A Dozen Problems, Questions and Conjectures about Positive Scalar Curvature

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Contents

Abstract

Unlike manifolds with positive sectional and with positive Ricci curvatures which aggregate to modest (roughly) convex islands in the vastness of all Riemannian spaces, the domain $\{\mathcal{SC} > 0\}$ of manifolds with *positive scalar curvatures* protrudes in all direction as a gigantic octopus or an enormous multi-branched tree. Yet, there are certain rules to the shape of $\{\mathcal{SC} > 0\}$ which limit the spread of this domain but most of these rules remain a guesswork.

In the present paper we collect a few "guesses" extracted from a longer article, which is still in preparation: *100 Questions, Problems and Conjectures around Scalar Curvature.*

Some of these "guesses" are presented as questions and some as conjectures. Our formulation of these conjectures is not supposed to be either most general or most plausible, but rather maximally thought provoking.

1   Definition of Scalar Curvature.

The scalar curvature of a C^2-smooth Riemannian manifold $X = (X, g)$, denoted $Sc = Sc(X) = Sc(X, g) = Sc(g)$ is a continous function on X, written as

$Sc(X)(x)$ and $Sc(g)(x)$, $x \in X$, which is uniquely characterised by the following four properties.

$\bullet_1$ *Additivity under Cartesian-Riemannian Products .*

$$Sc(X_1 \times X_2, g_1 \oplus g_2) = Sc(X_1, g_1) + Sc(X_2, g_2),$$

where this equality is understood point-wise,

$$Sc(X_1 \times X_2)(x_1, x_2) = Sc(X_1)(x_1) + Sc(X_2)(x_2).$$

$\bullet_2$ *Scale covariance.*

$$Sc(X, \lambda^2 \cdot g) = \lambda^2 \cdot Sc(X).$$

Thus, for instance, since $(\mathbb{R}^n, g_0)$ is isometric to $(\mathbb{R}^n, \lambda^2 \cdot g_0)$ for the Euclidean metric g_0,

$$Sc(\mathbb{R}^n) = 0 \text{ for all } n = 1, 2, 3, \dots.$$

$\bullet_3$ *Volume Comparison.* If the scalar curvatures of n-dimensional manifolds X and X' at some points $x \in X$ and $x' \in X'$ are related by the strict inequality

$$Sc(X)(x) < Sc(X')(x'),$$

then the Riemannian volumes of small balls around these points satisfy

$$vol(B_x(X, \varepsilon)) > vol(B_{x'}(X', \varepsilon))$$

for all sufficiently small $\varepsilon > 0$.

This volume inequality, in agreement with $\bullet_1$, is *additive under Riemannian products*: if

$$vol(B_{x_i}(X, \varepsilon)) > vol(B_{x'_i}(X'_i, \varepsilon)), \text{ for } \varepsilon \leq \varepsilon_0,$$

and *for all points* $x_i \in X_i$ and $x'_l \in X'_i$, $i = 1, 2$, then

$$vol_n(B_{(x_1, x_2)}(X_1 \times X_2, \varepsilon_0)) > vol_n(B_{(x'_1, x'_2)}(X'_1 \times X'_2, \varepsilon_0))$$

for all $(x_1, x_2) \in X_i \times X_2$ and $(x'_1, x'_2) \in X'_1 \times X'_2$.

This follows from the Pythagorean formula

$$dist_{X_1 \times X_2} = \sqrt{dist^2_{X_1} + dist^2_{X_2}}.$$

and the Fubini theorem applied to the "fibrations" of balls over balls:

$$B_{(x_1, x_2)}(X_1 \times X_2, \varepsilon_0)) \to B_{x_1}(X_1, \varepsilon_0) \text{ and } B_{(x'_1, x'_2)}(X'_1 \times X'_2, \varepsilon_0)) \to B_{x_1}(X'_1, \varepsilon_0),$$

where the fibers are balls of radii $\varepsilon \in [0, \varepsilon_0]$ in X_2 and X'_2.

$\bullet_4$ *Normalisation/Convention for 2-spheres.* The unit sphere $S^2 = S^2(1)$ has constant scalar curvature 2 (twice the sectional curvature).

It is an elementary exercise to prove the following.

$\star_1$ *The function $Sc(X, g)(x)$ which satisfies $\bullet_1$-$\bullet_4$ exists and unique;*

$\star_2$ *The unit spheres and the hyperbolic spaces with sect.curv $= -1$ satisfy*

$$Sc(S^n(1)) = n(n-1) \text{ and } Sc(H^n(-1)) = -n(n-1).$$

Thus,

$$Sc(S^n(1) \times H^n(-1)) = 0 = Sc(\mathbb{R}^n),$$

which implies that the volumes of the small balls in $S^n(1) \times H^n(-1)$ are "very close" to the volumes of the Euclidean $2n$-balls.

$\star_3$ *The scalar curvature of a Riemannin manifold X is equal to the sum of the values of the sectional curvatures at the bivectors of an orthonormal frame*[1] *in X,*

$$Sc(X)(x) = \sum_{i,j} c_{ij}, \ i,j = 1,...,n.$$

For example, all *compact Riemannin symmetric spaces X*, except for the n-torus $\mathbb{T}^n$, have $Sc(X) > 0$, while $\mathbb{T}^n$, being covered by $\mathbb{R}^n$, has $Sc(\mathbb{T}^n) = 0$.

It may be tempting to take the above $\bullet_1 - \bullet_4$ for a definition of scalar curvature for singular metric spaces X. In fact, it may work for X with moderate singularities, e.g. for *Alexandrov's spaces with sectional curvatures bounded from below* (see [1][2]), where the properties of the so defined scalar curvature must be comparable to what is observed in the smooth case (see section 7).

Yet, volumes of balls to not touch the heart of the scalar curvature; we suggests an alternative in section 7.

2   Soft and Hard Facets of Scalar Curvature.

We are not so much concerned with the scalar curvature $Sc(X)$ per se, but rather with the effect of *lower scalar curvature bounds* on the geometry and the topology of X, where, for instance, the inequality "$Sc(X) > 0$" can be **defined** by saying that

all sufficiently small balls $B_x(\varepsilon) \subset X$, $\varepsilon \le \varepsilon_0(x) > 0$, have the volumes smaller than the volumes of the equividimensional Euclidean ε-balls.

Then "$Sc(X) \ge 0$" is defined as

$$Sc(X) > -\varepsilon" \text{ for all } \varepsilon > 0.$$

Similarly

"$Sc(X) \ge \sigma$", $\sigma > 0$, is equivalent the volumes of $B_x(\varepsilon)$ in X being smaller than the volumes of the ε-balls in the Euclidean spheres $S^n(R)$ of radii $R > \sqrt{(n(n-1)/\sigma)}$,

and $Sc(X) \ge -\sigma$ is expressed by

the bound on the volumes of $B_x(\varepsilon)$ by those of the ε-balls in the hyperbolic spaces with constant the sectional curvatures $< -\sigma/n(n-1)$.

Alternatively, "$Sc(X) \ge -\sigma$" can be defined with *no reference* to hyperbolic spaces by *the reduction to the case $\sigma = 0$* and appealing to the relation

$$Sc(X \times S^m(R)) \ge 0 \text{ for } R = \sqrt{(m(m-1)/\sigma}, $$

where one may use any $m \ge 2$ one likes.

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[1]Remarkably, this sum is independent of the frame by the Pythagorean theorem.

[2][1] S.Alexander, V. Kapovitch, A.Petrunin, *Alexandrov geometry,*
http://www.math.psu.edu/petrunin/

Although the key role of the scalar curvature in general relativity was established by Hilbert's variational derivation of the Einstein equation more than a century ago (see [2][3]) the significance of $Sc(X)$ in the global geometry and in topology remained obscure until 1963, when André Lichnerowicz (see [3][4]) showed that the inequality $Sc(X) > 0$ imposes non-trivial constraints on the topology of X.

For instance, Lichnerowicz' theorem implies that

if m is even, then smooth complex projective hypersurfaces $X \subset \mathbb{C}P^{m+1}$ (these have real dimension $dim(X) = 2m$) of degrees $\geq m + 2$, e.g. $X \subset \mathbb{C}P^3$ given by the equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0,$$

admit no metrics with $Sc > 0$.

This follows from the Atiyah-Singer formula for the (Atiyah-Singer)-Dirac operator D confronted with (what is now called) the *Schroedinger-Lichnerowicz-(Weitzenboeck-Bochner) identity.*

In fact, the index formula implies that the index of D on these manifolds *does not vanish,*[5] and, consequently, *there are non-zero harmonic spinors on these X* (i.e. solutions s of $D(s) = 0$), while the *Schroedinger-Lichnerowicz-(Weitzenboeck-Bochner) identity*

$$D^2 = \nabla\nabla^* + \frac{1}{4}Sc,$$

shows that closed manifolds with $Sc > 0$ admit *no harmonic spinors.*

Eleven years later, Nigel Hitchin(see [4][6]) used a more sophisticated 1971 version of the Atiyah-Singer index theorem which yields harmonic spinors on some *exotic spheres* Σ^n (which are homeomorphic but not diffeomorphic to the ordinary spheres S^n) of dimensions $n = 8k+1$ and $n = 8k+2$ and which, together with the Schroedinger-Lichnerowicz' identity, implies that

there is no metrics with $Sc > 0$ on these Σ^n.

Then Stefan Stolz, elaborating on the earlier work by several authors, showed that there are

no further obstructions to the existence of metrics with $Sc > 0$ on simply connected *manifolds of dimension ≥ 5 besides those delivered by the index theorem* [5].[7]

For instance

all simply connected manifolds of dimensions $n = 3, 5, 6, 7 \mod 8$ admit metrics with positive scalar curvatures.

The proof of this theorem, which relies on *surgery of manifolds with $Sc > 0$* and on the *cobordism theory*, suggests that manifolds with positive scalar curvature are almost as *soft* as smooth manifolds with no geometric constraints

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[3][2] David Hilbert, The Foundations of Physics", 1915.

[4][3] A. Lichnerowicz, Spineurs harmoniques. C. R. Acad. Sci. Paris, Série A, 257 (1963), 7-9.

[5]This formula says in the present case that $Ind(D) = \hat{A}(X)$ where $\hat{A}(X)$ is a particular Pontryagin number of X.

[6][4] N. Hitchin, Harmonic Spinors , Adv. in Math. 14 (1974), 1-55.

[7][5] S. Stolz. *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) 136 (1992), 511-540.

imposed on them. But the grand picture of scalar curvature in all its beauty unravels when one looks beyond this "almost".

(The opposite inequality $Sc(X) < 0$ is truly and fully soft and, unlike $Sc > 0$, has no influence on the topology and global geometry of X what-so-ever (see [6][8]).

A manifestly *rigid* property of $Sc > 0$ can be already seen in the following corollary to Schoen-Yau solution of the Riemannin positive mass conjecture in relativity (see[7][9]).

Solution of the Geroch Conjecture.[10] *The Euclidean metric g_0 on $\mathbb{R}^3$ (which has $Sc(g_0) = 0$) admits no non-trivial compactly supported perturbations g with $Sc(g) \geq 0$.*

Namely, if a smooth Riemannin metric g on the Euclidean space $\mathbb{R}^3$ has $Sc(g) \geq 0$ and if g is equal to g_0 outside a compact subset in $\mathbb{R}^3$, then $Sc(g) = 0$; moreover, g is Riemannin flat, that is $(\mathbb{R}^3, g)$ is isometric to $(\mathbb{R}^3, g_0)$.

This result has been refined and generalised in a variety of directions (see below and also [13] and [21] at the end of the next section and references therein) but the rigidity of $Sc > 0$ we are after, albeit related to the above, is of different nature. In fact what we look for is

a structurally organised set of (desirably sharp) *geometric inequalities satisfied by manifolds with $Sc > 0$, more generally, with $Sc \geq \sigma$.*

Also, we search for a general category (or categories) of spaces, or other kind of objects, which would satisfy (certain classes of) such inequalities.

Additional Remarks and References.

Geroch conjecture has been validated in all dimensions:

The Euclidean metrics on $\mathbb{R}^n$ for all n admit no non-trivial compactly supported perturbations with $Sc \geq 0$.

This (trivially) follows, for instance, from *non-existence of metrics with $Sc > 0$ on the n-tori* where the latter can be most easily proved by applying the index theorem to suitably "twisted" Dirac operators.

Witten suggested a different way of using the Dirac operator in the context of the positive mass problem, where the index theorem is replaced by a direct proof of *harmonic stability of parallel spinors* on $\mathbb{R}^n$ under certain perturbations of the Euclidean metric.

By a similar method, Min-Oo (see [8][11]) proved that

the hyperbolic metric g_0 on the real hyperbolic space $H^n_{\mathbb{R}}$ admits non non-trivial compactly supported perturbations g with $Sc(g) \geq -n(n-1) = Sc(g_0)$.

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[8][6] J. Lohkamp, *Metrics of negative Ricci curvature*, Annals of Mathematics, 140 (1994), 655-683.

[9][7] R. Schoen and S.-T. Yau, On the proof of the positive mass conjecture in general relativity, Commun. Math. Phys. 65, (1979). 45-76.

[10]Attribution of this simplified podescribessitive mass conjecture to Robert Geroch is made in the above cited paper by Schoen and Yau.

In fact, the full Riemannin positive mass conjecture which describes possible asymptotic behaviours of metrics with $Sc > 0$ on $\mathbb{R}^3$ (and on $\mathbb{R}^n$ for this matter) which are close (rather than equal) to the Euclidean metric at infinity follows from this Geroch conjecture according to *J. Lohkamp, Scalar curvature and hammocks, Math. Ann. 313 (1999), 385-407.*

[11][8] M. Min-Oo, Scalar curvature rigidity of asymptotically hyperbolic spin manifolds,. Math. Ann. 285, 527?539 (1989)

Apparently, it is unknown

if other symmetric spaces of non-compact types admit compactly supported perturbations of their Riemannian metrics which would increase scalar curvature.

3 Bounds on the Uryson Width, Slicing Area and Filling Radius.

A. Conjecture. *Let X be an n-dimensional Riemannin manifold with scalar curvature bounded from below by*

$$Sc(X) \geq n(n-1) = Sc(S^n).$$

Then the $(n-1)$-dimensional Uryson width of X is bounded by a universal constant.

This means that there exists a continuous map from X to an $(n-1)$-dimensional polyhedral space P,

$$f : X \to P = P^{n-1},$$

such that the pullbacks of all points have controllably bounded diameters, namely,

$$diam_X(f^{-1}(p)) \leq const \text{ for all } p \in P.$$

for some universal constant $const > 0$ possibly (and undesirably) depending on n.

This conjecture says, in effect, that that n-dimensional manifolds X with $Sc(X) \geq \sigma > 0$ "topologically spread" in at most $n-1$ directions.

In fact, one expects that these X spread only in $n-2$ direction which can be formulated as follows.

A$_+$. Conjecture.*The above X admits a a continuous map f to an $(n-2)$-dimensional polyhedral space P, such that $diam_X(f^{-1}(p)) \leq const_+$ for all $p \in P$.*

But the most attractive (and least tenable) is the conjecture **A$_{++}$** below which claims that *closed* manifolds with $Sc \geq \sigma > 0$ can be *sliced by surfaces with small areas* according the following definition.

Slicings and Waists. An *m-sliced n-cycle*, $m \leq n$, is an n-dimensional psedomanifold $P = P^n$ partitioned into *m-slices* $P_q \subset P$, which are the pullbacks of the points of a simplicial map $\varphi : P \to Q$ where Q is an $(n-m)$-dimensional pseudomanifold and where all pullbacks $P_q = \varphi^{-1}(q) \subset P$ have $dim(P_q) \leq m$, $q \in Q$.

(Sometimes one insists that φ must be *proper*, hence, with compact pullbacks $\varphi^{-1}(q)$, even if P is non-compact.)

The m-waist (mod 2), denoted $waist_m(h)$, of a homology class $h \in H_n(X;\mathbb{Z}_2)$ is

the infimum of the numbers w,

such that X receives a Lipschitz map from a compact m-sliced cycle, $\phi : P^n \to X$, which represent h, i.e.

$$\phi_*[P] = h$$

and the

$$the\ images\ of\ all\ slices\ in\ X\ have\ m\text{-}volumes \leq w,$$

where these "volumes of the images" are counted with multiplicities (which is unneeded for generically 1-1 maps.)

A$_{++}$. Conjecture. Let X be a *closed n-dimensional Riemannin manifold* the scalar curvature of which is bounded from below as earlier:

$$Sc(X) \geq n(n-1)(= Sc(S^n)).$$

Then the slicing area of the fundamental homology class $[X] \in H_n(X; \mathbb{Z}_2)$ is bounded by

$$waist_2[X] \leq const_{++}.$$

(Ideally, one expects

$$const_{++} = waist_2(S^n)$$

where $waist_2[S^n] = area(S^2) = 4\pi$ by an Almgren's theorem.)

The above conjectures can be interpreted as saying that X contains "many" small subsets of dimensions 1 and/or 2.

For instance, **A** implies that that X contains a *topologically significant/representative* family of 1-*dimensional* subsets (graphs) with diameters $\lessapprox \frac{1}{\sqrt{\sigma}}$.

This suggests the following.

(a) Conjecture. *If $Sc(X) \geq \sigma > 0$ and if X is a* closed *(compact without boundary) manifold, then X contains a closed minimal geodesic of length $\leq \frac{const_n}{\sqrt{\sigma}}$, or, at least, a stationary one-dimensional $\mathbb{Z}_2$-current of diameter (better length) $\leq \frac{const_n}{\sqrt{\sigma}}$.*

And **A$_{++}$** actually implies the following.

(a$_{++}$) Conjecture. *Closed manifolds X with $Sc(X) \geq \sigma > 0$ contain closed minimal surfaces (i.e. stationary two-dimensional $\mathbb{Z}_2$-currents) of areas $\leq \frac{const_n}{\sigma}$.*

Below is a weaker version of **A** which already imposes non-trivial topological constraints on X.

A$_-$. Conjecture. *If $Sc(X) \geq n(n-1)$ then the* filling radius *of X is bounded by*

$$fil.rad(X) \leq const_-.$$

Definition of fil.rad. If $X = (X, g)$ is *closed* Riemannian manifold then the *filling radius* is equal to the infimum of $R > 0$, such that the cylinder $X^{\times} = X \times [0, 1)$ admits a Riemannin metric $g^{\times}$ with the following three properties.

$\bullet_1$ the restriction of $\hat{g}$ to $X = X_0 \times \{0\} \subset X \times [0, 1) = X^{\times}$ is equal to g; moreover,

$$dist_{g^{\times}}|X = dist_g.$$

This means that the g-shortest curves in X between all pairs of points in X minimise the $g^{\times}$-lengths of such curves in $X^{\times} \supset X$.

$\bullet_2$ All points in $X^{\times}$ lie within distance at most R from X,

$$dist_{g^{\times}}(x^{\times}, X) \leq R \text{ for all } x^{\times} \in X^{\times}.$$

$\bullet_3$ The n-dimensional volumes of the submanifolds $X \times \{t\} \subset X \times [0,1) = X^\times$, $t < 1$, with respect to $g^\times$ vanish in the limit for $t \to 0$,

$$vol(X \times \{t\}) \to 0 \text{ for } t \to 1.$$

(The equivalence of this definition to the usual one follows from the the *filling volume inequality* see [9][12] and references therein).

Then the filling radius of a compact manifold X with boundary – our manifolds may, a priori, have boundaries and/or to be incomplete – is defined as $fil.rad$ of the double of X along the boundary and $fil.rad$ of an open X is defined via exhaustions of X by compact submanifolds.

It is obvious that $\quad \mathbf{A_+} \Rightarrow \mathbf{A} \Rightarrow \mathbf{A_-}$ and that $\mathbf{A_+}$ is optimal in a way.

Indeed, the product $X_r = X_0 \times S^2(r))$, where X_0 is, a compact manifold and $S^2(r))$ is the 2-sphere of small radius $r \to 0$, (these spheres have $Sc(S^2(r)) = \frac{2}{r^2}$), has $Sc(X_r) \geq (\frac{2}{r^2} - const_{X_0}) \to +\infty$, while the $(n-2)$-dimensional size/spread of X_r is as large as that of X_0.

Also one knows (see [17] at the end of this section and references therein) that

$$\mathbf{A_{++}} \Rightarrow \mathbf{A_-}.$$

(It is plausible in view of [18] that $\mathbf{A_{++}} \Rightarrow \mathbf{A}$.)

On the other hand, it is not hard to show that if the *if the isometry group of a Riemannin manifold $\hat{X}$ acts* cocompactly *on $\hat{X}$, i.e $\hat{X}/isom(\hat{X})$ is compact, and if $\hat{X}$ is* contractible, *then*

$$fil.rad(\hat{X}) = \infty.$$

Therefore, $\mathbf{A_-}$ yields the following topological $Sc > 0$-non-existence corollary.

B. Conjecture. *Closed manifolds X with contractible universal coverings $\tilde{X}$ admit no metrics with $Sc > 0$.*

(Granted **B**, the non-strict inequality $Sc(X) \geq 0$ implies that X *Ricci flat* by Kazdan-Warner's perturbation theorem (see [10] [13]). And since $\tilde{X}$ is contractible, the universal covering $\tilde{X}$ is *isometric to the Euclidean space $\mathbb{R}^n$, $n = dim(X)$*, by the *Cheeger-Gromoll splitting theorem*.)

Remarks and References.

However plausible, none of the **A**-conjectures (above dimension 2) has been confirmed except for $\mathbf{A_+}$ for 3-manifolds X with (apparently non-sharp) constant $const_+ = 2\pi\sqrt{6}$ (see [14] below).

On the other hand **B** is known to hold for many manifolds X, starting from the case of n-tori due to Schoen and Yau. Later B was proven by a use of *twisted Dirac operators*[14] for several classes of manifolds with "large" universal coverings including those X which admit metrics with non-positive sectional curvatures.

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[12] [9] L. Guth, *Notes on Gromov's systolic estimate*, Geom Dedicata (2006) 123:113-129.

[13] *If a metric g_0 with $Sc \geq 0$ can't be perturbed to g with $Sc(g) > 0$, then $Ricci(g) = 0$.*

[10] J Kazdan, F. Warner, Existence and Conformal Deformation of Metrics With Prescribed Gaussian and Scalar Curvatures, Annals of Mathematics, 101,# 2. (1975), pp. 317-331.

[14] This means: *Dirac operators with coefficients in some (possibly infinite dimensional) vector bundles.*

Below are a few relevant papers where one can find further references.

[11] Yau, S.T., and Schoen, R. *On the Structure of Manifolds with positive Scalar Curvature.* Manuscripta mathematica 28 (1979): 159-184.

[12] J. Lohkamp, *The Higher Dimensional Positive Mass Theorem II*, (2016) arXiv:1612.07505.

[13] R. Schoen, S.T. Yau, *Positive Scalar Curvature and Minimal Hypersurface Singularities*, (2017) arXiv:1704.05490.

In [11], the authors introduced their method of *induction descent by minimal hypersurfaces* and proved non-existence of metrics with $Sc > 0$ on the n-tori[15] and, more generally, on n-dimensional manifolds X which admit smooth maps $X \to \mathbb{T}^{n-2}$, such that the homology classes in $H_2(X)$ represented by the pull backs of generic points are *non-spherical.*

Originally, this method was limited to $n \leq 7$, but the techniques developed in [12] and [13] apparently remove this limitation.

[14] M. Gromov, H. B. Lawson, Jr., *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. IHS 58 (1983), 295-408.

In this paper besides above mentioned **A**$_+$ for 3-manifolds, we *rule* out complete metrics with $Sc > 0$ on certain classes of manifolds, including

closed orientable n-dimensional spin[16] manifolds X which admit continuous maps to complete manifolds Y with non-positive sectional curvatures, such that the fundamental classes $[X] \in H_n(X)$ go to non-zero classes in $H_n(Y)$ under these maps.

[15] M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures,* in Proc of 1993 Conf. in Honor of of the Eightieth Birthday of I. M. Gelfand, Functional Analysis on the Eve of the 21st Century: Volume I Progress in Mathematics, (1996) pp 1-213, Vol. 132,

This paper presents a geometric perspective on the Dirac operator and soap bubble methods in the study of scalar curvature and related problems.

[16] S. Markvorsen, M. Min-Oo, *Global Riemannian Geometry: Curvature and Topology*, 2012 Birkhuser.

A chapter in this book [17] offers a friendly introduction to the Dirac operator methods in the $Sc > 0$ problems.

[17] L. Guth, *Metaphors in systolic geometry.* In: Proceedings of the International Congress of Mathematicians. 2010, Volume II, pp. 745?768.

[18] L. Guth,*Volumes of balls in Riemannian manifolds and Uryson width.* Journal of Topology and Analysis, Vol. 09, No. 02, pp. 195-219 (2017).

These two papers and references therein give a fair idea of results and ideas around the filling radius.

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[15]This trivially implies non-existence of compactly supported perturbations with $Sc > 0$ of the Euclidean metric on $\mathbb{R}^n$.

[16]A manifold of dimension $n \geq 3$ is spin if the restrictions of the tangent bundle $T(X)$ to all immersed surfaces in X are trivial bundles.

Most (all?) known non-existence results for $Sc > 0$ obtained for *spin manifolds* more or less automatically generalise to manifolds whose *universal coverings are spin*, i.e where $T(X)$ trivialises on all immersed 2-spheres in X.

[17]Also see MinOo, *K-Area, mass and asymptotic geometry,*
http://ms.mcmaster.ca/minoo/mypapers/crm

[19] D.Bolotov, A. Dranishnikov. *On Gromov's conjecture for totally non-spin manifolds*, (2015) arXiv:1402.4510v6.

[20] M. Marcinkowski, *Gromov positive scalar curvature conjecture and rationally inessential macroscopically large manifolds.* Journal of topology 9, 1; 105-116 (2016). Oxford University Press

The authors of these two papers are concerned with topological versions of **A**$_+$ for certain classes of manifolds X.

[21] J. Rosenberg, *Manifolds of positive scalar curvature: a progress report*, in: Surveys on Differential Geometry, vol. XI: Metric and Comparison Geometry, International Press 2007.

This is survey of topological obstructions to metrics with $Sc > 0$ on spin manifolds X expressed in terms of indices of *Dirac operators twisted with C^*-algebras* of $\pi_1(X)$.

Also obstructions for 4-*dimensional* manifolds X with non-vanishing *Seiberg-Witten invariants* due to Taubes and Le Brun are described in this paper.

[22] M. Gromov, *Morse Spectra, Homology Measures, Spaces of Cycles and Parametric Packing Problems.* `www.ihes.fr/~gromov/PDF/Morse-Spectra-April16-2015-.pdf`

This is an overview of waists and related invariants which may bear some relevance to $Sc \geq \sigma$.

4 Extremality and Rigidity with Positive Scalar Curvature.

The proof(s) of the above A-conjectures (let them be only approximately true) would require *constructions* of certain maps or spaces which makes these conjectures difficult.

What is easier is getting upper bounds on the "size" of an X with $Sc(X) \geq \sigma > 0$ by proving *lower bounds* on dilations of *topologically significant* maps from X to (more or less) standard manifolds Y.

The first *sharp* bound of this kind was proved in

[23] M. Llarull, *Sharp estimates and the Dirac Operator*,

Math. Ann. 310 (1998), 55-71,

followed by

[24] M. Min-Oo, *Scalar Curvature Rigidity of Certain Symmetric Spaces*,

Geometry, Topology and Dynamics (Montreal, PQ, 1995), CRM Proc.

Lecture Notes, 15, Amer. Math. Soc., Providence, RI, 1998, pp. 127-136.

and by

[25] S. Goette and U. Semmelmann, *Scalar curvature estimates for*

compact symmetric spaces, Differential Geom. Appl. 16(1), (2002) 65-78,

where further references can be found.

What is proven in these papers can be expressed in the the following terms.

Extremality/Rigidity. A Riemannian metric $\underline{g}$ on a manifold Y is called *length extremal* if it

can't be enlarged without making the scalar curvature smaller somewhere.
Namely, the inequalities

$$Sc(g) \geq Sc(g_0) \text{ and } g \geq g_0$$

for a Riemannian metric g on Y imply

$$Sc(g) = Sc(\underline{g}).$$

Then the stronger implication

$$[Sc(g) \geq Sc(g_0)] \& [g \geq g_0] \Rightarrow [g = \underline{g}]$$

is qualified as *length rigidity* of $\underline{g}$.[18]

CY-Example. If a closed manifold Y admits no metric with $Sc > 0$, then all g_0 with $Sc(g_0) = 0$ [19] are extremal according to this definition.

Instances of such *scalar flat* manifolds are flat Riemannin manifolds (with universal coverings $\mathbb{R}^n$) and also (simply connected) hypersurfaces $Z \subset \mathbb{C}P^{n+1}$ of degree $n+2$ and even n, with *Ricci flat Calabi-Yau metrics,* where non-existence of metrics with $Sc > 0$ on these Z follows from the Lichnerowicz, theorem.

Next, define *area extremality* and *area rigidity* by relaxing the inequality $g \geq g_0$, which says in effect that $lenght_g(C) \geq lenght_g f(C)$ for all smooth curves $C \subset Y$), to

$$area_g(\Sigma) \geq area_{\underline{g}}(f(\Sigma))$$

for all smooth surfaces $\Sigma \subset Y$, where the extremality and rigidity requirements remains the same: $Sc(g) = Sc(\underline{g})$ and $g = \underline{g}$.

Stronger versions of these extremalities and rigidities allow modifications of the topology as well as geometry of Y, where the role of "topologically modified" Y are played by a Riemannin manifold $X = (X, g)$ and a map $f : X \to Y$, where the above inequalities are understood as

$$Sc(g)(x) \geq Sc(\underline{g})(f(x)), lenght_g(C) \geq lenght_{\underline{g}} g(f(C))$$

and
$$area_g(\Sigma) \geq area_{\underline{g}}(f(\Sigma))$$

correspondingly.

Accordingly, the required conclusion for *extremality* is

$$Sc(g)(x) = Sc(\underline{g})(f(x)),$$

while both, the *length* and the *area rigidities*, signify that

$$lenght_g(C) = lenght_{\underline{g}}(f(C)).$$

for all smooth curves $C \subset X$.

Of course, these definitions makes sense only for particular topological classes of manifolds X and maps f, such for instance as the class $\{\mathcal{DEG} \neq 0\}$ of orientable manifolds of dimension $n = dim(Y)$ and C^2-*smooth* maps with *non-zero degrees*.

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[18] Extremal manifolds define, in a way, the boundary of the domain $\{\mathcal{SC} \geq 0\}$ of manifolds with $Sc \geq 0$.

[19]The condition $Sc(g_0) = 0$ implies g_0 $Ricci(g_0) = 0$ on these Y by the Kazdan-Warner perturbation theorem, see [10] in section 3

C. Problem. Find verifiable criteria for extremality and rigidity, decide which manifolds admit extremal/rigid metrics and describe particular classes of extremal/rigid manifolds.

For instance,

do all closed manifolds which admits metrics with $Sc \geq 0$ also admit (length) extremal metrics?

More specifically, prove (disprove?) the following.

C$_1$. Conjecture. *All compact Riemannin symmetric spaces are area extremal in the class $\{\mathcal{DEG} \neq 0\}$ and those which have Ricci > 0 (this is equivalent to absence of local $\mathbb{R}$ factors, and to is finiteness of fundamental group) are area rigid in this class.*

This conjecture was proved by Llarull (see [23] above) in the case $Y = S^n$, under the additional assumption of X being *spin*.[20]

Then Min-Oo [24] proved *area extremality* for *Hermitian symmetric spaces* in the class $\{\mathcal{SPIN}, \mathcal{DEG} \neq 0\}$, where the maps $f : X \to Y$, besides having degrees $\neq 0$, must be *spin*.[21]

This was generalised by Goette and Semmelmann [25] who proved

area extremality in $\{\mathcal{SPIN}, \mathcal{DEG} \neq 0\}$ of compact (here it means closed) Kähler manifolds with Ricci ≥ 0, rigidity for Ricci ≥ 0.

Moreover, they establish

area rigidity in $\{\mathcal{SPIN}, \mathcal{DEG} \neq 0\}$ of certain (non-Hermitian) compact symmetric spaces including those with non-vanishing Euler characteristics and also of Riemannian metrics on S^{2m} with positive curvature operators.

These extremality and rigidity theorems are proven in the non-Kählerian cases by *sharply evaluating* the contribution from $f^!(\mathbb{S}^+(Y))$ in the Schroedinger-Lichnerowicz formula for the Dirac operator on X twisted with the f-pullback of the *spinor*$^+$ *bundle* $\mathbb{S}^+(Y)$ which is, in the case where $\chi(Y) \neq 0$ is confronted with the index theorem.

(The case of odd dimensional spheres S^{2m-1}, which depends on an additional argument(s) applied to maps $X \times S^1 \to S^{2m}$ [22] seems to apply only to metrics on S^{2m-1} with constant sectional curvatures.)

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[20]Since $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$, there are at most two isomorphism classes of vector bundles with $rank \geq 3$ over connected surfaces Σ (exactly two for closed Σ), where the trivial bundle is called *spin* and where bundles of $rank < 3$ are spin if their Whitney sums with trivial bundles are spin. An orientable vector bundle V of over a topological space B is *spin* if the pullbacks of V under continuous maps $\phi : \Sigma \to B$ for all surfaces Σ are spin. A manifold X is spin if its tangent bundle is spin.

The spin condition is necessary for the definition of the Dirac operator on X but some *twisted Dirac operators* make sense on non-spin manifolds.

[21]A map $f : X \to Y$ is *spin* if the pullbacks $\phi^!(T(X))$ for maps of surfaces, $\phi : \Sigma \to X$, satisfy

$$[\phi^!(T(X)) \text{ is spin}] \Leftrightarrow [(\phi \circ f)^!(T(Y)) \text{ is spin}]$$

for all Σ and f. Equivalently, a map f between orientable manifolds is spin if the Whitney sum $T(X) \oplus f^!(T(Y))$ is spin.

Obviously, the identity map $id : Y \to Y$ is spin and if Y is spin, e.g. $Y = S^n$, then

$$[f : X \to Y \text{ is spin}] \Leftrightarrow [X \text{ is spin}].$$

[22]Llarull uses the product metric on $X \times S^1$, where his calculation applies even though the scalar curvature $Sc(X \times S^1)$, which is $\geq Sc(S^{2m-1})$, may be smaller than $Sc(S^{2m})$.

Alternatively, one can use the *spherical suspension metric* g^S (of g on X) on (the bulk of) $X \times S^1$, which has $Sc(g^S) \geq Sc(S^{2m})$ and thus allows a formal reduction of the $2m - 1$ case

And in the Kähler case, this is done with the *"virtual square root" of the canonical* (complex) *line bundle* on Y instead of $\mathbb{S}^+(Y)$.

Spin or non-Spin? In all of the above cases one can replace the spin condition for $f : X \to Y$ by this condition for the corresponding map between the universal coverings, $\tilde{f} : \tilde{X} \to \tilde{Y}$, where a version of Atiyah's L_2-index theorem applies.

Probably,

"spin" can be removed all together in these theorems

but this seems beyond reach of the present day methods.[23]

On the other hand, the spin condition is essential for the extremality in the class $\{\mathcal{SPIN}, \mathcal{DEG}_{\hat{A}} \neq 0\}$ where the dimension of X can be greater than $n = dim(Y)$ and where the condition $deg(f) \neq 0$ is replaced by $deg_{\hat{A}}(f) \neq 0$, where the $\hat{A}$-*degree* $deg_{\hat{A}}(f)$ stands for the $\hat{A}$-genus of the f-pull back of a generic point $y \in Y$,

$$deg_{\hat{A}}(f) = \hat{A}(f^{-1}(y)).$$

(Here, strictly speaking, f must be smooth; if f is just continuous, this applies to a smooth approximation of f, where the so defined $\hat{A}$-degree does not depend on a choice of approximation.).

This implies for instance, that

the products of the above Y, e.g. of $Y = S^n$ by the Calabi-Yau manifolds with $\hat{A} \neq 0$, e.g with Z from the above CY-example are area extremal in the class $\{\mathcal{SPIN}, \mathcal{DEG}_{\hat{A}} \neq 0\}$ as well s in the class $\{\overline{\mathcal{SPIN}}, \mathcal{DEG}_{\hat{A}} \neq 0\}$ where spin condition is delegated to $\tilde{f} : \tilde{X} \to \tilde{Y}$.

Notice, however, that *neither* simply connected Calabi-Yau manifolds Z themselves *nor* their products by Y are extremal in the class $\{\mathcal{SPIN}, \mathcal{DEG} \neq 0\}$, at least if $dim(Z) \geq 5$.

Indeed the connected sums $X = Z\#(-Z)$, where "−" stands for the reversal of orientation and where the obvious map $Z\#(-Z) \to Z$ has degree 1, admit metrics with $Sc > 0$ by Stolz' theorem mentioned in section 2. sleeker

It seems that the there are two divergent, yet interconnected by bridges, branches in the tree of $Sc(X) \geq 0$, where a smoother and sleeker one involves differential structure and depends on spin, while the other one is made of rougher staff such as the homotopy classes of X. [24]

Probably, the second branch can be transplanted to a harsh world inhabited by singular spaces but fully cleaning off spin from this branch is by no means easy even for smooth X.

Extremality and Rigidity of Products. It seems not hard to show[25] that the Riemannian products of the area extremal/rigid manifolds in the above examples are area extremal/rigid which suggests to the following.

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to that of $2m$.

[23]Apparently, no single case of extremality of a closed simply connected manifold X of dimension $n \geq 3$ is amenable to the the minimal hypersurface techniques, except, may be(?) for $X = S^3$.

[24]The smooth branch is manifested by $\hat{A}$ and the mod 2 α-invariant in the index formula while the rough branch is represented by the Chern character and supported by minimal hypersurfaces.

[25]I have not verified the proof in detail at this point.

C₂. **Question**. Are the Riemannian products of all area extremal/rigid manifolds area extremal/rigid

Smoothing Lipschitz Maps. The length extremal/rigid manifolds in some *homotopy* class of smooth maps remain extremal/rigid in the corresponding class of Lipschitz maps f.

This can be proven by a smooth approximation of these f with a minor change of their length dilations.

But

       this is unclear for the *area* extremality and/or area rigidity,

since, conceivably(?) all smooth approximation f' of a Lipschitz map $f : X \to Y$ may have $area(f'(\Sigma)) >> area(f(\Sigma))$ for some Σ.

Normalisation by Scalar Curvature: Extremality/Sc and Rigidity/Sc. A map $f : X \to Y$ between Riemannin manifolds $X = (X, g)$ and $Y = (X, g_0)$ with positive scalar curvatures, $Sc(g), Sc(g_0) > 0$, is called *length decreasing/Sc* if it decreases the length of the curves measured in the metrics $Sc(X)^{-1}g$ and $Sc(g_0)^{-1}g_0$, i.e. if it decreases the integrals of $\sqrt{Sc}$ over all curves in X. Similarly one understand *decrease/Sc of areas* of surfaces $\Sigma \subset X$ under maps $X \to Y$, etc.[26]

Accordingly, one defines *length/area extremality/Sc* of a Y as non existence of strictly length/area decreasing/Sc maps $X \to Y$ in a given class of manifolds and maps, while the rigidity/Sc signifies that all length/area non-increasing/Sc maps $f : X \to Y$ are homotheties (similarities) with respect to the original metrics, i.e. $f^*(g_0) = const \cdot g$.

Since the "contribution of the twist" to the Schroedinger-Lichnerowicz formula for the twisted Dirac opertor on X scales as $Sc(X)^{-1}$, the arguments from the above cited papers based on this formula deliver the corresponding extremality/Sc and rigidity/Sc results. (This was pointed out in [26][27])

Category $\mathcal{R}_+/_{Sc}$. Let this be the category of Riemannian manifolds with $Sc > 0$ and length (alternatively, area) non-increasing/Sc maps.

C₃. **Question**. *How much of the geometry of spaces with $Sc > 0$ can be reconstructed in the category theoretic language of $\mathcal{R}_+/_{Sc}$?*

Extremality beyond $Sc \geq 0$. The condition $Sc(g) \geq 0$ may be not indispensable for extremality of g.

For instance, the double of the unit hyperbolic disk is (kind of) extremal for the natural C^0-continuous metric on it and there are similar high dimensional examples. But it is unclear if such metrics are ever smooth.

Relativisation of Non-existence Theorems for $Sc > 0$. Let Y be a closed length or area *extremal or rigid* manifold in some class of smooth manifolds X and smooth maps $f : X \to Y$, where this class is invariant under homotopies of maps.

Then, most (all?) known *Dirac operator obstructions* to the existence of metrics with $Sc > 0$ on closed manifolds X_0 naturally extend to *similar obstructions* to the existence of (strict) *area decreasing/Sc* maps in certain homotopy invariant classes of maps $X \to Y$, including $X = X_0 \times Y \to Y$ for $(x_0, y) \mapsto y$.

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[26] It may (or may not) be worthwhile to normalise by $g \rightsquigarrow n(n-1)Sc(X)^{-1}g$, $n = dim(X)$, and see what happens for $n \to \infty$.

[27] [26] M.Listing, *The Scalar curvature on compact symmetric spaces*, arXiv:1007.1832, 2010 - arxiv.org.

For instance, one knows that (co)*homologically symplectic manifolds* X_0 with $\pi_2(X_0) = 0$ admit no metrics with $Sc > 0$ and the proof of this (see [15] cited in the previous section) also implies that words

if Y is the above area-extremal manifold, e.g. $Y = S^n$, then no homologically symplectic[28] *map $f : X \to Y$, which, moreover, induces an isomorphism $\pi_2(X) \to \pi_2(Y)$, can be strictly area decreasing/Sc.*

This suggests the following.

C_4. **Conjecture.** Let g be a metric on X and $f_0 : X \to Y$ be a (smooth?) strictly length (area?) decreasing/Sc map in this class.

Then there exists a smooth map f homotopic to f_0 transversal to a point $y_0 \in Y$, such that the f-pullback submanifold $f^{-1}(y_0) \in X$ admits a metric with $Sc > 0$.

Also other properties, e.g. *extremality*, of *manifolds X with $Sc(X) > 0$* may have counterparts for length and area decreasing/Sc *maps $X \to Y$* and, furthermore, for *foliations* on X.

C_5. **Question.** *Are infinite dimensional counterparts of compact symmetric spaces, e.g. the Hilbert sphere S^∞, extremal/rigid in some class(es) of perturbations of their metrics?*

5 Extremality and Gap Extremality of Open manifolds.

Let $U \subset Y$ be an open subset in a extremal or rigid Riemannin manifold Y where the extremality/rigidity for this Y follows by the twisted Dirac operator argument from the previous section. Then the same argument yields the following.

$\star$ *If the complement $Z = Y \smallsetminus U$ is non-empty, yet* LC-negligible *(explained below) then* no complete *orientable Riemannian manifold admits a smooth area non-increasing/Sc map $f : X \to U$, which has non-zero degree*[29] *and the lift of which to to the universal coverings, $\tilde{f} : \tilde{X} \to \tilde{U}$, is* spin.

LC-negligible Sets . A piecewise smooth polyhedral subset Z in a Riemannin manifold Y is called LC-negligible if the Levi-Civita connection on the tangent bundle of X restricted to Z is *split trivial.* For instance,

• finite subsets in Y are LC-negligible;
• piecewise smooth graphs $Z \subset Y$ with trivial monodromies around the cycles, e.g. disjoint unions of trees, are LC-negligible;
• simply connected *isotropic* (e.g.*Lagrangian*) submanifolds in Kähler manifolds are LC-negligible.

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[28] A smooth proper map between orientable manifold, $f : X \to Y$, is *homologically symplectic* if the difference of the dimensions $n_0 = n - m$ for $n = dim(X)$ and $m = dim(Y)$ is *even* and if there exists a closed 2-form ω on X such that the integrals of $\omega^{\frac{n_0}{2}}$ over the f-pullbacks of generic points $y \in Y$ do not vanish.

In other words, the real fundamental cohomology class $[X]^\circ \in H^n_{comp}(X; \mathbb{R})$ with compact support is equal to the $\smile$-product of the f-pullback of $[Y]^\circ \in H^m_{comp}(Y, \mathbb{R})$ and the $\frac{n_0}{2}$th-$\smile$-power of the class $[\omega] \in H^2(X; \mathbb{R})$,
$$[X]^\circ = f^*([Y]^\circ) \smile [\omega]^{\frac{n_0}{2}}.$$

[29] Maps $f : X \to Y$ of non-zero degree, by definition, must be equidimensional and proper.

This definition extends to general closed subsets Z, such as *Cantor sets*, for instance, by requiring that the monodromies along smooth curves C in the ε-neighbourhoods of Y are $o(\varepsilon \cdot \text{length } (C))$ as $\varepsilon \to 0$ but the geometry behind this definition needs to be clarified.

D$_1$. Problem. *Study essential properties, such as the Hausdorff dimensions, of these subsets $Z \subset Y$ and find cases (if there are any) where $\star$ remains valid for small, yet non-LC-negligible $Z \subset Y$, e.g. for (generic) smooth curves Z in Y.*

Notice in this regard that a simple surgery type argument (see Stolz' paper [5] cited in section 2 and references therein) shows that

• if Z is equal to the k-skeleton $\mathcal{T}^k$ of a smooth triangulation $\mathcal{T}$ of a compact Riemannin manifold (Y, g_0), for $k \geq 2$, then $U = Y \smallsetminus Z$ admits a complete metric $g \geq g_0$ with $Sc(g) \geq \sigma_0 = \sigma_0(Y, Z) > 0$.

Moreover, it is easy to show that

the complements $U_\varepsilon = Y \smallsetminus \mathcal{T}^k_\varepsilon$ of the k-skeleta of the "standard fat" ε-refinements[30] of $\mathcal{T}$ admit complete Riemannin metrics $g_\varepsilon \geq g$ the scalar curvatures of which for $k \geq 2$ satisfy

$$Sc(g_\varepsilon) \geq const \frac{1}{\varepsilon^2}$$

for some constant $const = const(Y, \mathcal{T}) > 0$.

Thus $\star$ fails to be true, for $Z = \mathcal{T}^k_\varepsilon$, $k \geq 2$, and small (how small?) ε.

On the other hand, the *torical band width inequality* from the next section shows that if, for instance, Z is a codimension two torus in Y, e.g. $Z = \mathbb{T}^2 \subset S^4$, then the complement $U = Y \smallsetminus Z$ admits no complete metrics with $Sc \geq \sigma > 0$ whatsoever and the same applies to a large (how large) class of codimension two polyhedra $Z \subset Y$ with contractible universal coverings.

Non-existence of *complete* metrics $g \geq g_0$ with $Sc > \sigma_0$ on the above $U = (U, g_0)$ with $Sc(g_0) = \sigma_0$ may be interesting in its own right but this can't be regarded as extremality of g_0, since a comparison of the manifolds (U, g_0), which have *bounded diameters* with their competitors (U, g) of *infinite size* is patently unfair. The true extremity issue for these U, thus, remains unresolved.

D$_2$. Question. Do there ever exist length extremal domains $U \subset Y$, $U \neq Y$, in closed connected Riemannin manifolds Y of dimensions ≥ 3?

For instance, is the the sphere S^3 minus a point (or the 3-torus minus a point) extremal?

We still do not know the answer but, on the other hand, the following *warped product* construction sometimes delivers examples of both complete and non-complete extremal and rigid manifolds (compare §12 in [14] cited in section 3 and [27] cited below).

Let $Y_0 = (Y_0, g_0)$ be a Riemannian manifold with constant scalar curvature σ_0 and let $g_1 = \varphi^2 g_0 + dt^2$ be a Riemannin metric on $Y_1 = Y \times (l_-, l_+)$ for $-\infty \leq l_- < l_+ \leq \infty$, for some smooth function $\varphi = \varphi(t) > 0$ for $l_- < t < l_+$.

---

[30]It is more practical to start with a *cubilation* $\mathcal{T}$ of Y which can be canonically ε-refined for $\varepsilon = \frac{1}{i}$, $i = 2, 3, \ldots$, by subdividing each m-cube into i^m-sub-cubes in an obvious way.

Then, by elementary calculation,

◑ $\qquad Sc(g) = \dfrac{\sigma_0}{\varphi^2} - 2n\dfrac{\varphi''}{\varphi} - n(n-1)\dfrac{\varphi'^2}{\varphi^2}$, where $n = dim(Y_0)$.

Now, let g have *constant scalar curvature*, say $Sc(g_1) = \sigma_1$ for a given $\sigma_1 \geq 0$, and prescribe: $\varphi(0) = 1$ and $\varphi'(0) = 0$.

Then ◑, regarded as an ODE and rewritten as

$$f'' = -\frac{1}{2}(n+1)f'^2 + \frac{\sigma_0}{2ne^{2f}} - \frac{\sigma_1}{2n} \quad \text{for } f = \log\varphi,$$

admits a unique solution f on some maximal (extremal) open interval (l_-^{ext}, l_+^{ext}) beyond which the solution does not extend.

Examples.(a) If $Y_0 = S^n$ and $\sigma_1 = n(n+1)$, then Y_1 is equal to S^{n+1} minus two opposite points.

(b) If $Y_0 = \mathbb{R}^n$ and $\sigma_1 = 0$, then $Y_1 = \mathbb{R}^{n+1}$.

(c) If $Y_0 = \mathbb{R}^n$, $\sigma_1 = n(n+1) = Sc(S^{n+1})$ and $n = 1$, then Y_1 is equal the universal covering of S^2 minus two opposite points.

In general, the manifold (Y_1, g_1) is uniquely characterised by the following three properties.

$[\bigcirc_{n(n+1)}]$ The scalar curvature of Y_1 is everywhere equal to $n(n+1)$ for $n = dim(Y_1) - 1$.

$[\bigcirc_{O(n) \ltimes \mathbb{R}^n}]$ The isometry group of Y_1 is $Iso(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ times $\mathbb{Z}_2$. (This $\mathbb{Z}_2$ corresponds to the involution $t \leftrightarrow -t$.)

$[\bigcirc_{2\pi/n+1}]$ The *band width* of Y_1 is $\frac{2\pi}{n+1}$, where this width is understood in the present case as the distance between the two (one point) boundary components of Y_1 in the metric completion $\bar{Y}_1 \supset Y_1$.

(The band-like shape of Y_1 is best seen for $dim(Y_1) = 2$, where this Y_1 is equal to the universal covering of the doubly punctured sphere S^2.)

Alternatively, one might say that the in-radius of Y_i is equal to $\frac{\pi}{n+1}$:

> there are closed compact balls in Y_1 of all radii $R < \frac{\pi}{n+1}$ but no ball of radius $\geq \frac{\pi}{n+1}$ is compact.

Gap Extremality. We do not know if the above spheres minus pairs of points are extremal for $n \geq 2$ but the Euclidean spaces $\mathbb{R}^m$ are definitely *not length extremal* starting from $m = 2$.

In fact, there are (obvious, $O(m)$-invariant) metrics $g \geq g_{Eucl}$ on $\mathbb{R}^m$ with $Sc(g_1) > 0$ for all $m \geq 2$.

On the other hand,

$(\ast)$ *no metric* $g \geq g_{Eucl}$ *on* $\mathbb{R}^m$ *may have* $Sc(g) \geq \varepsilon > 0$.(See [15] cited in section 3.)

This suggests the following weaker version of extremality for non-compact manifolds which we call *gap extremality*.

A metric g_0 on Y is *ε-gap length extremal* if *no* $g \geq g_0$ on Y satisfies

$$Sc(g) - Sc(g_0) > \varepsilon.$$

Then g_0 is called *gap length extremal* if it is *ε-gap length extremal* for all $\varepsilon > 0$ (0-gap extremal=extremal).

Similarly one defines *area gap extremality* and *gap extremality for classes of maps* $f : X \to Y$. (But I am not certain what a *workable* definition of normalised gap extremality/Sc should be.)

Whenever the twisted Dirac operator argument from the previous section yields area extremality of a closed manifold Y, e.g. if $Y = S^n$ or $Y = \mathbb{C}P^n$, this argument, combined with that from [15] (cited in section 3) for $\mathbb{R}^m$, also delivers

(✷✷) *gap area extremality of* $Y_m = Y \times \mathbb{R}^m$ *for all* $m = 1, 2, ...,$ *as well as this extremality for smooth proper spin maps* $f : X \to Y_m$ *of non-zero degrees.*

if a smooth proper spin map $f : X \to Y_m$ *of non-zero degree decreases the areas of all surfaces* $\Sigma \subset X$, *then, given* $\varepsilon > 0$, *there exists a point* $x \in X$, *such that*

$$Sc(X)(x) - Sc(Y')(f(x)) < \varepsilon.$$

D$_3$. Question. Does gap extremality is always stable under $Y \rightsquigarrow Y \times \mathbb{R}^m$? (Beware of $dim(Y) = 4$.)

One can't discard of ε for $m \geq 2$ but the true area (or, at least length) extremality of $Y' = Y \times \mathbb{R}$ (that allows $\varepsilon = 0$) may be provable by some twisted Dirac operator argument. For instance, if $Y = \mathbb{T}^n$ this follows from theorem 6.12 in [14] (cited in section 3). Alternatively, one might use minimal hypersurfaces and soap bubble in X the f-images of which separate the two ends in $Y' = Y \times \mathbb{R}$ but then onr would face a possibility of non-compact minimal hyper surfaces in X and would be obliged to resort to imposing extra assumptions on X, e.g. uniform two sided bounds on the sectional curvatures of X.

Finally, let us look at the manifold Y_1, which has the band width $\frac{2\pi}{n+1}$, in the above Example (c).

It is plausible that this Y_1 is length gap extremal but not length extremal starting from $D = dim(Y_1) = 3$.

And what we definitely know is that

the quotient space $Y_1/\mathbb{Z}^n = \mathbb{T}^n \times (-\frac{\pi}{n+1}, \frac{\pi}{n+1})$, $n + 1 = dim(Y_1)$, *is length extremal.*

We shall see the reason for this in the next section, where we shall also explain the current status of the rigidity problem for these manifolds.

6 Bounds on Widths of Bands with Positive Curvatures.

Let us start with the following question which, on the surface of things, has nothing to do with scalar curvature.

Given a smooth n-dimensional manifold X immersed[31] into a complete Riemannian manifold Y denote by $rad^\perp(X \looparrowright Y)$ the maximal R, such that the normal exponential map

$$exp^\perp : T^\perp(X) = T(Y)_{|X} \ominus T(X) \to Y,$$

---

[31] A smooth map $X \to Y$ is an immersion if it is a diffeomorphism of small neighbourhoods in X to smooth submanifolds in Y.

is locally injective on the subbundle $B^\perp(R)(X) \subset T^\perp(X)$ of open normal R-balls $B_x^{N-n}(R) \subset T^\perp(X)$, $x \in X$.

(If the ambient space $Y = \mathbb{R}^n$, then $rad^\perp(X; \mathbb{R}^n)$ is equal to the reciprocal of the supremum of the principal curvatures of X.)

Take the supremum of these radii over all immersions $f : X \hookrightarrow Y$, set

$$suprad^\perp(X; Y) = \sup_f rad^\perp(X \underset{f}{\hookrightarrow} Y)$$

and let

$$suprad_N^\perp(X) = \sup_{f_\circ} rad^\perp(X \underset{f_\circ}{\hookrightarrow} \mathbb{R}^N),$$

where the latter "sup" is taken over all immersion $f_\circ$ from X to the *unit ball* $B^N(1) \subset \mathbb{R}^N$.

(The notation $suprad^\perp(X; B^N(1))$ would be unjustified, since the image of the exponential map may be not contained in $B^N(1)$.)

E_1. **Problem**. Evaluate $suprad_N^\perp(X)$ in terms of the topology of X.

Examples. (a) It is obvious that $suprad_N^\perp(X) \leq 1$ for all closed manifolds X, where the equality holds if and only if X is diffeomorphic to S^n and $N > n$.

(b) Let X_k is diffeomorphic to the product of k spheres,

$$X_k = S^{n_1} \times ... \times S^{n_k}, n_k \geq 1.$$

Then

$$suprad_N^\perp(X) \geq \frac{1}{\sqrt{k}} \text{ for all } N \geq (n_1 + 1) + ... + (n_k + 1).$$

But we do not know, for instance, whether

$$suprad_N^\perp(X_k) \to 0 \text{ for } N = dim(X_k) + 1 \text{ and } k \to \infty.$$

or, on the contrary, if

$$suprad_N^\perp(X) \geq \rho_0$$

for *all* manifolds X, (e.g. for all X_k) all sufficiently large $N \geq N(X)$ and some universal constant $\rho_0 > 0$, say $\rho_0 = 0.001$.

All known upper bounds on $suprad_N^\perp(X)$ – *am I missing something obvious?* exclusively apply to manifolds X which admit no metrics with $Sc > 0$.

A simple way to obtain such a bound is as follows.

1.Scale $B^N(1) \to B^N(\frac{1}{2})$, project $B^N(\frac{1}{2})$ to S^N from the south pole of S^N and observe that this distorts the curvatures of submanifolds X in the ball $B^N(1)$ by a finite amount independent of X and N.

2. Apply the Gauss formula to $X \hookrightarrow S^N$ and thus show that the supremum of the principal curvatures of X in S^N satisfies

$$supcurv(X \hookrightarrow S^N) \geq \frac{\sqrt{n-1}}{N-n}$$

and therefore,

$$suprad_N^\perp(X) \leq const \cdot \frac{N-n}{\sqrt{n-1}}$$

for all n-dimensional manifolds X which admit no metrics with $Sc > 0$ and for some constant $const \leq 100$. (See [27] cited below for details.)

It follows, for instance, that there are

exotic spheres Σ^n of dimensions $n = 9, 17, 25, 33, ...,$ such that

$$suprad_{n+1}^{\perp}(\Sigma^n) \leq \frac{100}{\sqrt{n-1}},$$

but one has no idea how sharp this inequality is and if there are similar inequalities for exotic spheres which admit metrics with $Sc > 0$.

The above also applies to tori $\mathbb{T}^n$, since these admit no metrics with $Sc > 0$ either, but here the following better (but, probably, still very far from being sharp) inequality is available.

$$suprad_{n+1}^{\perp}(\mathbb{T}^n) \leq \frac{2\pi}{n+1}.$$

This is proven again by passing to S^{n+1}, where all we use of the geometry of S^{n+1} is the inequality $Sc(S^{n+1}) \geq n(n+1)$. (Isn't it amazing that there is no apparent *direct* proof of a *much stronger* bound on $rad^{\perp}(\mathbb{T}^n \subset B^{n+1}(1))$.))

Namely, the above bound on $suprad_{n+1}^{\perp}(\mathbb{T}^n)$ trivially follows from the following.

Torical Band Width Inequality. *Let g be a metric with $Sc(g) \geq n(n+1) = Sc(S^{n+1})$ on the torical band (cylinder) $\mathbb{T}^n \times [-1, 1]$. Then the distance between the two boundary components of this band satisfies*

$$\left[\,\mathbf{O}_{\pm} < \tfrac{2\pi}{n+1}\,\right] \qquad\qquad dist_g(\mathbb{T}^n \times \{-1\}, \mathbb{T}^n \times \{1\}) < \frac{2\pi}{n+1}.$$

This is proven in $[27]$[32] with a relative version of the Schoen-Yau minimal hypersurface method.

Besides a bound on $suprad_{n+1}^{\perp}(\mathbb{T}^n)$, the inequality $\left[\,\mathbf{O}_{\pm} < \tfrac{2\pi}{n+1}\,\right]$ (trivially) implies that

the warped product metric $\varphi^2(t)g_{\mathbb{T}^n} + dt^2$ on $\mathbb{T}^n \times \left(-\frac{\pi}{n+1}, \frac{\pi}{n+1}\right)$ with $Sc = n(n+1)$, which was introduced in the previous section, is length extremal.

Also, the argument in $[27]$ yields *length rigidity* of this metric for $n \leq 6$, while the general case needs an elaboration on recent results on "irrelevance of singularities" of minimal hypersurfaces proved in the papers $[12]$ and/or $[13]$ cited in section 3.

7 Extremality and Rigidity of Convex Polyhedra.

Let $P \subset \mathbb{R}^n$ be a compact convex polyhedron with non-empty interior, let $Q_i \subset P$, $i \in I$, denote its $(n-1)$-faces and let

$$\angle_{ij}(P) = \angle(Q_i, Q_j)$$

denote its dihedral angles.

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[32][27] M.Gromov, *Metric Inequalities with Scalar Curvature.*
http://www.ihes.fr/~gromov/PDF/Inequalities-July%202017.pdf.

Say that P is *extremal* if all convex polyhedra P' which are combinatorially equivalent to P and which have

$$\angle_{ij}(P') \le \angle_{ij}(P) \text{ for all } i, j \in I,$$

necessarily satisfy

$$\angle_{ij}(P') = \angle_{ij}(P).$$

It is known – the proof is elementary – that

the simplices and the rectangular solids are extremal and also all P with $\angle_{ij}(P) \le \frac{\pi}{2}$, are extremal.

But it is unclear (at least to the present author) what are (if any) non-extremal P.

What we are truly interested in, however, is extremality (and rigidity) of P under transformations which keep the faces Q_i convex (rather than flat) or, even better, *mean convex*, i.e. keeping their mean curvatures non-negative.

Thus, we say that P is *mean convexly extremal* if *there is no $P' \subset \mathbb{R}^n$ diffeomorphic to P* and such that

• the faces $Q'_i \subset P'$ corresponding to all $Q_i \subset P$ have $mean.curv(Q'_i) \ge 0$,

• the dihedral angles of P', that are the angles between the tangent spaces $T_{p'}(Q'_i)$ and $T_{p'}(Q'_j)$ at the points p' on the $(n-2)$-faces $Q'_{ij} = Q'_i \cap Q'_j$, satisfy

$$\angle_{ij}(P') \le \angle_{ij}(P),$$

• this angle inequality is strict at some point, i.e. there exits $p'_0 \in Q'_{ij}$ in some Q'_{ij}, such that

$$\angle (T_{p'_0}(Q'_i), T_{p'_0}(Q'_j)) < \angle_{ij}(P).$$

F$_1$. Question. *Are all extremal convex polyhedra P are mean convexly extremal?*

It is not even known if the regular 3-simplex is mean convexly extremal, but

the mean convex extremality of the n-cube

follows by developing the cube P into a complete (orbi-covering) manifold $\hat{P}$ homeomorphic to $\mathbb{R}^n$ by reflecting P in the faces, approximating the natural continuous Riemannin metric metric on $\hat{P}$ by a smooth one with $Sc \ge \varepsilon > 0$ (see [28][33]) and appealing to gap extremality of $\mathbb{R}^n$ stated in section 5.

And the same argument yields (see [28]) the following

[∗] *Let a Riemannin metric g on the n-cube P satisfy:*
    $*_0$ $Sc(g) \ge 0$.
    $*_1$ $mean.curv_g(Q_i) \ge 0$,
    $*_2$ $\angle_{ij}(P, g) \le \frac{\pi}{2}$.
Then, necessarily, $Sc(g) = 0$, $mean.curv_g(Q_i) = 0$ and $\angle_{ij}(P, g) = \frac{\pi}{2}$.

Probably, these equalities imply that P is *isometric to a Euclidean rectangular solid* but the approximation/smoothing is no good for proving this kind of rigidity.

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[33][28] M.Gromov *Dirac and Plateau Billiards in Domains with Corners*, Cent. Eur. J. Math. 12(8) pp 1109-1156, (2014).

The main merit of [✱] is that it provides a test for $Sc \geq 0$ in all Riemannin manifolds X:

$Sc(X) \geq 0$ *if and only if no cubical domain* $P \subset X$ *satisfies*

$$[mean.curv_g(Q_i) > 0]\&[\angle_{ij}(P,g) \leq \frac{\pi}{2}].$$

This suggests a possibility of defining $Sc(X) \geq 0$ for some singular spaces, X. e.g. for *Alexandrov spaces* with sectional curvatures bounded from below.

F$_2$. Conjecture *All known (and expected) properties of Riemannian manifolds with $Sc \geq 0$, which have no "spin" attached to their formulations, generalise to Alexandrov's spaces.*

For instance, most probably,

if an n-dimensional Alexandrov space X with curvatures bounded from below has $Sc > 0$ at all regular points $x \in X$, (or if the volumes of all infinitesimally small balls in X are bounded by the volumes of such Euclidean balls) then

every continuous map from X to a space Y with $CAT(0)$ universal covering (i.e. an Alexandrov's space with non-positive sectional curvatures) contracts to an $(n-1)$-dimensional subset in Y.

If true, this would imply that (suitably defined) *harmonic maps* $X \to Y$ must necessarily have $(n-1)$-*dimensional* images, which suggests a (non-local?) Weitzenboeck-Bochner type formula in this context and a definition of $Sc > 0$ via spectral properties of small (large?) balls (cubes?) in X.

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