defined for $\rho > 0$ and satisfying $\lim_{\rho\to 0}\ \varphi(\rho) = \infty$ may have $\int_V \|d(\varphi(\mathrm{dist}(v_0, v)))|H\|^N dv < \infty$. Yet one can sometimes understand pretty well the ways a functions f with $\|df|$ $H\|_{L_q} < \infty$, where $q \leqslant N$, goes to infinity. For example, if V is compact Riemannian (i.e. $H = T(V)$) and $q > N - 1$, then f is "zero-dimensional at infinity" in the following sense. *Every connected component of the ε^{-1}-level $\{v \in V||f(v)| > \varepsilon^{-1}\}$ has diameter $\leqslant \delta$ where $\delta \to 0$ for $\varepsilon \to 0$ (and, in fact, one may estimate δ in terms of ε and $q - N + 1$).* More generally if $q > N - i$, $i = 1, \dots N - 1$, then f is "at most i-dimensional at infinity". Namely, *the i-width of the ε^{-1}-level goes to zero with ε*, i.e. this level can be retracted to an $(i-1)$-dimensional subset by a δ-move in V.

This is well known and, I guess, is due to Karen Uhlenbeck. The proof follows by observing that the restriction of f to the $(N-i)$-skeleton of a generic δ-triangulation of V is bounded in a controlled way which allows a δ-push of our ε^{-1}-level to the complementary $(i-1)$-skeleton (compare 3.4).

(codim 1)-*Stability*. In order to apply the above argument to a C-C manifold (V, H) and $i = 1$ one needs sufficiently many hypersurfaces $V' \subset V$, such that the Hölder estimates from 2.4.F′ apply to $f \mid V'$ with $N-1$ in place of N. For this it would suffice to have $(V', H' = H \cap T(V'))$ equiregular of Hausdorff dimension $N - 1$ which is essentially the same as Lipschitz equivalence of the metrics $\mathrm{dist}_{H'}$ and dist_H on V'.

Definition. We say that V is (codim 1)-*stable* if for each $v \in V$ and every hyperplane $T' \subset T_v(V)$ there exists (a germ of) a smooth hypersurface $V' \subset V$ passing through v with $T_v(V') = H'$ such that the *formal Hausdorff dimension* $N' = \max_{v' \in V'} \Sigma_{i=1}^{d} i(n_i'(v) - n_{i-1}'(v))$, for n_i' being the ranks of the spaces $H_i' \subset T'(V')$ spanned by k-th order commutators of H'-horizontal fields on V', satisfies $N' \leqslant N - 1$ for the formal Hausdorff dimension N of (V, H).

Examples. The Riemannian manifolds (where $H = T(V)$) are, obviously, (codim 1)-stable. Contact manifolds of dimension $n = 2m + 1$ are stable for $n \geqslant 5$ and are (codim 1)-unstable for $n = 3$. In fact, generic H of rank $n-1$ on n-dimensional manifolds V are (codim 1)-stable starting from $n = 4$. On the other hand, H's of rank 2 are unstable. In general, a generic H with rank $H \gg$ corank H is (codim 1)-stable but if corank $\gg$ rank then H is typically unstable.

Now we see with this definition that *smooth functions f on (codim 1)-stable manifolds with $\|df \mid H\|_q < \infty$ for $q > N - 1$ have high levels of "zero dimension at infinity"* as in the Riemannian case.

Conformal interpretation. The connectedness results for ε^{-1}-levels can be equivalently expressed in the conformal language. Namely, let dist_φ be conformal to the original C-C metric dist on (V, H) with the conformal factor $\varphi = \varphi(v)$ (corresponding to $\|df(v) \mid H\|$) which is a positive Borel function on V with $\|\varphi\|_{L_q} < \infty$. If $q > N$ the metric dist_φ is equivalent to dist by the above discussion but for $q \leqslant N$ dist_φ may become infinite on certain pairs of points. However there is a well defined maximal subset $V_\varphi \subset V$ where dist_φ is finite and one can reformulate the above properties of $f(v)$ (corresponding to $\mathrm{dist}_\varphi(v, v_0)$) as certain connectedness at infinity of the metric space $(V_\varphi, \mathrm{dist}_\varphi)$. Here the case $q = N$ appears especially attractive as the condition $\|\varphi\|_{L_N} < \infty$ is equivalent to $\mathrm{mes}_N(V_\varphi, \mathrm{dist}_\varphi) < \infty$.

2.5. Homotopy bounds by $\|\mathcal{D}f \mid H\|_{L_N}$, taut maps and the bubbling phenomenon. Despite the failure of the compactness, the homotopy finiteness conclusion may hold true for certain $q \leqslant N$ as the homotopy invariants of a map $f : V \to W$ may be estimated, under favorable topological assumptions on V and W, in terms of the L_q-norm of the differential $\mathcal{D}f$ on H. For example, if W is a compact aspherical manifold, then the homotopy class of f is determined by the restriction of f to the 1-skeleton S_1 of a triangulation of V. If we choose this S_1 horizontal and sufficiently generic, then the length of $f(S_1)$ will be (obviously) bounded in terms of $\|\mathcal{D}f \mid H\|_{L_1}$ and so we conclude that there are *at most finitely many homotopy classes of maps* $f : V \to W$ *with* $\|\mathcal{D}f \mid H\|_{L_1} \leqslant \text{const}$. *Furthermore, if* const $\leqslant c_0$ *for some sufficiently small* $c_0 > 0$, *then* f *is null-homotopic.* (All this is standard for V Riemannian.) Similarly, one may treat the case where $\pi_i(W) = 0$ for $i > i_0$, and V contains sufficiently many i_0-dimensional submanifolds $V' \subset V$ having $\dim_{\mathrm{Hau}} V' < q$ with respect to the polarization $H' = H \cap T(V)$.

A somewhat different argument shows, in the case V is Riemannian, that the number of the homotopy classes of maps $f : V \to W$ for an arbitrary W is bounded by $\|\mathcal{D}f\|_{L_N}$ for $N = \dim V$. For example, $\|\mathcal{D}f\|_{L_N}$ obviously bounds the topological degree of f in the case $\dim W = \dim V$ and there are similar bounds for the other (rational) homotopy invariants of f (compare 1.4.E′). However, such a "bound on degree" consideration does not immediately yield null homotopy of an f with small $\|\mathcal{D}f\|_{L_N}$ (e.g. for maps $f : S^4 \to S^3$ with small $\|\mathcal{D}f\|_{L_4}$); nor it extends to general C-C manifolds. But there is an alternative approach borrowed from the (quasi)conformal geometry and based on a suitable form of the Schwarz lemma (compare 2.6). The Schwarz lemma does not directly apply to general (non-quasi-conformal) maps $f : V \to W$ but the following modification (probably, first pointed out by Uhlenbeck) works just as well. First we observe that in homotopy problems the map f can be assumed *taut* in the following sense.

Definition. A map f of a smooth manifold into a Riemannian one is called *taut* if no homotopy of f can make the induced (possibly singular) Riemannian metric strictly smaller.

Example. A taut map into $\mathbb{R}^n$ is necessarily (and obviously) constant. Similarly, every taut map which lands inside a small (and hence convex) ball in a Riemannian manifold is also constant.

Now, if V and W are compact, then every map f is obviously homotopic to a taut one, say f_{tau}, such that $\|\mathcal{D}f_{\mathrm{tau}}(v)\| \leqslant \|\mathcal{D}f(v)\|$ everywhere on V and, in particular, $\|\mathcal{D}f_{\mathrm{tau}}\|_{L_N} \leqslant \|\mathcal{D}f\|_{L_N}$.

Proposition. *Let V be compact and W has locally bounded geometry. Then there exists $c_0 > 0$, such that every taut map $f : V \to W$ with $\|\mathcal{D}f\|_{L_N} \leqslant c_0$ is constant.*

Proof. Normalize W so that Conv Rad $W = 1$. Then if some small ball $B_\varepsilon \subset V$ lands in a unit ball in V then, the tautness of f obviously yields the following

Key relation. $\mathrm{Diam} f(B_\varepsilon) = \mathrm{Diam} f(S_\varepsilon)$ for the boundary sphere S_ε of B_ε.

The second fact we are going to use is that for every small sphere $S_\varepsilon \subset V$ there is a concentric sphere S_δ for $\varepsilon \leqslant \delta \leqslant 2\varepsilon$ such that $\mathrm{Diam} f(S_\delta) \leqslant d_0 = d(c_0)$ where $d_0 \to 0$ for $c_0 \to 0$. (This follows from the bound on $f \mid S^{N-1}$ by $\|\mathcal{D}f \mid S^{N-1}\|_{L_N}$, as we have already seen.) If we knew that $\mathrm{Diam} f(B_\delta) \leqslant 1$, we could conclude that, in fact, $\mathrm{Diam} f(B_\delta) \leqslant d_0$ which for small c (and hence d_0) would make $\mathrm{Diam} f(V) = 1$ and yield the desired constancy of f. Thus we are lead to look at the ball $B_\varepsilon \subset V$ of the minimal radius ε which has $\mathrm{Diam} f(B_\varepsilon) = 1/2$. Then the concentric ball B_δ for every $\delta \leqslant 2\varepsilon$ has $\mathrm{Diam} f(B_\delta) \leqslant 1$ as for each $v \in B_\delta$ there is a ball B'_ε containing v and intersecting B_ε (which, as we know, has $\mathrm{Diam} f(B'_\varepsilon) \leqslant 1/2$). But then for a suitable $\delta \geqslant \varepsilon$ we have $\mathrm{Diam} f(B_\delta) \leqslant \delta$ and so $\mathrm{Diam} f(B_\varepsilon) \leqslant d_0$. If $d_0 < \frac{1}{2}$ this leads to a contradiction and thus proves the desired constancy of f. ∎

Relative case. Let us try to generalize the above to the case where f is *taut relative* to a certain subset $V_0 \subset V$, i.e. the homotopy involved in the definition of tautness is required to be constant on V_0. If V_0 consists of a single point $v_0 \in V$ then such an f with small $\|\mathcal{D}f\|_{L_N}$ is still constant and maps all of V to $f(v_0) \in W$, but for several points the situation is different as for every finite subset V_0 in V there obviously exists a map f with prescribed values on V and having arbitrarily small energy

$\|\mathcal{D}f\|_{L_N}$. (A similar property is enjoined by all closed subsets V_0 in V of conformal capacity zero.) However, if such an f is taut relative to V_0 it must have "one-dimensional shape". Namely, a *small perturbation of f factors through a map of V into a tree with j extremities for $j = \operatorname{card} V_0$, and the implied map of this tree to W sends the i-th extremity to $f(v_i)$ for $V_0 = \{v_1, \dots, v_i, \dots, v_j\}$.*

Proof. One sees as earlier that there are small disjoint balls $B_{\rho_i}(v_i) \subset V$, $i = 1, \dots, j$, such that the map is nearly constant on the complements of these balls, i.e. has small $\operatorname{Diam} f(V - \underset{i}{\cup} B_{\rho_i}(v_i))$, and also f is nearly constant on the concentric spheres $S_\varepsilon(v_i) = \partial B_\varepsilon(v_i)$ for $i = 1, \dots, j$ and $\varepsilon \leqslant \rho_i$.

Remark. The "one-dimensionality" conclusion remains valid for V_0 consisting of the union of small disjoint spheres $S_{\rho_i}(v_i) = \partial B_{\rho_i}(v_i)$, provided the map f is already known to be nearly constant on these spheres.

The case where c is large. Now we do not expect that f with $\|\mathcal{D}f\|_{L_N} \leqslant c$ is constant but we still can control it on a domain $U \subset V$ if the integral of $\|\mathcal{D}f\|^N$ over (some ε-neighbourhood of) U is sufficiently small. Then we may try to cover V by a finite (depending over c) number of such domains and thus show f is piecewise controlled in a suitable sense.

Here is the relevant (trivial)

Covering lemma. *Let V be a metric space and μ a Borel measure of total mass $c < \infty$ on V without atoms. Then for arbitrary positive constants c_0, ρ and λ there exist a positive integer $k \leqslant k_0(c, c_0, \rho, \lambda)$ and $\varepsilon \geqslant \varepsilon_0(c, c_0, \rho, \lambda) > 0$, such that V can be partitioned into k pieces, $V = \cup_{i=1}^k V_i$ with the following three properties.*

(1) *The (major) piece V_1 is obtained from V by removing at most k disjoint balls of radii $\leqslant \rho$. Each of the remaining pieces V_i, $i = 2, \dots, k$, is either a ball or a ball B minus several balls which are strictly contained in B. The total number of balls involved is bounded by k and their radii are bounded by ρ. (As we insist that the number of pieces V_i is exactly k some of them may be taken empty.)*

(2) *For a sphere $S \subset V$ let $[1, \lambda]S \subset V$ denote the annulus consisting of concentric spheres of radii μ rad S for $\mu \in [1, \lambda]$. Then every two among the above balls, say B_1 and $B_2 \neq B_1$, have the boundary spheres S_1 and S_2 λ-disjoint in the sense that $[1, \lambda]S_1 \cap [1, \lambda]S_2 = \varnothing$.*

(3) *For every point* $v \in V_1$ *the ball of radius* ε *in* V_1 *has* mass $\leqslant c_0$, i.e. $\mu(B_\varepsilon(v) \cap V_1) \leqslant c_0$ *and for each* $i \geqslant 2$, *every ball* $B_\delta(v)$, $v \in V_i$, *of radius* $\delta = \varepsilon \operatorname{diam} V_i$ *has* $\mu(B_\delta(v) \cap V_i) \leqslant c_0$.

Now we apply this lemma to the measure $\|\mathcal{D}f\|^N\, dv$ on V. By moving the boundary spheres within their respective λ-annuli, we can make the integrals of $\|\mathcal{D}f\|^N$ over these spheres small which makes small the diameters of the f-images of these spheres. Then, on each connected component $V_i \subset V$ of the complement to these spheres, the map $f_i = f \mid V_i \to W$ falls into one of the two categories.

(1) *Prebubbles.* f_i is (c_0, ε)-controlled on V_i endowed with the metric $\operatorname{dist}_i = \operatorname{dist}/\operatorname{Diam} V_i$ where dist denotes the original metric in V and where the control means that the f_i-image of the ε-balls from V_i have diameters $\leqslant c_0$ in W, where c_0 is given beforehand and $\varepsilon = \varepsilon(c_0, \|\mathcal{D}f\|_{L_N}) \to 0$ for $c_0 \to 0$.

(2) *Bridges.* f_i has small energy, i.e. $\int_{V_i} \|\mathcal{D}f\|^N \leqslant c_0$.

(Notice that (1) and (2) are not mutually exclusive.) We apply the previous remark to the bridges and see what can be called a *prebubbling decomposition* of f (see Fig. 4) as the actual (Uhlenbeck's) bubbling occurs in the limit for sequences of maps $f_\nu : V \to W$ with $\|\mathcal{D}f_\nu\|_{L_N} \leqslant c$ where $\operatorname{diam} f_\nu(S_{\nu,i}) \to 0$ for the boundary spheres $S_{\nu,i}$ of the $V_{\nu,i}$'s and the bubbles appear as the limits of the maps $f_{\nu,i} : (V_{\nu,i} \operatorname{dist}/\operatorname{Diam} V_{\nu,i}) \to W$ for the prebubbles $f_{\nu,i}$ and $\nu \to \infty$. (Sometimes one excludes from the ranks of bubbles the mainland piece $V_{\nu,1}$ where $\operatorname{Diam} V_{\nu,1}$ does not go to zero for $\nu \to \infty$.)

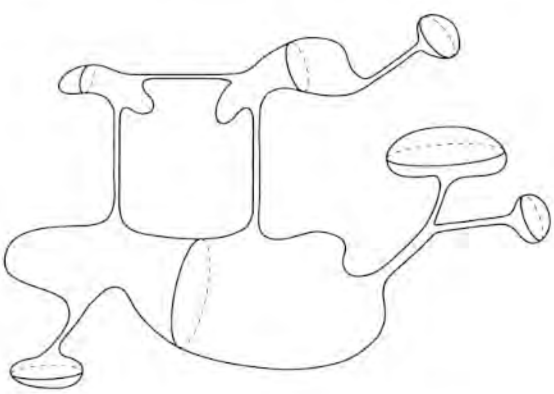


Figure 4

As a consequence of the prebubbling decomposition we see again that the number of the homotopy classes of maps $f : V \to W$ is bounded in terms of $\|Df\|_{L_N}$ provided V and W are compact and W is *simply connected* or rather $\pi_1(W)$ *trivially* acts on $\pi_n(W)$ for $n = N = \dim V$. Furthermore, *all this extends to* (codim 1)*-stable* C-C *manifolds such as contact manifolds* V *of dimension* $\geqslant 5$, *for example*. So we conclude for such V's, that *the maps* $f : V \to W$ *with small norms* $\|\mathcal{D}f \mid H\|_{L_N}$ *are null-homotopic and if* $\pi_1(W)$ *acts trivially on* $\pi_n(W)$, $n = \dim V$, *then there are at most finitely many homotopy classes of maps with* $\|\mathcal{D}f \mid H\|_{L_N} \leqslant c$ *for every* $c \geqslant 0$ (where the implied number of the classes of maps depends on c).

Question. If V is Riemannian then all rational homotopy invariants of maps $f : V \to W$ (e.g. $\deg f$ for $N = n = \dim W = \dim V$) admit polynomial bounds by $\|\mathcal{D}f\|_{L_N}$ (see 3.6 and compare 1.4.E$'$). We want to know if this is also true for the general C-C case. For example, what is the actual bound on $\deg f$, for $\dim W = \dim V < N = \dim_{\mathrm{Hau}} V$ in terms of $\|\mathcal{D}f \mid H\|_{L_N}$? Is it $\approx \|Df \mid H\|^N_{L_N}$?

A natural approach to this problem consists in seeking a "nice" covering of V by standard domains V_i, $i = 1, \ldots, k$, (e.g. balls $B(r_i)$ of some radii r_i) such that the number k is bounded by something like const $\int_V \|\mathcal{D}f(v) \mid H\|^N dv$ and the f-images of all V_i have $\mathrm{Diam}(f(V_i)) \leqslant \varepsilon$ for a fixed small positive ε. The latter inequality would follow if somewhat enlarged V_i, say $2V_i$, (e.g. doubled concentric balls $B(2r_i)$) satisfied $\int_{2V_i} \|\mathcal{D}f \mid H\|^N \leqslant c_0$ for a fixed small $c_0 > 0$. The difficulty here stems from the possibility of fast variation of the integrant $\|\mathcal{D}f(v) \mid H\|^N$ on V (compare [Kor]).

2.5.A. Weak stability of homotopy classes. Let a sequence f_i of smooth maps converges to a smooth map f in some weak topology, e.g. almost everywhere and we want to have all f_i for large i homotopic to f. Then the above shows this is indeed so if the norm $\|\mathcal{D}f(v) \mid H\|$ is bounded on V in the following sense. There is $\rho > 0$, such that the integral of $\|\mathcal{D}f(v) \mid H\|^N$ over every ρ-ball B in V is universally bounded by a sufficiently small positive constant, i.e. $\int_B \|\mathcal{D}f(v) \mid H\|^N dv \leqslant c_0$, (where we assume as earlier that V is (codim 1)-stable.

Corollary. *The homotopy class of* f *is stable under small perturbation in the* F^H_N*-topology (defined in* 2.5.E*).*

2.5.B. On the topology of the space F_c of maps f with $\|\mathcal{D}f \mid H\|_{L_N} \leqslant c$. The above discussion only touches the zero-homotopies (i.e. connected components) of F_c or rather the image of these in the space of all maps, $F = \cup_{c<\infty} F_c$. Yet the full homotopy structure of F_c and its image in F is an equally interesting matter and our argument provides some information on the higher dimensional homotopy of F_c. Namely, if V is (codim 1)-stable we have the following

Homotopy finiteness proposition. *The homotopy image of F_c in F is bounded in terms of c provided W is compact simply connected.* [6]

2.5.C. Surface maps of small area and related questions. The above is well known in the Riemannian case. For example, the space F_{c_0} of smooth maps f of a compact surface Σ into a complete Riemannian manifold W with bounded geometry, defined by $F_{c_0} = \{f \in F \mid \|\mathcal{D}f\|_{L_2} \leqslant c_0\} \subset F$, contracts to (the space of) constant maps provided $c_0 > 0$ is small enough.

Let us take the sphere S^2 for the above Σ and observe that a generic smooth map $f : V \to W$, for $\dim W \geqslant 2$, can be made *conformal* onto its image by composing with a self-homeomorphism of S^2. Since $\|\mathcal{D}f\|^2_{L_2} =$ Area $f(S^2)$ for such f, we obtain the above contractibility property for Area f in place of $\|\mathcal{D}f\|_{L^2}$. Similarly, we see that *the space of based maps $f : S^2 \to W$ with bounded area, $\{f \mid f(s_0) = w_0$, Area $f \leqslant c\}$ has bounded homotopy image in the space of all based maps (i.e. with $f(s_0) = w_0$), provided W is compact simply connected, and is contractible for sufficiently small $c \leqslant c_0 > 0$.* (Here the area of f is counted with due multiplicity, i.e. Area $f \underset{\text{def}}{=} \int_{S^2} |\text{Jacobian } f| = \int_{S^2} \|\Lambda^2 \mathcal{D}f\|$.)

If Σ is a general compact surface ($\neq S^2$), it seems not hard to show that the maps f of small area simultaneously contract to maps with 1-dimensional images. Furthermore, one should be able to understand the (space of) maps with Area $f \leqslant c$ properly taking into account $\pi_1(W)$ and the action of $\pi_1(W)$ on $\pi_i(W)$ for $i \geqslant 2$.

[6] The boundedness means that the inclusion $F_c \subset F$ factors up to a homotopy through a map of F_c into a finite polyhedron with the number of cells bounded by some function of c.) *Furthermore, if $c < c_0$, then the space F_c homotopy retracts (within itself, not only in $F \supset F_c$) to the subspace of the constant maps.* (Here we allow non-trivial $\pi_1(W)$). Moreover, W may be non-compact, but yet with bounded geometry.

Questions. Let $n = \dim V \geqslant 3$, e.g. $V = S^3$. What is the homotopy structure of (the space of) maps $V \to W$ with $\mathrm{Vol}_n f \leqslant c$? Do these maps for small c (and with suitable homotopy restrictions on V and W) simultaneously contract to $(n-1)$-dimensional maps? If so, what is the algebro-topological structure of spaces of such maps?

2.5.D. On the homotopy role of the L_q-norm of the differential $\mathcal{D}f$ on H for $q < N$. Let us look at the maps f with $\|\mathcal{D}f \mid H\|_{L_q} \leqslant c$ for some $q < N$. Now, we do not have the full finiteness-contractibility result but one obtains a weaker homotopy conclusion by restricting f to the k-skeleton $V^k \subset V$ brought to a sufficiently general position with respect to f. Then the bound $\|Df \mid H\|_{L_q} \leqslant c$ implies a similar bound on V^k, namely

$$\|\mathcal{D}f \mid T(V^k) \cap H\|_{L_q} \leqslant c', \qquad (*)'$$

where $c' = \text{const} \cdot c$, where the "tangent bundle" $T(V^k)$ is understood as the set of the vectors tangent to the (smooth!) simplices in V^k and where the L_q-norm of the differential $\mathcal{D}f$ on the intersection $T(V^k) \cap H$ is obtained by integration over V^k. Now, if the intersection $T(V^k) \cap H$ induces a C-C structure of formal Hausdorff dimension $N' \leqslant q$ on each k-face of V^k, then the bound $(*)'$ has non-trivial homotopy effect on $f' = f \mid V^k$ and hence on f. Namely, if $\|\mathcal{D}f \mid H\|_{L_q}$ is sufficiently small, then the restriction $f \mid V^k$ is contractible whenever V^k can be *stably* brought into a position where $\dim_{\mathrm{Hau}}(V^k, T(V^k) \cap H)) \leqslant q$. Here the stability means the existence of a measure μ on the space of embeddings (i.e. positions) V^k in V having $\dim_{\mathrm{Hau}} \leqslant q$ and such that the push-forward of the measure $\mu \times dv^k$ to V is absolutely continuous with respect to Lebesgue (or equivalently C-C Hausdorff) measure in V.

If V is Riemannian (i.e. $\dim V = \dim_{\mathrm{Hau}} V$) then the above stability is trivially satisfied with $N' = k$ and so every smooth map $f : V \to W$ with $\|\mathcal{D}f\|_{L_q}$ small is null-homotopic on some (and hence every) k-skeleton $V^k \subset V$ with $k \leqslant q$. In fact the converse is also known to be true. Every smooth map $f : V \to W$ sending a small neighbourhood of V^k to a point can be composed with a suitable diffeotopy $\varphi_t : V \to V$, $0 \leqslant t < \infty$, such that $\|\mathcal{D}f \circ \varphi_t\|_{L_q} \underset{t\to\infty}{\to} 0$ for every $q < k + 1$. (If $V = S^n$ and $V^0 \subset V$ is the south pole, one uses the north pole south pole push for φ_t. In general one uses such pushes in the cells in $V - V^k$. First every n-cell B is radially pushed from the center $b_0 \in B$ toward the boundary, so that in the limit for $t \to \infty$ all of $B - \{b_0\}$ goes to ∂B. Then one composes the above push toward V^{n-1} with a similar push of a small neighbourhood of V^{n-1} toward V^{n-2} and so on, see p.388 in [E-L] and references therein. Probably, a similar construction can be carried over for certain C-C manifolds, e.g. the contact ones.

Now, let us look at the maps f with $\|\mathcal{D}f \mid H\|_{L_q} < c$ with a possibly large c and observe that there are at most finitely many homotopy classes of restrictions of f to V^k in the following two cases, (i) $q > N'$ and (ii) $q = N'$ and $\pi_1(W)$ acts trivially on $\pi_k(W)$, where N' is, as earlier, the minimal integer so that V^k can be made stably of formal Hausdorff dimension N' for the C-C metric associated to $T(V^k) \cap H$. For example, $N' = k$ in the Riemannian case. If V is a contact C-C manifold, then $N' = k$ for $2k < \dim V$ and $N' = k+1$ for $2k \geqslant \dim V$ as follows from the discussion in 3.4.B, and see 4.? for the general C-C case.

The space $F_{q,c}$. Let us look at the homotopy property of the space $F_{q,c}$ of the maps f with $\|\mathcal{D}f \mid H\|_{L_q} \leqslant c$. We have just seen that the zero-dimensional homotopy (i.e. connected components) of $F_{q,c}$ are strongly affected by q and c but, probably, there is no additional link between the geometry of maps f (encoded into q and c) and higher homotopies of spaces of these f for $q < N$. Namely, if $f_a : V \to W$ is a family of smooth map parametrized by a compact polyhedron $A \ni a$ such that each f_a can be individually contracted to $F_{q,c}$ then, conjecturally, the whole family can be continuously moved to $F_{q,c}$ (or, possibly, to $F_{q,c'}$ for $c' = c'(c, A)$) in the case where $q < N$. This appears easy in the Riemannian case. For example, the above diffeotopy φ_t of V works for families of maps $f_a : V \to W$ which send a *fixed* (i.e. independent of a) skeleton V^k to a point. In general, if $\|\mathcal{D}f_a\|_{L_q}$ is small for all a, we can only have f_a almost constant on V_a^k depending on a. In fact we can make V_a^k constant in a on each simplex of a suitable subdivision of A and then $\varphi_{a,t}$ can be probably build using some induction on skeletons (or partition of unity) in A.

Exercise. Determine the homotopy structure of the space of maps $f : S^1 \times [0,1] \to S^2$ with $\|\mathcal{D}f\|_{L_q} \overset{\text{def}}{=} \left(\int \|\mathcal{D}f\|^q\right)^{\frac{1}{q}} \leqslant c$ for given $c > 0$ and $1 \leqslant q < 2$.

2.5.E. The space F_q^H of measurable maps f with $\|\mathcal{D}f \mid H\|_{L_q} < \infty$. The norm $\|\mathcal{D}f \mid H\|_{L_q}$ makes sense for those measurable (possibly discontinuous) maps $f : V \to W$ where the derivative $\partial_\tau f \in T(W)$ exists for almost all $\tau \in H$. Then $\|\mathcal{D}f \mid H\|_{L_q}$ may be defined by integrating $\|\partial_\tau f\|^q$ over the (unit sphere) bundle of the unit vectors in H and the space of maps with $\|\mathcal{D}f \mid H\|_{L_q} < \infty$ is denoted by F_q^H. If $q > N$, then our earlier argument works equally well for maps $f \in F_q^H$ and show these are, in fact, continuous and even C^β for $\beta = \frac{q-N}{q}$. It is not so, of course, for $q \leqslant N$ and, moreover, continuous maps in F_q^H are not necessarily dense in this space (see (c_1) below). In fact, one know exactly for which q continuous maps are dense when V is Riemannian (see [Beth]) but the corresponding result in the C-C category remains conjectural. A closely related (apparently more global) question is that of the homotopy content of F_q^H and the homotopy structure of the inclusions $C^\infty \cap F_q^H \subset F_q^H$ and $C^\infty \cap F_{q,c}^H \subset F_{q,c}^H \subset F_q^H$ where $F_{q,c}^H = \{f \in F_q^H \mid \|\mathcal{D}f \mid H\|_{L_q} \leqslant c\}$, where the space F_q^H is given L_q^H-*topology* arizing from the norm $\|\mathcal{D}f \mid H\|_{L_q}$ via an embedding of W into some Euclidean space. One asks in this regard, for example, which homotopy invariants of smooth maps extends to F_q^H and are F_q^H-continuous (or continuous in some weaker topology on F_q^H).

2.5.E′. Examples of F_q^H-non-density of smooth maps in F_q^H. ([7]) Let f_0 be the radial projection of the n-ball $V = B^n$ to the boundary $S^{n-1} = \partial B^n$. This map is in F_q^H for $H = T(V)$ and all $q < n$ but for $q > n-1$ it is *not* an F_q^H-limit of smooth maps $f : V \mapsto S^{n-1}$ since the F_q^H-closeness between maps, say $\mathrm{dist}_{L_q^H}(f, f_0) \leqslant \varepsilon$, implies, for $q > n-1$, the uniform closeness of the (continuous!) maps on a sphere $S_\rho^{n-1} \subset B^n$ concentric to ∂B^n and, hence, contractibility of the map $f_0 \mid S_\rho^{n-1}$. The same applies to composed maps $f = \varphi \circ f_0$ for non-null-homotopic maps $\varphi : S^{n-1} \to W$. Moreover, this is also valid for $q = n-1$ as follows from corollary in 2.5.A applied to concentric spheres in the ball $B = B^n$.

Now let V be a smooth (topological) ball around the origin in a nilpotent Lie group with a one-parametric self-similarity such that each orbit of this self-similarity transversally meets $S^{n-1} = \partial V$ at a single point. Then the radial projection $f_0 : V \to S^{n-1}$ along the orbits is in F_q^H for all $q < N = \dim_{\mathrm{Hau}} V$ and neither this f_0 nor any $f = \varphi \circ f_0$ for a non-null-homotopic φ can be approximated by smooth maps in F_{N-1}^H (under the standing assumption of (codim 1)-stability of V).

[7] Compare [Sh-Uhl], [Beth] and [$\mathrm{Haj}_{\mathrm{ASM}}$].

2.5.F. Space F_N^H. Let us look at the homotopy structure of a map $f \in F_N^H$. We start with the (well known) Riemannian case (where $H = T(V)$) and observe that if $f \in F_q^H$ then the restriction $f \mid V'$ has $\|\mathcal{D}f \mid TV'\|_{L_q(V')} < \infty$ for a generic hypersurface V' in V and so for $q > \dim V - 1$ this restriction in continuous. However, the homotopy class of $f \mid V'$ may jump under small perturbations of V' in V for $q < \dim V$. Let us show this does not happen for $q = n = \dim V$. We observe that for each point $v \in V$ and every small positive ε there is a sphere S_δ around v of radius δ in the interval $\varepsilon \leqslant \delta \leqslant 2\varepsilon$, such that the L_n-norm of f on this sphere S_δ with the normalized metric ($=\operatorname{dist}/\delta$) is bounded by the L_n-norm of f on the ball $B_{2\varepsilon}$ (by integrating $\|\mathcal{D}f\|^n$ over the annulas between S_ε and $S_{2\varepsilon}$). It follows, that $\operatorname{Diam} f(S_\delta)$ is bounded in terms of $\int_{B_{2\varepsilon}} \|\mathcal{D}f(v)\|^n \, dv$, as we, in fact, have seen earlier and therefore for every $\varepsilon > 0$ one can cover V by balls of radii between ε and 2ε, such that every ball among these at most $\nu = \nu(n)$ neighbours. We assume without loss of generality that the union Σ of the boundary spheres of these balls is connected and we partition V into the connected components of the complement $V - \Sigma$, say $V = \cup U_i$, where the boundary of each U_i has $\operatorname{Diam} f(\partial U_i) \leqslant \nu \operatorname{Diam} f(S_\delta)$ which uniformly (in i) goes to zero as $\varepsilon \to 0$. Then one can regularize f by using some standard continuous extension of $f \mid \partial U_i$ for each i to all of U_i within the (small) ball of radius $\rho = \operatorname{Diam} f(\partial U_i)$ in W. Furthermore, given a hypersurface $V' \subset V$, one can do the same to the (finer) partition into the connected components U'_i of $V - (\Sigma \cup V')$ as all $\operatorname{Diam} f(U'_i)$ are necessarily small for small (now, depending also on V') ε. Thus the (continuous) restriction $f \mid V' : V' \to W$ admits a *continuous* extension to all of V which, in particular, imply the contractibility of $f \mid V'$ for small spheres in V (which are contractible in V). Then, obviously, the homotopy type of $f \mid V'$ is invariant under the deformations of V' in V and also under homotopies of f in the space F_n^H for $H = T(V)$.

The same reasoning applies to (codim 1)-stable C-C manifolds V. For example, *if V is a contact C-C manifold, then every map $f \in F_N^H$, $N = n+1$, has a well defined homotopy class of the restriction of f to the $(n-1)$-skeleton of V, provided $n = \dim V \geqslant 5$.* (It is unclear what happens for 3-dimensional contact manifolds.)

2.5.F′. Regularization of F_N^H-maps. The above process of filling small U_i-holes by continuous maps (extending $f \mid \partial U_i$) allows us to approximate every $f \in F_N^H$ by a continuous map, say $f' : V \to W$, such that any two such approximation to f are mutually homotopic. More precisely, we have the following proposition (which is well known in the Riemannian case and is due, I believe, to K. Uhlenbeck).

Let V be a compact (codim 1)-*stable* C-C *manifold (i.e. with sufficiently many $(N-1)$- dimensional hypersurfaces for $N = \dim_{\mathrm{Hau}} V$, e.g. V is Riemannian or contact of dimension $\geqslant 5$) and $f : V \to W$ be a map with $\|\mathcal{D}f \mid H\|_{L_N} \leqslant c < \infty$. Then for every $\varepsilon > 0$ there is a decomposition of V, say $V = V_\varepsilon \cup V_{1-\varepsilon}$ with the following three properties.*

(1) *V_ε is an open subset in V of $\mathrm{mes}_N V_\varepsilon \leqslant \varepsilon$ and $V_{1-\varepsilon} = V - V_{1-\varepsilon}$; furthermore, each connected component U of V_ε has $\mathrm{Diam}(U) \leqslant \delta$ where $\delta \to 0$ for $\varepsilon \to 0$.*

(2) *The restriction $f \mid V_{1-\varepsilon}$ is a continuous, moreover, C^β-Hölder map for $\beta = 1/N$ (where the implied Hölder constant may depend on ε). Furthermore the image $f(\partial U)$, of the boundary of every component U of V_ε, has $\mathrm{Diam} f(\partial U) \leqslant \delta'$ where $\delta' \to 0$ for $\varepsilon \to 0$.*

(3) *If W has locally bounded geometry (e.g. compact) and ε is sufficiently small, then $f \mid V_{1-\varepsilon}$ admits a continuous extension say $f_\varepsilon : V \to W$ which is contained in F_N^H and, moreover, has $\|\mathcal{D}f_\varepsilon \mid H\|_{L_N} \leqslant \|\mathcal{D}f \mid H\|_{L_N}$. (In fact this f_ε may be chosen taut on V_ε relative to the boundary $\partial V_\varepsilon \subset V_{1-\varepsilon}$.)*

The proof follows by our earlier argument and is left to the reader.

Notice that the maps f converge to f in F_N^H for $\varepsilon \to 0$ and so they are all mutually homotopic for small ε by the weak homotopy stability observed in 2.5.A. Also notice that the regularization $f \mapsto f_\varepsilon$ applies to families of maps and shows that *the space F_N^H is homotopy equivalent to the space of continuous maps $V \to W$* (where V and W are compact and V is (codim 1)-stable, i.e. has many "nice" hypersurfaces as we always assume). In fact *the inclusion of the space C^1 of smooth maps $V \to W$ into F_N^H is a homotopy equivalence. Furthermore the space F_N^H (as well as $C^1 \subset F_N^H$ with the induced topology) is locally contractible.* (Of course, this all is well known for Riemannian manifolds V.)

Example. Let V be homeomorphic to the sphere S^n and W be an n-dimensional Riemannian manifold with locally bounded geometry. Suppose there exists a L_N^H-map $f : S^n \to W$ which has degree 1 in the following strong sense: there is an open subset $U \subset S^n$ such that the map is one-to-one on U and moreover is a homeomorphism of U onto some $U' \subset W$ with $f^{-1}(U') = U$. Then f can be approximated by a continuous map of degree 1 and so W is a homotopy sphere.

Remarks

(a) The degree 1 condition can be more succinctly expressed if V is Riemannian by $\int_V f^*(\omega) = 1$ for the normalized oriented volume form ω on W but I do not know how to do this in the general C-C case.

(b) Here and in future N refers to the Hausdorff dimension of V if it is *equiregular* and to the *formal dimension* $\max \Sigma i(n_i - n_{i-1})$ otherwise. But in fact many of our results hold true for $N = \dim_{\text{Hau}}$ under milder (genericity) assumptions than equiregularity.

2.5.G. Restriction of L_q^H-maps to k-dimensional submanifolds in codim-stable manifolds for $q < N$ and $k < q$. If V is Riemannian then the restriction $f \mid V$ is obviously continuous for generic $V' \subset V$ of dimension $k < q$ (where $f \in F_q^{T(V)}$). Furthermore if $k \leqslant q - 1$, then the homotopy type of this restriction is well defined. This is derived from the case codim $V' = 1$ as follows. First, let codim $V' = 2$, take two generic hypersurfaces V_1' and V_2' in V and observe that the restrictions $f \mid V_i' \in F_q^{T(V)}$, $i = 1, 2$, are continuous and if $q > n - 2 = \dim V_i' - 1$, then the restriction of f to the (transversal!) intersection $V' = V_1' \cap V_2'$ is also continuous. Furthermore, if $q \geqslant n - 1$ then the homotopy type of $f \mid V'$ is stable under deformations of V_1' and of V_2' by (c_2) applied to V_2' and V_1' correspondingly. Next, this intersection argument applies to all V' which are transversal intersections of hypersurfaces, e.g. to those with trivial normal bundles and then we localize in a standard fashion to make the intersection trick work for all V'. Moreover, one easily shows with a properly localized intersection argument that f defines the homotopy type of the restriction of f to the k-skeleton of V for $k = \text{ent}(q - 1)$ (i.e. k is biggest integer $\leqslant q - 1$).

Codim-stability. Call V *stable in codimension* $n - k$ if for each tangent k-plane $T' \subset T(V)$ there exists a germ of smooth submanifold V' tangent to T' such that the formal Hausdorff dimension of V' at most equals that of V minus $n - k$ (compare 2.4.B). This means, intuitively, that generic k-dimensional submanifolds V' have $\text{codim}_{\text{Hau}} V' = \text{codim}_{\text{top}} V'$.

Example. Contact manifolds are $(n-k)$-stable for $k \geqslant 3$.

If V is $(n-k)$-stable then the restriction of $f \in F_q^H$ to generic V' of dimension $k < q$ is continuous and if $k \leqslant q-1$ the homotopy class of this restriction is well defined by the above Riemannian argument. This applies in particular to k-dimensional submanifolds in contact manifolds for $k \geqslant 3$ (where the situation for $k \leqslant 2$ remains unclear).

Restriction to horizontal submanifolds V' in a contact V. Such V' are plentiful for $k < n/2$, $n = \dim V$, and the restriction $f \mid V'$ is continuous for generic horizontal V' provided $q > k$ (compare 3.1). Now we claim that *if $k \leqslant q-1$ and $k+1 < \dim V/2$, then the homotopy class of this (continuous) restriction $f \mid V'$ is stable under (horizontal!) deformations of V'. This implies (via the horizontal triangulation of V, see* 3.4.B*) that each $f \in F_q^H$ has well defined homotopy class of the restriction of f to the k-skeleton of V for every $k < (\dim V/2) - 1$ and $q \geqslant k+1$.*

Proof. To grasp the idea we start with the Riemannian case and indicate another way of reducing the case codim $V' > 1$ to that of codim $V' = 1$. For example, let V' be a small k-dimensional sphere in V and let us show that, generically, the restriction $f \mid V'$, which is continuous for $q > k$, is contractible for $q \geqslant k+1$. We make this sphere V' the boundary of a small $(k+1)$-ball $V_1' \subset V$ and observe that the restriction $f \mid V_1'$ is genericly contained in the space $F_q^{T(V_1')}$ on V_1'. Then we restrict further, to a generic sphere in V_1' concentric to $\partial V_1'$, and obtain contractible (as well as continuous) f on a *generic* small k-sphere in V. This argument easily generalizes to all (generic) $V' \subset V$ and then it extends into the contact framework with the provisions of 3.1, 3.4.A.

2.5.G$'$. On singularities of $f \in F_q^H$. One can imagine, following Karen Uhlenbeck, every map $f \in F_q^H$ as being regular (e.g. continuous) away from a certain (pole-like) singularity $\Sigma_f \subset V$ which, in the case where V is Riemannian, has "dim" $\Sigma_f \leqslant \operatorname{ent}(n-q)$. This is justified by the following three facts

(1) generic V' of dimension $k < q$ misses Σ_f as $f \mid V'$ is continuous;

(2) if $k \leqslant q-1$, then generic 1-parameter families of V''s miss Σ_f and so the homotopy class of $f \mid V'$ is well defined;

(3) the blow-up construction in 2.5.E′ applied to the balls $B^{k+1} \times b \in B^{k+1} \times B^{n-k-1} = V$, $b \in B^{n-k+1}$, gives us $f \in F_q$, with $q = k+1-\varepsilon$ for all $\varepsilon > 0$, and with $(n-k-1)$-dimensional singularity.

Now, in the C-C case, one should think of Σ_f as some virtual subset (in V or in some auxiliary jet space over V) so that generic submanifolds V' and generic *horizontal* submanifolds of the same topological dimension have different chances to meet Σ_f. (Notice that both, "generic" and "generic horizontal" submanifolds V' of *Hausdorff* dimension ℓ, miss Σ_f, $f \in F_q^H$, if $\ell < q$, and the restriction $f \mid V'$ is homotopically sound if $\ell \leqslant q-1$.)

We conclude by observing that the restriction problem is still not solved in full generality for C-C manifolds. Namely, when does every map $f \in F_q^H$ restrict to a continuous map on a generic submanifold $V' \in V$ of dimension k? When is the homotopy class of the restriction of f to the k-skeleton of V well defined? When does there exist at least one submanifold $V' \subset V$ of dimension k for which $f \mid V'$ is continuous?

2.5.H. On local estimates for taut L_N^H-maps. Consider a taut map f on an ε_0-ball $B(\varepsilon_0) \subset V$ where $\int_B \|\mathcal{D}f(v) \mid H\|^N \, dv \leqslant c_0$ for a small $c_0 > 0$ and let us evaluate the diameters of the f-images of the concentric ε-ball $B(\varepsilon)$ for $\varepsilon \leqslant 1/2\ \varepsilon_0$. Since f is taut, this diameter does not exceed the infimum of those for the concentric spheres $S(\rho)$ for $\rho \in [\varepsilon_0, \varepsilon]$ and Diam $f(S(\rho))$ is bounded by const $\rho^{-1} \int_{S(\rho)} \|Df(v) \mid H\|^N \, dv$, provided the induced C-C geometry on these spheres has formal Hausdorff dimension $N-1$. (In fact, the C-C spheres are usually non-smooth and so the formal dimension makes no sense. The true condition we need is, of course, (codim 1)-stability which allows us to approximate the spheres by piecewise smooth hypersurfaces of formal dimension $N-1$.) Then we integrate over $\rho \in [\varepsilon_0, \varepsilon]$ and conclude to the uniform continuity of f on $B(\varepsilon^0/2)$ with the logarithmic modulus of continuity,

$$\operatorname{dist}(f(v), f(v')) \leqslant \text{const}_0\ \varepsilon_0^{-1}(-\log \operatorname{dist}(v, v'))^{-\frac{1}{N}}$$

for all v, v' in $B(\varepsilon^0/2)$.

2.5.H′. Hölder estimates. Let us additionally assume that our f minimizes the energy $f \mapsto \|Df \mid H\|^N_{L_N}$ and prove that then f is Hölder. It is well known (and obvious by the previous discussion) that the Hölder bound for taut maps issues from the following monotonicity inequality

$$\int_{B(2\varepsilon)} \|\mathcal{D}f(v) \mid H\|^N \, dv \geqslant \lambda \int_{B(\varepsilon)} \|\mathcal{D}f(v) \mid H\|^N \, dv \qquad (*)$$

for $\varepsilon \leqslant \varepsilon_0/2$ and some $\lambda > 1$ independent of ε. Now, since f is minimizing, no extension of f from the sphere $S(2\varepsilon) = \partial B(2\varepsilon)$ to a map f' on $B(2\varepsilon)$ may have $\int_{B(2\varepsilon)} \|\mathcal{D}f'(v) \mid H\|^N \, dv < \int_{B(2\varepsilon)} \|\mathcal{D}f(v) \mid H\|^N \, dv$, and so the following lemma yields $(*)$.

Modification lemma. Every taut map $f_0 : B(2\varepsilon) \to W$ which lands in a (small) ball within the range of the convexity radius of W can be modified to a map $f' : B(2\varepsilon) \to W$ agreeing with f_0 on $S(2\varepsilon)$, landing in the same (small) ball in W and having

$$\int_{B(2\varepsilon)} \|Df'(v) \mid H\|^N \, dv \leqslant C \int_{A(\varepsilon)} \|Df_0(v) \mid H\|^N \, dv, \qquad (*+)$$

where $A(\varepsilon)$ denotes the annulus $B(2\varepsilon) - B(\varepsilon)$ in V.

Proof. Let B_0 be the minimal (convex!) ball in W containing the image $f_0\left(B\left(\frac{3}{2}\,\varepsilon\right)\right)$ and w_0 be the center of W_0. Observe that the radius of B_0 is bounded by $R_0 = \operatorname{Rad} B_0 \leqslant C_0 \left(\int_{A(\varepsilon)} \|\mathcal{D}f_0(v) \mid H\|^N \, dv\right)^{\frac{1}{N}}$ by our earlier argument (using (codim 1)-stability and the tautness of f_0). Then we use the notation $w \mapsto \delta w$, $\delta \in [0,1]$, for the geodesic scaling of the ball B_0 toward the center (i.e. δw stands for the geodesic convex combination $\delta w + (1-\delta) w_0$) and define f' on $B(2\varepsilon)$ by "compressing" f_0 by means of the cut-off distance function. Namely, take

$$d(v) = \begin{cases} 1 \text{ for } v \in B(2\varepsilon) - B\left(\frac{3}{2}\,\varepsilon\right), \\[2ex] (\varepsilon/2)^{-1} \operatorname{dist}(v, B(\varepsilon)) \text{ for } v \in B(\frac{3}{2}\,\varepsilon) \end{cases}$$

and $f'(v) = d(v)\, f_0(v)$. The horizontal differential of f' clearly satisfies

$$\|\mathcal{D}f'(v) \mid H\| \leqslant \varepsilon^{-1} R_0 + \|\mathcal{D}f_0(v) \mid H\|$$

all $v \in A(\varepsilon)$ and $\mathcal{D}f'(v) = 0$ on $B(\varepsilon)$. Thus $(*+)$ follows from the above bound on R_0 and the volume bound $\operatorname{mes}_N B(2\varepsilon) \lesssim \varepsilon^N$.

Remarks

(a) The minimizing property of f can be replaced by "quasi-minimizing". This means that, for every relative compact domain $U \subset V$ and every map $f' : V \to W$ obtained by a homotopy of V fixed outside U, the total energy of f' on U can not significantly smaller than that of f, i.e.

$$\int_U \|\mathcal{D}f'(v) \mid H\|^N \geqslant C \int_U \|\mathcal{D}f(v) \mid H\|^N \, dv$$

for a fixed constant $C > 0$ given beforehand (and defining our class of C-quasi-minimizing maps). Notice that quasi-minimizing maps need not to be taut but the Hölder bounds holds true all the same by a simple additional argument. Also observe that quasi-conformal (sometimes called "quasi-regular") maps between Riemannian manifolds are taut as well as quasi-minimizing for the energy $E_n = \|\mathcal{D}f\|^n_{L_n}$, $n = N = \dim_{\mathrm{Hau}} V$ and, similarly, contact quasi-conformal maps are quasi-minimizing for E_{n+1}. Thus we recapture the (well known, compare [Ko-Re]) Hölder estimate for quasi-conformal maps.

(b) One knows, thanks to K. Uhlenbeck, much more than mere Hölder for minimizing maps in the Riemannian case but the corresponding analysis is yet to be developed on C-C manifolds. In fact, our monotonicity argument is borrowed from the (well known) Riemannian situation, where, in fact, one can use a straightforward radial extension of maps from the spheres $S(\rho) = \partial B(\rho)$ to the balls $B(\rho)$. Such an extension implicitly uses the fact that the radial projection of the ball $B(\rho)$ to the sphere $S(\rho)$ is Lipschitz away from the center. Nothing of the kind will work for general C-C manifolds where smooth hypersurfaces are *not* neighbourhood Lipschitz retracts.

(c) Our estimates became global in certain cases, e.g. for *non-constant* quasi-minimizing maps f of a nilpotent group V with a natural C-C geometry into a Riemannian manifold W with non-positive curvature. Here the monotonicity argument provides a lower bound

$$\int_{B(R)} \|\mathcal{D}f \mid H\|^N \, dv \geqslant \mathrm{const}\, R^\alpha \qquad (**)$$

for some const > 0 and $\alpha > 0$ and all $R \geqslant 1$, where $B(R)$ stands for the R-ball around the origin. (Probably, honest *minimizing* maps satisfy $(**)$ with $\alpha = N = \dim_{\mathrm{Hau}} V$.) Furthermore, if f is bounded one can easily replace const R^α by const $\exp \alpha R$, but it is unclear if non-constant bounded (quasi)minimizing maps exist at all (Liouville problem).

(d) Let us indicate an integrated version of our "compressing" argument in the proof of the modification lemma which sometimes applies to the energy $E_q = \int \|\mathcal{D}f \mid H\|^q$ for $q < N$. Now we use variable $w_0 = f_0(v_0)$, where v_0 runs over the annulas $A(\varepsilon) = B(2\varepsilon) - B(\varepsilon)$. Then we estimate the average of the energies of the extended maps $f'_{v_0}(v)$ obtained by "compression" toward $w_0 = f_0(v_0)$. We write this as if W were a Euclidean space, namely

$$f'_{v_0}(v) = d(v)(f_0(v) - f_0(v_0))$$

and have, after the normalization making $\varepsilon = 1$,

$$\|\mathcal{D}f'_{v_0}(v)\| \leqslant 2(f_0(v) - f_0(v_0)) + \|\mathcal{D}f_0\|.$$

Then our average is bounded by,

$$(\operatorname{Vol} A(\varepsilon))^{-1} \int_{A(\varepsilon)} dv_0 \int_{B(2\varepsilon)} \|\mathcal{D}f'_{v_0}(v) \mid H\|^q \, dv \leqslant$$

$$\text{const}\left(\int_{A(\varepsilon)} \|Df_0 \mid H\|^q \, dv + \iint\limits_{A(\varepsilon)\times A(\varepsilon)} |f_0(v) - f_0(v_0)|^q \, dv \, dv_0 \right).$$

In order to make this work we need an estimate for the second integral on the right hand side by the first one (where $|f_0(v) - f_0(v_0)|$ should be understood as $\operatorname{dist}_W(f_0(v, f_0(v_0))$. If $\dim V \geqslant 2$ and $A(\varepsilon)$ is connected, the desired estimate is a trivial case of the Sobolev inequality (which is discussed in sharper form below) and so the monotonicity comes along if f_0 lands within the convexity radius of W, for example if W is complete simply connected with non-positive curvature. Unfortunately, this does not yield Hölder for $q < N$ although this likely to be true for many E_q-minimizing maps for $q > 1$. (One may try here the standard Riemannian trick of infinitesimal deformations of f along radial fields similar to our "compression".)

2.5.I. On semicontinuity of the energy $E_q(f)$. We want to know that if f_i weakly converge to f for $i \to \infty$ then

$$E_q(f) \leqslant \liminf_{i\to\infty} E_q(f_i),$$

which is important for the calculus of variations. If the limit map is a.e. smooth this semicontinuity can be sometimes derived from the E_q-minimizing property of linear maps between Euclidean spaces or some other standard maps between relevant spaces approximating our f at the regular points. Here we indicate another approach based on a regularization of energy functionals (compare [Jo]). First we introduce a class of energies similar (and somewhat more general) than E_q. Such an energy will be constructed with some auxiliary space M endowed with the following structures

(1) Projection $p : M \to V$

(2) A measure $d\mu$ on M

(3) A vector field X on M.

Our main requirement is that each vector X_m, $m \in M$, is sent by p to a *horizontal* vector at $v = p(m) \in V$. Here we require just enough regularity of M and p make sense of this. For example, M may be a smooth manifold and p a smooth map, but, in general, p should be smooth only along the orbits of X and just measurable in the transversal directions. Now we can define the energy $E_q^\mu(f)$ by

$$E_q^\mu(f) = \int_M \|X(p \circ f)\|^q \, d\mu$$

where $X\ \varphi \underset{\mathrm{def}}{=} (\mathcal{D}\varphi)(X)$. If the push-forward of the measure $d\mu$ under the map $M \to H$ for $m \mapsto X_m\ p$ dominates in an obvious way the measure of the unitary subbundle of H, then E_q^μ is equivalent to E_q, i.e.

$$C^{-1}\ E_q \leqslant E_q^\mu \leqslant C\ E_q,$$

and for suitable $(M, d\mu)$ one may have $E_q^\mu = E_q$.

In what follows we add an extra assumption on X, and $d\mu$

(4) X integrates to a flow $X(t)$ on M and this flow preserves the measure $d\mu$.

Examples

(a) If V is Riemannian (i.e. $H = T(V)$), then the unit tangent bundle with the Liouville measure and the geodesic flow provides a model example.

(b) Fix a measure on V (e.g. the Hausdorff measure), let A be a family of measure preserving horizontal fields X_a on V (these, as we know, are plentiful) and let $M = V \times A$ with the product measure for some measure on A and the field $(v, a) \mapsto X_a$ at v in $V = V \times a$.

Now we introduce the following ε-regularization of the energy.

$$E_q^{\mu,\varepsilon} = \varepsilon^{-1} \int_M (\mathrm{dist}_W(p \circ f(m), p \circ f(X(\varepsilon)m)))^q \; d\mu.$$

It is clear (without (4)) that $E_q^{\mu,\varepsilon} \to E_q^\mu$ for $\varepsilon \to 0$. What is more interesting (albeit obvious), is the inequality

$$E_q^{\mu,\varepsilon} \leqslant E_q^\mu \quad \text{for all } \varepsilon > 0$$

which follows by integration and use of (4) from the corresponding inequality on a single X-orbit in M.

Finally we observe that every energy $E_q^{\mu,\varepsilon}(f)$ for $\varepsilon > 0$ is continuous with respect to the uniform topology in the space of maps f and therefore, E_q^μ is semicontinuous in this topology. Now we can apply it to E_N^μ when it is equivalent to E_N and prove *the existence of the energy minimizing Hölder map in every homotopy class of maps $V \to W$, provided V is* (codim 1)-*stable (e.g. being contact of dimension $\geqslant 5$) and $\pi_n(W) = 0$.* (If $\pi_n(W) \neq 0$ one realizes the homotopy classes modulo the action of $\pi_1(W)$ on these classes as is seen by looking at the bubbling picture.)

On V's which are not (codim 1)-*stable.* If rank $H \geqslant 3$, then, genericly, the polarization $H' = H \cap T(V')$ Lie-generates $T(V')$ (at least, apart from a stratified subset of positive codimension) and so the (formal) Hausdorff dimension of (V', H') is finite, say N'. Then our regularity arguments for $f \in F_N^H$ remain valid with N replaced by $N' + 1$ but this does not seem to do us any good as we have Hölder estimates for $F_{N+\varepsilon}^H$ anyway.

2.6. Isoperimetric inequalities and quasi-conformal mappings. The theory of quasi-conformal (= quasi-regular) mappings between *Riemannian* manifolds, say $f : W \to V$, relies on isoperimetric inequalities in V for large domains $D \subset V$ which for the nilpotent groups V are governed by the corresponding Carnot-Carathéodory isoperimetric inequalities. (In fact, this motivated the study of such inequalities, see [G-L-P] and [$\mathrm{Pan}_{\mathrm{InIs}}$].) Let us spell it out in details. Suppose every domain D in a Riemannian manifold V of volume $\geqslant \mu_0$ satisfies the isoperimetric inequality with the exponent $\alpha = \frac{N}{N-1}$ for $N > n = \dim V$, i.e.

$\mathrm{Vol}_n\, D \leqslant C(\mathrm{Vol}_{n-1}, \partial D)^\alpha$, provided $\mathrm{Vol}_r\, D \geqslant \mu_0$. Then *quasi-conformal mappings $f : W \to V$ are uniformly continuous (and even Hölder) if* $|K(W)| \leqslant \mathrm{const} < \infty$ by the following standard argument. Consider concentric R-balls in W mapped into V, say $B(R) \subset W$, and let $p(w)$ denotes the Jacobian of f. Then $\mathrm{Vol}_n\, f(B(R)) = \int_{B(R)} p(w)\, dw$ and by quasi-conformality, the volume of the sphere $S(R) = \partial B(R)$ in the image satisfies

$$\mathrm{Vol}_{n-1} f(S(R)) \leqslant C_f \int_{S(R)} p^{\frac{n-1}{n}}(w)\, dw.$$

If for some $R_0 < R$ the volume $\mu(R) = \mathrm{Vol}_n\, f(B(R))$ satisfies $\mu(R_0) \geqslant \mu_0$, then the isoperimetric inequality applied to $f(B(R))$ provides a lower bound on $\sigma(R) = \int_{S(R)} p^{\frac{n-1}{n}}(w)\, dw$, that is

$$\sigma(R) \geqslant C_1(\mu(R))^\beta \quad , \quad \beta = \frac{N-1}{N}.$$

On the other hand, since

$$\mu(R) = \int_{B(R)} p(w)\, dw = \int_0^R dR \int_{S(R)} p(w)\, dw,$$

the derivative $\mu'(R) = \int_{S(R)} p(w)\, dw$ satisfies by Hölder inequality,

$$\mu'(R) \geqslant (\sigma(R))^{\frac{n}{n-1}}\ (\mathrm{Vol}_{n-1}\ S(R))^{-\frac{1}{n-1}},$$

which implies with the above that

$$\mu'(R) \geqslant C_2(\mu(R))^\gamma\, \mathrm{Vol}_{n-1} S(R)^{-\frac{1}{n-1}}\ , \text{ for } \gamma = \frac{\beta n}{n-1} > 1.$$

Now, if $\mathrm{Vol}_{n-1} S(R)$ grows no faster than $C_0\, R^{n-1}$, the ratio R/R_0 must be bounded by

$$R/R_0 \leqslant \mathrm{const} = \mathrm{const}(\mu_0, C_2, \gamma, C_0) = \mathrm{const}(\mu_0, N, C, C_0),$$

since the above differential inequality for $\mu(R)$ with the initial condition $\mu(R_0) \geqslant \mu_0$ predicts the blow-up of $\mu(R)$ at some moment after R_0. Thus we obtain a bound on $\int_{B(R_0)} p(w)\, dw$ and, hence, by quasi-conformality, a similar bound on the L_n-norm of the differential,

$$\int_{B(R)} \|\mathcal{D}f(w)\|^n\, dw = \int_0^R dR \int_{S(R)} \|\mathcal{D}f(w)\|^n\, dw.$$

If the spheres $S(R)$ have (roughly) the standard geometry, the bound on the integral $\int_{S(R)} \|\mathcal{D}f(w)\|^n$ implies, by the Sobolev inequality, the bound on Diam $f(S(R))$, and therefore (by openness of the map f) the required bound on Diam $f(B(R_0))$. Finally, in order to use all this for bounded curvature $|K(W)| \leqslant$ const, we notice that up to the last moment we only needed the bound on $\mathrm{Vol}_{n-1} S(R) \lesssim R^{n-1}$ ensured by Ricci $W \geqslant -$const, while the standardization of the geometry of $S(R)$ is achieved for $|K(W)| \leqslant$ const by using small balls in $T_w(W)$ immersed into W by the exponential map (which may be non-injective).

Remarks and corollaries

(a) Since the isoperimetric inequality applies to multiple domains (see 2.3.D (c)) the above argument allows non-injective maps f which may, moreover, have ramification points.

(b) The uniform continuity property remains valid whenever V satisfies an isoperimetric inequality which is asymptotically (for large domains) stronger than the Euclidean one). In this case one obtains a differential inequality $\mu' \geqslant \psi(\mu, R)$ which makes μ grow faster than R^n though not forcing a blow up of $\mu(R)$ for $R < \infty$. Then one uses the (obvious in this case) fact that the ball $B(R)$ can be covered by at most $k = c(R/R_0)^n$ of balls of radius $R_0/2$ and therefore some of these balls, say $B_1(R_0/2)$ around some $v_1 \in B(R)$ satisfies

$$\mathrm{Vol}\, f(B_1(R_0/2)) \geqslant 2\,\mathrm{Vol}\, f(B(R_0)),$$

since $\mu(R) = \mathrm{Vol}\, f(B(R))$ grows faster than R^n and we choose R/R_0 large. Then we find next ball $B_2(R_0/4)$ with center $v_2 \in B_1$ and so on. These balls B_i, $i = 1, 2, \ldots$, have infinite volume altogether and they are contained in $B(2R)$. Thus $\mathrm{Vol}\, f(B(R))$ does become infinite in finite time unless $\mathrm{Vol}\, B(R_0)$ was rather small, exactly as in the case considered earlier. We conclude by noticing that even the Euclidean inequality $\mathrm{Vol}_n D < C(\mathrm{Vol}_{n-1}\ \partial D)^{\frac{n}{n-1}}$ may be used for this purpose if the constant C is sufficiently small with respect to the quasi-conformality constant of f.

(c) The key use of the quasi-conformality in our argument is the inequality between the norms of the differential $\mathcal{D}f$ on the forms of degree n and $n-1$, i.e.

$$\left(C_f^{-1} \| \wedge^{n-1}\ \mathcal{D}f(w)\|\right)^{\frac{n}{n-1}} \leqslant \| \wedge^n\ \mathcal{D}f(w)\| \underset{\mathrm{def}}{=} p(w).$$

This suggests introducing more general classes of *pinched* maps satisfying

$$\| \wedge^{n-1} \ \mathcal{D}f(w)\| \leqslant A\| \wedge^{n} \ \mathcal{D}f(w)\|^{a} + B\| \wedge^{n} \ \mathcal{D}f(w)\|^{b}$$

for certain a and b close to $\frac{n-1}{n}$. One knows that such maps do share some quasi-conformal properties (see [Deg], and § 7.C in [Gro$_{\mathrm{FPP}}$]) and they seem to be relevant in the Hölder geometry of C-C spaces. (Another avenue of generalizing quasi-conformality is suggested by taut E_N-quasi-minimizing maps mentioned earlier in 2.5.H$'$.)

(d) If V is a simply connected non-Abelian nilpotent groups it satisfies, as we know, an asymptotically N-dimensional isoperimetric inequality for $N > n = \dim V$ and so quasi-conformal maps $f : W \to V$ are uniformly continuous if $|K(W)| \leqslant \text{const} < \infty$. Furthermore, every quasi-conformal map $f : \mathbb{R}^n \to V$ is constant. Now, if W also such a non-Abelian nilpotent group and f is bijective as well as quasi-conformal than f is *quasi-isometric*, i.e. bi-Lipschitz on the large scale. Then the corresponding *asymptotic tangent cones* $V^\infty = \lim\limits_{\varepsilon\to 0} \varepsilon V$ and $W^\infty = \lim\limits_{\varepsilon\to 0} \varepsilon W$, which are certain nilpotent groups with self-similarities, are bi-Lipschitz equivalent with respect to their (limit) C-C metrics and by Pansu theorem they are isomorphic as Lie groups (see [Pan$_{\mathrm{QIR1}}$]). In particular, if V and W admit dilating automorphisms (e.g. being of nilpotency degree two) then the existence of a quasi-conformal homeomorphism $V \leftrightarrow W$ makes them isomorphic.

(d') The above is probably not hard to generalize to some "pinched" homeomorphisms $V \leftrightarrow W$ but it seems more difficult to decide when there exists a non-injective non-constant quasi-conformal map of a non-Abelian nilpotent group into another (possibly Abelian) nilpotent group (compare [Hol], [Hol-Rick]).

(e) According to Varopoulos, every discrete group Γ which grows faster than $\mathbb{Z}^n$ satisfies the isoperimetric inequality with the exponent $\alpha = \frac{N}{N-1}$ for $N \geqslant n+1$. It follows, that if Γ serves as the fundamental group of a closed manifold $\overline{V}$ of dimension $n < N = N(\Gamma)$, then quasi-conformal maps $W \to \overline{V}$ which lift to the universal covering V of $\overline{V}$ are uniformly continuous for $|K(W)| \leqslant \text{const} < \infty$. Furthermore, if W also appears as the universal covering of a closed manifold whose fundamental group Γ' grows faster than $\mathbb{Z}^n$, then the existence of a quasi-conformal homeomorphism makes Γ and Γ' quasi-isometric. Probably, this remains true with no extra assumption on the dimension.

Conjectures

(i) *If the universal coverings of two closed manifolds are quasi-conformally homeomorphic then the fundamental groups are quasi-isometric.*

(ii) *If the group $\Gamma = \pi_1(\overline{V})$ is not virtually Abelian, then quasi-conformal maps $W \to \overline{V}$ are uniformly continuous under the standing assumptions $|K(W)| \leqslant \text{const} < \infty$.*

(iii) **Example.** Let $V = V_1 \times V_2$, where V_1 is a simply connected non-Abelian nilpotent Lie group and V_2 is a closed manifold. Then (ii) implies that every quasi-conformal map $\mathbb{R}^n \to V$ is constant. (This example as well as the above conjectures can be probably solved with the techniques developed in [Hol-Rick].

(iv) The notion of a quasi-conformal map makes sense for Carnot-Carathéodory spaces. Such maps, however, are rather rare species for general C-C spaces (see [Pan_{QIR1}] where quasi-conformal maps are shown to be conformal in many cases) but they are plentiful in the contact case due to the abundance of contact maps. Our uniform continuity proof easily extends to the C-C category and all applications have C-C counterparts. (The reader may replace everywhere "manifold" by "contact manifold" and ponder over the significance of resulting theorems and conjectures. Then we suggest the paper [Kor-Rei] for a more systematic and profound study of contact quasi-conformal maps. Notice that these maps were discovered by Mostow [Mos] in his remarkable work on the rigidity of locally symmetric spaces of rank one.) Finally, we want to bring reader's attention to "pinched" maps in the C-C category which are (at least) as abundant as Hölder maps and to which some quasi-conformal techniques still apply.

3. Carnot-Carathéodory geometry of contact manifolds

Recall that a contact structure on V is given by a codimension one polarization $H \subset T(V)$ with non-degenerate *curvature form* $\Omega : H \wedge H \to T(V)/H$ which can be defined in the following two equivalent fashions.

(1) Represent H locally as the kernel of a 1-form, say η on V, identify $T(V)/H$ with the trivial line bundle and then define Ω as $d\eta|H$. Notice that this can be done globally on V if H is *coorientable*, i.e. if the line bundle $T(V)/H$ is trivial.

(2) Define $\Omega(X,Y)$ on pairs of vector fields tangent to H by $\Omega(X,Y) = [X,Y] \bmod H$ and verify this is indeed a 2-form, i.e. $\Omega(aX,bX) = ab\,\Omega(X,Y)$ for arbitrary smooth functions a and b on V.

To simplify the matter we assume below that H is coorientable, we fix η and write ω for $d\eta|H$. This is an ordinary 2-form on the bundle H and "non-degenerate" applies to this form in the usual sense. Notice that the non-degeneracy of ω makes $\operatorname{rank} H$ even and so $n = \dim V$ is odd, say $n = 2m+1$. Also observe that if $\omega(=\Omega)$ does not vanish (which is the case for contact structures and $n \geq 3$) then the commutators of degree ≤ 2 span $T(V)$ and so C-C balls in V look as $(\underbrace{\varepsilon\times\varepsilon\times\cdots\times\varepsilon}_{2m}\times\varepsilon^2)$-boxes, where the $(\varepsilon\times\varepsilon\times\cdots\times\varepsilon)$-face is (roughly) tangent to H and the ε^2-edge is transversal to H. In particular, the Hausdorff dimension of V equals $n+1$ for $n = \dim_{\mathrm{top}} V$ and this remains valid for more general (non-contact) H where the non-degeneracy of ω is weakened to mere non-vanishing of ω.

The equality $\dim_{\mathrm{Hau}} V = n+1$ implies that there exists no C^α-Hölder homeomorphism (or just surjective map)

$$f : (V, \mathrm{Riem\,.\,metric}) \to (V, \text{C-C metric})$$

for $\alpha > \frac{n}{n+1}$. On the other hand, the identity map is C^α for $\alpha = \frac{1}{2}$ and one may suspect that there are no V^α-homeomorphisms for $\alpha > \frac{1}{2}$. We shall prove below the inequality $\alpha \leq \frac{m+1}{m+2}$ with $m = (n-1)/2$ for C^α-homeomorphisms $(V, \mathrm{Riem}) \to (V, \text{C-C})$ which improves the above bound $\alpha \leq \frac{n}{n+1}$. Here is the basic C-C feature of contact manifolds that makes this possible.

3.1. Abundance of contact horizontal submanifolds. *If $k \leq m = \frac{n-1}{2}$, then there are plenty of k-dimensional H-horizontal submanifolds in V. In particular, every continuous map $\varphi_0 : \mathbb{R}^k \to V$ can be uniformly approximated by smooth immersions φ everywhere tangent to H.*

This is proven in full generality in § 3.4.3 of [Gro$_{\mathrm{PDR}}$] on the basis of a suitable h-principle but the abundance of H-horizontal spaces can be seen quite elementarily as follows. Every contact manifold admits by Darboux' theorem local coordinates (near each point), say, $x_1, \ldots, x_m, y_1, \ldots, y_m, z$, such that $H = \operatorname{Ker} \eta$ for

$$\eta = dz + x_1 dy_1 + \cdots + x_m dy_m.$$

Now every function f on $\mathbb{R}^m$ with coordinates $y_1, \ldots, y_m$ defines the following (jet) map $J_f : \mathbb{R}^m \to \mathbb{R}^{2m+1}$,

$$J_f : (y_1, \ldots, y_m) \mapsto (x_i = \frac{\partial f}{\partial y_i}, y_i = y_i,\ z = -f),$$

which is H-horizontal, since

$$J_f^*(\eta) = -df + \sum_{i=1}^{m} \frac{df}{dy_i} dy_i = 0.$$

The images $J_f(\mathbb{R}^m) \subset V$ for various f and their contact transforms in V constitute the bulk of H-horizontal manifolds in V needed for C-C geometry, but it is faster to refer to the general results in [$\mathrm{Gro}_{\mathrm{PDR}}$].

3.1.A. A lower bound on the Hausdorff dimension of k-dimensional subsets in V for $\mathbf{k > n - 1/2}$. The above "horizontal abundance" has two somewhat opposite C-C uses. First it tells us that there are plenty of injective Lipschitz maps $\mathbb{R}^k \to V$ for $k \leq m = (n-1)/2$ which provide k-dimensional submanifolds having

$$\text{C-C-}\dim_{\text{Hau}} = k = \dim_{\text{top}}.$$

Secondly, one easily shows with "horizontal abundance" that *every compact k-dimensional subset $V' \subset C$ for $k \geq m+1$ satisfies* C-C-$\dim_{\text{Hau}} V' \geq m + 2$.

Proof. Since $\dim_{\text{top}} V' = k$ there exists a continuous map $\mathbb{R}^{n-k} \to V$, such that every nearby map meets V (compare Alexandroff theorem in 4.5). Now, by 3.1, we may assume that this map is H-horizontal and moreover, we can easily arrange a "parallel family" of such maps that define a submersion of some neighbourhood $U \subset V$ onto $\mathbb{R}^k$, say $\psi : U \to \mathbb{R}^k$, such that the levels $\psi^{-1}(y) \in U$ are H-horizontal for all $y \in \mathbb{R}^k$ and such that $\psi(U \cap V') = \mathbb{R}^k$. Then we use the ball-box theorem as in 2.1 and conclude that $\dim_{\text{Hau}} V' \geq k + 1$. ∎

Corollary. *Every C^α-embedding $\mathbb{R}^k \to V$ for $k \geq m + 1$ has $\alpha \leq \frac{m+1}{m+2}$. In particular there is no C^α-homeomorphism (and even no C^α-map of locally non-zero degree) $\mathbb{R}^n \to V$ for $\alpha > \frac{m+1}{m+2}$.*

Question. Can one improve upon this bound on α?

3.2. Polarizations with degenerate curvature ω. Suppose the (curvature) form $\omega(=\Omega)$ on H has constant rank $2r > 0$. Then every field X in Ker ω satisfies $[X, Y] = 0 \bmod H$ for all Y in H. This implies that the subbundle Ker $\omega \subset H \subset V$ is integrable of dimension $2m - 2r$ and so, locally, the polarization H is induced by a smooth map $V \to V_0^{2r+1}$ from some contact structure H_0 on V_0^{2r+1}. Now we see that V contains (quite a few) of H-horizontal submanifolds of dimension $n - r - 1$ which are pull-backs of r-dimensional H_0-horizontal manifolds in V_0. These give us Lipschitz embeddings $\mathbb{R}^k \to V$ for $k \le n - r - 1$ and topological k-dimensional submanifolds $V' \subset V$ with

$$\dim_{\mathrm{top}} V' = \dim_{\mathrm{Hau}} V' = k.$$

But such V' are rather exceptional for $k > r$, because if the projection of V' to V_0 is not totally degenerate, i.e. has $\dim_{\mathrm{top}} = k > r$, then $\dim_{\mathrm{Hau}} V' \ge k + 1$. It (obviously) follows that V receives no C^α map $\mathbb{R}^n \to V$ of locally positive degree for $\alpha > \frac{r}{r+1}$. It is also clear that the $r = \operatorname{rank} \omega$ metrically distinguishes the corresponding C-C manifolds. Moreover, the same argument shows that

every proper C^α-map $V_{r_1} \to V_{r_2}$ of non-zero degree has

$$\alpha < \frac{n - r_2}{n - r_2 + 1}, \text{ if } r_1 < r_2,$$

and

$$\alpha < \frac{r_2 + 1}{r_2 + 2}, \text{ if } r_1 > r_2 (> 0).$$

(Here r_1 and r_2 denote the $\frac{1}{2}$ranks of the implied curvature forms on V_{r_1} and V_{r_2} while $\dim V_{r_1} = \dim V_{r_2} = n$ and basic examples of maps of locally positive degree are given by homeomorphisms $V_{r_1} \to V_{r_2}$.)

In fact, the first inequality is detected with $(n - r_2)$-dimensional horizontal submanifolds in V_{r_1}: these have $\dim_{\mathrm{Hau}} = n - r_2$, in V_{r_1} but the images (of some of them) in V_{r_2} must have $\dim_{\mathrm{Hau}} = n - r_2 + 1$. In the second case, where $r_1 > r_2$, one uses horizontal $(r_2 + 1)$-dimensional manifolds. These are "dense" in V_{r_1} but not in V_{r_2} and so the Hausdorff dimension (of some of them) jumps from $r_2 + 1$ to $r_2 + 2$ under our map $V_{r_1} \to V_{r_2}$.

Question. Let $r_1 = r_2 = r$. Then one expects that C^α-homeomorphisms $V_r \to V_r$ with α close to one preserve the foliation defined by Ker ω. Intuitively, this foliation can be visualized in the C-C geometry of V_r by observing that the leaf through a given point $v \in V_r$ equals the intersection of all "sufficiently regular" (or "generic") submanifolds $V' \subset V_r$ passing through v and having

$$\dim_{\mathrm{Hau}} V' = \dim_{\mathrm{top}} V' = n - r - 1.$$

(Another characteristic property of this foliation is the existence of "many" Lipschitz homeomorphisms of V_2 preserving the leaves.) The question is how to make a *rigorous* description of this foliation in C-C C^α-Hölder terms for some $\alpha = 1 - \varepsilon$.

3.3. Differential forms and straight Alexander-Spanier cocycles. Instead of H-horizontal submanifolds in V one may use H-horizontal exterior forms a on V where "horizontality" means "vanishing on H" or, equivalently, representability by $\eta \wedge b$ for the 1-form η defining H (which we assume coorientable in this section). Such forms can be either obtained with measures on (sufficiently rich) families of H-horizontal submanifolds (viewed as currents, compare 2.2.A.) or by a purely algebraic consideration based on the following well-known property of *non-degenerate* forms ω on H.

Lefschetz lemma. *The operator $\Lambda^k H \to \Lambda^{k+2} H$ defined by the exterior product with ω (where $h \mapsto \omega \wedge h$) is injective for $k \le m-1$ and surjective for $k \ge m-1$ (where $m = \frac{1}{2}\operatorname{rank} H$).*

Using this lemma we obtain a complete description of closed H-horizontal k-forms on V. Namely

Every closed H-horizontal k-form vanishes if $k \le m$ On the contrary, if $k \ge m+1$, then such forms are plentiful, e.g. *every* k-dimensional de Rham cohomology class can be represented by a closed H-horizontal form. This is proven by inverting the exterior differential d on horizontal forms. More explicitly, let $H\Lambda^k(V) \subset \Lambda^k(V)$ denote the bundle (as well as the sheaf) of horizontal forms on V and observe that $\Lambda^k H = \Lambda^k(V)/H\Lambda^k(V)$. Then we denote by $\bar{d} : \Lambda^k(V) \to \Lambda^{k+1} H$ the composition of d with the projection (restriction homomorphism) $\Lambda^{k+1}(V) \to \Lambda^{k+1}(H)$ and let d' stand for the restriction of d to $H\Lambda^k(V) \subset \Lambda^k(V)$.

Algebraic inversion lemma. ([8]) *There exist homomorphisms (i.e. differential operators of order zero)*

$$\bar{\delta} : \Lambda^k(H) \to \Lambda^{k-1}(V) \quad \text{for} \quad k \geq m+1$$

and

$$\delta' : \Lambda^{k+1}(V) \to H\Lambda^k(V), \quad \text{for} \quad k \leq m,$$

such that

$$\bar{d}\,\bar{\delta} = \mathrm{Id} \text{ and } \delta' d' = \mathrm{Id}\,.$$

Proof. First we construct $\bar{\delta}$ by defining $x = \bar{\delta}\,\bar{a}$ as a (canonically chosen) solution of the equation $\bar{d}x = \bar{a}$ which is equivalent to the equation $dx|H = \bar{a}$. We solve this by taking $x = \eta \wedge y$ for the form η defining H and observe that $d(\eta \wedge y)|H = d\eta \wedge y|H$ where the 2-form $\omega = d\eta|H$ is non-singular and the equation $d\eta \wedge y|H = \bar{a}$ is solvable for $\deg y \geq \frac{1}{2}\operatorname{rank} H - 1$ by the Lefschetz lemma. In fact, since the operator $y \mapsto d\eta \wedge y$ is surjective it admits a right inverse, say ω_r^{-1}, such that $d\eta \wedge \omega_r^{-1} y = y$, and then the operator $\bar{\delta} : \bar{a} \mapsto \eta \wedge \omega_r^{-1}\,\bar{a}$ serves as the required right inverse for $\bar{d}$. Next, we turn to d' where we have to show that the equation $d'x' = 0$ implies $x' = 0$. This equation is equivalent to $d(\eta \wedge x) = 0$ or $d\eta \wedge x = \eta \wedge dx$ which says that $d\eta \wedge x|H = 0$, and so $x|H = 0$ according to the Lefschetz lemma. Hence, $x' = \eta \wedge x = 0$ and the required left inverse δ' of d' may be given by $\delta'(a) = \eta \wedge (\omega_{\mathrm{left}}^{-1} a|H)$.

Remarks. (a) These operators $\bar{d}$ and d' are mutually formally adjoint and so are their inverses $\bar{\delta}$ and δ'.

3.3.A. Rumin complex. The above properties of differential forms on contact manifolds trivially follow from the elegant *contact de Rham theorem* discovered by Michel Rumin in his theses (see [$\mathrm{Rum}_{1;2}$]). To state this theorem we denote by $I^* \subset \Lambda^*(V)$ *the differential ideal* generated by η, i.e. I^* consists of the differential forms on V representable as $\eta \wedge x + d\eta \wedge y$ for some forms x and y. Then we observe that the exterior differential d sends I^k to I^{k+1} for $i = 0, \dots, 2m$ which gives us operators $\Lambda^k(V)/I^k \to \Lambda^{k+1}(V)/I^{k+1}$, denoted by d_H. Notice that these operators (and the spaces they act upon) depend only on $H = \ker \eta$ and so they are contact invariant. The operators d_H work well below the middle dimension while above the middle dimension one uses d^H, that is the

[8] Compare 2.A, 4.11.

restriction of d to the *annihilator* J^* of I^* with respect to the exterior product. (*Closed* forms in J^* are the same as closed H-horizontal forms). The following operator crosses the middle line.

Rumin operator. $D : \Lambda^m(V)/I^m \to J^{m+1}$. *There exists a unique operator* $\tilde{D} : \Lambda^m(H) \to H\Lambda^{m+1}$, *such that* $\tilde{D}(\alpha) = d\tilde{\alpha}$ *for every m-form $\tilde{\alpha}$ on V satisfying $\tilde{\alpha}|H = \alpha$ and $d\alpha \in H\Lambda^{m+1}$. Furthermore $\tilde{D}$ passes to the quotient* $\Lambda^m(V)/I^m = \Lambda^m(H)/\{\omega \wedge x \mid x \in \Lambda^{m-2}H\}$, $\omega = d\eta|H$, *i.e.* $\tilde{D}(\omega \wedge x) = 0$ *and D is defined as the resulting operator*

$$\Lambda^m(V)/I^* \to H\Lambda^{m+1} \subset J^{m+1}.$$

The proof is straightforward with the Lefschetz lemma. Notice, following Rumin, that D is a second-order differential operator, since the lift $\alpha \mapsto \tilde{\alpha}$ is a first-order operator as is clear from the computation needed to define $\tilde{D}$.

Rumin-de Rham theorem. *The sequence*

$$0 \longrightarrow \mathbb{R} \longrightarrow \Lambda^0(V) \xrightarrow{d_H} \Lambda^1(V)/I^1 \xrightarrow{d_H} \cdots \longrightarrow$$
$$\Lambda^m(V)/I^m \xrightarrow{D} J^{m+1} \xrightarrow{d^H} \cdots \longrightarrow J^{2m+1} \longrightarrow 0$$

is a locally exact complex. Its cohomology is isomorphic to the usual de Rham cohomology.

In fact, Rumin's proof (essentially explained above) provides a chain homotopy equivalence between Rumin's and de Rham complexes where all homomorphisms (especially chain homotopies) are given by *differential* operators. (See [$\mathrm{Vin}_{1;2;3}$], [Br-Gr] and [$\mathrm{Ge}_{\mathrm{BNCC}}$] for further algebraic results of this kind.)

3.3.B. Construction of straight Alexander-Spanier cocycles with a controlled growth at the diagonal. Recall that the real k-dimensional cohomology of V can be represented by *straight* (Alexander-Spanier) cochains that are functions $c(v_0, v_1, \dots, v_k)$ defined arbitrarily near the principal diagonal in $\underbrace{V \times V \times \cdots \times V}_{k+1}$. If V is given a metric then we may restrict c to the ε-neighbourhood U_ε of the diagonal for small $\varepsilon > 0$ (here V is compact) and the supremum of $|c|$ on U_ε for $\varepsilon \to 0$

is a relevant characteristic of c. (I have picked up this idea from Alain Connes).

Example. Let V be Riemannian and c represent a non-zero class. Then $\|c\|_\varepsilon \underset{\text{def}}{=} \sup |c| \big| U_\varepsilon \gtrsim \varepsilon^k$. To see this we fix a homology class c' on which $[c]$ does not vanish and then, for each $\varepsilon > 0$, represent c' by a cycle built of $N_\varepsilon \approx \varepsilon^{-k}$ geodesic Riemannian simplices of diameter $\leq \varepsilon$. Now the lower bound on $\|c\|_\varepsilon$ follows from the obvious inequality which is valid for all sufficiently small $\varepsilon > 0$,

$$[c](c') \leq \operatorname{const} N_\varepsilon \|c\|_\varepsilon .$$

Now let V be a manifold with an equiregular polarization H of arbitrary codimension. Then every $(k+1)$-tuple of points $(v_0, \dots, v_k)$ in the C-C ball $B(v, \varepsilon) \subset V$ can be canonically spanned by an actual smooth simplex Δ_ε with vertices $v_0, \dots, v_k$ of C-C diameter $\leq \operatorname{const} \varepsilon$. This can be done locally, for example, by using (local) exponential maps corresponding to (locally defined) full frames of vector fields on V formed by the commutators of suitable H-horizontal fields. If ω is a smooth k-form on V which vanishes on H, then

$$\int_{\Delta_\varepsilon} \omega \leq \varepsilon^{k+1}.$$

In fact, the ball-box theorem allows one to reduce this inequality to the obvious special case where $V = \mathbb{R}^n$ and Δ_ε is spanned by k vectors contained in the box

$$B_\varepsilon = \{|x_i| \leq \varepsilon,\ i = 1, \dots, n_1,\ |x_i| \leq \varepsilon^r,\ i = n_1 + 1, \dots, n\} \subset \mathbb{R}^n$$

for $n_1 = \operatorname{rank} H$ and $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where $i_k \geq n_1 + 1$. (See 4.1.C$'$ for a more general result.)

Corollary. *Let V be a contact $(2m+1)$-dimensional Carnot-Carathéodory manifold. Then every k-dimensional cohomology class with compact support for $k \geq m+1$ can be represented by a straight cochain c satisfying $\|c\|_\varepsilon \leq \operatorname{const} \varepsilon^{k+1}$ for all $\varepsilon > 0$.*

Notice that if $H^k(V; \mathbb{R}) = 0$ this Corollary may be applied to open subsets $U \subset V$ with $H^k(U; \mathbb{R}) \neq 0$. Thus one proves once more that the identity map $(V, \text{Riem.}) \to (V, \text{C-C})$ cannot be uniformly approximated by C^α-maps with $\alpha > \frac{m+1}{m+2}$.

One can get more mileage from Alexander Spanier cochains by observing that the norm $\|c\|_\varepsilon$ is semimultiplicative under the cup-product

$$\|c_1 \smile c_2\|_\varepsilon \le \operatorname{const} \|c_1\|_\delta \, \|c_2\|_\delta\,, \quad \delta = 2\varepsilon.$$

It follows that certain classes in the product of contact C-C manifolds $V = V_1 \times \cdots \times V_i$ in the dimension $k = m_1+1+\cdots+m_i+1$ are representable by cochains c with $\|c\|_\varepsilon \lesssim \varepsilon^{i+k}$. Therefore, *every proper C^α-map of non-zero degree*

$$(V, \text{Riem.metric}) \to (V, \text{product C-C metric})$$

must have $\alpha \le \frac{k}{k+1}$. In particular, if $\dim V_1 = \dim V_2 = \cdots = \dim V_i = 3$ (i.e. $m_1 = m_2 = \cdots = m_i = 1$), then $\alpha \le \frac{2}{3}$.

Question. Does the latter estimate $\alpha \le \frac{2}{3}$ remain valid for $i = \infty$? (Of course, one should be more specific here about the geometry and topology of infinite products.)

3.4. Width and filling radius. Let V be the $(2m+1)$-dimensional Heisenberg group with the standard (contact) C-C metric. We want to bound the width (and the filling radius) of subsets (and cycles) $V' \subset V$ in terms of their Hausdorff measures where the width $\operatorname{wid}_k(V' \subset V)$ is defined as the infimum of the numbers $\delta > 0$ for which there exists a continuous map $f : V' \to V$ having the topological dimension of the image at most k and $\operatorname{dist}_V(v', f(v')) \le \delta$ for all $v' \in V'$.

3.4.A. A bound on wid by mes. *Every closed subset $V' \subset V$ satisfies*

$$\operatorname{wid} V'_{k-1} \le \operatorname{const}_m (\operatorname{mes}_k V')^{\frac{1}{k}} \quad \text{for } k = 1, \ldots, m, \qquad (*)$$

$$\operatorname{wid}_{k-1} V' \le \operatorname{const}_m (\operatorname{mes}_{k+1} V')^{\frac{1}{k+1}} \quad \text{for } k = m+1, \ldots, 2m+1. \; (**)$$

Proof. The inequality $(*)$ follows from the corresponding Riemannian inequality. Namely, we first scale V' (by a C-C self-similarity of V) to have $\operatorname{mes}_k V' = 1$, observe that C-C $\operatorname{mes}_k \ge$ Riem-mes_k and then use the implication

$$\text{Riem-}\operatorname{mes}_k V' \le 1 \Rightarrow \operatorname{wid}_{k-1} V' \le \operatorname{const}_V$$

which is valid for all contractible Lie groups V with left-invariant Riemannian metrics by the Federer-Fleming isoperimetric argument (compare p. 17 in [Gro$_{\mathrm{FRM}}$]). Now we turn to the (more interesting) inequality $(**)$ where we shall use a contact version of the method of Federer-Fleming. Namely we need the following

3.4.B. Contact triangulation. *There exists a triangulation* Tr *of* V *into piecewise smooth simplices which is invariant under some discrete cocompact subgroup* Γ *(in the Heisenberg group* V*) and such that all simplices of dimensions* $\leq m$ *are* H*-horizontal, i.e. piecewise tangent to the implied contact structure* $H \subset T(V)$.

This can be derived from the h-principle for Legendre maps (see [$\mathrm{Gro}_{\mathrm{PDR}}$]) or proved by an elementary (albeit cumbersome) contact argument which we leave to the reader (compare 3.5 and 4.2). Next we come to the following

3.4.B′. Integral-geometric intersection inequality. *Let* $V'' \subset V$ *be a compact smooth* H*-horizontal submanifold of dimension* $\ell = 2m+1-k$ *and* $V' \subset V$ *be a closed* k*-dimensional subset with finite* C-C *Hausdorff measure* mes_{k+1}*. Then for almost all* $v \in V$ *the Heisenberg translate (written additively)* $V'' + v$ *intersects* V' *at finitely many points and the integral of the intersection number over* V *is bounded by*

$$\int_V \#(V' \cap V'' + v)dv \leq \mathrm{const}_{V''} \,\mathrm{mes}_{k+1} V'.$$

In fact,

$$\int_V \#(V' \cap V'' + v)dv \leq \mathrm{const}_V \,\mathrm{mes}_\ell V'' \,\mathrm{mes}_{k+1} V'$$

as follows by our argument in 2.1 and in 3.1.A.

Corollary. *For every contact* Γ*-invariant triangulation* Tr *of* V *and every* $k = m+1, \ldots, 2m$ *there exists a positive number* $\varepsilon = \varepsilon(Tr) > 0$*, such that for every* $V' \subset V$ *with* $\mathrm{mes}_{k+1} V' \leq \varepsilon$ *there exists a translate* $V' + v$ *of* V' *which does not intersect the* ℓ*-skeleton* $\mathrm{Tr}^\ell \subset \mathrm{Tr}$ *for* $\ell = 2m+1-k$.

Proof. In fact, for every fundamental domain $U \subset V$ of Γ, the integral over U of the intersection number between Tr^ℓ and the translates of V' is bounded by

$$\int_U \#(\mathrm{Tr}^\ell \cap V' + u)du \leq \mathrm{const}\,\mathrm{mes}_{k+1} V'$$

which makes this number zero for some $u \in U$ for small $\mathrm{mes}_{k+1} V'$.

Now the proof of $(**)$ is immediate. We rescale V' to have small fixed $\mathrm{mes}_{k+1} = \varepsilon > 0$ and then bring it to the position where it misses Tr^ℓ.

Then it can be mapped to the $(k-1)$-skeleton of the dual partition of V (as $k-1 = 2m+1-\ell-1$ for our $\ell = 2m+1-k$) by a map f within a fixed distance from the identity. ∎

3.4.C. Asymptotic Riemannian version of 3.4.A. *Let the Heisenberg group V be equipped with a left-invariant Riemannian metric. Then every smooth k-dimensional submanifold $V' \subset V$ for $k \geq m+1$ satisfies*

$$\operatorname{Riem\,wid}_{k-1} V' \leq \operatorname{const}_V (\operatorname{Vol}_k V')^{\frac{1}{k+1}}. \qquad (++)$$

Proof. Indeed,

$$\text{Riem-Vol}_k \geq \text{C-C } \operatorname{mes}_{k+1},$$

which reduces $(++)$ to $(**)$. (Notice that $(++)$ is truly interesting for large $\operatorname{Vol}_k V' \to \infty$.)

Exercise. State and prove a $(+)$-version of $(*)$.

Remarks

(a) There are two other asymptotic versions of 3.4.A: one concerns families of Riemannian metrics on (compact) manifolds approximating C-C metrics (as in 0.8.G and 1.4.D) and the other deals with certain infinite subsets in the complex hyperbolic space where the ideal boundary carries a contact C-C geometry (compare 7.C in [Gro$_{\mathrm{AI}}$]).

(b) For every contractible Lie group V with a left-invariant Riemannian metric one has certain bound on the widths of k-dimensional subsets $V' \subset V$ in terms of their volumes,

$$\operatorname{wid}_{k-1} V' \leq \psi(\operatorname{Vol}_k V)$$

for some function $\psi = \psi_V(X)$, but one knows little about the (asymptotic) behaviour of this ψ for general contractible groups V. (Notice that this ψ is essentially a quasi-isometric invariant which can be also defined for discrete groups in the spirit of [Gro$_{\mathrm{AI}}$].)

Filling Radius. This is defined for k-dimensional *cycles* V' in V as the infimum of those $\delta \geq 0$, such that V' becomes homologous to zero in the δ-neighbourhood of (the support of) V'. It is obvious that $\operatorname{Fill\,Rad} \leq \operatorname{wid}_{k-1}$ and so the theorem 3.4.A yields a bound on Fill Rad in terms of the measure of (the support of) V'. Unfortunately, the constant in the inequalities $(*)$ and $(**)$ depends on $\dim V$ while one is inclined to allow this constant to depend only on $\dim V'$. Such a bound on the filling radius in $\mathbb{R}^n$, namely

$$\operatorname{Fill Rad} V' \leq \operatorname{const}_k (\operatorname{Vol}_k V')^{\frac{1}{k}},$$

is even known with the *sharp* constant $\operatorname{const}_k = \big(\operatorname{Vol} B(1)\big)^{-\frac{1}{k}}$ for the unit Euclidean ball $B(1) \subset \mathbb{R}^k$ thanks to a result by Bombiery and Simon proven by the variational techniques (see [Bom], and p. 106 in [$\mathrm{Gro}_{\mathrm{FRM}}$]). It seems plausible that the k-cycles V' in the Heisenberg group V with the *standard* C-C metric (which is Hermitian with respect to the curvature from ω on H) also satisfy the filling inequality *independent* of $\dim V$, i.e.

$$\operatorname{Fill Rad} V' \leq \operatorname{const}_k (\operatorname{mes}_k V')^{\frac{1}{k}}, \quad \text{for } k \leq m,$$

and

$$\operatorname{Fill Rad} V' \leq \operatorname{const}_k (\operatorname{mes}_{k+1} V')^{\frac{1}{k+1}}, \quad \text{for } k \geq m+1,$$

but the determination of the best const_k does not appear realistic.

3.5. Lipschitz maps of Riemannian manifolds into contact C-C ones. Let V be a contact C-C manifold of dimension $n = 2m+1$ and W be a Riemannian manifold of dimension k. If $f : W \to V$ is a C^1-smooth Lipschitz map then f is (obviously) horizontal, i.e. tangent to the implied contact subbundle $H \subset T(V)$ (of rank $2m$) and moreover, if f is C^2-smooth it is Ω-*isotropic*, i.e. the curvature form Ω of H pulls back to zero by the differential $\mathcal{D}_f$, as

$$\mathcal{D}_f^* \Omega = \mathcal{D}_f^* d\eta = d\mathcal{D}_f^* \eta = 0$$

where η is the 1-form defining H by $\operatorname{Ker} \eta = H$ and the equality $\mathcal{D}_f^* \eta = 0$ expresses the H-horizontality of f. (All this remains valid for C^1-maps f if the differential $d\mathcal{D}_f^* \eta$ is understood as a distribution.) Since the 2-form Ω is non-singular on H, the map f has $\operatorname{rank} \mathcal{D}_f \leq m$ and so it is everywhere singular if $k = \dim W > m$. On the other hand if $k \leq m$, the map f can be very well *regular* which means "immersion", i.e. the injectivity of $\mathcal{D}_f$ all over W. Such a map is, locally, a diffeomorphism onto its image which is a horizontal k-dimensional submanifold in V. (Horizontal submanifolds are called *Legendre* for $k = m$.) The local geometry of horizontal manifolds in V is well understood. If V' is such a submanifold in V, then for each point $v \in V'$ there are local coordinates $x_i, y_i, z,\ i = 1, \ldots, m$, in V at v, such that $H = \operatorname{Ker}(\eta = dz - \sum_{i=1}^m x_i dy_i)$ and the projection Y of V' to $\mathbb{R}^m$ given by the y_i's coordinates is regular at v. Then, if $k = m$, the map $Y : V' \to \mathbb{R}^m$ is locally onto some neighbourhood $U \subset \mathbb{R}^m$ and V' is represented by the (graph of the) 1-jet of the function $\zeta : u \mapsto z \circ Y^{-1}(u)$

on U as the relation $dz - \sum_i x_i dy_i = 0$ makes $x_i = \frac{\partial \varsigma}{\partial y_i}$, $i = 1, \ldots, m$. Thus we obtain a 1-1 correspondence between (germs of) H-horizontal m-dimensional submanifolds in V close to V' at v and (germs of) functions on $\mathbb{R}^m$. In general, if $p \leq m$, we represent horizontal submanifolds in V by 1-jets of functions in $\mathbb{R}^m$ along p-dimensional submanifolds in $\mathbb{R}^m$. It follows that the local geometry in the space of horizontal immersions $W \to V$ is essentially the same as in the space of functions on W. In particular, the sheaf of horizontal immersions $f : W \to V$ is *microflexible* in the sense of [Gro$_{\text{PDR}}$]. This means, roughly speaking, that every deformation of f initially defined near some compact subset $W_0 \subset W$, say ψ_t, $t \in [0,1]$, where $\psi_{t=0}$ equals f on some neighbourhood $U_0 \supset W_0$ and where ψ_t is H-horizontal and regular on U_0 for each t, can be extended to an H-horizontal deformation f_t of $f = f_{t=0}$ for $t \in [0, \varepsilon]$ for some *positive* $\varepsilon \leq 1$ (depending on ψ_t) and where "extension" means $f_t \mid W_0 = \psi_t \mid W_0$ for $t \in [0, \varepsilon]$.

Moreover, one knows that the horizontal immersions are, in fact, *flexible* which means one can take the above ε equal to 1, but this is a rather difficult theorem (see [Gro$_{\text{PDR}}$]) which is not truly needed for our present purpose. What we need, however, is a *piecewise* smooth extension f_t for $t \in [0,1]$ which is horizontal and regular on each piece for all t. The existence of such an extension is achieved with the *Poenaru pleating* (or *folding*) *lemma* which delivers such homotopy where f_1 for $\dim W = 1$ looks as in Fig. 5 below (compare 4.4, also see pp. 51 and 112 in [Gro$_{\text{PDR}}$]).

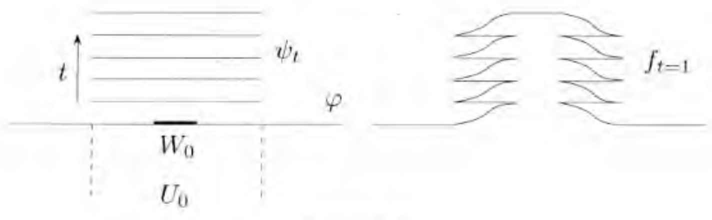


Figure 5

Notice that piecewise smooth horizontal maps are Lipschitz as well as the smooth ones and, if one wishes, one can make them smooth by applying a smooth self-mapping $W \to W$ collapsing a neighbourhood of the non-smoothness locus. But the regularity cannot be recaptured once it is lost. In fact, piecewise regular maps are not difficult to deal with as we know how to deform them (using microflexibility) over the regular pieces. On the other hand, smooth horizontal maps without any a priori control over the domain of regularity, admit no nice 1-jet representation and they cannot be so easily deformed.

Now we state (and indicate the proofs of) several theorems which ensures sufficiently many piecewise smooth Lipschitz maps as in the case of maps into a Riemannian (rather than C-C) manifold V.

3.5.A. First Lipschitz approximation theorem. *Every continuous map $f : W \to V$ admits a fine C^0-approximation (which amounts to the uniform approximation for compact V) by piecewise smooth and piecewise H-horizontal (and hence locally Lipschitz) maps* (where we assume throughout $k \leq m$, for $k = \dim W$ and $2m + 1 = \dim V$).

Sketch of the proof. First we recall the necessary and sufficient condition for an approximation of f by *smooth* horizontal maps f_{hor}. This reads,

there exists a continuous injective homomorphism $\varphi : T(W) \to f^*(H)$ *with an* Ω*-isotropic image* (i.e. with $\Omega|\varphi\big(T(V)\big) = 0$ for the curvature form Ω on H corresponding to $\omega = d\eta$ for the 1-form η defining H).

The necessity is easy as one can make φ out of the differential $\mathcal{D}f_{\mathrm{hor}}$: $T(W) \to f^*_{\mathrm{hor}}(H)$ for f_{hor} close to f. The sufficiency follows from the (dense) h-principle for horizontal maps (see p. 339 in [Gro$_{\mathrm{PDR}}$]). Notice that the proof of the h-principle is based upon flexibility of the sheaf of horizontal immersions. As we are content here with *piecewise* horizontal maps, we may use Poenaru's pleating (or folding) trick and thus obtain an alternative (more elementary) construction of piecewise horizontal maps approximating f under the assumption of the existence of φ. This works, for example, in the case where W is homeomorphic to $\mathbb{R}^k$ and so φ exists being a section of a certain bundle over $W (= \mathbb{R}^k)$.

Now, to grasp the idea suppose the map f decomposes as

$$\begin{tikzcd} W \arrow[r, "e"] \arrow[rr, bend right, "f"'] & W' \arrow[r, "f'"] & V \end{tikzcd}$$

where f' is a smooth horizontal immersion and e is some continuous map. Then, assuming $\dim W' \geq k = \dim W$, the required approximation of f

can be achieved by approximating e by piecewise smooth and piecewise regular maps $W \to W'$. These are constructed by using sufficiently fine triangulations of W, approximating e on the vertices and then "linearly" interpolating to the simplices in W.

Next suppose there exists a vector bundle T^* over W of rank $\ell \geq k = \dim W$ which admits an injective Ω-isotropic homomorphism $\varphi : T^* \to f^*(H)$ (where Ω-isotropic means $\varphi^*(\Omega) = 0$). Then φ on each fiber T^*_v, $v \in W$, can be turned into a germ, say T'_v, of a ℓ-dimensional H-horizontal submanifold in V through $f(v) \in V$ continuously depending on v. Then, by microflexibility of such germs, one can, after slightly perturbing them, construct, for each $\varepsilon > 0$, a non-Hausdorff (or branched as in 4.4) manifold W'_ε horizontally immersed into V, say by $f'_\varepsilon : W'_\varepsilon \to V$, such that f can be ε-approximated by the composed map $f'_\varepsilon \circ e$ for a suitable $e : W \to W'_\varepsilon$. Compare D - D″ in 2.2.7 of [Gro$_{\text{PDR}}$] and see Fig. 6 below.

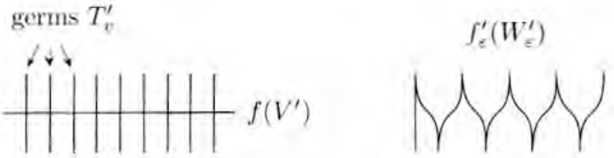


Figure 6

Then e is made piecewise "linear" as in the case of an ordinary W'.

Example. If V is contractible, then the required T^* and φ obviously exist and the above applies.

Notice that we have used so far only the microflexibility of horizontal immersions and no specific contact geometry.

General case. Suppose W is triangulated and we want to approximate $f : W \to V$ by continuous maps which are smooth horizontal and regular on all simplices. The obvious necessary condition requires the existence of Ω-isotropic injective homomorphisms $\varphi_\Delta : T(\Delta) \to f^*(H)$ for all simplices Δ in the triangulation of W, such that $\varphi_{\Delta'}$ over each face Δ' of Δ is obtained by restricting φ_Δ to $T(\Delta') \subset T(\Delta)$. Then the h-principle in 3.4.3 of [Gro$_{\text{PDR}}$] implies that this condition is necessary as well as sufficient and the approximation theorem reduces to finding a triangulation of W for which the homomorphism φ_Δ exist. In fact, these φ_Δ exist for every sufficiently fine triangulation of W which is most clearly seen if

$$(V,H) = \left(\mathbb{C}^m \times \mathbb{R},\ H = \ker\left(\sum_{i=1}^{m} x_i dy_i - dz\right)\right).$$

Here one may use any triangulation of W and use a generic piecewise linear map $W \to V$ whose projection to $\mathbb{C}^m$ is regular and *totally real* on all simplices of W, where an affine subspace in $\mathbb{C}^m$ is called totally real if it contains no $\mathbb{C}$-line. The space of totally real subspaces in $\mathbb{C}^m$ contracts (by an easy argument) to the space of the ω-isotropic subspaces for the form $\omega = \sum_i dx_i \wedge dy_i$ which essentially equals the curvature Ω of H. Furthermore, this contraction can be made compatible with the inclusions between totally real (respectively, ω-isotropic) subspaces of different dimension. Thus the differential of our piecewise linear map contracts to the desired piecewise linear Ω-isotropic homomorphism $\{\varphi_\Delta\}$. (A more logical proof of the existence of $\{\varphi_\Delta\}$ should use certain connectivity of the *Tits' building* which is a polyhedron with the vertices corresponding to linear ω-isotropic subspaces in $\mathbb{C}^m$ and where the simplices correspond to flags of such subspaces in $\mathbb{C}^m$.)

3.5.A′. Lipschitz approximation of families of maps $\mathbf{W} \to \mathbf{V}$. Suppose we start with a family of continuous maps $f_p : W \to V$ where p runs over some compact space and we want an approximation by a P-family of maps where all members are Lipschitz. This can be done again with piecewise horizontal maps but now the implied triangulation of W must depend on P. (A generic family of piecewise linear maps of a fixed triangulation into $\mathbb{R}^m$ may become non-regular on some simplices for certain values $p \in P$ if $\dim P$ is large.) For example, if P is a manifold, one may use some triangulation of $W \times P$ and use generic piecewise linear maps $W \times P \to \mathbb{C}^m \times \mathbb{R}$. This idea can be extended to general compact (and even locally compact) spaces Q foliated by k-dimensional smooth manifolds which allows for $k \leq m$ an *approximation of every continuous map $f : Q \to V$ by leafwise Lipschitz maps*. This conclusion is by no means deep (not to say plain trivial). Yet, I do not see a short elementary proof of it.

3.5.B. Extension of piecewise horizontal maps. Take a submanifold or, more generally, a piecewise smooth subcomplex $W_0 \subset W$ and let $f_0 : W_0 \to V$ be a piecewise regular horizontal map. Then the above arguments allow an extension of f_0 to a similar map of all of W to V provided f_0 admits a continuous extension to W. In the case where W_0 is a smooth submanifold in W and f_0 is smooth this can be done by a more or less straightforward application of the flexibility (or microflexibility + pleating) of the sheaf of smooth horizontal maps. In general, one should note that even a local extension of f_0 at a non-smooth point is a global problem. For example, if $f_0(W_0)$ is a piecewise horizontal curve in V, then an extension to an ambient $W \supset W_0$ of dimension two at a breaking point v of f_0, amounts to joining two points (corresponding to two tangent vectors τ_1 and τ_2, see Fig. 7) in the standard contact sphere $S_v^{2m-1} \subset H_v = \mathbb{R}^{2m}$ by a horizontal path in S_v^{2m-1}.

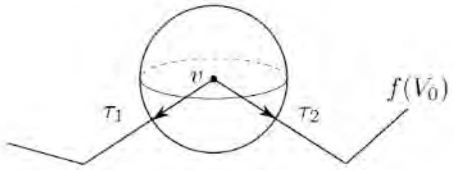


Figure 7

Compactness and complexity. We want to introduce a suitable notion of *complexity* (or size) of a piecewise smooth map so that the complexity of the above extension of f_0 from W_0 to W would admit a bound in terms of the complexity of f_0 on W_0. An equivalent formulation can be achieved with some topology in the space of our maps so that for each precompact subset F_0 of maps $W_0 \to V$ there would exist of a precompact set F of maps $W \to V$, where each $f_0 \in F_0$ could be extended to an $f \in F$. (Eventually, we want to reduce complexity to the Lipschitz constant but this will be done later on the basis of more complicated preliminary notion of complexity.) In what follows we assume the manifolds V and W are compact and we fix Riemannian metrics in both of them. Then we may speak of the norms of the differentials $\|\mathcal{D}^r f\|$ for C^r-maps $f : W \to V$, and of the norm of $(\mathcal{D}f)^{-1}$. The latter is finite iff f is regular (i.e. an immersion). Now we introduce the r-complexity $|\!|\!| f |\!|\!|_r$ of an immersion $f : W \to V$ as the supremum over W of the sum of the norms

$$\|(\mathcal{D}f)^{-1}\| + \|\mathcal{D}f\| + \|\mathcal{D}^2 f\| + \cdots + \|\mathcal{D}^r f\|.$$

Warm-up exercise. *If* $\dim V > \dim W$ *and an immersion* $f_0 : W_0 \to V$ *admits an extension to an immersion* $W \to V$, *then, for each* $r = 1, 2, \dots,$ *there is such an extension* f *whose complexity* $\|f\|_{r-1}$ *is bounded by* $(\Phi\|f_0\|_r)$ *for some function* $\Phi(c) = \Phi_{V,W,W_0}(c)$.

The main ingredient of the proof is the flexibility of immersions (Smale-Hirsch theory) which implies that if some C^1-immersion $f_\infty : W_0 \to V$ is a C^1-limit of immersions $f_i : W_0 \to V$, extendable by immersions $\tilde{f}_i : W \to V$, then f_∞ also extends to a C^1-immersion $\tilde{f}_\infty : W \to V$. (This is the only place where we need $\dim W < \dim V$.) Next, assuming f_∞ is C^{r-1}, the extension $\tilde{f}_\infty$ can be easily smoothed to C^{r-1} and then, if the convergence $f_i \to f_\infty$ is C^{r-1}, each f_i for $i \geq i_0$ obviously extends to $\tilde{f}_i : W \to V$ which is C^{r-1}-close to $\tilde{f}_\infty$ and thus have $\|\tilde{f}_i\|_{r-1} \leq \text{const}$. All this yields what we need as the set of immersions $W_0 \to V$ with a given bound on $\|\cdot\|_r$ is C^{r-1}-precompact.

Remark. One could easily recapture the loss of one derivative by a simple smoothing argument but this is unnecessary for our ultimate purpose (where such a smoothing becomes problematic).

Horizontal version. The above discussion extends to the case where V is a contact manifold and "immersion" is everywhere replaced by "horizontal immersion" and where we assume as earlier that $\dim V \geq 2\dim W + 1$. (This inequality is necessary for the existence of a horizontal immersion $W \to V$.) In fact, horizontal immersions have the same flexibility and approximation properties as ordinary immersions. Here, as earlier, one could estimate $\|f\|_r$ by $\|f_0\|_r$ rather than by $\|f_0\|_{r+1}$ but we allow the loss of smoothness to make the present discussion suitable for a generalization to non-contact C-C manifolds V.

Complexity made piecewise. If W is smoothly triangulated then each simplex comes along with a smooth map $\sigma : \Delta \to V'$ where Δ is the unit Euclidean simplex of dimension $i = 0, 1, \dots,\ k = \dim W$. Then the C^r-complexity of the triangulation is defined as the sum of all $\|\sigma\|_r$ and the C^r-complexity of a piecewise regular map f which is actually smooth on each simplex in W is defined as the sum of that of the triangulation with the complexities $\|f\|_r$ on all simplices Δ of the triangulation. In general, if no triangulation is specified, the complexity of f refers to the infimum of those with respect to all triangulation of W for which f is smooth and regular on the simplices.

Admission. This definition suffers from a variety of defects, where the major one is the lack of the scale invariance. This will be taken into account later on.

Extension Lemma. *If a horizontal piecewise regular map $f_0 : W_0 \to V$ admits a continuous extension to $W \supset W_0$ then it also admits a horizontal piecewise regular extension f whose piecewise complexity $\|f\|_r$ can be bounded by some function of $\|f_0\|_{r+r_0}$, where $r_0 \geq 0$ depends on $\dim W$ (where, as earlier, V is contact of dimension $2m+1$ and $\dim W \leq m$).*

Idea of the proof. The construction of f is essentially the same as that of the approximation in the previous section. It is useful to keep the following points in mind.

(a) The standard induction by the skeletons of a suitable triangulation of W reduces the general problem to the case where W is a unit Euclidean simplex Δ and W_0 is the boundary of Δ. This explains why the total loss of regularity, i.e. the number r_0 depends only on $\dim W$. (In fact, one can, with some extra effort, make $r_0 = 0$.)

(b) As we are allowed to subdivide W as much as needed, we do not have to use the flexibility of horizontal maps but only microflexibility augmented by Poenaru's pleating (folding) construction. This becomes especially important for more general (non-contact) C-C geometries where the flexibility is less available.

(c) The *local* extension of f_0 at the faces of dimension i leads to the corresponding *global* extension problem with the dimension shifted down by $i+1$.

(d) The steps of the extension are completely constructive in nature which ensures the required bound

$$\|f\|_r \leq \Phi(\|f\|_{r+r_0}). \qquad (*)$$

(e) All ingredients of the extension construction are geometrically rather trivial with the exception of pleating (or the flexibility which contains the pleating as the major geometric component) but the overall proof becomes lengthy and boring if one tries to write down the details. (This is a good excuse for us not to do it here.)

(f) Even if the original map f_0 was everywhere smooth and regular, on W_0 it does not always admit a regular horizontal extension to W' and the division into pieces is unavoidable.

Scale invariance of Φ. If we replace V and W by λV and $\lambda' W$, i.e. we multiply the underlying metrics in V and W by the constants λ and λ', then $\Phi = \Phi_{V,W}$ in $(*)$ may change; however, if $\lambda, \lambda' \geq 1$ then these $\Phi_{\lambda,\lambda'} = \Phi_{\lambda V, \lambda' W}$ are bounded independently of λ and λ'. This follows from the fact that λV converges to the Heisenberg group for $\lambda \to \infty$ and $\lambda' W$ converges to $\mathbb{R}^k$, $k = \dim W$.

3.5.C. Smoothing Lipschitz maps. Let f_0 be a Lipschitz map of a Riemannian manifold W into a C-C contact manifold V. We want to *smooth* f_0, i.e. to approximate it by smooth Lipschitz maps $f_\varepsilon : W \to V$ whose Lipschitz constants $L(f_\varepsilon)$ converge to $L(f_0)$ for $\varepsilon \to 0$. (This is easy and well known when V is Riemannian.) Unfortunately, we are able to do it only for $m \geq k$, where $2m + 1 = \dim V$ and $k = \dim W$, and also we cannot achieve the sharp bound of $L(f_\varepsilon)$ by $L(f_0)$. Yet we have the following

3.5.C$'$. Second Lipschitz approximation theorem. *If $f_0 : W \to V$ is Lipschitz, then, for $m \geq k$, it admits a uniform approximation by smooth horizontal maps $f : W \to V$ whose Lipschitz constants $L(f) = \sup_{v \in V} \|\mathcal{D}f\|(v)$ are bounded by*

$$L(f) \leq CL(f_0)$$

for some universal constant $C = C_m$.

Proof. First enlarge V and W by rescaling $V \mapsto \varepsilon^{-1} V$ and $W \mapsto L(f_0)\varepsilon^{-1} W$ for a small $\varepsilon > 0$. Then triangulate $L(f_0)\varepsilon^{-1} W$ into simplices Δ with the complexities bounded by a universal constant $\mathrm{const}'_k > 0$ (which is possible for small ε as everybody knows). Restrict f_0 to the 0-skeleton of this triangulation and then extend it into each Δ inductively by the skeletons inside of a ball in $\varepsilon^{-1} V$ of radius $R \leq \mathrm{const}_m$. This extension may be chosen with the complexity bounded by a universal constant C_m according to the scale invariance of Φ mentioned earlier. Thus we obtain a map $f_\varepsilon : L(f_0)\varepsilon^{-1} W \to \varepsilon^{-1} V$ whose complexity and, hence, the Lipschitz constant are bounded by C_m. This very map f_ε is $CL(f_0)$-Lipschitz for the original (unscaled) metrics in W and V and it converges to f_0 for $\varepsilon \to 0$ for these metrics. This gives us a piecewise smooth approximation of f_0 which can be made smooth by composing f_ε with a self-mapping of W retracting a small neighbourhood of the singular locus of f_ε onto this locus. ∎

Remarks

(a) Our argument applies, strictly speaking, only to *compact* manifolds W but an obvious modification of the above yields, for all W, a fine C^0-approximation of f_0 by smooth maps f with $\|\mathcal{D}f\|(w)$ controlled at each $w \in W$ by the Lipschitz constant of f_0 on a small ball in W around w.

(b) The above argument also yields certain information when f_0 is a C^α-Hölder map, for example if f_0 is C^1-smooth and hence $C^{\frac{1}{2}}$-Hölder. Namely, *such an f_0 can be ε-approximated (with respect to the* C-C *metric in V) by f_ε with*

$$L(f_\varepsilon) \leq C_m \varepsilon^{\alpha-1} L_\alpha(f_0),$$

where L_α is the Hölder constant of f_0.

(c) The second approximation theorem, as the first one, should extend to families and to maps of foliations into V.

3.5.D. Construction and extension of non-piecewise smooth Lipschitz maps. We start with the following

Trivial Example. Let $v_0, v_1, \ldots$ be a sequence of points in V (where V is contact Carnot-Carathéodory as earlier) such that

$$\operatorname{dist}(v_0, v_1) < \frac{1}{2}, \qquad \operatorname{dist}(v_1, v_2) < \frac{1}{4}, \qquad \operatorname{dist}(v_2, v_3) < \frac{1}{8}, \ldots .$$

Then we join v_0 and v_1 by a smooth horizontal curve of length $\leq \frac{1}{2}$, we join v_1 and v_2 by such a curve of length $\frac{1}{4}$ and so on. Thus we obtain in the limit a 1-Lipschitz map $f : [0,1] \to V$ with $f(0) = v_0$, $f(\frac{1}{2}) = v_1$, $f(\frac{3}{4}) = v_3, \ldots f(1) = \lim\limits_{i\to\infty} v_i$.

One can think of the points $1, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}$ as vertices of an infinite triangulation of [0,1] and interpret the above construction as a Lipschitz extension of a map from the zero-skeleton $= \{1, \frac{1}{2}, \ldots\}$ to all of $[0,1]$. In fact, such an extension is possible according to 3.5.B for higher dimensional (finite or infinite) triangulations of Riemannian manifolds where all simplices Δ are *small and fat*. This means that the diameters $d(\Delta)$ are uniformly bounded and the rescaled simplices $d^{-1}\Delta$ for $d = d(\Delta)$ have uniformly bounded complexities.

Here, in Fig. 8, is an example of such a triangulation of the unit square $\square$ which is obtained by refining a (more natural) partition of $\square$ into (small and fat) pentagons.

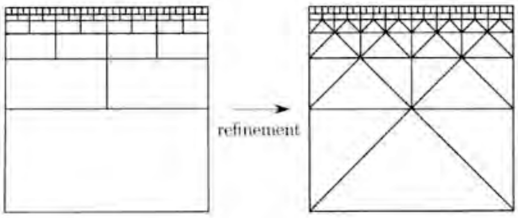


Figure 8

Every Lipschitz map of the set of the vertices of such a triangulation into V extends to all of the square $\square$ provided $\dim V \geq 5$. This is done by induction on skeletons where at each step we use the (rescaled) Extension Lemma of 3.5.B. Notice that this lemma provides at each stage maps of bounded complexity (on the scale of each simplex) which makes the induction possible. For example, suppose we are given some Lipschitz map f_0 of the top (limit) segment I of $\square$ into V. This map, composed with the orthogonal projection of $\square$ to I gives us a Lipschitz map of the whole square $\square$, and hence of the zero skeleton into V. Then the above gives a new map of $\square$ to V which is smooth and regular on each triangle and which equals (by continuity) f_0 on the top segment I of $\square$. We conclude that for every Lipschitz map $f_0 : I \to V$ there exists a map $f : \square = [0,1]\times I \to V$ which is piecewise smooth and regular on $\square - I$, which extends f_0 from $I = 1\times I \subset \square$ and whose Lipschitz constant can be bounded by that of f_0 (provided V is compact of dimension at least 5).

Now let us do the same to the partition of the square $\square$ in Fig. 9 below where the whole boundary plays the role of the limit segment I.

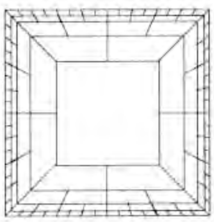


Figure 9

Here again, every Lipschitz map f_0 of the boundary $\partial\square$ into V extends to the zero skeleton with the central projection $\square \to \partial\square$ and then f_0 admits a Lipschitz extension f to $\square$, provided it admits just a continuous extension (where $\dim V$ is assumed ≥ 5 as earlier). What is most important here is the control of the Lipschitz constant of f by that of f_0 which makes our infinite construction relevant even if the original map f_0 were smooth. Namely we have the following

Disk extension theorem. *Let V be a compact simply connected contact C-C manifold of dimension ≥ 5. Then every Lipschitz map f_0 of the boundary circle S^1 of the unit disk $D \subset \mathbb{R}^2$ (which is bi-Lipschitz to $\square$) extends to a Lipschitz map $f : D \to V$, where the Lipschitz constant of f is bounded by*

$$L(f) \leq CL(f_0) \text{ for some } C = C(V).$$

As an immediate corollary we have the following

Isoperimetric inequality. *Every closed C-C rectifiable curve in V bounds a disk of the area (i.e. 2-dimensional C-C Hausdorff measure) satisfying*

$$\text{Area} \leq C_V(\text{length})^2. \tag{$*$}$$

Remarks

(a) If length $\gg 1$ the above inequality can be (obviously) improved to

$$\text{Area} \leq C_V \text{ length}.$$

(b) The inequality $(*)$ as well as the Disk extension theorem remain valid for certain non-compact manifolds V. For example, this is the case for the Heisenberg groups H^n for $n \geq 5$. In fact, everything for H^n reduces to the compact case by the obvious use of the self-similarities of H^n.

(c) The present proof of $(*)$ simplifies and conceptualizes the argument in 5.A_3 of [$\mathrm{Gro}_{\mathrm{AI}}$]. Also Thurston has been claiming a variety of isoperimetric inequalities in the spirit of $(*)$ and I suspect the above proof must be close to what he had in mind as he insisted (hearsay) on the notion of the *Lipschitz homotopy groups* in the metric and asymptotic geometries. The inequality $(*)$ also appears in [Lee] in the study of minimal Lagrange surfaces in Kähler manifolds.

Our extension argument obviously generalizes to higher dimensions and yields the following

Lipschitz extension theorem. *Let W be a compact k-dimensional Riemannian manifold, $W_0 \subset W$ be a submanifold and let V be a compact contact* C-C *manifold of dimension $n \geq 2k+1$. Then, if V is $(k-1)$-connected, every Lipschitz map $f_0 : W_0 \to V$ extends to a Lipschitz map $f : W \to V$ where the Lipschitz constant of f satisfies*

$$L(f) \leq CL(f_0) \text{ for } C = C(V, W, W_0).$$

Remarks and questions

(a) It seems not difficult to generalize the above theorem to families and to leafwise Lipschitz maps of foliations into V.

(b) Most results in this section probably extend to the case where the source manifold W is C-C rather than Riemannian. The first case to check is where W is C-C *contact* of dimension $< \dim V$. (It is less clear what are *equidimensional non-injective* Lipschitz maps between contact manifold. For example, one does not know if and how such maps fold. On the other hand, such maps may easily collapse domains to points and to more general horizontal submanifolds. Then, using the collapse to points one may, for instance, construct equidimensional Lipschitz maps of a given degree between closed contact manifolds where the receiving one is the sphere S^{2k+1} with the standard contact structure.)

(c) Recall that the Nash-Kuiper theorem (see [$\mathrm{Gro}_{\mathrm{PDR}}$], for instance) allows one to approximate under favorable topological conditions, $(1-\varepsilon)$-Lipschitz maps between Riemannian manifolds by isometric C^1-maps where "isometric" means preservation of the length of the curves. The Nash part of the theorem was transplanted by D'Ambra to the contact C-C manifolds (see [$\mathrm{DA}_{\mathrm{C1}}$]). On the other hand, the Riemannian immersion theory has a Lipschitz (non-C^1) counterpart (which allows, in particular, equidimensional isometric maps, see 2.4.11 in [$\mathrm{Gro}_{\mathrm{PDR}}$]). This suggests similar results for (Lipschitz) isometric maps into contact C-C manifolds.

(d) Our Lipschitz extension theorem does not seem to yield any higher dimensional isoperimetric inequality. Nobody knows yet (except, possibly, Bill Thurston) whether every (closed horizontal) surface S in the

Heisenberg group H^n for $n \geq 7$ bounds something 3-dimensional of the volume (i.e. the 3-dimensional C-C Hausdorff measure) satisfying

$$\text{Volume} \leq C(\text{Area})^{\frac{3}{2}}.$$

(e) There are many other dilation characteristics of maps $f : W \to V$ besides the Lipschitz constant and the volume, such as the *p-energy* $\int_W \|\mathcal{D}f(w)\|^p dw$, which is defined whenever f is almost everywhere horizontal. If the receiving space V is Riemannian rather than C-C, one knows how to control the energy of extension of f_0 from $W_0 \subset W$ to W by a suitable energy of f_0 on W_0 (see [G-E] and p. 388 in [E-L]). Some of these extension results may be valid with values in contact C-C manifolds.

Besides the p-energies which measure the dilation of f along curves one may measure the dilation of the volumes of k-dimensional submanifolds by using the action of the differential on the k-th exterior power of $T(W)$. Then one defines the (k,p)-energies $\int_W \|\Lambda^k \mathcal{D}f(w)\|^p dw$ and raises the extension problem for these. Most generally, one may try an extension with a control over several such energies which amounts to controlling the joint distribution of the functions $T_k(w) = \operatorname{Trace}\Lambda^k \mathcal{D}^*\mathcal{D}(w)$, for $\mathcal{D} = \mathcal{D}f$, with respect to the Riemannian measure dw.

3.6. Controlled integration of differential forms and bounds on the rational homotopy invariants of maps. We start with a recollection of some known Riemannian facts which we then will extend to contact C-C manifolds.

Let V be a compact Riemannian manifold. Then every exact k-form α on V can be *integrated* to a $k-1$ form β, which means $d\beta = \alpha$, such that

$$\|\beta\|_{L_p} \leq C\|\alpha\|_q \qquad (*)_q$$

for all q in the interval $1 < q < N$, $\frac{1}{p} = \frac{1}{q} - \frac{1}{\dim V}$ and some constant $C = C(V,p)$ for $k = 1$. This is just the Sobolev inequality of 2.4 and the proof for $k \geq 2$ is as follows. Choose β with $d^*\beta = 0$ as well as $d\beta = \alpha$ and write, using the elliptic theory, $\beta = K\alpha$, where K is a singular integral operator with the kernel $K(v,v')$ satisfying $\|K(v,v')\| \leq \text{const}\big(\text{dist}(v,v')\big)^{-(\dim V-1)}$. (In fact, the general case can be reduced to that of $V = \mathbb{R}^n$ where K is *the same* convolution kernel as in 2.4 applied to each scalar component of α.) Then the above $(*)_q$ follows from $(**)_q$ in 2.4. (Probably $(*)_1$ is not true for $k \geq 2$.)

Remarks

(a) The above proof shows, in fact, that

$$\|J^1\beta\|_q \leq C_1\|\alpha\|_q \qquad (*)'_q$$

where J^1 denotes the 1-jet of β, and $(*)'_q$ implies $(*)_q$ via the Sobolev inequality.

(b) Inequalities like $(*)_q$ are dual to *filling inequalities* evaluating the minimal k-chain filling in a given k-cycle, but the precise relation between the two classes of inequalities is not quite clear apart from the cases $k = 1$ and $k = \dim V$ (compare 2.4 and 3.6.B).

The inequality $(*)_q$ is useful for bounding rational homotopy invariants of smooth maps $f : V \to W$ in terms of a suitable $L_q\Lambda^k$-energies, $E_{k,q}(f) = \int_V \|\Lambda^k \mathcal{D}f(v)\|^q dv$, mentioned earlier. If W is simply connected then every such invariant is obtained, according to D. Sullivan, by the following procedure.

Step 1. Take some closed forms $\alpha_1, \ldots, \alpha_i, \ldots$ in W and let $\alpha_i^* = f^*(\alpha_i)$. Notice that the ($L_p$-norm of $\alpha_i^*)^p$ can be bounded by const $E_{k,q}(f)$ for each $k \leq \deg \alpha_i$ and $q = (k^{-1} \deg \alpha_i)p$.

Step 2. Take among the closed forms α_i^* the non-exact ones and evaluate (i.e. integrate) them against some cycles in V. The resulting numbers are the invariants of the first level (encoding the action of f on the real cohomology).

Step 3. Integrate the exact forms among α_i^* and call the integrals β_i. Consider the products $\gamma_{ij} = \beta_j \wedge \alpha_i^*$ and observe that γ_{ij} is closed whenever $\alpha_j \wedge \alpha_i = 0$. Then the values of the closed non-exact γ_{ij} on the cycles in V constitute the set of the invariants of the second level.

Step 4, etc. Integrate exact γ_{ij}. Consider their products with forms obtained at the previous stages and go on. Notice that the integrated forms are estimated by $(*)_q$ while the L_1-norms of products are controlled by the Hölder inequality. Finally the value of a closed k-form ω on a k-cycle V' in V, that is $\int_{V'} \omega$ is bounded by

$$\int_{V'} \omega = \text{const } \|\omega\|_{L_1}$$

for const depending on V and the homology class of V'.

Example. Let $W = S^2$ and $f : V \to W$ send the fundamental cohomology class of S^2, represented by the oriented area form α, to zero. Then α^* integrates to some 1-form β on V and the set of the values $\int_{V_i} \beta \wedge \alpha^*$ over the 3-cycles V_i in V constitutes the *Hopf invariant* of f. To bound this we need a bound on the $(L_q\Lambda^2)$-energy of f for $q = \dim V/2$. In fact, if $\|\Lambda^2\mathcal{D}f\|_{L_q} < A$, then $\|\alpha^*\|_{L_q} \le CA$ and $\|\beta\|_p \le C'A$ for $\frac{1}{p} = \frac{1}{q} - \frac{1}{\dim V}$. Then the L_1-norm of $\beta \wedge \alpha^*$ is bounded by $C''A^2$ provided $\frac{1}{p} + \frac{1}{q} = 1$, i.e. $\frac{1}{q} - \frac{1}{\dim V} + \frac{1}{q} = 1$, which makes $q = \dim V/2$.

Generalization. Let W be an arbitrary $(k-1)$-connected manifold. Then the forms α_i on W have degree $\ge k$ and a simple computation similar to the above shows that every rational homotopy invariant of $f : V \to W$ is bounded by $\|\Lambda^k\mathcal{D}f\|^m_{L_q}$ for $q = \dim V/k$ and a certain positive integer $m \le m_0(\dim V)$. This refines the homotopy finiteness property for maps f with $\|\mathcal{D}f\|_{L_N} \le \text{const}$, $N = \dim V$, (see 2.5) as the norm $\|\mathcal{D}f\|_{L_N}$ bounds the above $\|\Lambda^k\mathcal{D}f\|_{L_q}$.

Remarks

(a) Notice that our $L_q\Lambda^k$-energy for $q = \dim V/k$ is invariant under conformal changes of the Riemannian metric in V. Thus, for a fixed W, the infimum of this energy on the maps in a given homotopy class provides a conformal invariant of V (which measures, in a way, the conformal distance from V to W).

(b) Lower energies also provide non-trivial homotopy information. For example, if a map $f : V \to W$ has a sufficiently small (but yet positive) $L_1\Lambda^k$-energy, then it can be easily homotoped to a map $f' : V \to W$ which sends the k-skeleton of V to the $(k-1)$-skeleton of W. For example, for $k = 2$ we conclude that the induced homomorphism between the fundamental groups factors through a free group,

$$\begin{tikzcd} \pi_1(V) \arrow[r] \arrow[rr, bend right, "f_*"'] & \text{Free group} \arrow[r] & \pi_1(W) \end{tikzcd}$$

This raises the problem of a homotopy characterization (of the space) of maps f with some bound on $\|\Lambda^k\mathcal{D}f\|_{L_q}$ for given k and q and suggests the study of discontinuous maps f with $\|\Lambda^k\mathcal{D}f\|_{L_q} < \infty$. (See 3.6.C on some information on these matters.)

Making $q = \frac{3}{2}$ for maps $f : V \to S^2$. If in the above example we choose a sufficiently generic 3-cycle V_i in V we get the $L_q\Lambda^2$-energy of f restricted to V_i of the same order of magnitude as the corresponding energy of f, where the "generic" V_i depending on f is taken from a *compact family* (of 3-cycles or 3-submanifolds in V) *independent* of f. Then the above applies to f on V_i and gives the bound

$$\|\text{Hopf invariant of } f\| \leq \text{const}\left(\|\Lambda^2\mathcal{D}f\|_{L_{\frac{3}{2}}}\right)^2,$$

for every smooth map $f : V \to S^2$.

This restriction argument works in general and improves the bound on an invariant of f obtained by the evaluation of a product of certain forms on an m-cycle V' in V. Namely, it is more efficient to restrict f to a "generic" V' and apply $(*)_q$ to the induced forms on V' rather than on all of V.

3.6.A. $(\mathbf{L_q\Lambda^k})$-energies in the contact case. Now let V be a contact C-C manifold and see what happens if we evaluate $\Lambda^k\mathcal{D}f$ on the (horizontal) contact subbundle $H \subset T(V)$. We already know that the horizontality condition does not essentially restrict m-cycles for $m < \dim V/2$ and so the above restriction argument works as well with $\mathcal{D}f|H$.

Examples

(a) If $\dim V \geq 2m + 1$, then a smooth map $f : V \to W$, where W is a compact Riemannian manifold, has the norm of the induced cohomology homomorphism $f^* : H^m(W; \mathbb{R}) \to H^m(V; \mathbb{R})$ bounded by $\text{const}\,\|\Lambda^m\mathcal{D}f|H\|_{L_1}$.

(b) If $W = S^2$ as in the previous example, then (the norm of) the Hopf invariant of a smooth map $f : V \to W$ is bounded by $\text{const}\,\|\Lambda^2\mathcal{D}f|H|^2_{L_{\frac{3}{2}}}$, provided $\dim V \geq 7$.

Let us indicate a different instance where the restriction of f to horizontal cycles provides a non-trivial homotopy information.

Let V be a compact contact C-C *manifold of dimension $2m + 1$ and W be compact Riemannian. Then there exists an $\varepsilon > 0$, such that every smooth map $f : V \to W$ with $\|\Lambda^m\mathcal{D}f|H\|_{L_\infty} \leq \varepsilon$ has trivial homology homomorphism $H_i(V) \to H_i(W)$ for all $i \geq m$.*

Idea of the proof. This is obvious for $i = m$ as every m-cycle in V can be made horizontal. Then every i-cycle V' in V for $i = m + m_0$ can be sliced into m_0-dimensional family of horizontal m-cycles. If the f-images of these have small volumes, then $f(V')$ is homologous to zero in V (compare Appendix 1 in [Gro$_{\text{FRM}}$]). For example, if V' is homeomorphic to S^{2m+1}, the slicing of the fundamental cycle can be realized by a degree one map $\alpha : S^m \times S^{m+1} \to S^{2m+1}$ where the slices are the α-images of the spheres $S^m \times s \in S^m \times S^{m+1}$ as s runs over the parameter space, sphere S^{m+1}. One can make this map horizontal on each m-sphere $S^m \times s$ and if $\|\Lambda^m \mathcal{D}f|H\|_{L_\infty}$ is sufficiently small the f-images of these spheres in W have small volumes which makes f (as well as $\alpha \circ f$) homologous to zero.

Question. Does the above remain true for L_∞ replaced by L_q for some $q < \infty$?

The essence of the problem is seen in the following example. Suppose we have a function φ on the unit 2-disk D such that $\|\varphi\|_{L_q}$ is small. We want to slice the disk into curves c_t as in Fig. 10 so that $\|\varphi|c_t\|_{L_1}$ is small for *all* t.

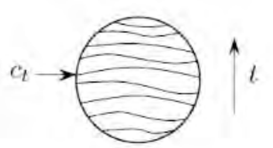


Figure 10

If we try to do it naively with families of parallel straight segments we may run into the "Kakeya" set of Besicovitch: the subset $\{v \in D \mid \varphi(v)| > \lambda\}$ may have small area and yet contain rather long segments in *all* directions.

To conclude the list of the known homotopy bounds on f in terms of $\mathcal{D}f|H$ we recall that the (C-C conformally invariant) norm $\|\mathcal{D}f\|_{L_N}$, $N = \dim V + 1 = \dim_{\text{Hau}} V$, bounds the number of homotopy classes of maps $f : V \to W$ provided $\dim V \geq 5$ (see 2.5) and we may only repeat the questions we stated earlier for V Riemannian.

Another class of questions arises where V is Riemannian, W is contact C-C of dimension $> 2 \dim V$ and we ask if the horizontality assumption on

$f : V \to W$ reinforces the bound on $\|\Lambda^k \mathcal{D}f\|$ in a homotopically significant way. But the discussion in 3.5.D makes the positive answer unplausible. The same seems to apply to contact maps between contact manifolds with bounds on $\mathcal{D}f|H$. Moreover it may be interesting to translate the above discussion into the intrinsic metric C-C language, where, in fact, $\|\Lambda^k \mathcal{D}f|H\|$ may have several non-equivalent C-C metric counterparts.

3.6.A′. Controlled integration of the Rumin complex. Let V be a compact contact manifold of dimension $n = 2m + 1$. We know (see 3.3) that the composition of the exterior d with the restriction to H, i.e. $x \mapsto dx|H$, $\deg x = k - 1$ is an *injection* of the quotient space $\big((k-1)\text{form}\big)/\big(\text{closed } (k-1)\text{-forms}\big)$ into the space of k-form on H (i.e. sections of $\Lambda^k H$), provided $k < m$. So we want to relate the norm on this quotient space coming from the L_p-norm on $(k-1)$-forms with L_q of k-forms on H. In other words, given an exact form α on V, we want to integrate it to a form β (which means $d\beta = \alpha$), such that $\|\beta\|_{L_p} \le \text{const}\,\|\alpha|H\|_{L_q}$. We know how to do this for $\deg\alpha = 1$, but in general it seems hardly possible without modifying the setting (see Remark (a) below). An attractive option is to divide the space of $(k-1)$-forms by a bigger space, namely by $(\text{closed } (k-1)\text{-forms}) + \big((k-1)\text{-forms vanishing on } H\big)$. Then the problem reduces to inverting the operator $d_H : \Lambda^{k-1}(V)/I^{k-1} \to \Lambda^k(V)/I^k$, where I^* is the differential ideal generated by the contact form and I^k stands for $I^* \cap \Lambda^k(V)$ (see 3.3). Now, Rumin shows (see [Rum$_{1;2}$]) that for $k < m$ the operator d_H is *hypoelliptic* in the following sense: the Rumin-Laplace operator $\Delta = (m-k)d_H d_H^* + (m-k+1)d_H^* d_H$ for a suitable adjoint operator d_H^* satisfies the following L_2-estimate

$$\|L_X L_Y f\|_{L_2} \le \text{const}(\|\Delta f\|_{L_2} + \|f\|_{L_2}),$$

where X and Y are arbitrary H-horizontal vector fields and the constant depends on X, Y, as well as on the metric on V used in the definition of d_H^*. Then, as in the Riemannian case, we have the (L_2-optimal) solution β of the equation $d\beta = \alpha$ which has $d_H^*\beta = 0$ and one can show that the operator $\alpha \mapsto \beta$ is given by a kernel $K(v, v')$ with $\|K(v, v')\| \le \text{const}\big(\text{dist}(v, v')\big)^{-(N-1)}$ for $N = \dim_{\text{Hau}} V$ (see [Fol]) and therefore (see 2.4)

$$\|\beta\|_{L_p} \le C\|\alpha\|_{L_q} \qquad (*)_q$$

for all q in the interval $1 < q < N$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{N}$. This $(*)_q$ can be used to bound the homotopy of a map f by observing that d_H-closed forms $\omega \bmod I^k$ on V of degree k for $k < m$ represent the cohomology classes in $H^k(V;\mathbb{R})$ by integrating over *horizontal* k-cycles, or by pairing ω with closed ℓ-forms ω' for $\ell = \dim V - k$ from the annihilator J^ℓ of I^k, where the pairing is the usual one, $(\omega, \omega') \mapsto \int_V \omega \wedge \omega'$. The actual bound on the homotopy invariants obtained this way are weaker than those using restriction of f to horizontal cycles but the above $(*)_q$ brings along a somewhat different perspective.

Question. What is the geometric significance of the controlled integration in the Rumin complex above the middle dimension?

Remarks

(a) Let us comment on the original problem where we to not mod away the horizontal part of β. According to $(*)_q$ every form α on V below the middle dimension integrates to $\bar\beta = \beta + \eta \wedge \beta'$, where $d\bar\beta = \alpha$, and $\|\beta\|_{L_p} \leq C\|\alpha|H\|_{L_q}$. On the other hand, $d(\eta \wedge \beta')|H = d\eta \wedge \beta'|H$, where, recall, η is the 1-form defining H and so $d\eta$ is non-singular on H. Thus, by Lefschetz lemma, $\|\eta\wedge\beta'\| \leq C'\|d(\eta\wedge\beta')|H\|$ everywhere on V. For example, if $\beta = 0$ we have the bounds $\|\eta \wedge \beta'\|_{L_p} \leq \|d(\eta \wedge \beta') \mid H\|_{L_p}$ for all p but for a *generic* L_p-form β' we do not expect anything better, i.e. we cannot significantly change $\eta \wedge \beta'$ by adding to it a *closed* form β'' which would make $\|\eta \wedge \beta' + \beta''\|_{L_p} \leq C\|d(\eta \wedge \beta')|H\|_q$ for $p > q$. (We suggest the reader would prove the non-existence of such β''.) Finally, even if $\beta \neq 0$, the way β is constructed seems to imply that it is "as smooth as possible", and in particular, $\|d\beta|H\|_{L_q} \leq \text{const}\,\|\alpha|H\|_{L_q}$. (To prove this one needs a suitable bound on $d_H K$ for the kernel $K = K(v, v')$ which, probably, is well known in the hypoelliptic theory.) Thus we conclude to an integration $\alpha \rightsquigarrow \bar\beta$ with $\|\bar\beta\|_{L_p} \leq \text{const}\,\|\alpha|H\|_{L_q}$ with $p = q$ which cannot be improved to $p > q$.

(b) Probably, much of the above discussion can be given an intrinsic C-C metric meaning in terms of straight cochains and/or (piecewise) Riemannian spaces V_ε approximating V. We suggest the reader would look into this.

3.6.B. Controlled integration and filling in Riemannian manifolds V. Let us start with several remarks on the relation between the controlled integration of forms and filling-in cycles S in V (compare Remark (b) at the beginning of 3.6). First we notice that the controlled integration inequality $(*)_q$, on a compact Riemannian manifold V, i.e.

$$\|\beta\|_{L_p} \leq C\|\alpha\|_{L_q}, \qquad (*)_q$$

remains valid, for $q = \dim V$ and $p = \infty$ if $\deg \alpha = \dim V$. That is, *for every exact form α on V of the top degree, there exists β with $d\beta = \alpha$, such that* $\sup \|\beta\| \leq C\|\alpha\|_{L_n}$, $n = \dim V$.

Proof. The form α defines the values of β on all $(n-1)$-*cycles* S in V which are boundaries via the Stokes formula

$$\beta(S) \underset{\text{def}}{=} \int_S \beta = \int_D \alpha \quad \text{for } \partial D = S.$$

Since $(\mathrm{Vol}_n D)^{\frac{n-1}{n}} \leq \mathrm{const}\,\mathrm{Vol}_{n-1} S$ for a suitable D filling-in S by the isoperimetric inequality, the values of β on S is bounded by

$$\beta(S) = \int_D \alpha \leq (\mathrm{Vol}\, D)^{\frac{n-1}{n}} \cdot \int_D \|\alpha\|^n \leq \mathrm{const}\,\|\alpha\|_{L_n} \mathrm{Vol}_{n-1} S.$$

Then, by the Hahn-Banach theorem the linear functional β can be extended from $(n-1)$-boundaries to all $(n-1)$-chains T in V such that the extended β satisfies the same inequality $\beta(T) \leq \mathrm{const}\,\|\alpha\|_{L_n} \mathrm{Vol}_{n-1} T$. This makes the pointwise norm of β, thought of as an n-current on V, bounded by $\mathrm{const}\,\|\alpha\|L_n$ and with a little fuss this current can be smoothed to an actual differential n-form with the same bound on the norm.

Remark. The above argument is dual (and essentially equivalent to) to the usual proof of the Sobolev inequality $\|f\|_{L_{n/n-1}} \leq \mathrm{const}\,\|df\|_{L_1}$ via the isoperimetric inequality (where the clarifying role of Hahn-Banach was much emphasized by Denis Sullivan).

3.6.B$'$. Thick families of filling-in codimension > 1 in Riemannian manifolds V. Now let $\deg \alpha = k < n$ and let us estimate $\beta(S)$ on a $(k-1)$-cycle in V homologous to zero in terms of α and chains D's filling-in S. For example, if $\deg \alpha = 1$, then S may consist of a pair of points $(v, v') \in V$ and one uses a suitable family of segments between v and v' for D's. (For example for v and v' being the opposite poles of a round sphere one takes all geodesic segments between the poles.) Thus one estimates the difference $\beta(v) - \beta(v')$ for the function β with $d\beta = \alpha$, by the integrals of α along these segments. In particular, one estimates $|\beta(v) - \beta(v')|$ in terms of the norms $\|d\beta\|_{L_{n+\varepsilon}}$ for $\varepsilon > 0$, which leads to the (rather obvious) Sobolev inequality for the C^δ-Hölder constant for $\delta = \varepsilon/n + \varepsilon$ of β by $\|d\beta\|_{L_{n+\varepsilon}}$ (which remains valid with $N = \dim_{\mathrm{Hau}} V$

in place of n for all C-C manifolds, see 2.3.E). The same logic applies to $\dim D$'s ≥ 2, where the suitable families of fillings must be sufficiently *thick* as D's approach S. Namely, at each point $v \in S$ the measure of the tangent spaces to the D's at v must be positive in the full linear "pencil" (which is S^{n-k} in this case) of the k-planes in $T_v(V)$ containing $T(S)$. Then one bounds $\beta(S)$ by

$$|\beta(S)| \leq \operatorname{const} \|\alpha\|_{L_q}, \; q = n - k + 1 + \varepsilon, \tag{+}$$

where the constant depends on S and on $\varepsilon > 0$. (If $n = k$ this is valid with $\varepsilon = 0$ as well.) In order to make (+) truly useful, one should reduce the dependence const(S) to something like const(Vol S) or const(Vol S, Diam S). This, probably, can be done in certain cases by refining the filling inequality of Federer-Fleming. Namely, instead of a single filling D of S one may look for a sufficiently thick family of these or (which is easier) to construct a D , such that a *given* positive function φ on V (playing the role of $\|\alpha\|^s(v)$) has $\int_D \varphi \leq \operatorname{const}(\|\varphi\|_{L_q}, \operatorname{Vol}_{n-1} S)$. Another possibility is to bound $\beta(S)$ only for certain "standard" cycles S keeping in mind that only a small portion of α matters in (+) for S of a small diameter ρ. In fact, the relevant family of D's may be taken inside a ball B_ρ of radius ρ containing S and then one can use $\|\alpha | B_\rho\|_{L_q}$ in (+) instead of the full L_q-norm. Such a localized version of (+) may suffice for $(*)_q$, $1 < q < n$. (This geometric approach to $(*)_q$ is motivated by possible generalizations where appropriate analytic techniques are not available.)

Remark on the thickness. Let us make more precise our thickness condition on a family of k-dimensional chains (e.g. submanifolds) D in V. We parametrize our family by a measure space M with a probability measure dm and, in fact, "a family of D's" means for us such a measure on the space of all D's.

Next, given a Borel function φ on V, we first integrate φ over each D in V, call this integral $\varphi(D)$, then integrate $\varphi(D)$ over (M, dm) for a given family D_m and call the result of the integration $\varphi(M)$. Notice that $\inf_M \varphi(D_m) \leq \varphi(M)$.

Definition. We say that a family (M, dm) of D_m has q-thickness $\geq C^{-1}$, iff $\varphi(M) \leq C\|\varphi\|_{L_q}$ for all positive Borel functions φ on V. (One usually uses a word like "modulus" but "thickness" seems less confusing.)

Thus our sentence "a standard S can be filled by a q-thick family" means that for each S from a certain given set of "standard" cycles there exists a family of chains D_m filling in S, such that $\{D_m\}$ has q-thickness $\geq C^{-1}$ for a constant $C > 0$ independent of S (from our set). Here is a simple existence theorem for such fillings which has been already used several times.

Let $q > n - k + 1$ and $\{S\}$ be a compact family of piecewise smooth $(k-1)$-cycles in V. Then every $S \in \{S\}$ can be filled by a q-thick family of D's. In particular, an individual cycle S built of finitely many smooth (i.e. smoothly embedded) simplices in V can be filled in by a q-thick family of D's.

3.6.C. Filling-in curves in Riemannian manifolds and $\|\Lambda^2\mathcal{D}\|$. Let us apply the above filling ideas directly to a smooth map $f : V \to W$ with a bound $\|\Lambda^2\mathcal{D}f\|_{L_q} \leq c$ for a given $q > n - 1$ and some $c > 0$. Then every piecewise smooth contractible curve S in V bounds a disk D with a control of the integral of $\|\Lambda^2\mathcal{D}f(v)\|$ over D and so the image $\Sigma = f(S) \subset W$ bounds a disk Δ in W of area $\leq c\,\mathrm{const}(S)$. (Actually one can reduce the dependence to const(length S) but this is not relevant at the moment.) Let us express the inequality area $\Delta \leq a$ by writing $\mathrm{Fill\,area}\,\Sigma \leq a$ (corresponding to the minimal disk filling-in Σ) and let us ask ourselves what is the homotopy structure of closed curves Σ in W with Fill area $\leq a$. Intuitively, curves Σ with small Fill area look "narrow" like the boundaries of small neighbourhoods of trees in a surface and one expects them to be *simultaneously* contractible. But the mere "narrowness" of curves does not suffice as, for example, the standard sphere S^2 can be sliced into arbitrary narrow curves, all looking like the one in Fig. 11 below.

Figure 11

However, one of the slices must necessarily have Fill area $\geq$ area $S^2/2$ and the following proposition shows that this is unavoidable.

3.6.C′. Narrow curves proposition. *Let W be a compact Riemannian manifold. If $\pi_1(W)$ acts trivially on $\pi_i(W)$, $i = 2, 3, \ldots$, then the space of closed curves $\{\Sigma\}_a = \{\Sigma \mid \text{Fill area}\, \Sigma \leq a\}$ has finite homotopy type in the space of all closed curves for every $a \geq 0$. That is, the inclusion of $\{\Sigma\}_a$ to the space $\{\Sigma\}$ of all curves factors through a map into a finite polyhedron with the number of cells bounded in terms of a. Furthermore, if $a \leq a_0$ for some sufficiently small $a_0 > 0$ then the subspace of based curves $\{\Sigma_{w_0}\}_a \subset \{\Sigma\}_a$ is contractible in the space of all based curves (i.e loops) in W (where we do not make any assumption on the action of $\pi_1(W)$ on $\pi_i(W)$.*

Sketch of the proof. We use the fact that the space of the maps $S^2 \to W$ with area $\leq a$ has finite homotopy type in the space of all maps $S^2 \to W$, and if a is small the space of based maps is contractible (see 2.5.C). This implies the corresponding properties of the space of fillings Δ of a fixed curve Σ with area $\Delta \leq a/2$ (where "filling" is a map of the disk D^2 to W which restricts on the boundary ∂D^2 to Σ). Finally, one passes from fillings to curves along the standard homotopy route.

Remarks

(a) The above seems to apply to annuli in W joining different curves and yield the uniform local contractibility of the space of curves where the distance is given by the area of the minimal annuli between curves.

(b) If $\pi_1(W)$ acts trivially on $\pi_2(W), \ldots, \pi_k(W)$ then $\{\Sigma\}_a$ has finite homotopy type up to dimension $k-1$.

Corollary. *If V is a compact simply connected n-dimensional Riemannian manifold, then for each $q > n-1$ the space of maps $F_c = \{f : V \to W \mid \|\Lambda^2 \mathcal{D} f\|_{L_q} \leq c\}$ has finite homotopy image in all of F, provided $\pi_1(W)$ trivially acts on $\pi_i(W)$, $i \geq 2$. Furthermore, the space F_{c_0} for a small $c_0 > 0$ contracts in F to (the space of) constant maps.*

Sketch of the proof. Join each point $v \in V$ with v_0 by a standard path π where the non-uniqueness is measured by the loops $\pi \circ (\pi')^{-1}$ in V. These loops form a compact family and so their images in W have filling areas controlled by c. In particular, if $c \leq c_0$, these filling areas are small which makes all these loops simultaneously contractible. Then our maps f are simultaneously contractible as well. Similarly, for large c we obtain a homotopy bound on F_c where the details are left to the reader.

Remark. The above seems to generalize to non-simply connected manifolds V, where F_c for small $c > 0$ should be contractible to the space of maps with 1-dimensional images.

Warning. *A map $f : V \to W$ with arbitrarily small norm $\|\Lambda^2 \mathcal{D} f\|_{L_\infty}$ and also with $\|\mathcal{D} f\| \leq$ const may easily have the image of dimension ≥ 2. In fact, there are Lipschitz maps of balls* onto *balls, e.g. $f : B^3 \to B^2$ which have* rank $\mathcal{D} f \leq 2$ *almost everywhere on B^3 (see* [Bat$_{1;2}$] *and* [Da-Se]*). But these maps can be uniformly approximated by maps with 1-dimensional images as our argument shows.*

Questions. It seems plausible that the above corollary remains valid for every $q > n/2$ and even for $q = n/2$. In fact this is so for maps into classifying spaces of compact Lie groups as follows from a theorem by Uhlenbeck (see 5.4). Then one may go below $n/2$ by looking on the restriction of f to some k-skeleton $V^k \subset V$ and applying the above corollary to $f|V_k$. Thus one sees, for example, that if $q > k - 1$ and $k \geq 2$, then *the bound $\|\Lambda^2 \mathcal{D} f\|_{L_q} \leq c_0$ for a small $c_0 > 0$ forces the restriction $f|V^k$ to be a contractible map $V^k \to W$*, where we assume as earlier that V is simply connected (compare 2.5.G). The question is what happens for smaller q?

One can define "narrow manifolds" of dimension > 1 but one does not expect the narrowness has an effect on homotopy comparable to that for curves. Yet our corollary may have non-trivial generalizations for Λ^j where $j > 2$ as the norm $\|\Lambda^j \mathcal{D} f\|_{L_q}$ does affect the homotopy for $q = n/j$ (see 3.6) and even stronger effect may be expected for $q > n - j + 1$. Yet we do not even know what happens for maps between spheres. For example, for which j, q, n and m is a map $f : S^n \to S^m$ with a sufficiently small norm $\|\Lambda^j \mathcal{D} f\|_{L_q}$ necessarily null-homotopic? Finally, one may allow certain discontinuous maps with $\|\Lambda^j \mathcal{D} f\|_{L_q} < \infty$ and raise the questions similar to those in 2.5.E - H. For example, one may look at the space of measurable maps f for which $\Lambda^j \mathcal{D} f$ is defined on almost all j-vectors (i.e. vectors in $\Lambda^j T(V)$) and $\|\Lambda^j \mathcal{D} f\|_{L_q} < \infty$. Here one should note that each functional $\|\Lambda^j \mathcal{D} f\|_q$ is semicontinuous in the space $F_1^{T(V)}$ (with the topology defined by the norm $\|\mathcal{D} f\|_{L_1}$), i.e. if $f_i \to f$ in this space then

$$\|\Lambda^j \mathcal{D} f\|_{L_q} \leq \liminf_{i \to \infty} \|\Lambda^j f_i\|_{L_q}.$$

Another encouraging sign comes from isosystolic inequalities (see [Gro$_{\mathrm{FRM}}$]) which may be applied to (the Riemannian metrics induced by) maps f with $\operatorname{Vol} f \underset{\deg}{=} \|\Lambda^n \mathcal{D} f\|_{L_1} \leq c$ and also to maps with $\|\Lambda^k \mathcal{D} f\|_{L_1} \leq c$ restricted to k-dimensional subspaces in V.

3.6.D. Thick filling of horizontal curves in contact manifolds. Let V be a contact manifold of dimension $n \geq 3$. Then the space of closed piecewise regular horizontal curves (where "regular" means "smoothly immersed" and the horizontality refers to the implied subbundle $H \subset T(V)$ of rank $n-1$) is homotopy equivalent to the space of all closed curves in V. Furthermore, if $n \geq 5$ then *each contractible piecewise regular horizontal curve S in V can be filled in by a q-thick family of horizontal disks D in V, for a given $q > n+2$.*

Idea of the proof. First let $n = 5$ and V be the space $J^1(\mathbb{R}^2, \mathbb{R})$ of the 1-jets of functions on the (x, y)-plane $\mathbb{R}^2$. Let S, locally, be given by the 1-jet of the zero function at the line $x = 0$ in $\mathbb{R}^2$ and f, g, h be smooth functions on $\mathbb{R}^2$ with 1-jets vanishing on the line $x = 0$. Thus we obtain a 3-parametric family of (xy)-planes in V containing S, say

$$\pi : (x, y, a, b, c) \mapsto \big(x, y, J^1_{(x,y)}(af + bg + ch)\big) \in V = J^1(\mathbb{R}^2, \mathbb{R}).$$

(Here and below we do not specify the measure implicit in our notion of "family" but this is always clear from the context.) We use the standard coordinates in $J^1(\mathbb{R}^2, \mathbb{R})$, where $J^1_{(x,y)}(\psi) = \big(\psi(x,y),\ \psi_x(x,y),\ \psi_y(x,y)\big)$ and then the Jacobian of the above map $\pi : \mathbb{R}^5 \to \mathbb{R}^5 = J^1(\mathbb{R}^2, \mathbb{R})$ equals

$$\det \begin{pmatrix} f & g & h \\ f_x & g_x & h_x \\ f_y & g_y & h_y \end{pmatrix}. \tag{$*$}$$

Since the 1-jets of the functions f, g, h vanish at $x = 0$, they are divisible by x^2 and, hence, this determinant is divisible by x^6. It follows that our family is q-thin (i.e. has q-thickness zero) for $q \leq 7$. On the other hand, for generic f, g and h (e.g. for $f = x^2,\ g = x^2 y$ and $h = x^3$) this determinant decays as x^6 (and not faster than that !) for $x \to 0$ and so $(\det)^{-1}$ is locally p-summable for $p < 6$. This implies that our family of planes has *positive* q-thickness for $q > 7$. Then, obviously, every smooth *regular* curve in V (which is locally equivalent to $J^1 \mid \{x = 0\} = 0$) can be filled by a q-thick (i.e. having q-thickness > 0) family of disks for every $q > 7$ which proves our claim for the regular case and $n = 5$.

Next, we reduce the case $n > 5$ to $n = 5$ as follows. Again we argue locally and we take a generic m-parametric family of germs of 5-dimensional submanifolds V_α containing our curve S at a given point $s \in S$, where $m = n - 5$. Each V_α is contact for the structure $H_\alpha = T(V_\alpha) \cap H$ and so $S \subset V_\alpha$ can be filled by a 3-dimensional $(7+\varepsilon)$-thick family of planes as earlier. Then the resulting $(n-2)$-dimensional family of planes in V is q-thick for every $q > 7 + (n-5) = n+2$ as a straightforward computation shows. This proves our claim for *regular* curves S in V for all dimensions ≥ 5.

Finally, let us look at a piecewise regular curve S at a non-smooth point (vertex) $s_0 \in S$. The basic example is where $V = J^1(\mathbb{R}^2, \mathbb{R})$ and S at s_0 is given by the 1-jet of the zero function on the boundary of the positive quadrant $\{x \geq 0,\ y \geq 0\} \subset \mathbb{R}^2$ where s_0 corresponds to $(0,0)$. Now, as earlier, we use a 3-parametric family $J^1(af + bg + ch)$ where the 1-jets of the functions f, h and h must vanish on S, i.e. on the lines $\{x = 0\}$ and $\{y = 0\}$. Here we take

$$f = \frac{x^2y^2}{x^2+y^2}\ ,\ g = \frac{x^2y^3}{(x^2+y^2)^{\frac{3}{2}}}\ ,\ h = \frac{x^3y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

and observe that the determinant $(*)$ equals $x^6y^6(x^2+y^2)^{-4}$ for these f, g, h. Thus $(\det)^{-1}$ is locally p-summable for $p < 6$ which implies the desired q-thickness of our family for $q > 7$.

This model case applies to $\dim V = 5$ whenever the two tangent vectors to S at the vertex s_0 span an *isotropic* plane in H_{s_0} (for the (curvature) forms $\omega = d\eta | H$ where $\ker = H$) and the general case easily reduces to this situation. Alternatively, one may use generic "conical" families at s_0 filling S (which are *not* C^1-smooth at s_0) similar to those in 3.5.B. (We leave the details to the reader.)

Corollary. *Let W be compact Riemannian, V be a compact contact simply connected C-C manifold of dimension $n \geq 5$ and $q > n+2$. Then the space of maps $F_c = \{f : V \to W \mid \|\Lambda^2 Df | H\|_{L_q} < c\}$ has finite homotopy image in the space F of all smooth maps $V \to W$, provided $\pi_1(W)$ acts trivially on $\pi_i(W)$ for $i \geq 2$. Furthermore, if $c_0 > 0$ is sufficiently small then $F_{c_0} \subset F$ contracts to (the space of) constant maps (with no assumption on $\pi_1(W)$, compare 3.6.C′).*

Counter-example for dim = 3. The Hopf map $f_0 : S^3 \to S^2$ can be easily homotoped to an f with $\Lambda^2 \mathcal{D} f | H = 0$ for the standard (horizontal) contact bundle $H \subset T(S^3)$.

Problems. Much of the questions raised in the Riemannian case extend to contact manifolds V, as we want to know what is the homotopy role of the norms $\|\Lambda^j \mathcal{D} f | H\|_{L_q}$ for given j and q, e.g. for $j = 2$ and $q \leq n+2$.

3.6.E. On the global contact geometry. Our study of (V, H) was local and essentially perpendicular to the global contact explosion of the last decade (see [Ben], [El$_{1\text{-}4}$], [Gir], [Ho]). It is unclear at the present moment if our C-C theory can be non-trivially globalized. Namely, we do not know which global contact invariants of (V, H) survive C-C bi-Lipschitz (or quasi-conformal) homeomorphisms. The simplest invariants where the question is already of interest are the Chern classes of the symplectic bundle $(H, d\eta)$. Then come the fillability and overtwist defined by Eliashberg, Hofer-Floer homology etc. Notice, that the behaviour of these invariants is unclear even if the homeomorphisms in question are smooth away from a *finite* subset in V and the contact geometry at such singular points looks very appealing.

4. Pfaffian geometry in the internal light

There are several simple geometric differential objects associated to a polarization (i.e. Pfaffian system) H on V which one wants to visualize in the (internal) C-C metric terms. Eventually one wishes to make the basic Pfaffian invariants and constructions independent of the differential background. But this goal is far from fulfillment.

4.1. A brief metricly guided Pfaffian tour. The basic characteristic of a polarization $H \subset V$ is its rank but even this still can not be recaptured by a robust (e.g. $C^{1-\varepsilon}$-Hölder) metric invariants of the corresponding C-C structure, unless the (first) commutators of the H-horizontal fields span $T(V)$ (as in contact manifolds, for instance) where rank H is determined by the equation

$$\operatorname{rank}(H) + 2(\dim V - \operatorname{rank} H) = \dim_{\mathrm{Hau}}(V, H)$$

(see 1.3.A). Notice that the Hausdorff dimension is sensibly behaved under Hölder maps and so the above metric characterization of rank H is $C^{1-\varepsilon}$-robust.

4.1.A. The H_i-filtration and the type numbers n_i. We denote, as earlier, by $H_i \subset T(V)$ the span of the commutators of order $\leqslant i$ of the H-horizontal fields,

$$H = H_1 \subset H_2 \subset \cdots \subset H_d = T(V)$$

where we assume the situation equiregular, i.e. $n_i = \operatorname{rank} H_i(v)$ independent of $v \in V$. Then the integer vector with the components n_i, where $n_1 < n_2 < \cdots < n_{d-1} < n_d = n = \dim V$ is called the *type* of H and d is the *depth* of H. We know (see 1.3.A) that the sum $\Sigma_{i=1}^{d} i(n_i - n_{i-1})$ is a $C^{1-\varepsilon}$-invariant as it equals the Hausdorff dimension of (V, H) but the metric meaning of individual n_i and of d remains obscure. (Notice that V is locally C^α-Hölder equivalent to $\mathbb{R}^n$ for $\alpha = d^{-1}$ and one might think d^{-1} equals the *maximal* α with this property.)

Remark. It seems to be unknown which sequences of numbers n_i may appear as ranks of H_i but these are easily computable for *generic* H where, for example, the inequality $n_1 + \frac{n_1(n_1-1)}{2} \geqslant n$ implies $n_d = n$ for $d \geqslant 3$. The first example with $n - n_2 > 0$ is that of an *Engel structure*, i.e. a generic 2-field H on a 4-space where $n_1 = 2$, $n_2 = 3$, $n_3 = n = 4$. Notice that all Engel structures are mutually locally isomorphic being similar in this respect to contact structures and there are the only generic polarizations with the local uniqueness property (see [Ger$_{\mathrm{EES}}$] about geometry of Engel structures).

4.1.A′. On local connectedness of smooth submanifolds. Naively, one could identify d as the maximum of the Hausdorff dimensions of *smooth* curves c in V. Unfortunately one lacks an adequate C-C metric characterization of smoothness albeit some aspects of smoothness are internally visible. For example, if a C^1-curve c is transversal to H_{d-1}, then the C-C metric on c is Lipschitz equivalent $\sqrt[d]{\text{Euclidean}}$ and so, in particular, $(c, \operatorname{dist}_{\text{C-C}})$ is C^1-*locally connected*, i.e. every two points within distance ε are contained in a (connected!) segment in c of diameter $\leqslant \operatorname{const} \varepsilon$ but this condition is not strong enough to rule out (non-smooth!) curves of large Hausdorff dimension even in the ordinary Euclidean space.

For more general (non-H_{d-1}-transversal) C^k-smooth curves one has some C^α-*connectedness* where the latter inequality is replaced by $\leqslant \operatorname{const} \varepsilon^\alpha$ with α depending on the smoothness of c and its tangency to H_i (compare 4.9). Furthermore, smooth submanifolds $V' \subset V$ also enjoy some local C^α-connectedness which means *contractibility* of each small ε-ball inside the concentric δ-ball for $\delta \leqslant \text{const}\, \varepsilon^\alpha$. (We suggest the reader would make it more precise and specific.) Unfortunately, no such property can distinguish "smooth".

***H_i*-filtration on curves.** The subbundles H_i filter the space $\mathcal{C}$ of smooth (and even Lipschitz) curves in V by $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_d = \mathcal{C}$ for $\mathcal{C}_i$ equal the space of H_i-horizontal curves. The essential metric property of smooth curves $c \in \mathcal{C}_i$ is having the Hausdorff dimension $\leqslant i$ and/or being C^α-Euclidean for $\alpha \geqslant i^{-1}$. This suggests the filtration of $\mathcal{C}$ by $\mathcal{C}_\nu$, $1 \leqslant \nu < \infty$, where ν refers the Hausdorff dimension of $c \in \mathcal{C}_\nu$ and/or the reciprocal of the Hölder exponent α allowing C^α-Hölder parametrization by $t \in \mathbb{R}$. One expects that $\mathcal{C}_\nu$ undergoes particular jumps as ν passes through the integer values. (One tends to think that C^α-curves for $\alpha > \frac{1}{2}$ are H_1-horizontal in a suitable sense, $C^{\frac{1}{3}+\varepsilon}$-curves are H_2-horizontal etc. Maybe, this can be shown if not for individual curves but for suitable families compare 4.9 below.)

Remarks on curvature. Besides the type numbers which manifest the anisotropic nature of the Carnot-Carathéodory geometry there are less apparent infinitesimal invariants but their metric effects may be sometimes more visible than those of the type numbers. For example, for every i_1 and $i_2 \geqslant i_1$ the commutator pairing defines a bilinear form

$$\Omega : H_{i_1} \otimes H_{i_2} \to H_{i_1+i_2}/H_{i_2}$$

generalizing the curvature form for the contact structure (see 3 and 3.2). We saw in 3.2 for type $H = (n-1, n)$ (i.e. for corank $H = 1$) then this curvature may metricly distinguish some H's via the Hausdorff dimension of submanifolds in V.

4.1.B. Submanifolds V' in V of a given type and Thom horizontal homology. Let $V' \subset V$ be a smooth *equiregular* submanifold which means the constancy of the ranks $n'_i = n'_i(V', H) = \operatorname{rank}(H'_i = H \cap T_{v'}(V'))$, $i = 1, \dots, d$, on V'. These *type numbers* of V' determine its Hausdorff dimension for the metric $\operatorname{dist}_H \mid V'$ by

$$\dim_{\mathrm{Hau}} V' = \sum_{i=1}^{d} i(n'_i - n'_{i-1})$$

and this is the only relation we know (besides $n'_d = n' = \dim V'$, of course). These numbers take the minimal values for *generic* V' transversal to all H_i, that are $m_i = \max(0, n' - (n - n_i))$ and for every integer vector $\{m'_i > m_i\}$ the inequalities

$$n'_i(V', H) \geqslant m'_i \quad , \quad i = 1, \dots, d, \tag{$*$}$$

impose a non-trivial system of partial differential equation on V'. An instance of this is the relation

$$\sum_{i=1}^{d} i(n'_i - n'_{i-1}) \leqslant M' \tag{$**$}$$

which has a metric interpretation via the Hausdorff dimension. For example, the relation $(**)$ for $(\dim_{\mathrm{Hau}} V =)$ $M' = n' = \dim V'$ is equivalent to the horizontality of V', i.e. to $n'_i(V', H) = n'$, $i = 1, \dots, d$, and so the non-smooth n'-dimensional subsets V' (e.g. n'-dimensional topological submanifolds) can be viewed as *generalized integral* (i.e. horizontal) *manifolds* of our Pfaffian system (polarization) H.

Let us count the number of P.D.E. (partial differential equations) encoded by $(*)$ and $(**)$. At every point $v \in V$ $(*)$ defines a certain (Schubert) subvariety Σ in the Grassmannian $\mathrm{Gr}_{n'} \mathbb{R}^n$ (for $\mathbb{R}^n = T_v(V)$) and the number in question equals, by definition, to the codimension of this subvariety. Now, clearly,

$$\dim \Sigma = \sum_{i=1}^{d} n_i(m'_i - m'_{i-1}) - (n')^2$$

and

$$\operatorname{codim} \Sigma = nn' - \sum_{i=1}^{d} n_i(m'_i - m'_{i-1}).$$

For example, the horizontality of V' is expressed by $(\dim V')$ $(\operatorname{corank} H)$ $= n'(n - n_1)$ equations, which corresponds to $(**)$ with $M' = n' = \dim V'$. Next, $(**)$ with $M' = n' + 1$ is given by $nn' - n_1(n' - 1) - n_2$ equations, for $M' = n' + 2$ we have m-equations where $m = \min(nn' - n_1(n' - 1) - n_3, nn' - n_1(n' - 2) - 2n_2)$, and so on. (Notice that the numbers n_i are not arbitrary, e.g. $n_2 - n_1 \leqslant \frac{n_1(n_1 - 1)}{2}$, $n_{i+1} - n_i \leqslant n_i\ n_1$, etc.)

Now we observe that V' in V is locally given by $n - n'$ functions on V' and so the relation $(*)$ is

(-) *underdetermined* : if $\operatorname{codim} \Sigma < n - n'$

(0) *determined* : if $\operatorname{codim}\Sigma = n - n'$

(+) *overdetermined* : if $\operatorname{codim}\Sigma > n - n'$.

Thus, in the case $(-)$, we typically expect plenty of C^∞ solutions and this is corroborated by the results in 4.2. Next, in the determined case, $(*)$ may have a reasonably large space of solutions but the existence of these is sometimes hard to prove. Finally, in the truly generic overdetermined case there should not be any C^∞-solutions and under relaxed genericity conditions the solutions must be very special. This will be explained better in 4.? but we should notice here that there is no satisfactory result limiting non-sufficiently smooth solutions. For example, there is no general principle prohibiting integral C^1-manifold of a generic C^∞-Pfaff system in the overdetermined case. In particular, our generic obstructions for C^∞-solutions of the relation $(**)$ (expressing $\dim_{\mathrm{Hau}} V' \leqslant M'$) do not exclude generalized solutions. Yet some of these can be ruled out under special favourable conditions (see 4.5 and 4.11).

Examples

(a) *Horizontality for* $\operatorname{corank} H = 2$. The horizontality of $V' \subset V$ is expressed here (i.e. for $n_1 = n - 2$) by $2n'$ equations and so this is an overdetermined condition for $n' > n/3$ and underdetermined for $n' < n/3$. So, for a generic H integral manifolds are expected (only) of dimension $n' \leqslant n/3$. Yet some non-generic H may have higher dimensional horizontal submanifolds. For instance, the complex holomorphic contact structure viewed as a real polarization of rank $2m-2$ on a $2m$-dimensional manifold has plenty of $2k$-dimensional horizontal submanifolds for $k = (m-1)/2$ (which, moreover, are complex holomorphic).

(b) *Horizontal surfaces.* If $\dim V' = 2$ then horizontality is overdetermined for $\operatorname{rank} H < \frac{1}{2}n + 1$, where $n = \dim V$, and underdetermined for $\operatorname{rank} H > \frac{1}{2}\, n + 1$. So (generic) H's below middle dimension are not expected to have integral manifolds of dimension > 1.

Remark on the numbers n_i and genericity. Let L be the free Lie algebra on n_1 generators and Δ_i, $i = 1, 2, 3, \ldots$ denote the rank of the space spanned by the commutators of degree exactly i (so, e.g. $\Delta_1 = n_1$). One knows that

$$\Delta_i = \frac{1}{i} \sum_{j|i} \mu(j)\, n_1^{\frac{i}{j}} \quad , \quad i = 1, 2, \ldots .$$

where μ is the Möbius function. Now, for every H of rank i, we have $n_i - n_{i-1} \leqslant \Delta_i$ and if H is generic then, clearly, $n_i = n_i^{\max} \underset{\text{def}}{=} \min\left(n, \Sigma_{j=1}^{i} \Delta_j\right)$. If we look at the polarizations H with the type numbers n_i prescribed in advance where $n_i < n_i^{\max}$ for some $i \geqslant 2$, we must be aware that these H are subjects to some system of P.D.E. (expressed by the relation type $H = \{n_i\}$) and the notion of genericity among these H should be treated with a respect due to possible complications arizing from this system.

Example where n_2 is prescribed. Fix $n_1 \geqslant 2$ and look at the polarizations H of rank n_1 with given $n_2 = \operatorname{rank} H_2$ written as $n_2 = n_2^{\max} - \delta$ where we assume $n = \dim V \geqslant \Delta_1 + \Delta_2 = n_1 + \frac{n_1(n_1-1)}{2}$. The relation $\operatorname{rank} H_2 \leqslant n_2^{\max} - \delta = n_1 + \frac{n_1(n_1-1)}{2} - \delta$ may be expressed by a system of $\delta(n - n_1 - \delta)$ P.D.E.'s on H. Since H is given by $n_1(n - n_1)$ functions on V, this is an underdetermined system for $\delta n, \dots, n$ and one expects a reasonable genericity theory for its solutions H (see p.121 in [Gro$_{\text{PDR}}$]). But for $\delta \gg n_1$ this system becomes overdetermined and the solutions should form a rather small space where the idea of genericity may be applied with a caution (if at all).

4.1.B$'$. On the type of a morphisms. Given two Pfaffian systems, i.e. polarized manifolds (V, H) and (V', H'), the *type* of a *morphisms*, i.e. of a smooth map $f : V' \to V$ is given by the numbers

$$n_{ij}(f, v') = \operatorname{rank}_{v'}\ H'_j \cap (\mathcal{D}^{-1} f)(H_i),$$

which reduce to the above n'_i for the case where $H' = T(V')$ and f is an embedding. In general, it is easy to determine the Hölder exponent of f with respect to the C-C metrics dist_H and $\operatorname{dist}_{H'}$ and this give us some metric extract from n_{ij}; but the full metric meaning of the totality of n_{ij} remains unclear.

Exercise. Count the number of P.D. equations on f expressing the inequalities $n_{ij}(f) \geqslant n_{ij}$ for given numbers n_{ij}. (Notice that there are certain inevitable relations between $n_{ij}(f)$, e.g. $n_{11}(f) = n'_1 \Rightarrow n_{ii}(f) = n'_i$ for $i \geqslant 2$.) Find out when the condition "f is C^α-Hölder" is under/overdetermined.

4.1.C. Pfaffian systems in jet spaces. Take a smooth m-dimensional manifold V_0 and observe that the Grassmann manifold $V^1 = \mathrm{Gr}_k V_0$ of k-planes in $T(V_0)$ carries a natural polarization $H^1 \subset T(V^1)$ of corank $m-k$ which is uniquely characterized by the following condition: *the tangential lift* (1-*jet*) *of every C^2-smooth k-dimensional submanifold $V_0' \subset V_0$ to V^1 is H^1-horizontal.* (For $\dim V_0' = \dim V_0 - 1$ this is the standard contact structure but for $\operatorname{codim} V_0' \geqslant 2$ this H^1 is not so symmetric and looks less beautiful.) This generalizes to the space $V^r = \mathrm{Gr}_k^r(V_0)$ of r-th jets of germs of k-dimensional submanifolds $V_0' \subset V_0$ by observing that V^r carries a natural polarization H^r which is *the minimal subbundle in $T(V^r)$ containing the tangent vectors of the r-jet lifts to V^r of all C^{r+1}-submanifolds $V_0' \subset V_0$ of dimension k.* It is easy to see that this H^r has corank equal the dimension of the space of $(r-1)$-jets of the maps $\mathbb{R}^k \to \mathbb{R}^{m-k}$ at $0 \in \mathbb{R}^k$, where $m = \dim V_0$. Thus

$$\text{corank } H^2 = (m-k)\left(1 + k + \frac{k(k+1)}{2} + \cdots + \frac{(k+r-2)!}{(k-1)!(r-1)!}\right).$$

If an integral manifold $V' \subset V^r$ of H^r *diffeomorphically* projects onto some submanifold $V_0' \subset V_0$ then V' equals the r-jet lift $J^r(V_0') \subset V^r$. But there are certain smooth integral manifolds $V' \subset V^r$ where the projection $V' \to V_0$ has a singularity (i.e. not a C^1-immersions).

Examples

(a) *Foliation by the graphs of polynomials.* The space V^r is locally isomorphic to the space of the r-jets of maps $\mathbb{R}^k \to \mathbb{R}^{m-k}$, called $\mathcal{J}^r$, where the graph (image) of the r-jet of each smooth map $f : \mathbb{R}^k \to \mathbb{R}^{m-k}$, denoted $V_f' = J_f^r(\mathbb{R}^k) \subset \mathcal{J}^r$, can be included as a leaf into a H^r-horizontal foliations, namely the one with the leaves V_{f+p}' where p runs over the space of polynomial maps $\mathbb{R}^k \to \mathbb{R}^{m-k}$ of degree r. Now let f be also a polynomial but of degree $> r$ and consider the one parameter family of foliations corresponding to $\lambda f + p$ for $\lambda \in \mathbb{R}$. The tangent space to this foliation at each point $j \in \mathcal{J}^r$, say $T_j(\lambda f + p) \in T_j(\mathcal{J}^r)$, converges as $\lambda \to \infty$ to some k-dimensional subspace $S_j(f)$ in $T_j(\mathcal{J}^r)$ (this is obvious and valid for every C^∞-map f) and the limit field $S(f)$ of k-dimensional subspaces in $T(\mathcal{J}^r)$ is C^∞-smooth away from a proper algebraic subset in $\mathcal{J}^r$. This $S(f)$ is obviously integrable with H^r-horizontal leaves which do not project to smooth k-dimensional submanifolds in $V_0 = \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ anymore. In fact, if f is a homogeneous polynomial map of degree $r+1$

and rank $\geqslant p$ (as defined below) then, clearly, the (k-dimensional) leaves of $S(f)$ project to $(k-p)$-dimensional subvarieties in $\mathcal{J}^{r+1}$ (and hence the projections to $\mathbb{R}^m$ under $\mathcal{J}^{r-1}$ are at most $(k-p)$-dimensional).

Definition. A homogeneous polynomial map $f : \mathbb{R}^k \to \mathbb{R}^{m-k}$ of degree $r+1$, viewed as a $\mathbb{R}^{m-k}$-valued symmetric $(r+1)$-form on $\mathbb{R}^k$, $f(x_0, x_1, \ldots, x_r)$, defines a linear map from $\mathbb{R}^k$ to the linear space of such r-forms, say $f' : x \mapsto f(x, x_1, \ldots, x_r)$, and

$$\operatorname{rank} f \underset{\text{def}}{=} \operatorname{rank} f'.$$

For instance, if $\operatorname{rank} f = k$, then the leaves of $S(f)$ are (affine) subspaces in the fibers of the projection $\mathcal{J}^r \to \mathcal{J}^{r-1}$ and as we vary f the tangents to these $S(f)$ span the tangent spaces to the fibers. It follows that the *fibers of the projection* $\mathcal{J}^r \to \mathcal{J}^{r-1}$, *and hence of* $V^r \to V^{r-1}$, *are* H^r*-horizontal.* (Notice that these fibers have dimension $> k$ unless $r = 1$ and $m - k = 1$.)

(b) There are many singular k-dimensional subvarieties $V_0' \subset V_0$ which become desingularized by lifting their non-singular loci to V^r and then taking the topological closure of these lifts in V^r. In fact this desingularization is conjectured (by J. Nash) to work for all algebraic subvarieties V_0' in $V_0 = \mathbb{R}^m$ and sufficiently large $r = r(V_0')$ but even without establishing this conjecture the non-singular loci of the r-jet lifts of singular algebraic varieties $V_0' \subset V_0 = \mathbb{R}^m$ provide us with a pool of smooth k-dimensional H^r-horizontal submanifolds in V^r having their projections to V_0 singular.

(c) The examples (a) and (b) can be brought to an equal footing by considering families of algebraic varieties $V_\lambda' \subset V_0$ where the dimension of V_λ' may jump down at certain values of λ (e.g. V_λ' may degenerate to a single point at $\lambda = 0$) but where the dimension of the r-jet lift of V_λ' to V^r remains constant at these λ.

One gets a better view on integral manifolds in V^r by looking at the natural embedding

$$V^r \subset \underbrace{\mathrm{Gr}_k(\mathrm{Gr}_k(\cdots \mathrm{Gr}_k(V_0)\cdots)}_{r}$$

and by taking the closure $\overline{V}^r$ in this iterated Grassmannian. (Notice that $V^1 = \overline{V}^1$.) Then the s-jets of (potential) integral manifolds $V' \subset V^r$ of H^r are represented by points in $\overline{V}^{r+s}$ where some complication may arise from the possible singularity of $\overline{V}^{r+s}$ at $\Sigma = \overline{V}^{r+s} - V^{r+s}$ (which is, probably, well understood by formal P.D.E. people, e.g. A. Vinogradov & co.).

On C-C regularity of continuous jets. If V_0' is a C^r-submanifold of V_0 its r-jet $V' \subset V^r$ is just continuous but the C-C geometry of (V^r, H^r) suggests certain ways of measuring the tangency of V' to H^r which lead to the following.

Questions. What is the relation between the ordinary $C^{r+\alpha}$-Hölder features of $V_0' \subset V_0$ and the Hölder exponent of the r-jet (lift) map $V_0' \to V' \subset V^r$ with respect to the C-C metric dist_{H^r} on V^r? What is the meaning of the C-C Hausdorff dimension of V'?

4.1.D. Horizontal chains and cycles. Consider singular chains $\Sigma_i c\sigma_i$ in a manifold V where the singular simplices σ_i are *smoothly immersed* into V. If V is polarized by some $H \subset T(V)$ then one filters (the space of) these chains by the types of σ_i and studies the arizing homology theories. The first instance of this was treated by Thom in 1959 (see [Th] who was interested in *horizontal chains* in the above jet spaces (V^r, H^r). Thom viewed such chains, and especially k-cycles, in open subsets $\mathcal{R} \subset V^r$ (where k refers to the dimension of submanifolds $V_0' \subset V_0$ whose r-jets form V^r) as generalized solution of the partial differential relation imposed on k-dimensional submanifolds $V_0' \subset V_0$ by the requirement $J^r(V_0) \subset \mathcal{R}$ (compare [Gro$_{\mathrm{PDR}}$]) and he indicated the idea of the proof of the following statements

(A) *every i-cycle in $\mathcal{R}$ for $i \leqslant k$ is homologous to a horizontal one*

(B) *every horizontal i-cycle for $i < k$ homologous to zero bounds a horizontal $(i+1)$-chain.*

Some aspects of Thom's idea (maps "à fort gradient" and singularities "dents de scie") have been already presented in our contact § 3 in the "pleated" disguise of Poenaru. This idea suffices to complete the proof of (A) (see 4.4.A) but (B) appears more difficult. In fact, the (B)-part of the singular homology theory based on smoothly immersed or embedded simplices (without any H in the picture) is a subtle matter only recently settled by F. Lalonde in a satisfactory fashion (see [Lal]. I am not certain Thom insists on *immersed* simplices but this point of view is taken by Briant and Griffiths in [Br-Gr] where the authors indicate a generalization of Thom's theorem, also see [$\mathrm{Ge}_{\mathrm{BNCC}}$] on this matter).

4.1.E. Horizontal forms and cohomology. Every finite measure μ on (a compact family of) H-horizontal oriented submanifolds V' of dimension k in V defines a *k-current* which, for a sufficiently "smooth and ample" measure, is represented by a (unique) differential $(n-k)$-form ω' on V which is obviously *H-horizontal* in the following sense

ω' annihilates every 1-form η vanishing on H, i.e.

$$\eta \mid H = 0 \Rightarrow \eta \wedge \omega' = 0.$$

In other words, ω' vanishes on each hyperplane in $T_v(V)$ containing H_v, $v \in V$, or equivalently, on every $(n-k)$-plane *non-transversal* to H. To see what it means, we represent H locally as the common kernel of $n - n_1$ linear forms, say $\eta_1, \dots, \eta_{n-n_1}$, for $n_1 = \operatorname{rank} H$ and observe that the above condition is equivalent to divisibility of ω' by $\zeta = \eta_1 \wedge \eta_2 \wedge \dots \wedge \eta_{n-n_1}$. Thus the H-horizontality condition distinguishes certain subbundle $H\Lambda^{n-k} \subset \Lambda(V) = \Lambda^{n-k}(T(V))$ of rank $n_1!/k!(n_1-k)!$ (where k should be $\leqslant n_1$ and $H\Lambda^{n-k} \underset{\text{def}}{=} 0$ otherwise).

Next we observe that if the boundaries of the submanifolds V' in the support of μ miss an open subset $U \subset V$, then the resulting form ω is *closed* on U. Thus *de Rham (co)homology of the horizontal forms "corresponds" to Thom's homology of the horizontal cycles* (compare 4.11).

Notice that closed H-horizontal forms annihilate the differential ideal $I^* = I^*(H)$ generated by the 1-forms vanishing on H and so the horizontal cohomology and homology are dual to those of the quotient complex Λ^*/I which are extensively studied in [$\mathrm{Ge}_{\mathrm{BNCC}}$] and [Br-Gr] (where the authors point out the duality between their $H^*(\Lambda^*/I)$ and Thom's homology).

Example. *Pure forms.* Let ω be induced from a volume form ω_0 on our $(n-k)$-dimensional manifold V_0 by a smooth map $\varphi : V \to V_0$ of rank $n-k$. Then the horizontality of ω amounts to horizontality of the fibers $\varphi^{-1}(v_0)$, $v_0 \in V_0$. Notice that closed 1-forms and $(n-1)$-forms are locally generically pure in the above sense and so the passage from (closed) horizontal submanifolds (or cycles) to closed horizontal forms does not essentially enlarges the picture.

Parameter count for closed horizontal forms. Let us evaluate the expected "functional dimension" (of the space) of closed horizontal forms. First, we do that without horizontality by observing that, by definition, the "functional dimension" of the space of k-forms on V equals $\lambda_i = \operatorname{rank} \Lambda^i(T(V))$, i.e. $n!/i!(n-i)!$, while the de Rham cohomology groups have "functional dimension" zero. Therefore, closed i-forms have

$$\text{“}f\dim\text{''}(\ker d^i) = \lambda_{i-1} - \lambda_{i-2} + \lambda_{i-3} - \cdots (-1)^{i-1}\,\lambda_0 = \\ \lambda_i - \lambda_{i+1} + \lambda_{i+2} - \cdots (-1)^{n-i}\,\lambda_n.$$

Then we come to the space of *closed horizontal* forms written as the intersection $H\Lambda^{n-k} \cap \ker d^{n-k}$ where the expected "functional dimension" in the generic case of "transversal" intersection is

$$\begin{gathered}\text{“}f\dim\text{''}(H\Lambda^{n-k} \cap \ker d^{n-k}) = \\ \operatorname{rank} H\Lambda^{n-k} + \text{“}f\dim\text{''}\ker d^{n-k} - \lambda_{n-k} = \\ n_1!/k!(n_1-k)! + \sum_{j=0}^{n-k} (-1)^{j+n-k-1}\,\lambda_j = \\ n_1!/k!(n_1-k)! + \sum_{j=n-k+1}^{n} (-1)^{j-n+k}\,\lambda_j.\end{gathered}$$

Notice that this "$f\dim$" is (significantly) greater than the one of the space of H-horizontal k-dimensional manifold (which equals $n-k-k(n-n_1)$) for $2 \leqslant k \leqslant n-2$. For example, if $k=2$ the "functional dimension" of closed horizontal $(n-2)$-forms equals $\frac{n_1(n_1-1)}{2} - n + 1$ and so one generically expects plenty of closed horizontal $(n-2)$-forms on V if $n_1(n_1-1) > 2n-2$ while the horizontal surfaces generically need $2n_1 \geqslant n+2$ for their existence. *Thus, closed horizontal $(n-k)$-forms may be present in abundance without the existence of a single horizontal k-submanifold for* $2 \leqslant k \leqslant n-2$. (But I have no convincing examples actually exhibiting such abundance, compare 4.11.)

Remarks

(a) *Closed forms of a given type.* One could start with submanifolds of given (horizontality) type $(n_1, n_2, \ldots)$ and arrive at similar filtration on de Rham cohomology (compare 4.1.C′ and 4.11).

(b) *Hodge theory.* The relation between the notions of horizontality (and type, in general) for submanifolds and forms is quite similar to what happens in the classical Hodge theory where, for example, closed (k,k)-forms correspond to complex submanifolds of real dimension $2k$. In both cases the linearization of the space of submanifolds is achieved via the Plücker embedding of the Grassmann manifold into exterior forms.

(c) *Linearization of* P.D.E. *and Bochner formulae.* A system of s partial differential equations of order r imposed on k-dimensional submanifolds $V_0' \subset V_0$, e.g. on (graphs of) maps $V_0' \to \mathbb{R}^{m-k}$, can be represented by a subvariety $V \subset V^r = \mathrm{Gr}_k^r(V_0)$ of codimension s. Then the solution of the system are represented by H-horizontal k-dimensional submanifolds $V' \subset V$ for $H = T(V) \cap H^r \subset T(V)$ where H^r is the canonical polarization of the jet space V^r. Thus an arbitrary P.D.E. system reduces to a Pfaffian one. (This point of view was emphasized by Thom.) Notice that in many cases one should differentiate our equations several times which corresponds to passing to $V^{(i)} \subset V^{r+i}$ consisting of $(r+i)$-jets of formal solutions of our P.D.E.'s. Only then the Pfaffian system expressed by $H^{r+i} \cap T(V^{(i)})$ becomes truly representative of the original P.D.E.'s (which is essential when the canonical projection $V^{(i)} \to V$ is not surjective). Next we linearize our equations by considering closed H-horizontal $(n-k)$-forms on V, for $n = \dim V$, instead of horizontal submanifolds. (This idea is presented in [Br-Gr] in the dual language of the characteristic homology.) This may greatly enlarge the space of solutions globally as well as locally. The undue enlargement of the space of solutions can be somewhat contained by passing to $V^{(i)}$ over V and even better by replacing the linear span by the *convex hull* of the (space of) actual solutions in the space of (closed) differential forms or currents.

(Not quite Pfaffian) example. Let V be a $2m$-dimensional manifold endowed with an almost complex structure $J : T(V) \to T(V)$. Then J-*holomorphic* submanifolds $V' \subset V$ of real dimension $2k$ are defined by the condition $J(T(V')) = T(V')$ which can be represented by $2k(2m-2k) - 2k(m-k) = 2k(m-k)$ partial differential equations imposed on $2m - 2k$ functions (on V'). This P.D.E. system is overdetermined for $2 \leqslant k \leqslant m-1$ and, generically, there is no J-holomorphic submanifolds in

V of dimension $2k \neq 0, 2, 2m$. On the other hand the corresponding bundle of $(2m-2k)$-forms, which is spanned by the monomials $\Lambda_{i=1}^{k} \eta_i \wedge J\eta_i$ for arbitrary 1-forms η_i on V, has rank $\left(\frac{m!}{k!(m-k)!}\right)^2$, since after the complexifications these monomials span the bundle of $(m-k, m-k)$-forms. For example, J-holomorphic *curves* (i.e. $k = 1$) which themselves has "functional dimension" zero ($2m-2$ equations against $2m-2$ functions) give rise to (the space of) closed $(m-1, m-1)$-forms having "functional dimension" $m^2 - 2m + 1$ which is strictly positive for $m \geqslant 2$. Next, for $k = 2$ this dimension becomes

$$\left(\frac{m(m-1)}{2}\right)^2 - \frac{2m(2m-1)(2m-2)}{6} + \frac{2m(2m-1)}{2} - 2m + 1$$

which is > 0 for $m \geqslant 4$, while the "functional dimension" of J-holomorphic surfaces is negative.

Now, if we pass to higher order jets, the corresponding variety $V^{(i)}$ with large i may become empty for $k \geqslant 2$ and, generically, no enlargement of the original (empty) space of J-holomorphic submanifolds takes place. More interestingly, the (formal) *convex hull* of the J-holomorphicity condition is strictly smaller than the linear span. Namely the $(n-k, n-k)$-forms ω in this hull are *positive* in the sense that $\omega(x_1, \ldots, x_{n-k}, Jx_1, \ldots, Jx_{n-k}) \geqslant 0$ for arbitrary tangent vectors x_i, $i = 1, \ldots, n-k$, in V.

Example. The space $\mathbb{C}^m$ contains no compact complex submanifolds of positive dimension k but has plenty of closed (and exact) $(n-k, n-k)$-forms with compact supports. Yet every *positive* closed $(n-k, n-k)$-forms ω with compact support is necessarily zero as on one hand, $\omega \wedge \omega_0 \geqslant 0$ for every positive (k,k)-forms ω_0 with constant coefficients and on the other hand, $\int \omega \wedge \omega_0 = 0$ since ω_0 is exact.

Similar positivity (or convex hull) phenomenon appears in the presence of non-trivial Bochner-Weitzenböck formulae given by certain "positive" forms in the jet spaces of submanifolds (or maps), but an appropriate (non-Pfaffian?) formalism is yet to be developed.

Example. *Holomorphic Pfaffian systems and* $\mathbb{C}$*-polarizations.* We want to think of complex k-dimensional integral submanifolds as real $2k$-dimensional submanifolds satisfying the Cauchy-Riemann equations as well as the horizontality equations. Then we have the notion of horizontality on form together with the Hodge (p,q)-decomposition. The complex structure provides us with the notion of positivity and closed positive horizontal (k,k)-forms constitute a satisfactory "convex hull" of complex horizontal submanifolds. A particular interesting case is where H is a holomorphic contact structure, possibly with singularities, on a complex projective variety.

Finally we observe that one can formally bring horizontality and holomorphicity on an equal footing with the notion of *complex polarization* which is a $\mathbb{C}$-subbundle H in the complexified tangent bundle $\mathbb{C}T(V)$. But the geometric significance of complex polarizations remains obscure in general.

4.1.E′. Intrinsic metric evaluation of horizontality of forms. Let H be an equiregular polarization on V of rank n_1 and $H = H_1 \subset H_2 \subset \cdots \subset H_d$ be the commutator filtration as earlier. Then each (type) integer vector $m_1 \leqslant m_2 \leqslant \cdots m_d = n - k$ defines a subbundle in $\Lambda^{n-k}(V)$ of $(n-k)$*-forms* ω *of cotype* $\{m_i\}$, $i = 1, \dots d-1$ which vanish on a $(n-k)$-vector $x_1 \wedge \cdots \wedge x_{n-d}$ whenever $x_1, \dots, x_{m_i+1} \in H_i$ for every $i = 1, \dots, d-1$. For example, horizontal forms ω have cotype $\{m_i\}$ for $m_1 = m_2 = \cdots = m_{d-1} = n_1 - k$ (where we assume $n_1 \geqslant k$ as horizontal forms are zero otherwise). Then we set $M = m_1 + 2(m_2 - m_1) + \cdots + d(m_d - m_{d-1})$, with the convention $m_d = n - k$, and let $M^*(\omega)$ be the minimum of $M = M\{m_i\}$ over all $\{m_i\}$ serving as cotype for ω. (If $\{m_i\}$ is a cotype for ω then so is $\{m'_i\}$ with $m'_i < m_i$ and the minimum for M is achieved for the maximal m_i for $i < d$, as m_d is fixed and equals $n-k$.) Finally, for every $(n-k)$-dimensional de Rham cohomology class h define $M^*(h)$ as the maximum of $M^*(\omega)$ over all closed form ω representing h.

Proposition. (Compare 3.3.B.) *Every cohomology class* $h \in H^{n-k}(V;\mathbb{R})$ *can be represented by a straight (Alexander-Spanier) cocycle* c*, such that* $\|c\|_\varepsilon \lesssim \varepsilon^M$*, for* $M = M^*(h)$*, which means* $c(v_0, \dots, v_{n-k}) \leqslant \mathrm{const}(\mathrm{diam}\{v_0, \dots, v_{n-k}\})^M$ *for all* $(n-k+1)$*-tuples of points of diameter* $\varepsilon \to 0$*, where "diameter" refers to the* C-C *metric* dist_H *on* M.

Proof. Let us first visualize the effect of $M^*(\omega)$ on the norm $\|\omega\|_{g^*_\varepsilon}$ for the anisotropically blown up metric g^*_ε (see 1.4.D). This blow up depends on a choice of a Riemannian metric g on V and g^*_ε is characterized by the equality $\|X\|_{g^*_\varepsilon} = \varepsilon^{-i}$ for each unit vector $X \in H_i \ominus_g H_{i-1}$. Thus we see that $\|\omega\|_{g^*_\varepsilon} = O(\varepsilon^{M^*(\omega)})$ for all ω and this property uniquely defines $M^*(\omega)$.

Now we construct c by integrating ω representing h over some "standard" (straight) ε-simplex Δ_ε in V (compare 3.3.B). Notice that, according to 1.4.D-D$''$ simplices of diameter ε in the C-C metric become roughly of the unit size with respect to $\text{dist}^*_\varepsilon$ corresponding to g^*_ε. Thus, in order to have $\int_{\Delta_\varepsilon} \omega \leqslant$ const, our Δ_ε must be represented by a map σ_ε of the unit simplex Δ to V, such that the Lipschitz constant of σ_ε with respect to $\text{dist}^*_\varepsilon$ is bounded by a constant independent of ε. This is immediate if V is a nilpotent Lie group with a self-similarity $A_\varepsilon : V \to V$, where one may take $\sigma_\varepsilon = A_\varepsilon\ \sigma_1$ for some standard smooth map σ_1 (representing $\Delta_1 \subset V$ with given vertices $v_0, \ldots, v_{n-k}$). Then in the general case, as we know, V can be identified near each point $v_0 \in V$ with a nilpotent Lie group N_{v_0} such that the Riemannian metric of N_{v_0} approximates that of V such that the distance between the two metrics goes to zero in the unit ball around v_0 after the anisotropic ε-blow up of V and N_{v_0} (see 1.4.A, 1.4.D, 1.4.D$'$). It follows that the above simplex $\Delta_\varepsilon = \sigma_\varepsilon(\Delta)$ is good enough for V as well as for N_{v_0}. ∎

Exercises

(a) Evaluate the "functional dimension" of the space of closed form ω of given cotype $\{m_i\}$ and, in particular, with $M^*(\omega)$ equal to a given number M_0.

(b) Determine the degree of the exterior differential relative to cotype, i.e. determine the precise rule $\{m_i\} \mapsto \{m'_i\}$ so that the differential of a form of cotype $\{m_i\}$ has cotype $\{m'_i\}$. Then find the largest M_0 (in terms of n, k and n_i), such that V admits a *non-zero exact* form ω of degree $n-k$ with $M^*(\omega) \geqslant M_0$. (Of course, the truly interesting problem is finding *non-exact closed* forms ω on V and on open subsets $U \subset V$ with $M^*(\omega) \geqslant M_0$ for large M_0. Such forms are expected to exist for a *generic* polarization H whenever the "functional dimension" of the space of closed forms with $M^*(\omega) \geqslant M_0$ is positive).

(c) Extend (a) to the P.D.E. system defined by the relations

$$\text{cotype}(\omega) = \{m_i\} \quad , \quad \text{cotype}(d\omega) = \{m'_i\}$$

for given m_i and m_i'. Do this, in particular, to the relations

$$M^*(\omega) = M_0 \quad , \quad M^*(d\omega) = M_0'.$$

(d) Filter the de Rham complex on V by $M = M^*(\omega)$, define the corresponding graded complex and relate this to the filtration on the (straight) Alexander-Spanier cochains c with $\|c\|_\varepsilon \lesssim \varepsilon^M$ according to the exponent M.

4.2. Analytic techniques for local construction of integral (*H*-horizontal) submanifolds. We represent our $H \subset T(V)$ (locally) as the common zero of $n - n_1$ 1-forms $\eta_1, \dots, \eta_{n-n_1}$ on V for $n_1 = \operatorname{rank} H$, which represent the tautological $H^\perp$-valued form $\eta : T(V) \to H^\perp \underset{\text{def}}{=} T(V)/H$, i.e. the quotient homomorphism, and let $\Omega : \Lambda^2 H \to H^\perp$ be the curvature form of H represented by the differentials $\omega_i = d\eta_i$ on H, $i = 1, \dots, n - n_1$ (compare § 3). We express the horizontality (integrality) condition for an immersion $f : W \to V$ by the system

$$f^*(\eta_i) = 0 \quad , \quad i = 1, \dots, n - n_1 \ , \tag{$*$}$$

which, in fact, contains $k(n - n_1)$ partial differential equations for $k = \dim W$, as every (induced) 1-form on W has k components. (Notice in passing that $(*)$ obviously implies the vanishing of the induced curvature forms, i.e.

$$f^*(\omega_i) = 0,$$

for C^2-maps f and then this conclusion extends to all C^1-maps f satisfying $(*)$ by a straightforward approximation argument. This rules out k-dimensional integral C^1-manifolds through a given point $v \in V$ whenever the forms ω_i on H_v have no common isotropic k-dimensional subspace. If $\omega = \{\omega_i\}$ is generic and $k \leqslant \frac{n_1+1}{2}$, then k-dimensional isotropic subspaces of ω forms a subvariety of codimension $\frac{k(k-1)}{2}$ in the Grassmann manifold $\mathrm{Gr}_k\ H_v$ having $\dim \mathrm{Gr}_k\ H_v = k(n_1 - k)$ and so for $n - n_1$ generic ω_i's appearing for a generic H there is no common isotropic subspace in H_v for $(k-1)(n - n_1) > n_1 - k$, and hence, no k-dimensional integral C^1-submanifold through v).

Linearization of $(*)$. If we deform an immersion f along a field ∂ on $f(W) \subset V$ then the derivative (variations) of the induced form $f^*(\eta_i)$ clearly equals $\omega_i(\partial, \cdot) + d\eta_i(\partial)(\cdot)$ (compare 3.4.1 in [$\mathrm{Gro}_{\mathrm{PDR}}$] and [$\mathrm{DA}_{\mathrm{ISB}}$])

and, in particular, if ∂ is horizontal the linearized equations $(*)$ become linear *algebraic* in the unknown ∂,

$$\omega_i(\partial, X_j) = \sigma_{ij} \quad , \quad i = 1, \dots, n - n_1 \ , \ j = 1, \dots, k \qquad (*)'$$

where X_j is a full frame of vector fields (locally) on W.

4.2.A. Ω-regularity and infinitesimal invertibility. A k-dimensional linear subspace $S \subset T_v(V)$, $v \in V$, is called *Ω-regular* if the above linear system $(*)'$ with $X_j \in S$ is non-singular and hence, admits a solution $\partial \in H_v$ for arbitrary σ_{ij}. If S is H-horizontal, i.e. contained in H_v, then this definition is truly correct being independent of the choice of η_i, and hence of the 2-forms ω_i on $T_v(V)$ representing Ω on H_v. Namely, this regularity amounts to surjectivity of the linear map $\Omega_\bullet : H_v \to \mathrm{Hom}(S, H_v^\perp)$ naturally associated to Ω. In general, the notion of Ω-regularity depends on a choice of η_i but this will cause no difficulty as we shall only need horizontal S at the crucial moment.

Denote by $\mathcal{I}$ the differential operator which assigns to each smooth map $f : W \to V$ the induced forms $\{f^*(\eta_i)\}_{i=1,\dots,n_1-k}$. If f is *Ω-regular*, i.e. is an immersion with Ω-regular tangent spaces in $T(V)$, then the linearized equations $(*)'$ are algebraically solvable on W and so $\mathcal{I}$ is *infinitesimally invertible* (in the sense of 2.3.1 in [Gro$_{\mathrm{PDR}}$]) on Ω-regular immersions (compare [DA$_{\mathrm{ISB}}$]). Therefore main theorem 2.3.2 of [Gro$_{\mathrm{PDR}}$] yields the following.

4.2.A$'$. Local h-principle. (see [DA$_{\mathrm{ISB}}$]) *If H is C^∞-smooth then the sheaf of H-horizontal Ω-regular C^∞-immersions $W \to V$ is microflexible and satisfies the local h-principle. In particular, for every Ω-isotropic and Ω-regular linear subspace $S \subset H_v$, $v \in V$, there exists a germ of integral C^∞-submanifold $W \subset V$ at v with $T_v(W) = S$.*

The needed result from [Gro$_{\mathrm{PDR}}$] is a version of *Nash implicit function theorem* which also applies to C^r-smooth H with sufficiently large r, say $r \geqslant 10$ (probably, less than 10 but 10 is what I am able to see without much thinking) and then the above proposition is valid for C^{r-6}-immersions. Furthermore, if H is real analytic, then the local h-principle (but not microflexibility) remains valid for Ω-regular horizontal C^{an}-immersions (see 2.3.6 in [Gro$_{\mathrm{PDR}}$]). On the other hand germs of (non-Ω-regular) integral manifolds can be sometimes obtained via the Cauchy-Kovalevskayo theorem which may apply in some cases where the C^∞-techniques fail.

Proposition. (compare [DA$_{\mathrm{IC}}$].) *If the system* $(*)'$ *is* $(k+1)$*-underdetermined, i.e.* $n_1 - k(n-n_1) \geqslant k+1$*, then every* Ω*-regular* H*-horizontal germ of a* k*-dimensional* C^{an}*-submanifold in* V *extends to a horizontal* $(k+1)$*-dimensional* C^{an}*-germ.*

In fact, the Ω-regularity allows one to resolve $(*)$ with respect to the derivative $\partial = \frac{\partial}{\partial u_{k+1}}$ and the underdeterminancy condition gives room to a ∂ independent of X_i so that the resulting solution becomes an *immersion* of a $(k+1)$-dimensional manifold into W.

4.2.A″. Dimension count for regular isotropic subspaces. Since the map $\Omega_\bullet : H_v \to \mathrm{Hom}(S, H_v^\perp)$ vanishes on $S \subset H_v$ as S is Ω-isotropic, the inequality

$$n_1 - k = \dim H_v/S \geqslant \dim \mathrm{Hom}(S, H_v^\perp) = k(n-n_1)$$

is *necessary* for (the possibility of) S being Ω-regular. Next, we claim that this inequality is also *sufficient* for the existence of an Ω-regular isotropic k-dimensional subspace S in H_v, for a *generic* 2-form Ω on H_v. This is proven in two steps.

Step 1. The forms Ω for which the required $S \subset H_v$ exists constitute a Zariski *open* subset in the space of all forms $\Omega : \Lambda^2 H_v \to H_v^\perp$.

The proof follows by induction on k as the Ω-regularity of a $(k-1)$-dimensional isotropic subspace $S' \subset H_v$ reduces the Ω-isotropic condition on $S \supset S'$ to a *non-singular* system of *linear* equations, namely to

$$\Omega(X, X_i') = 0,$$

for a basis $X_1', \ldots, X_{k-1}'$ in S' and some fixed $X \in S \ominus S'$.

Step 2. If $n_1 - k \geqslant k(n-n_1)$ then the space of Ω's admitting an Ω-regular isotropic S of dimension k is *non-empty*.

To see that, take some $S \subset H_v$, take a bilinear form on $S \oplus (H_v/S)$ corresponding to a *surjective* linear *map* $H_v \to \mathrm{Hom}(S, H_v^\perp)$ vanishing on S and extend this form to an antisymmetric form on $H_v = S \oplus (H_v/S)$ using the natural embedding $A \otimes B \to \Lambda^2(A \oplus B)$ (for $A = S$ and $B = H_v/S$).

Corollary. *(Compare* $[\mathrm{DA}_{\mathrm{ISB}}]$ *and* $[\mathrm{DA}_{\mathrm{IC}}]$*.)* If H is a generic C^∞-polarization of rank n_1 and $k \leqslant n_1/(n-n_1+1)$ then H admits germs of Ω-regular horizontal k-dimensional C^∞-submanifolds at all $v \in V$ away from a stratified subset $\Sigma \subset V$ of positive codimension. If, moreover, H is C^{an} and $k \leqslant (n_1-1)/(n-n_1+1)$, then there is an analytic horizontal $(k+1)$-dimensional germ at every $v \in V - \Sigma$.

Remark. The inequality $k \leqslant (n_1-1)/(n-n_1+1)$ is equivalent to $n-(k+1) \geqslant (k+1)(n-n_1)$ which expresses the (under)determinancy of the P.D.E.-system for the horizontality of $(k+1)$-dimensional submanifolds in V. Therefore, this system is *overdetermined* for $k > (n_1-1)/(n-n_1+1)$ and then there is no (even C^∞-smooth) horizontal $(k+1)$-dimensional submanifolds in V for *generic* (C^∞ or C^{an}) H. In other words, the condition $k \leqslant (n_1-1)/(n-n_1+1)$ (for the existence of $(k+1)$-dimensional C^{an}-germs) is sharp. This can not be said about our bound on k in the C^∞-case. Yet some version of the local h-principle and microflexibility for a suitably "regular" class of k-dimensional horizontal germs is expected (for generic H) whenever $n-k > k(n-n_1)$, i.e. where the corresponding system of P.D.E. is underdetermined (compare (2) in (E) of 2.3.8 in $[\mathrm{Gro}_{\mathrm{PDR}}]$). On the other hand the existence of integral submanifolds remains highly problematic in the (generic) determined C^∞-case, i.e. for $n-k = k(n-n_1)$.

Exercises

(a) Evaluate the codimension of those (non-generic) Ω which admit no Ω-regular isotropic S of given dimension k and thus find $\operatorname{codim}\Sigma$ for Σ in the above corollary.

(b) Let A, B and S be linear spaces. Study (the space of) pairs $\Omega : \Lambda^2 A \to B$ and $\varphi : S \to A$, where φ is injective Ω-regular and Ω-isotropic. Decide when for a fixed (generic) Ω the space of the above φ's is i-connected for given i and the dimensions of A, B and S (compare 3.3.1 in $[\mathrm{Gro}_{\mathrm{PDR}}]$).

4.2.B. Calculus of variations for regular horizontal submanifolds. Let us make more precise the above remarks on the general notion of regularity which applies to an arbitrary system of partial differential equations imposed on maps $f : W \to V$. Such a system of m equations of order r is represented by an m-codimensional subvariety in the space of r-jets of maps $W \to V$, denoted $\mathcal{R} \subset \mathcal{J}^r$, and our equations are expressed by the inclusion $J^r(f)(W) \subset \mathcal{R}$. (We prefer to work here with maps rather than submanifolds and so use $\mathcal{J}^r$ rather than $\mathrm{Gr}_k^r(V)$. This makes no essential difference as our discussion is local and submanifolds $W^k \subset V^n$ can be represented by maps $\mathbb{R}^k \to \mathbb{R}^{n-k}$, compare 4.1.C). Actual equations appear if we write $\mathcal{R}$ as the zero set of s functions $\Delta_1, \dots, \Delta_m$ on $\mathcal{J}^r$ and our system becomes $\Delta_i \circ J^r(f) = 0$, $i = 1, \dots, m$. Differentiating these equations corresponds to lifting (prolongations) of our $\mathcal{R}$ to higher order jet space, say $\mathcal{R}^j \subset \mathcal{J}^{r+j}$ and it is convenient to stabilize $\mathcal{R}$, i.e. to take $\mathcal{R}^\infty \subset \mathcal{J}^{r+\infty}$ where ∞ stands for a large non-specified j. The notion of regularity will apply first to individual jets $x \in \mathcal{R}^\infty$ and then to solutions f of $\mathcal{R}$ where the regularity of f means that for the jets $J^{r+\infty} f(w) \in \mathcal{R}^\infty$ for all $w \in W$.

So we represent our $\mathcal{R}^\infty$ near a point $x \in \mathcal{R}^\infty$ in question by a (non-linear) differential operator $\Delta = \{\Delta_1, \dots, \Delta_m\}$ and denote by $L = L_f$ the linearization of Δ at f with $J^\infty f(v) = x$. This is a linear differential operator on W defined near the point w under x acting on n-tuples of functions ($n = \dim V$) with the range in m-tuples. Then we look for the right inversion M of L, i.e. a differential operator of certain order s, sending m-tuples to n-tuples, such that $L \circ M = \mathrm{Id}$. It is shown in 2.3.8 of [Gro$_{\mathrm{PDR}}$] that such an M, of a sufficiently high order s, does exist in the generic underdetermined (i.e. $m < n$) case by reducing the identity $L \circ M = \mathrm{Id}$ to a linear system of algebraic equations on (the coefficients of) M. This system, for a sufficiently large s, has more unknown than equations (as we assume $m < n$) and it is proven in [Gro$_{\mathrm{PDR}}$] that this system is non-singular for generic $L = L_f$ and hence has the desired solution M. Notice that the coefficients of this system are made out of coefficients of L and their derivative and so singularity or non-singularity of this system at a given jet $x \in \mathcal{R}^\infty$ makes perfect sense. Thus we call x *regular* if (for a suitable choice of Δ) the above mentioned system on M is non-singular for some (sufficiently large) s.

Observe that the regularity gives us somewhat more than invertibility of L, namely it allows us to "uniformize" (or parametrize) the solutions of the homogeneous P.D.E. system $L(g) = 0$, by taking (locally) the full system of solutions of the system $L \circ M = 0$, say $M_1, \dots, M_t$. Then every g satisfying $L(g) = 0$ can be written as $\Sigma_{i=1}^t \; M_i \; h_i$ for some m-tuples of functions $h_1, \dots, h_t$.

Example. Let L act on pairs of functions in one variable by

$$L : (g_1, g_2) \mapsto g_1' + \ell g_2'$$

where $\ell = \ell(t)$ and g' stands for $\frac{dg}{dt}$. If $g_1' + \ell g_2' = 0$, then

$$g_2 = h'/\ell' \quad \text{and} \quad g_1 = h - \ell h'/\ell' \quad \text{for} \quad h = g_1 + \ell g_2,$$

and so the uniformization (of g_1 and g_2 by h) is achieved whenever ℓ' does not vanish.

Warning. *There are certain non-regular situations where the uniformization is still present. For example, every closed forms on $\mathbb{R}^k$ is exact but the exterior differential is* not *invertible by a differential operator.*

Now we turn to the calculus of variations where we extremize the value of certain (energy) functional $E(f)$ on the solutions f of some system $\mathcal{R}$ and we want to write down the Euler-Lagrange equations. In order to write these equations we use infinitesimal variations of f supported near a given point $w \in W$. These variations g are just solutions of the system $L_f\ g = 0$ and, in general, it is hard to generate these with the vanishing conditions away from w. But if these g are uniformized by (unrestricted) strings of functions $h_1, \dots, h_t$, we apply the usual derivation of the Euler-Lagrange to h_i and thus obtain the desired equations on extremal f (restricted by $\mathcal{R}$).

Example. All smoothly immersed horizontal curves in a contact manifold are regular by a trivial computation (essentially reproduced for $n = 3$ in the previous example). Thus C-C geodesics satisfy the geodesic equation and, in particular, are C^∞-smooth. Similarly, the immersed horizontal submanifolds of dimension k are regular for all k, and if they are extremal (i.e. area minimizing) they satisfy the respective Euler-Lagrange equations. (*Warning*. A smooth horizontal map $W \to V$ extremizing some energy may fail to be an immersion and then neither the regularity nor Euler-Lagrange are automatic. Furthermore, there are certain generic polarizations where some immersed horizontal *curves* are non-regular despite claims to the contrary by several people including the present author in exercise (a) p. 84 in [$\mathrm{Gro}_{\mathrm{PDR}}$]. This was pointed out to me by L. Hsu, see [$\mathrm{Ge}_{\mathrm{VP}}$], [$\mathrm{Hsu}_{\mathrm{CVG}}$], [Mont], [Suss], [Bou] and [Pel-Bou] on this matter).

Fat polarizations. A polarization H is called *fat* if every 1-dimensional subspace in H is Ω-regular and 1-dimensional variational problems has been most extensively studied in the fat case (see [Ge$_{\mathrm{CVP}}$], [Ge$_{\mathrm{HPS}}$] and references therein). Here we only notice that "fat" = "contact" for corank $H = 1$ but for corank $H \geqslant 2$ the fatness is locally non-generic (albeit open) condition which seems highly restrictive especially if the underlying manifold is compact without boundary. Standard examples of fat H are provided by the complex contact structure (corank $H = 2$) and the horizontal bundle of the Hopf fibrations $S^{4n+3} \to \mathbb{H}P^n$ (where corank $= 3$).

Question. What could be a meaningful notion of *k-fatness* allowing many (but not all) Ω-regular isotropic k-plane in H for corank $H \geqslant 2$?

Horizontal submanifolds from the infinite dimensional point of view. The space of (smooth) maps $f : W \to V$ can be viewed as an infinite dimensional manifold, say $\mathcal{F}$, where each tangent space $T_f(\mathcal{F})$, $f \in \mathcal{F}$, equals the space of sections of the induced bundle $f^*(T(V)) \to W$. It may be hard (?) to speak of vector fields on $\mathcal{F}$ but one can define (the space of) jets of such "fields" at each point $f \in \mathcal{F}$ by the following (well known) recipe. Given an infinite dimensional vector bundle $p : \mathcal{S} \to \mathcal{F}$, where both $\mathcal{S}$ and $\mathcal{F}$ are spaces of maps between finite dimensional manifolds, one defines the fiber of the 1-jet bundle $J^1(\mathcal{S})$ over $\mathcal{S}$ at s as the space of homomorphisms $h : T_f(\mathcal{F}) \to T_s(\mathcal{S})$ satisfying $(\mathcal{D}p) \circ h = \mathrm{Id}$. For example, if $\mathcal{S} = T(\mathcal{F})$ for the above $\mathcal{F} = \mathrm{Maps}\ (W, V)$, then every such homomorphism h is essentially the same as a homomorphism $T_f(\mathcal{F}) \to T_f(\mathcal{F})$ which is, in turn, given by a "kernel" $K(w, w')$ which is a section of a certain (finite dimensional) vector bundle over $W \times W$. Thus, the functor $\mathcal{S} \to J^1(\mathcal{S})$ keeps us within the category of infinite dimensional manifolds which are spaces of maps between finite dimensional ones. Then, once we have $\mathcal{S}^1 = J^1(\mathcal{S})$, we define $J^2(\mathcal{S})$ as a suitable subspace in $J^1(\mathcal{S}^1)$, etc.

Next we can speak of the Lie bracket between jets of vector "fields" on $\mathcal{F}$ which is a pairing

$$J^r(T(\mathcal{F})) \otimes J^r(T(\mathcal{F})) \to J^{r-1}(T(\mathcal{F}))$$

and if $\mathcal{H} \subset T(\mathcal{F})$ is a polarization in our category, the expression "$\mathcal{H}$ Lie spans $T(\mathcal{F})$" makes perfect sense. For example, if $\mathcal{H}$ corresponds to $H \subset T(V)$, then $\mathcal{H}$ Lie spans $T(\mathcal{F})$ whenever H spans $T(V)$.

Now, let $\mathcal{G} \subset \mathcal{F}$ be a submanifold, for example the subspace of H-horizontal maps $f : W \to V$. We consider the induced polarization $\mathcal{H}' = \mathcal{H} \cap T(\mathcal{F})$ on $\mathcal{G}$ and ask ourselves when the Chow connectivity theorem holds for $\mathcal{H}'$. For example, if $\mathcal{G}$ consists of H-horizontal maps $f : W \to V$, then $\mathcal{H}'$-horizontal curves $\mathbb{R} \to \mathcal{G}$ correspond to H-horizontal cylinders $W \times \mathbb{R} \to V$ and one would like to interpret the existence results for these (e.g. our implicit function theorem) as connectivity theorems for $\mathcal{H}'$.

4.2.C. Partially horizontal submanifolds in V. We want to extend our analysis to k-dimensional submanifolds $W \subset V$ which have rank $(T(W) \cap H) = m$ for a given $m \leqslant k = \dim W$. The corresponding system of P.D.E. now applies to pairs (f, G) where f is a map $W \to V$ and $G \subset T(W)$ is a subbundle of rank m, and the equations express the tangency of the f-image of G to H by

$$f^*(\eta_i) \mid G = 0 \quad , \quad i = 1, \ldots, n - n_1, \tag{$+$}$$

for the forms η_i defining H (compare $(*)$ at the beginning of 4.2). We linearize $(+)$ (as we did it with $(*)$) using a frame $X_1, \ldots, X_m$ in G and the complementary bundle $G^\perp = T(W)/G$ so that the tangent vectors to the space of G's can be represented by homomorphisms $G \to G^\perp$ or by m-tuples of sections $\partial_1, \ldots, \partial_m$ of $G^\perp$. These ∂_j, $j = 1, \ldots, m$, serve together with the field ∂ (tangent to V along $f(W) \subset V$) as the unknowns of the linearized system which algebrizes for *horizontal* fields ∂ on W (i.e. sections of $H \mid f(W) \underset{\text{def}}{=} f^*(H)$) and reads

$$\omega_i(\partial, X_j) + \eta_i(\partial_j) = \sigma_{ij}. \tag{$+$}'$$

Now, we want to express non-singularity of $(+)'$ and (partial) isotropy of G in an invariant language and we simplify the notations by identifying W with $f(W) \subset V$. Then the inclusion $(T(W), G) \hookrightarrow (T(V), H)$ induces an embedding $G^\perp \hookrightarrow H^\perp$ and consequently a homomorphism

$$\mathrm{Hom}(G, H^\perp) \to \mathrm{Hom}(G, H^\perp / G^\perp).$$

We compose this with $\Omega_\bullet : H \to \mathrm{Hom}(G, H^\perp)$ and denote by $\Omega'_\bullet$ the composed homomorphism $H \to \mathrm{Hom}(G, H^\perp/G^\perp)$. We easily see that

(1) G is *partially Ω-isotropic* in the sense that $G \subset \operatorname{Ker} \Omega'_\bullet$;

(2) *non-singularity* of $(+)'$ is equivalent to *surjectivity* of $\Omega'_\bullet$.

We apply the generalized Nash implicit function theorem as earlier and arrive at the *microflexibility and the local h-principle for immersions f which are m-horizontal and partially Ω-regular*, i.e. having rank $(\mathcal{D}f)^{-1}(H) = m$ and $\Omega'_\bullet$ surjective. In particular, *if H is C^∞-generic we have a C^∞-germs of m-horizontal partially regular k-dimensional submanifold $W \subset V$ at each $v \in V - \Sigma$*, $\operatorname{codim}\Sigma > 0$, *provided* $\operatorname{rank} G \leqslant \operatorname{rank} H$, $\operatorname{rank} G^\perp \leqslant \operatorname{rank} H^\perp$ and

$$\operatorname{rank} H - \operatorname{rank} G \geqslant \operatorname{rank} G \operatorname{rank}(H^\perp / G^\perp),$$

i.e. $n_1 - m \geqslant m(n - n_1 - k + m)$. Furthermore, if H is C^{an}, $n_1 \geqslant m + 1$ and $n_1 - m \geqslant m(n - n_1 - k + m) + 1$, *then there is a $(k+1)$-dimensional $(m+1)$-horizontal C^{an}-germ through every $v \in V - \Sigma$.*

Remark. Recall that the $(m+1)$-horizontality of a $k+1$-dimensional submanifold in V is expressed by $(m+1)(n - n_1 - (k+1) + (m+1))$ P.D. equations against $n - k - 1$ unknown functions and note that the (under)determinancy of these equations, expressed by the inequality $n - k - 1 \geqslant (m+1)(n - n_1 - k + m)$, is equivalent to $n_1 - m \geqslant m(n - n_1 - k + m) + 1$.

Exercises

(a) Generalize the above to submanifolds $W \subset V$ of a given horizontality type, i.e. with prescribed ranks of the intersections $T(W) \cap H_i$ (compare 4.1.B).

(b) Generalize the exercises from 4.2.A″ to the present context.

On higher order regularity. One expects that m-horizontal k-dimensional germs in generically polarized manifolds (V, H) becomes amenable to the above techniques in the underdetermined case (i.e. for $n - k > m(n - n_1 - k + m)$) with the general notion of regularity indicated in 4.2.B (which implies the infinitesimal invertibility). In particular, one may think there is an m-horizontal germ at each point $v \in V$, if H is C^∞-generic and $n - k > m(n - n_1 - k + m)$.

4.3. The global h-principle for smooth horizontal submanifolds. The microflexibility and local h-principle do not suffices by themselves to construct (global regular) horizontal immersions of k-dimensional manifolds W into V but the theory of *continuous sheaves* (see 2.2 in [Gro$_{\mathrm{PDR}}$]) provides global constructions in the presence of "sufficiently many" $(k+1)$-dimensional (regular) horizontal germs.

Overregularity. Call an isotropic linear subspace $S \subset H_v \subset T_v(V)$ overregular if it is contained in an Ω-regular Ω-isotropic subspace $S' \subset H_v$ with $\dim S' > \dim S$. Then a fiberwise injective homomorphism $T(W) \to T(V)$ is called *overregular* if it sends each $T_w(W) \subset T(W)$, $w \in W$ onto an overregular subspace in H. Finally, an immersion $f : W \to V$ is called *overregular* if the differential $\mathcal{D}f : T(W) \to T(V)$ is overregular. (Notice that our overregularity implies horizontality).

Approximation theorem. *Smooth overregular immersions $W \to V$ satisfy the C^0-dense h-principle. It follows that for every overregular homomorphisms $T(W) \to T(V)$ the underlying map $W \to V$ admits a fine C^0-approximation (which amounts to the uniform approximation if W is compact) by smooth overregular (and hence, horizontal) immersions $W \to V$*(compare 3.5).

The proof follows from the local analysis of the previous section and the microextension theorem in 2.2.4 of [Gro_{PDR}].

In order to apply the theorem to a k-dimensional manifold W one needs at least one Ω-regular isotropic subspace $S'_v \subset H_v$ of dimension $> k$. What is even better is a continuous fields of such subspaces, say $S'_v \in H_v$ for all $v \in V$, forming a vector bundle S over V. Then every injective homomorphism $T(V) \to S \subset H \subset T(W)$ is overregular. Notice that every S'_v at a given point $v \in V$ extends to such a field in a small neighbourhood $U \subset V$ of v. Furthermore, every S'_v extends to a field on all V if (V, H) admits a transitive locally free action of a Lie group L, i.e. $V = L/\Gamma$ for a discrete group Γ and the polarization H is left-invariant. In both cases the resulting bundle S is trivial and so it receives an injective homomorphism from $T(W)$ whenever the bundle $T(W)$ is also trivial (i.e. W is parallelizable), e.g. if $W = \mathbb{R}^k$.

Corollary. *Every continuous map $\mathbb{R}^k \to U$ for the above U admits a C^0-fine (in particular uniform) approximation by horizontal immersions. This remains true for $U = V$ in the case of V being a Lie group with a left-invariant H admitting a single Ω-regular Ω-isotropic $S' \subset H$ of dimension $k+1$, and, more generally, for $V = L/\Gamma$.*

Remark. The flagrant deficiency of the above results is an appeal to $(k+1)$-dimensional integral (i.e. horizontal) manifolds in order to produce k-dimensional ones which unpleasantly reduces the range of dimensions where these results may be applied. There are some cases (e.g. contact manifolds see 3.4.3 in [$\mathrm{Gro}_{\mathrm{PDR}}$]) where one needs no extra dimension at all and, in general, one probably need much less than overregularity.

Conjecture. *If $n-(k+1) \geqslant (k+1)(n-n_1)$, then Ω-regular horizontal immersions satisfy the C^0-dense h-principle (where the above inequality expresses (under)determinancy of the horizontality condition for $(k+1)$-dimensional germs).*

Exercise. Extend the above theorem to m-horizontal immersions and then to immersions with a prescribed tangency to H_i, $i=1,\ldots,d$.

***h*-principle for horizontal curves.** It is known (see [$\mathrm{Ge}_{\mathrm{HPS}}$] and [Sar]) that *the space of smooth (possibly non-immersed) horizontal paths between two given points in (V,H) is weakly homotopic to the space of all continuous paths between these points.* This generalizes Chow connectivity theorem. Furthermore, (the orbits of) suitably chosen (sufficiently twisted) horizontal vector fields generate (and regularize) enough horizontal curves (compare 1.2.B) to ensure *the dense h-principle for immersed horizontal paths with given ends.* An appropriate form of such h-principle allows, for example, a fine C^0-approximation of a continuous map of 1-dimensional foliations S into V, say $f_0 : S \to V$, by smooth maps $f : S \to V$ which are *horizontal immersions on the leaves*, provided f_0 lifts to an *injective* homomorphism of the (1-dimensional) tangent bundle of the foliation to H (compare [$\mathrm{Hsu}_{\mathrm{GHP}}$]).

Warning. *The commutator generation condition for H is not sufficient for the (even higher order) regularity of immersed horizontal curves. In fact there are* rigid, *i.e. non-microflexible curves (along which the relevant operator is not infinitesimally invertible).*

Example. (borrowed from [Mont]) Let H on $\mathbb{R}^3$ be defined by the form $\eta = y^2dx+dz$ and c be the unit segment given by $\{x\in[0,1], y=0, z=0\}$. Then every C^1-small perturbation of c can be presented by (the graphs of) functions $y(x)$ and $z(x)$ where the derivative of z is *negative* as $z'(x) = -y^2(x) \leqslant 0$. Since every non-trivial perturbation of c with $z(0)=0$ has $y^2(x)>0$ for some $x\in[0,1]$, the second end has $z(1)<0$ and so no non-trivial deformation with fixed ends exists. (Yet one can deform $c = c_0$ by reparametrizing the x-interval with maps $\kappa_t : [0,1] \to [0,1]$ fixing the ends, starting from $\kappa_0 = \mathrm{Id}$ and terminating with a smooth map $\kappa_1(x)$ having many critical points $x_i \in [0,1]$. Then the resulting curve $c_1(x) = c_0(\kappa_1(x)) = (x = \kappa_1(x), y = 0, z = 0)$ can be "bent" at the critical points and C^∞-continuously moved keeping horizontality, away from the original position in $\mathbb{R}^3$. In fact, the proof of the above mentioned h-principle for *non-immersed* horizontal curves is based on reparametrizations creating critical points.) Such rigid curves may appear at the first sight impossible paradoxal monsters (only reluctantly, under the continuous pressure from Lucas Hsu and Richard Montgomery I accepted their generic appearance) but, in fact, they are ubiquitous in the nonholonomic geometry (see [$\mathrm{Mont_{SSC}}$]).

Remarks

(a) Generic horizontal curves for the above H meet the "dangerous" hypersurface $y = 0$ at finitely many points and (since η is contact away from H) are microflexible. Probably, *generic* horizontal curves in every H Lie generating $T(V)$ are microflexible[9].

(b) Construction of horizontal curves as orbits of suitable vector fields is in the spirit of *convex integration* (see [$\mathrm{Gro_{PDR}}$] and [$\mathrm{Gro_{CIDR}}$]) which suggests a similar approach to k-dimensional horizontal manifolds for $k \geqslant 2$ viewed as horizontal curves in an infinite dimensional space (see 4.2.B).

4.3.A. On the h-principle for morphisms of a given type. Suppose W is given a polarization $G \subset T(W)$ and we look for maps $f : W \to V$ with $n_{ij}(f) \geqslant n_{ij}$ for given n_{ij} where $n_{ij}(f)$ are defined as rank $G_j \cap \mathcal{D}^{-1}f(H_i)$. One can work out a suitable notion of regularity and prove the local h-principle as earlier. Then one takes $W' = W \times \mathbb{R}$ with $G' = G \times \mathbb{R}$ and call (a germ of) a map $f : W \to V$ *overregular* if it decomposes as $f = f' \circ \rho$ where $\rho : W \to W'$ is the graph of a smooth function $W \to \mathbb{R}$, and $f' : W' \to V$ is "regular" with $n_{ij}(f') \geqslant n'_{ij} = n_{ij} + 1$, where the "regularity" matches the n'_{ij}-inequalities such that the local h-principle and microflexibility hold for f'. Then the continuous sheaves deliver the global h-principle for overregular maps $f : W \to W$ satisfying $n_{ij}(f) \geqslant n_{ij}$. (We suggest the reader would look at this more specifically.)

[9] In fact, this is proven in [$\mathrm{Cor_1}$], also see [$\mathrm{Cor_2}$].

4.4. Folded integral submanifolds. For many purposes, e.g. for making horizontal cycles, one may use *piecewise* smooth integral submanifolds which are easier to come by than the smooth ones. In fact, the local h-principle and microflexibility suffice to produce certain *folded* (or branched) integral k-dimensional submanifolds without resorting to a use of $(k+1)$-dimensional horizontal germs. An example of such a "manifold" supporting a 1-dimensional cycle is exhibited on Fig. 12 below (compare Poenaru pleating lemma and Fig. 5 and 6 in 3.5).

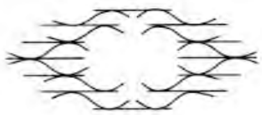


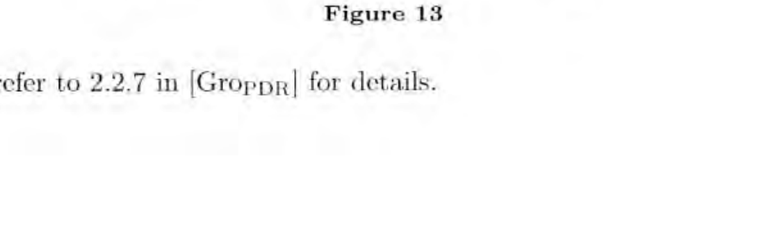
Figure 12

One may think of folded submanifolds W in V as finite or countable unions of actual smooth k-dimensional submanifolds, $W = \cup_i W_i$, where the intersection between W_i and W_j, whenever non-empty, is also a k-dimensional manifold with piecewise smooth boundary. Such a W is called integral (or horizontal) if such are the pieces W_i for all i. In fact one can define abstract branched (or folded) spaces W built of manifolds and speak of their smooth (horizontal) immersions into V. In this case microflexibility implies the h-principle which we apply to Ω-regular horizontal folded manifolds. Here is a specific result which follows from the folded h-principle (see (D) and (C) in 2.2.7 of [Gro$_{\mathrm{PDR}}$]).

We start with an arbitrary continuous map of a locally finite (at most) k-dimensional polyhedron into V, say $f_0 : W \to V$. We want to approximate f_0 by horizontal *branched* (or *folded*) immersions of another locally finite k-dimensional polyhedron, say $f' : W' \to V$, where f' is called a *branched* (or *folded*) C^∞*-immersion* if W' admits a locally finite covering by compact subpolyhedra, $W' = \cup_i W'_i$, such that f' is smooth on each simplex in W and homeomorphicly sends each W'_i onto a compact C^∞-smooth submanifold in V with boundary. It makes now perfect sense to attribute the properties of H-horizontality and/or Ω-regularity to f' as these apply to the images $f'(W'_i)$. Furthermore, we say that the maps f' in question *approximate* f_0 if for every neighbourhood $U \subset V \times W$ of the graph $\Gamma_{f_0} \subset V \times W$ of f_0 and every neighbourhood $U' \subset W \times W$ of the diagonal there exist proper homotopy equivalences $\varphi : W \to W'$, $\varphi' : W' \to W$ and one of our $f' : W' \to V$, such that the graph of $\varphi \circ \varphi'$ is contained in U' and the graph of $f \circ \varphi$ is contained in U.

Folded approximation theorem. *A map $f_0 : W \to V$ admits an approximation by branched (folded) H-horizontal Ω-regular immersions $f' : W' \to V$ if and only if there is a continuous map $w \mapsto S_w \subset H_v$ for $v = f(w)$ which assigns to each $w \in W$ an Ω-regular Ω-isotropic k-dimensional subspace S_w in $H_{f(w)}$.* (These S_w, $w \in W$ form a k-dimensional vector bundle over W which injects into H by an Ω-isotropic and regular homomorphism.)

Idea of the proof. The "only if" claim is obvious as every folded immersion which is Ω-regular and isotropic comes along with such "tangent bundle" S' over W' which then transports to W by $\varphi : W \to W'$. To prove "if" we first use the local h-principle and associate to each S_w a k-dimensional Ω-regular and Ω-isotropic germ in V at $f_0(w)$ tangent to S_w. We take representatives of such germs, called $W_i(S) \subset V$, at some points $v_i = f_0(w_i) \in V$, such that the points $w_i \in W$, $i = 1, 2, \ldots$ form a sufficiently dense discrete subset in W, and then we slightly perturbe $W_i(S)$ using microflexibility to new germs, say $W'_i \subset V$, which intersect according to certain pattern, so that the polyhedron W' *defined* as the abstract union $\cup_i W'_i$ with the identifications corresponding to our pattern is homotopy equivalent to W and then the tautological map $f' : W' \to \cup_i W'_i \hookrightarrow V$ satisfies our specifications. The passage from W_i to W'_i is similar to what we have already seen in Fig. 5, 6 and 12 and is reproduced below in Fig. 13 with W_0 being a circle embedded to the plane.



$\bigcup_{i=1}^{12} W_i(S)$ $\qquad$ $\bigcup_{i=1}^{12} W'_i$

Figure 13

We refer to 2.2.7 in [Gro$_{\mathrm{PDR}}$] for details.

Corollary. *Granted "if", the map $f_0 : W \to V$ admits a fine C^0-approximation by piecewise smooth and piecewise horizontal maps $f : W \to V$.*

Proof. Recall $\varphi : W \to W'$ and use $f' \circ \varphi$ for f. Notice that we do not speak here of the Ω-regularity (of f) as this is only needed to oil the microflexibility gear in the machinery of the continuous sheaves but not truly relevant for our applications where we are concerned with the horizontality alone. In fact the notion of regularity can be relaxed in many cases and, probably, brought to a point where it could be (genericly) used for $n - k > k(n - n_1)$ instead of $n_1 - k \geqslant k(n - n_1)$ associated to the Ω-regularity (compare 4.2.A).

The above approximation theorem can be used (in the same way as the one in the previous section) in the presence of a single Ω-regular Ω-horizontal k-dimensional subspace $S \subset H_v$. Namely, *every continuous map of W into a small neighbourhood $U \subset V$ of v admits an approximation by folded horizontal maps $W' \to U$ and this is valid for $U = V$ in the Lie group theoretic case* (compare Corollary in the previous section).

4.4.A. The proof of Thom's theorem (A) in the jet bundle. Let $V = V^r = \mathrm{Gr}_k^r(V_0)$ be the space of r-jets of germs of k-manifolds in V_0 (compare 4.1.B$'$) and $H = H^r$ be the canonical polarization on this V. If we declare *regular* the horizontal submanifolds in V which *regularly* project (i.e. immerse) to V_0 and hence, equal the r-jets of k-dimensional submanifolds in V_0, we arrive at the situation where the folded h-principle may be applied since the microflexibility and the local h-principle are (trivially) valid in this case. It is also obvious that the "if" of the folded approximation theorem is satisfied and so *every $f_0 : W \to V$ can be approximated by folded regular horizontal immersions $f : W' \to V$*. Thus every ℓ-cycle for $\ell \leqslant k$ can be made folded horizontal in every open $\mathcal{R} \subset V$ and (A) of Thom's theorem (see 4.1.B$''$) follows.

4.4.B. Extension of folded and subfolded maps. A map $f : W \to V$ is called *subfolded of rank k* if it is the composition of a folded map $f' : W' \to V$, $\dim W' = k$, and a piecewise linear $\varphi : W \to W'$ (as in the above corollary but now we do not assume $\dim W \leqslant k$). If f' is horizontal and regular, these properties are attributed to f and also the map φ brings the "tangent bundle" of W' to the "if" bundle S over W. We want to decide when f extends to a regular horizontal subfolded map of rank k of a larger polyhedron $W_1 \supset W$.

Extension theorem. *Suppose (compare "if") the bundle S extends to S_1 on W_1 and the tautological homomorphism ("differential" of f) $S \to T(V)$ extends to a fiberwise injective regular horizontal bundle homomorphism of S_1 to $T(V)$. Then f extends to a regular horizontal subfolded map of rank k of W_1 to V where, moreover, this map can be chosen arbitrarily close to the continuous map $W_1 \to V$ underlying the homomorphism $S_1 \to T(V)$.*

The proof is similar to that of the folded approximation theorem.

On folded maps f satisfying $n_{ij}(f) \geqslant n_{ij}$. (Compare 4.3.A.) If W comes with $G \subset T(W)$ and maps $f : W \to V$ take G into account, then the notion of folding becomes problematic unless (W, G) has a sufficient symmetry. A happy case is where $(W, G) = (W_0 \times \mathbb{R}, G_0 \times \mathbb{R})$, as this G is invariant under the diffeomorphisms preserving $\mathbb{R}$-lines in W and one may fold along the graphs $\Gamma \subset W$ of functions $W_0 \to \mathbb{R}$. Then one has the full folded map package (with a suitable notion of regularity) by 2.2.2 and 2.2.7 in [$\mathrm{Gro}_{\mathrm{PDR}}$].

4.4.C. Deformations of regular horizontal folded maps. An Ω-*regular* horizontal folded map $f' : W' \to V$ admits "many" horizontal deformations as if they were not restricted by the horizontality condition. This immediately follows from the generalized Nash implicit function theorem in [$\mathrm{Gro}_{\mathrm{PDR}}$] and the specific feature of the ("many") deformations we need is expressed in the following.

Proposition. *There exists a continuous map $F' : W' \times \mathbb{R}^q \to V$ for some q, (e.g. for $q = 2n$) such that*

(i) $F' \mid W' = W' \times 0 = f'$,

(ii) the restriction of F' to each "branch" W'_i of W' is smooth, i.e. $F' \mid W'_i \times \mathbb{R}^q$ is smooth for the smooth structure on W'_i induced by the embedding $f' \mid W'_i : W'_i \to V$,

(iii) each map $F' \mid W'_i \times \mathbb{R}^q$ is a (smooth) submersion $W'_i \times \mathbb{R}^q \to V$.

4.5. Lower bounds on the Hausdorff dimension of subsets in (V, H). The folded (and subfolded) horizontal manifolds W mapped into V (whenever they exist) provide us with k-dimensional subsets in V of Hausdorff dimension k. Similarly, m-horizontal submanifolds W (i.e. having rank $H \cap T(W) \geqslant m$) satisfy a certain non-vacuous bound on the Hausdorff dimension. For example, if $H_2 = T(V)$, i.e. the commutators of the horizontal fields span $T(V)$, then these W have $\dim_{\mathrm{Hau}} \leqslant m + 2(k - m)$, and, in general, if $H_d = T(V)$ the inequality becomes $\dim_{\mathrm{Hau}} \leqslant m + d(k - m)$. What is more interesting, however, is a non-trivial *lower* bound on $\dim_{\mathrm{Hau}} V'$ for subsets $V' \subset V$ of a given topological dimension which follows from the global h-principle for folded horizontal (and partially horizontal) submanifolds in V of the complementary dimension (compare 2.3).

Theorem. *If each horizontal space $H_v \subset T_v(V)$, $v \in V$, contains an Ω-regular and Ω-isotropic subspace $S_v \subset H_v$ of dimension k, then every closed $(n-k)$-dimensional subset $V' \subset V$ has $\dim_{\mathrm{Hau}} V' \geqslant \dim_{\mathrm{Hau}} V - k$, where we assume V is equiregular. More generally, if we have m-horizontal and partially Ω-isotropic and Ω-regular $S_v \subset T_v(V)$ at all $v \in V$, then $\dim_{\mathrm{Hau}} V' \geqslant \dim_{\mathrm{Hau}} V - m - d(k - m)$.*

Proof. We use the following characterization of closed subsets V' in V of topological dimension $n - k$.

Alexandroff theorem. ([10]) *There exist $(k-1)$-dimensional cycles of arbitrary small diameter in the complement $V - V'$ which bound no chains of small diameter in $V - V'$.*

It follows, there exists a k-dimensional polyhedron $W \subset V$ which *stably intersects* V', i.e. every continuous map $f' : W' \to V$ *approximating* the inclusion $f_0 : W \hookrightarrow V$ in the sense of the definition in 4.4 has *non-empty* pull-back $(f')^{-1}(V')$.

Next, we observe that V can be localized in the present situation to a small neighbourhood $U \subset V$ of a single point $v' \in V' \subset V$ and so the corollary to the folded approximation theorem (see 4.4) provides a branched (folded) H-horizontal (and Ω-regular) immersion $f' : W' \to V$ which stably intersects V'. This f' gives rise to a "large" family of

[10] See VIII.5 in [Nag] and references therein.

such maps, i.e. $F' : W' \times \mathbb{R}^q \to V$ of 4.4.C where each $F' \mid W'_i \times \mathbb{R}^q$ is a submersion. The stable intersection property of f' implies that for the restriction of f' to some of the branches W'_i of W say to W'_0 and so the projection of the pull-back $V'_0 \subset W'_0 \times \mathbb{R}^q$ of the submersion map $F' \mid W'_0 \times \mathbb{R}^q : W'_0 \times \mathbb{R}^q \to V$ to $\mathbb{R}^q$ has non-empty interior. Then, by an obvious argument, there is a small ball $B_0 \subset W'_0$ and some $\mathbb{R}^k \subset \mathbb{R}^q$, such that F' is a submersion on $B_0 \times \mathbb{R}^k$ and such that the projection of $V''_0 = V'_0 \cap B_0 \times \mathbb{R}^k$ still has non-empty interior. Now $\dim B_0 \times \mathbb{R}^k = n = \dim V$, so our submersion $B_0 \times \mathbb{R}^k \to V$ is an equidimensional immersion and we may assume $V = B_0 \times \mathbb{R}^k$, where $V' = V''_0$ has non-empty interior when projected to $\mathbb{R}^k$ by $\psi : V = B_0 \times \mathbb{R}^k \to \mathbb{R}^k$. Now we argue as in 2.1 and 3.1.A (where the reader should beware of slight discrepancy between the present notations and those in 3.1.A) by using the ball-box theorem and horizontality of the fibers $\psi^{-1}(y)$, $y \in \mathbb{R}^k$ which gives us the desired bound $\dim_{\mathrm{Hau}} V' \geqslant \dim_{\mathrm{Hau}} V - k$, since small ε-balls in V (used in determination of $\dim_{\mathrm{Hau}} V'$) project to approximate boxes in $\mathbb{R}^k$ with the sides

$$\underbrace{\varepsilon \times \cdots \times \varepsilon}_{n_1 - k} \times \underbrace{\varepsilon^2 \times \cdots \times \varepsilon^2}_{n_2 - n_1} \times \cdots \times \underbrace{\varepsilon^d \times \cdots \times \varepsilon^d}_{n_d - n_{d-1}}$$

and thus with volume $\approx \varepsilon^{N-k}$ for $N = \Sigma_{i=1}^d \, i(n_i - n_{i-1}) = \dim_{\mathrm{Hau}} V$. The details of this, as well as the generalization to the m-horizontal case is left to the reader. We also suggest to the reader to try the further generalization to *M-horizontality* for $M = (m_1, \ldots, m_d)$ which refers to submanifolds $W \subset V$ with rank $(T(W) \cap H_i) \geqslant m_i$.

Remarks

(a) It is hard to believe that a Nash type implicit function theorem is indispensable for the lower bound on $\dim_{\mathrm{Hau}} V'$.

(b) It would be interesting to use the above projection argument to bound the Euclidean Hausdorff dimension of V' in terms of the C-C one. One knows in this regard (this was explained to me by J. Bourgain) that the normal projection of a subset $V' \subset \mathbb{R}^n$ with $\dim_{\mathrm{Hau}} V' \geqslant n - k$ onto a *generic* $\mathbb{R}^{n-k} \subset \mathbb{R}^n$ has $\mathrm{mes}_{n-k} > 0$ (see [Fal]) and one wants to extend this result to (generic members of) more general families of smooth maps $\mathbb{R}^n \to \mathbb{R}^{n-k}$. (Actually, even in the topological category one may ask such a question. For example, let $V' \subset \mathbb{C}^n$ have topological dimension $\geqslant 2(n-k)$. Does there exist a *complex* linear map $\mathbb{C}^n \to \mathbb{C}^{n-k}$ so that the image of V' has non-empty interior?) Notice that the argument in [Fal] uses a potential theoretic definition of $\dim_{\mathrm{Hau}}$ in $\mathbb{R}^n$ which seems to nicely fit the C-C framework.

4.6. Horizontal triangulations and a bound on the width of subsets in nilpotent Lie groups. Let V be a simply connected two step nilpotent group with a left-invariant polarization $H \subset T(V)$ of rank n_1 complementary to the center $V_1 \subset V$ (of dimension $n - n_1$). Recall that such a group is determined up to an isomorphism by the 2-cocycle on $\mathbb{R}^{n_1} = V/V_1$ with values in $\mathbb{R}^{n-n_1} = V_1$ as such cocycles govern $\mathbb{R}^{n-n_1}$-central extensions over $\mathbb{R}^{n_1}$. This cocycle is nothing else but our curvature form $\Omega = \Omega_H : \Lambda^2 H \to H^{\perp} = T(V)/H$ restricted to the $T_{\mathrm{id}}(V)$ and, in fact, every Ω on $\mathbb{R}^{n_1}$ correspond to some Lie group.

Theorem. *If Ω admits an Ω-regular and Ω-isotropic subspace in $\mathbb{R}^{n_1} = V/V_1 = H_{\mathrm{id}}$ of dimension k then every closed subset $V' \subset V$ satisfies*

$$\text{C-C } \mathit{wid}_{n-k-1}\, V' \leqslant \mathrm{const}_V (\text{C-C } \mathit{mes}_{N-k}\, V')^{\frac{1}{N-k}}, \qquad (*)$$

where the width (as defined in 3.4) and the Hausdorff measure refer to the C-C metric associated to H and some left-invariant Riemannian metric g in V, where $n = \dim_{\mathrm{top}} V$, $N = n_1 + 2(n - n_1) = \dim_{\mathrm{Hau}} V$ and "const_V" signifies "$\mathrm{const}_{H,g}$". Furthermore, the Riemannian width and the Hausdorff measure are related by the inequality

$$\mathit{Riem\text{-}wid}_{n-k-1}\, V' \leqslant \mathrm{const}_g (\mathit{Riem\text{-}mes}_{n-k}\, V')^{\frac{1}{N-k}}. \qquad (**)$$

Proof. First we observe that $(**)$ follows from $(*)$ and notice that $(**)$ is interesting only for *large* mes_{n-k} as for mes_{n-k} small this follows from the purely Riemannian inequality $\text{Riem-wid}_{n-k-1}\, V' \leqslant \mathrm{const}_V (\text{Riem-mes}_{n-k}\, V')^{\frac{1}{n-k}}$ valid for all Riemannian manifolds V of locally bounded geometry, provided $\mathrm{mes}_{n-k} V' \leqslant \varepsilon$ for some positive $\varepsilon = \varepsilon_V$.

Now, we turn to the proof of $(*)$ and observe, with an obvious generalization of the intersection inequality 3.4.B$'$ to the present context, that $(*)$ would follow from the existence of a suitable "horizontal" triangulation of V (compare 3.4.B). Notice that we do not truly need the Γ-invariance of such triangulation as was required in 3.4.B but we want all simplices of a given dimension to be roughly of unit size (as spelled out below). Then we can make the simplices of any size we want, since V being a *two*-step group, admits a non-trivial self-similarity.

Horizontal triangulation lemma. *There exists a triangulation* Tr *of* V, *such that*

(i) every i-simplex of *Tr* is the image of the standard simplex Δ^i under a piecewise smooth map $\Delta^i \to V$ with the Lipschitz constant (with respect to the metric g in V) bounded by a constant $C = C_V$,

(ii) for every ball in V the number of simplices in *Tr* meeting this ball is bounded by a constant depending on the radius (but not on the position) of the ball,

(iii) every k-simplex in *Tr* is (piecewise) H-horizontal.

Proof. First, notice that (i) and (ii) express the idea of "unit size" indicated above while the guts of the lemma are in (iii) where we shall use our basic assumption of the existence of Ω-regular and Ω-isotropic subspace of dimension k. The presence of such subspace, allows one an approximation of the k-skeleton Tr^k of any triangulation Tr_0 satisfying (i) and (ii) by a subfolded regular horizontal map $\varphi_k : \mathrm{Tr}_0^k \to V$. If $2k \leqslant n-2$, this map can be made an (horizontal!) embedding by a small perturbation provided by the generalized Nash theorem (compare (E) in 2.3.2 of [Gro$_{\mathrm{PDR}}$]) and the image $\varphi_k(\mathrm{Tr}^k) \subset V$ can be easily extended to the desired triangulation Tr with $\varphi_k(\mathrm{Tr}_0^k)$ serving as the k-skeleton of Tr.

Finally, if $k \geqslant 2n-2$, one could make φ_k an embedding extendable to Tr by sharpening the folded approximation theorem but this is not necessary as the only cases where $2k > n-2$ are those of the Heisenberg groups of dimensions 3 and 5 (and k equal 1 and 2 respectively) where our contact discussion in 3.4 applies anyway. (We suggest the reader would check the details of this proof.)

Exercise. Generalize the above theorem to d-step nilpotent groups for all $d \geqslant 2$.

4.7. Lipschitz maps into C-C spaces. We shall extend here the results from 3.5 to general equiregular C-C manifolds (V, H), where we assume that H admits an Ω-regular Ω-isotropic subbundle $S \subset H$ of rank k (e.g. being a nilpotent group of the previous section). Our Lipschitz maps $W \to V$ for $\dim W \leqslant k$ will appear as uniform limits of (Ω-regular) horizontal *subfolded* maps of rank k replacing piecewise horizontal maps of 3.5. As we mentioned earlier, even the *local* extension problem for general (piecewise) horizontal maps into (V,H) appears quite difficult and solved only in the contact case. On the other hand, the extension is easy in the category of subfolded maps of a fixed rank k where all folded maps into V in question have their "tangent" bundles induced from a fixed S on V, as the "if" conditions of the folded approximation (see 4.4) and extension (see 4.4.B) theorem is automatic with such an S. It should be noticed, however, that the contact case is not covered by the present discussion as we did not assume there the existence of S which, in fact, does not necessarily exist even if there is an Ω-regular isotropic $S_v \subset H_v \subset T_v(V)$ for every $v \in V$. (Probably, our present Lipschitz results generalize to the case of a discontinuous field of Ω-regular isotropic subspaces $v \mapsto S_v \subset H_v$.)

The notion of complexity for piecewise smooth maps in 3.5.B obviously extends to the subfolded category and everything remains intact. When this notion applies to the construction and extension of Lipschitz maps one crucially uses rescaling of V and so if (V, H) is a nilpotent group with a self-similarity (or is locally isomorphic to such a group) everything from 3.5.B immediately generalizes to such (V, H). In the general case, as one needs maps of bounded complexity produced on an arbitrary small scale, one needs the following additional argument using the nilpotent Lie group N_v approximating (V, H) at a given point $v \in V$ (see 1.4). The Lie algebra L_v of N_v is, in the present case, a two step nilpotent algebra, $L_v = H_v \oplus H_v^\perp$, where $H_v^\perp = T_v(V)/H_v$, such that $H_v^\perp$ lies in the center of L_v and the commutator rule $H_v \otimes H_v \to H_v^\perp$ is given by the form Ω_v. It follows that N_v has Ω_v-regular isotropic subspaces at all points and so we do have folded maps of bounded complexity in N_v at each scale as N_v comes with a self-similarity. But since N_v approximates (V, v) at the small scale these maps can be moved by small perturbation to corresponding maps into V. (It is useful to think of N_v as a deformation of V. Namely, there exists a smooth family of polarizations H_ε on V at v for $\varepsilon \in [0, \rho]$, $\rho > 0$, such that $H_{\varepsilon>0}$ on the ρ-ball around v is isomorphic to H on the ε-ball and H_0 equals the implied polarization on N_v, where N_v is locally diffeomorphically mapped to V by $E_v \circ E_0^{-1}$ (see 1.4). Then we have folded ε-families of maps $f_\varepsilon : W \to (V, H_\varepsilon)$, $\varepsilon \in [0, \rho]$, which can be treated as individual maps and which specialize for every ε to the required maps of bounded complexity on the ε-scale.)

Now, everything is ready for the proof of

Lipschitz extension theorem. (Compare 3.5.D.) *Let W be a compact simplicial polyhedron of dimension $\leqslant k$, and $W_0 \subset W$ be a subpolyhedron. Then a Lipschitz map $f_0 : W_0 \to (V, H)$ Lipschitz extends to W if and only if it extends continuously.*

Exercise. State and prove all results from 3.5 in the present situation.

4.7.A. Construction and extension of Hölder maps. If we apply our (folded H-horizontal) extension procedure to a C^α-Hölder map $f_0 : W_0 \to V$ then the resulting extension $f : W \to V$ is also C^α-Hölder. This works for all $\alpha > 0$ if $\dim W \leqslant \operatorname{rank} S$ but for smaller α we can do better using suitable S_i in the i-th commutator bundle $H_i \supset H = H_1$ for $i \geqslant 2$. For example, every H_2-horizontal map is $C^{\frac{1}{2}}$-Hölder and if H_2 admits a subbundle $S_2 \subset H_2$ which is regular isotropic for the curvature form $\Omega_2 : \Lambda^2 H_2 \to H_2^\perp$, then $C^{\frac{1}{2}}$-Hölder maps $W \to V$ becomes available for $\dim W \leqslant k_2 = \operatorname{rank} S_2$ which is an improvement if $k_2 > k = k_1$. Similarly, an S_i of rank k_i gives us $C^{\frac{1}{i}}$-Hölder maps $W \to V$ for $\dim W \leqslant k_i$. Finally, one may wonder what is the role of m-horizontal (and M-horizontal for $M = (m_1, \ldots, m_d)$) maps into (V, H) and some aspect of the problem will be adressed in the next section.

4.8. Dehn isoperimetry in nilpotent Lie groups. (Compare 5.A in [Gro$_{\mathrm{AI}}$].) Let V be a simply connected nilpotent Lie group and $H \subset T(V)$ be a left-invariant polarization corresponding, on the Lie algebra level, to a subspace $H(\mathrm{id}) \subset L$ complementary to $[L, L] \subset L$ (and the curvature form Ω on $H(\mathrm{id}) \subset T_{\mathrm{id}}(V)$ can be identified with the form $\Omega_0 : \Lambda^2(L/[L, L]) \to [L, L]/[L, [L, L]])$ corresponding to the Lie bracketing in the Lie algebra $L = L(V)$. We fix some left-invariant Riemannian metric g on V and we observe that the Lipschitz extension theorem of the previous section specializes to the following.

Quadratic isoperimetric inequality. (C-C *version.*) *Let Ω admits a two dimensional Ω-regular and Ω-isotropic linear subspace in* $H(\mathrm{id})$. *Then every closed curve in V of finite* C-C *length bounds a disk of* C-C *area (i.e.* C-C *Hausdorff measure* mes_2*) satisfying*

$$\mathit{area}(\mathit{disk}) \leqslant \mathrm{const}_V(\mathit{length}(\mathit{curve}))^2. \qquad (*)$$

In fact there exists a C-C Lipschitz map of the unit disk D to V such that the boundary circle of D parametrizes our curve in V and the implied Lipschitz constant of the map satisfies

$$Lip \leqslant \mathrm{const}'_V \ \ length(curve).$$

Now we recall that the Riemannian geometry of (V, g) on the large scale is equivalent to the Carnot-Carathéodory one and conclude to the following.

Riemannian isoperimetric inequality. *Every closed curve in V of finite g-length bounds a disk of finite g-area such that*

$$g\text{-area}(disk) \leqslant \mathrm{const}_g (g\text{-}length(curve))^2. \qquad (**)$$

Furthermore, our curve can be parametrized by the boundary of a g-Lipschitz map $D \to V$ with

$$Lip \leqslant \mathrm{const}'_g \ \ length.$$

Discrete corollary. *Every discrete nilpotent group Γ for which the corresponding Lie algebra admits an Ω-regular isotropic plane* (in $L/[L, L]$) *satisfies the quadratic isoperimetric inequality.*

Notice that the existence of such a plane is a generic phenomenon for *two* step nilpotent groups with $3n_1 \geqslant 2n + 2$, where n_1 is the rank of the group and n_1 the rank of its abelianization. But d-step groups for $d \geqslant 3$ admit no Ω-regular planes at all and one is faced with a more difficult problem of finding (high order) regular jets in the sense of 4.2. This problem remains open and one does not know if there are any 3-step nilpotent groups with quadratic isoperimetry.

4.8.A. Hölder maps $D \to V$ and isoperimetric inequalities of degree $2i$. We note that H_i-horizontal maps to V are $C^{\frac{1}{i}}$-Hölder (where $H_i \subset T(V)$ is the i-th commutator bundle) and have finite C-C Hausdorff measure mes_{2i} for dist_H in (V, H). We denote by Ω_i he curvature form on the bundle H_i defined by the Lie bracketing as earlier,

$$\Omega_i : \Lambda^2 H_i \to H_i^{\perp} = T(V)/H_i$$

and observe the following.

Theorem. *If H_i contains an Ω_i-regular and Ω_i-isotropic plane then V (and Γ) satisfies the isoperimetric inequality of degree $2i$,*

$$\text{area}_g(\text{filling disk}) \leqslant \text{const}_g(\text{length}_g(\text{curve}))^{2i}.$$

Proof. Lipschitz curves in (V, H) extends to $C^{\frac{1}{i}}$-Hölder disks by the discussion in 4.7.A.

4.8.B. Regular (i, j)-surfaces in V and isoperimetric inequalities of degree $i + j$. A smooth (or folded) map f of a disk D to V is called (i, j), where $j \geqslant i$, if it is H_j-horizontal and also has non-trivial tangency to H_i at all points in D. In other words,

$$\operatorname{rank}(\mathcal{D}f)^{-1}H_j \geqslant 2 \quad \text{and} \quad \operatorname{rank}(\mathcal{D}f)^{-1}H_i \geqslant 1$$

everywhere on D. Images of such maps have C-C $\text{mes}_{i+j} < \infty$ and if these are sufficiently abundant they may be used to fill in closed horizontal curves in V. In order to formulate a sufficient condition for such abundance we use the abstract language of 4.2.B and denote by $\mathcal{R}^{\infty}_{i,j}$ the stabilized differential condition expressing the (i, j)-property of smooth maps $f : D \to V$.

$(i + j)$-isoperimetric inequality. ([11]) *If $\mathcal{R}^{\infty}_{i,j}$ contains a regular jet (in the sense of 4.2.B) then (V, g) satisfies the isoperimetric inequality of degree $i + j$.*

Proof. We proceed as earlier with folded (i, j)-surfaces in (V, H) constructed with the scaling pattern expressed by Fig. 8 and Fig. 9 in 3.5.D. It is easy to see that the C-C Hausdorff measure mes_{i+j} of such a surface filling in our closed curve is finite and, moreover

$$\text{mes}_{i+j} \leqslant \text{const}_V(\text{length})^{i+j}.$$

This C-C isoperimetric $(i + j)$-inequality yields the Riemannian one as earlier and the proof is concluded.

In order to use the $(i + j)$-inequality one needs specific criteria for (the existence of) regular jets in $\mathcal{R}^{\infty}_{i,j}$ some of which are indicated in 5.A_2-A_5 of [Gro_{AI}]. Here is another such criterion in the spirit

---

[11] Compare 5.A_4''' in [Gro_{AI}].

of Ω-regularity. Let $\eta_1, \dots, \eta_{n-n_j}$, be 1-forms on V defining $H_j \subset T(V)$ and $\eta_1, \dots, \eta_{n-n_j}, \dots, \eta_{n-n_i}$ be the form defining $H_i \subset H_j$. Let $\omega_1, \dots, \omega_{n-n_j}, \dots, \omega_{n-n_i}$ be the differentials of these forms and X_1, X_2 two independent tangent vectors at id $\in V$, where $X_1 \in H_i$ and $X_2 \in H_j$. We associate to X_1 and X_2 the following equations

$$\left.\begin{aligned} \omega_\nu(\partial, X_1) &= \sigma_{\nu,1}\ ,\ \nu = 1, \dots, n - n_i, \\ \omega_\nu(\partial, X_2) &= \sigma_{\nu,2}\ ,\ \nu = 1, \dots, n - n_j, \end{aligned}\right\} \tag{$*$}$$

and call the plane spanned by X_1 and X_2 *regular* if this system in non-singular, for the unknown vector ∂ from $(H_i)_{\mathrm{id}}$. In other words the homomorphism $\Omega^0_{i,j} : H_i \to H_i^\perp \oplus H_j^\perp$ defined at id $\in V$ by

$$\partial \mapsto \{\omega_1(\partial, X_1), \dots, \omega_{n-n_i}(\partial, X_1), \omega_1(\partial, X_2), \dots, \omega_{n-n_j}(\partial, X_2)\}$$

is surjective. The above (X_1, X_2)-plane is called *isotropic* if X_1 and X_2 extend to *commuting* fields, the first in H_i and the second in H_j, which is equivalent to $\omega_\nu(X_1, X_2) = 0$, $\nu = 1, \dots, n - n_j$. Then one easily sees as earlier that an isotropic regular plane gives rise to a regular jet in $\mathcal{R}^\infty_{i,j}$, since the linearization of the form inducing system (imposed on a map $f : D \to V$ by $f^*(\eta_\nu) = \theta_\nu$) reduces to $(*)$.

It is worthwhile to work out specific examples where this regularity takes place and to find other (practical) criteria for the existence of regular jets.

Remark. Regularity of an H-horizontal map $f : D \to V$ is essentially equivalent to algebraic solvability of the system $L_f\ \partial = \sigma$ where L_f is the linearization of the differential operator D which assigns to each $f : D \to V$ the $(n - n_1)$-tuple of forms on D by $D(f) = \{f^*(\eta_i)\}$, where η_i, $i = 1, \dots, n - n_1$ are some forms on V defining H. In fact one needs algebraic solvability of $L_f\ \partial = \sigma$ not only for a fixed f but for all nearby f's which do not have to be horizontal. It may happen that $L_f\ \partial = \sigma$ is algebraically solvable for a horizontal f, but not (at least not apparently so) for nearby f's. For example let V be a 3-step group, so that $H_3 = T(V)$ and let L° be the associated graded Lie algebra, (isomorphic to the tangent Lie algebra to (V, H) at id $\in V$, see 1.4), written as $L^\circ = L_1 \oplus L_2 \oplus L_3$, so that L_1 corresponds to H_1, and $L_1 \oplus L_2$ to H_2. Take two independent vectors X_1 and X_2 in L_1 and define a linear map $\Omega^0 : L_1 \oplus L_2 \to (L_2 \oplus L_3) \oplus (L_2 \oplus L_3) = \mathrm{Hom}(\mathbb{R}^2, L_2 \oplus L_3)$ by

$\partial \mapsto ([\partial, X_1], [\partial, X_2])$. It is easy to see that commuting vectors X_1 and X_2 (should) correspond to horizontal surfaces (as the tangent frame on such surface can be made of commuting fields) and that the surjectivity of Ω^0 is sufficient for algebraic solvability of $L_f\, \partial = \sigma$ for a horizontal f. Yet it is unclear if the existence of commuting X_1 and X_2 with surjective Ω^0 yields (the local h-principle and microflexibility for) horizontal surfaces and/or the quadratic isoperimetric inequality in V. I also must admit I have not worked out any example where such X_1 and X_2 are actually present.

4.8.C. On filling in dimension $\geqslant$ 3. The general question is as follows. Let S be a $(k-1)$-dimensional cycle in (V, H) of finite Hausdorff measure mes_ℓ for some $\ell \geqslant k-1$. When can it be filled in by a k-dimensional chain D with finite Hausdorff measure mes_m for a given $m \geqslant \max(\ell, k)$? Furthermore, we want some specific bounds on mes_m in terms of mes_ℓ, e.g. an inequality

$$\mathrm{mes}_m D \leqslant \mathrm{const}(\mathrm{mes}_\ell S)^\alpha. \qquad (?)$$

More geometrically, we may start with a piecewise smooth *horizontal* cycle S and look for a *horizontal* filling D of a controlled k-dimensional volume. This could give us the above (?) for $\ell = k$, $m = k+1$ and $\alpha = k + 1/k = m/\ell$. Similarly, one may start with a *partially horizontal* cycle having prescribed ranks of the intersections of $T(S)$ with H_i and try to fill it in by a partially horizontal chain D (possibly, with weaker tangencies to H_i than S) with controlled k-volume. Of course, in order to do that, we need "many" horizontal (or partially horizontal) k-dimensional submanifolds in V. But even when such manifolds are abundant, e.g. in contact manifolds V of dimension $n \geqslant 2k+1$, we do not know how to fill in horizontal $(k-1)$-dimensional cycles by horizontal k-chains with controlled k-volume. Moreover, our telescoping filling used for $k = 2$ (see Fig. 9 in 3.5.D) may work for $k = 3$ and S being a surface with genus $(S) \leqslant$ const, e.g. for S homeomorphic to S^2. Here is the idea. First, the general problem can be reduced to the case where $S \subset V$ is *sufficiently regular* in the following sense: every domain $S_0 \subset S$ with area $S_0 \leqslant \frac{1}{2}$ area S has length $\partial S_0 \geqslant \mathrm{const}_V(\text{area } S_0)^{\frac{1}{2}}$. In particular, the metric balls in S of radii $(, \ldots, (\ S)^{\frac{1}{2}}$ have at least quadratic area growth and, consequently, $\mathrm{Diam}\, S \leqslant \mathrm{const}(\text{area } S)^{\frac{1}{2}}$. This is done with the usual cut-and-paste (inductive) regularization techniques which are presented in a sufficient generality in [$\mathrm{Gro}_{\mathrm{FRM}}$] (compare 2.5 of the present paper). These techniques work in all dimension (and reduce the problem to S with

$\mathrm{vol}_{k-2}\,\partial S_0 \geqslant \mathrm{const}_V\ (\mathrm{vol}_k\, S_0)^{\frac{k-2}{k-1}}$ for all $S_0 \subset S$ with $\mathrm{vol}\, S_0 \leqslant \frac{1}{2}\mathrm{vol}\, S)$ and if $k-1 = \dim S = 2$ the cut-and-paste procedure does not increase the genus of S and we assume below S is homeomorphic to S^2. Now we want to fill such H-horizontal S in by an H-horizontal 3-ball (or something topologically similar) of bounded volume. Notice that it is relatively easy to construct a Hölder map of the ball into V extending S on the boundary for a suitable *conformal* parametrization $f : S^2 \to S$ since the regularity of S allows a *Hölder* parametrization of S with the implied Hölder exponent depending on const_V in the isoperimetric inequality for length ∂S_0 (where "Hölder" would turn into Lipschitz if that inequality was sharp with const_V asymptotic to $2\sqrt{\pi}$ for domains $S_0 \subset S$ with area $S_0 \to 0$, compare 2.5.H$'$). But Hölder (unlike Lipschitz) is not sufficiently good for us and we shall indicate below a possible construction of the filling using intrinsic geometry of S. We start by observing that our disk filling S^1 exhibited in Fig. 9 of 3.5.D arises from a sequence of diadic partitions of S^1, where S^1 is divided first into four segments of equal length and then we keep dividing in two as usual. Now we want to produce a similar sequence of partitions of the sphere S (with the intrinsic metric induced from V) and the key property of the partitions we need is expressed in the following.

Tentative proposition. *Let (V, H) be a nilpotent Lie group admitting a regular jet of a 3-dimensional horizontal submanifold (e.g. an Ω-regular Ω-isotropic 3-plane in H) and $S \subset V$ be a (piecewise) smooth horizontal surface. Suppose S admits a sequence of finite partitions denoted P_i with the parts $S_{i,j} \subset S$ which intersect (only) across their boundaries and where P_{i+1} refines P_i for all $i = 1, \ldots$. Assume that*

(i) $\mathrm{diam}\, S_{i,j} \leqslant \varepsilon_i \underset{i\to\infty}{\longrightarrow} 0$,

(ii) *every $S_{i,j}$ is divided into at most q parts of P_{i+1} for a fixed q independent of i. Furthermore, this division (partition) of each $S_{i,j}$ into parts of P_{i+1} can be refined to a cell division of S_{ij} with the total number of cells bounded by some constant q',*

(iii) *the totality of the diameters of $S_{i,j}$ is bounded by*

$$\sum_{i=1}^{\infty}\sum_{j}(\mathrm{diam}\, S_{i,j})^3 \leqslant \mathrm{const}(\mathrm{area}\ S)^{\frac{3}{2}}. \tag{$+$}$$

Then S can be filled in by a horizontal ball D in (V, H) with

$$\mathrm{Vol}_3\, D \leqslant \mathrm{const}'(\mathrm{area}\ S)^{\frac{3}{2}} \tag{$*+$}$$

for $\mathrm{const}' = \mathrm{const}'(V, q')$.

The proof is easy, guided by Fig. 9 in 3.5.D.

Now we need good partitions of S where we are allowed to assume S is sufficiently regular in the above sense. Then one obtains sensible partitions of S, by using, for example, maximal δ-separated nets in S and dividing S into the corresponding (nearest point) Dirichlet domains. The trouble is that the local complexity of such partitions may be unbounded as a small δ-ball in V intersects too many other (disjoint) δ-balls. On the other hand, as the genus of S is assumed bounded (e.g. $S \approx S^2$), this concentration of balls is not significant on the average due to the Gauss-Bonnet formula, and with a suitably generalized averaged version of (ii) one expects some filling D of S exists (where one should not insist on D being the topological ball anymore).

Remark. The above tentative proposition admits a variety of generalizations and refinements. Its role consists in reducting the general filling problem to two subproblems.

(1) Filling (horizontal or partially horizontal) cycles of *bounded complexity* in V.

(2) Abstract combinatorial filling of S (constructed with partitions of S).

Here S may be a general $(k-1)$-cycle with a metric, e.g. a closed (oriented) Riemannian manifold of dimension $k-1$ and an "abstract combinatorial filling" refers to a k-dimensional metric space D isometrically containing S as a closed nowhere dense subset, such that S is homologous to zero in D and such that $D - S$ admits an infinite triangulation into simplices Δ_i, $i = 1, \ldots,$ such that

$$\sum_{i=1}^{\infty} (\operatorname{diam} \Delta_i)^k \leqslant \mathrm{const}_* (\mathrm{Vol}_{k-1} S)^{\frac{k}{k-1}}, \tag{$\star$}$$

and where simplices in $D - S$ approaching S have diam $\to 0$. (Notice we do not assume D compact and so the metric completion of $D - S$ may contain apart from S other pieces but of dimension $< k - 1 = \dim S$.) Our, conjecture is that such a D exists for "sufficiently regular" S homeomorphic to S^2 (with the implied const_* depending on this regularity and nothing else). Similar result may hold for higher dimensional S *conformal* to S_0 with a bound on geometry. Finally, we observe that the infimum of

$\Sigma_i(\Delta_i)^\ell$ over all triangulated metric fillings D of S appears an interesting geometric invariant of S. (Compare Fill Vol and Fill Rad in [Gro$_{\mathrm{FRM}}$] and filling on the large scale in [Gro$_{\mathrm{AI}}$].)

Remarks. The "inductive" argument in [Gro$_{\mathrm{AI}}$] for filling circles by disks suggests a slightly different (?) approach to filling 2-spheres by balls.

4.9. Metric properties of submanifolds partitions and maps. The basic invariant of a subvariety $V' \subset V = (V, H)$ is the Hausdorff dimension $\dim_{\mathrm{Hau}} V'$ and $\inf \dim_{\mathrm{Hau}} V'$ over all V' with $\dim_{\mathrm{top}} V' = k$ is a basic metric Hölder-robust invariant of $V = (V, H)$. The major problem in evaluating this invariant, denoted $d_{H|t}(V, k)$, is finding a lower bound on $\dim_{\mathrm{Hau}} V'$ without assuming any smoothness of V' in V. This problem becomes somewhat easier if we replace the Hausdorff dimension by the *Minkowski* (or *entropy*) *dimension* which is defined similar to $\dim_{\mathrm{Hau}}$ with covering by balls but in the definition of $\dim_{\mathrm{Min}}$ all ball covering V' must be of the same radius ε (where eventually $\varepsilon \to 0$ and the number of balls is roughly asymptotic to $(\varepsilon^{\dim_{\mathrm{Min}}})^{-1}$ for $\varepsilon \to 0$ by the definition of $\dim_{\mathrm{Min}}$. Notice that smooth equiregular submanifolds V' have $\dim_{\mathrm{Min}} V' = \dim_{\mathrm{Hau}} V'$ and in all cases $\dim_{\mathrm{Min}} \geqslant \dim_{\mathrm{Hau}}$. Furthermore, if V is equiregular, then the Minkowski dimension can be defined as $\dim_{\mathrm{Min}} = \dim_{\mathrm{Hau}} V - \operatorname{codim}_{\mathrm{Min}} V'$, where the Minkowski codimension is defined with the Hausdorff (which is equivalent to Lebesgue) measure on $V = (V, H)$ as the *critical exponent* α for $\varepsilon^\alpha \operatorname{mes}(V' + \varepsilon)$, where $V' + \varepsilon$ refers to the ε-neighbourhood of V' in V with respect to dist_H and "critical α" is the infimum of those α' for which

$$\varepsilon^{\alpha'} \operatorname{mes}(V' + \varepsilon) \to 0 \quad , \quad \text{for } \varepsilon \to 0.$$

One can construct further invariants of V by applying $d_{H|t}$ and $d_{M|t}$ to $V' \subset V$ (where "M" stands for "Minkowski") by the following inductive scheme. Let inv be an invariant of metric spaces with values in some set R. For example $\mathrm{inv} = \dim_{\mathrm{Hau}}$ takes value in $\mathbb{R}_+$ while $d_{H|t}$ values in $\mathbb{R}_+^{\mathbb{Z}_+} = \operatorname{maps}(\mathbb{Z}_+, \mathbb{R}_+)$ where $\mathbb{Z}_+$ corresponds to the topological dimension $k \in \mathbb{Z}_+$. Now, for a general "inv" we consider *all* topological submanifolds $V' \subset V$ of a given dimension k and take the set $\mathrm{in}_k \subset R$ of the values $\mathrm{inv}\, V' \in R$. Thus, for variable k, we obtain a new invariant $\mathrm{inv}_{\mathrm{new}}(V)$ with values in $\mathbb{Z}_+^{2^R}$, namely the one which assigns to each

$k \in \mathbb{Z}_+$ the set $\mathrm{in}_k \in 2^R$. (Of course one could consider more general subsets V' in V and replace $\mathbb{Z}_+$ by the set of topinv$(V' \subset V)$.) Notice that the smooth counterparts of such invariants with "C^∞-submanifolds" instead of "topological submanifolds" are relatively easy to evaluate and the basic unsolved problem is computing the "topological" metric invariants of (V, H) in terms of the "smooth" ones.

Example (where this does not work). Let $V' \subset V$ be a compact connected smooth submanifold of codimension $n_1 = \operatorname{rank} H$ which is transversal to H. Then the metric dist_H on V' is greater than $\sqrt{\text{Euclidean}}$ which can be expressed in one of the following six ways (where the properties (1)-(6) below are immediate with the ball-box theorem).

(1) There is a Riemannian metric dist' on V' such that $\mathrm{dist}_H \geqslant \sqrt{\mathrm{dist}'}$ on V'.

(2) There is a Riemannian metric dist' on V' and a proper continuous map $f : V' \to V'$ of degree one, such that every dist_H-ball of radius ε goes to an ε^2-ball for dist' for all $\varepsilon \geqslant 0$.

(3) An *ε-chain* in a metric space is a sequence of points

$$v_0, \dots, v_i, v_{i+1}, \dots, v_k$$

where $\mathrm{dist}(v_i, v_{i+1}) \leqslant \varepsilon$ for $i = 0, 1, \dots, k-1$, and the length of an ε-chain is defined as $k\varepsilon$. Then we define $\mathrm{dist}'_\varepsilon$ on V' as the infimum of the lengths of the ε-chains in V' between pairs of points for the metric $\mathrm{dist}_H \mid V'$. Then the transversality of V' to H implies that

$$\mathrm{dist}'_\varepsilon(v_1, v_2) \geqslant \mathrm{const}\max(\mathrm{dist}_H(v_1, v_2), \varepsilon^{-1}\,\mathrm{dist}^2_H(v_1, v_2)). \qquad (*)$$

(4) Let $\mathrm{dist}_\varepsilon$ be the Riemannian (or piecewise Riemannian) ε-approximation to dist_H (see 1.4.D). Then $\mathrm{dist}_\varepsilon \mid V'$ is bounded from below by the right hand side of $(*)$.

(5) Given a subset $c \in V$, let

$$\mathrm{length}_\varepsilon\, c = \varepsilon(\text{minimal number of } \varepsilon\text{-balls needed to cover } c).$$

Then define $\mathrm{dist}''_\varepsilon$ as the infimum of the ε-length of curves c joining pairs of points in V'. Then this $\mathrm{dist}''_\varepsilon$ satisfies $(*)$.

(6) Every subset $c \in V'$ of topological dimension $\geqslant 1$ has $\dim_{\mathrm{Hau}} c \geqslant 2$.

We suggest the reader at this point would ponder an inter-relation between these six metric properties of V' and, in particular, show that $(3) \Rightarrow (2)$ by adopting the proof of the "cubical" Besikovič' lemma in § 7 of [Gro$_{\mathrm{FRM}}$].

Next we observe, that if V is a *two* step space, i.e. $H_2 = T(V)$, which means the commutators of H-horizontal fields span $T(V)$, then, in fact, $\mathrm{dist}_H \mid V' \approx \sqrt{\mathrm{Euclidean}}$, and in particular, $\dim_{\mathrm{Hau}} V' = 2\dim_{\mathrm{top}} V'$. On the other hand, if V is d-step for $d \geqslant 3$, then our V' necessarily has $\dim_{\mathrm{Hau}} V' > 2\dim_{\mathrm{top}} V'$.

Now, we want a similar result *without* assuming V' is smooth but just satisfying (some of) (1)-(6). Unfortunately, this does not work in general. Namely, if H admits an integral (i.e. H-horizontal) submanifold $W \subset V$ (on which the metric dist_H is Riemannian) of $\dim W \gg n - n_1$, then one can find a (highly non-smooth) submanifold $V' \subset W$ of dimension $n - n_1$ with $\mathrm{dist}_H \mid V' \approx \sqrt{\mathrm{Euclidean}}$, (as was pointed out to me by S. Semmes).

Question. Are there cases where (1)-(6) imply $\mathrm{dist}_{\mathrm{Hau}} V' > 2\dim_{\mathrm{top}} V'$ or every V admits a topological submanifold V' of codimension n_1 satisfying (1)-(6) and having $\dim_{\mathrm{Hau}} V' = 2\dim_{\mathrm{top}} V'$ (and moreover, having $\mathrm{dist}_H V' \approx \sqrt{\mathrm{Euclidean}}$)?

Local contractibility of V'. We start by observing (with the ball-box theorem) that every smooth *equiregular* $V' \subset V$ (i.e. having rank $(T(V') \cap H_i)$ constant on V' for all $i = 1, \dots, d$) is *locally contractible* in the sense that every ε-ball in V' (for $\mathrm{dist}_H V'$) is contractible within C_ε-ball for some constant $C = C(V')$ (compare 1.4.B). This remains valid for C^2-smooth not necessarily equiregular V' if V is 2-step, (as again follows from the ball-box theorem and the Taylor remainder formula) but C^∞-submanifolds in d-step spaces for $d \geqslant 3$ do not have, in general, this property. For example, let V be 3-step nilpotent group with the graded Lie algebra, $L = L_1 \oplus L_2 \oplus L_3$ and $V_0' \subset V$ be a 2-dimensional Abelian subgroup corresponding to a subalgebra spanned by some $x \in L_1$ and $y \in L_3$. The metric dist_H on V_0' (isomorphic to $\mathbb{R}^2$) has the ε-balls at the origin equivalent to the boxes $\{|x| \leqslant \varepsilon\ ,\ |y| \leqslant \varepsilon^3\}$ and the parabola $V' = \{y = x^2\}$ in $V_0' \subset V$ has highly disconnected balls at the points close to $(x = 0\ ,\ y = 0)$ as a simple consideration shows. On the other hand one can easily show that every C^∞-smooth V' in a d-step manifold V has the local contractibility with the exponent $\alpha = (d-1)^{-1}$, i.e. every ε-ball is contractible within a concentric ball of radius $C\varepsilon^\alpha$.

Including V' into a family. The transversality of V' to H seems hard to characterize in terms of $\mathrm{dist}_H V'$ (as we saw above) and one may try to use more (metric) information on the relative position of such a V' in V. We observe that every smooth V' of codimension n_1 transversal to H, locally, appears as a fiber of a smooth map $\Pi : V \to \mathbb{R}^{n_1}$, say $V' = \Pi^{-1}(0)$ and we have "parallel" submanifolds $V' = V'(x) = \Pi^{-1}(x) \in V$ for all $x \in \mathbb{R}^{n_1}$. Let us enumerate the metric properties of the family $V'(x)$ (which easily follow from the ball-box theorem). To simplify the matter, we assume that our p topologically is a trivial fibration with compact fibers V' over the unit ball $B_1 \subset \mathbb{R}^{n_1}$ around the origin.

(1) The map Π is Lipschitz.

(2) The map Π is *co-Lipschitz*, i.e. the Π-image of every ε-ball around $v \in V$ contains a ball of radius $\mathrm{const}\,\varepsilon$ around $\Pi(v) \in B$.

(2′) The adjoint map $\Pi^{\#} : 2^B \to 2^V$ for $\Pi^{\#} : B' \mapsto p^{-1}(B') \subset V$, for $B' \subset B$, is Lipschitz where the spaces of subsets 2^B and 2^V are given their respective Hausdorff metrics.

(3) The partition (foliation) of V into $V'(x)$ is *transversally Lipschitz* which may be expressed in two slightly different ways.

(3a) $\mathrm{dist}_{\mathrm{Hau}}(V'(x), V'(y)) \leqslant \mathrm{const}\,\mathrm{dist}(V'(x), V'(y))$, for all $V'(x)$, where dist on the right hand side refers to the infimum of the distances $\mathrm{dist}(v_1, v_2)$, $v_1 \in V'(x)$, $v_2 \in V'(y)$.

(3b) Let $W \subset V$ be the graph of a continuous section $q : B \to V$. Such a V defines a metric on B (and hence on $W = q(B)$), namely the maximal (or supremal) among metrics dist such that the Π-image of every ε-ball in V with the center in W is contained in the concentric ε-ball for the maximal dist. Then the transversally Lipschitz property claims the Lipschitz equivalence of such metrics for different sections $B \to V$.

(4) There is a (multiply) transitive group of homeomorphisms acting on V which sends fiber of Π to fibers and such that

(i) the induced homeomorphisms on B are bi-Lipschitz,

(ii) if V is 2-step that the homeomorphisms are bi-Lipschitz on the fibers,

(iii) if V is d-step, the homeomorphisms are C^α-Hölder on the fibers for $\alpha = (d-1)^{-1}$.

These homeomorphisms of V are constructed with H-horizontal lifts of vector fields on B.

Unfortunately, the above discussion, metrically speaking, hangs in the air as we do not know what kind of partitions (fibrations) may arise without the assumption of smoothness. Here is a specific

Question. When can a given (V, H) be fibered (foliated) by (topological!) submanifolds V_x of a prescribed dimension k, such that one (or all) of the following three conditions is satisfied?

$(*)$ The foliation of V into V' is transversally C^α-Hölder, for a given α, e.g. for $\alpha = 1$;

$(**)$ every $V'(x)$ has Hausdorff dimension in a given interval, for example the metric dist_H on each fiber $V'(x)$ is C^β-Hölder equivalent to (Euclidean)$^\gamma$ for given β and γ in the interval $0 < \beta, \gamma \leqslant 1$;

$(***)$ the space of V'-fibers (with the Hausdorff distance) has the Hausdorff dimension in a given interval, e.g. being C^δ-Hölder equivalent to a Riemannian space for a given positive $\delta \leqslant 1$.

Now we return from topological dreams to C^∞-reality and indicate some extensions of the above (1)-(4) to more general families of equiregular submanifolds V'_x arising as fibers of a smooth fibrations $\Pi : V \to B$. Such a Π remains Lipschitz but "co-Lipschitz" must be replaced by "co-Hölder" with a suitable exponent $\alpha \geqslant d^{-1}$ (which precise evaluation we leave to the reader). The space of the leaves (fibers) with the Hausdorff metric now may be a more complicated space than Carnot-Carathéodory albeit it has many C-C features (e.g. the finite Hausdorff dimensions; we leave to the reader to decide when it has finite Hausdorff measure). The partition into the leaves is transversally Hölder (find the exponent!) and the geometry of the maximal metric dist (depending on q) also seems less regular (?) than C-C. On the other hand (ii) in the above (4) may be extended to d-step spaces which we explain only for $d = 3$ as follows. Locally, there are two smooth partitions (foliations) of V, one into fibers V' of codimension n_1 transversal to $H = H_1$ as earlier and the second partition with fibers (leaves) V'' of codimension $n_2 (= \mathrm{rank}\, H_2)$ associated to a smooth map $\Pi_2 : V \to B_2 \subset \mathbb{R}^{n_2}$ (where our old Π must be now christened $\Pi_1 : V \to B_1 \subset \mathbb{R}^{n_1}$) such that the fibers V'' are contained in the fibers V' (and, in fact, Π_1 is obtained from Π_2 by composing with a linear projection $\mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$). Now we observe that

(a) each fiber V'' has $\sqrt[3]{\text{Euclidean}}$ geometry;

(b) for each fiber V' the space of fibers V'' in V' with the Hausdorff metric has $\sqrt[2]{\text{Euclidean}}$ geometry and the partition of V' into the fibers V'' is transversally Lipschitz. Furthermore, the implied Lipschitz constant is uniform for all V' in V;

(c) the fibers V' form a transversally Lipschitz foliation with (essentially) Euclidean quotient space (i.e. our old $B \subset \mathbb{R}^{n_1}$).

Now we observe that every diffeomorphism φ of V preserving both partitions is *para-Lipschitz* in the sense that

(a_1) φ maps every fiber V'' onto another such fiber by a *bi-Lipschitz* map;

(b_1) as $V'(x)$ goes onto another fiber, say $V'(x_1)$ the space of V''-fibers in $V'(x)$ is send to the corresponding space in $V'(x_1)$ by a *bi-Lipschitz* map;

(c_1) the map induced on the space of V'-fibers, (i.e. on our $B_1 \subset \mathbb{R}^{n_1}$) is *bi-Lipschitz*.

4.9.A. Parabolic metric spaces. Let us axiomatize the above situation and introduce a class of metric spaces generalizing Carnot-Carathéodory ones. For this we need an auxiliary *parabolic structure* P on a space V, namely a *flag* (i.e. a sequence) of partitions $P_1 > P_2 > \cdots > P_{d-1}$ where "$P_i > P_{i+1}$" means "P_{i+1} refines P_i". The "parts" of P_i are called *fibers* or *leaves* and denoted by V^i, where V^i_v means the fiber passing through a point $v \in V$.

A standard example of P is the *affine* flag of *type* $(n_1 < n_2 < \cdots < n_{d-1})$ on $\mathbb{R}^n$ where V^i's are the affine subspaces of codimension n_i parallel to the linear space $\{x_j = 0, j = 1, \dots, n_i\} \subset \mathbb{R}^n$. Parabolic structures *homeomorphic* to the affine ones are called in sequel *flat* and only flat structures are relevant for the C-C geometry. Observe that the group of homeomorphisms preserving a flat flag (parabolic structure) is quite large, e.g. transitive on $V \approx \mathbb{R}^n$ but yet is somehow bounded in complexity by the numbers $n_{i+1} - n_i$ as this group is built in a certain way out of Homeo $\mathbb{R}^{n_{i+1}-n_i}$. This limitation of Homeo (V, P) is especially clear for a *full* flat flag P where $n_i = i = 1, \dots, n-1$, and Homeo (V, P) is "built of" Homeo $\mathbb{R}$.

Finally, we use the notation V/P_i for the space of V^i-leaves (homeomorphic to $\mathbb{R}^{n_i}$ in the flat case) and V^i/P_j, $j > i$, for the space of V^j-leaves inside some $V^i = V_v^i \subset V$. Observe that these V^i/P_j are the fibers of the natural projection $\Pi_i^j : V/P_j \to V/P_i$.

Now, let V, besides a flag P, is given a metric. The idea of *parabolicity* of this metric with respect to P must incorporate (at least) the following three features.

(1) The fibers V^{i+1} inside each V_v^i must be *metrically parallel* in Lipschitz (or at least some Hölder) sense which can be expressed, for example, by requiring the maps $\Pi_i^{i+1} \mid V_v^i \to V_v^i/P_{i+1}$ to be Lipschitz (or Hölder) with the implied Lipschitz (Hölder) constant uniform on (compact subset in) V, where the space V_v^i/P_j is given the Hausdorff distance (between the fibers $V^{i+1} \subset V_v^i$).

(2) The geometry of each quotient space V_v^i/P_{i+1} must be *standard* in a suitable sense. For example we may require each V_v^i/P_{i+1} to be bi-Lipschitz (or bi-Hölder) to $\mathbb{R}^{n_{i+1}-n_i}$ with the metric (Euclidean)$^{\alpha_i}$, (where $\alpha = (i+1)^{-1}$ in the C-C case) and with implied Lipschitz constant uniform on V.

(3) There must exist a sufficiently large (in particular transitive) group of homeomorphisms f of (V, P) which are *para-Lipschitz* for our metric, i.e. bi-Lipschitz $V_v^i/P_{i+1} \leftrightarrow V_{v'}^i/P_{i+1}$, with $v' = f(v)$, for all $i = 1, \dots, d-1$ and $v \in V$. For example, one may ask for the existence of many refinements of P by *full* flat flags P', such that our metric has all of the above properties with respect to P' and there are many homeomorphisms preserving P' and being para-Lipschitz with respect to P' and P. Furthermore, these refinements must be all mutually para-Lipschitz equivalent. (In the C-C case these "many" P' are taken from the pool of the smooth ones as "para-Lipschitz" is automatic for C^1-smooth homeomorphisms.)

Then a metric space V may be called parabolic if it (locally) admits a compatible flag P with some specific (Hölder) exponents as indicated in (1) and (2). The basic problem is the invariance of these exponents under changes of P. The only invariant one can easily reconstruct with a P is the Hausdorff dimension of V and everything else remains problematic.

Let us indicate further properties of (possible) parabolic structures associated to C-C manifolds which reflect the idea of H_i-horizontality. We observe that the projection $\Pi_v^i : V_v^i \to V_v^i/P_{i+1}$ is bi-Lipschitz on H_{i+1}-horizontal submanifolds in V_v^i and that V_v^i (when it is smooth) contains "many" horizontal curves, so that for each (smooth) curve $c \in V_v^i/P_{i+1}$ the pull-back $\left(\Pi_v^i\right)^{-1}(c) \subset V_v^i$ splits by the horizontal lifts of c to V_v^i.

This suggests the following notion of a *polarization* of a full flat flag P which is a system of n (local) free one parameter groups $X_i(t)$ of homeomorphisms of (V, P), such that $X_{i+1}(t)$ maps every V^i into itself for $i = 1, \dots, n$. Such polarization is called horizontal for a given metric on V if the actions are para-Lipschitz and the induced metric on each orbit is (uniformly) bi-Lipschitz to $|t_1 - t_2|^{\alpha_i}$ and also if we take the induced action of $X_i(t)$ on V/P_i, $i = 1, \dots, n-1$, then again the induced metrics on the orbits are $\approx |t_1 - t_2|^{\alpha_i}$ for some positive exponents $\alpha_i \leqslant 1$. We leave to the reader to think over the meaning (and the existence) of these P and X_i for C-C manifolds.

We conclude by indicating the Riemannian counterpart of metric parabolicity. Such Riemannian manifolds are defined inductively so that if V_1 and V_2 are parabolic and V is a Riemannian fibration over V_1 with the fiber V_2 and an Iso V_2-connection, then V is parabolic (where one may additionally assume that the curvature of the connection is bounded on V_1). Then one obtains a class of (parabolic) manifolds by specifying the building blocks. One possibility is to use for this purpose the Euclidean spaces. A more narrow class appears if we start with $\mathbb{R}$ and allow only oriented fibrations $V \to V_1$ with $\mathbb{R}$-fibers. (Here the curvature is an ordinary q-form on V_1 which is necessarily exact, $\omega = d\eta$, and much depends on the possible rate of growth of η on V_1, where, for example, ω may be assumed bounded.) Parabolic Riemannian manifolds have a variety of distinctive asymptotic metric features (e.g. concerning the isoperimetric profile and the spectrum of Δ) which we shall not discussed here. (It is tempting to think of such Riemannian manifolds as approximations to parabolic metric spaces where the latter may appear as asymptotic tangent cones of the former.)

The above definition of parabolicity parallels nilpotent Lie groups. One may generalize from "nilpotent" to "solvable" which leads to another inductively defined class of spaces, where V belongs to the class if $V_1 = V/\mathrm{Iso}\ V$ does, and where the bottom space reduces to a single point. (Alternatively, one may retain the previous definition of parabolicity with *all homogeneous* spaces as building blocks.)

Example. Let V be built of n copies of $\mathbb{R}$,

$$V = V_n \xrightarrow{\mathbb{R}} V_{n-1} \xrightarrow{\mathbb{R}} V_{n-2} \xrightarrow{\mathbb{R}} \dots \xrightarrow{\mathbb{R}} V_1 = \mathbb{R},$$

where the corresponding (curvature) q-forms ω_i on V_i, $i = 1, \dots, n-1$ are bounded. Then an easy induction shows that there are diffeomorphisms $\varphi_i : V_i \leftrightarrow \mathbb{R}^i$, $i = 1, \dots, n$ where $\|\mathcal{D}\varphi_i\|$ and $\|\mathcal{D}\varphi_i^{-1}\|$ have (at most) polynomial growth and $\omega_i = d\eta_i$ for $\|\eta_i\|$ of polynomial growth. Consequently, V has (at most) polynomial volume growth. (Probably, this remains true for V's built of $\mathbb{R}^{n_i}$'s.)

4.10. Anosov endomorphisms. Let A be an Anosov self-mapping of an infra-nil-manifold V. Such an A comes from an automorphism of a nilpotent Lie group, say $\widetilde{A} : \widetilde{V} \to \widetilde{V}$. We look at the corresponding automorphism a acting on the Lie algebra $L = L(\widetilde{V})$ and we measure the dilation of a on linear subspaces $L' \subset L$ as follows. First, for an individual vector $\ell \in L$ we set

$$\lambda(\ell) = \lambda_a(\ell) = \limsup_{i \to \infty} \ \|a^i \ \ell\|^{\frac{1}{i}}$$

for a fixed norm $\| \ \|$ in L. Clearly the resulting $\lambda(\ell)$ does not depend on a choice of the norm and for a *generic* $\ell \in L$

$$\lambda(\ell) = \text{spec rad } a \underset{\text{def}}{=} \max |\text{eigenvalues of } a|.$$

Next, we consider the natural action of a on the exterior powers $\Lambda^k L$ and apply the above definition to k-vectors acted upon by $\Lambda^k a$. In particular, we take $\lambda = \lambda_{\Lambda^k a}$ of the k-vectors defined by k-dimensional subspaces $L' \subset L$ and thus define our *dilation* $\lambda(L')$.

Example. If L' is a-invariant, (e.g. $L' = L$) then

$$\lambda(L') = |\text{Det}(a \mid L')|.$$

Finally, we define

$$\text{ent } a = \sup_{L' \subset L} \ \log \lambda(L')$$

and, more generally,

$$\text{ent } a \mid L' = \sup_{L'' \subset L'} \ \log \lambda(L'').$$

This entropy can be computed in terms of the eigenvalues of a as follows. Denote the eigenvalues of a by $\lambda_1, \dots, \lambda_n$, $n = \dim L$, where each eigenvalue appears with the multiplicity equal to the dimension of the corresponding invariant subspace. Then, obviously,

$$\text{ent } a = \sum_{|\lambda_i| \geqslant 1} \log |\lambda_i| = \sum_{i=1}^{n} \max(0, \log |\lambda_i|) = \sum_{i=n_-+1}^{n} \log |\lambda_i|$$

where $\lambda_{n_-+1}, \dots, \lambda_n$ are the eigenvalues with $|\lambda_i| \geqslant 1$. Next, let $0 < \mu_1 < \mu_2 < \cdots < \mu_p$, $p \leqslant n$, be the absolute values of the λ_i's, $i = 1, \dots, n$, and let $L_1 \subset L_2 \subset \cdots \subset L_p = L$ be the corresponding a-invariant subspaces, i.e. L_j is generated by the vectors "belonging" to λ_i with $|\lambda_i| \leqslant \mu_j$. Notice that

$$\lambda(L_j) = |\mu_1|^{d_1} \ |\mu_2|^{d_2} \cdots |\mu_j|^{d_j}$$

for $d_k = \dim(L_k / L_{k-1})$. Finally, for an arbitrary $L' \subset L$ we clearly have

$$\text{ent } a \mid L' = \sum_{\mu_j > 1} d'_j \log \mu_j$$

for $d'_j = \dim(L' \cap L_j / L' \cap L_{j-1})$ under a.

4.10.A. Entropy in codimension 1. It is well known that the topological entropy of A can be computed by the formula

$$\text{ent } A = \text{ent } a = \sum_{i=n_-+1}^{n} \log |\lambda_i|.$$

(This is obvious if you know the definition of ent). We want to evaluate the entropy of A on a compact (possibly non-invariant) subset $V' \subset V$ of codimension one. To do this we take the minimal integer $m \leqslant n_+$, such that the Lie span of the subspace $L_m \subset L$ corresponding to $\lambda_1, \dots, \lambda_m$ equals L. Now we claim

$$\text{ent}(A \mid V') \geqslant \left(\sum_{i=n_-+1}^{n} \log |\lambda_i| \right) - \log |\lambda_m|. \qquad (+)$$

As in the case of $(*)$ in 2.1 this $(+)$ is obvious for smooth hypersurfaces V' since these must be a.e. transversal to the subbundle $H = H_m \subset T(V)$ corresponding to L_m. In the general case, one may just repeat the proof of $(*)$ in 2.1, which says, in effect, that the above transversality property generalizes in a suitable sense to an arbitrary compact subset

V' of dimension $n-1$. A more suggestive approach reducing the entropy to some generalized Hausdorff dimension is due to Ya. Pesin (see [Pes]) and it runs as follows. Fix a metric in V and $\delta > 0$ and then define ε_k-balls (secretly for $\varepsilon_k = \delta \exp -k$) with respect to A at a point $v \in V$ by

$$B(v, \varepsilon_k, A) = \bigcap_{i=1}^{k} A^{-i}(B(A^i(v), \delta))$$

where $B(A^i(v), \delta)$ is the δ-ball for the given metric. Then for every $V' \subset V$ define the Hausdorff measure of dimension d with coverings of V' by the ε_k-balls and with sums $\Sigma_k\ \varepsilon_k^{\log d}$ (where k numerates balls rather than numbers). Then the Hausdorff dimension defined via the measure as usual. (A more geometric viewpoint on this definition is suggested by the hyperbolic discussion in 0.9 and 1.4.D$'$.)

Notice that in our infranil case these balls look like boxes of size

$$\delta \left(\underbrace{1 \times 1 \times \quad \times 1}_{n_-} \times \underbrace{\lambda_{n_-+1}^{-k} \times \cdots \times \lambda_n^{-k}}_{n_+} \right)$$

for $n_+ = n - n_-$ and so the resulting Hausdorff dimension does not depend on δ. (For general dynamical systems one eliminates the dependence of δ by letting $\delta \to 0$, see [Pes] for details.)

Remark. The inequality $(+)$ is sharp, as there exists a hypersurface $V' \subset V$ having $\mathrm{ent}(A \mid V') = \left(\Sigma_{i=n_-+1}^{n} \log \lambda_i\right) - \log|\lambda_m|$. Namely, take a smooth V' in V which is everywhere tangent to the subbundle corresponding to $L_{m-1} \subset L$. Thus we have computed $\mathrm{ent}_{n-1}(A)$ that is the infimum of the entropies of A on all compact subsets of codimension one in V.

Notice that from the dynamics point of view the basic reason for $(+)$ is the existence of sufficiently many curves c in V for which $\mathrm{ent}(A \mid c) \leqslant \log \lambda_m$.

4.10.B. Entropy in high codimension. One defines the entropy spectrum of an endomorphism A of a metric space V by setting

$$e_k = \inf_{V'}\ \mathrm{ent}\ (A \mid V'),$$

where the infimum is taken over all compact subsets $V' \subset V$ of topological dimension k. Notice that if $\widetilde{A}$ covers A as earlier then A and $\widetilde{A}$ have equal spectra. Now, we concentrate on the case when $\widetilde{A} : \widetilde{V} \to \widetilde{V}$ is an *expanding* automorphism of the Lie group $\widetilde{V}$ (covering V) which necessarily serves as a self-similarity of some C-C metric in $\widetilde{V}$ (which is unique up to bi-Lipschitz equivalence). Let $\widetilde{\lambda}$ be the scaling done by $\widetilde{A}$, i.e.

$$\operatorname{dist}(\widetilde{A}(v_1), \widetilde{A}(v_2)) = \widetilde{\lambda} \operatorname{dist}(v_1, v_2),$$

for $\operatorname{dist} = \operatorname{dist}_{\text{C-C}}$ and observe that

$$\operatorname{ent} A = N \log \widetilde{\lambda}$$

for $N = \dim_{\text{Hau}}(\widetilde{V}, \operatorname{dist}_{\text{C-C}})$. Similarly, for every compact subset $\widetilde{V}' \subset \widetilde{V}$,

$$\operatorname{ent}(\widetilde{A} \mid \widetilde{V}') = \dim_{\text{Min}} \widetilde{V}'$$

and for $V' \subset V$,

$$\operatorname{ent}(A \mid V') = \dim_{\text{Min}} V'$$

as well. Thus the entropy spectrum reduces to measuring the Minkowski dimension of k-dimensional subsets in $\widetilde{V}$ and our results from 3.1.A and 4.5 apply. For example, *if $\widetilde{V}$ is the Heisenberg group of dimension $n = 2m+1$, then $e_n = \operatorname{ent} A$ and*

$$e_k/e_n = k/2m+2 \quad \text{for} \quad k \leqslant m$$

and

$$e_k/e_n = (k+1)/2m+2 \quad \text{for} \quad k > m.$$

It would be appealing to make such a computation for every Anosov-Bowen hyperbolic systems in terms of a Markov partition. In fact, it would be equally interesting to extend in full our C-C metric discussion to combinatorially defined (metric and quasi-conformal) spaces, e.g. semi-Markov of [Gro$_{\text{HG}}$], or those underlying *finitely presented* dynamical systems called "hyperbolic" in [Gro$_{\text{HMGA}}$].

Here is a typical question arising in the combinatorial framework.

What is the arithmetic structure of the numbers $e(k)$ and $e(k)/e(\ell)$?

4.11. Horizontal forms on polarized manifolds. We return to the discussion in 4.1.E on horizontal forms and cohomology of (V, H). We write locally H as $\operatorname{Ker}\{\eta_i\}$ for some 1-forms η_i on V, $i = 1, \ldots, m = n - n_1$, $n_1 = \operatorname{rank} H$, and denote by $H_j\ \Lambda^{n-k} \subset \Lambda^{n-k}(V) = \Lambda^{n-k}\ T(V)$ the bundle of *j-horizontal* $(n-k)$-forms α on V, which have, by definition, degree $\geqslant j$ in η_i, i.e. $\alpha = \Sigma_\mu\ \beta_\mu \wedge \zeta_\mu$, where each ζ_μ is the product of j forms among η_i, say $\eta_{i_1} \wedge \eta_{i_2} \wedge \cdots \wedge \eta_{i_j}$. In particular, we are interested in the bundle $H\Lambda^{n-k} = H_m\ \Lambda^{n-h}$ of *horizontal* forms which are divisible by the m-form $\zeta = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m$ on V. (Notice that ζ is globally defined up to a scalar multiple on V. More invariantly, this is a form with values in the line bundle $\Lambda^m H^\perp$ for $H^\perp = T(V)/H$.)

Now we want to figure out when V supports "many" *closed* (and exact) horizontal forms α. In particular, we wish to realize a given (absolute or relative) de Rham cohomology class by such a form (compare linear lemma in 2.2., contact discussion in 3.3 and 4.1.E). We already know that if V contains "many" regular horizontal jets of k-dimensional submanifolds, e.g. Ω-regular isotropic k-planes in $H \subset T(V)$, then there are "many" (possibly folded) horizontal k-submanifolds in V which can be smoothed (as currents in families) to closed horizontal $(n-k)$-forms. This corresponds to the "non-linear proof" of linear lemma while the "linear proof" should proceed as follows (compare algebraic inversion lemma in 3.3). Given a closed $(n-k)$-form a on V (representing a de Rham class) we want to make it horizontal by adding an exact form. Thus we try to solve the equation $dx = a \bmod \zeta$ on V where $a = b \bmod \zeta$ signify that $a - b$ is divisible by ζ. In other words we denote by $\bar{d}$ the composition of $d : \Lambda^{n-k-1} \to \Lambda^{n-k}$ with the quotient map $\Lambda^{n-k} \to I_\zeta^\perp \underset{\text{def}}{=} \Lambda^{n-k}/H\Lambda^{n-k}$ (where the notation Λ^i sometimes applies to the sheaf of forms as well as the corresponding vector bundle), and want to solve the equation $\bar{d}x = \bar{a}$ for $\bar{d} : \Lambda^{n-k-1} \to I_\zeta^\perp$ and $\bar{a}$ being a section of $I_\zeta^\perp$. Ideally, we want a *differential* operator $\bar{\delta} : I_\zeta^\perp \to \Lambda^{n-k-1}$ inverting $\bar{d}$, i.e. satisfying $\bar{d}\,\bar{\delta} = \mathrm{Id}$. We know such a $\bar{\delta}$ exists for all H (Lie generating $T(V)$) for $k = 1$ (i.e. for $\deg a = n - 1$, (see "linear proof" of linear lemma) and also for *contact* structures H above the middle dimension, i.e. $k < \frac{1}{2}n$, $n = \dim V$ (see 3.3). On the other hand such a $\bar{\delta}$ may be expected for generic H if

$$\text{“}f\dim\text{”}(H\Lambda^{n-k} \cap \operatorname{Im} d) \underset{\text{def}}{=} n_1!/k!\,(n_1 - k)! + \sum_{j=n-k+1}^{n} (-1)^{j-n+k}\,\lambda_j > 0$$

for $n_1 = \operatorname{rank} H$ and $\lambda_j = n!/i!\,(n-i)!$ (see 4.1.E).

Questions. Is it actually true that $\bar{d}$ is differentially invertible for generic H when "f dim" > 0? If not, what is the correct relation between n, $m = n - n_1$ and k generically needed for the existence of $\bar{\delta}$?

The passage from horizontal k-submanifolds to closed $(n-k)$-forms suggests that the existence of a regular jet (of a horizontal k-germ) at every point $v \in V$ should imply the existence of $\bar{\delta}$. This is confirmed by the linear proof of linear lemma in 2.2 and algebraic inversion lemma in 3.3. But one should not neglect the essential appearance of non-purity in differential forms in the degrees strictly between 1 and $n-1$ and keep in mind that the formal linearization of the horizontality for submanifolds may lead to something more restrictive than the horizontality of forms, since the ideals in the Grassmann algebra do not always satisfy the Hilbert zero theorem. Namely, let I be the ideal in $\Lambda^*(\mathbb{R}^{n_1})$ generated by some q-forms $\omega_1, \dots, \omega_m$ (corresponding to the differentials $d\eta_1, \dots, d\eta_m$, $m = n - n_1$, restricted to $\mathbb{R}^{n_1} = H_v$, where $\eta_1, \dots, \eta_m$ are the I-forms defining H). This I defines, for each $k = 2, 3, \dots, n_1$, a subvariety $Z_k = Z_k(I) \subset \mathrm{Gr}_k(\mathbb{R}^{n_1})$ consisting of those k-dimensional subspaces in $\mathbb{R}^{n_1}$ on which all ω_i, $i = 1, \dots, m$, vanish. But (unlike the polynomial case) these $Z_k(I)$ do not, in general, uniquely determine I, and it may happen that the ("radical") ideal I_V consisting in each degree k of the form vanishing on Z_k is significantly greater than I_V and then the annihilator J_V of I_V may be viewed as the true linearization of Z_k. (Recall that $\omega_i = d\eta_i$ vanish on horizontal submanifold $V' \subset V$ and so the above Z_k contain the possible tangent spaces to such V' at a given point $v \in V$. Then one defines the differential ideal I_V consisting of forms vanishing together with their differentials on the tangents to the horizontal submanifolds in V and the annihilator of I_V may play the role of horizontal forms.) This discussion suggests the following.

Algebraic question. Let $G_k \subset \Lambda^k(\mathbb{R}^{n_1})$ be the subvariety of the *pure* k-forms, i.e. the Plücker image of $\mathrm{Gr}_k(\mathbb{R}^{n_1})$ in $\Lambda^k(\mathbb{R}^{n_1})$ and $L \subset \Lambda^k(\mathbb{R}^{n_1})$ is a linear subspace of dimension ℓ. When does the intersection $L \cap G_k$ linearly span all of $\Lambda^k(\mathbb{R}^{n_1})$? Is this true, for instance, for generic $L \subset \Lambda^2(\mathbb{R}^{n_1})$ where n_1 is large and $\operatorname{codim} L$ is small?

Künneth type question. Let (V_i, H_i), $i = 1, 2$ be contact manifolds of equal dimension. Does the above mentioned $\overline{\delta}$ exist for $(V, H) = (V_1 \times V_2, H_1 \times H_2)$ and $\dim V - k \geqslant (\dim V_1 + 1)/2$? (If $m = \dim V - k = 2$ and one tries to solve $dx = a \bmod \zeta$ for $\zeta = \eta_1 \wedge \eta_2$ with $x = \zeta \wedge y$ one does get an algebraic equation, namely $d\zeta \wedge y = a \bmod \zeta$, where, moreover, one may assume $a = a_1 \wedge \eta_1 + a_2 \wedge \eta_2$. But as $d\zeta \wedge d\zeta = 0$, this equation is not solvable unless $d\zeta \wedge a = 0 \bmod \zeta$ which is a non-vacuous relation on a one should reckon with. Notice that according to Z. Ge (see [Ge$_{\mathrm{BNCC}}$]) the vanishing of $H^i(I(H), d)$ for the differential ideal $I(H)$ generated by η_i and $d\eta_i$ follows from the injectivity of the linear map $\underbrace{\Lambda^i H \oplus \cdots \oplus \Lambda^i H}_{m} \to \Lambda^{i+2} H$ given by $(a_1, \ldots, a_m) \mapsto (\omega_1 \wedge a_1, \ldots, \omega_m \wedge a_m)$. Ge shows that for $i = 1$ this injectivity condition is satisfied by fat H (see 4.2.B) and possibly (?) there are meaningful examples for $i \geqslant 2$ and $m \geqslant 2$. Also the Poincaré duality suggests something similar for small $n - i$.)

Stabilization and characteristic cohomology. If we want to linearize not only tangents to horizontal submanifolds by higher order jets of these, we should pass to the manifold V^r of r-jets of germs of k-dimensional submanifolds in V which comes along with a polarization $H^r \subset T(V^r)$. Now we take the (open) subset $V^r_{\mathrm{reg}} \subset V^r$ consisting of regular jets and ask if, for sufficiently large (stable) r, the local cohomology of the complex $(\Lambda^i(V^r_{\mathrm{reg}})/I(H^r), d)$ for $i < k$, where $I(H^r)$ is the differential ideal generated by the 1-forms in V^r_r vanishing on H^r and by their differentials. (If "yes", the proof must come by a gentle algebraic hand waving but finding the minimal "stable" r may require a specific calculation, compare [Vin$_{1;2;3}$] and [Br-Gr].)

Remark. We shall return to this question for $V^r = V$ and $i = 2$ in the framework of *H-connections* over V with possibly non-Abelian structure groups (see § 5).

4.11.A. Horizontality via the anisotropic blow-up. We recall the (cotype) horizontality degree $M = M^*(\omega)$ of an $(n-k)$-form ω on (V, H) measuring an averaged horizontality of ω with respect to $H = H_1 \subset H_2 \subset \cdots \subset H_d = T(V)$ (see 4.1.E′). For example, if $d = 2$, and ω is m_1-horizontal but not $(m_1 + 1)$-horizontal, then $M^*(\omega) = m_1 + 2(n - k - m_1)$ and vice versa. Also recall that forms ω of horizontality M integrate to straight cochains c with $\|c\|_\varepsilon \lesssim \varepsilon^M$ (see 4.1.E′ and 3.3.B) and the pointwise norm of ω decays as $\mathcal{O}(\varepsilon^M)$ under the anisotropic blow-up g^*_ε of a background metric g on V (which expands the vectors in $H_i \ominus H_{i-1}$ with the rate ε^{-i} for $\varepsilon \to 0$, see 4.1.E$'$). The latter (characteristic) property of M suggests the following approach to defining interesting horizontalities of tensorial (and non-tensorial) fields on (V, H) as well as constructing such horizontal fields. We consider a distinguished class of fields $\{\omega_\varepsilon\}$ on (V, g_ε^*) (e.g. harmonic forms or spinors, or these belonging to eigenvalues of certain level, or something non-linear like a Yang-Mills field) and try to select among them those which have certain rate of decay (or growth) for $\varepsilon \to 0$ with respect to a suitable norm in the corresponding function space. Such a selection process of ω_ε can be often expressed in terms of H as a horizontality type condition for (possibly non-existant) limit field ω_∞ on V. Then one may ask if (or when) a (suitably renormalized) ω_ε actually converges for $\varepsilon \to 0$ to some field on V and whether the limit field is horizontal in a desirable sense.

Example. If we start with g_ε^*-harmonic *functions* (scalars) on V we obtain in the limit for $\varepsilon \to 0$ Hörmander harmonic functions (i.e. satisfying $\Delta_H\, \omega = 0$ for the scalar Hörmander-Laplace operator Δ_H (compare [Fuk], [Ge$_{\text{CRM}}$]).

The convergence $\omega_\varepsilon \to \omega_\infty$ looks most promising (especially for linear fields) if (V, H, g) is homogeneous under an action of a Lie group, preferably a compact one, where everything reduces to the representation theory. But we do not indulge ourselves in the luxury of linear algebra but rather indicate a couple of direct analytic consequences of the symmetry allowing a simple comparison between the L_∞ and L_2-norms of the fields in question. Our objective is constructing L_∞-slow growing (or rather L_∞-fast decaying) g_ε^*-harmonic forms ω_ε on V eventually giving rise to "small" straight cocycles on V.

4.11.B. Horizontal cohomology on nilpotent Lie groups and algebras. Let $L = \oplus_{i=1}^d L_i$ be a graded nilpotent Lie algebra and (V, H) the corresponding Lie group with the left-invariant polarization H arizing from $L_1 = H_{\text{id}} \subset T_{\text{id}}(V)$ (where we assume L_1 Lie-generates L as usual). Let $A^t : L \to L$ act on L_i by $\ell \mapsto t^i \ell$, $0 < t < \infty$ and observe that the induced action of A^t on the cohomology $H^*(L)$ also decomposes into eigenspaces, $H^* = \oplus_{M=0}^N H_M^*$, where A^t acts on the M-th component by t^M, where $N = \Sigma_{i=1}^d i \operatorname{rank} L_i = \dim_{\text{Hau}}(V, H)$ and where $H_N^* = H^n(L) = \mathbb{R}$ (and where H of the polarization should not be confused with H^* of the cohomology). We observe that the cohomology H_M^*

is represented by (closed left-invariant) M-horizontal forms on V and we are interested in finding such forms ω with possibly large $M = M^*(\omega)$. To do this we observe that the action of A^t on the cohomology commutes with the cup-product and that the cup-product pairing between H^k and H^{n-k} is faithful by the Poincaré duality (proven on L with an obvious use of the Hodge $*$-operator for some positive quadratic form on L). It follows that there are dual bases h_μ in H^k and h'_ν in H^{n-k}, i.e. with $h_\mu \cup h'_\nu = \delta_{\mu\nu}$, where h_μ and h'_ν are eigenvectors, and consequently

$$M^*(h_\mu) + M^*(h'_\mu) = N, \tag{$*$}$$

for all $\mu = 1, \ldots, \operatorname{rank} H^k = \operatorname{rank} H^{n-k}$, (with $M = M^*(h)$ defined by $h \in H^*_M$).

Corollary. *If n is even and the middle cohomology $H^{n/2}(L)$ does not vanish, then there is a closed left-invariant $n/2$-form ω on V non-homologous to zero in $H^*(L)$ with $M^*(\omega) = N/2$.*

4.11.B′. Horizontal cycles and cocycles on nil-manifolds V/Γ. The above becomes (geometrically) more interesting if V admits a cocompact lattice Γ and we can pass to the (compact nil-manifold) quotient V/Γ. The cohomology $H^*(L)$ obviously (by Poincaré duality in $H^*(L)$) *injects* to $H^*(V/\Gamma;\mathbb{R})$ and thus the relation $(*)$ remains valid in V/Γ, where $M^*(h)$ may be now understood as the maximal M, so that $h \in H^*(V/\Gamma;\mathbb{R})$ is representable by a straight cocycle c with $\|c\|_\varepsilon \leqslant \varepsilon^M$ (compare 4.1.E). In fact, one knows that the homomorphism $H^*(L) \to H^*(V/\Gamma)$ is a bijection and so the eigen splitting $H^*(L) = \oplus_0^N H^*_M$ applies to $H^*(V/\Gamma)$ and carries full information on the M-horizontality of (forms representing) the cohomology of V/Γ. This also gives as a non-trivial lower bound on the *Minkowski dimension* (see 4.9.) of the homology classes $c \in H_*$ which is defined as the infimum of these dimensions of the cycles representing c. To formulate this, define the function M_* on H_* as the dual of M^* on H^*, i.e. let $M_*(c) = \max M^*(h)$ over all $h \in H^*$ for which $\langle h, c\rangle \neq 0$.

Proposition. *The Minkowski dimension of every $c \in H_*(V/\Gamma;\mathbb{R})$ is bounded from below by*

$$\dim_{\mathrm{Min}} c \geqslant M_*(c). \tag{$+$}$$

Proof. If a cycle $c_\bullet$ can be covered by j balls of C-C radius ε, and a straight cocycle $c^\bullet$ has $\|c^\bullet\|_\varepsilon \leqslant \varepsilon^M$ then $c^\bullet(c_\bullet) \lesssim j\ \varepsilon^M$ and the proof follows.

Remarks

(a) The above proposition applies to all compact C-C manifolds, where M^* (and the issuing M_*) can be understood either as the (best) degree of horizontality of forms representing $h \in H^*$ or as the exponent M in the (best) bound on the ε-norms of the straight cochains representing h.

(b) Probably, one can replace $\dim_{\mathrm{Min}}$ in $(+)$ by $\dim_{\mathrm{Hau}}$.

Next, we combine $(+)$ and $(*)$ and conclude to the following.

Corollary. *If two cycles (or homology classes) c and c' in V/Γ have non-zero index of intersection, then*

$$\dim_{\mathrm{Min}} c + \dim_{\mathrm{Min}} c' \geqslant N = \dim_{\mathrm{Hau}} V. \tag{$*+$}$$

This shows, in particular, that if $H_k(V/\Gamma;\mathbb{R}) \neq 0$ then the maxima $M_+(k) = \max\limits_{c \in H_k} \dim_{\mathrm{Min}} c$ and $M_+(n-k) = \max\limits_{c \in H_{n-k}} \dim_{\mathrm{Min}} c$ satisfy

$$M_+(k) + M_+(n-k) \geqslant N. \tag{$++$}$$

Remarks

(a) It is likely that $(*+)$ and $(++)$ remain true for all compact equiregular (or generic) C-C manifold and we shall prove it below for compact homogeneous spaces.

(b) The Minkowski dimension $\dim_{\mathrm{Min}} c$ is not, a priori, an integer. Yet the *proof* of $(*+)$ implies the existence of positive *integers*, say $d(c_\mu)$, satisfying

$$d(c_\mu) + d(c'_\mu) \geqslant N, \tag{$*+$}'$$

and such that

$$\dim_{\mathrm{Min}} c_\mu \geqslant d(c_\mu) \quad \text{and} \quad d_{\mathrm{Min}}(c'_\nu) \geqslant d(c'_\nu).$$

Consequently, $(++)$ improves to

$$M'_+(k) + M'_+(n-k) \geqslant N, \tag{$++$}'$$

where M' signifies the greatest integer below M.

(c) The number $M_+(k)$ of a C-C manifold V bounds the Hölder exponent α of a possible "homotopically surjective" (e.g. homeomorphic onto) C^α-map $(V, \mathrm{Riem}) \to (V, \text{C-C})$ by $\alpha \leqslant k/M_+(k)$ which may happen to be better than the general bound $\alpha \leqslant n/N$, for $n = \dim V$ and $N = \dim_{\mathrm{Hau}} V$, or even the slightly improved bound $\alpha \leqslant n-1/N-1$ (which is always valid since all hypersurfaces in V have $\dim_{\mathrm{Hau}} \geqslant N-1$, see 2.1). For example, if n is even, say $n = 2m$ and N is odd, $N = 2M+1$, then $(++)'$ gives us $M_+(m) \geqslant M+1$ (provided $H_m(V;\mathbb{R}) \neq 0$) and so $\alpha \leqslant m/M+1$ which is better than $\alpha \leqslant 2m-1/2M$ if $M < 2m-1$.

One can get extra milage from $(++)'$ by deriving a lower bound on $M_+(n-k)$ from an upper bound on $M_+(k)$ in the case where V has sufficiently many horizontal k-submanifolds. For example, such submanifolds are there for generic $H \subset T(V)$ and $k \leqslant n_1/n-n_1+1$, where $n_1 = \operatorname{rank} H$ (see 4.2.A″). This makes with $(++)'$

$$M'_+(n-k) \geqslant N-k$$

for $N = n_1 + 2(n-n_1)$ and all $k \leqslant n_1/n - n_1 + 1$ for which the k-th homology of V does not vanish, and so the (best) estimate for α one may obtain this way is, roughly,

$$\alpha \leqslant 1 - \frac{N-n}{N-(n_1/n-n_1)}.$$

Example. (inspired by [Pit]) Let $V' \subset V$ be a connected subgroup, such that the intersection $V' \cap \Gamma$ is cocompact in V' and the submanifold $V'/V' \cap \Gamma \subset V/\Gamma$ is not homologous to zero which is equivalent to the existence of a closed left-invariant form ω on V extending the oriented volume form on V'. Then the Minkowski dimension M' of the homology class $[V'/V' \cap \Gamma]$ equals the exponent of the volume growth of V' in V. This means that the R-balls in V' for the distance induced from V have the Riemannian k-dimensional volumes $\approx R^{M'}$ where $k = \dim V'$. This is especially clear if V' is invariant under A^t but, in fact, it is true (and follows by our arguments) for arbitrary $V' \subset V$ without even assuming that the (nilpotent) group V admits any dilation. Furthermore, one can exclude Γ from this discussion and show that if the volume form of V' extends to a closed equivariant form on V, then every infinite k-cycle V'' lying within finite distance from V' and homologous to V' inside some

ρ-neighbourhood of V' has $\dim_{\mathrm{Min}} V'' \geqslant M'$, where $\dim_{\mathrm{Min}}$ refers to the supremum of Minkowski dimensions of the compact parts of V'' (with respect to dist_H) and M' denotes the exponent of the volume growth of V'. Further generalizations concern a submanifold $V' \subset V$ which is not a subgroup but, preferably, of polynomial growth in V and which should be asymptotically non-homologous to zero. The latter may be ensured by the existence of a closed k-form ω on V, $k = \dim V'$, such that ω is positive and does not decay fast on V' (e.g. being equal on V' to the volume forms of V') and on the other hand, ω does not grow too fast (e.g. being bounded) on V. Then one can show in some cases that the volume growth of V' can not be significantly decreased by "small" moves of V' and the Minkowski dimension of the "homology class" $[V']$ can be estimated from below in terms of the rate of the volume growth of V'.

Applications to the lower bound on the filling exponent. Suppose every closed curve in V of length ℓ can be filled in by a disk of area $\lesssim \ell^p$. Then every 2-dimensional homology class in V/Γ grows, under the anisotropic blow-up g^*_ε of g, with the rate $\lesssim \varepsilon^{-p}$ (which is essentially equivalent to $M_+(2) \leqslant p$), as is seen with triangulated cycles where the edges are (short) horizontal curves and the triangles are filled g^*_ε-minimally. It follows that the 2-dimensional filling exponent p of V is bounded from *below* by $M_+(2)$ which implies the following.

Corollary. ([12]) *Let a simply connected nilpotent group V contain a 2-dimensional subgroup $V' \subset V$ which is not homologous to zero in the sense that the area form of V' extends to a closed left-invariant 2-form ω on V. Then the 2-dimensional filling exponent p of V is bounded from below by the exponent of the area growth of R-balls in V', (i.e. $\mathrm{Area}B(R) \lesssim R^p$ for the balls in V' for the metric $\mathrm{dist}_V \mid V'$).*

Remarks

(a) We do not assume in this Corollary the presence of Γ as we know how to dispose of it. Furthermore, one may relax the invariance assumption on the extended form ω. Namely, it is enough to require ω is *bounded* or even growing on V but rather slowly, say $|\omega(v)| \lesssim (\mathrm{dist}(v_0, v))^\varepsilon$, so that the integral of $|\omega|$ over the boundaries of balls $B(R)$ in V' and near lying manifolds V'' is dominated by area $B(R)$ for $R \to \infty$.

(b) Notice that the fillings involved in the proof of the inequality $M_+(2) \leqslant p$ apply only to some "standard" curves in V. For example if V admits

[12] See [Pit].

a dilation A, then we only need to fill the curves $A^t(c)$ $t \to \infty$ where c runs over a finite set of closed curves. This allows an effective generalization to the k-dimensional fillings for $k \geqslant 3$ via triangulations of cycles and induction by skeletons. For example, if V contains $(k-1)$-regular jets we can make our cycle to be controllably small on the $(k-1)$-skeleton and then the exponent of the k-th filling (inequality) can be estimated from below by $M_+(k)$. (There is much here to be made more precise and specific and this we leave to the reader.)

4.11.B″. Harmonic forms and anisotropic blow-up of V/Γ. Since the (self-similarity) operators $A^t : L \to L$, being automorphisms of the Lie algebra $L = L(V)$, commute with the exterior differential d (on the Grassmann algebra $\Lambda^* L =$ the algebra of left-invariant forms on the Lie group V) so do the projectors on the eigenspaces of A^t acting on $\Lambda^* L$. Furthermore, for every metric g on L with respect to which different eigenspaces of A on L are mutually orthogonal, the eigenspaces of A on $\Lambda^* L$ are also mutually orthogonal and therefore the space of g-harmonic forms, i.e. $(\ker d) \ominus_g (\mathrm{imd}) \in \Lambda^* L$, is A-invariant. Finally we observe that the anisotropic blow-up g_ε^* of the metric g^* left-invariantly extended from $L = T_{\mathrm{id}} V$ to V can be realized by the (expanding) automorphisms $V \to V$ corresponding to A^t, since A^t for $t = \varepsilon^{-1}$ pulls-*back* g to g_ε^*. It follows that *every g-harmonic form on V (and hence on V/Γ) is also g_ε^*-harmonic for all $\varepsilon > 0$.*

Now let us take a metric g on L which does not necessarily agree with A and look at g_ε^*-harmonic forms for $\varepsilon \to 0$. It is easy to see that the quadratic forms $g'_\varepsilon = (A^\varepsilon)^* \, g_\varepsilon^*$ on L converge for $\varepsilon \to 0$ to some positive definite quadratic form g^1 which does agree with A (i.e. the eigenspaces of A are g^1-orthogonal) and consequently the space $\mathcal{H}_{g_\varepsilon^*}$ of g_ε^*-harmonic form converge to $\mathcal{H}_{g^1}$ for $\varepsilon \to 0$.

Example. Let L be the Heisenberg algebra of rank $n = 2m + 1$ where the de Rham complex of L (i.e. of left-invariant forms on the Heisenberg group) is generated by x_i, y_i and z, $i = 1, \dots, m$ with the relations $dx_i = dy_i = 0$, $dz = \Sigma_{i=1}^m \, x_i \wedge y_i$. Then one easily sees that every cohomology class $h \in H^k(L)$, $k \leqslant m$ can be represented by an (exterior) polynomial in x_i and y_i, i.e. by some $\lambda \in \Lambda^k(\mathbb{R}^{2m})$ and $h = 0 \Leftrightarrow \lambda$ is divisible by $\omega = \sum_{i=1}^m x_i \wedge y_i$. In other words $H^{\leqslant m}(L) = \Lambda^{\leqslant m}(\mathbb{R}^{2m})/I_\omega$, where I_ω denotes the ideal in the Grassmann algebra $\Lambda^*(\mathbb{R}^{2m})$ spanned by ω.

On the other hand, every $h \in H^{>m}(L)$ can be represented by $z \wedge \gamma$, $\gamma \in \Lambda^*(\mathbb{R}^{2m})$, where $d(z \wedge \gamma) = \omega \wedge \gamma = 0$. Thus $H^{>m}(L)$ identifies with the kernel of the multiplication operator $\wedge\omega : \Lambda^*(\mathbb{R}^{2m}) \to \Lambda^*(\mathbb{R}^{2m})$, i.e. the annihilator $J_\omega \subset \Lambda^*(\mathbb{R}^{2m})$ of I_ω. It follows that $H^*(L)$ equals the space of equivariant Rimin's forms on the Heisenberg group (see 3.3.A).

Exercise. Prove the above using Gysin exact sequence,

$$\cdots \to H^{k-1}(L) \to \Lambda^k(\mathbb{R}^{2m}) \overset{\wedge\omega}{\to} \Lambda^{k+2}(\mathbb{R}^{2m}) \to H^{k+2}(L) \to \cdots$$

and Lefschetz lemma from 3.3.

We conclude by observing that A^t acts on L by $\{x_i, y_i, z\} \mapsto \{tx_i, ty_i, t^2 z\}$ and $M^*(h) = \deg h$ for $\deg h = k \leqslant m$ and $M^*(h) = k + 1$ for $\deg h = k > m$.

Questions

(a) It remains unclear what happens to harmonic spaces $\mathcal{H}_{g_\varepsilon^*}(V/\Gamma)$ where g on V/Γ *does not* descend from a *left-invariant* metric on V[13].

(b) What is the behaviour of (harmonic) spinors on L under the action of A and the blow-up of g? (Beware that the spinors themselves, not only Dirac, depend on g.)

4.11.C. Lower bound on the Minkowski dimension of cycles in compact homogeneous spaces (V, H). We want to show that *every two cycles c and c' in V have*

$$\dim_{\mathrm{Min}} c + \dim_{\mathrm{Min}} c' \geqslant N \qquad (*+)$$

whenever $c \cap c' \neq 0$ and thus

$$M_+(k) + M_+(n-k) \geqslant N \qquad (++)$$

if $H_k(V; \mathbb{R}) \neq 0$ (compare 4.10.B$'$). To prove this we invoke our anisotropic blow-up g_ε^* of a back-ground metric g on V where we observe that the k-dimensional volume of every k-dimensional homology class c in V satisfies

$$\dim_{\mathrm{Min}} c < M \Rightarrow \mathrm{Vol}_{g_\varepsilon} c = o(\varepsilon^{-M}).$$

Now $(*+)$ follows from the following.

[13] This was clarified in [$\mathrm{Ge}_{\mathrm{ALRC}}$].

Homogeneous antisystolic inequality. *Let V be a compact homogeneous Riemannian manifold and c and c' be two cycles of complementary dimensions in V. Then their intersection index is bounded by*

$$c \cap c' = \mathrm{const}_n (\mathrm{Vol}\, c)(\mathrm{Vol}\, c')/ \mathrm{Vol}\, V. \qquad (\star)$$

Proof. Let us translate c and c' by the isometry group G of V and average them as currents over the Haar measure of G. The resulting currents are G-invariants, hence smooth, and are given by closed differential forms, say ω and ω' representing dual cohomology classes of c and c'. In particular, $[\omega \wedge \omega'](V) = c \cap c'$. On the other hand the pointwise norm of ω is (obviously) bounded by

$$\|\omega\| \leqslant \mathrm{const}'_n \,\mathrm{Vol}\, c/ \mathrm{Vol}\, V$$

and similarly,

$$\|\omega'\| \leqslant \mathrm{const}'_n \,\mathrm{Vol}\, c'/ \mathrm{Vol}\, V.$$

This implies the desired bound on $c \cap c' = [\omega \wedge \omega'](V)$.

4.11.C$'$. Small harmonic forms and a lower bound on $\dim_{\mathrm{Min}}$ for non-compact homogeneous spaces. We want to make the above argument more constructive by exhibiting invariant forms ω with large $M^*(\omega)$, (or at least with small pointwise norms with respect to g^*_ε) without any "ad absurdum" assumptions on cycles in V. Recall that the closed forms ω with minimal *L_2-norms* in their respective cohomology classes are the harmonic ones and since V is homogeneous the harmonic ω are invariant and so their pointwise norms are related to the L_2-norms by

$$\|\omega\|_{L_2} = \|\omega\| (\mathrm{Vol}\, V)^{\frac{1}{2}}.$$

Furthermore, the L_2-norms on the spaces harmonic forms of complementary dimensions are *Poincaré dual* which means the existence of L_2-*orthonormal* bases $h_\mu \in \mathcal{H}^k_{g^*_\varepsilon}$ and $h'_\nu \in \mathcal{H}^{n-k}_{g^*_\varepsilon}$, such that $h_\mu \cup h'_\nu = \delta_{\mu\nu}$. This applies to all metric g^*_ε for $\varepsilon \to 0$ and gives us sufficiently many "small" forms ω to prove $(\star)$. This also gives us a bound on $\|c\|_\varepsilon$ for straight cocycles c's representing h's, as every ω on V with $\|\omega\| \leqslant \varepsilon^{-M}$ integrates to a straight cochain (cocycle if $d\omega = 0$) c with $\|c\|_\varepsilon \leqslant \mathrm{const}\, \varepsilon^{-M}$ for "const" independent of ε. Unfortunately, the harmonic forms $h \in \mathcal{H}^*_{g^*_\varepsilon} = H^*(V)$ where we have good bound on $\|h\|$ may (?) a priori strongly vary with ε and so we can not get a (interesting) bound $\|c\|_\varepsilon \lesssim \varepsilon^{-M}$ for a fixed c and all $\varepsilon \to 0$. But yet we have the following weak form of $(*)$ from 4.10.B.

There exist Poincaré dual bases $h_\mu \in H^k$ and $h'_\nu \in H^{n-k}$, a sequence $\varepsilon_i \to 0$ and sequences of straight cocycles $c_\mu(i)$ and $c'_\nu(i)$, $i = 1, 2, \dots$, such that

$$(i) \quad [c_\mu(i)] \to h_\mu \quad \text{and} \quad [c'_\nu(i)] \to h'_\nu \quad \text{for} \quad i \to \infty,$$

$$(ii) \quad \|c_\mu(i)\|_\varepsilon \leqslant \operatorname{const} \varepsilon^{-M_\mu(i)} \quad \text{and} \quad \|c'_\nu(i)\| \leqslant \operatorname{const} \varepsilon^{-M'_\nu(i)}$$

for some $M_\mu(i)$ and $M'_\nu(i)$ satisfying

$$M_\mu(i) + M'_\mu(i) \leqslant N \ , \ i = 1, 2, \dots \tag{$*-$}$$

and converging to some constants, say $M_\mu(i) \to M^-(h_\mu)$ and $M'_\nu(i) \to M^-(h_\nu)$ for $i \to \infty$.

Now we observe that the above applies to equivariant forms on arbitrary (not necessarily compact) homogeneous space (V, H, g) in-so-far as the equivariant cohomology satisfies the Poincaré duality. For example, this is the case if V is a *unimodular* Lie group, i.e. Trace $ad_x = 0$ for every $x \in L = L(V)$. Then "small" forms and (straight cycles) descend to the quotients V/Γ for all discrete subgroups Γ in V and lead to non-trivial lower bound on the Minkowski dimension on the homology $H_*(V/\Gamma)$. For example, if $H^k(L) \neq 0$ (where L is the Lie algebra of V), *then the maximal Minkowski dimensions of the k-cycles in $H^k(V/\Gamma)$ and $H_{n-k}(V/\Gamma)$ satisfy, as earlier, the "anti-systolic" inequality*

$$M_+(k) + M_+(n-k) \geqslant N (= \dim_{\mathrm{Hau}}(V, H)).$$

In particular, if $k = n/2$, then there exists a class $h \in H_k(V/\Gamma)$ with $\dim_{\mathrm{Min}} h \geqslant N/2$.

Remark. It is unclear with this argument if $M_+(k)$ is an integer and/or if the numbers $M_+(k)$ and $M_+(n-k)$ can be minorized by integers $M'_+(k)$ and $M'_+(n-k)$ satisfying $M'_+(k) + M'_+(n-k) \geqslant N$ (compare $(++)'$ in 4.10.B$'$).

On the upper systolic bound. Let $M_-(k)$ denotes the infimum of the Minkowski dimensions of the non-zero homology classes in $H_k(V; \mathbb{R})$. If $k = 1$ and $H_1(V; \mathbb{R}) \neq 0$, then, obviously,

$$M_-(1) + M_-(n-1) \leqslant N$$

as $M_-(1) = 1$ and $M_-(n-1) = N-1$, but this inequality seems unlikely to be true in general (even with $\dim_{\mathrm{Hau}}$ instead of $\dim_{\mathrm{Min}}$) for $2 \leqslant k \leqslant n-2$.

In fact, one expects a strong failure of intersystolic inequalities for the (anisotropically blown-up) manifolds $V_\varepsilon^* = (V, g_\varepsilon^*)$, where, conjecturally,

$$\mathrm{syst}_k(V_\varepsilon^*)\ \mathrm{syst}_{n-k}(V_\varepsilon^*)/\operatorname{Vol}(V_\varepsilon^*) \to_{\varepsilon\to 0} \infty,$$

for most (V, H) and $2 \leqslant k \leqslant n-2$, (contrary to what happens to the *stable systoles* where the volume of the cycles is replaced by the $\mathbb{R}$-*mass*, compare [Gro$_{\mathrm{SISI}}$]). Unfortunately, we have no way of evaluating the volumes of the homology classes in $H_k(V_\varepsilon^*)$ for $\varepsilon \to 0$ even for homogeneous (V, H) (while the $\mathbb{R}$-mass is computable in the homogeneous case by averaging over $G = \operatorname{Iso} V_\varepsilon^*$).

Remark. The systolic situation in V_ε^* is somewhat similar to that in *almost Kähler manifolds* V where the underlying complex structure is non-integrable (while the structure 2-form is closed). There , one easily sees the failure on the sharp isosystolic inequality for even $k \geqslant 4$ but does not know if non-sharp inequality remains valid (compare [Gro$_{\mathrm{SISI}}$] and [Gro$_{\mathrm{MCKM}}$]).

4.11.C$''$. On the limit of $\mathcal{H}_{g_\varepsilon^*}$ for $\varepsilon \to 0$. Let L be a Lie algebra with a (horizontal) subspace $H \subset L$ which Lie generates L and $H = H_1 \subset H_2 \subset \cdots \subset H_d = L$ be the filtration by the commutators, i.e. $H_i = [H, H_{i-1}]$. We denote by $L^\bullet$ the corresponding graded space $L^\bullet = \oplus_{i=1}^d H_i/H_{i-1}$ with the natural structure of a graded nilpotent Lie algebra. We give L a metric g, identify $L^\bullet$ with L via $H_i/H_{i-1} = H_i \ominus_g H_{i-1}$, introduce the linear operators $A^t : L^\bullet \to L^\bullet$ acting by t^i on $H_i \ominus H_{i-1}$ and let $g_\varepsilon^* = (A^t)^* g$ for $t = \varepsilon^{-1}$. Then we observe that the space of g_ε^*-harmonic forms of $L = L^\bullet$ say $\mathcal{H}_\varepsilon \subset \Lambda^*(L)$ converges, as $\varepsilon \to 0$, to a space $\mathcal{H}_0 \subset \Lambda^*(L)$ which is contained in the space $\mathcal{H}(L^\bullet) \subset \Lambda^*(L^\bullet = L)$ of the harmonic form of the algebra $L^\bullet$. This leads to the following.

Questions

(1) When (if) does $\mathcal{H}_0$ non-trivially intersect the image of the differential $d = d_L$ on $\Lambda^*(L)$?

(2) In which case does there exist a natural boundary operator on the cohomology of $L^\bullet$ say $d^\bullet$ on $H^*(L^\bullet)$, such that the cohomology of the complex $(H^*(L^\bullet), d)$ equal $H^*(L)$? (We know $d^\bullet$ does exist for $\operatorname{rank} L/H = 1$ by a Lie algebraic adaptation of the Rumin complex, compare example in 4.10.B$''$, but in the general case one may need a spectral sequence rather than a single complex, compare [Br-Gr].)

Next we pass to the anisotropic blow-up of a manifold $(V, H, g) \to (V, H, g_\varepsilon^*)$ and observe that for each point $v_0 \in V$ there is a (pointed) limit of $(V, v, H, g_\varepsilon^*)$ for $\varepsilon \to 0$ to a graded nilpotent Lie group $(V_v, v, H(v), g(v))$ where harmonic forms on V subconverge to left-invariant harmonic forms on V_v. This suggests (generalizing the Rumin complex) some operator $d^\bullet$ on sections of the bundle of the cohomologies of the Lie algebras $L_v = L(V_v)$ but I have nothing to say in this regard. (There is Vinogradov's spectral sequence in this picture, see [$\mathrm{Vin}_{1;2;3}$], [Br-Gr], but its role in our game is unclear.)

5. Anisotropic connections

Let V be a smooth manifold with a polarization $H \subset T(V)$ and $p : X \to V$ be a G-bundle for some Lie group G. Geometrically speaking, an *H-connection* in X is given by parallel transport in X along *H-horizontal* curves in V or, equivalently, rectifiable curves for the C-C geometry in V corresponding to H. Infinitesimally, an H-connection ∇ in a principal bundle X is defined by a G-invariant subbundle $\tilde{\nabla} \subset T(X)$ such that the differential $\mathcal{D}_p : T(X) \to T(V)$ sends each $\tilde{\nabla}_x$ *isomorphically* onto H_v for $v = p(x) \in V$.

5.1. Curvature Ω_∇. Recall the curvature $\Omega_H : H \wedge H \to H^\perp = T(V)H$ (which can be represented by k scalar 2-forms $\omega_1, \ldots, \omega_k$ for $k = \operatorname{rank} H^\perp$ for $\omega_i = d\eta_i$ where $\eta_1, \ldots, \eta_k$ are 1-forms on V locally defining H, i.e. $H = \bigcap_i \operatorname{Ker} \eta_i$) and let $K \subset H \wedge H$ denote the kernel of Ω_H. We denote by $L(X) \to V$ the adjoint Lie algebra bundle over V and we think of linear forms on K with values in $L(X)$ as of partially defined $L(X)$-valued 2-forms on H. In particular, we have the curvature tensor $\Omega_\nabla : K \to L(X)$ which is such a form defined as follows. Let $k \in K_v \subset K$ be represented by $k = (\sum_i h_i \wedge h_i')(v)$ for some H-horizontal vector fields h_i near v. Then, by the definition of Ω_H, its vanishing at k means that

$$\sum_i [h_i, h_i'] \in H.$$

Then we lift all h_i and h_i' to $\tilde{\nabla}$-horizontal G-invariant fields $\tilde{h}_i$ and $\tilde{h}_i'$ on X and take the sum of their Lie brackets, $\sum_i \left[\tilde{h}_i, \tilde{h}_i'\right]$. Since the bracketing agrees with the lift, the field $\sum_i \left[\tilde{h}_i, \tilde{h}'_i\right]$ lies in $\tilde{H} = (\mathcal{D}_p)^{-1}(H) \subset T(X)$ and as $\tilde{\nabla}$ also lies in $\tilde{H}$ we may project to the vertical bundle $T_{\mathrm{ver}}(X) = \tilde{H}/\tilde{\nabla}$ consisting of the vectors tangent to the G-fibers of X. This projection clearly is G-invariant and thus defines a vector in $L(X)$ at v which is just the desired value of Ω_∇ at k.

Remarks

(a) Partial connections in this context were introduced by Z. Ge in [Ge$_{\mathrm{BNCC}}$] who equivalently defines Ω_∇ as an $L(X)$-valued 2-form on V modulo the differential ideal generated by H i.e., by the 1-forms vanishing on H.

(b) Intuitively, the value of the curvature on an infinitesimal disk $D \subset V$ tangent to a bivector equals the holonomy around the boundary circle S of D. In our case this only makes good sense for H-horizontal disks D which necessarily have $\Omega_H|D = 0$ and thus the domain of Ω_∇ is restricted to the kernel of Ω_H.

5.1.A. On the equation $\Omega_\nabla = 0$. To grasp the meaning of Ω_∇ let us analyze the equation $\Omega_\nabla = 0$. Unlike the usual case this equation does not always imply local splitting (triviality) of (X, ∇). For example, every connection ∇ over a three-dimensional contact manifold (V, H) has $\Omega_\nabla = 0$, since the form Ω_H on $H \wedge H = \mathbb{R}$ has trivial kernel K. But *if every small smooth generic H-horizontal closed curve S bounds an H-horizontal disk D in V, then the vanishing of the curvature makes the holonomy over every small horizontal curve trivial.* In fact, the connection ∇ splits over D whenever Ω_∇ vanishes on D.

On the other hand, *if ∇ has trivial holonomy over all smooth closed horizontal curves in some domain V_0 in V then ∇ splits over V_0.* To construct a splitting, i.e. a horizontal frame field over V_0, we take such a frame f_0 at a point $v_0 \in V_0$ and then parallelly transport it to other $v \in V_0$ along horizontal paths in V_0 (where we assume V_0 is connected). Since the holonomy is trivial, the result of such a transport is independent of the underlying path and defines, indeed, the required frame field over V_0.

Corollary. *Let H admit a regular H-horizontal jet of a 2-germ at each point $v \in V$ (see 4.2). Then the equation $\Omega_\nabla = 0$ makes ∇ locally split.*

Proof. The existence of regular jets ensures filling curves by disks (see 4.7).

Remark. The above Corollary shows, in particular, that the complex $(\Lambda^*(V)/I(H), d)$ is locally exact in degree one, where $I(H)$ denotes the differential ideal spanned by H. One may try a similar argument in higher degrees, using, for example, *h-horizontal* triangulations Tr of V, where all simplices up to dimension k are horizontal. Then every i-form ω with $d\omega = 0 \operatorname{mod} I(H)$ for $i < k$ gives us a closed cochain which is locally a coboundary, say $\omega_\nabla = \delta u$ and then one may try to go back from cochains in Tr to forms in Λ^*/I by either some smoothing process, or by scaling Tr to Tr_ε with simplices of size $\varepsilon \to 0$.

Example. Suppose we want to find a 1-form $\lambda \operatorname{mod} I(H)$ so that $d\lambda = \omega \operatorname{mod} I(H)$ where $d\omega = 0| \operatorname{mod} I(H)$. If we have in our possession local filling of circles by disks and 2-spheres by 3-balls, the given information on λ is encoded (via $d\lambda$) in the integrals of λ over all (small) closed H-horizontal circles. To have the full $\lambda \operatorname{mod} I(H)$ we need the integrals of λ on all (not only closed) horizontal curves and so we are faced with an extension problem à la Hahn-Banach and the above triangulation may suggest a suitable functional set-up (compare our abortive discussion in 4.11).

Questions

(a) Are there examples of polarizations H with high dimensional kernels $K \subset H \wedge H$ of Ω_H admitting non locally split connections ∇ with $\Omega_\nabla = 0$ (on K)? (Somewhat non-convincing examples are suggested by *integrable* polarizations H and by Cartesian products of 3-dimensional contact manifolds.)

(b) Recall that the implication

$$\Omega_\nabla = 0 \Rightarrow (\nabla \text{ locally splits}) \qquad (\Rightarrow)$$

for $\mathbb{R}$-bundles is equivalent to the local exactness of the complex $(\Lambda^*(V)/I(H), d)$ in degree one. Does this local exactness yield $(\Rightarrow)$ for all ∇ (with possibly non-Abelian structure groups)?

Notice that ∇ is locally trivial if and only if it has trivial holonomy along all small closed H-horizontal curves and so a trivialization of ∇, if it exists, is essentially unique. This shows that $\Omega_\nabla = 0$ is an integrable system in the sense of Frobenius and so verification of ($\Rightarrow$) reduces to an infinitesimal computation. To see this more explicitly, let us observe that ($\Rightarrow$) is equivalent to a possibility of extending our H-connection ∇ over (V, H) with $\Omega_\nabla = 0$ to an actual connection (define over all of $T(V)$), say ∇^+ with $\Omega_{\nabla^+} = 0$. Such an extension, if it exists, is unique and can be constructed as follows. We recall the filtration $H = H_1 \subset H_2 \subset \cdots \subset H_j \subset \cdots \subset H_d = T(V)$ where H_j is generated by the commutators of H-horizontal fields of order j. Then we extend ∇ from $H_1 = H$ to $\nabla^{(2)}$ on H_2 according to the formula

$$\nabla^{(2)}_{[h_1,h_2]} = \nabla_{h_1}\nabla_{h_2} - \nabla_{h_2}\nabla_{h_1} \tag{+}$$

for h_1, h_2 in H and then proceed in the same way for H_3, H_4, etc up to $\nabla^+ = \nabla^{(d)}$. Notice that the vanishing of Ω_∇ makes (+) correctly defined, and, in order to proceed further, $\nabla^{(2)}$ must be an actual orthogonal connection (which is by no means automatic) and its H_2-curvature must be zero. Then one can extend $\nabla^{(2)}$ to $\nabla^{(3)}$ on H_3 etc. Thus the local triviality of ∇ amounts to a chain of differential relations incorporating the equations $\Omega_\nabla = 0$, $\Omega_{\nabla^{(2)}} = 0, \ldots$, as well as the "orthogonal connection" properties of $\nabla^{(i)}$ for $i \geq 2$. Notice that in this chain each following relation makes sense due to the preceding ones.

Characteristic forms. Z. Ge points out in [Ge$_{\mathrm{BNCC}}$] that the curvature Ω_∇ can be used to define in the obvious way the Chern-Weil characteristic classes of our bundle X over V in the cohomology of the complex $(\Lambda^*(V)/I(H), d)$. For example, if V is a contact manifold, then this cohomology below the middle dimension is canonically isomorphic to the de Rham cohomology by Rumin theorem and so these classes land in the ordinary cohomology of V where they coincide with the classical characteristic classes. It is unclear what applications these classes may have.

Remark. If one does not insist on the curvature of ∇ but allows higher order infinitesimal formulae one may hope to recapture *ordinary* de Rham representatives of the characteristic classes of X in terms of an H-connection ∇ for an arbitrary H (Lie spanning $T(V)$). This was pointed out to me by A. Šwarc who was interested in quantization of Chern-Simons kind of invariants over 3-dimensional contact manifolds.

5.2. Norms and metric associated to orthogonal connections. From now on we assume G is a compact group and $X \to V$ is a G-vector bundle. In fact, we make no essential use of G anymore and work with an arbitrary Euclidean bundle with a Euclidean H-connection ∇ on V (i.e. $G = O(r)$ for $r = \operatorname{rank} X$ and the standard representation of $O(r)$ on $\mathbb{R}^r$). Now we may speak of the pointwise norm $\|f(v)\|$ of a section $f : V \to X$ and, if H is endowed with a Riemannian metric, which we always assume, we also have the norm $\|\nabla f(v)\|$ (which reduces to $\|\mathcal{D}f|H\|$ for the trivial (split) bundles (X, ∇)).

The local Sobolev inequalities (see 2.4) for sections with (small) compact support extend to these norms,

$$\|f\|_{L_p} \leq \text{const}\,\|\nabla f\|_{L_q} \qquad (*)_q$$

for all q in the interval $1 \leq q < N = \dim_{\mathrm{Hau}} V$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{N}$. This follows from 2.4 applied to the function $\|f(v)\|$ on V. An interesting feature here is the independence of const of ∇.

Next we want to study the (Hölder) continuity of f with $\|\nabla f\|_{L_q} < \infty$ for $q \geq N$. For this we need some metric on X compatible with ∇. An obvious candidate is obtained as follows. Let $\tilde{H} = (\mathcal{D}p)^{-1}(H) \subset T(X)$ and observe that $\tilde{H} = T_{\mathrm{ver}}(X) \oplus \tilde{\nabla}$. The subbundle $\tilde{\nabla}$ inherits a metric from H while $T_{\mathrm{ver}}(X)$ has a metric from the Euclidean structure on X. Thus $\tilde{H}$ also has a Riemannian metric and we give X the associated C-C metric.

Warning. *We have not made any regularity assumptions on ∇ and the above* C-C *metric on X may degenerate for a general Borel measurable ∇. For example, if arbitrarily short loops at some point $v \in V$ have definite holonomy, then this metric may become zero for some pairs of distinct points in the fiber $X_v \subset X$. Yet this does not happen for continuous (or even bounded) connections ∇. (Recall that every H-connection ∇ can be written locally as (trivial connection) + ($L(G)$-valued* 1*-form on H) and the continuity of ∇ refers to this* 1*-form viewed as a section of a bundle over V.)*

Observation. *Every section f with $\nabla f \in L_q$ for $q > N$ is C^α with $\alpha = (q - N)/N$ where the implied bound on the Hölder constant $L_\alpha(f) \leq$ const $\|\nabla f\|_{L_q}$ has const independent of ∇.*

This is shown by the same straightforward computation as was used in 2.3.E, 2.4 and 2.5 with "standard pencils" of H-horizontal curves between given points v_1 and v_2 in V. A generic curve in such a pencil, say $\gamma = \gamma(t)$, has well controlled integral $\int_\gamma \|\nabla_t f(\gamma(t))\| dt$ which implies the desired bound on L_α.

The case $q = N$. If our H-connection ∇ is continuous (or at least Borel measurable bounded) then the metric in X is locally equivalent to the product metric on $V \times \mathbb{R}^m$, $m = \operatorname{rank} X$, and so our regularization from 2.5 applies here. Thus every section f with $\|\nabla f\|_{L_N} < \infty$ can be approximated by continuous sections f_ε, $\varepsilon \to 0$, where "approximation" means a.e. convergence as well as $\|\nabla f - \nabla f_\varepsilon\|_{L_N} \to 0$ for $\varepsilon \to 0$. Furthermore, if f is *taut*, in the sense that no deformation with small support diminishes $\|\nabla f(v)\|$, then the logarithmic modulus of continuity of f is bounded in terms of $\|\nabla f\|_{L_N}$. Furthermore, if f is locally $\|\nabla f\|_{L_N}$-minimizing (or only quasi-minimizing) then it is Hölder.

(codim 1)-*Reminder*. All these properties of $\|\nabla f\|_N$ were established earlier (for split connections) under the (codim 1)-stability assumption on (V, H) and this assumption is needed in the present (non-split) case as well.

Generalization. The above Hölder continuity results for $q > N$ as well as for $q = N$ make sense for an arbitrary (non-vector) G-bundle where the fiber W is a G-manifold with a G-invariant metric (and where we do not have to assume G is compact). Our Hölder observation for $q > N$ obviously extends to this case with const independent of W and ∇; furthermore, the above statements (and their proofs) for $q = N$ also generalize with no difficulty if W has locally bounded geometry. This leads to the following

Theorem. *Let (V, H) be* (codim 1)*-stable. Then*

(i) *the existence of a measurable section $f : V \to X$ with $\|\nabla f\|_{L_N} \leq c < \infty$ implies the existence of a continuous section $f' : V \to X$ with $\|\nabla f'\|_{L_N} \leq c + \varepsilon$ for an arbitrary small $\varepsilon > 0$;*

(ii) *if V and W are compact and $\pi_1(W)$ acts trivially on $\pi_n(W)$ for $n = \dim V$, then there are at most finitely many homotopy classes of sections $f : V \to X$ with $\|\nabla f\|_{L_N} \le c$ for every constant c;*

(iii) *if V and W are compact and $\pi_1(W)$ acts trivially on $\pi_i(W)$ for $i = n, n+1, \ldots$, then the space F_c of continuous sections $f : V \to X$ with $\|\nabla f\|_{L_N} \le c$ has finite homotopy type in the space $F = F_\infty \supset F_c$ of all continuous sections.*

Notice that all this is well known in the Riemannian case (i.e. where $H = T(V)$) and due to K. Uhlenbeck.

Remark on Kac-Feynman formula and Kato inequality. One knows that the heat flow in a vector bundle with a Euclidean connection ∇ over a Riemannian manifold V decays the slowest if the bundle is split, which gives one a lower bound on the spectrum of $\nabla^*\nabla$ in terms of the Laplacian spectrum of V. Apparently, this generalizes to our H-connections. Furthermore one expects similar bounds for the (non-linear) L_p- and L_{pq}-spectra in the sense of [$\mathrm{Gro}_{\mathrm{DNLS}}$].

5.3. L_q-distance in the space of connections. Let X_1 and X_2 be Euclidean vector bundles of the same dimension over V and $Y = \mathrm{Iso}(X_1, X_2)$ be the (non-vector) bundle of isometric homomorphisms $X_1 \to X_2$. If X_1 and X_2 come along with H-connections ∇_1 and ∇_2 then these give rise to a natural H-connection ∇ on Y and so one may speak of the norms $\|\nabla f\|_{L_q}$ of our (smooth, continuous or measurable) homomorphisms $f : X_1 \to X_2$ viewed as (smooth, continuous or measurable) sections $f : V \to Y$.

Example. A smooth homomorphism f has $\|\nabla f\|_{L_q} = 0$ if and only if it sends the connection ∇_1 to ∇_2.

To get a clearer picture of $\|\nabla f\|$ we identify X_1 and X_2 by f and thus define the difference $\nabla_1 - \nabla_2$ which is an $L(G)$-valued form on V. Thus we see that $\|\nabla f\| = \|\nabla_1 - \nabla_2\|$ which implies that

(i) $\|\nabla f\|_{L_q} = \|\nabla f^{-1}\|_{L_q}$,

(ii) $\|\nabla(f_1 \circ f_2)\|_{L_q} \le \|\nabla f_1\|_{L_q} + \|\nabla f_2\|_{L_q}$, for $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$.

It follows that $\inf_f \|\nabla f\|_{L_q}$ over all measurable $f : X_1 \to X_2$ defines a metric (which may be sometimes infinite) on the set of isomorphism classes of bundles with H-connections over V. This metric is called L_q and denoted dist_{L_q}.

Example. If V is (codim 1)-stable and $\mathrm{dist}_{L_N}(X_1, X_2) < \infty$ for $N = \dim_{\mathrm{Hau}} V$, then X_1 and X_2 are isomorphic as vector bundles since the existence of a measurable section $f : V \to Y = \mathrm{Iso}(X_1, X_2)$ with $\|\nabla f\|_{L_N} < \infty$ implies the existence of a continuous one.

5.3.A. On $\mathbf{L_q}$-non-flatness of $\boldsymbol{\nabla}$. Let X be a Euclidean bundle with an H-connection ∇ over (V, H) and let us discuss several invariants of ∇ measuring its L_q-distance from a (locally) split connection.

Example: connections over $\boldsymbol{S^n}$. Let V be homeomorphic to the n-sphere covered by two balls B_+ and B_- and d_+ and d_- be the L_q-distances from the bundles $(X, \nabla)|B_+$ and $(X, \nabla)|B_-$ to the split bundles over these balls. This means we have orthonormal frames f_+ over B_+ and f_- over B_- with $\|\nabla f_\pm\|_{L_q} \le d_\pm + \varepsilon$. Then we have the usual map of the annulus $A = B_+ \cap B_-$ to the structure group $G = O(r)$, $r = \mathrm{rank}\, X$, call it $\varphi : A \to G$ (defined by the relation $\varphi f_+ = f_-$), which clearly satisfies

$$\|\mathcal{D}\varphi|H\|_{L_q} \le d_+ + d_- + 2\varepsilon.$$

Then the same inequality is satisfied (up to a constant) on some $(n-1)$-sphere of this annulus $A = I \times S^{n-1}$, and so, for example, if $d_+ + d_- \le c_0$ for some small $c_0 > 0$, then this map $\varphi : S^{n-1} \to G$ is *contractible* provided $q > N - 1$ and (V, H) is $(n-1)$-stable (where N is the Hausdorff dimension of V and V is assumed equiregular, compare 2.5). Now, if our frames f_+ and f_- where *continuous*, this would make the bundle $X \to V$ *topologically trivial* but this is not so for general measurable f_+ and f_-. Yet, if $q \ge N$, we can always make f_+ and f_- continuous (see 5.2.) and then the triviality of $X \to V$ is ensured whenever $d_+ + d_- \le c_0$. Furthermore, if $q \ge N$, we also see that the number of mutually non-equivalent bundles over $V = S^n$ which may have $d_+ + d_- \le c$ for some H-connection ∇ on X is finite and bounded by some constant depending on c.

Let us generalize the above to an arbitrary compact V which is covered by finitely many balls $B_i \subset V, i = 1, \dots, m$. We measure the non-flatness of ∇ by the L_q-distances of $(X,\nabla)|B_i$ from split bundles, call them d_i, and claim that in the (codim 1)-stable case and for $q \geq N$ the sum $\sum_{i=1}^m d_i$ gives us a bound on a possible (topological) type of X. Namely

(I) *There are at most finitely many mutually non-equivalent bundles X over V with $\sum_{i=1}^m d_i < c$, for every fixed $c > 0$, and their number is bounded by some constant depending on c (as well as on V, B_i, H and rank X).*

(II) *There exists a positive constant $c_0 = c_0(V, B_i, H, \operatorname{rank} X)$, such that the inequality $\sum_{i=1}^m d_i \leq c_0$ implies that X admits a locally split (flat) connection. In particular, if V is simply connected then $X \to V$ is topologically trivial.*

Proof of (I). Every frame f_i on B_i is given by $r = \operatorname{rank} X$ sections, say f_i^j, $i = 1, \ldots, m$, $j = 1, \ldots, r$, and each of these sections can be cut off near the boundary of B_i and thus extended to all of V. These sections define a homomorphism of the split bundle of rank mr into X, say $\pi : Y \to X$, which is easily seen to be surjective and having $\|\nabla\pi\|_{L_q} < c'$, where $c' \leq c_1 c$ for some $c_1 = c_1(V, B_i)$. We assign to each $v \in V$ the orthogonal complement of the kernel of π in the fiber $Y_v = \mathbb{R}^M$, $M = mr$, that is $v \mapsto (\ker \pi_v)^\perp \subset \mathbb{R}^M$, and thus map V to the Grassmann manifold, say by $\psi : V \to Gr_r\mathbb{R}^M$. As we can assume the frames f_i continuous (for $q \geq N$) this ψ is continuous and obviously is a classifying map for $X \to V$, i.e. it induces on V an r-bundle isomorphic to X. In fact, our extended sections f_i^j define a fiberwise injective morphism of X to the canonical bundle over $\mathrm{Gr}_r\,\mathbb{R}^M$, say $\tilde\psi : X \to X_{\mathrm{can.}}$ (To see this, one should actually use the dual frames of linear functions on X which give us an injective homomorphism $X \to Y$ but this duality may be absorbed by the Euclidean structure in the fibers.) Then it is easy to see that $\tilde\psi$ and ψ are L_q-controlled over V by the L_q-norms of ∇f_i^j and for ψ; this means

$$\|\mathcal{D}\psi|H\|_{L_q} \leq c''$$

for some c'' determined by c. This bounds the homotopy class of ψ (see 2.5.B) and thus the equivalence class of X. Q.E.D.

Remark. The above argument appeals to the L_q-non-flatness of (X,∇) measured by injective homomorphisms $h : X \to Y$, where Y is a split bundle, and the non-flatness is measured by

$$\|\nabla h\|_{L_q} + \|h\|_{L_q} + \|h^{-1}\|_{L_q}, \tag{$*$}$$

where h^{-1} denotes the inversion of h on its image in Y. If $\operatorname{rank} Y = \operatorname{rank} X$ this non-flatness is essentially the same as the L_q-distance between X and Y.

Question. Let V be compact and X be (known) topologically trivial, e.g. V be contractible. Suppose there is an injective $h : X \to Y$, where $\operatorname{rank} Y > \operatorname{rank} X$, with $\|\nabla h\|_{L_q} + \|h\|_{L_q} + \|h^{-1}\|_{L_q} \le c$. Can one bound the L_q-distance of X to the split bundle of rank $r = \operatorname{rank} X$ in terms of c?

Exercise. Assume the frames f_i *continuous* and prove (I) for all $q > N - 1$.

Proof of (II). Our frames, which we may assume continuous, define a 1-cocycle φ on the nerve of the covering of V by B_i with values in the sheaf of continuous maps of V into the structure group $G = O(r)$, where each implied map $\varphi_{ij} : B_i \cap B_j \to G$ is defined by the formula $\varphi_{ij} f_i = f_j$ on $B_i \cap B_j$. If f_i and f_j have small norms $\|\nabla\|_{L_q}$, then the norm $\|\mathcal{D}\varphi_{ij}|H\|_{L_q}$ is also small and then, for $q > N$, this φ_{ij} is uniformly ε-close to a constant map, where $\varepsilon \to 0$ for $c_0 \to 0$. Then a simple limit argument (left to the reader) allows us to construct a new cocycle $\bar{\varphi}$, where all $\bar{\varphi}_{ij}$ are *constant* maps $B_i \cap B_j \to G$, which gives us the required flat structure in X. Finally, the case $q = N$ in the (codim 1)-stable case is handled with a (taut) regularization as earlier (where the details are left to the reader).

Remarks

(a) If one applies the proof of (I) to a flat bundle, one concludes to the (well known) finiteness of equivalence classes of such bundles over V (which may be derived from the finiteness of the number of connected components of representations $\pi_1(V) \to G = O(r)$).

(b) The proof of (II) suggests the infimum of L_q-norms of cocycles defining X over V (covered by B_i) as a measure of non-flatness of X where ∇ is not present any more.

Exercise. Prove (I) by the cocycle argument employed in the proof of (II).

5.3.B. Non-flatness measured by horizontal monodromy and curvature. The *monodromy* of an H-connection ∇ at a closed H-horizontal curve S in V is the conjugacy class of the ∇-parallel transport along S, denoted $\nabla(S) \in G/\mathrm{Conj}$ for the structure group G $(=O(r))$. The function $S \mapsto \nabla(S)$ is the basic invariant of ∇ which can be made numerical by composing with some $\sigma : G/\mathrm{Conj} \to \mathbb{R}$ and integrating over a measure on the space of horizontal curves S in V. An important function σ in this regard is the "norm" $\|g\| \underset{\mathrm{def}}{=} \mathrm{dist}(g, \mathrm{id})$ for the standard bi-invariant metric on $G = O(r)$ corresponding to the Killing form. This "norm" is subadditive on closed curves: if S is decomposed into S_i, $i = 1, \dots, m$, then

$$\|\nabla(S)\| \leq \sum_{i=1}^{m} \|\nabla S_i\|,$$

where a typical decomposition is as in Fig. 14

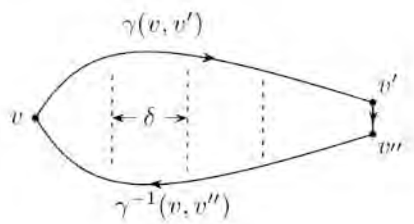


Figure 14

The values of $\nabla(S)$ on infinitesimal curves at a point $v \in V$ can be expressed in terms of suitable curvatures of ∇ at v and their jets. In particular, if S bounds our H-horizontal disk D, then $\|\nabla(S)\| \leq C \int_D \|\Omega_\nabla\|$ for a universal constant C depending on how we normalize (the norm of) the curvature. In what follows, we absorb C into such a normalization, i.e. make it 1.

If the function $\|\nabla(S)\|$ vanishes on all closed horizontal curves, then, obviously, the bundle (X, ∇) splits and, in particular, X is topologically trivial. Furthermore, if $\|\nabla(S)\| = 0$ on all sufficiently short curves, then (X, ∇) locally splits and so X is a flat bundle coming from some representation $\pi_1(V) \to G$. Then an obvious limit argument shows that if V is compact then there exists $\varepsilon > 0$, such that

(I) *if $\|\nabla(S)\| \leq \varepsilon$ for all closed horizontal curves S in V then X is topologically trivial*;

(II) *if* $\|\nabla(S)\| \leq \varepsilon$ *for all short horizontal curves, to be specific for all* S *of length* ≤ 1, *then* X *is equivalent to a flat bundle.*

Corollary. *Let* H *admit an* Ω*-regular* Ω*-isotropic jet of 2-germs at each point* $v \in V$. *Then the inequality* $\|\Omega_\nabla\| \leq c_0$ *for a small* $c_0 = c_0(V) > 0$ *makes the conclusion of (II) hold true.*

Proof. One can fill in short curves S by horizontal disks D of small area (see 4.7) and bound $\|\nabla(S)\|$ by $\int_D \|\Omega_\nabla\| \leq \|\Omega_\nabla\| \operatorname{area} D$.

Remark. The conclusions of (I) and (II) remain valid if the bound $\|\nabla(S)\| \leq \varepsilon$ is assumed only for "standard" closed curves, e.g. having their complexity (see 3.5, 4.7.) bounded by a fixed large constant. Thus one does not need in the above Corollary the full geometric force of the isoperimetric inequality but only the analytic part (filling curves of bounded complexity) depending on the implicit function theorem. Probably, the latter can also be removed from the present discussion with some kind of "almost horizontal disks" filling in "very standard infinitesimal curves".

5.3.C. Radial gauge fixing. Let us make the above remarks clear by using "standard short segments" in V between near points. Namely, we associate to each $v \in V$ a horizontal vector field Y_v in a small ball $B_v(\rho) \subset V$, such that the forward orbits of Y_v converge to v and such that the length of an orbit from $v' \in B_v(\rho)$ to v is bounded by const dist (v, v') (compare 2.3.A). Thus we join every two close by points v and v' in V by a horizontal curve, say $\gamma(v, v')$ (where, in general, $\gamma(v', v) \neq \gamma^{-1}(v, v')$), such that

$$\operatorname{length} \gamma(v, v') \approx \operatorname{dist}(v, v').$$

A *radial gauge* in $B_v(\rho)$ is a frame f of X over $B_v(\rho)$ obtained by parallel transport of a frame at v along the orbits of some radial field Y_v. The values of such f at two near points v' and v'' can be compared with the parallel transport along some (standard) path $\gamma(v', v'')$ and the result of this transport, (which is an element of the group G) equals (up to conjugation) to the monodromy around the triangle $S = \gamma(v, v')\gamma(v'v'')\gamma^{-1}(v, v'')$, see Fig. 15 below.

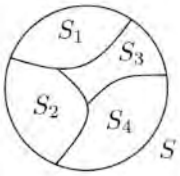


Figure 15

This monodromy can be estimated under the assumptions of the above Corollary by

$$\mathrm{const}\,\|\Omega_\nabla\|(\mathrm{length}\,S)^2 \le \mathrm{const}'\,\|\Omega_\nabla\|\left(\mathrm{dist}(v,v')+\mathrm{dist}(v',v'')\right)^2 .$$

This can be improved for $v'' \to v'$ since for small $\mathrm{dist}(v'',v')$ the triangle becomes narrow and can be filled-in more efficiently. Namely, the distance between the Y_v-orbits of v' and v'' is bounded by $\mathrm{const}_\rho\,(\mathrm{dist}(v',v''))^{\frac{1}{d}}$ where d is the depth of the filtration $H = H_1 \supset H_2 \supset \cdots H_d = T(V))$, see 4.9) and one can subdivide S into m pieces S_i of size $\delta \approx (\mathrm{dist}(v',v''))^{\frac{1}{d}}$ for $m \approx \rho/\delta$. Then we fill in each S_i by a disk D_i of area $\lesssim \delta^2$ and thus obtain a filling D of S with area $\lesssim \rho\,(\mathrm{dist}(v',v''))^{\frac{1}{d}}$. This shows that the radial frame (gauge) is $C^{\frac{1}{d}}$-Hölder with the implied Hölder constant bounded by $C\rho\|\Omega_\nabla\|$ which yields the Corollary by the argument of 5.3.A and also gives a topological bound on X in terms of $c = \sup\|\Omega_\nabla\|$ for every (not only small) $c > 0$ (compare 5.3.A).

5.4. Geometric and topological effects of the bound $\|\Omega_\nabla\|_{L_q} \le c$. We saw in the previous section that the bound $\|\Omega_\nabla\|_{L_\infty} \le c_0$ for small c_0 makes X geometrically almost flat and topologically flat and we want to prove this for some $q < \infty$. The corresponding result for the ordinary connections (where $H = T(V)$) due to Karen Uhlenbeck (see [Uhl]) reads

(I) *Let V be a compact simply connected Riemannian manifold. Then there exists a constant $c_0 > 0$ (depending on V and $r = \mathrm{rank}\,E$) such that every Euclidean bundle (E,∇) of rank r over V with $\|\Omega_\nabla\|_{L_{n/2}} \le c_0$ admits a continuous orthonormal frame f over V satisfying*

$$\|\nabla f\|_{L_n} \le \mathrm{const}\,c_0 \qquad (*)$$

for const = const (V,r).

Remarks

(a) The norm $\|\nabla f\|$ is essentially the same thing as the norm of the $L(G)$-valued 1-form ∇_f representing ∇ in the frame f. The full Uhlenbeck theorem says that the gradient (i.e. the 1-jet) of this form for a suitable f satisfies

$$\|\operatorname{grad} \nabla_f\|_{L_{n/2}} \leq \operatorname{const} c_0 \qquad (**)$$

and then $(*)$ follows from $(**)$ by the Sobolev inequality.

(b) If V is not simply connected, the Uhlenbeck theorem applies to coverings of V by balls and yields a local information on X which can be globalized by the argument employed in 5.2.A. In fact, the proof of the theorem in the general simply connected case is obtained with such a covering.

(c) If the structure group G is Abelian, the Uhlenbeck theorem reduces to the controlled integration of exact 2-forms (with the corresponding Green kernel, see 3.6). Then the general case follows by an implicit function argument.

(d) The Uhlenbeck theorem implies (see 5.2.A) that every bundle (X, ∇) over a compact simply connected Riemannian manifold V with small norm $\|\Omega_\nabla\|_{L_{n/2}}$ is topologically trivial. Consequently every map φ of V into the classifying space $\mathrm{Gr}_r \mathbb{R}^M$ with small $\|\Lambda^2 \mathcal{D}\varphi\|_{L_{n/2}}$ is contractible. (We mentioned earlier that the similar property is unknown for more general simply connected manifolds W in place of $\mathrm{Gr}_r \mathbb{R}^M$.)

(e) Uhlenbeck's theorem has non-trivial implications for the bundles (X, ∇) where $\|\Omega_\nabla\|_{L_{n/2}} \leq c$ for a fixed but not necessarily small $c > 0$. In fact, the norm $\|\Omega_\nabla\|_{L_{n/2}}$ is invariant under scaling the metric in V and so (X, ∇) can be studied over small balls in V rescaled to the unit size. Then almost all of V can be covered by balls B_i with $\|\Omega_\nabla | B_i\|_{L_{n/2}} \leq c_0$ and the remaining "bad part" of V reduces to (arbitrarily small) neighbourhoods of finitely many points in V where (only) Uhlenbeck's bubbling may occur (compare 2.5). The topological corollary of this is *the finiteness of the topological equivalence classes of bundles with* $\|\Omega_\nabla\|_{L_{n/2}} \leq c$. But this corollary can be easily derived by pure topology by observing that the $L_{n/2}$-norm of the curvature bounds the rational characteristic classes of E via the Chern-Weil formula and these classes determine E up to a finite number of possibilities.

(f) Finally we point out (following Uhlenbeck) an (immediate) asymptotic corollary of her theorem.

Let (X, ∇) be a bundle over $\mathbb{R}^n$ with $\|\Omega_\nabla\|_{L_{n/2}} < \infty$. Then X extends to a bundle $X^\bullet$ on the sphere S^n obtained by the 1-point compactification of $\mathbb{R}^n$ and ∇ extends to a measurable connection $\nabla^\bullet$ on $X^\bullet$ which admits a continuous frame f near the infinity $v_\infty \in S^n$ with $\|\nabla^\bullet f\|_{L_n} < \infty$.

Now we can state our basic problem for H-connections ∇ on bundles X over (V, H). We want to know for which q (depending on (V, H)) the bound on $\|\Omega_\nabla\|_{L_q}$ has an effect similar to the conclusion of Uhlenbeck's theorem and issuing corollaries. For example, when does the bound $\|\Omega_\nabla\|_{L_q} < c$ allow at most finitely many topological non-equivalent bundles X and imply triviality of X for small $c = c_0$? What is the integral curvature condition on an H-connection (or an ordinary one) over a simply connected nilpotent Lie group which would allow a good one-point compactification? (This question makes sense for more general open Riemannian and C-C manifolds.)

5.4.A. Gauge theory over contact manifolds. Since Rumin theory provides controlled integration of 2-forms on contact manifolds V of dimension $n \geq 5$ (see 3.6), the full Uhlenbeck package seems to generalize to this case with L_N in place of L_n for $N = n + 1 = \dim_{\mathrm{Hau}} V$. Namely, *every (X, ∇) with small $\|\Omega_\nabla\|_{L_{N/2}}$ must admit continuous frames f with small $\|\nabla f\|_{L_N}$ over simply connected (regions in) V, etc.* (We suggest the reader would check that Uhlenbeck's proof indeed transplants to Rumin's hypoelliptic framework of contact manifolds.)

An especially attractive case is that of $\dim V = 5$, where $\operatorname{rank} H = 4$ and one can define the Yang-Mills equation. In particular, if H is given by a connection in an S^1-bundle over a symplectic 4-manifold V_0, one may lift the Yang-Mills fields from V_0 to V, the total space of the S^1-bundle $V \to V_0$, and try to express Donaldson invariants of V_0 in terms of contact geometry of V. (But this is just a wishful thinking not corroborated by a serious evidence.)

5.4.B. Monodromy control by thick fillings. In order to bound the monodromy over a curve $S \subset V$ it suffices to find a horizontal filling D of S with small integral of $\|\Omega_\nabla\|$ over this D and if we have a bound on $\|\Omega_\nabla\|_{L_q}$ such a D can be chosen as a member of a given q-thick family of fillings of S (see 3.6.D). If we look for such a family consisting of Ω-regular disks, the question becomes essentially (differential) algebraic as jets of such families at S extend to germs by the generalized Nash implicit function theorem but the specific evaluation of the minimal $q = q(H)$ requires a computation in particular cases which was performed so far only for contact manifolds (see 3.6.D). Recall that if our family is given by a smooth map $f : D \times B^{n-2} \to V$ which is regular (i.e. immersive) on the disks and if the Jacobian of f decays as (or no faster than) ρ^j where $\rho = \rho(d, b)$ is the distance of $d \in D$ to the boundary $S = \partial D$, then this family is q-thick for every $q > j + 1$. For example, the obvious $(n-2)$-dimensional pencil of disks in $\mathbb{R}^n$ filling the standard circle is q-thick for $q > n - 1$. We have constructed in 3.6.D contact pencils f with $j = n + 1$ and hence q-thick for $q > n + 2$ and now we make three extra points.

(1) Start with a generic H of rank $n_1 > (2n + 2)/3$ which ensures the existence of regular Ω-isotropic plane at a generic point in V (see 4.2.A″). In fact, this also gives us such a plane through a generic horizontal vector. Assuming this genericity one can produce families $f : D \times B^{n-2} \to V$ with decay (at most) $a_1\ \rho^j$ for some $j < \infty$. This gives us q-thick fillings, for $q = j + 1 + \varepsilon < \infty$, if sufficiently many horizontal curves S in order to derive basic topological finiteness results. For example, *there are at most finitely many topologically non-equivalent bundles* $X \to (V, H)$ *with* $\|\Omega_\nabla\|_{L_q} \leq c$. It would be interesting to have a specific evaluation of j and q for general H but we indicate this below only in two cases close to the contact one.

(2) Let $(V, H) = (V_1 \times V_2, H_1 \times H_2)$ for contact (V_i, H_i) of dimensions ≥ 5. Then H_i-horizontal disks $\varphi_i : D \to V_i$, $i = 1, 2$. Define a horizontal bi-disk in V, namely $\varphi = \varphi_1 \times \varphi_2 : D \times D \to V$ and then families of disks in V_i with the Jacobian decays of degrees j_i, $i = 1, 2$, give us a family in V with $j = j_1 + j_2 + 1$. Thus we obtain q-thick fillings of horizontal closed curves S in V for every $q > n + 3$. Furthermore, since the Ω-regularity etc is stable under small perturbation of H, these q-thick fillings also exist for polarizations H on V close to $H_1 \times H_2$ and so the basic topological finiteness results hold true for these H and q.

(3) Let (V, H) be the 1-jet bundle of maps $V_0 \to \mathbb{R}^2$. This H is very much similar to the above and the evaluation of q is left to the reader. We also suggest the reader would look at the thick filling of k-dimensional cycles for $k > 1$.

Final Remarks. The above results are of preliminary nature as we strive for $q = N/2$ which would give the best possible result with the (scale invariant) norm $\|\Omega_\nabla\|_{L_{N/2}}$. One way to do this is offered by an extension of hypoelliptic theory to the complex $\left(\Lambda^*(V)/I(H), d\right)$ between degrees 1 and 2 which is needed for an analytic proof à la Uhlenbeck. A step in this direction is made by Z. Ge in [Ge$_{\text{BNCC}}$] but his hypoellipticity criterion (in degree one) seems rather restrictive. Another (less realistic but geometrically attractive) possibility is to combine the thick filling idea with some geometric "controlled integration" (over suitable measures in the space of curves) which would lead us directly to bounds on ∇ in terms of Ω_∇.

Inverse Kato inequalities. The existence of frames and sections f with small integral norms of ∇f is opposite to Kato inequality (see 5.2) and one may expect further upper bounds on the spectrum of the operator $\nabla^*\nabla$ in terms of the Hörmander Laplacian ∇_H on (scalars on) V and integral bounds on $\|\Omega_\nabla\|$. For example, one wishes to estimate the number λ, such that the eigensections of $\nabla^*\nabla$ below λ span each fiber of our bundle $X \to V$. It would be interesting to estimate this λ in terms of $\|\Omega\|_{N/2}$ (and V) by a purely linear argument which would provide an alternative to Uhlenbeck's approach.

References

[Bass] Bass H., The degree of polynomial growth of finitely generated nilpotent group. Proc. Lond. Math. Soc. (3) 25, pp. 603–614, 1972.

[Bell] Bellaïche A., The tangent space in sub-Riemannian geometry. This volume.

[Be-Ve] Berestovskii V.N., Vershik A.M., Manifolds with intrinsic metrics and nonholonomic spaces. Advances in Sov. Math. vol (**7**) AMS 1992.

[Beth] Bethuel F., The approximation problem for Sobolev maps betwen two manifolds. Acta. Math., **167** (1991), pp. 153–206.

[Ben] Bennequin D., Entrelacements et équations de Pfaff. Soc. Math. France, Astérisque **107–108** (1983), 83–161.

[Bou] Valère Bouche L., The geodesics' problem in Sub-Riemannian geometry. About the R. Montgomery's example simplified by I. Kupka. Preprint du L.A.M.A. 92-06 (1992).

[Br-Gr] Bryant R.L. and Griffiths P.A., Characteristic Cohomology of Differential systems (I): General Theory. J.-Amer.-Math.-Soc. **8** (1995), no. 3, 507–596.

[Brock] Brockett R.W., Nonlinear Control Theory and Differential Geometry, Proc. of the Int. Congress of Mathematicians, Warszawa, (1983).

[Bu-Za] Burago Y.D. and Zalgaller V.A.: Geometric inequalities. Springer, Berlin, Heidelberg 1988. [Russian edn.: Nauka, Moscow, 1980]

[Cho] Chow W.L., Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung, Math. Annalen **117** (1939), pp. 98–105.

[Cor] Corlette K., Hausdorff dimensions of limit sets I, preprint (1989).

[D-G] D'Ambra G. and Gromov M., Lectures on transformation groups: geometry and dynamics, Surveys in Differential Geometry (Supplement to the Journal of Differential geometry), **1** (1991), pp. 19–111.

[DA_{C1}] D'Ambra G., Nash C^1-embedding theorem for Carnot-Carathéodory metrics. Differential-Geom.-Appl. **5** (1995), no. 2, 105–119.

[DA_{IC}] D'Ambra G., Construction of connection inducing maps between principal bundles. J. Diff. Geom. **26**, (1987), pp. 67–79.

[DA_{IS}] D'Ambra G., Induced subbundles and Nash's implicit function theorem. Differential-Geom.-Appl. **4** (1994), no. 1, 91–105.

[Da-Se] David G., Semmes S., Lipschitz mappings between dimensions, Unpublished manuscript.

[Dek] Dektjarev I.M., Problems of value distribution in dimension higher than one, Uspeki Mat. Nauk 25 : **6** (1970), pp. 53-84.

[E-L] Eells J. and Lemaire L., Another report on harmonic maps. Bull. London. Math. Soc. **20** (1988) 385–524, pp. 388.

[El_1] Eliashberg Ya., Classification of overtwisted contact structures on manifolds, Invent. Math. **98** (1989), pp. 623–637.

[El_2] Eliashberg Ya., Contact 3-manifolds, twenty years since J. Matinet's work. Ann. Inst. Fourier **42** (1992), pp. 165–192.

[El_3] Eliashberg Ya., New invariants of open symplectic and contact manifolds, J. Amer. Math. Soc. **4** (1991), pp. 513–520.

[El_4] Eliashberg Ya., Filling by holomorphic disks and its applications, London Math. Soc. Lect. Notes Ser. **151** (1991), pp. 45–67.

[Fal] Falconer K., Fractal Geometry. Mathematical Foundations and Applications. John Wiley and Sons. 1990.

[Fef-Ph] Fefferman C.L. and Phong D.H., Subelliptic eigenvalue problems, Proceedings of the Conference on Harmonic Analysis in Honor of Antoni Zygmund, Wadsworth Math. Series, pp. 590–606 (1981).

[Fol] Folland G.B., Subelliptic estimates and function spaces on nilpotent Lie groups, Arkiv för Mat. **13**, pp. 161–207 (1975).

[Fuk] Fukawa K., Collapsing of Riemannian manifolds and eigenvalues of Laplacian operators, Invent. Math. **87** (1987), pp. 517–547.

[G-E] Gromov M. and Eliashberg J., Construction of nonsingular isoperimetric films, Trudy Steklov Inst. **116**, (1971), pp. 18–33.

[G-L-P] Gromov M., Lafontaine J., Pansu P., Structures métriques pour les variétés Riemanniennes, Cedic-Fernand Nathan, Paris (1981).

[Ge_{BNCC}] Ge Z., Betti numbers, characteristic classes and sub-Riemannian geometry, Illinois J. of Mathematics, **36** (1992), pp. 372–403.

[Ge_{CRM}] Ge Z., Collapsing Riemannian metrics to sub-Riemannian metrics and Laplacians to sub-Laplacians, Canadian J. Math., 1993, pp. 537–552.

[$\mathrm{Ge}_{\mathrm{GG3}}$] Ge Z., On the Global Geometry of Three-dimensional Sub-Riemannian Manifolds. II On Sub-Riemannian Metrics and $\widetilde{SL_2R}$-geometry. The Fields Institute for Research in Mathematical Sciences, Preprint 1994.

[$\mathrm{Ge}_{\mathrm{HPS}}$] Ge Z., Horizontal paths space and Carnot-Carathéodory metrics, Pacific J. of Mathematics, **161** (1993), pp. 255–286.

[$\mathrm{Ge}_{\mathrm{VP}}$] Ge Z., On a variational problem and the spaces of horizontal paths, Pacific J. of Mathematics, **149** (1991), pp. 61–93.

[Gir] Giroux E., Topologie de contact en dimension 3. Séminaire Bourbaki, 1992-93, n° 760.

[Good] Goodman R.W., Nilpotent Groups, the Structure, and the Application to Analysis, Lectures Notes in Math., vol. 562, Springer, 1977.

[$\mathrm{Gro}_{\mathrm{AI}}$] Gromov M., Asymptotic invariants of infinite groups. Geometric group theory, Vol. 2 (Sussex, 1991), 1–295. London Math. Soc. Lecture Note Ser., **182**, Cambridge Univ. Press, Cambridge, 1993.

[$\mathrm{Gro}_{\mathrm{AI}}$] Gromov M., Asymptotic geometry of homogeneous spaces, Proceedings Rend. Sem. Mat. Univ. Politec. Torino (1983), Conférence on Homogeneous Spaces in Symp. Math., pp. 59–60.

[$\mathrm{Gro}_{\mathrm{CIDR}}$] Gromov M., Convex integration of differential relations, Izv. Akad. Nauk. S.S.S.R. 33, **2**, (1973), pp. 329–343.

[$\mathrm{Gro}_{\mathrm{DNLS}}$] Gromov M., Dimension, non-linear spectra and width, Springer-Verlag, Lecture Notes in Mathematics, **1317** (1988), pp. 132-1-85.

[$\mathrm{Gro}_{\mathrm{FPP}}$] Gromov M., Foliated plateau problem, parts I, II, Geometric and Functional Analysis 1 : **1** (1991), pp. 14–79 ; (1991), pp. 253–320.

[$\mathrm{Gro}_{\mathrm{FRM}}$] Gromov M., Filling Riemannian manifolds, Journal of Differential Geometry **18** (1983), pp. 1–147.

[$\mathrm{Gro}_{\mathrm{GPG}}$] Gromov M., Groups of polynomial growth and expanding maps, Publications Mathématiques IHES **53** (1981), pp. 53–73.

[$\mathrm{Gro}_{\mathrm{HED}}$] Gromov M., Homotopical effects of dilatation. Journal of Differential Geometry **13** (1978) 303–310.

[$\mathrm{Gro}_{\mathrm{HG}}$] Gromov M., Hyperbolic groups, Essays in Group Theory, S. Gersten editor, MSRI Publications n° **8**, Springer (1987), pp. 75–265.

[$\mathrm{Gro}_{\mathrm{HMGA}}$] Gromov M., Hyperbolic manifolds, groups and actions, in "Riemannian Surfaces and Related Topics", Ann. Math. Studies **97** (1981), pp. 183–215.

[$\mathrm{Gro}_{\mathrm{MIKM}}$] Gromov M., Metric invariants of Kähler manifolds, Differential geometry and topology (Alghero, 1992), 90–116. World Sci. Publishing, River Edge, NJ, 1993.

[$\mathrm{Gro}_{\mathrm{PDR}}$] Gromov M., Partial differential relations, Springer-Verlag (1986).

[$\mathrm{Gro}_{\mathrm{PLI}}$] Gromov M., Paul Levy's isoperimetric inequality, Preprint IHES (1980).

[$\mathrm{Gro}_{\mathrm{SAP}}$] Gromov M., Stability and pinching. Sessions on Topology and Geometry of Manifolds (Italian) (Bologna, 1990), 55–97. Univ. Stud. Bologna, Bologna, 1992.

[$\mathrm{Gro}_{\mathrm{SISI}}$] Gromov M., Systoles and intersystolic inequalities. Preprint IHES (1993).

[$\mathrm{Gro}_{\mathrm{WRI}}$] Gromov M., Width and related invariants of Riemannian manifolds, Astérisque **163–164** (1988), pp. 93–109.

[Hart] Hart P., Ordinary differential equations. John Wiley and Sons. New York-London-Sydney. 1964.

[Ham] Hamenstädt U., Some regularity theorems for Carnot-Carathéodory metrics, J. Diff. Geom. **32** (1991), pp. 192–201.

[Herm] Hermann R., Differential Geometry and the Calculus of Variations, Math. Sci. Eng. **49**, Academic Press, New-York, 1968.

[Ho] Hofer H., Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three, Inv. Math. **114** (3), pp. 515–585, 1993.

[Hol] Holopainen I., Positive solutions of quasilinear elliptic equations on riemannian manifolds. Proc. London Math. Soc. (3) **65** (1992), pp. 651–672.

[Hol-Rick] Holopainen I. and Rickman S., Quasiregular mappings of the Heisenberg group. Preprint (1991).

[Hör] Hörmander L., Hypoelliptic second order differential equations, Acta Math. **119**, pp. 147–171 (1967).

[$\mathrm{Hsu}_{\mathrm{CVG}}$] Hsu L., Calclus of Variations via the Griffiths formalism, Journal of Differential Geometry **36** (1992), pp. 551–589.

[$\mathrm{Hsu}_{\mathrm{GHP}}$] Hsu L., Gromov's h-principle for strongly bracket-generating distributions (IAS preprint: 1993)

[$\mathrm{Hsu}_{\mathrm{SRCT}}$] Hsu L., Sub-Riemannian Comparison Theorems for Contact Manifolds (IAS preprint: 1993).

[Je-Sa] Jerison D. and Sánchez-Calle A., Subelliptic, second order differential operators. Lect. Notes in Math. **1277**, pp. 46–77, Springer-Verlag 1987.

[Jo] Jost J., Equilibrium Maps between Metric Spaces. Ruhr-Universität Bochum, Preprint.

[Kar] Karcher H., Riemannian center of mass and mollifier smoothing. Comm. Pure and Appl. Math. **30** (1977), 509–541.

[Kari] Karidi R., Geometry of balls in nilpotent Lie groups. Duke-Math.-J. **74** (1994), no. 2, 301–317.

[Kor] Korevaar N., Upper bounds for eigenvalues of conformal metrics. J. Differential Geometry **37** (1993), pp. 73–93.

[Kor-Rei] Koranyi A. and Reimann H.M., Foundations for the theory of quasiconformal mappings on the Heisenberg group. Preprint 1991.

[La] Lalonde F., Homologie de Shih d'une submersion (homologies non singulières des variétés feuilletées). Supplément au Bulletin de la Société Mathématique de France. 1978, Tome **115**, Fascicule 4.

[Lee] Lee Y.I., The metric properties of Lagrangian surfaces. Dissertation of Stanford University for the degree of doctor of philosophy. 1992.

[Mar] Margulis G., Discrete groups of motions of manifolds of non-positive curvature (Russian), ICM 1974. Translated in A.M.S. Transl. (2) **109** (1977), pp. 33–45.

[Mit_1] Mitchell J., A Local Study of Carnot-Carathéodory Metrics. Ph.D. Thesis. Stony Brook 1982.

[Mit_2] Mitchell J., On Carnot-Carathéodory metrics, J. Differ. Geom. **21** (1985), pp. 35–45.

[Mont] Montgomery R., Survey of singular geodesics, this volume.

[Most] Mostow G.D., Strong rigidity of symmetric spaces, Ann. Math. Studies **78**, Princeton (1973).

[N-S-W] Nagel A., Stein E.M. and Wainger S., Balls and metrics defined by vector fields I : Basic properties, Acta Math. **155**, pp. 103–147 (1985).

[Nag] Nagata J-I., Modern dimension theory, North-Holland, 1965.

[$\mathrm{Pan}_{\mathrm{CBN}}$] Pansu P., Croissance des boules et des géodésiques fermées dans les nil-variétés, Ergod. Th. dynam. Syst. **3** (1983), pp. 415–445.

[$\mathrm{Pan}_{\mathrm{InIs}}$] Pansu P., Une inégalité isopérimétrique sur le groupe d'Heisenberg, C.R. Acad. Sci. Paris **295** (1982), pp. 127–131.

[$\mathrm{Pan}_{\mathrm{QIR1}}$] Pansu P., Métriques de Carnot-Carathéodory et quasi-isométries des espaces symétriques de rang un, Annals of Maths. **129**: 1 (1989), pp. 1–61.

[$\mathrm{Pan}_{\mathrm{QM}}$] Pansu P., Quasiconformal mappings and manifolds of negative curvature, in "Curvature and Top of Riemannian Manifolds" (Shiohama et al., eds), Lect. Notes in Math. **1201** (1986), pp. 212–230, Springer-Verlag.

[$\mathrm{Pan}_{\mathrm{Th3}}$] Pansu P., Géometrie du groupe d'Heisenberg. Thèse Univ. Paris VII, 1982.

[Pel-Bou] Pelletier F., et Valére Bouche L., Le problème des géodésiques en géométrie sous-riemannienne singulière-II, C.-R.-Acad.-Sci.-Paris-Ser.-I-Math. **317** (1993), no. 1, 71–76.

[Pes] Pesin Ya. B, Dimension type characteristics for invariant sets of dynamical systems. Uspekhi Mat. Nauk **43**: 4 (1988), pp. 95–128.

[Pit] Pittet C., Isoperimetric Inequalities in Nilpotent Groups. Preprint 1994.

[Rash] Rashevski P.K., About connecting two points of complete nonholonomic space by admissible curve, (in Russian), Uch. Zapiski ped. inst. Libknexta, no. **2**, pp. 83–94, (1938).

[Ru] Ruh E.A., Almost Lie groups, Proc ICM-1986, Berkeley, pp. 561–564, AMC 1987.

[Rum_1] Rumin M., Formes différentielles sur les variétés de contact. Thèse de Doctorat, Orsay 1992.

[Rum_2] Rumin M., Un complexe de formes différentielles sur les variétés de contact, C.R. Acad. Sci. Paris, t. **310** (1990), serie I., pp. 401–404.

[Sar] Sarychev A.A.V., On the homotopy type of the spaces of trajectories of nonholonomic dynamic systems, Dokl. Acad. Sci. USSR, v. **314** (6), 1990.

[Sch-Uhl] Schoen R. and Uhlenbeck K., A regularity theory for harmonic maps. J. Differential Geom., **17** (1982), pp. 307–335.

[Sin] Singer I.M., Infinitesimally homogeneous spaces, Comm. Pure Appl. Math. **13**, pp. 685–697, 1960.

[Ste] Stein E.M., Singular integrals and differentiability properties of functions, Princeton Univ. Press (1970).

[Stri] Strichartz R.S., Sub-Riemannian Geometry, J. Differential Geometry **24** (1986), pp. 221–263.

[Suss] Sussmann H.J., Abnormal sub-Riemannian minimizers, this volume.

[Th] Thom R., Remarques sur les problèmes comportant des inégalités différentielles globales, Bull. Soc. Math. France **87** (1959), pp. 455–461.

[Uhl] Uhlenbeck Karen K., Connections with L^p Bounds on Curvature. Commun. Math. Physics. **83**, pp. 31–42 (1982).

[Var] Varopoulos N. Th., Small Time Gaussian Estimates of Heat Diffusion Kernels. II. The Theory of Large Deviations. J. of Functional Analysis, Vol **93**, No 1. (1990).

[Var-Sa-Co] Varopoulos N. Th., Saloff-Coste L., Coulhon Th., Analysis and Geometry on Groups, Cambridge University Press (1993).

[Ver-Ger] Vershik A.M., Gershkovich V. Ya., nonholonomic Manifolds and Nilpotent Analysis, J. Geom. and Phys. vol. **5**, no. 3, (1988).

[Ver-Ger] Vershik A.M., Gershkovich V. Ya., nonholonomic Geometry and Nilpotent Analysis, J. Geom. and Phys. 5 3 (1989), pp. 407–452.

[Vin$_1$] Vinogradov A.M., Geometry of nonlinear differential equations, J. Soviet Math. **17** (1981), pp. 1624–1649.

[Vin$_2$] Vinogradov A.M., The $\mathcal{C}$-spectral sequence, Lagrangian formalism and conservation laws I, II, J. Math. Anal. Appl. **100** (pp. 1–129), 1984.

[Vin$_3$] Vinogradov A.M., Local symmetries and conservation laws, Acta Applicandae Mathematicae **2** (21–78), 1984.