Math Currents in the Brain.

Misha Gromov

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cogito ERGO **sum.**

What is mathematics and how has it originated?

Where does the stream of mathematical ideas flow from?

What is the ultimate source of mathematics in the brain?

These are reminiscent of the ancient question

"What does the Earth rest on?

with our instincts pushing us toward "*On a* Giant Turtle" answers.

Rather than rushing to say something clever about mathematics, let us search for a *general context* for these questions. Our candidate for such a context is a class of

mathematical models[1] of *universal learning processes* that we call *ergo-systems.*

Without a theory of such or similar "systems" a discussion on the *"nature of mathematics"* will remain a rattle words.

(In science, nothing can be understood within itself: particular notions, objects and phenomena are almost invariably defined and analysed within general contexts. What of worth can you say about *Earth* if you are oblivious to *"stellar evolutions"*, *"nuclear fusion"*, *"planetary systems"*, *"carbon chemistry"*, *heteropolymers, etc*?[2])

The existence of such "systems" is manifested by the ability of the brain to build coherent structures, such as visual images and mathematical theories from seemingly chaotic flows of electrochemical signals that the brain receives.

There is further evidence in favour of such "systems"; yet, their existence remains conjectural.[3]

A lively objection to a possibility of a mathematical resolution of the problem of mind was articulated by Haldane:[4]

> *If my opinions are the result of the chemical processes going on in my brain, they are determined by the laws of chemistry, not those of logic.*

[1] "*Mathematical model*" is understood here in a physicists' sense with mathematical rigour being a secondary issue.

[2] Those who are not attuned to science would find all this more farfetched than the idea of Giant Turtle. An intelligent Cro-Magnon hunter-gatherer, for instance, would lough at a learned scientist who will try to teach him/her what *his/her Earth* is.

[3] We explain this in our two "ergo-articles" indicated at the end of this text.

[4] J.B.S. Haldane (1892 – 1964) was a mathematically minded evolutionary biologist and a famous science populariser.

Convincing?... unless you realise that the persuasive power of the above *"determined", " laws of chemistry", "logic"* depends on a metaphoric use of these notions out of their proper contexts.

But ants, for instance, make no such epistemological mistake: their collective mind employs the "laws of chemistry" to "logically determine" shortest paths between locations in a rugged terrain:

> *a busy ant highway between an anthill and a source of food usually implements a nearly shortest possibility.*

(If you fail to guess how it works do not blame your brain. Much of it, similarly to the brains of ants, was configured by brutal chopping branches from the potentially exponentially growing *Tree of Life*,[5] where Nature had less time and opportunities tinkering with our genomes than with genomes of insects.[6])

Solution. Ants mark their trails with pheromones and themselves tend to choose the routes that have stronger pheromone odors. All things being equal,

> *the number of ants that pass back and forth on some track, say during 1h, is inverse proportional to the length of this track*;

hence, the shortest track becomes the smelliest one, thus, eventually preferred by the ants.[7]

What has made this algorithm evolutionary attainable is its simplicity and universality. And the basic programs running within our minds, just in order to exist at all, must be comparably universal, simple and beautiful.

Psychology of Mathematics and Mathematics of Psychology.

Mathematicians, as much as everybody else on Earth, marvel at their own selves.

Henri Poincaré, for instance, speaks of random dance of glimmering specks of dust in his mind that coalesce into mathematical ideas in eureka moments.

> *Of the very large number of combinations which the subliminal ego blindly forms almost all are without interest and without utility. But, for that very reason, they are without action on the aesthetic sensibility; the consciousness will never know them....*
>
> *A few only are harmonious, and consequently at once useful and beautiful, and they will be capable of affecting the geometrician's special sensibility I have been speaking of; which, once aroused, will direct our attention upon*

[5] This mutilation process, euphemistically called *natural selection*, serves to curb rather than to foster evolutionary diversity.

[6] Probably, the evolutionary development of most complicated and interesting patterns in behaviour of *social* insects, similarly how it is with human brains, followed the routes *transversal* to the (stochastic) gradient of unrestricted selection.

[7] Richard Feynmann, while explaining how the phase cancellation in his integral implies the least action principle, jokes of particles that *"smell" the neighboring paths to find out whether or not they have more action.*

them, and will thus give them the opportunity of becoming conscious...

In the subliminal ego, on the contrary, there reigns what I would call liberty, if one could give this name to the mere absence of discipline and to disorder born of chance. Only, this very disorder permits of unexpected couplings.[8]

Jacques Hadamard collects poetic accounts of mental experiences by scientists, including those by Poincare and Einstein, in his book

The Mathematician's Mind:
The Psychology of Invention in the Mathematical Field.[9]

The discouraging upshot of Hadamard's book, in accord with Poincaré, is that the essential mental processes are *unconscious*[10] and run *in parallel* along several lines. (Of course, the latter implies the former: our conscious mind is almost fully ordered by the time coordinate.) All by itself, introspective self analysis, even by brilliant minds, can not elucidate the nature of mathematics.

(Indeed, can fish develop the theory of liquids?
Does experiencing gargantuan passions in eating advance one toward understanding metabolism?[11]
Do waves of artistic feelings through the heart of a performing dancer reveal the principles of mechanical motion?)

But unlike searching our own soles, our experience with building elaborate mathematical/mental structures may help.

Coming from a different angle, psychologists have been trying to use mathematics for the study of psychological phenomena, but this does not include modelling higher levels of learning, e.g. of mother tongue by a child or a mathematical theory by a mathematician.[12]

Universality and Evolution.

Every connected graph decomposes into its *core* and *periphery*,

$$G = G_{core} \cup G_{peri},$$

where G_{core} is a subgraph with *no vertices of degrees one* and G_{peri} is a disjoint union of *trees*, each attached to G_{core} at a single vertex.

[8] A corresponding *neural Darwinism* model of the brain functions was suggested by Gerald Eidelman, probably, motivated by the immunological selection mechanism of antibody proteins.

On the other hand, the *subliminal ego* of Poincaré serves as a precursor of what we call *"ergo-brain"*. But "ergo" is, albeit stochastic, entails a high level of structural organisation unlike this "ego".

[9] Also see: *How Mathematicians Think* by William Byers, *The Mathematician's Brain* by David Ruelle, *The number sense* by Stanislas Dehaene, *The Math Instinct* by Keith Devlin, *Where Mathematics Comes From* by George Lakoff and a Rafael Núñez.

[10] Do not confuse this with *subconscious* that is usually understood as a part of consciousness.

[11] This, for instance in the case of eating candies, is an elaborate chain of chemical reactions of the oxidation of acetate derived from carbohydrates into carbon dioxide and intracellular chemical energy in the form of adenosine triphosphate.

[12] I must admit I only briefly browsed through a few randomly chosen papers, e.g. *The mathematics used in mathematical psychology* by Robert Duncan Luce, *Logical and mathematical psychology* by Nicolae Margineanu and *Mathematical Psychology: An Elementary Introduction* by Front Cover Clyde Hamilton Coombs, Robyn M. Dawes, Amos Tversky, Mathematical psychology: *Prospects for the 21st century* by James T. Townsend (2008). Also see http://www.indiana.edu/~psymodel/publications_all.shtml.

Human/animal psyche is like such a G, where "G_{peri}" corresponds to what is directly observable in the human/animal behaviour and/or what is accessible to the human conscious mind.

Much of "G_{peri}" depicts evolutionary selected programs that control behaviour of an individual and his/her conscious thinking. These programs stay on guard of one's personal survival and of conservation of relevant genes in the population.

Our cherished ideas about ourselves, about, our thinking, our intelligence, our intuition, etc. are products of these programs running our minds.[13] Irreplaceably useful? – Yes; but practical usefulness of these ideas does not make them scientifically valid[14] nor does it brings structural beauty, unity and universality to "G_{peri}".

It is up to politicians, educators, psychologists and writers of psychological fiction books to explore and to look after the wild forest of trees in "G_{peri}" that had resulted from a series of biological/historical accidents; this is not the business of mathematicians.

What we want to understand and to mathematically model is

> *the invisible interface* between the *electro-chemical neurophysiology* of the brain and the *psychology* of basic learning processes.

This interface symbolised by "G_{core}", that we expect being organised according to general semi-mathematical principles, plays the role vaguely similar to that of the machinery of

molecular cell biology + embryology

that transforms/translates genetic information into the dynamical architectures of living organisms.[15]

Unlike "G_{peri}", much of "G_{core}" is of *universal nature*, that was not specifically selected by evolution but was chosen out of sheer logical necessity, similarly to how

one-dimensionality + 3d-folding of polypeptides[16]

was promoted by Nature to the principal role in the cellular biochemistry.

The two instructive instances of "psychological universality" are the following.

1. *Imprinting in Young Animals.* How does a baby animal know who its mother is? Who to trust and who to love?

The illuminating answer was suggested and experimentally verified by *Douglas Spalding* as recorded in his short note *On instinct.* Nature, 6, 485-486

[13] Most "very human ideas", are driven by the core behaviour programs that originated in the nerves systems of the worm-like ancestors of animals about 500 billion years ago. These programs are invisible to our inner eye.

[14] These ideas are much further removed from "the true laws of thinking" than the motion perception installed into our motor control system is from the Newtonian laws of mechanics.

[15] Embryogenesis remains an unresolved mystery of Life. How does a developing organism implement the design that is encoded in the genome?

[16] Polypeptides are polymeric chains of amino acids (typically, with 100-300 units in them) that, upon being synthesised in cells, *fold* into definite 3d-conformations. (This happens essentially spontaneously in accordance with attraction/repulsion forces between residues; yet, no present day mathematical theory is able to fully account for the dynamics of *protein folding* that is a "baby version" of embryogenesis.) The folded conformations, called *proteins*, perform most functions in cells, including the polypeptide synthesis itself – that is the most elaborate chemical process taking place in our Universe.

(1872).[17]

The first moving object.

The baby brain has no idea of *mother, love, trust* but operates with *universal* mathematical concepts:

first, change/motion, object

that were not subjected to evolutionary selection.[18]

2. *Hawk/Goose effect.* A baby chick does not have any bult-in image of "deadly hawk" in its head but distinguishes *frequent*, hence, harmless shapes, sliding overhead from potentially dangerous ones that appear *rarely*.

Similarly to "*first*", "*frequent*" and "*rare*" are *universal concepts* that were not specifically designed by evolution for distinguishing hawks from geese.

This kind of universality is what, we believe, turns the hidden wheels of the human thinking machinery.

Learning Languages and Learning Mathematics.

It is counter productive to attempt to even define what "thinking" and "intelligence" are, but *learning* is a different matter. *Learning* is a clearly observable phenomenon, where the following three instances of learning are, probably, run by essentially identical programs.

1. *Learning native tongues.*
2. *Learning playing chess.*
3. *Learning Mathematics.*

As far as languages are concerned, almost every child learns one, this is the most common instance of "deep structural learning" by humans. No one has a constructive idea of what lies at the bottom of it and how it may work.

In mathematics, a brilliant example is that of

Srinivasa Ramanujan.

Ramanujan, upon reading a book with 5000 theorems and formulas in it, has written down 4000 new formulas himself, where one of the first was

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}=3.$$

Learning appears here in a purest form as a

a process of "construction" of an "operator" in the brain

that manifestly transforms one set of formulas to another such set.

No general learning theory can be taken seriously unless it indicates, at least in an outline form, universal rules of such a "construction".

(A misuser of statistics, may reject *Ramanujan phenomenon* as "a fluke of chance", but, in fact, this *miracle of Ramanujan* forcefully points toward

[17]The contribution by Spalding to fundamental psychology was forgotten for years and revived relatively recently. It remains overshadowed by hordes of experiments, answering "profound questions" of the kind:

What percentage of people would steal if certain of impunity?

See something more amusing on http://list25.com/25-intriguing-psychology-experiments/.

[18]It is unlikely that Nature tried and rejected "second moving", "third unmoving"...

the same universal principles that make possible mastering native languages by billions of children.[19])

Playing Chess is a model thinking process. It has been examined from different angles by philosophers, psychologists, computer programmers and mathematicians.

According to Freud, the interest in playing chess by human males is driven by their subconscious urge of killing their fathers[20] while Wittgenstein championed the idea of chess being governed to greater extent by the *conventional relations* between pieces rather than by their *internal composition*. Mature players, he convincingly argues, would not consume chess pieces as food, even if these are made of chocolate.[21]

In 1836, *Edgar Poe* argued that no automaton designed similarly to *the Babbage automaton*[22] can play good chess.

In 1957, a simple minded program implemented on a computer by Alex Bernstein and his collaborators defeated Hubert Dreyfus – one of the 20th century opponents of the existence of such a program.

In 1997, the *Deep Blue*, that could evaluate 200 million positions per second, defeated the world champion Kasparov, 3.5-2.5.[23]

In 2014, no human would even dream of competing in chess with computers, but... the following "chess learning problems" remain as widely open today as they were two hundred years ago.

Level 1. Design a universal algorithm/program that, upon observing a few thousand (rather than hundreds of million) chess game, would reconstruct the rules of chess.

Level 2. Design a universal algorithm/program that, after some period of learning would be able to distinguish games played by masters from those by beginners.

Level 3. Design a universal algorithm/program, that after a brief exposure to chess, will start teaching itself to play and, eventually, will play by orders of magnitude better than any conceivable knowledge-based chess program with comparable computational resources (and/or initial access to the chess literature).

In all three instances, "*universal*" means that the corresponding algorithms should not be specific to chess, but be meaningfully applicable to a class of *input signals flows*, desirably, including those originated from natural languages

[19] *Supernovae* seem very different from slow burning stars. They have enormous intensities of energy outputs, some as bright as 100 billion suns. And they are as rare in the skies as Ramanujans are on Earth – none was observed in our galaxy with 300 billion stars since October 9, 1604. Yet both processes depend on the same general principles of gravitation and nuclear fusion; probably, about a billion stars in our galaxy will eventually explode as supernovae.

[20] We present a futuristic perspective on *Freudian complexes* in section 6.7 of our *Structures, Learning and Ergosystems*.

[21] The philosopher does not describe any experiment verifying his idea.

[22] In his article *Maelzel's Chess-Player* about the fake chess playing machine invented by Wolfgang von Kempelen in 1769, Poe, apparently, refers to *Difference Engine* described by Charles Babage in 1822 rather than to the universal computer (*Analytic Engine*) proposed by Babbage in 1837 (that was 99 years before Turing).

[23] This notwithstanding, Poe's scepticism, that unlike the argument of Dreyfus was based on lucid thinking, can be justified: Poe clearly saw limitations of sequential computing devices available/imaginable in the 19th century.

and/or from mathematical texts.[24] For example, a *Level 3* universal program, when applied to a flow of *informally presented* mathematical theorems and formulas, should work as a mathematician's brain does and generate an output flow of new theorems and formulas.

Such high level learning algorithms operate in the unconscious minds of all human beings on Earth and we conjecture that the potential resources of the present day mathematics can help to bring these algorithms to the open and to design the corresponding computer programs.

On the other hand, if you look at the items on the pages of Wikipedia concerning learning, such as

educational psychology, behaviorism, conditioning, cognitivism,
instructional theory, multimedia learning theory, social cognitive theory,
connectivism, constructivism, transformative learning theory,
educational neuroscience, a brain-based theory of learning,

machine learning, decision tree learning, association rule learning,
artificial neural networks, inductive logic programming,
support vector machines, clustering, bayesian networks,
reinforcement learning, representation learning,
similarity and metric learning, sparse dictionary learning,

you hardly find ideas that direct you toward solving the problem of high level learning; yet, some bits and pieces may be of help.

Our guiding principle of fundamental learning, both of natural and artificial, reads:

THE CORE PROCESSES OF LEARNING ARE UNIVERSAL, GOAL FREE AND ESSENTIALLY INDEPENDENT OF AN EXTERNAL REINFORCEMENT.

This idea is (almost) equivalent to that of

curiosity-driven learning

suggested by robotises Jürgen Schmidhuber, Frédéric Kaplan and Pierre-Yves Oudeyer, who developed algorithms for a robot's behaviour depending on the *information/prediction profile*[25] of the flow of signals the robot receives.[26]

What we see as another key ingredient of the future theory is a description of combinatorial structures that would imitate multi-level architectural arrangement of "ideas in the brain".

Essential (but not the only) "interatomic" constituents of this architecture, as we see it, are the following:

- *equivalence-like relations of various kinds and strengths,* $x_1 \sim_\kappa x_2$*;*
- *partly composable classifier/reduction arrows, of various kinds,* $x \to_\mu y$*;*
- *cofunction collaboration associations of various kinds,* $x_1 \smile_\phi x_2$*;*

$\star$ *analogous relations "$\sim$", "$\to$" and "$\smile$" between "the kinds"* κ, μ, ϕ *themselves.*

[24] No algorithm can be efficiently applicable to *all* flows of signals; in fact, our framework of learning does not even admit the mathematical concept of unrestricted "ALL".

[25] Also see the lecture about the brain by Jeff Hawkins on https://www.youtube.com/watch?v=G6CVj5IQkzk.

[26] See references at the end of this text.

But it is not, a priori, clear how to properly define such *"self referential labeled polygraph structures"* that would encompass the above ingredients consistently with the following provisions.

- Our "polygraph" must incorporate some features of *n-categories* (for n=2?, 3?) and of *self-similar fractal sets* at the same time.
- In order to define the desirable class of "polygraph structures", one has to depart from the traditional logic and operate in terms of what we call *ergo-logic*; in particular, one needs to rethink the ideas of *"there exists"*, *"all"*, *"equality"*, *"number"*, *"set"*, *"infinity"*.
- Learning algorithms for building these "polygraphs" must tackle large volumes of data, where the applicable in our case concept of "statistics" does not fit into the frame of the traditional probability theory. The latter needs to be modified along with "sets" and "numbers".

Currently, I am struggling with these issues; I wrote down 20-30% of the intended article: *Understanding Languages and Making Dictionaries.*

Comments, Links, References.

If you are a mathematician you ought to look at everything around, including mathematics itself, from a mathematical viewpoint. But to see something interesting, something new, something you had no preconception of, you have to distance yourself from what you try to discern.

Prior turning to Math one may think of science. I collected some ideas expressed by scientists through the ages and indicated what a mathematician can make out of these in two partly overlapping short essays:

Introduction aux mystères (2012)

and

Allure of Quotations and Enchantment of Ideas,
www.ihes.fr/ gromov/PDF/quotationsideas.pdf

Dazzlingly interesting ideas come from Poincaré. For instance, one finds in his

Science and Hypothesis (1905)

among many other things, a mathematician's perspective on fundamental problems in visual perception.[27] This is the starting point of what we call *"ergo-thinking"*.

Another source of inspiration for what we call *"ergo"* originates in the overall structure of biology, especially of molecular biology:

mathematics of mathematics is closer to mathematics of Life than to mathematics of a physicist's non-Life.

An enjoyable book for a mathematician to read is

The Logic of Chance by Eugine Koonin (2011)

that is about statistics and evolution of genomes, where the author demonstrates how the telescopic power of *sequence alignment techniques* enables one to discern outlines of Life on Earth as it was $3\frac{1}{2}$ billion years ago.[28]

[27] Only recently, comparable general ideas were developed by people in the vision community.

[28] Reading some sections in this book requires a minimal prerequisite in molecular biology. Such a prerequisite, we believe, is also needed for understanding the nature of mathematics by mathematicians.

In our articles

Structures, Learning and Ergosystems
www.ihes.fr/ gromov/PDF/ergobrain.pdf

and

Ergostructures, Ergologic and the Universal Learning Problem
www.ihes.fr/ gromov/PDF/ergologic3.1.pdf

we present an ergo perspective on the natural and artificial learning processes.

This came very close to what has been already understood by some robotists quite a while ago and exposed under the heading of

INTRINSICALLY MOTIVATED AND/OR CURIOSITY DRIVEN LEARNING:

Formal Theory of Fun and Intrinsic Motivation and Creativity (1990–2010) by Jürgen Schmidhuber,
http://www.idsia.ch/~juergen/,
www.ece.uvic.ca/~bctill/papers/ememcog/Schmidhuber_2010.pdf,
http://www.idsia.ch/~juergen/,
http://www.idsia.ch/~juergen/interest.html.

Intrinsic Motivation Systems for Autonomous Mental Development
by Pierre-Yves Oudeyer, Frdéric Kaplan, and Verena V. Hafner
. www.pyoudeyer.com/ims.pdf,
https://flowers.inria.fr/,
www.pyoudeyer.com/, https://flowers.inria.fr/ICDL12-MoulinFrier-Oudeyer.pdf,
https://flowers.inria.fr/IMCleverWinterSchool-Oudeyer.pdf.
http://csl.sony.fr/publications.php?keyword=curiosity

Also the following two books promote ergo-like ideas.

Sparse distributed memory (1988):
by Pentti Kanerva that is a stochastically homogeneous model of memory based on the law of large numbers.

and

Aux sources de la parole. Auto-organisation et évolution (2013)
by Pierre-Yves Oudeyer, who suggests a simple mathematical model for formation of different "species of languages".

Ergo within Math. Some mathematicians instinctively follow the guidelines of what we call *ergo-logic* in doing math, with Alexander Grothendieck being ahead of the rest of us.

I tried ergo in math, starting with the article

Mendelian Dynamics and Sturtevant's Paradigm (2008)
(Contemporary mathematics - American Mathematical Society, 469).

My progress is slow with many projects remaining a dream. I explain some of it in

In a Search for a Structure, Part 1: On Entropy,
www.ihes.fr/~gromov/PDF/structre-serch-entropy-july5-2012.pdf.

Psychology, Science, Ergo. Our "ergo" originates in ideas about human mind which makes mathematicians edgy. Can psychology be taken seriously?

Is it a true science? Isn't it too slippery to be be grasped by a mathematician's mind?

We can not answer these questions, since we are even less ergo-prepared to define what a science is than it is the case with mathematics. And, mathematically speaking, the interesting question is that of *classification of levels of structural organisations*[29] of different bodies of knowledge rather than of assigning complimentary or derogatory labels to them, such as "science" and "pseudoscience".

Even though much of what is poured into our brains under the name of *"psychology"* is indigestible by mathematicians, there are quite a few intellectual gems that we may appreciate.

Historically the first(?) of these was the *universality of imprinting* revealed by Spalding in 1872; below is another instance of something non-trivial.

Synesthesia. Suppose somebody, call the person X, claims that he/she perceives different *graphemes* e.g. figures 5 and 7 in different colours, e,g. 5's as yellow and 7's as red. Can you verify that the person tells you the truth rather than making fun of you?

It looks impossible, after all "colours" in one's mind is not something that has a status of "objective existence", unless you believe in *red fairies* with *yellow qualia*. However the following ingenious experiment[30] shows that you "logical intuition" fails you.

Let X out of the room and randomly draw 5's and 7's on a blackboard B, densely spread and 50:50 distributed everywhere except for a *region* $D \subset B$ where you draw 30-60% more 7's than 5's.

If X indeed sees figures in colours, than upon entering the room, he/she will instantaneously notice a *reddish spot* against an orange background on the blackboard.

Apparently, our visual system systematically teaches itself to make new *units of perception* from the old ones,[31] but no present day universal man-made algorithm would come up with anything like the "ergo-formula" $5 \times 18 \simeq 7$:

$$7 \sim \overbrace{\qquad\qquad}^{55555555} = 5 \times 18.$$

[29] Representations of these "bodies" by networks G can be characterised by their *normalised connectivities* that are the ratios:

[*the first Betti number of* G]/[*the number of nodes in* G].

[30] If you've already guessed how this should be designed, you must have heard about it and forgot.

[31] Isolating and/or creating such *units* is the main and often most difficult task faced by an ergo-system.