HYPERBOLIC MANIFOLDS, GROUPS AND ACTIONS

M. Gromov

0. Introduction

This lecture gives an outline of basic geometric notions and ideas associated with the conception of hyperbolicity. Very little is said here about the hyperbolic space itself (the main source of knowledge is Thurston's lectures [21]), but it is shown how the phenomenon of hyperbolicity appears in Riemannian geometry, topological dynamics, combinatorial group theory and geometrical theory of mappings. Our presentation is expository, proofs are only sketched, but constructions and definitions are illustrated by examples. The comprehensive theory of hyperbolicity is yet to be built and my purpose here is to provide motivation for future research. (Questions and conjectures are italicized.)

1. Length Spaces

1.1. *Riemannian spaces with singularities*

Our main example is the following. Take a piecewise smooth polyhedron $X \subset \mathbf{R}^N$. The space X carries the metric d induced from $\mathbf{R}^N$. We get another metric if we use *the length function* on curves in X and define $R(x,y)$, $x,y \in X$, as the length of the shortest curve in X joining x and y. Such an R is called the *induced length metric.*

We obviously have $R \geq d$ and the equality holds iff X is a convex set.

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DIGRESSION. The quantity $\sup(R(x,y)\cdot d^{-1}(x,y))$ measures the distortion of X. One can easily show that the distortion of any topological circle is at least $\pi/2$ and consequently, $\text{Distortion}(X) < \pi/2$ implies that X is simply connected. *Probably this inequality implies that* X *is contractible*, but I can show it only under the stronger assumption $\text{Distortion}(X) < \pi/2\sqrt{2}$.* A different approach to the distortion can be found in [10].

DEFINITION. A length structure in a space X is given by a metric and a length function on curves, such that the distance between any two points is equal to the length of the shortest curve (supposed to exist) joining these points. We shall also assume all metric spaces to be complete unless stated otherwise.

FURTHER EXAMPLES. Let V be a Riemannian manifold with boundary. Its length structure is more complicated than that of a manifold due to the fact that the shortest curves between interior points can touch the boundary.

Take a Riemannian manifold and divide it by a group of isometries. When the action is not free then the natural length structure again has new features.

DEFINITIONS. A map from one length space (space with length structure) to another is called *isometric* if it preserves the distance function. Such maps are always embeddings. A map preserving the lengths of the curves is called *path-isometric*. Isometric maps from $[a,b]$, $\mathbf{R}_+$, $\mathbf{R}$ into X are called *straight segment, ray* and *straight line* correspondingly. A locally isometric map $\mathbf{R} \to X$ is called a *geodesic*. There is a natural $\mathbf{R}$-action in the set of all geodesics called the *geodesic flow*.

1.2. *Horofunctions and the ideal boundary*

Let X be a complete metric space. The distance function determines an isometrical embedding $x \to \text{dist}(x,y)$ from X into the space $C(X)$ of

*Unpublished.

the continuous functions on X. Consider the factor space $C' = C(X)/(\text{constant functions})$ with the topology of uniform convergence on bounded sets in X. The space X is now embedded into C' and we can define the closure $C\ell(X)$ and the boundary $\partial X = C\ell(X) \setminus X$. The space X is assumed further to be countably compact. In this case, $C\ell(X)$ and $\partial(X)$ are compact.

A function $h \in C(X)$ that projects into a point $b \in \partial X \subset C'$ is called a horofunction centered at b. The sets $h^{-1}(-\infty, c) \subset X$ are called (open) *horoballs* centered at c; the levels $h^{-1}(c) \subset X$ are called *horospheres* and the sets $h^{-1}(c, \infty)$ are called *horospaces*.

The *limit set* ∂X_0 (or the set of the limit points) of a closed $X_0 \subset X$ is defined as the intersection of its closure in $C\ell(X_0)$ with ∂X.

When X is a length space, every ray has exactly one limit point. (We identify rays and straight lines with their images.) The corresponding horofunction is called the *ray function* or *the Busemann function*.

1.3. *Isometries*

With an isometry $\gamma: X \to X$, one associates the *displacement function* $\delta_\gamma: X \to \mathbf{R}_+$, $\delta_\gamma(x) = \operatorname{dist}(x, \gamma(x))$. An isometry is called *elliptic* if its displacement $X \to \mathbf{R}_+$ is proper (i.e., the sets $\delta_\gamma^{-1}[0, a] \subset X$ are compact). The isometry γ is called *hyperbolic* if it generates a free cyclic group Γ, the action of Γ in X is discrete and δ_γ is proper on X/Γ (δ_γ is Γ-invariant function). When γ is elliptic or hyperbolic, δ_γ assumes its minimum.

In a length space, every hyperbolic or a fixed point free elliptic isometry has an invariant geodesic, (an easy argument).

An isometry γ without fixed points is called *parabolic* if each set $\delta_\gamma^{-1}(0, \varepsilon) \subset X$, $\varepsilon > 0$, is unbounded. Every such isometry has, obviously, a fixed point at ∂X. (All isometries can be continuously extended to $C\ell X$.)

1.4. *The word metric*

Consider a group Γ generated by $\beta_1, \cdots, \beta_k$ and denote by $\|\gamma\|$ the

length of the minimal word (in β_i and β_i^{-1}) representing γ. Obviously $\|\gamma\| = \|\gamma^{-1}\|$ and $\|\gamma_1\gamma_2\| \leq \|\gamma_1\| + \|\gamma_2\|$. The length function gives rise to the left invariant metric $\|\gamma_1^{-1}\gamma_2\|$ on Γ.

Quasi-isometry. A map f from a metric space X to Y is called quasi-isometric if the ratio $\mathrm{dist}(x_1, x_2)/\mathrm{dist}(f(x_1), f(x_2))$ $x_1, x_2 \in X$, is pinched between C and C^{-1}. Sometimes one specifies C and calls such an f a C-quasi-isometric map.

Obviously, two different word metrics, corresponding to different choices of generators, are quasi-isometric.

Coarse quasi-isometries. A coarse quasi-isometric map $X \to Y$ is a map f_0 from a subset $X_0 \subset X$ to Y such that X_0 is ε-dense* in X (i.e., its ε-neighborhood coincides with X) and f_0 is C-quasi-isometric. We call f_0 a coarse equivalence if its image is ε-dense* in Y. This is an equivalence relation if the constants C and ε are allowed to vary.

The following obvious fact plays an important role in connecting geometry of a space with its fundamental group.

Let X be a compact length space and $\tilde{X}$ be its universal covering. Then $\tilde{X}$ (with the induced length structure) is coarse equivalent to the fundamental group $\pi_1(X)$. (We suppose here X to be topologically a polyhedron or a more general space admitting covering space theory.)

1.5. *Convex sets*

A set A in a length space is called *convex* if its intersection with every geodesic segment is connected. A set is called *locally convex* if for any $a \in A$ there is an ε such that the intersection of A with any geodesic segment of the length ε passing through a is connected.

2. Convex Spaces

2.1. *Definition*

A length space X is called *convex* if the distance function is convex; namely, for any two geodesic segments $x : [a, b] \to X$, $y : [c, d] \to X$ $\mathrm{dist}(x(t_1), y(t_2))$ is convex on $[a, b] \times [c, d]$.

*In this paper ε is positive but not always small.

X is called *locally convex* if it can be covered by open U_i such that $\mathrm{dist}(x, y)$ is convex when x, y belong to U_i.

EXAMPLES. A Riemannian manifold is locally convex iff $K \leq 0$ (i.e., sectional curvature ≤ 0). It is convex iff $K \leq 0$ and $\pi_1 = 0$. In the context of length spaces one has by adapting the classical arguments,

THE CARTAN HADAMARD THEOREM. *A simply connected locally convex space is convex. Every convex space is straight (any two points can be joined by the unique geodesic segment) and hence contractible.*

QUESTION. *Are there convex length spaces which are topological manifolds different from* R^n?

2.2. *Manifolds with boundary*

Let V be a Riemannian manifold with smooth boundary B. The boundary is called convex when the second quadratic form is non-negative; it is called concave when this form is non-positive. The boundary is called k-*convex* if the second quadratic form looks infinitesimally as

$$\sum_{1}^{n-1} a_i x_i^2, \quad n-1 = \dim B,$$

where among a_i there are at least k non-negative numbers. When $k = n-2$, we call B *next-to-convex*. ($k = n-1$ corresponds to convexity.)

REMARK. *Riemannian manifolds with* $K \leq 0$ *and with next-to-convex boundary* B *have locally convex length structure.* For example, surfaces with boundary are locally convex when $k \leq 0$.

More generally, take a 2-plane τ tangent to B and denote by K'_τ the sectional curvature of the induced metric in B. The following property is necessary and sufficient for local convexity of V (viewed as a length space):

$K \leq 0$ and for any τ where the second quadratic form is negative K'_τ is non-positive. The proofs of all these points are straightforward.

2.3. *Digression*: k-*convex hypersurfaces in* $\mathbf{R}^n$

Let $V \subset \mathbf{R}^n$ be a compact domain with smooth boundary B.

The classical fact stating convexity of a locally convex connected set can be formulated in the following fancy fashion:

B is (n–1)-convex iff for any straight line $\ell < \mathbf{R}^n$ the homomorphism $H_0(\ell \cap V) \to H_0(V)$ is injective.

By using rudimentary Morse theory one generalizes this theorem:

B is k-convex iff for any (n–k)-dimensional plane $P \subset \mathbf{R}^n$ the homomorphism $H_{n-k-1}(P \cap V) \to H_{n-k-1}(V)$ is injective. In this case V has the homotopy type of an (n–k)-dimensional polyhedron.

The most regular behavior is shown by V's with the next-to-convex boundary: Let V have unit volume. Denote by A the (n–1)-dimensional volume of its boundary and by b_1 the first Betti number of V. *Then* V *contains a ball of radius* $\varepsilon > F(A, b_1, n)$, where for F one can take $\exp(-\exp(A+b_1+n)$.

Sketch of the proof. We actually find a cube C of the size ε contained in V: moving an ε-cube by parallel translation, we first inscribe its 1-skeleton $C^{(1)}$ into V such that the homomorphism $H_1(C^{(1)}) \to H_1(V)$ is trivial. Because V is next-to-convex, the inclusion $C^{(1)} \subset V$ implies $C \subset V$.

2.4. *The double construction*

Take a space X and a closed set $A \subset X$. Take two copies of X and glue them at A.

When X is a Riemannian manifold with $K \leq 0$ and A is a closed locally convex set, then the resulting space Y (with the natural length structure) is locally convex.

The proof is simple.

EXAMPLE. Suppose that the interior Int A is dense in A. The geodesic flow in X Int A with reflection at the boundary is called *Sinai's billiard in* X *with obstacle* A. Each of the two natural embeddings X Int $A \to Y$

sends the orbits of Sinai's billiard into geodesics in Y (this is easy to show). So we have:

The geodesic flow in Y contains the Sinai's billiard as an invariant set.

2.5. *Branched covers*

Take a Riemannian manifold V with $K \leq 0$ and a codimension 2 submanifold $V_0 \subset V$. When V_0 is locally convex (i.e., totally geodesic) any branched covering of V with the branching locus V_0 is a convex length space. (Easy to show.)

Branched torus. Let V be a flat n-dimensional torus and V_0 is a union of flat codimension 2 subtori with transversal intersections. The notion of branched covering still makes sense here, and when all intersections are not only transversal but orthogonal, the branched cover $\tilde{V}$ is a convex length space.

When $\dim V$ is odd, a generic linear map $V \to S^1$ gives rise to a fibration $\tilde{V} \to S^1$ where the fiber F is again a branched torus but the subtori forming its branch-locus have non-normal (though transversal) intersections (this was pointed out to me by Thurston).

When $\dim V = 3$, for an appropriate $\tilde{V}$ one can change its metric to a Riemannian metric of constant negative curvature (Jorgensen example, see [21], [23]). In this case, F is a surface of genus ≥ 2.

When $\dim V \geq 5$, the fiber F does not carry a metric of negative curvature. It follows from the following unpublished theorem of Thurston: *Let F be a compact Riemannian manifold with $K < 0$ and $\dim F \geq 3$. Let $\alpha : \pi_1(F) \circlearrowleft$ be an automorphism. Then the orbits of α acting on the conjugacy classes of $\pi_1(F)$ are finite.*

2.6. *Parabolic isometries*

Consider an isometry γ of a length space X with displacement function $\delta_\gamma = \mathrm{dist}(x, \gamma(x))$. Denote by δ_γ^∞ the limit

$$\lim_{n\to\infty} \frac{1}{n}\, \delta_{\gamma n}(x), \quad x \in X .$$

The limit obviously exists and does not depend on x because $\delta_{\gamma^{m+n}}(x) \leq \delta_{\gamma^m}(x) + \delta_{\gamma^n}(x)$.

Let X *be convex and let* γ *have no fixed points. Then*

$$\delta_\gamma^\infty = \inf_{x \in X} \delta_\gamma(x) .$$

Proof. The inequality $\delta_\gamma^\infty \leq \inf_{x \in X} \delta_\gamma(x)$ is trivial and it holds for all length spaces. When X is convex, we obviously have:

$$\inf_{x \in X} \delta_{\gamma^2}(x) \geq 2 \inf_{x \in X} \delta_\gamma(x) ;$$

this implies the theorem.

COROLLARY. *If* γ *is fixed point free and* $\delta_\gamma^\infty = 0$, *then* γ *is parabolic.*

2.7. *Straight subgroups and asymptotic torsion*

Let Γ be any group and $\Gamma^0 \subset \Gamma$ be a finitely generated subgroup. We call Γ^0 *straight* in Γ if for any finitely generated $\Gamma' \subset \Gamma$ containing Γ^0, the inclusion $\Gamma^0 \hookrightarrow \Gamma'$ is a quasi-isometric imbedding with respect to the word metrics in Γ^0 and Γ'.

We call a $\gamma \in \Gamma$ an asymptotic torsion element if for some finitely generated Γ' containing γ, we have $\lim_{n \to \infty} \frac{1}{n} \|\gamma^n\| = 0$. This happens exactly when the group generated by γ is finite, or when it is infinite cyclic but not straight.

An example. Let Γ be a nil-potent group without torsion. Then Γ has asymptotic torsion unless it is Abelian.

Combining this with the theorem from the previous section, we conclude.

THEOREM. *Let* Γ *be a group of isometries of a convex space. If* Γ *has no elliptic or parabolic elements, then every nil-potent subgroup in* Γ *is Abelian.*

An analogous argument shows:

If there is an $\varepsilon > 0$ such that $\delta_\gamma(x) \geq \varepsilon$. $x \in X$, $\gamma \in \Gamma^*$ $(= \Gamma \setminus \text{identity})$ then any Abelian subgroup in Γ is straight. Observe also that an Abelian group with $\delta_\gamma(x) \geq \varepsilon$ must be finitely generated and torsion free.

As a corollary, we get a generalization of the Gromoll-Wolf-Lawson-Yau theorem (see [5]):

Let Γ be a soluble group of isometries of a convex space X. If $\delta_\gamma(x) \geq \varepsilon > 0$, $\gamma \in \Gamma^*$, $x \in X$, then Γ contains an Abelian subgroup of finite index.

Proof. If every Abelian subgroup in a soluble group Γ is finitely generated and straight, then Γ obviously contains the required Abelian subgroup.

Observe that for the branched torus the subgroup $\pi_1(F) \subset \pi_1(\tilde{V})$ is usually not straight.

2.8. *The uniformization problems*

Let X be a finite dimensional polyhedron of $K(\pi; 1)$ type. *Does there exist a locally convex space (non-compact but preferably finite dimensional) homotopy equivalent to* X? The Thurston theory [21] says "yes" for a vast class of 3-manifold containing all the known examples.

Let X be any finite dimensional polyhedron. *Does there exist a convex space $\tilde{X}$ and a discrete group (with fixed points) Γ of isometries of $\tilde{X}$ such that $\tilde{X}/\Gamma$ is homeomorphic to* X? When $\dim X = 2$, we have an easy "yes".

3. Strict Convexity

3.1. *Definition*

We shall assume below that our length space X satisfies the following property: for any $x \in X$ there is an ε such that every sphere centered at x of radius $\leq \varepsilon$ contains no geodesic segments. (This is always so for Riemannian manifolds.) When X is convex, the above property implies that large spheres contain no geodesic segments as well.

δ-convexity. We call a convex X *strictly δ-convex* if there is a positive ν such that for any convex set $A \subset X$ the normal projection $P: X \to A$ (P sends x to the nearest point from A; this is a well-defined map due to the absence of geodesic segments in spheres) satisfies: for any curve $C \subset X$ with $\operatorname{dist}(C, A) \geq \delta$, we have length $(P(C)) \leq (1-\nu)$ length (C).

Observe that $(\delta+\delta_1)$-convexity follows from δ-convexity.

X is called *strictly convex* if it is δ-convex for a positive δ.

EXAMPLE. A simply-connected Riemannian manifold (possibly with next-to-convex boundary) with $K \leq -\kappa$, $\kappa > 0$, is strictly convex (it is actually δ-convex for all $\delta > 0$).

We call X *locally δ-convex* (strictly convex) if its universal covering is δ-convex (strictly convex). This is really a "local" notion, because δ-convexity follows from δ-convexity of the balls of radius δ.

The double construction from section 2.4 gives a strictly convex space when the underlying manifold has $K \leq -\kappa$, $\kappa > 0$. As a pleasant example, take a manifold X with $-\kappa_1 \leq K \leq -\kappa_2 < 0$ and with finite total volume. In this case, one can find a locally convex $A \subset X$ such that the complement $X \setminus A$ is bounded and homeomorphic to X itself. The double space Y in this case has the same homotopy type as the usual double of $X' = X \setminus \operatorname{Int} A$. When X has constant curvature, the double of X' can be itself equipped with a metric of non-positive curvature. This metric can be chosen real analytic and with negative Ricci curvature (The last possibility was pointed out to me by Ernst Heintze.) Observe that there are compact manifolds that have C^∞-metrics with $K \leq 0$ but have no real analytic metrics with $K \leq 0$. The proofs of these facts are easy (see also [11]).

3.2. *Closure*

Let X be strictly convex. In this case every horofunction is a ray function. Moreover, there is a natural one-to-one correspondence between

the points from ∂X and the rays starting from a fixed point $x_0 \in X$. Intersection between any two horoballs with different centers is compact, and thus we have the V-*property*: any two distinct points from ∂X can be joined by a straight line.

For any straight line $\ell \subset X$, the normal projection $X \to \ell$ can be continuously extended to $C\ell(X)$ and the only fixed points of the extension are the "ends" ℓ^+, $\ell^- \subset \partial X$ of this line.

All these facts are classical for Riemannian manifolds and the classical proofs work in our case with no problems.

When X is the hyperbolic space $(K=-1)$, there is another important property:

Let $A \subset \partial X$ be a closed set and C be its convex hull. Then the limit set of C is equal to A.

This feature is probably shared by all strictly convex spaces, but I could prove it only when X is a Riemannian manifold with $K \leq -1$ and $A \subset \partial X$ is a finite set. (Recall that the convex hull of A is defined as the minimal set $C \subset C\ell X$ containing A such that $A \cap X \subset X$ is a convex set.)

3.3. *Isometries*

Let X be strictly convex and $\gamma: X \to X$ be an isometry. There are only three possibilities:

a) γ is elliptic. In this case it has a fixed point and the topological group generated by γ (in the group $Is(X)$ of all isometries) is compact.

b) γ is parabolic. Then γ has no fixed points in X but has a *unique* fixed point x in ∂X. This point is called the center of γ. All horospheres centered at x are invariant under γ. The group generated by γ is isomorphic to $\mathbf{Z}$ and its action in $C\ell(X) \setminus \{x\}$ is discrete.

c) γ is hyperbolic. In this case γ has two fixed points γ^+ and γ^- in ∂X, it generates $\mathbf{Z}$ that acts freely and discretely in

$C\ell(X)\setminus\{\gamma^+,\gamma^-\}$. The point γ^+ is an attractor for γ; moreover, for every compact set A, $A \subset C\ell(X)\setminus\{\gamma^-\}$ and any neighborhood $U \subset C\ell(X)$ of γ^+ there is an n such that $\gamma^n(A) \subset U$. The point γ^- is an attractor for γ^{-1}.

This classification is well known for Riemannian manifolds, and the proof depends only on the properties stated in the previous section.

3.4. *Special groups of isometries*

X is again supposed to be strictly convex. Consider a group Γ acting by isometries on X. Γ is called special when one of the following three cases occur:

1) Γ keeps fixed a point $x \in X$. In this case the closure of Γ in the group $Is(X)$ of all isometries is compact. This is a Lie group when X is a Riemannian manifold (possibly with boundary). When X is a polyhedron, this closure is usually a profinite group; this is always so when X is one-dimensional.

2) There is a straight line $\ell \subset X$ invariant under X. In this case Γ factors as follows: $0 \to \Gamma' \to \Gamma \to \Gamma_\ell \to 0$, where $\Gamma' \subset \Gamma$ is the subgroup keeping ℓ fixed and Γ_ℓ is a subgroup in $Is(\ell)$.

3) Γ keeps fixed a point $x \in \partial X$. The group Γ can be very complicated even when X is a Riemannian manifold, but when the curvature is pinched, i.e., $-\kappa_1 \le K(X) \le -\kappa_2 < 0$, the closure of Γ is isomorphic to an extension of a soluble group by a compact Lie group.

This follows from the Margulis lemma (see [11]).

3.5. *Non-special groups*

Let Γ be a discrete non-special group of isometries of a strictly convex space X.

THEOREM. *There is a closed set* $L \subset C\ell X$ *with the following properties:*

a) L *is an infinite set without isolated points;*

b) *the action of* Γ *in* L *is minimal, i.e., there are no invariant closed subsets;*

c) *the action of* Γ *in* $C\ell(X)\setminus L$ *is discrete.*

Denote by $L^{(2)}$ *the set of distinct pairs* $\ell_1, \ell_2 \in L$. *Such a pair is called axial if there is a hyperbolic isometry from* Γ *keeping this pair fixed.*

d) *Axial pairs are dense in* $L^{(2)}$, *unless all isometries from* Γ *are elliptic.*

Denote by $L^{(3)}$ *the space of pair-wise distinct triples from* L.

e) *The action of* Γ *in* $L^{(3)}$ *is discrete. If the action of* Γ *is uniform in* X *(i.e., the translates of a compact set cover* X*) then* $L = \partial X$ *and* Γ *acts uniformly in* $L^{(3)}$.

All these properties are well known when Γ is a Kleinian group. (Many of them I have learned from Dennis Sullivan.) Classical proofs use only the properties stated in sections 3.3 and 3.4. (See [6] for more information.)

There is another important property of Γ that can be established by Klein's argument:

Let $\gamma_1, \cdots, \gamma_m \subset \Gamma$ be hyperbolic elements where no two of them are powers of a third element from Γ. Then there is a number k such that powers $\gamma_i^{k_i}$ for $k_i \geq k$, $i = 1, \cdots, m$, generate a free group of rank m.

To complete the list of basic properties of Γ, we must mention another one, obvious but quite important:

Every hyperbolic $\gamma \in \Gamma$ can be uniquely written as γ_0^p where γ_0 is not a proper power.

3.6. *Remarks on not strictly convex manifolds*

There are some conditions weaker than strict convexity that lead to the conclusions analogous to the theorem from 3.5.

Consider first the case when X is a Riemannian manifold of non-positive curvature. We suppose that X is simply connected and the group of isometries $Is(X)$ acts on X uniformly, for example, X is the universal covering of a compact manifold. Then X shares all global

properties of a strictly convex space (in particular Theorem 3.5) iff it contains no flats. (A flat is a totally geodesic submanifold isometric to $\mathbf{R}^k$, $k \geqq 2$.)

As an example, one can take a not flat surface with $K \leqq 0$. When this surface has a closed geodesic such that in a neighborhood of this geodesic curvature vanishes, then the universal covering has no strict convexity (because some horospheres contain geodesic segments).

The "no-flats" condition is also satisfied by the universal coverings of branched tori from 2.5 when V_0 intersects all 2-dimensional flat subtori in V. Though $\tilde{V}$ is not a Riemannian manifold, its metric can be smoothed to a Riemannian metric with $K \leq 0$ and with no flats in the universal covering. It is unclear whether in the "no-flats" case $\tilde{V}$ carries a Riemannian metric with $K < 0$.

QUESTION. *Let V be a compact Riemannian manifold with $K \leq 0$ and its universal covering contains a flat. Does it follow that $\pi_1(V)$ contains $\mathbf{Z} + \mathbf{Z}$?*

When the universal covering of V has no flats, the periodic geodesics are dense in the unit tangent bundle (because Theorem 3.5, in particular, d) is applicable to the universal covering of V), but if there is a flat, it is unclear whether V has more than one simple closed geodesic. The positive answer to the question would provide infinitely many such geodesics.

Of course, there are many compact manifolds with $K \leq 0$ where the density of closed geodesic in the unit tangent bundle is established. This is the case when V is compact and locally symmetric (see [17]).

Another (very easy) example is provided by compact connected manifolds having a tangent vector such that the sectional curvature is negative on all tangent 2-planes containing this vector.

Observe in the end that the whole discussion can be extended to a certain general class of locally convex spaces.

4. Isoperimetric Inequalities

4.1. *Openness at infinity*

Consider a domain Ω in an n-dimensional Riemannian manifold V. We denote by $\mathrm{Vol}(\Omega)$ its volume and by $\mathrm{Vol}(\partial\Omega)$ the (n-1)-dimensional volume of the boundary.

V is called *open at infinity* (see [19]) if there is a constant C such that any $\Omega \subset V$ satisfies $\mathrm{Vol}(\Omega) \leq C\,\mathrm{Vol}(\partial\Omega)$. When V is orientable, openness at infinity is equivalent (see [19]) to any one of the following three conditions:

a) The volume form in V is the differential of a bounded form ω (i.e., $\|\omega\|_v \leq \mathrm{const}$, $v \in V$).

b) Any bounded n-form on V is the differential of a bounded form.

c) There is a bounded vector field on V such that its divergence is greater than a fixed positive number.

EXAMPLES. A closed manifold is not open at infinity.
$\mathbf{R}^n$ is not open at infinity.
Strictly convex Riemannian manifolds (possibly with boundary) are open at infinity.

THEOREM (Avez [3]). *Let V be a simply connected manifold (without boundary) of non-positive curvature such that the group $\mathrm{Is}(V)$ acts uniformly in V. If V is not open at infinity, then it is isometric to the Euclidean space.*

4.2. *Amenable groups*

A group Γ is called amenable if any action of Γ on a compact space has an invariant measure. Basic properties of such groups can be found in [9]. Here are a few simple facts: A finite extension of a soluble group is amenable. If a group Γ contains a free subgroup of rank ≥ 2, then Γ is non-amenable. Every finitely generated non-amenable group has exponential growth: the number N of words with $\| \ \| \leq R$ satisfies $N \geq (\mathrm{const})^R$.

The following simple fact gives a geometric interpretation of amenability:

Let V *be a compact Riemannian manifold with fundamental group* Γ. *This group is non-amenable iff the universal covering* $\tilde{V}$ *is open at infinity.*

We shall use this fact in a slightly more general situation: take a submanifold $V_0 \subset V$. If $\pi_1(V \setminus V_0)$ is non-amenable, then the universal covering $\widetilde{V \setminus V_0}$ is open at infinity.

4.3. *Quasiconformal maps*

Let V be a complete Riemannian manifold of dimension n. Take a point $v_0 \in V$ and denote by $s(r)$ the $(n-1)$-dimensional volume of the radius r sphere centered at v_0.

AHLFORS' LEMMA. *If the integral* $\int_1^\infty (s(r))^{-\frac{1}{n-1}}\, dr$ *diverges, then* V *cannot be made open at infinity by any conformal change of metric.*

Proof. Take an arbitrary function $f > 0$ on V, multiply the metric by f, and show that there is a ball Ω (with respect to the old metric) such that $\mathrm{Vol}^{new}(\Omega)/\mathrm{Vol}^{new}(\partial\Omega)$ becomes arbitrarily large. When $\Omega = \Omega_r$ is a ball centered at v_0 of radius r, we have

$$\mathrm{Vol}^{new}(\Omega_r) = \int_\Omega f^n = \int_0^r dr \int_{\partial\Omega} f^n$$

$$\geq \int_0^r dr \left(\int_{\partial\Omega} f^{n-1} \right)^{\frac{n}{n-1}} \cdot s(r)^{-\frac{1}{n-1}}$$

$$= \int_0^r dr\, (\mathrm{Vol}^{new} \partial\Omega_r)^{\frac{n}{n-1}} (s(r))^{-\frac{1}{n-1}} ,$$

A straightforward calculation shows now that the divergence of

$\int_1^\infty (s(r))^{-\frac{1}{n-1}} n$ makes the ratio $\mathrm{Vol}^{new}(\Omega_r)/\mathrm{Vol}^{new}(\partial\Omega_r)$ unbounded when

$r \to \infty$.

Let V, W be two orientable Riemannian manifolds of dimension n, where V is complete and W is open at infinity. Let $h: V \to W$ be a map with the following properties generalizing the notion of quasiconformality:

a) The map h factors as

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[rr, bend left, "h"] & V^1 \arrow[r, "h'"] & W \end{tikzcd}$$

where V^1 is a Riemannian manifold, the map g is a conformal equivalence and h' is a path-quasi-isometric map, i.e., for any curve $\ell \subset V^1$, we have

$$\text{const} \leq \frac{\text{length } \ell}{\text{length } h'(\ell)} \leq \text{const}^{-1} .$$

b) Consider the balls $\Omega_r \subset V$ as above and denote by w^* the pull-back $h^*(w)$ of the volume form in W. Our second condition is the following:

$$\limsup_{r \to \infty} \frac{\int_{\Omega_r} |w^*|}{\int_{\Omega_r} w^*} < \infty ,$$

This condition is automatically satisfied when h has non-negative Jacobian and such an h is usually called a quasiconformal map.

THEOREM (Picard-Ahlfors [2]). *If* h *satisfies* a); b) *then the integral*

$$\int_1^\infty \mathrm{Vol}(\partial\Omega_r)^{-\frac{1}{n-1}} dr$$

converges. In particular, there is no (non-constant) quasiconformal maps from $\mathbf{R}^n$ *into a compact non-flat manifold with* $K \leq 0$ *or into a manifold* W *containing a non-abelian free subgroup in its fundamental group. For example, there is no quasiconformal map from* $\mathbf{R}^2$ *into the* 2*-sphere minus three points, because the sphere with three punctures has a free fundamental group of rank* 2.

The proof is obvious in view of the lemma.

Conformal hyperbolicity. There is an analogue of the Kobayashi metric for the quasiconformal situation: this is a metric that must increase no more than k-times under k-quasiconformal maps. No such metric, though, leads to a reasonable length structure, because there are non-Lipschitz quasiconformal maps. (This was pointed out to me by Dennis Sullivan.) One way to start constructing this metric is the following:

For a set $A \subset W$, its conformal capacity $\mathrm{Cap}A$ is defined as

$\inf_f \int_W \|\mathrm{grad}\, f\|^n$ where f runs over all compactly supported non-negative functions on W which are equal to 1 on A, $n = \dim W$. For two points $x, y \in V$, we define $\mathrm{dist}(x, y)$ as the lower bound of capacities of the connected sets containing both points. Of course, such a metric is always zero on a closed manifold, but we use this construction only for open balls, and then we use the functorial Kobayashi-type argument. When W is a closed manifold, the resulting metric vanishes iff there is a (non-constant) quasiconformal map $\mathbf{R}^n \to W$, $n = \dim W$. (Easy to show.)

QUESTION. *Let* W *be a closed simply-connected Riemannian manifold. Does there always exist a non-constant quasiconformal map* $\mathbf{R}^n \to W$?

The locally homeomorphic case. There are relatively few locally homeomorphic quasiconformal maps. For example, for such maps the answer to our question is "No", unless W is homeomorphic to the sphere.

When $W = S^n$ and $n \geq 3$, every locally homeomorphic quasiconformal map $\mathbf{R}^n \to S^n$ is injective, and the complement to its image consists of a single point. This is the Zorich Theorem (see [15]).

The Zorich theorem (or rather the proof) has a wider range: *Let* V *be a complete manifold with finite volume and* W *be a simply-connected manifold, both of dimension* $n \geq 3$. *Then any locally homeomorphic quasiconformal map* $V \to W$ *is injective, and the complement to its image is zero-dimensional.*

4.4. *Euclidean inequality*

For an $\Omega \subset \mathbf{R}^n$, one classically has $V\ell \leq A_n(S)$ where $V\ell = \mathrm{Vol}(\Omega)$, $S = \mathrm{Vol}(\partial\Omega)$ and A_n denotes the volume of the Euclidean ball with the boundary sphere of volume one.

Let V be a Riemannian manifold, possibly with boundary, such that the length structure is convex. *Does a domain* $\Omega \subset V$ *satisfy the above inequality* $V\ell \leq A_n(S)$? What is easy to prove by standard integral geometry (see [18]) is $V\ell \leq \mathrm{const}_n A_n(S)$ and the problem is to make $\mathrm{const}_n = 1$. That can be done [18] when $n = 2$. Observe also that the inequality $V\ell \leq A_n(S)$ is classically known when V is a simply-connected manifold of constant negative curvature. On the other hand, this inequality is "stable" under direct product. (Easy to show.) As a consequence we get:

If the manifold V metrically splits into the product $V_1 \times \cdots V_i \times \cdots V_k$ such that each V_i is a simply-connected manifold satisfying one of the two conditions: a) V_i has non-positive curvature and $\dim V_i = 2$; b) V_i has constant negative (or zero) curvature, then $V\ell \leq A_n(S)$ for any $\Omega \subset V$.

Intermediate inequalities. Consider a (k–1)-dimensional cycle C in $\mathbf{R}^n$. Denote by S its volume (mass) and by $V\ell$ the volume of the minimal chain spanning C.

THE FEDERER-FLEMING THEOREM.

$$V\ell^{k-1} \leq \mathrm{const}_n S^k .$$

See [14] for a geometrically oriented proof.

QUESTIONS. *Does the Federer-Fleming inequality hold for all convex manifolds?*

Can one take $\mathrm{const}_n = 1$ or at least replace it by const_k? This is not quite known even for $\mathbf{R}^n$ (see [22] for some information).

We shall see below how to handle $k = 2$.

4.5. *Taut surfaces*

Let X be any length space and D be a 2-dimensional manifold. A map $f: D \to X$ is called *taut* if no compactly supported deformation decreases the induced length in D, i.e., any compactly supported deformation $f_t: D \to V$, $f_0 = f$, satisfying: $\mathrm{length}(f_t(\ell)) \leq \mathrm{length}(f(\ell))$ for all curves $\ell \subset D$, does not change length at all.

When $V = \mathbf{R}^n$, the taut surfaces are essentially the same as saddle-surfaces. In particular, their Gauss curvature is non-positive.

Under very mild restrictions on X, every contractable curve $\ell \subset X$ with $\mathrm{length}(\ell) < \infty$ can be spanned by a taut disk. As a consequence, we have: *if the induced structure in every taut disk* $D \to X$ *is convex and* X *is simply connected, then* X *is convex.*

Taut surfaces allow us to reduce the study of negatively curved spaces to the special case of dimension 2 where much information is available (see [1]). In particular we have:

The Ionin-Burago inequality: Let V be a two-dimensional manifold with boundary. Denote by A its total area; by L the lengths of the boundary, and by $V_+ \subset V$ the set, where the curvature $K(v)$ is greater than -1. Set $\omega_+ = \int_{V_+} (K(v)+1)\,dv$. We have:

$$\frac{1}{2}L^2 \geq A\left(\frac{A}{2} - \omega_+ + 2\pi\chi\right),$$

where χ is the Euler characteristic of V.

See [12] for a short proof.

4.6. *Metric inequalities*

The following two theorems link volumes with the distance function.

The Almgren inequality. Let X be an n-dimensional cube with a Riemannian metric. Denote by ℓ_i the length distance between i-th pair of opposite (n–1)-faces. We have:

$$\mathrm{Vol}X \geq \prod_{1}^{n} \ell_i$$

(This form of the Almgren inequality was communicated to me by Iu Burago.)

Proof. Let $F_i \subset X$, $i = n$, $i \cdots n$, be (n–1)-faces in X and none of two F_i are opposite. Consider the map $\phi : X \to \mathbf{R}^n$ given by the functions $\mathrm{dist}(x, F_1)$, $\mathrm{dist}(x, F_2), \cdots$.

The Jacobian of ϕ does not exceed 1 and, by an obvious topological reason, the image of ϕ contains the direct product $[0, \ell_1] \times [0, \ell_2] \cdots \subset \mathbf{R}^n$. Q.E.D.

The simplex inequality. Let X be an n-dimensional simplex with a Riemannian metric. Suppose that for any $x \in \partial X$ the sum of distances from x to all (n–1)-dimensional faces is not less than one, then the volume of X is not less than the volume of the regular Euclidean simplex Δ_1 of unit height.

Proof. Using distances from an $x \in X$ to all (n–1)-faces, we get a map $X \to \mathbf{R}^{n+1}$. Use now the projection from $\mathbf{R}^{n+1}$ onto the affine hyperplane spanned by the basis vectors. The image of the resulting map $\gamma : X \to \mathbf{R}^{n+1}$ contains the standard n-simplex Δ, and the Jacobian of γ does not exceed the ratio $\frac{\text{Volume } \Delta}{\text{Volume } \Delta_1}$. Q.E.D.

An application. Let V be a complete simply-connected Riemannian manifold such that every taut disk $D \to V$ satisfies:

$$\mathrm{Area}(D) \leq \mathrm{const} \cdot \mathrm{length}(\partial D)$$

where "const" depends on V but not on $D \to V$.

Let $\ell_1, \ell_2 \subset V$ be straight lines such that

$$\sup_{x \in \ell_1} \operatorname{dist}(x, \ell_2) \leq C < \infty .$$

Then

$$\inf_{x \in \ell_1} \operatorname{dist}(x, \ell_2) \leq \text{const} ,$$

and this is the constant from above.

Proof. Take $x, y \in \ell$ with $\operatorname{dist}(x,y) = d$ and point $x^1, y^1 \in \ell_2$ with $\operatorname{dist}(x, x^1) \leq C$ $\operatorname{dist}(y, y^1) \leq C$. Span the geodesic tetragon $x y y^1 x^1$ by a taut disk. Its area A must be less than $\text{const}(2d + 4C)$. Applying the Almgren inequality, we have $A \geq (d - 2C)(\operatorname{Dist}(\ell_1, \ell_2))$. When $d \to \infty$, we get the required estimate.

5. Hyperbolic Systems

5.1. *Shifts*

Let Γ be any countable group. Consider the space A of the functions on Γ with values in a finite set S. (A is equipped with the product topology and homeomorphic to a Cantor set.) The left translations induce an action of Γ in A which is called the *Bernoulli shift.*

An invariant set $B \subset A$ is called *Markov* (or of finite type) if there is a continuous map $\Delta : A \to A$ commuting with the action of Γ such that B consists of all $a \in A$ satisfying $\Delta(a) = a$. The induced action in such a B is called a *Markov shift.*

Markov shifts and continuous maps commuting with the action of Γ form a nice category. In particular, the product of two Markov shifts with the diagonal action carries a natural structure of a Markov shift.

5.2. *Markov relations*

A Markov relation over B is, by definition, a Markov set $R \subset B \times B$. When R is an equivalence relation (i.e., it is reflexive, symmetric and transitive) we have the quotient B/R with a natural action of Γ. The

dynamical systems arising in this way are exactly all *hyperbolic systems* (over Γ).

For example, the basic sets of an axiom A diffeomorphism are hyperbolic with $\Gamma = \mathbf{Z}$ (see [4]).

Look now at an example of algebraic origin.

5.3. *Hyperbolic automorphisms*

Let G and Γ be finitely generated groups and let $\{\gamma_1, \cdots, \gamma_k\}$ generate Γ. Suppose that Γ acts on G by automorphisms $g \to g^\gamma$, $g \in G$, $\gamma \in \Gamma$. Consider the space F of all functions $\Gamma \to G$.

A function $f : \Gamma \to G$ is called an *orbit* if $f(\gamma_i \gamma) = (f(\gamma))^{\gamma_i}$, $\gamma \in \Gamma$, $i = 1, \cdots, k$.

A function f is called an ε-orbit, $\varepsilon \geq 0$, if

$$\sup_{\gamma \in \Gamma} \operatorname{dist}(f(\gamma_i \gamma), (f(\gamma))^{\gamma_i}) \leq \varepsilon ,$$

$$i = 1, \cdots, k ,$$

where "dist" means a fixed word distance. When ε is not specified, we call an ε-orbit a *pseudo-orbit*.

We say that $f_1, f_2 \in F$ are δ-close if $\operatorname{dist}(f_1, f_2) \leq \delta$. When δ is not specified, we just say they are close.

The action is called *hyperbolic* if there are ε_0, $\delta_0 \geq 0$ such that:

a) Every pseudo-orbit is close to an ε_0-orbit;

b) When two ε_0-orbits are close, then they are δ_0-close.

The space F carries the product topology. Take the set F_{ε_0} of all ε_0-orbit and identify close ε_0-orbits. The resulting space we denote by $\overline{G}$. The group G is embedded in F as follows: g goes to the orbit $\{g^\gamma\}$, $\gamma \in \Gamma$, and hence, we get an action of G in $\overline{G}$ coming from multiplication of functions $\Gamma \to G$. The group Γ naturally acts in $\overline{G}/G$ and under the conditions a), b) this action is hyperbolic. (This is easy to check.)

Classical example (Franks [8]). Take $\Gamma = \mathbf{Z}$ and consider an Anosov action of $\mathbf{Z}$ on an infranilmanifold X. Then the induced action of $\mathbf{Z}$ in $G = \pi_1(X)$ is hyperbolic and $\overline{G}/G$ with the action of $\mathbf{Z}$ is canonically isomorphic to the original action of $\mathbf{Z}$ in X. (In particular, X is homeomorphic to $\overline{G}/G$.)

5.4. *The boundary action*

Let Γ act by isometries on a strictly convex space. If the action is discrete, free and uniform, then the associated action on ∂X is hyperbolic. This is an unpublished result of Sullivan (who stated it in a slightly different form). We discuss the proof in the next chapter. Observe that the action in ∂X is minimal (see 3.5), and the dynamical systems that are both hyperbolic and minimal appear seldom. For example, $\mathbf{Z}$ has no such (non-trivial) actions.

6. Coarse Hyperbolicity

6.1. *Coarse invariance of the boundary*

The boundary ∂X of a metric space is naturally invariant under isometries, i.e., an isometry $X \xrightarrow{f} Y$ can be continuously extended to $C\ell(X)$. When f is a quasi-isometry, or more generally a coarse equivalence (see 1.4), such extension is usually impossible. For example, quasi-isometries of $\mathbf{R}^n$, $n \geq 2$, do not generally extend to the $\partial \mathbf{R}^n = S^{n-1}$.

The following fundamental theorem is due essentially to Efremovitch-Tichomirova (see [7]).

If X and Y are strictly convex spaces, then every quasi-isometry (and even a coarse quasi-isometry) $X \xrightarrow{f} Y$ can be continuously extended to a homeomorphism $\partial X \to \partial Y$.

To prove the theorem, we have to provide a coarse invariant construction of the boundary. The following definition is analogous to the one from section 5.3.

6.2. *Coarse boundary*

Let X be a complete metric space. We view a set $F \subset X \times X$ as a set of arrows $x_1 \rightarrowtail x_2$ for $(x_1, x_2) \in F$. We call F a λ-field, $\lambda \geq 0$ if

$$\sup_{(x_1,x_2)\in F} \operatorname{dist}(x_1, x_2) \leq \lambda .$$

We write $x > y + k$, $x, y \in X$, $k = 0, 1, \cdots$, if there are $x_0, x_1, \cdots, x_m \in X$, $m \geq k$, such that $(x_i, x_{i+1}) \in F$ and $x_0 = x$, $x_m = y$.

A field is called straight if for some $\varepsilon < 1$, it satisfies: $x > y + k$ implies $\operatorname{dist}(x, y) \geq (1-\varepsilon)k$.

Denote by $x + \{k, k+1, \cdots\}_F$ the set of all points $z \in X$ satisfying $z > x + k$.

We call a λ-field *compressing* if there are $\delta \geq 10\lambda$ and $p = 0, 1, \cdots$, such that for any $x, y \in X$ with $\operatorname{dist}(x, y) \leq 10\delta$ one can find $i, j \leq p$ such that the Hausdorf distance between $x + \{i, i+1, \cdots\}_F$ and $y + \{j, j+1, \cdots\}_F$ is less than δ.

We call two fields F_1 and F_2 *equivalent* if the Hausdorf distance between sets $x + \{1, 2, \cdots\}_{F_1}$ and $x + \{1, 2, \cdots\}_{F_2}$ is bounded (i.e., finite) for any $x \in X$.

If X is a strictly convex length space, there is a canonical homeomorphism between the boundary ∂X and the set of the equivalence classes of the straight compressing λ-fields. (The last set carries a natural topology.)

Proof. Given a horofunction h on X, we have a λ-field $F \subset X \times X$ defined as follows: $\{x, y\} \in F$ iff $\operatorname{dist}(x, y) \leq \lambda$ and $(h(x) - h(y)) \geq \lambda/10$.

On the other hand, a field F defines a point in ∂X: this is the limit point of the set $x + \{1, 2, \cdots\}_F$. It is easy to check that this correspondence gives the required homeomorphism.

When Γ is a group with the word metric, the set of all λ-fields can be identified with the set of all maps $\Gamma \to S$, where S is the set of all

subsets from the radius λ ball in Γ around the identity. When Γ is as in section 5.4, the straight compressing λ-fields (with a fixed λ) form a Markov subset and the equivalence relation of fields is Markov. This explains the Sullivan theorem on hyperbolicity of the action of Γ on ∂X.

As an application of the coarse invariance of ∂X and the action of Γ on ∂X we have:

The geodesic flow on a compact strictly locally convex space $\overline{X}$ is uniquely determined by the fundamental group $\pi_1(\overline{X})$, i.e., if $\pi_1(\overline{X}) = \pi_1(\overline{X}')$ then the (one-dimensional) orbit foliations of the geodesic flows over $\overline{X}$ and $\overline{X}'$ are homeomorphic.

The proof given in [13] for Riemannian manifolds works in the general case.

6.3. *Coarse geometry of symmetric spaces*

We have seen so far that coarse equivalence is strong enough to recover the topology at infinity. The boundary ∂X also carries some geometric structure: a generalized quasiconformal structure (in the terminology of Mostow and Sullivan, see [17]). This structure is best understood in the case of symmetric spaces, in particular for the constant curvature manifolds. In the last case, this is the usual quasiconformal structure on S^{n-1}.

The generalized quasiconformal structure is a coarse invariant. When it comes to symmetric spaces, the Mostov theorem shows [17] that is is a very strong invariant.

If two symmetric spaces (of noncompact type) of rank 1 are coarse equivalent, then they are "essentially" the same (i.e., isometric up to the choice of the normalizing constants).

This result can probably be generalized to more general spaces having large groups of isometries. Here is a sample of such generalization:

Let X_0 *be a compact space covered by a convex space* X. *Suppose that one of the following two conditions is satisfied:*

a) *The space* X *(or, equivalently the group* $\pi_1(X_0)$*) is coarse equivalent to the Lobachevskyi space of dimension* n,

b) X *is a Riemannian manifold which is quasiconformally equivalent to the standard open ball in* $\mathbf{R}^n$.

Then X_0 *is homotopically equivalent to an* n*-dimensional manifold of constant curvature.*

Proof. According to Mostow [17], conditions a) and b) make the group $\Gamma = \pi_1(X_0)$ act on S^{n-1} by quasiconformal transformations with uniformly bounded dilatation. Using a remark by Sullivan from [20], we get a measurable conformal structure on S^{n-1} invariant under Γ. When $n = 2$, the proof is concluded (see [20]) by applying the measurable Riemannian mapping theorem. When $n \geq 3$, we observe that the measurable conformal structure is, almost everywhere, close to a flat structure. Using the action of Γ, one can "extend the flatness" to the whole sphere S^n. We shall discuss details at another opportunity.

6.4. *Coarse hyperbolicity*

As we have already seen that the essential features of strictly convex spaces are coarse invariant. That suggests a "coarse" counterpart of the notion of strict convexity. Unfortunately, the needed definitions and the proofs of the basic properties are fairly lengthy though quite trivial. Here we mention only one important property.

We say that a Riemannian manifold X has Is_2*-property* if there is a constant C such that every contractable closed-curve in X of the length ℓ can be spanned by a disk of area $\leq C \circ \ell$.

Observe that this property is a coarse invariant of the universal covering $\tilde{X}$. In particular, if one of two compact manifolds with the same fundamental group satisfies Is_2, so does the other. This allows us to attribute Is_2 to finitely presented groups.

It is not hard to show that fundamental groups of compact spaces with strictly convex universal coverings satisfy Is_2 (compare with Chapter 4). For example, free groups, fundamental groups of compact manifolds of negative curvature, and finite extensions of such groups satisfy Is_2.

An application to complex geometry. Let V be a compact Kähler manifold such that $\pi_1(V)$ satisfies Is_2 and $\pi_2(V) = 0$. Then V is hyperbolic in the sense of Kobayashi, i.e., every holomorphic map $\mathbf{C} \to V$ is constant.

Proof. We pass to the universal covering $\tilde{V}$ and use minimality of the images of maps $\mathbf{C} \to V$. The rest immediately follows from the Ahlfors Lemma (section 4.3).

6.5. *Attaching cells*

Let X be a Riemannian manifold and $X_0 \subset X$ be a submanifold. We denote by $R(X_0)$ the maximal number ε such that the function $\operatorname{dist}(x, X_0)$ is smooth in the open ε-neighborhood of X_0.

Suppose now that $K(X) \leq -1$ and X_0 is the union of disjoint closed geodesics $g_1, \cdots, g_p$ of lengths $\ell_1, \cdots, \ell_p$, and attach 2-disks to each of g_i.

We denote by Y the resulting space and denote by $\overline{Y}$ the manifold with boundary obtained by "thickening" Y, $\dim\overline{Y} = \dim X + 1$. When $\ell_i \geq 10$, $i = 1, \cdots, p$, $R(X_0) \geq 10$, one can construct a Riemannian metric in $\overline{Y}$ such that the universal covering of Y is a strictly convex space. (Easy to show.) This implies that Y itself (i.e., its universal covering) is coarse hyperbolic and, in particular, satisfies Is_2. (Actually, it is much easier to check coarse hyperbolicity than to construct convex length structures. We consider an important example in the next section.)

Using the attaching of 2-cells, one can prove the following:

Let X be a closed manifold of negative curvature. Then the fundamental group $\pi_1(X)$ contains a free non-Abelian normal subgroup.

6.6. *The small cancellation groups*

Let F be a free group freely generated by $f_1, \cdots, f_k$. Consider a

system of words $W = \{w_1, \cdots, w_s\}$ with the following properties:

1) if $w \in W$ then $w^{-1} \in W$;

take a $w \in W$ and an arbitrary $w' \in F$ conjugate to w,

2) $w' \in W$ iff $\|w'\| \leq \|w\|$. (Of course, we use the norm associated with the generators f_i.)

We call a $p \in F$ a *common piece* of two words $f_1, f_2 \in F$ if $\|p^{-1}f_1\| = \|f_1\| - \|p\|$ and $\|p^{-1}f_2\| = \|f_2\| - \|p\|$.

The system W is said (see [16]) to satisfy 1/6-*property* if for any two $w_i, w_j \in W$, $i \neq j$ and their common piece p we have $\|p\| < \frac{1}{6}\|w_i\|$, $\|p\| < \frac{1}{6}\|w_j\|$.

We say that the group presented by W has 1/6-property if W has this property.

Consider the two-dimensional polyhedron X corresponding to a 1/6-presentation. It is not hard to show that its universal covering is coarse hyperbolic, in particular, it satisfies Is_2.

Observe, also, that usual algebraic treatment of the small cancellation groups (i.e., 1/6-group and alike) is based on a combinatorial version of Is_2.

The geometric point of view described here is essentially due to Dehn. As it has been realized (see [16]), his original ideas lead to a better understanding of the small cancellation groups than the algebraic approach.

For example, the word and conjugacy problems (see the next section) or existence of free non-Abelian subgroups are obvious from Dehn's point of view.

6.7. *Algorithmical problems*

Consider a group Γ presented by $\{f_1, \cdots, f_k; w_1, \cdots, w_s\}$ and the presenting homomorphism $F \xrightarrow{\sqcap} \Gamma$. Denote by $\| \ \|_{\sqcap}$ the pullback to F of the word norm of Γ. The function $\| \ \|_{\sqcap} : F \to \mathbf{Z}_+$ cannot be, in general, effectively computed. For example, deciding when $\|f\|_{\sqcap} = 0$ is equivalent to the word problem.

There is the following geometric interpretation of our problem. Take a finite polyhedron X with a length structure and with $\pi_1(X, x_0) = \Gamma$. Computing $\|\gamma\|$ is essentially the same as estimating the length of the minimal loop at x_0 representing $\gamma \in \Gamma$.

When X is locally convex, the function $\| \ \|$ *is effectively computable.*

The proof immediately follows from the uniqueness of a closed geodesic loop in each homotopy class.

An analogous argument solves the conjugacy problem as well.

The commutator norm. Let $\gamma \in [\Gamma, \Gamma]$, $\Gamma = \pi_1(X_1)$. Denote by $[\gamma]$ the minimal possible genus of a surface spanning γ. When X is a compact manifold of negative curvature, the genus of a minimal surface can be estimated by use of the inequality from section 4.5. That implies computability of the $[\ ]$-norm.

Unfortunately, the extension of those arguments to the coarse hyperbolic case requires more work (or, rather, more terminology) but the basic conclusions are the same.

Observe in the end, that the geometrical interpretation of computability can be used the other way around: when the fundamental group $\pi_1(X)$ is "logically complicated", X has many geodesic loops, closed geodesics, minimal surfaces, etc., (see [13]).

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