DYNAMICAL MORSE ENTROPY

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1. INTRODUCTION

Consider a "crystal", that is, the standard lattice $\Gamma = \mathbb{Z}^3$ in $\mathbb{R}^3$ with identical "molecules" positioned at all sites (points) γ in Γ. Denote by M the configuration space of such a molecule which is assumed to be a smooth finite dimensional manifold and let $X = M^\Gamma$ be the configuration space of the crystal, that is, the infinite product of Γ copies of M. Suppose adjacent molecules interact via a potential (energy) which is, by definition, a smooth function of two variables, say $f : M \times M \to \mathbb{R}$. Then the total energy of the crystal could be thought of as the (infinite !) sum of copies of f over all adjacent pairs of sites (γ, γ'), called edges of Γ (with six edges at each site) :

$$F(x = (x_\gamma)) = \sum_{\text{edges}(\gamma,\gamma')} f(x_\gamma, x_{\gamma'}). \tag{$*$}$$

Such an F is clearly almost everywhere infinite for non-trivial f but its gradient (differential) is obviously well defined and finite at all points x in X. Thus one may speak of the critical points of F, also called the stationary states of the crystal. Observe that the set S of stationary states make a closed subset in X invariant under the obvious (shift) action of Γ on X. The basic question is that of evaluating the entropy of the dynamical system (S, Γ) in terms of the topology of M and/or some generic features of f. One still does not have a satisfactory criterion for non-vanishing of this entropy except for a few specific cases, such as the discretized geodesic flow on a Riemannian manifold for instance, but one can give a lower bound on the asymptotic distribution of the critical values of F as follows.

Exhaust Γ with some standard subsets Ω_i, e.g. by concentric cubes of edge size $2i$, and let F_i denote the "restriction" of F to Ω_i, that is, the sum of the terms in $(*)$ corresponding to edges in Ω_i. This sum is regarded as a function on M^{Ω_i}, call it $F_i : M^{\Omega_i} \to \mathbb{R}$. It is further normalized by letting $F'_i = 1/\operatorname{card}(\Omega_i)\, F_i$. The functions F'_i take values in a fixed interval, namely in $[f- = 6\inf(f), f+ = 6\sup(f)]$. We count the number $\#_i(I)$ of critical values in each subinterval I of $[f-, f+]$, and set

$$\operatorname{cri}_i(I) = \frac{1}{\operatorname{card}(\Omega_i)} \log \#_i(I).$$

The purpose of this paper is to provide a Morse theoretic lower bound for $\liminf \operatorname{cri}_i(I)$, $i \to \infty$, in terms of a certain (strictly positive !) concave function (entropy) on $[f-, f+]$ capturing the homological behaviour of functions F'_i for $i \to \infty$.

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2. FRAMEWORK

Consider a countable group Γ endowed with a left-invariant metric $d : \Gamma \times \Gamma \to \mathbb{R}^+$. Given a finite subset $\Omega \subset \Gamma$, its cardinality is denoted by $|\Omega|$. The set of finite subsets of Γ is denoted by

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$B(\Gamma)$. The distance d is extended to a map $d : B(\Gamma) \times B(\Gamma) \to \mathbb{R}^+$ (not a distance) as follows :

$$d(\Omega, \Omega') = \inf\{d(\gamma, \gamma'); \gamma \in \Omega, \gamma' \in \Omega'\}.$$

Given a nonnegative number N, the N-boundary and the N-interior of $\Omega \in B(\Gamma)$ are the sets

$$\begin{aligned} \partial_N \Omega &= \{\gamma \in \Gamma; d(\gamma, \Omega), d(\gamma, \Gamma - \Omega) \leq N\}, \\ \mathrm{int}_N \Omega &= \Omega - \partial_N \Omega. \end{aligned}$$

When reference to N is clear the set $\mathrm{int}_{N/2}\Omega$ will be denoted by $\widetilde{\Omega}$. Given Ω_o and Ω, two finite subsets of Γ, we denote their amenability ratio by $\alpha(\Omega, \Omega_o)$, that is

$$\alpha(\Omega, \Omega_o) = \frac{|\partial_{D_o} \Omega|}{|\Omega|},$$

where $D_o = \sup\{d(\gamma, \gamma'); \gamma, \gamma' \in \Omega_o\}$ is the diameter of Ω_o. Let us recall that a countable group is said to be *amenable* if it admits an amenable sequence (Ω_i), i.e. an increasing sequence of finite subsets exhausting Γ such that for any nonnegative number N,

$$\lim_{i \to \infty} \frac{|\partial_N \Omega_i|}{|\Omega_i|} = 0.$$

Let X be a compact topological space endowed with a Γ-action $\rho : \Gamma \times X \to X : (\gamma, x) \mapsto \gamma x$.

Let $f_o : X \to \mathbb{R}$ be any continuous function with $f_o(X) = [0, 1]$. For Ω in $B(\Gamma)$, we define the average of f_o along Ω to be the function

$$f_\Omega(x) = \frac{1}{|\Omega|} \sum_{\gamma \in \Omega} f_o(\gamma^{-1} x).$$

3. PRODUCT-LIKE ACTIONS

We will impose on the group action ρ a restrictive assumption of homological nature expressing abundance of multiplicative structure. It will ensure that the homological measure defined in the next section will have a well-defined exponential growth. Its statement requires the introduction of some elements of notation.

Let F be a field and let $H^*(X; F)$ denote the singular cohomology of X with coefficients in F. Given a finite-dimensional subalgebra $A \subset H^*(X; F)$ and a finite subset Ω of Γ, we denote by A_Ω the (finite-dimensional) subalgebra of $H^*(X; F)$ generated by the translates of A along Ω, i.e.

$$A_\Omega = \mathrm{Alg}\Big\langle \bigoplus_{\gamma \in \Gamma} \gamma_* A \Big\rangle,$$

where $\gamma_* = (\gamma^{-1})^*$ denotes the induced (left) action of γ on $H^*(X; F)$.

Assumption 3.1. There is a nontrivial subalgebra $\mathcal{A} \subset H^*(X; F)$ for which any finite-dimensional subalgebra $A \subset \mathcal{A}$ admits a number $N = N(A) \geq 0$ such that if $\Omega, \Omega' \in B(\Gamma)$ satisfy $d(\Omega, \Omega') > N(A)$, then the cup product map is injective

$$(\times) \qquad A_\Omega \otimes A_{\Omega'} \hookrightarrow A_{\Omega \cup \Omega'} : a \otimes a' \mapsto a \wedge a'.$$

Remark 3.2. Assumption 3.1 The word *nontrivial* in the statement above should be given the meaning that the algebra $\mathcal{A}$ contains some nonzero finite-dimensional algebra.

When this assumption is satisfied, the action ρ is said to be a *product-like action*, in reference to the following example.

Example 3.3. (Products) Let M be a manifold. Consider $X = M^\Gamma$, the infinite product of Γ copies of M, or equivalently, the set of maps

$$\Gamma \to M : \gamma \to x_\gamma,$$

with the topology of pointwise convergence (or product topology). Assumption 3.1 holds. Indeed, an algebra $\mathcal{A}$ satisfying the condition $(\times)$ is the direct limit of the direct system of subalgebras described hereafter. To each finite subset $\Omega \subset \Gamma$ is associated a finite-dimensional subalgebra $A(\Omega)$ of $H^\star(M^\Gamma; F)$:

$$A(\Omega) = p_\Omega^*(H^\star(M^\Omega; F)),$$

where $p_\Omega : M^\Gamma \to M^\Omega$ is the canonical projection. When $\Omega' \subset \Omega$, there is a map $p_{\Omega,\Omega'} : M^\Omega \to M^{\Omega'}$ and hence a pullback $i_{\Omega,\Omega'} : A(\Omega') \to A(\Omega)$. The algebra

$$\mathcal{A} = \varinjlim_{\Omega \in B(\Gamma)} A(\Omega)$$

is a subalgebra of $H^\star(M^\Gamma; F)$ that satisfies $(\times)$. Indeed, let $A \subset \mathcal{A}$ be a finite-dimensional subalgebra. There exists a finite subset $\Omega_o \subset \Gamma$ such that $A \subset A(\Omega_o)$. Given Ω, the space A_Ω is contained in $A(\Omega \cdot \Omega_o)$ and, as the Künneth formula implies, it suffices to use $N(A) = \operatorname{diam}(\Omega_o \cdot \Omega_o^{-1})$.

A class of examples of Γ-spaces not of the product type, but enjoying the product-like property is described below in Section 12.

Remark 3.4. Assumption 3.1 is not satisfied when X is a manifold, or when $H^\star(X; F)$ has finite rank.

4. HOMOLOGICAL MEASURE OF THICKENED LEVEL SETS

We will define homological invariants associated to a continuous function f_o on X. They can be interpreted, roughly speaking, as a measure of the amount of cohomology supported in the various thickened level sets of the averages of f_o over the finite subsets of Γ, and therefore could be called *homological measures* of slices. If X was a manifold and if f_o wasa smooth function, these invariants would provide a measure for the number of "homologically-detectable" critical points of the various functions f_Ω located in the various thickened level sets of f_Ω (cf. Section 8). The real purpose is to consider the exponential growth of this invariant as the finite subset becomes large. The resulting object will depend upon two variables : the level and the normalized degree in cohomology. It is called hereafter the *homological entropy* of the function f_o, in analogy with the traditional entropy of an observable ([4]). The entropy is well-defined provided these invariants satisfy certain properties. Classically these properties are submultiplicativity and Γ-invariance. In contrast, the homological measure is invariant as well (Lemma 5.2), but **super**multiplicative (Lemma 5.1). To define entropy in this situation necessitates the introduction of an additional assumption on the group, called here *tileability* (cf. Section 6).

Notation 4.1. *If a is a cohomology class and if O is an open subset of X, the expression* $\operatorname{supp} a \subset O$ *("a is supported in O"), means that for some open set O' such that $X = O \cup O'$, the restriction of a to O' vanishes. Observe that if* $\operatorname{supp} a \subset O$ *and* $\operatorname{supp} b \subset U$ *then* $\operatorname{supp} a \wedge b \subset O \cap U$.

Given $A \subset \mathcal{A}$ a finite-dimensional subalgebra, an open set O, a non-negative number ℓ, a positive number ν and a finite subset $\Omega \subset \Gamma$, we consider the subalgebras

$$\begin{aligned} H^\star_{A_{\widetilde{\Omega}}}(O) &= \{a \in A_{\widetilde{\Omega}}; \operatorname{supp} a \subset O\}, \\ H^{\ell,\nu}_{A_{\widetilde{\Omega}}}(O) &= \{a \in A_{\widetilde{\Omega}}; \operatorname{supp} a \subset O \;\&\; (\ell - \nu)|\Omega| < \deg a < (\ell + \nu)|\Omega|\}. \end{aligned}$$

Recall the convention that $\widetilde{\Omega}$ denotes $\operatorname{int}_{N/2} \Omega$, where $N = N(A)$ is the number associated to $A \subset \mathcal{A}$ from Assumption 3.1. The inequalities involving the degree of a have to be verified by each component of pure degree. Finally, the open sets considered hereafter will be sublevel, superlevel or thickened level sets of the function f_Ω, typically :

$$O = f_\Omega^{-1}(-\infty, c+\delta) \text{ or } f_\Omega^{-1}(c-\delta, +\infty) \text{ or } f_\Omega^{-1}(c-\delta, c+\delta),$$

for some $c \in [0,1]$ and $\delta > 0$. Then consider the map

$$\varphi^{\ell,\nu}_{A,\Omega,c,\delta} : H^{\ell,\nu}_{A_{\widetilde\Omega}}(f_\Omega^{-1}(-\infty, c+\delta)) \to \operatorname{Hom}\Big(H^{\star}_{A_{\widetilde\Omega}}(f_\Omega^{-1}(c-\delta, +\infty)), H^{\star}_{A_{\widetilde\Omega}}(f_\Omega^{-1}(c-\delta, c+\delta))\Big)$$

$$\Big(\varphi^{\ell,\nu}_{A,\Omega,c,\delta}(a)\Big)(b) = a \wedge b.$$

Its rank is denoted hereafter by

$$b^{\ell,\nu}_{A,\Omega}(c,\delta) = \operatorname{rank} \varphi^{\ell,\nu}_{A,\Omega,c,\delta}.$$

Definition 4.2. *$b^{\ell,\nu}_{A,\Omega}(c,\delta)$ is called the $(\ell-\nu, \ell+\nu)$-th homological measure of the thickened level $f_\Omega^{-1}(c-\delta, c+\delta)$ with respect to A_Ω.*

5. PROPERTIES OF THE HOMOLOGICAL MEASURE

We prove in this section the two properties – supermultiplicativity and Γ-invariance – necessary to obtain a well-defined homological entropy.

Lemma 5.1. *The map $B(\Gamma) \to \mathbb{R} : \Omega \mapsto b^{\ell,\nu}_{A,\Omega}(c,\delta)$ is supermultiplicative. More generally, let $\Omega, \Omega' \in B(\Gamma)$ be disjoint finite subsets, let $c, c' \in [0,1]$ and let $\ell, \ell' \geq 0$, then*

$$b^{\alpha\ell+(1-\alpha)\ell',\nu}_{A,\Omega\cup\Omega'}(\alpha c + (1-\alpha)c', \delta) \geq b^{\ell,\nu}_{A,\Omega}(c,\delta) \cdot b^{\ell',\nu}_{A,\Omega'}(c',\delta), \tag{1}$$

where $\alpha = \frac{|\Omega|}{|\Omega\cup\Omega'|}$ and thus $1-\alpha = \frac{|\Omega'|}{|\Omega\cup\Omega'|}$.

Proof. The argument relies on the few simple observations listed below :

- Let Ω and Ω' be disjoint finite subsets of Γ, then
  $$f_{\Omega\cup\Omega'} = \tfrac{|\Omega|}{|\Omega\cup\Omega'|} f_\Omega + \tfrac{|\Omega'|}{|\Omega\cup\Omega'|} f_{\Omega'} = \alpha f_\Omega + (1-\alpha) f_{\Omega'}.$$
  Thus $f^{-1}_{\Omega\cup\Omega'}(\alpha I + (1-\alpha)I') \supset f_\Omega^{-1}(I) \cap f_{\Omega'}^{-1}(I')$ for intervals I and I'.
- If a class a is supported in $f_\Omega^{-1}(I)$ and if a class a' is supported in $f_{\Omega'}^{-1}(I')$, then the class $a \wedge a'$ is supported in $f_\Omega^{-1}(I) \cap f_{\Omega'}^{-1}(I') \subset f^{-1}_{\Omega\cup\Omega'}(\alpha I + (1-\alpha)I')$.
- If Ω and Ω' are disjoint then the distance between $\widetilde\Omega$ and $\widetilde\Omega'$ is greater than N and $\widetilde\Omega \cup \widetilde\Omega' \subset \widetilde{\Omega\cup\Omega'}$. Therefore Assumption 3.1 provides us with an injective map
  $$A_{\widetilde\Omega} \otimes A_{\widetilde\Omega'} \to A_{\widetilde{\Omega\cup\Omega'}}.$$
  This explains the choice of $\widetilde\Omega$ instead of Ω.
- The degree of $a \wedge a'$ if the sum of the degree of a and that of a'. Thus
  $$\left.\begin{array}{l} a \in H^{\ell,\nu}_{A_{\widetilde\Omega}} \\ a' \in H^{\ell',\nu}_{A_{\widetilde\Omega'}} \end{array}\right\} \quad \Rightarrow \quad a \wedge a' \in H^{\alpha\ell+(1-\alpha)\ell',\nu}_{A_{\widetilde{\Omega\cup\Omega'}}}.$$

Combining the previous observations we obtain an injection :

$$\Psi_{I,I'} : H^{\ell,\nu}_{A_{\widetilde\Omega}}(f_\Omega^{-1}(I)) \otimes H^{\ell',\nu}_{A_{\widetilde\Omega'}}(f_{\Omega'}^{-1}(I')) \to H^{\alpha\ell+(1-\alpha)\ell',\nu}_{A_{\widetilde{\Omega\cup\Omega'}}}(f^{-1}_{\Omega\cup\Omega'}(\alpha I + (1-\alpha)I')).$$

Now consider the following sequence of maps. We will abbreviate $\alpha c + (1-\alpha)c'$ to $\bar c$ and $\alpha\ell + (1-\alpha)\ell'$ to $\bar\ell$.

$$\begin{gathered}
H^{\ell,\nu}_{A_{\tilde{\Omega}}}(f_\Omega^{-1}(-\infty, c+\delta)) \bigotimes H^{\ell',\nu}_{A_{\tilde{\Omega}'}}(f_{\Omega'}^{-1}(-\infty, c'+\delta)) \\
\downarrow \\
\mathrm{Hom}\Big(H^\star_{A_{\tilde{\Omega}}}(f_\Omega^{-1}(c-\delta, +\infty)), H^\star_{A_{\tilde{\Omega}}}(f_\Omega^{-1}(c-\delta, c+\delta))\Big) \bigotimes \\
\mathrm{Hom}\Big(H^\star_{A_{\tilde{\Omega}'}}(f_{\Omega'}^{-1}(c'-\delta, +\infty)), H^\star_{A_{\tilde{\Omega}'}}(f_{\Omega'}^{-1}(c'-\delta, c'+\delta))\Big) \\
\downarrow \\
\mathrm{Hom}\Big(H^\star_{A_{\tilde{\Omega}}}(f_\Omega^{-1}(c-\delta, +\infty)) \otimes H^\star_{A_{\tilde{\Omega}'}}(f_{\Omega'}^{-1}(c'-\delta, +\infty)), \\
H^\star_{A_{\tilde{\Omega}}}(f_\Omega^{-1}(c-\delta, c+\delta)) \otimes H^\star_{A_{\tilde{\Omega}'}}(f_{\Omega'}^{-1}(c'-\delta, c'+\delta))\Big) \\
\downarrow \\
\mathrm{Hom}\Big(H^\star_{A_{\widetilde{\Omega\cup\Omega'}}}(f_{\Omega\cup\Omega'}^{-1}(\tilde{c}-\delta, +\infty)), H^\star_{A_{\widetilde{\Omega\cup\Omega'}}}(f_{\Omega\cup\Omega'}^{-1}(\tilde{c}-\delta, \tilde{c}+\delta))\Big).
\end{gathered}$$

The first arrow stands for the map $\varphi^{\ell,\nu}_{A,\Omega,c,\delta} \otimes \varphi^{\ell',\nu}_{A,\Omega',c',\delta}$. The second arrow is a classical isomorphism, indeed,

$$\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(C, D) \simeq \mathrm{Hom}(A \otimes C, B \otimes D)$$

for finite-dimensional vector spaces A, B, C, D. The third one is the injection induced by a choice of complementary subspaces to the images of the maps $\Psi_{(c-\delta,+\infty),(c'-\delta,+\infty)}$ and $\Psi_{(c-\delta,c+\delta),(c'-\delta,c'+\delta)}$ respectively. We will denote the composition of second and third map by Φ. There is also another sequence obtained from composing

$$\Psi = \Psi_{(-\infty,c+\delta),(-\infty,c'+\delta)}$$

with

$$\varphi^{\alpha\ell+(1-\alpha)\ell',\nu}_{A,\Omega\cup\Omega',\alpha c+(1-\alpha)c',\delta}.$$

These two sequences commute :

$$\Phi \circ \left[\varphi^{\ell,\nu}_{A,\Omega,c,\delta} \otimes \varphi^{\ell',\nu}_{A,\Omega',c',\delta}\right] = \varphi^{\alpha\ell+(1-\alpha)\ell',\nu}_{A,\Omega\cup\Omega',\alpha c+(1-\alpha)c',\delta} \circ \Psi.$$

Since Φ and Ψ are both injective, this implies that

$$\mathrm{rank}\Big(\varphi^{\ell,\nu}_{A,\Omega,c,\delta} \otimes \varphi^{\ell',\nu}_{A,\Omega',c',\delta}\Big) \leq \mathrm{rank}\, \varphi^{\alpha\ell+(1-\alpha)\ell',\nu}_{A,\Omega\cup\Omega',\alpha c+(1-\alpha)c',\delta}$$

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Lemma 5.2. *The map*

$$B(\Gamma) \to \mathbb{R} : \Omega \to b^{\ell,\nu}_{A,\Omega}(c, \delta)$$

is Γ*-invariant.*

Proof. The proof follows from the simple facts stated below. Let $\Omega \in B(\Gamma)$, let $\gamma_o \in \Gamma$ and let $I \subset [0, 1]$ be any interval. Then

- $f^{-1}_{\gamma_o\Omega}(I) = \gamma_o f_\Omega^{-1}(I)$.
- $A_{\widetilde{\gamma_o\Omega}} = (\gamma_o)_* A_{\tilde{\Omega}}$.
- $\widetilde{\gamma_o\Omega} = \gamma_o\widetilde{\Omega}$.
- $(\gamma_o)_*$ induces an isomorphism between $H^{\ell,\nu}_{A_{\tilde{\Omega}}}(f_\Omega^{-1}(I))$ and $H^{\ell,\nu}_{\gamma_{o*}A_{\tilde{\Omega}}}(\gamma_o f_\Omega^{-1}(I))$.

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6. SUPERADDITIVE ORNSTEIN-WEISS LEMMA FOR TILEABLE GROUPS

The ℓ-th Betti number entropy of f_ν will be defined from the exponential growth, with respect to the index i, of the sequence $(b^{\ell,\nu}_{A,\Omega_i}(c,\delta))_{i\geq 1}$, where Ω_i is an amenable sequence in Γ, that is to say, from the limit

$$\lim_{i\to\infty}\frac{\ln(b^{\ell,\nu}_{A,\Omega_i}(c,\delta))}{|\Omega_i|},$$

when it exists. Lemma 5.1 implies that the map $\Omega \mapsto \ln(b^{\ell,\nu}_{A,\Omega}(c,\delta))$ is superadditive on disjoint sets, while the Ornstein-Weiss lemma [3] provides convergence of the sequence of averages $h(\Omega_i)/|\Omega_i|$ under the hypotheses that the map $h : B(\Gamma) \to \mathbb{R}^+$ is Γ-invariante and **sub**additive. The proof of the Ornstein-Weiss Lemma requires the construction of ε-quasi-tilings that any amenable group admits. In contrast, a proof of the superadditive version of this lemma seems to necessitate the construction of *disjoint* such tilings which might not exist in general (although we do not know of any counterexample). Whence the following definition.

Definition 6.1. *An amenable group Γ is said to be tileable if it admits a tiling amenable sequence, that is, an amenable sequence (Ω_i) such that given $\varepsilon > 0$ and a subsequence (Ω_{i_n}) of (Ω_i), there exists a finite subsequence of (Ω_{i_n}), denoted $\Omega_1, \dots, \Omega_s$, such that any finite subset Ω with sufficiently large amenablility ratios $\alpha(\Omega, \Omega_j)$, $j = 1, \dots, s$ can be disjointly ε-tiled by translates of the Ω_j's, i.e. there exists center $\gamma_{j,k}$, $1 \leq j \leq s$, $1 \leq k \leq r_j$ in Γ such that*

- $\gamma_{j,k}\Omega_j \subset \Omega$,
- $\gamma_{j,k}\Omega_j \cap \gamma_{j',k'}\Omega_{j'} = \emptyset$ *for* $(j,k) \neq (j',k')$,
- $|\bigcup_{j,k}\gamma_{j,k}\Omega_j| \geq (1-\varepsilon)|\Omega|$.

Examples 6.2. Weiss introduces in [5] the notion of *monotileable amenable groups*. Those are groups admitting an amenable sequence (Ω_i) consisting of monotiles. This means that for each index i there exists a set $C_i \subset \Gamma$ for which the various translates $\Omega_i c$ of Ω_i along C_i form a partition of Γ. Such groups belong to the class of tileable amenable groups. Moreover, Weiss proves that any residually finite amenable group is monotileable, implying that the following amenable groups are also tileable.

- Abelian and solvable groups.
- Amenable linear groups, i.e. linear groups not containing F_2 as a subgroup.
- Grigorchuk's groups of intermediate growth.[1]

Lemma 6.3. *(Superadditive Ornstein-Weiss lemma) Let Γ be a tileable amenable group. Let h be a nonnegative function defined on $B(\Gamma)$ and satisfying the following two conditions :*

- *superadditivity : $h(\Omega \cup \Omega') \geq h(\Omega) + h(\Omega')$ for disjoint subsets Ω and Ω',*
- *Γ-invariance : $h(\gamma\Omega) = h(\Omega)$ for any $\gamma \in \Gamma$.*

Then, given a tiling amenable sequence (Ω_i), the following limit exists

$$\lim_{i\to\infty}\frac{h(\Omega_i)}{|\Omega_i|}.$$

Remark 6.4. Observe that under the hypotheses of the previous lemma, the limit is independent of the choice of a tiling amenable sequence in Γ.

Proof. Let $\varepsilon > 0$ and let (Ω_i) be a tiling amenable sequence. Extract a subsequence $\Omega_{i_1}, \dots, \Omega_{i_s}$ with which we can ε-tile any element Ω_i of the initial sequence with sufficiently large index. Suppose also that if h^+ stands for the limsup of the sequence $h(\Omega_i)/|\Omega_i|$, then $h(\Omega_{i_j})/|\Omega_{i_j}| \geq h^+ - \varepsilon$ for all

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[1] Grigorchuk, Rotislav I., Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), no. 5, 939–985.

j. Let $\gamma_{j,k}$, $1 \leq j \leq s$, $1 \leq k \leq r_j$ denote the centers of a disjoint ε-tiling of Ω_i by translates of the Ω_{i_j}'s and let $\Omega_i' = \cup_{j,k} \gamma_{j,k}\Omega_{i_j}$. Then

$$\begin{aligned}\frac{1}{|\Omega_i|}h(\Omega_i) &\geq \frac{1}{|\Omega_i|}\Big(h(\Omega_i') + h(\Omega_i - \Omega_i')\Big) \geq \frac{1}{|\Omega_i|}h(\Omega_i') \\ &\geq \frac{1}{|\Omega_i|}\sum\nolimits_{j,k} h(\Omega_{i_j}) \geq \frac{1}{|\Omega_i|}\sum\nolimits_{j,k}(h^+ - \varepsilon)|\Omega_{i_j}| \\ &\geq \frac{1}{|\Omega_i|}(h^+ - \varepsilon)(1-\varepsilon)|\Omega_i| = (h^+ - \varepsilon)(1-\varepsilon).\end{aligned}$$

Hence

$$\liminf_{i\to\infty} \frac{1}{|\Omega_i|}h(\Omega_i) \geq (h^+ - \varepsilon)(1-\varepsilon).$$

Since this holds for arbitrary ε, the limit $\lim_{i\to\infty} h(\Omega_i)/|\Omega_i|$ exists.

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7. HOMOLOGICAL ENTROPY OF FUNCTIONS

Let $(\Omega_i)_{i\geq 1}$ be a tiling amenable sequence in the tileable amenable group Γ and consider the exponential growth of the sequence $b^{\ell,\nu}_{A,\Omega_i}(c,\delta)$:

$$b^{\ell,\nu}_{A}(c,\delta) = \lim_{i\to\infty} \frac{1}{|\Omega_i|} \ln\Big[b^{\ell,\nu}_{A,\Omega_i}(c,\delta)\Big]. \tag{2}$$

As implied by Lemma 5.1, Lemma 5.2 and Lemma 6.3, this limit indeed exists. Observing that the function $b^{\ell,\nu}_{A}(c,\delta)$ is increasing in ν and δ, we let δ and ν approach 0 :

$$b^{\ell}_{A}(c) = \lim_{\nu\to 0}\lim_{\delta\to 0} b^{\ell,\nu}_{A}(c,\delta).$$

Independence on A is obtained by considering the supremum over all possible choices of a finite-dimensional subalgebra A of $\mathcal{A}$:

$$b^{\ell}(c) = \sup\{b^{\ell}_{A}(c); A \subset \mathcal{A} \ \& \ \dim A < \infty\}.$$

Definition 7.1. *The function $b^{\ell} : [0,1] \to \mathbb{R} : c \mapsto b^{\ell}(c)$ is called the ℓ-th Betti number entropy of f_o.*

Remark 7.2. One may also define the *sum of the Betti number entropy of f_o* by the same process except that the cohomological degree is not restricted. The two functions are related as follows :

$$b(c) = \sup_{\ell} b^{\ell}(c).$$

(This is a consequence of the general fact that the exponential growth of the sum of two sequences coincides with the exponential growth of the maximum sequence.)

Remark 7.3. The condition that Γ be tileable is only used to guarantee existence of the limit (2).

8. RELATION WITH CLASSICAL MORSE THEORY

This section is devoted to showing how, in the setting of a manifold M endowed with a Morse function $f : M \to \mathbb{R}$, the sum of the Betti number entropy essentially coincides with the sum of the Betti numbers of M and provides a lower bound for the number of critical points of f (cf. Proposition 8.1).

Let M be a connected, closed, oriented smooth manifold. Consider a Morse function f on M. Let F be a field. We recall that if O is some subset of M and if a is a cohomology class in $H^\star(M;F)$, the expression $\operatorname{supp} a \subset O$ means that $a|_{O'} = 0$ for some open subset O' containing $M - \operatorname{int} O$. We denote by $\operatorname{Crit}_c(f)$ the set of critical points of f at level c. Define

$$H^*(O) = \Big\{a \in H^*(M;F);\, \operatorname{supp} a \subset O\Big\},$$

$$\begin{aligned} \varphi_{c,\delta} : H^*(f^{-1}(-\infty, c+\delta)) &\to \operatorname{Hom}\Big(H^*(f^{-1}(c-\delta, +\infty)), H^*(f^{-1}(c-\delta, c+\delta))\Big), \\ a &\mapsto \Big[b \mapsto a \wedge b\Big], \end{aligned}$$

$$b(c,\delta) = \operatorname{rank} \varphi_{c,\delta},$$

$$b(c) = \lim_{\delta\to 0} b(c,\delta).$$

We might sometimes denote $b(c,\delta)$ by $b(c-\delta, c+\delta)$ or consider $b(I)$ when I is some interval.

Proposition 8.1.

(a) $\sum_{c\in\mathbb{R}} b(c) = SB(M)$.

(b) $b(c) \leq \operatorname{Crit}_c(f)$.

Proof.

(a) The main ingredient is the specific version of Poincaré duality mentioned below in Section 10.1. Indeed, if $a \in H^*(M;F)$, define

$$c_a = \inf\{c; \operatorname{supp} a \subset f^{-1}(-\infty, c)\}.$$

Since for all $\delta > 0$ the restriction of a to $f^{-1}(c_a - \delta, +\infty)$ does not vanish, there exists a class b with $\operatorname{supp} b \subset f^{-1}(c_a - \delta, +\infty)$ such that $a \wedge b \neq 0$ (cf. Proposition 10.2). Hence a provides a contribution to $b(c_a)$. Here follows a more precise argument taking into account the following difficulty : there might exist two classes that have same c_a and are independent, but who do not generate a 2-dimensional space of classes with same c_a.

Decompose the range I of f_o into intervals as follows :

$$I \subset \bigcup_{k=1}^{K} (I_k = [a_k, a_{k+1})) \qquad a_k < a_{k+1}.$$

If $J \subset \mathbb{R}$, let $r_J : H^*(M;F) \to H^*(f^{-1}(J);F)$ denote the restriction map. Then consider the following increasing sequence of subspaces

$$\{0\} = \operatorname{Ker} r_{I_1 \cup \ldots \cup I_K} \subset \ldots \subset \operatorname{Ker} r_{I_{K-1} \cup I_K} \subset \operatorname{Ker} r_{I_K} \subset H^*(M;F).$$

Choose a corresponding sequence of spaces $V_1, \ldots, V_K$ such that

$$\operatorname{Ker} r_{I_k \cup \ldots \cup I_K} \oplus V_k = \operatorname{Ker} r_{I_{k+1} \cup \ldots \cup I_K} \qquad k = 1, \ldots, K.$$

(For $k = K$, we mean $\operatorname{Ker} r_{I_K} \oplus V_K = H^*(M;F)$.) If $0 \neq a \in V_k$ then $c_a \in I_k$. Hence $b(I_k) = \operatorname{rank} V_k$ and

$$\sum_{k=1}^{K} b(I_k) = SB(M). \tag{3}$$

This is true for arbitrarily fine subdivisions of I. Now observe that for each c, either $b(c) = 0$, in which case $b(c-\delta, c+\delta) = 0$ for all sufficiently small $\delta > 0$, or $b(c-\delta, c+\delta) \neq 0$ for all $\delta > 0$. Relation (3) implies that there are finitely many numbers c with $b(c) \neq 0$. So $\sum_k b(I_k)$ is constant, equal to $\sum_c b(c)$, for all sufficiently fine subdivisions of I.

(b) Given $a \in H^*(M;F)$, c_a must be a critical value, otherwise we would be able to move $f^{-1}(-\infty, c+\delta)$ below level $c-\delta$ by an ambient isotopy, disjointifying the supports of the classes a and b. In consequence, $a \wedge b$ would vanish. This alone implies (b) when $b(c) \leq 1$ for all c. We will argue that if $b(c) = 2$ then f cannot have a single critical point at level c (the general case can be handled in a similar way). Suppose on the contrary that $\{x\} = \mathrm{Crit}_c(f)$. Let $x_1, \dots, x_m$ be coordinates on M, centered at x, for which f has the canonical form

$$f(x) = -x_1^2 - \dots - x_n^2 + x_{n+1}^2 + \dots + x_m^2.$$

Let a_1 and a_2 be two independent classes with $c_{a_1} = c_{a_2} = c$. Consider piecewise smooth cycles α_1 and α_2 representing their Poincaré dual homology classes. We will make the following assumptions on α_i, $i = 1, 2$:

- α_i is supported in $f^{-1}(-\infty, c]$,
- $\alpha_i \cap f^{-1}(c) = \{x\}$,
- x is a regular value of (each of the simplices composing) α_i,
- α_i intersects the local unstable manifold $\mathcal{W}^u(x) = \{x; x_1 = \dots = x_n = 0\}$ of x transversely at x.

It is long but not difficult to verify that these hypotheses are not restrictive.

Now, we will show that the degree of α_i must equal the index n of x. The degree of α_i can certainly not exceed n, otherwise α_i would not be supported in $f^{-1}(-\infty, c]$. If the degree of α_i was less than n, one could slide α_i down the stable manifold of x (in a direction transverse to that of $T_x\alpha_i$) below level c.

Then, one can subdivide all the simplices of α_i containing x in such a way that x lies in the interior of each simplex to which it belongs and that each such simplex can be isotoped to a fixed simplex that coincides with the stable manifold $\mathcal{W}^s(x)$ of x in a neighborhood of x. Thus,

$$\alpha_i = f_i^0 \sigma_i^0 + \sum_{j \geq 1} f_i^j \sigma_i^j,$$

where $f_i^0, f_i^j \in F$, where σ_i^0 is a piece of $\mathcal{W}^s(x)$ containing x and where σ_i^j avoids x. It follows that $f_1^0\alpha_2 - f_2^0\alpha_1$ vanishes near x, hence that $c_{f_1^0 a_2 - f_2^0 a_1} < c$. So a_1 and a_2 do not generate a space contributing to $b(c)$, a contradiction.

■

Remark 8.2. The previous lemma implies the Morse theoretic lower bound announced in the introduction. Referring to the notation used therein, one observes that the previous proof implies in particular that if $I \subset \mathbb{R}$ is some interval, then $\mathrm{Crit}_I(F_i') \geq b_{A,\Omega_i}(I)$, where $A \simeq H^*(M; F)$. Hence

$$\liminf_{i \to \infty} \mathrm{cri}_i(I) \geq b_A(I).$$

(The function F_i' defined in the introduction does not quite coincide with the function f_{Ω_i} defined in Section 2, but the difference will not affect the asymptotic behavior of the objects considered here). Moreover, as proved later on, the function $b_A(I)$ is concave and strictly positive.

9. CONCAVITY OF THE ENTROPY

The above-defined function $b : \mathbb{R}^+ \times [0, 1] \to \mathbb{R}$ is concave. This follows mainly from Lemma 5.1, with a slight help from the following fact.

Lemma 9.1. *The function b is upper semi-continuous.*

Proof. Let (ℓ_k, c_k) be a sequence converging to some pair (ℓ, c) in $\mathbb{R}^+ \times [0, 1]$. Since $b_{A,\Omega}^{\ell,\nu}(c, \delta)$ is increasing with respect to both intervals $(\ell - \nu, \ell + \nu)$ and $(c - \delta, c + \delta)$,

$$b_{A,\Omega}^{\ell,\nu}(c, \delta) \geq b_{A,\Omega}^{\ell_k, \frac{\nu}{2}}(c_k, \tfrac{\delta}{2})$$

for sufficiently large k and for all A and Ω. Hence $b^\ell(c) \geq b^{\ell_k}(c_k)$ and $b^\ell(c) \geq \limsup_{k\to\infty} b^{\ell_k}(c_k)$.

■

Proposition 9.2. *The function b is concave. That is to say, for any ℓ, $\ell' \in \mathbb{R}^+$, any $c, c' \in [0, 1]$, and any $\alpha \in [0, 1]$,*

$$b^{\alpha\ell+(1-\alpha)\ell'}(\alpha c + (1-\alpha)c') \geq \alpha\, b^{\ell}(c) + (1-\alpha)\, b^{\ell'}(c'). \tag{4}$$

Proof. Let (Ω_i) be an amenable sequence. For each i, let $\Omega_i' = \gamma_i \cdot \Omega_i$ be disjoint from Ω_i. Then the sequences (Ω_i') and $(\Omega_i \cup \Omega_i')$ are amenable as well. Besides, Lemma 5.1 implies that

$$b^{\alpha\ell+(1-\alpha)\ell',\nu}_{A,\Omega_i\cup\Omega_i'}(\alpha c + (1-\alpha)c', \delta) \geq b^{\ell,\nu}_{A,\Omega_i}(c,\delta) \cdot b^{\ell',\nu}_{A,\Omega_i'}(c',\delta),$$

with $\alpha = \frac{1}{2}$. Hence

$$b^{\frac{1}{2}\ell+\frac{1}{2}\ell'}\left(\tfrac{1}{2}c + \tfrac{1}{2}c'\right) \geq \tfrac{1}{2}b^{\ell}(c) + \tfrac{1}{2}b^{\ell'}(c'). \tag{5}$$

This implies that the relation (4) holds for any dyadic rationnal α. The result for arbitrary α follows from the upper semi-continuity of b (Lemma 9.1). ∎

10. NONTRIVIALITY OF THE ENTROPY FOR PRODUCTS

Let F be a field. Let M be a closed F-orientable manifold, that is to say $H^m(M; F) \simeq F$ for $m = \dim M$. In other words, either M is orientable, or $F = \mathbb{Z}_2$. Let $f_o : X = M^\Gamma \to \mathbb{R}$ be a continuous function with range $[0, 1]$.

Proposition 10.1. *The associated homological entropy of f_o achieves a strictly positive value.*

This results holds because a products M^Γ inherits some Poincaré duality (cf. Lemma 10.3) from the manifold M.

10.1. **Poincaré duality on a closed orientable manifold.**

Here follows the specific version of Poincaré duality that is needed below.

Proposition 10.2. *If a is a class in $H^*(M; F)$ whose restriction to the open set O does not vanish, then there exists a class b with support in O such that $a \wedge b \neq 0$.*

Proof. Let $a \in H^i(M; F)$ with $a|_O \neq 0$. Then there exists a homology class $\beta \in H_i(O; F)$ such that $< a, \beta > \neq 0$. Let b be the Poincaré dual of β. Then $a \wedge b$ does not vanish since its evaluation on the fundamental class of M coincides with $< a, \beta >$. Moreover, if β is represented by a chain c, the class b can be represented by a form whose support is contained in any given neighborhood of the image of c. ∎

10.2. **Poincaré duality in a product M^Γ.**

Let $id \in \Omega_o \subset \Gamma$ be a finite subset and let $A = A(\Omega_o) = p^*_{\Omega_o}(H^*(M^{\Omega_o}; F))$, where $p_{\Omega_o} : M^\Gamma \to M^{\Omega_o}$ is the canonical projection. Let also $N = N(A)$ (cf. Assumption 3.1 and Example 3.3). If $\Omega \subset \Gamma$ is another finite subset, we can define the positive number

$$\delta_\Omega = \delta_\Omega(f_o, \Omega_o) = 4\ \sup\{\left|f_\Omega(x) - f_\Omega(y)\right|; x_\gamma = y_\gamma \text{ for } \gamma \in \Omega \cdot \Omega_o\}.$$

Observe that δ_Ω is decreasing in Ω. In fact δ_Ω approches 0 as Ω becomes large (cf. Lemma 10.4).

Lemma 10.3. *(Poincaré duality in M^Γ) Let a belong to A_Ω. Then there exists a level c_a^Ω and an element b in A_Ω such that*

- $\operatorname{supp} a \subset f_\Omega^{-1}(-\infty, c_a^\Omega + \delta_\Omega)$,
- $\operatorname{supp} b \subset f_\Omega^{-1}(c_a^\Omega - \delta_\Omega, +\infty)$,
- $a \wedge b \neq 0$.

Proof. The level c_a^Ω defined below obviously satisfies the first condition.

$$c_a^\Omega = \inf\{c \in [0,1]; \operatorname{supp} a \subset f_\Omega^{-1}(-\infty, c)\}. \tag{6}$$

As explained below, existence of the class b follows from Poincaré duality in any finite product M^Ω. Choose a point o in M^Γ and define new functions $g_\Omega : M^\Gamma \to [0,1]$ by $g_\Omega(x) = f_\Omega(\hat{x})$, with

$$\hat{x}_\gamma = \begin{cases} x_\gamma & \text{if } \gamma \in \Omega \cdot \Omega_o \\ o_\gamma & \text{otherwise.} \end{cases}$$

By definition of δ_Ω,

$$\sup_{x \in M^\Gamma} \left| f_\Omega(x) - g_\Omega(x) \right| \leq \frac{\delta_\Omega}{4}.$$

Now observe that the restriction of a to $g_\Omega^{-1}(c_a^\Omega - \frac{3}{4}\delta_\Omega, +\infty)$ does not vanish. Indeed, if it did vanish then $\operatorname{supp} a$ would be contained in $g_\Omega^{-1}(-\infty, c_a^\Omega - \frac{1}{2}\delta_\Omega)$ which itself is contained in $f_\Omega^{-1}(-\infty, c_a^\Omega - \frac{1}{4}\delta_\Omega)$. This contradicts the definition of c_a^Ω.

Since g_Ω depends only on the variables indexed by $\Omega \cdot \Omega_o$, the open set $g_\Omega^{-1}(c_a^\Omega - \frac{3}{4}\delta_\Omega, +\infty)$ coincides with the pullback $p_{\Omega \cdot \Omega_o}^{-1}(O)$ of some open subset O of $M^{\Omega \cdot \Omega_o}$. Combined with the fact that $a = p_{\Omega \cdot \Omega_o}^* \overline{a}$ for some class $\overline{a}$ in $H^*(M^{\Omega \cdot \Omega_o}; F)$, this implies that the restriction of the class $\overline{a}$ to O does not vanish. Poincaré duality in closed orientable manifolds yields a class $\overline{b} \in H^*(M^{\Omega \cdot \Omega_o}; F)$ with $\operatorname{supp} \overline{b} \subset O$ and such that $\overline{a} \wedge \overline{b} \neq 0$. The class $b = p_{\Omega \cdot \Omega_o}^*(\overline{b})$ satisfies the required conditions. ■

The following result implies that in Lemma 10.3 one can replace δ_Ω by any given $\delta > 0$ at the cost of considering only "large" Ω's.

Lemma 10.4. *Let $(\Omega_i)_{i \geq 1}$ be an amenable sequence. Then the sequence δ_{Ω_i} converges to 0.*

Proof. Let $\delta > 0$. By (uniform) continuity of f_o, there exists a $\eta = \eta(\delta) > 0$ such that $\hat{d}(x,y) < \eta \Rightarrow |f_o(x) - f_o(y)| < \delta$. The symbol $\hat{d}$ denotes one of the following (compatible) metrics on M^Γ :

$$\hat{d}(x,y) = \sum_{\gamma \in \Gamma} \frac{d_o(x_\gamma, y_\gamma)}{\lambda^{|\gamma|}},$$

where d_o is some Riemannian metric on M, where λ is some fixed number in $(1, +\infty)$ and where $|\gamma| = d(id, \gamma)$. In particular $\hat{d}(x,y) < \eta$ when sufficiently many components of x and y coincide. More precisely, there exists $\Omega_\delta \in B(\Gamma)$ with $\Omega_\delta \ni id$ such that $\hat{d}(x,y) < \eta(\delta)$ as soon as $x_\gamma = y_\gamma$ for all $\gamma \in \Omega_\delta$.

Now fix $\Omega = \Omega_i$ and let $x, y \in M^\Gamma$ be such that $x_\gamma = y_\gamma$ for $\gamma \in \Omega \cdot \Omega_o$. Then

$$\left| f_o(\gamma^{-1}x) - f_o(\gamma^{-1}y) \right| < \delta$$

when $\gamma \in \operatorname{int}_D(\Omega \cdot \Omega_o)$, where D denotes the diameter of Ω_δ. Set $\hat{\Omega} = \operatorname{int}_D(\Omega \cdot \Omega_o) \cap \Omega$ and decompose f_Ω into a convex linear combination as follows :

$$f_\Omega = \tfrac{|\hat{\Omega}|}{|\Omega|} f_{\hat{\Omega}} + \tfrac{|\Omega - \hat{\Omega}|}{|\Omega|} f_{\Omega - \hat{\Omega}}.$$

Then

$$\begin{array}{rcll} \left| f_\Omega(x) - f_\Omega(y) \right| & \leq & \tfrac{|\hat{\Omega}|}{|\Omega|} \left| f_{\hat{\Omega}}(x) - f_{\hat{\Omega}}(y) \right| & + \ \tfrac{|\Omega - \hat{\Omega}|}{|\Omega|} \left| f_{\Omega - \hat{\Omega}}(x) - f_{\Omega - \hat{\Omega}}(y) \right| \\ & \leq & \tfrac{|\hat{\Omega}|}{|\Omega|} \ \delta & + \ \tfrac{|\Omega - \hat{\Omega}|}{|\Omega|} \\ & \leq & 2\delta, & \end{array}$$

provided the index i is sufficiently large. Indeed, $\Omega - \hat{\Omega} \subset \partial_D \Omega = \partial_D \Omega_i$. ■

10.3. **Repartition of classes according to degree and support.** Now we are ready to prove the nontriviality of b. It follows from Lemma 10.3 and Lemma 10.4 and does not further use the assumption that X is a product.

Proof of Proposition 10.1 Let $A = A(\Omega_o)$ as before and let (Ω_i) be an amenable sequence in Γ. Lemma 10.3 implies that for each i and each $a \in A_{\Omega_i}$, there exists a $c_a^{\Omega_i} \in [0,1]$ such that

$$\varphi^*_{A,\Omega_i,c_a^{\Omega_i},\delta_{\Omega_i}}(a) \neq 0.$$

Thus any a in A_{Ω_i} contributes to $b^{\ell,\nu}_{A,\Omega_i,c,\delta_{\Omega_i}}$ for some c and ℓ. Using the pigeon-hole principle, in the spirit of Proposition 8.1, we will find some ℓ and some c for which exponentially many classes of degree around $\ell|\Omega_i|$ are supported around $f_{\Omega_i}^{-1}(c)$.

The degree of a is an integer number between 0 and $m|\Omega_o \cdot \Omega_i| \leq m\omega_o|\Omega_i|$, where $m = \dim M$ and $\omega_o = |\Omega_o|$. Fix $r \in \mathbb{N}_o$. Then for each i, there exists an interval $J_i = [\frac{s-1}{r}, \frac{s}{r}] \subset [0, m\omega_o]$ for which the rank of the space $A_{\Omega_i}^{J_i}$ of classes in A_{Ω_i} whose degree belongs to the interval $J_i|\Omega_i| = \left[\frac{(s-1)}{r}|\Omega_i|, \frac{s}{r}|\Omega_i|\right]$ satisfies

$$\operatorname{rank} A_{\Omega_i}^{J_i} \geq \frac{1}{rm\omega_o} \operatorname{rank} A_{\Omega_i}.$$

Lemma 10.5. *Fix $k \geq 1$ and $i \geq 1$. Then there exist an interval $I_i = [\frac{j-1}{k}, \frac{j}{k}] \subset [0,1]$ and a subspace $\overline{A_i} \subset A_{\Omega_i}^{J_i}$ such that*

- $a \in \overline{A_i} \Rightarrow c_a^{\Omega_i} \in I_i$,
- $\operatorname{rank} \overline{A_i} \geq \dfrac{1}{k} \operatorname{rank} A_{\Omega_i}^{J_i}$.

Proof. If $I \subset [0,1]$ and if a is a cohomology class in $H^*(M^\Gamma; F)$, denote by $r_I(a)$ the restriction of a to the open set $f_{\Omega_i}^{-1}(I)$. Let $[0,1] = \cup_{j=1}^k I_j$, with $I_j = [\frac{j-1}{k}, \frac{j}{k}]$. We decompose the space $A_{\Omega_i}^{J_i}$ into a direct sum

$$A_{\Omega_i}^{J_i} = A_1 \oplus \ldots \oplus A_k$$

in such a way as to satisfy the following properties :

$$A_k \oplus \left(\operatorname{Ker} r_{I_k} \cap A_{\Omega_i}^{J_i}\right) = A_{\Omega_i}^{J_i},$$

and for $j = 1, \ldots, k-1$,

$$\begin{cases} A_j \subset \left(\operatorname{Ker} r_{I_{j+1}\cup\ldots\cup I_k} \cap A_{\Omega_i}^{J_i}\right) \\ A_j \oplus \left(\operatorname{Ker} r_{I_j\cup\ldots\cup I_k} \cap A_{\Omega_i}^{J_i}\right) = \left(\operatorname{Ker} r_{I_{j+1}\cup\ldots\cup I_k} \cap A_{\Omega_i}^{J_i}\right). \end{cases}$$

Thus if $a \in A_j$ then $c_a^{\Omega_i} \in I_j$. Now there exists a $j = j(i)$ such that $\operatorname{rank} A_{j(i)} \geq \frac{1}{k} \operatorname{rank} A_{\Omega_i}^{J_i}$. Let $\overline{A_i} = A_{j(i)}$.

■

The collection of intervals J_i and I_i being finite, there exist

- a subsequence of (Ω_i), denoted (Ω_i) as well,
- an interval $J = [\ell - \frac{1}{2r}, \ell + \frac{1}{2r}]$, with $\ell = \ell(r)$,
- an interval $I = [c - \frac{1}{2k}, c + \frac{1}{2k}]$, with $c = c(k)$,

such that $\delta_{\Omega_i} \leq \frac{1}{4k}$, $J_i = J$ and $I_i = I$ for all i. Moreover, for each i and each $a \in \overline{A_i}$,

$$\operatorname{supp} a \subset f_{\Omega_i}^{-1}(-\infty, c_a^{\Omega_i} + \delta_{\Omega_i}) \subset f_{\Omega_i}^{-1}(-\infty, c + \tfrac{1}{k}).$$

Furthermore, Lemma 10.3 provides a class $b \in A_{\Omega_i}$ such that

$$\operatorname{supp} b \subset f_{\Omega_i}^{-1}(c - \tfrac{1}{k}, +\infty) \quad \text{and} \quad a \wedge b \neq 0.$$

Thus the map $\varphi^{\ell,\frac{1}{2r}}_{A,\Omega_i,c,\frac{1}{k}}$ is injective on $\overline{A_i}$ for all i. Therefore,

$$\begin{aligned}
b_A^{\ell,\frac{1}{2r}}(c,\tfrac{1}{k}) &\geq \lim_{i\to\infty}\frac{1}{|\Omega_i|}\ln\bigl(\operatorname{rank}\overline{A_i}\bigr)\\
&\geq \lim_{i\to\infty}\frac{1}{|\Omega_i|}\ln\Bigl(\frac{1}{k}\frac{1}{rm\omega_o}\operatorname{rank}A_{\Omega_i}\Bigr)\\
&= \lim_{i\to\infty}\frac{1}{|\Omega_i|}\ln\Bigl(\frac{1}{k}\frac{1}{rm\omega_o}\bigl(\operatorname{rank}H^\star(M;F)\bigr)^{\Omega_i\cdot\Omega_o}\Bigr)\\
&= \lim_{i\to\infty}\frac{|\Omega_i\cdot\Omega_o|}{|\Omega_i|}\ln\bigl(\operatorname{rank}H^\star(M;F)\bigr)\\
&\geq \ln\bigl(\operatorname{rank}H^\star(M;F)\bigr).
\end{aligned}$$

To conclude, observe that $\ell=\ell(r)$ and $c=c(k)$. Let $\nu,\delta>0$. There exists a subsequence $\ell(r_s)$ of $\ell(r)$ (respectively $c(k_l)$ of $c(k)$) such that $\ell(r_s)$ (respectively $c(k_l)$) converges to some $\ell\in\mathbb{R}^+$ (respectively $c\in[0,1]$). Since $b^{\ell,\nu}_{A,\Omega}(c,\delta)$ increases with the size of $(\ell-\nu,\ell+\nu)$ and that of $(c-\delta,c+\delta)$,

$$b_A^{\ell,\nu}(c,\delta)\geq b_A^{\ell(r_s),\frac{1}{2r_s}}(c(k_l),\tfrac{1}{k_l})\geq\ln\bigl(\operatorname{rank}H^\star(M;F)\bigr),$$

for l and s sufficiently large. Thus

$$b^\ell(c)\geq\ln\bigl(\operatorname{rank}H^\star(M;F)\bigr)>0.$$

■

Remark 10.6. At least for products, $b_A^\ell(c)=b^\ell(c)$ provided A contains $H^\star(M;F)$. Indeed, let $A^o=p_\gamma^* H^\star(M;F)$, some γ in Γ, and let A be another finite-dimensional subalgebra of $\mathcal{A}$ containing A^o, necessarily contained in some subalgebra $A^1=p^*_{\Omega_1}H^\star(M^{\Omega_1};F)$ with $\Omega_1\ni\gamma$. Thus $A^o_\Omega\subset A_\Omega\subset A^1_\Omega$ and

$$b^{\ell,\nu}_{A^o,\Omega}(c,\delta)\leq b^{\ell,\nu}_{A,\Omega}(c,\delta)\leq b^{\ell,\nu}_{A^o,\Omega}(c,\delta)+\bigl(\operatorname{rank}H^\star(M;F)\bigr)^{|\Omega\cdot\Omega_1-\Omega|}.$$

Thus

$$b^{\ell,\nu}_{A^o}(c,\delta)\leq b^{\ell,\nu}_A(c,\delta)\leq\max\Bigl\{b^{\ell,\nu}_{A_o}(c,\delta)\,,\,\lim_{i\to\infty}\tfrac{|\Omega_i\cdot\Omega_1-\Omega_i|}{|\Omega_i|}\ln\bigl(\operatorname{rank}H^\star(M;F)\bigr)\Bigr\}=b^{\ell,\nu}_{A^o}(c,\delta).$$

Example 10.7. Let $F_o:S^1\to[0,1]$ be a Morse function with two non-degenerate critical points, x_-, the minimum, and x_+, the maximum. Let $f_o=F_o\circ p_{id}$. The entropy of f_o can be computed explicitely (cf. [4], §A4). Indeed, let μ denote the fundamental class of $M=S^1$. Then any class in $H^\star(M^\Omega;F)$ is of the type

$$\sum_{\Omega'\subset\Omega}n_{\Omega'}\mu^{\Omega'},$$

where $n_{\Omega'}\in F$ and where $\mu^{\Omega'}=\wedge_{\gamma\in\Omega'}(p_\gamma^*\mu)$. Moreover, for a monomial $a=\mu^{\Omega'}$,

$$\begin{cases} c_a^\Omega=\dfrac{|\Omega-\Omega'|}{|\Omega|},\\ \deg a=|\Omega'|.\end{cases}$$

The first equality is a consequence of the fact that the fundamental class μ can be supported in an arbitrarily small neighborood of any point (e.g. x_-), implying that the class $\mu^{\Omega'}$ can be supported in any neighborhood of $M^{\Gamma-\Omega'}\times\{x_-\}^{\Omega'}$. It is now easy to convince oneself that if $A=p^*_{id}H^\star(M;F)$, then

$$b^{\ell,\nu}_{A,\Omega}(c,\delta)=\#\Big\{\Omega'\subset\Omega;\left\{\begin{array}{l}\dfrac{|\Omega-\Omega'|}{|\Omega|}\in(c-\delta,c+\delta),\\ \dfrac{|\Omega'|}{|\Omega|}\in(\ell-\nu,\ell+\nu).\end{array}\right.\Big\}.$$

Let $n=|\Omega|$. Then

$$b^{\ell,\nu}_{A,\Omega}(c,\delta)\ =\ \sum_{\substack{(c-\delta)n<j<(c+\delta)n\\(1-\ell-\nu)n<j<(1-\ell+\nu)n}}\binom{n}{j}.$$

Therefore

$$\sup_{\substack{(c-\delta)n<j<(c+\delta)n\\(1-\ell-\nu)n<j<(1-\ell+\nu)n}}\binom{n}{j}\ \le\ b^{\ell,\nu}_{A,\Omega}(c,\delta)\ \le\ 2\delta n\sup_{\substack{(c-\delta)n<j<(c+\delta)n\\(1-\ell-\nu)n<j<(1-\ell+\nu)n}}\binom{n}{j},$$

Besides, by Stirling's formula,

$$\begin{aligned}\frac{1}{n}\ln\binom{n}{j}\ &\sim\ \frac{1}{n}\ln\Big(\frac{n^{n+\frac{1}{2}}}{j^{j+\frac{1}{2}}\,(n-j)^{n-j+\frac{1}{2}}}\Big)\\ &\sim\ -\big(\frac{j}{n}\big)\ln\big(\frac{j}{n}\big)-\big(1-\frac{j}{n}\big)\ln\big(1-\frac{j}{n}\big).\end{aligned}$$

where we have removed terms that would produce a nul contribution in the limit. Now

$$b^{\ell,\nu}_{A}(c,\delta)=\sup_{\substack{c-\delta<x<c+\delta\\1-\ell-\nu<x<1-\ell+\nu}}\Big(-x\ln x-(1-x)\ln(1-x)\Big).$$

And thus (using Remark 10.6)

$$b^{\ell}(c)=b^{\ell}_{A}(c)=\left\{\begin{array}{ll}-\infty & \text{if}\ \ c\neq 1-\ell\\ -c\ln c-(1-c)\ln(1-c) & \text{if}\ \ c=1-\ell.\end{array}\right.$$

So b is concentrated along the diagonal $c=1-\ell$ and vanishes at the corners $(0,1)$ and $(1,0)$. The sum of the Betti number entropy is therefore given by

$$b(c)=-c\ln c-(1-c)\ln(1-c).$$

Remark 10.8. As suggested by the previous example, it is always true in the product case that, provided the functions f_Ω have constant range, the function $b(c)$ is nonnegative (i.e. does not achieve the value $-\infty$). This is a consequence of the presence of the fundamental class whose support can be concentrated around any given point in M.

11. GENERALIZED POINCARÉ DUALITY

It has been observed in the product case that a class a in A_Ω, whose restriction to an open subset O does not vanish, admits a nontrivial pairing with a class b in A_Ω provided "O is not too small", meaning is of type $p_\Omega^{-1}(O_o)$ for some O_o in M^Ω (in the case $A=p^*_{id}H^*(M;F)$). This suggests that a condition generalizing Poincaré duality (more precisely its Lemma 10.3 version) should involve a filtration $(\mathcal{T}_\Omega)_{\Omega\in B(F)}$ of the topology $\mathcal{T}$ of X such that Poincaré duality holds in A_Ω for open subsets of $\mathcal{T}_\Omega$ (a precise statement follows). In the product case the topology $\mathcal{T}_\Omega$ is the one generated by the supports of the classes belonging to A_Ω. It seems necessary to assume in addition that these topologies are induced by a family of (continuous) maps $(X,\mathcal{T}_\Omega)\to(X,\mathcal{T})$ converging uniformely to the identity map. In the product case, for $A = p_{id}^* H^*(M; F)$, these maps are the compositions $s_\Omega^o \circ p_\Omega$, where s_Ω^o is a section $M^\Omega \to M^\Gamma$ associated to a point $o \in M^\Gamma$ as follows :

$$\left(s_\Omega^o(x)\right)_\gamma = \begin{cases} x_\gamma & \text{if} \quad \gamma \in \Omega \\ o_\gamma & \text{if} \quad \gamma \notin \Omega. \end{cases}$$

A last detail : Lemma 10.3 holds when A is the full cohomology algebra of a finite product. If A is not of this type (e.g. $M = \mathbb{T}^2$ and A is generated by one cohomology class in $H^1(\mathbb{T}^2)$), then the class b does belong to B_Ω instead of A_Ω, where $B = H^*(M^{\tilde{\Omega}}; F)$ and $\tilde{\Omega}$ is the smallest set for which $H^*(M^{\tilde{\Omega}}; F) \supset A$.

Before stating the condition, we introduce the convention that whenever a sequence $something_\Omega$ (thus indexed by the set $B(\Gamma)$) is said to converge to $something$, it means that $something_{\Omega_i}$ converges to $something$ whenever (Ω_i) is an amenable sequence in Γ.

Condition 11.1. *For any finite-dimensional subalgebra $A \subset \mathcal{A}$, there exist another finite-dimensional subalgebra B with $A \subset B \subset \mathcal{A}$ and a family of continuous maps $(r_\Omega : X \to X)_{\Omega \in B(\Gamma)}$ such that if $\mathcal{T}_\Omega$ denote the topology obtained by pulling back that of X via r_Ω, then*

- $\gamma \circ r_\Omega = r_{\gamma \cdot \Omega} \circ \gamma$,
- *r_Ω converges uniformly and monotonously to the identity map,*
- *for any $a \in A_\Omega$ and $O \in \mathcal{T}_\Omega$ such that $a|_O \neq 0$, there exists a class $b \in B_\Omega$ with*

$$\begin{cases} \operatorname{supp} b \subset O \\ a \wedge b \neq 0. \end{cases}$$

When, in addition to Assumption 3.1, the previous condition is fulfilled, one may carry through the proofs of Lemma 10.3 and of Lemma 10.4, and hence that of Proposition 10.1.

Let A be a finite-dimensional subalgebra of $\mathcal{A}$. Let $\delta > 0$. Define $g_\Omega = f_\Omega \circ r_\Omega : (X, \mathcal{T}_\Omega) \to [0, 1]$ and let

$$\delta_\Omega = 4 \sup\{|f_\Omega(x) - g_\Omega(x)|; x \in X\}.$$

Lemma 11.2. *(Poincaré duality under Condition 11.1) If a belongs to A_Ω for some $\Omega \in B(\Gamma)$, then there exist a level c_a^Ω and a class b in B_Ω such that*

- $\operatorname{supp} a \subset f_\Omega^{-1}(-\infty, c_a^\Omega + \delta_\Omega)$,
- $\operatorname{supp} b \subset f_\Omega^{-1}(c_a^\Omega - \delta_\Omega, +\infty)$,
- $a \wedge b \neq 0$.

Proof. As in the proof of Lemma 10.3 let

$$c_a^\Omega = \inf\{c \in [0, 1]; \operatorname{supp} a \subset f_\Omega^{-1}(-\infty, c)\}.$$

Now define $O_1 = f_\Omega^{-1}(c_a^\Omega - \delta_\Omega, +\infty)$ and $O_2 = f_\Omega^{-1}(c_a^\Omega - \frac{1}{2}\delta_\Omega, +\infty)$. By construction,

$$\sup_{x \in X} \left| f_\Omega(x) - g_\Omega(x) \right| = \frac{\delta_\Omega}{4}.$$

Hence

$$O_2 \subset \left(g_\Omega\right)^{-1}(c - \tfrac{3}{4}\delta_\Omega, +\infty) \subset O_1.$$

Let O denote $(g_\Omega)^{-1}(c - \frac{3}{4}\delta_\Omega, +\infty)$. Then $a|_O \neq 0$. Since $O \in \mathcal{T}_\Omega$, there exists a class $b \in B_\Omega$ with $\operatorname{supp} b \subset O$ and $a \wedge b \neq 0$.

■

Lemma 11.3. *The sequence δ_Ω converges to 0. In other words, the sequence of functions $g_\Omega - f_\Omega$ converges uniformly to 0.*

Proof. Let $\delta > 0$. Since r_Ω converges uniformly and monotonously to the identity map and since X is compact, there exists a finite set $\Omega_o \subset \Gamma$ containing id such that for any $\Omega \supset \Omega_o$ for which the amenability ratio $\alpha(\Omega, \Omega_o)$ is sufficiently small,

$$\left|f(r_\Omega(x)) - f(x)\right| < \delta \qquad \forall x \in X.$$

Let $D_o = \operatorname{diam} \Omega_o$. If $\gamma \in \operatorname{int}_{D_o} \Omega$, then

$$\left|f(\gamma^{-1} r_\Omega(x)) - f(\gamma^{-1}x)\right| = \left|f(r_{\gamma^{-1}\Omega}(\gamma^{-1}x)) - f(\gamma^{-1}x)\right| < \delta.$$

Let $\widetilde{\Omega} = \operatorname{int}_{D_o} \Omega$. Then

$$\left|f_{\widetilde{\Omega}}(r_\Omega(x)) - f_{\widetilde{\Omega}}(x)\right| \le \frac{1}{|\widetilde{\Omega}|} \sum_{\gamma \in \widetilde{\Omega}} \left|f(\gamma^{-1} r_\Omega(x)) - f(\gamma^{-1}x)\right| < \delta.$$

Hence

$$\begin{aligned}
\left|g_\Omega(x) - f_\Omega(x)\right| &\le \frac{|\widetilde{\Omega}|}{|\Omega|}\left|f_{\widetilde{\Omega}}(r_\Omega(x)) - f_{\widetilde{\Omega}}(x)\right| \\
&\qquad + \frac{|\Omega - \widetilde{\Omega}|}{|\Omega|}\left|f_{\Omega - \widetilde{\Omega}}(r_\Omega(x)) - f_{\Omega - \widetilde{\Omega}}(x)\right| \\
&\le \frac{|\widetilde{\Omega}|}{|\Omega|}\delta + \frac{|\Omega - \widetilde{\Omega}|}{|\Omega|} \\
&\le 2\delta,
\end{aligned}$$

Where the very last equality holds when, once more, Ω and Ω_o have a sufficiently small amenability ratio.

■

Combining Lemma 11.2 and Lemma 11.3 we obtain the following result, whose proof is essentially the same as that of Proposition 10.1.

Proposition 11.4. *The function b achieves a strictly positive value.*

12. NON-PRODUCT EXAMPLE

Let M be a projective algebraic variety, e.g. the projective space $\mathbb{C}P^n$, and let Y be a symbolic algebraic subvariety in $X = M^\Gamma$ (in the sense of [1] and [2]), that is, a compact subset Y such that Y_Ω, the "restriction" of Y to each Ω in $B(\Gamma)$, defined as the image of the natural projection $M^\Gamma \to M^\Omega$, is an algebraic subvariety in M^Ω. So Y comes as the projective limit of the Y_Ω's, where one may (or may not) assume that Y is Γ-invariant. We assume that for large enough $d(\Omega, \Omega')$ (depending on Y), the projection $Y_\Omega \cup Y_{\Omega'} \to Y_{\Omega \cup \Omega'}$ is onto. Observe that surjective maps between projective (in general Kähler) varieties are injective on the top-dimensional cohomology with complex coefficients due to existence of multivalued algebraic sections. (In fact if the fibers of such a map have, in a suitable sense, degree d, the same injectivity holds for F_p-coefficients, provided p does not divide d). Therefore, if the target variety is non-singular, then the map is injective on all cohomology by Poincaré duality.

Subexample 12.1. Let $M = \mathbb{C}P^n$ and consider a hypersurface in $M \times M$ represented by an equation $h(x, x') = 0$. Then the infinite chain of equations $h(x_i, x_{i+1}) = 0, i = ..., -1, 0, 1, ...$ defines a subvariety Y in $X = M^{\mathbb{Z}}$ invariant under the $\mathbb{Z}$-action.

Remark 12.2. Unfortunately, even for generic h, it is unclear whether this Y is non-singular in the sense that the restrictions of Y to the intervals $[i, i+1, ..., i+k]$, denoted $Y_{[i,i+1,...,i+k]}$, are non-singular. However, a small (but non-$\mathbb{Z}$-invariant) perturbation Y' of such a Y, allowing different h's, i.e. equations $h_i(x_i, x_{i+1}) = 0$, is non-singular by a simple argument (see [1] and [2]). Furthermore, the non-singular pertubations of Y are all canonically homeomorphic and thus their cohomology can be attributed to Y (alternatively, one may speak of a random Y in X with a suitable $\mathbb{Z}$-invariant probability measure on the space of strings $\{h_i\}$ and similarly introduce random potentials on Y (and/or on X itself). This significantly adds to possible examples and needs only a minor modification of our setting (with a reference to the sub-additive ergodic theorem).

Continuation of the example. The cohomology of our (desingularized) Y enjoys the above product-like action on cohomology. In particular, for $\Gamma = \mathbb{Z}$, the homological entropy of a function exists.

Remark 12.3. It seems hard to compute the (co)homologies of the above $Y_{[i,i+1,...,i+k]}$ or even to elucidate the properties of (the analytic continuation of) their entropic limit. However, it is easy to calculate the Chern numbers and thus the Euler characterictics of the Dolbeaut (and thus the ordinary) cohomology of all (desingularized!) $Y_{[i,i+1,...,i+k]}$.

13. POINCARÉ POLYNOMIAL

This section consists of defining the entropic Poincaré polynomial of a Γ-space. It does not require the action to be product-like nor the group to be tileable. Amenability is the only condition required here.

Consider the A_Ω-Poincaré polynomial of X :

$$p_{A,\Omega}(t) = \sum_{d=1}^{\infty} t^d \operatorname{rank} A_\Omega^d,$$

where A_Ω^d denote the set of classes in A_Ω of (exact) degree d.

Lemma 13.1. *The Poincaré polynomial of X is Γ-invariant and subadditive, that is*

$$p_{A,\Omega\cup\Omega'}(t) \le p_{A,\Omega}(t)\, p_{A,\Omega'}(t) \qquad \text{for} \qquad t \ge 0. \tag{7}$$

Proof. First observe that for any $\Omega_1, \Omega_2 \in B(\Gamma)$, the map

$$\bigoplus_{d_1+d_2=d} A_{\Omega_1}^{d_1} \otimes A_{\Omega_2}^{d_2} \to A_{\Omega_1\cup\Omega_2}^d$$

is surjective. Thus

$$\operatorname{rank} A_{\Omega_1\cup\Omega_2}^d \le \sum_{d_1+d_2=d} \operatorname{rank} A_{\Omega_1}^{d_1} \operatorname{rank} A_{\Omega_2}^{d_2},$$

which immediately implies the relation (7).

■

Thus the limit

$$\lim_{i\to\infty} \frac{1}{|\Omega_i|} \ln\bigl(p_{A,\Omega_i}(t)\bigr)$$

exists whenever (Ω_i) is an amenable sequence in Γ (cf. [3]). We define the Poincaré polynomial of the group action $\rho : \Gamma \times M \to M$ to be

$$p(t) = \sup_A \lim_{i\to\infty} \frac{1}{|\Omega_i|} \ln\bigl(p_{A,\Omega_i}(t)\bigr).$$

Remark 13.2. This definition is analoguous to that in [1] §1.14. Indeed, the process of factoring away ε-fillable classes corresponds roughly to restricting to classes in A_Ω.

REFERENCES

[1] Misha Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps. *I. Math. Phys. Anal. Geom.* **2** (1999), no. 4, 323–415.

[2] Misha Gromov, Endomorphisms of symbolic algebraic varieties. *J. Eur. Math. Soc.* **1** (1999), no. 2, 109–197.

[3] Donald S. Ornstein, Benjamin Weiss, Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.* **48** (1987), 1–141.

[4] Oscar Lanford, Entropy and Equilibrium States in Classical Statistical Mechanics. *Statistical mechanics and mathematical problems. Battelle Rencontres, Seattle, Wash.*, 1971. *Lecture Notes in Physics*, **20**. Springer-Verlag, Berlin-New York, 1973.

[5] Weiss, Benjamin, Monotileable amenable groups. Topology, ergodic theory, real algebraic geometry, 257–262, *Amer. Math. Soc. Transl. Ser. 2*, **202**, AMS, Providence, RI, 2001.

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