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A GEOMETRICAL CONJECTURE OF BANACH

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This article is devoted to the following problem of Banach: Let B^n be a Banach space of finite or infinite dimension n and let k be a natural number satisfying the inequalities $1 < k < n$; if all the k-dimensional subspaces of B^n are isometric to each other, is B^n a Hilbert space? We give a positive answer to this question under certain restrictions on k and n.

§1. Statement of results

The following was conjectured by Banach: If B^n is an n-dimensional (where n is not necessarily finite) Banach space for which all the subspaces of a fixed finite dimension k are isometric to each other and $1 < k < n$, then B^n is a Hilbert space. (We say that a Banach space is a Hilbert space if we can define a scalar product $g(a, b)$ $(a, b \in B^n)$ in it so that $g(a, a) = \|a\|^2$.)

In the real case this conjecture was proved by S. Mazur when $k = 2$ and by A. Dvoretsky for arbitrary k and $n = \infty$ (cf. [5]).

In this article we obtain the following result.

Theorem 1. *Let B^n be a real or complex Banach space and suppose that all its k-dimensional subspaces $(1 < k < n)$ are isometric to each other. Then, in each of the following three cases, B^n is a Hilbert space*:

1. *k is even.*
2. *B^n is complex, k is odd, and $n \geq 2k$.*
3. *B^n is real, k is odd, and $n \geq k + 2$.*

Thus, Banach's conjecture remains unproved for odd k and $n = k + 1$ in the real case, and for odd k and $n < 2k$ in the complex case.

In the cases 1 and 2, Theorem 1 follows from the following:

Theorem 2. *Let L^n be a real or complex linear space of finite dimension n and suppose that in its k-dimensional spaces we have introduced a norm which is a continuous function on the total space of the canonical vector k-bundle with the (real or complex) Grassman manifold $G_{n,k}$ as base. If all the Banach spaces obtained in this way are isometric to each other, then they are Hilbert spaces in each of the following three cases*:

1. *k is even.*
2. *L^n is complex, k is odd, and $n \geq 2k$.*

3. *L^n is real, k is odd, and $n \geq 3k-2$.*

To deduce Theorem 1 from Theorem 2 in the first two cases it is sufficient to quote the well-known and obvious result that a Banach space is a Hilbert space if all its subspaces of some fixed dimension $k > 1$ are Hilbert spaces.

Remark 1. The case 3 of Theorem 2 is not used in the proof of Theorem 1.

Remark 2. The restrictions 1, 2, and 3 in Theorem 2 are essential, in general (cf. §5).

Theorem 2 will be obtained in §4 as a consequence of results in algebraic topology connected with the problem of reducing the structure groups of certain fiber bundles (cf. §3).

The proof of case 3 of Theorem 1 is also based on a result in algebraic topology which is proved in §3.

§2. Geometrical lemmas

Lemma 1. *Let L^k be a finite-dimensional (real or complex) linear space with a (real or complex) norm. Among the scalar products f (real or complex) defined on L^k and satisfying the condition $f(a, a) \leq \|a\|^2$ ($a \in L^k$) there exists a unique scalar product such that the Haar measure of the unit ball defined by this scalar product is minimal. Such a scalar product will be said to be extremal.*

(We note that the extremal scalar product does not depend on the choice of the Haar measure in L^k.)

This result is given in H. Busemann's book ([4], Russian pp. 123-124); however, we give a proof here since Busemann's presentation is only for the real case and is developed in terms somewhat different from ours.

Proof. The existence of an extremal scalar product follows trivially from the fact that L^k is finite-dimensional. We assume that there are two extremal scalar products f, f' for which the volume of the unit ball is V, and we consider the scalar product $g = {}^1/_2(f+f')$. It is clear that $g(a, a) \leq \|a\|^2$. Lemma 1 will have been proved if we can show that, when $f \neq f'$, the volume of the unit ball defined by g is strictly less than V.

As is well-known, there exists a basis in L^k with respect to which the forms f and f' are diagonal: $f = (f_1, \ldots, f_k)$ and $f' = (f_1', \ldots, f_k')$, where the f_i and f_i' are positive real numbers. With respect to this basis,

$$g = \left(\frac{f_1 + f_1'}{2}, \ldots, \frac{f_k + f_k'}{2}\right).$$

On the other hand, for any scalar product h which is reduced to the diagonal form $(h_1, \ldots, h_k)$, the volume of the unit ball is $C(\Pi h_i)^{-1/2}$ in the real case and $C(\Pi h_i)^{-1}$ in the complex case; here C is a positive constant independent of h. Thus, it is sufficient to prove that

$$2^{k/2}(\Pi(f_i + f_i'))^{-1/2} < (\Pi f_i)^{-1/2} \tag{1}$$

in the real case and that

$$2^k(\Pi(f_i + f_i'))^{-1} < (\Pi f_i)^{-1} \tag{2}$$

in the complex case, under the assumption that the f_i and f_i' are positive numbers and that

$$(f_1, \ldots, f_k) \neq (f_1', \ldots, f_k'), \tag{3}$$

$$\Pi f_i = \Pi f_i'. \tag{4}$$

The inequality (1) follows from (2) which, in turn, is obtained by the following calculations. It follows from (3) that

$$4^k < \prod \left(4 + \left(\frac{\sqrt{f_i}}{\sqrt{f_i'}} - \frac{\sqrt{f_i'}}{\sqrt{f_i}}\right)^2\right),$$

that is

$$4^k < \prod \frac{(f_i + f_i')^2}{f_i \cdot f_i'};$$

taking (4) into account we obtain

$$2^k < \prod \left(\frac{f_i + f_i'}{f_i}\right),$$

which is equivalent to (2).

Thus, on a linear space L^k of any finite-dimensional Banach space B^k there is a uniquely defined extremal scalar product which converts L^k into a Hilbert space Γ^k. In view of the uniqueness, this extremal scalar product is invariant under those linear transformations of L^k which are isometries of B^k.

Two obvious corollaries follow from this last remark; we shall need them in what follows.

Corollary 1. *The group $G(B^k)$ of all isometries of B^i is naturally imbedded as a subgroup in the group of isometries of the Hilbert space Γ^k.*

Corollary 2. *If the group $G(B^k)$ operates transitively on the unit ball in Γ^k, then B^k is a Hilbert space.*

To prove Corollary 2 it is sufficient to note that the intersection of the unit balls in the spaces B^k and Γ^k is nonempty.

Remark. We can establish in the same way that if the group $G(B^k)$ operates transitively on the unit ball of B^k, then B^k is a Hilbert space (cf. [4], Russian p. 124); however, this result will not be used in the sequel.

Next we consider the k-dimensional (real or complex) vector-bundle with base A and fibers L_a^k $(a \in A)$ in which there is a norm that is a continuous function on the total space of the bundle. In each fiber we introduce an extremal scalar product which is obviously continuous with respect to the base, and we consider the bundle Y of unit spheres (of dimension $k-1$ in the real case and of dimension $2k-1$ in the complex case) defined by this scalar product. The structure group of this bundle is $O(k)$ in the real case and $U(k)$ in the complex case.

Lemma 2. *If all the fibers of the fibering of X are isometric (with respect to the norm introduced above) to some space B^k, then either B^k is a Hilbert space or the structure group of the bundle Y can be reduced to a closed subgroup which is nontransitive in a fiber of the fibering of Y.*

Proof. We shall show that the bundle X, and so also the bundle Y, admits a reduction of the structure group to $G(B^k)$; then, in view of Corollary 2 of Lemma 1, Lemma 2 will have been proved. (The group $G(B^k)$ is compact and hence is closed in any Lie group.)

In the set of all those linear mappings of the space B^k into the fibers of the bundle X which preserve the extremal scalar product, we distinguish the subset of mappings which preserve the norm of B^k. These determine a cross-section (the continuity is obvious) of the fiber-bundle associated with X, with fiber $O(k)/G(B^k)$ in the real case, and with fiber $U(k)/G(B^k)$ in the complex case. Hence (cf. [9], Russian p. 56) the structure group of X is reducible to $G(B^k)$.

§3. Some criteria for the irreducibility of the structure groups of fiberings

Lemma 3. *Suppose we are given a principal m-universal fibering X_m whose fiber is a compact Lie group G. We denote by N the nontrivial connected simple normal divisors of G. If $m > 1$ and*

$$m \geqslant \max_{N \subset G} \left(\frac{2 \dim N}{\operatorname{rank} N} \right) - 2, \tag{5}$$

then the structure group G of X_m cannot be reduced to any of its closed subgroups distinct from G.

Proof. We note that a closed subgroup of a compact Lie group is a compact Lie group (see [7]). For any connected compact Lie group H we have the formula (cf. [8], Russian p. 155):

$$\Sigma i \operatorname{rank} \pi_i(H) = \dim H, \tag{6}$$

where rank $\pi_i(H) = 0$ when i is even.

If now H is an arbitrary connected closed subgroup of a compact connected Lie group G, then $\dim H < \dim G$ (since G is connected) and, by applying (6), we find that there is an $i > 0$ such that

$$\operatorname{rank} \pi_i(H) < \operatorname{rank} \pi_i(G). \tag{7}$$

We denote by $l_1(G)$ the minimum (nonzero) and by $l_2(G)$ the maximum of those dimensions in which the ranks of the homotopy groups of G are different from zero.

For a nontrivial commutative connected compact Lie group G we obviously have

$$l_1(G) = l_2(G) = 1. \tag{8}$$

For a connected compact simple Lie group G the following condition (cf. [8], Russian p. 158) holds:

$$l_1(G) = l_2(G) = \frac{2 \dim G}{\operatorname{rank} G}, \quad l_1(G) \geqslant 3,$$

and hence

$$l_2(G) \leqslant \frac{2 \dim G}{\operatorname{rank} G} - 3. \tag{9}$$

On the other hand, the universal covering of any connected compact Lie group is isomorphic to the direct sum

$$\hat{G} = R^q + \hat{N}_1 + \ldots + \hat{N}_p,$$

where the $\hat{N}_j$ are the finite-sheeted coverings of all the simple normal divisors of G (cf. [6], p. 12). This implies that for a compact connected noncommutative Lie group we have

$$l_2(G) = \max_{N \subset G} l_2(N)$$

and, by applying (9), we obtain

$$l_2(G) \leqslant \max_{N \subset G} \left(\frac{2 \dim G}{\operatorname{rank} G}\right) - 3. \tag{10}$$

Now suppose that when the conditions of Lemma 3 hold, the structure group of the bundle X_m is reducible to $H \subset G$. Then, since the fibering of X_m is m-universal, the structure group of any G-fibering over any sphere S^{i+1} of dimension $i + 1 \leq m$ is reducible to H. If we assume that the groups G and H are connected, then (cf. [9], Russian p. 117) for $i < m$ the imbeddings $\pi_i(H) \to \pi_i(G)$ are epimorphisms and consequently

$$\operatorname{rank} \pi_i(H) \geqslant \operatorname{rank} \pi_i(G) \qquad (i < m). \tag{11}$$

We now prove Lemma 3 for a connected group G. When G is connected and $m > 1$, the base A of the fibering of X_m is obviously simply-connected and, consequently, because G is reducible to a subgroup H, it follows that the structure group of the fibering of X_m is reducible to a connected component of the identity of the subgroup H.

On the other hand, $m > l_2(G)$; this follows from the fact that $m > 1$ and from (8) when G is commutative, and from (5) and (9) when G is noncommutative. By comparing (7) and (11) we obtain the proof of Lemma 3 for a connected group G.

If G is not connected, its irreducibility to the subgroup H consisting of whole connectedness components follows from the result that the total space of the fibering of X_m is connected. If H does not consist of whole components, the problem reduces to the case of a connected group G, if we consider the fibering induced from the fibering of X_m by the universal covering of the base of the fibering of X_m.

Lemma 4. *Suppose that we are given an orthogonal bundle of $(k-1)$-dimensional spheres ($k \geq 2$)*

$$\varphi:\ Y \xrightarrow{S^{k-1}} A$$

whose structure group is reducible to some subgroup $H \subset O(k)$ that is nontransitive on the sphere S^{k-1}. Then all the characteristic homomorphisms (homomorphisms that are the compositions of the boundary homomorphisms of the homotopy sequence of the fibering and the homomorphisms imbedding a fiber in the fiber space (cf. [9], Russian p. 115):

$$\chi:\ \pi_i(A, a_0) \to \pi_{i-1}(S^{k-1})$$

are trivial for $i > 1$.

Proof. Since the structure group of the bundle Y is reducible to H, in the set of orthogonal mappings of the sphere S^{k-1} into the space Y we distinguish a certain subset of mappings under which a specific point $s_0 \in S^{k-1}$ maps out a certain subspace Y' of Y.

We consider the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi'} & A \\ {\scriptstyle j}\downarrow & \nearrow_{\varphi} & \\ Y & & \end{array} \qquad (12)$$

where ϕ' is the restriction of the mapping ϕ to Y' and j is the natural imbedding of the space Y' in Y. It is clear that the mapping ϕ': $Y' \to A$ is turned naturally into a fiber-bundle (H associated with the original one), whose fiber F is the trajectory of the point s_0 on S^{k-1} under the action of the subgroup H. Because the diagram (12) is commutative and because of the functorial nature of the homomorphism χ, the following diagram is also commutative:

$$\chi:\ \pi_i(A)\begin{array}{l} \nearrow \pi_{i-1}(F) \\ \qquad\ \ \downarrow {\scriptstyle k_*} \\ \searrow \pi_{i-1}(S^{k-1}) \end{array} \qquad (13)$$

The homomorphism k_* corresponds to the imbedding $k : F \to S^{k-1}$. Since $F \neq S^{k-1}$, $k_* = 0$ for $i > 1$ and, since the diagram (13) is commutative, the homomorphism $\chi : \pi_i(A) \to \pi_i(S^{k-1})$ is also trivial when $i > 1$.

Proposition 1. *The structure group of the canonical fibering over the real or complex Grassman manifold $G_{n,k}$ $(n > k > 0)$ is not reducible to any of its proper closed subgroups in each of the following cases:*

1. *$G_{n,k}$ is complex and $n \geq 2k$.*
2. *$G_{n,k}$ is real, k is odd, and $n \geq 3k - 2$.*
3. *$G_{n,k}$ is real, k is even, and $n \geq 3k - 4$.*

Proof. Case 1. The structure group of the fibering over the complex manifold $G_{n,k}$ is $U(k)$. If $k = 1$, then $U(k)$ is commutative but it is noncommutative for $k > 1$. It is well-known that when $k > 1$, $U(k)$ has a unique nontrivial simple normal divisor which is the group $SU(k)$ of rank $k - 1$ and dimension $k^2 - 1$. (cf. [6], p. 12). The fibering over $G_{n,k}$ in the complex case is $(2(n - k) + 1)$-universal (cf. [9], Russian p. 162). Since $n \geq 2k$ it follows that

$$2(n-k)+1 \geqslant \frac{2(k^2-1)}{k-1} - 2,$$

so that it only remains to apply Lemma 3.

Case 2. In this case the structure group of the fibering is $O(k)$. When $k = 1$, the principal fibering is a two-sheeted covering $(n > k)$ so that the assertion is obviously valid. For $k > 1$, $O(k)$ has a unique nontrivial connected simple normal divisor which is the group $SO(k)$ of rank $p = (k - 1)/2$ and dimension $k(k - 1)/2$ (cf. [6], p. 12). In the real case the fibering over $G_{n,k}$ is $(n - k)$-universal (cf. [9], Russian p. 160). Since $n \geq 3k - 2$ it follows that

$$n-k \geqslant \frac{2k(k-1)}{2p} - 2,$$

so that again it only remains to apply Lemma 3.

Case 3. We assume that $k \neq 2$. The structure group of this fibering is again $O(k)$ which has the single nontrivial connected simple normal divisor $SO(k)$ of rank $p = k/2$ and of dimension $k(k-1)/2$ (cf. [6], p. 12). In this case the fibering over $G_{n,k}$ is $(n-k)$-universal and, since $n \geq 3k-4$, we find that

$$n-k \geqslant \frac{2k(k-1)}{2p}-2,$$

so that we need only apply Lemma 3.

If $k = 2$ we must prove that the structure group of the fibering over $G_{n,k}$ is irreducible when $n > 2$. The irreducibility of the structure group $O(2)$ to the subgroup $SO(2)$ follows trivially from the connectedness of the total space of the principal fibering over $G_{n,2}$ for $n > 2$. Any subgroup of $O(2)$, distinct from $SO(2)$, acts nontransitively in the fiber S^1 of the associated sphere bundle and, in this case, Proposition 1 follows from the following:

Proposition 2. *The structure group of the canonical fibering over the real or complex Grassman manifold $G_{n,k}$ $(n > k > 0)$ cannot be reduced, when k is even, to any of its closed subgroups that act nontransitively in the fiber of the associated sphere bundle (the spheres being $(k-1)$-dimensional in the real case and $(2k-1)$-dimensional in the complex case).*

Proof. We note that if the structure group of some fibering is reducible to a subgroup H, then the structure group of any induced fibering is also reducible to H.

In the real case we consider the fibering

$$\varphi\colon\ V_{k+1,2} \xrightarrow{S^{k-1}} S^k,$$

induced from the bundle of $(k-1)$-dimensional spheres over $G_{n,k}$ $(n > k)$ ($V_{k+1,2}$ is a real Stiefel manifold). The homomorphism

$$\chi\colon\ \pi_k(S^k) \to \pi_{k-1}(S^{k-1})$$

is nontrivial when k is even (cf. [9], Russian p. 146), so that it remains to apply Lemma 3.

In the complex case we consider the fibering

$$\varphi\colon\ W_{k+1,2} \xrightarrow{S^{2k-1}} S^{2k+1},$$

induced from the bundle of $(2k-1)$-dimensional spheres over the complex manifold $G_{n,k}$ $(n > k)$ ($W_{k+1,2}$ is a complex Stiefel manifold). The homomorphism

$$\chi\colon\ \pi_{2k+1}(S^{2k+1}) \to \pi_{2k}(S^{2k-1})$$

is nontrivial for even k (cf. [9], Russian p. 152, and also [3]), so that again it remains to refer to Lemma 3.

Remark. The condition that k is even in Proposition 2 is necessary when $n = k + 1$. In the real case this follows from the existence of nonzero vector fields on odd-dimensional spheres, and in the complex case from the possibility of introducing a symplectic structure (cf. [9], Russian p. 171).

We now consider the real euclidean space E^n with $n \geq 4$, and the associated space $V_{n,4}$ of orthonormal 4-frames (e_1, e_2, e_3, e_4), where the $e_i \in E^n$. We consider two fiberings

$$p_1\colon V_{n,4} \to V_{n,2} \text{ and } p_2\colon V_{n,4} \to V_{n,2},$$

where

$$p_1(e_1, e_2, e_3, e_4) = (e_1, e_2),$$

and

$$p_2(e_1, e_2, e_3, e_4) = (e_3, e_4).$$

The fibers of the fiberings p_1 and p_2 are homeomorphic to the Stiefel manifold $V_{n-2,2}$. We consider the imbedding $i: V_{n-2,2} \to V_{n,4}$ of the standard fiber of the fibering p_1 and use i_p^* to denote the homomorphism

$$i_p^*: H^p(V_{n,4}; Q) \to H^p(V_{n-2,2}; Q).$$

If n is odd we have the following results (cf. [2], Russian p. 192):

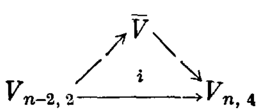
$$i_p^* \neq 0 \text{ when } p = \dim V_{n-2,2} = 2n-7, \tag{14}$$

$$H^p(V_{n-2,2}; Q) = 0 \text{ when } 0 < p < \dim V_{n,2} = 2n-3. \tag{15}$$

We consider a nonempty, closed subset $V \subset V_{n,2}$ and we denote the inverse image $p_1^{-1}(V)$ by $\bar{V}$.

Proposition 3. *If n is odd and if the restriction $p_2'\colon \bar{V} \to V_{n,2}$ of the projection p_2 is a fibering in in the sense of Serre, then the subset V coincides with the whole of $V_{n,2}$.*

Proof. From the commutativity of the diagram of imbeddings



and from (14) it follows that

$$H^p(\bar{V}; Q) \neq 0 \text{ when } p = \dim V_{n-2,2} = 2n-7. \tag{16}$$

We now consider an arbitrary fiber $V_{n-2,2}$ of the fibering p_2, and we prove that the intersection $\bar{V} \cap V_{n-2,2}$, which we denote by V', coincides with $V_{n-2,2}$; this will also prove Proposition 3. In fact, if $V' \neq V_{n-2,2}$, then $H^{2n-7}(V; Q) = 0$ and for $p + q = 2n-7$, by (15), the second term $E_2^{p,q}$ in the spectral sequence of the fibering $p_2'\colon \bar{V} \xrightarrow{V'} V_{n,2}$ is trivial. But then the group $H^{2n-7}(\bar{V}; Q)$ is trivial, contradicting (16).

§4. The proof of Theorems 1 and 2

The case 1 of Theorem 2 follows immediately from Lemma 2 and from Proposition 2.

The cases 2 and 3 of Theorem 2 are derived in the same way from Lemma 2 and Proposition 1.

As we have already remarked, the cases 1 and 2 of Theorem 1 reduce trivially to the cases 1 and 2 of Theorem 2.

To prove the case 3 of Theorem 1 it is sufficient to prove that Banach's conjecture is true for odd k and $n = k + 2$.

We define a scalar product in the real space B^n, independently of the norm, and choose an arbitrary two-dimensional subspace $B^2 \subset B^n$, which is then kept fixed. We use the notation of Proposition 3. We form the subset $V \subset V_{n,2}$ as follows: a pair of orthogonal vectors belongs to V if and only if the subspace spanned by them is isometric to B^2. The mapping $p_2': \widetilde{V} \to V_{n,2}$ is a fiber bundle with structure group $G(B^k)$. By Proposition 3 the subset V coincides with the whole space $V_{n,2}$ and, consequently, all the two-dimensional subspaces of B^n are isometric to each other; it only remains to refer to the case 1 of Theorem 1.

§5. Supplement

We shall show that, in general, the restrictions in Theorem 2 are essential. For this we remark that the following converse of Lemma 2 holds.

Lemma 2'. *If the structure group of the (orthogonal or unitary) k-bundle X is reducible to a subgroup G containing all products by* λ *(*λ *is real or complex and* $|\lambda| = 1$*) where G acts nontransitively on the unit vectors of each fiber of the fibering of X, then we can introduce a norm in all the fibers so that the Banach spaces obtained in this way are isometric to a fixed space* B^k *that is not a Hilbert space, and* $G(B^k) \supset G$.

By comparing this lemma with the remark to Proposition 2 we see that the restrictions 1, 2, and 3 in Theorem 2 are essential.

The following question arises in connection with Theorem 2: What can be said about the space B^k if all the fibers of the bundle over $G_{n,k}$ $(n > k > 1)$ are Banach spaces isometric to B^k? By using various theorems on the absence of cross-sections in these or other fiberings we can obtain information about the group $G(B^k)$. For example, it is easy to show that if B^k is real and if $k = 4l + 1$, then, when $1 < i < k - 1$, $G(B^k)$ does not have invariant i-dimensional subspaces. Using the results of Adams (cf. [1], Russian p. 49) it is not difficult to show that for real B^k, apart from the cases $n = k + 1 = 4$ and $n = k + 1 = 8$, we can find a two-dimensional Hilbert subspace of B^k, any isometry of which can be extended to the whole of B^k.

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