Metric invariants of Kähler manifolds.

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§1. Setting up the problem. A Kähler structure on a smooth manifold V is given by a triple of tensors (g, ω, J) which play quite different roles in our narrative. The main character is g which is a *Riemannian metric* on V, i.e. a positive definite symmetric bilinear form on the tangent bundle $T(V)$. The second tensor ω, called the *(symplectic) Kähler form*, is an antisymmetric bilinear form on $T(V)$ and finally, J is an *almost complex structure* on V, that is an endomorphism $J : T(V) \to T(V)$ satisfying $J^2 = -\mathrm{Id}$.

Triples (g, ω, J) of this kind do not appear especially attractive unless we subject them to the following three conditions (A),(B) and (C) which constitute the definition of Kähler.

(A) *Coherence.* This is the basic algebraic relation between g, ω and J which requires, the J-invariance of g and w, i.e. $g(J\tau_1, J\tau_2) = g_1(\tau_1, \tau_2)$ and $\omega(J\tau_1, J\tau_2) = \omega(\tau_1, \tau_2)$ for all pairs of tangent vectors $\tau_1, \tau_2 \in T_v(V)$, $v \in V$, as well as the following identity

$$\omega(\tau_1, \tau_2) = g(\tau_1, J\tau_2) . \tag{$*$}$$

Coherent g, ω and J define a *Hermitian structure* on every tangent space $T_v(V)$ and manifolds V carrying coherent g, ω and J are called *quasi-Hermitian* (or *almost Hermitian*) manifolds. It is clear in this case that each of the three tensors g, ω and J is uniquely determined by the remaining two. It is equally clear that the positive definiteness of g makes the form ω non-singular. In fact, the top exterior power ω^n for $n = \dim V/2$ (the presence of J makes $\dim V$ even) is related to the (oriented) Riemannian volume element of (V, g) by

$$\omega^n = n! d_g v \tag{$+$}$$

and, in particular,

$$\mathrm{Vol}(V, g) = \int_V \omega^n / n! .$$

Proof. There is a g-orthonormal frame of covectors x_i, $y_i \in T_v^*(V)$, $i = 1, \dots, n$, at each point $v \in V$, such that $\omega_v = \sum_{i=1}^{n} x_i \wedge y_i$. Then

$$d_g v = x_1 \wedge y_1 \wedge x_2 \wedge y_2 \wedge \dots \wedge x_n \wedge y_n$$

while

$$w_v^n = (x_1 \wedge xy_1 + x_2 \wedge y_2 + \dots + x_n \wedge y_n)^n = \\ n!(x_1 \wedge y_1 \wedge x_2 \wedge y_2 \wedge \dots \wedge x_n \wedge y_n) .$$

(B) *Closeness of ω.* The second Kähler requirement which we add to (A) is that the 2-form ω is *closed*, i.e. $d\omega = 0$. This condition brings forth the cohomology class $[\omega] \in H^2(V, \mathbb{R})$, called the *Kähler class* of $V = (V, g, \omega, J)$. If V is a *closed* manifold (i.e. compact without boundary) then $[\omega] \neq 0$. In fact, even $[\omega]^n \neq 0$ as the value of $[\omega]^n = [\omega^n] \in H^{2n}(V, \mathbb{R})$ on the fundamental class of V, i.e. $\int_V \omega^n$, equals $n! \operatorname{Vol}(Vg) > 0$ according to (A).

Recall, that non-singular closed 2-forms are called *symplectic structures* on V. Given such a structure ω, one can always find g and J, such that the triple (g, ω, J) is coherent. This is called an *almost Kähler structure subordinated* to ω. (Such Kähler structure is not unique but every two can be joined by a homotopy. Moreover, the space of almost Kähler structures subordinated to a given ω is contractible. This follows from non-singularity of ω while the condition $d\omega = 0$ becomes relevant at the later stages of the almost Kählerian approach to the symplectic geometry, see [Gro]$_4$.)

(C) *Integrability of J.* A smooth map between almost complex manifolds, say $f : (V, J) \to (V', J')$, is called *holomorphic* if the differential $\mathcal{D}f : T(V) \to T(V')$ commutes with $J's$, i.e.

$$(\mathcal{D}f) \circ J = J' \circ \mathcal{D}f \ .$$

The existence of such maps for $\dim V > 2$ is an exception rather than a rule and so the following condition is quite restrictive for $\dim V > 2$.

J is called *integrable* if for each point $v \in V$ there exists a neighbourhood $U \subset V$ of v which admits a holomorphic diffeomorphism onto an open subset U' in $\mathbb{C}^n$, $n = \dim V/2$, where $\mathbb{C}^n$ carries the standard (almost) complex structure $T(\mathbb{C}^n) \to T(\mathbb{C}^n)$ given by the multiplication by $\sqrt{-1}$. Finally, following the custom (which reflects the historical development of the notions), we cancel "almost" by "integrable" and use "complex" for "integrable almost complex".

Examples of Kähler manifolds. (a) An obvious example is the *flat Kähler structure* on $\mathbb{C}^n$. Here $g = \sum_{i=1}^{n} dx_i^2 + dy_i^2$, $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$, and $J = \sqrt{-1}$. This flat $\mathbb{C}^n$ may appear as boring as the flat $\mathbb{R}^n$. But a second look brings a refreshing surprise, plenty of non-flat Kähler submanifolds in $\mathbb{C}^n$. In fact, the restriction of a Kähler structure to a *complex* submanifold is Kähler on this submanifold! Thus complex submanifolds in $\mathbb{C}^n$ are naturally Kähler. For example, we may take a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ and thus obtain a new Kähler structure on $\mathbb{C}^n$ by the graph embedding

$$\Gamma_f : \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1} \ .$$

(b) **Riemann surfaces**. Let V be an oriented surface with an arbitrary Riemannian metric g. Then we define ω as the oriented area element of g and define J by rotating each tangent plane $T_v(V)$ counter-clockwise by 90°. This is indeed a Kähler structure as the (only non-trivial) integrality property of J is ensured by the (local) Riemann mapping theorem.

(c) **Cartesian products** of Kähler manifolds are, obviously, Kähler. In particular, products of Riemann surfaces are Kähler.

(d) $\mathbf{CP^n}$ **and algebraic varieties.** These are most important. First, we recall that the complex projective space $\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times$ carries a unique complex structure J for which the quotient map $\mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$ is holomorphic. Then we restrict this map to the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ and observe that there is a unique metric g on $\mathbb{C}P^n$ for which the differential of the (Hopf) map $\varphi : S^{2n+1} \to \mathbb{C}P^n$ is isometric on the (horizontal) subbundle in $T(S^{2n+1})$ normal to the fibers $\varphi^{-1}(x) \in S^{2n+1}$, $x \in \mathbb{C}P^n$. Finally, we *define* ω by $\omega(\tau_1, \tau_2) = g(\tau_1, J\tau_2)$.

In the case $n = 1$ we have $\mathbb{C}P^1 = S^2$ and the above g is the round spherical metric of constant curvature 4 and area π. To simplify future notations we rescale the above g to have this area $= 1$. From now on the *standard (rescaled) metric* g on $\mathbb{C}P^n$ gives area one to each projective line $\mathbb{C}P^1$ in $\mathbb{C}P^n$. This is equivalent to $\int_{\mathbb{C}P^1} \omega = 1$. The latter condition makes the class $[\omega]$ *integral* (i.e. contained in the subgroup $H^2(\mathbb{C}P^n; \mathbb{Z}) \subset H^2(\mathbb{C}P^n; \mathbb{R})$) and generating the group $H^2(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}$. Notice that the Riemannian volume of this standard $(\mathbb{C}P^n, g)$ equals $(n!)^{-1}$ by formula $(+)$. (The only non-trivial point in verifying the Kähler condition is $d\omega = 0$. This is seen by pulling back ω to S^{2n+1} and identifying it with $d\alpha$ where α is a 1-form on S^{2n+1} of constant length (i.e. $\|\alpha(s)\|$, $s \in S^{2n+1}$, independent of s) with $\operatorname{Ker} \alpha \subset T(S^{2n+1})$ normal to the fibers of the Hopf map $\varphi : S^{2n+1} \to \mathbb{C}P^n$.) Now we recall that $\mathbb{C}P^n$ harbours plenty of closed complex submanifolds called *complex projective (algebraic) manifolds* which come along with Kähler structures induced from $\mathbb{C}P^n$. The simplest truly interesting class of such manifolds consists of *algebraic hypersurfaces* in $\mathbb{C}P^n$. Every such hypersurface H is given by a single equation $Q(z_0, z_1, \dots, z_n) = 0$ for a homogeneous polynomial Q on $\mathbb{C}^{n+1}$ of certain degree $d \geq 1$. (Such H is non-singular, i.e. is an honest submanifold, if Q has no critical points on $\mathbb{C}^n$ apart from the origin, and it is easy to see that H is non-singular for *generic* Q.)

(e) **Variation of Kähler structures.** Given a Kähler manifold (V, g, ω, J) one may vary g and ω without changing J as follows. Let f_0 be a C^2-smooth function on V and let $\omega_0 = dJdf$, where J acts on 1-forms in an obvious way. One knows this form ω_0 is J invariant (and, obviously, exact) and therefore the triple $(g', \omega' = \omega + \omega_0, J)$, for $g'(\tau_1, \tau_2) = -\omega'(\tau_1, J\tau_2)$, is Kähler in-so-far as g' is positive definite. In particular, this is so if f is $\mathbb{C}^2$-small.

One also knows that every two Kähler forms in the same cohomology class differ by $dJdf$ for some f. In fact, every exact J-invariant 2-form ω_0 equals $dJdf_0$ for some function f_0 on V by the classical $\partial\bar{\partial}$-Lemma.

Warning. The above description of $\omega - \omega'$ as $dJdf_0$ does not immediately reveal the geometry of an arbitrary Kähler metric g' on (V, J) with $[\omega'] = [\omega]$ because of the required positivity of $\omega' = \omega + dJdf_0$ (i.e. of g') which is a tricky differential inequality on f_0. For example, if $(V, J) = \mathbb{C}^n$ then every Kähler form ω is $dJdf$ for some f where the positivity of ω (i.e. of the corresponding g) amounts to *plurisubharmonicity* (or $\mathbb{C}$-*convexity*) of f. This condition reduces to ordinary convexity if f is invariant under a sufficiently large symmetry group. Let, for instance, f be induced from a function $\underline{f}$ on $\mathbb{R}^n \subset \mathbb{C}^n$ by the normal projection $\mathbb{C}^n \to \mathbb{R}^n$. Then f is plurisubharmonic $\Leftrightarrow$ $\underline{f}$ is convex. (This simple observation leads to an unexpected connection between convex sets and Kähler geometry, see survey article $[\mathrm{Gro}]_6$.)

Formulation of the main problems. A Riemannian manifold (V, g) is called *Kähler(ian)* if g can be completed to a Kähler structure with suitable ω and J. Our problem consists, broadly speaking, in finding (many) (geo)metric invariants of (V, g) which take special values on Kählerian manifolds. We want these invariants to be sufficiently robust; at the very least they should be C°-continuous on the space of metrics.

Besides finding a (geo)metric characterization of Kähler manifolds we want to measure a deviation (or distance) of an arbitrary Riemannian manifold from the Kählerian locus in the space of all Riemannian manifolds. Here again, we want to use (geo)metric invariants and see how much non-Kählerian they may become on general Riemannian manifolds.

Remark. The full Kähler structure can be almost uniquely recaptured by g. To see this we first recall that the Kähler condition implies that the holonomy group G_v of (V, g) at every point $v \in V$, is contained in the unitary group $U(n)|T_v(V)$, i.e. the linear automorphism group of $(T_v(V), g_v, J_v)$. Thus J (and hence ω) is determined by J_v at a single point $v \in V$, provided V is connected. In fact, one may characterize Kähler manifolds by G_v being conjugate to a subgroup in $U(n)$. However, this does not help us in search for invariants since the holonomy (i.e. parallel transform) depends on the (first) derivatives of g and this does not meet our (geo)metric criteria.

Systoles, energy and spectrum. Let us indicate the invariants which appear in this paper. The first group of these appeals to the volumes of submanifolds in (V, g) of positive codimension. In particular, one defines, following M. Berger, $\mathrm{sys}_k(V, g)$ as the infimum of the volumes of the k-dimensional submanifolds (or more general cycles) in V which are not homologous to zero (see $[\mathrm{Ber}]_{1,2}$, $[\mathrm{Gro}]_{3,8}$). These systoles can be evaluated for some complex algebraic manifolds. Then we exhibit non-Kähler examples showing that the range of the systolic invariants drastically increases when we pass from Kähler manifolds to all Riemannian manifolds.

Our second group of invariants refers to continuous maps $\varphi : V \to W$ where $W = (W, h)$ is a standard Riemannian manifold (e.g. a flat Riemannian torus T^d) and where φ should belong to a suitable homotopy class Φ of maps $V \to W$. Then we define $\mathrm{Min}\, EnV = \mathrm{Min}\, En(V|W, \Phi)$ as the infimum of the Dirichlet energies of the maps φ in Φ. (Recall that the Dirichlet energy is defined by $En\,\varphi = \frac{1}{2}\int_V \|\mathcal{D}\varphi\|^2 dv$.) If V is Kähler, then the complex geometry often provides holomorphic maps φ which minimize the energy and, moreover, equate $\mathrm{Min}\, EnV$ to some purely topological invariant of V. On the other hand, we indicate non-Kähler examples where $\mathrm{Min}\, En$ may become arbitrarily large compared to the Kähler case.

Finally, we look at the spectrum $\{0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots\}$ of the Laplace operator on V. One knows by the work of Hersch, Yang-Yau, Li-Yau and Bourguignon-Li-Yau that λ_1 admits a non-trivial upper bound in the Kähler case and we shall indicate here similar bounds for all eigenvalues λ_k. On the other hand we recall examples showing that there are no restrictions on individual λ_k for general Riemannian manifolds (see [CdV]).

Our presentation in the following sections §2–4 (except for concluding remarks in §4) is shaped by an attempt to be generically understandable. Thus we carve our definitions in order to fit a few simple examples rather than to fulfil the intrinsic logical demands of the mathematical structure. We do not go after precise and general results and suppress most of the technical details but, in violation of the tradition, we spend time explaining elementary features of basic notions.

§2. Systolic invariants. Let (V, g) be a compact Riemannian manifold and $M \subset V$ a smooth closed k-dimensional submanifold. We denote by $\mathrm{Vol}_k M$ the Riemannian volume of M and we observe that if M is *non-homologous to zero* then $\mathrm{Vol}_k M$ is bounded from below by a positive constant $C = C(V, g)$. Here "non-homologous to zero" refers to the homology of V with real coefficients. Thus M is assumed oriented and the homology condition, expressed by $[M]_{\mathbb{R}} \neq 0$, is equivalent to the existence of a closed differential k-form Ω, such that $\int_M \Omega \neq 0$. Since this integral is bounded by

$$\int_M \Omega \leq \int_M \|\Omega\|_m \, dm \ ,$$

where $\|\Omega\|_m$ is a suitable norm of Ω at $m \in M \subset V$ with respect to g (see below), we may bound $\mathrm{Vol}\, M$ from below by

$$\int_M \Omega / \sup_M \|\Omega\|_m \, dm \ , \tag{$*$}$$

where one should observe that the integral $\int_M \Omega$ is a topological (even homological) invariant of M, that is $\langle [M]_{\mathbb{R}}, [\Omega] \rangle$ where $[\Omega] \in H^k(V; \mathbb{R})$ denotes the de Rham class of Ω. (The "suitable norm" $\|\Omega\|_v$, $v \in V$, is defined as $\sup \|\Omega(\tau_1, \ldots, \tau_k)\|$ over all g_m-orthonormal frames $\tau_1, \ldots, \tau_k$ of vectors in $T_v(V)$.) Then the bound $(*)$ implies the universal bound (independent of M)

$$\mathrm{Vol}_k M \geq C = C(V, g) > 0 \tag{$**$}$$

for all M non-homologous to zero since the homology and cohomology of V are finite dimensional and so there are finitely many closed k-forms Ω_i, $i = 1, \ldots, p$, for $p = \operatorname{rank} H^k(V)$, such that "non-homologous to zero" implies non-vanishing of at least one of the p integrals $\int_M \Omega_i$. Furthermore, notice that $H^k(V)$ is generated by *integral* cohomology classes which, by definition, are representable by closed forms Ω such that the *periods*, i.e. the integrals $\int_M \Omega$, are integers for *all* closed submanifolds M. Now, we may choose the above forms Ω_i with integral periods and thus obtain the following lower bound on the constant C in the inequality $(**)$

$$C \geq \min_{i=1,\ldots,p} \left(\sup_{v \in V} \|\Omega_i\|_v \right)^{-1}$$

In fact, if $[M]_{\mathbb{R}} \neq 0$, then $\int\limits_M \Omega_i \neq 0$ for some i and since this is an integer,

$$\left| \int\limits_M \Omega_i \right| \geq 1 \, ,$$

which makes

$$\operatorname{Vol} M \geq \left(\sup_{m \in M} \|\Omega_i\|_m \right)^{-1} \geq \left(\sup_{v \in V} \|\Omega_i\|_v \right)^{-1} .$$

Definition of $\mathbf{sys_k(V,g)}$. The k-dimensional systole of a Riemannian manifold (V,g) is the best possible constant C in $(**)$, i.e. the infimum of the volumes of all closed oriented k-dimensional submanifolds in V which are not homologous to zero. For example, if V is a closed connected orientable manifold, then $\operatorname{sys}_n V = \operatorname{Vol} V$, for $n = \dim V$.

The above discussion shows that $\operatorname{sys}_k V > 0$ for compact manifolds V and thus it provides a non-trivial (geo)metric invariant of V whenever $H_k(V;\mathbb{R}) \neq 0$.

Computation of the systoles of Kähler metrics on $\mathbb{C}\mathbf{P}^n$. (Compare $[\mathrm{Ber}]_1$.) *Let g be a Kählerian Riemannian metric on $\mathbb{C}P^n$ normalized by the condition*

$$\operatorname{Vol}(\mathbb{C}P^n, g) = (n!)^{-1} \, .$$

Then

$$\operatorname{sys}_{2k}(\mathbb{C}P^n, g) = (k!)^{-1}, \ k = 1, \ldots, n \, .$$

Proof. Recall that the odd dimensional homology of $\mathbb{C}P^n$ vanishes and $H_{2k}(\mathbb{C}P^n)$, $k = 1, \ldots, n$ is infinite cyclic generated by the fundamental class of $\mathbb{C}P^k \subset \mathbb{C}P^n$. Now we divide the proof by establishing separately the inequalities

$$\operatorname{sys}_{2k}(\mathbb{C}P^n, g) \geq (k!)^{-1} \, , \tag{+}$$

and

$$\operatorname{sys}_{2k}(\mathbb{C}P^n, g) \leq (k!)^{-1} \, . \tag{−}$$

The proof of $(+)$ is based on the *Wirtinger inequality* which claims that the exterior $2k$-form $\left(\sum\limits_{i=1}^{n} x_i \wedge y_i \right)^k$ on $\mathbb{C}^n$ has norm (in our "suitable" sense) no greater than (in fact, equal to) $k!$. (This is proven by straightforward linear algebra.) It follows, that the 2-form ω in an arbitrary almost Hermitian (in particular, Kähler) manifold satisfies

$$\|\omega^k\|_v \leq k! \quad , \quad v \in V \, ,$$

and so every oriented $2k$-dimensional submanifold $M \subset V$ satisfies

$$\int\limits_M \omega^k \leq k! \operatorname{Vol}_{2k} M \, .$$

Therefore, if $\operatorname{Vol} M < (k!)^{-1}$ then

$$\int_M \omega^k < 1 ,$$

and if ω is a *closed* form with *integral* periods and M is a *closed* submanifold, then

$$\int_M \omega^k = 0 .$$

Now we specialize to our Kähler case where the form ω on $V = \mathbb{C}P^n$ *is* closed and does have integral periods. The latter is ensured by our normalization condition $\operatorname{Vol} = (n!)^{-1}$ equivalent (compare (d) in §1) to the equality $\int_{\mathbb{C}P^1} \omega = 1$ which implies, in turn, the equalities $\int_{\mathbb{C}P^k} \omega^k = 1$ for all $k = 1, \ldots, n$. Finally, we observe that M is homologous to zero in $\mathbb{C}P^n$ if and only if $\int_M \omega^k = 0$, since $H_{2k}(\mathbb{C}P^n)$ is cyclic, and thus arrive at the desired conclusion

If $\operatorname{Vol}_{2k} M < (k!)^{-1}$ *then* M *is homologous to zero.*

This is equivalent to the inequality $\operatorname{sys}_{2k} \geq (k!)^{-1}$.

Notice that the above conclusion remains valid for an arbitrary closed *almost Kähler* manifold V of total volume $(n!)^{-1}$ whose $2k$-dimensional homology is cyclic. Thus the integrability of the complex structure does not play any role at this stage. Yet the inequality $\operatorname{sys}_{2k} \geq (k!)^{-1}$ is not quite trivial even for the standard metric on $\mathbb{C}P^n$. In fact an ε-variation of this inequality leads to the following

Open problem. Let M be a closed $2k$-dimensional submanifold in $\mathbb{C}P^n$ with the standard metric, such that the ($2k$-dimensional) volume of the ε-neighbourhood $U_\varepsilon(M) \subset \mathbb{C}P^n$ for a given fixed $\varepsilon > 0$ is strictly less than the volume of the ε-neighbourhood of the subspace $\mathbb{C}P^k \subset \mathbb{C}P^n$. Is M homologous to zero ?

Now we turn to the inequality $\operatorname{sys}_{2k} \mathbb{C}P^n \leq (k!)^{-1}$. The submanifold M we use is just a linear subspace $\mathbb{C}P^k \subset \mathbb{C}P^n$. If the underlying complex structure of our (non-standard) Kähler metric is standard then this $M = \mathbb{C}P^k$ is a *complex* submanifold in $V = \mathbb{C}P^n$. Therefore the form $\omega^k|M$ is related to the oriented Riemannian volume element dm of M by

$$\omega^k|M = k!dm .$$

This trivially follows from the coherence between g, ω and J restricted to the tangent subbundle $T(M) \subset T(V)$. (Notice, that the Wirtinger inequality claims that $\omega^k|M \leq k!dm$ for *all* submanifolds while the above gives us the equality for *complex* submanifolds $M \subset V$.) The essential feature of $\mathbb{C}P^n$ used in the above argument is the existence of a *complex* submanifold M representing the generator in the homology group $H_{2k}(\mathbb{C}P^n)$ and this property relies on the integrability of the complex structure (at least, for $k \geq 2$) but does not involve the metric.

Now we recall a deep result in Kähler geometry (where the final touch is furnished by Yau's solution of Calabi conjecture) which claims that every Kähler manifold homeomorphic to $\mathbb{C}P^n$ is, in fact, biholomorphic to $\mathbb{C}P^n$, i.e. isomorphic to $\mathbb{C}P^n$ as a complex manifold. Thus an assumption that the complex structure is standard becomes redundant.

Variation of $\mathrm{sys}_k(\mathbb{C}P^n, g)$ for non-Kähler deformations of g. Now we want to show that for certain *non-Kähler* metrics g on $\mathbb{C}P^n$ the ratio $(\mathrm{sys}_k)^{\frac{1}{k}}/(\mathrm{sys}_\ell)^{\frac{1}{\ell}}$ may go quite far from $(\ell!)^{\frac{1}{\ell}}/(k!)^{\frac{1}{k}}$. (The Riemannian manifolds $(\mathbb{C}P^n, g)$ for such g must be positioned far away from the locus of Kähler manifolds in the space of all Riemannian manifolds.) First we observe the following

Trivial Lemma. *Let M_0 be a compact i-dimensional submanifold in a Riemannian manifold (V, g) and g' be another Riemannian metric on V which equals g outside the ε-neighbourhood of M_0 with respect to g. Then for all $k > i$,*

$$\mathrm{sys}_k(V, g') \geq \mathrm{sys}_k(V, g) - \delta$$

for some $\delta = \delta(V, g, M_0, \varepsilon)$ such that $\delta \to 0$ for $\varepsilon \to 0$. Furthermore, if $k < \mathrm{codim}\, M_0$, then

$$\mathrm{sys}_k(V, g') \leq \mathrm{sys}_k(V, g) + \delta$$

where again $\delta \to 0$ for $\varepsilon \to 0$.

In other words a modification of g near an i-dimensional submanifold cannot substantially diminish sys_k for $k > i$ and enlarge sys_k for $k < \dim V - i$. On the other hand such modification diminishes sys_i if M_0 is non-homologous to zero and g' is small on M, e.g. $g' = \varphi_\varepsilon g$, where φ_ε is a positive function on V which is small on M_0 and equals one ε-far from M_0. Then, with a little effort, we arrive at the following

Corollary. *Let (V, g) be a closed Riemannian manifold and let $s_1, \ldots, s_m$, $m = \dim V$, be real numbers, such that*

$$(s_{i+1})^{\frac{1}{i+1}}/(s_i)^{\frac{1}{i}} \geq \left(\mathrm{sys}_{i+1}(V, g)\right)^{\frac{1}{i+1}} / \left(\mathrm{sys}_i(V, g)\right)^{\frac{1}{i}}, \text{ for } i = 1, \ldots, m-1 .$$

Then for an arbitrary $\varepsilon > 0$ there exists a metric g_ε on V such that

$$|\,\mathrm{sys}_i(V, g_\varepsilon) - s_i| \leq \varepsilon, \; i = 1, \ldots, m .$$

As we mentioned earlier, the Riemannian manifold (V, g_ε) goes infinitely far from the Kählerian (and also almost Kählerian) locus as $(s_{i+1})^{\frac{1}{i+1}}/(s_i)^{\frac{1}{i}} \to \infty$ and $\varepsilon \to 0$.

Now, we want to indicate examples where $(\mathrm{sys}_{i+1})^{\frac{1}{i+1}}/(\mathrm{sys}_i)^{\frac{1}{i}}$ becomes small rather than large in order to emphasize the role of the integrability of the complex structure needed for the inequality $(-)$ saying that $\mathrm{sys}_{2k}(\mathbb{C}P^n, g) \leq (k!)^{-1}$ for Kähler metrics g normalized by the condition $\mathrm{sys}_{2n}(\mathbb{C}P^n, g) = \mathrm{Vol}(\mathbb{C}P^n, g) = (n!)^{-1}$. Again we modify the metric near a fixed closed i-dimensional manifold $M_0 \subset V$ but now we do it in a more specific way. We take some tubular ε-neighbourhood U_ε of M_0, denote by $T_{\mathrm{vert}} \subset T(U_\varepsilon) \subset T(V)$ the kernel of the differential of the normal projection $U_\varepsilon \to M_0$ and let $T_{\text{hor}} = T(U_\varepsilon) \ominus T_{\text{vert}}$. Then we split the metric g on U_ε according to the orthogonal splitting $T(U_\varepsilon) = T_{\text{hor}} \oplus T_{\text{vert}}$, say

$$g|U_\varepsilon = g_{\text{hor}} + g_{\text{vert}} \ ,$$

(where g_{hor} vanishes on T_{vert} and g_{vert} vanishes on T_{hor}). We take the function $\varphi_{\varepsilon,\varepsilon'}$ on V which equals $\varepsilon^{-1}\,\text{dist}(v, M_0) + \varepsilon'$ for $\text{dist}(v, M_0) \leq (1-\varepsilon')\varepsilon$ (where we assume $\varepsilon' \in]0,1[$) and which extends by $\varphi_{\varepsilon,\varepsilon'} = 1$ for dist $\geq (1-\varepsilon')\varepsilon$. Then we modify g on U_ε by taking $g_{\varepsilon,\varepsilon'} = \varphi^{-1}_{\varepsilon,\varepsilon'}\, g_{\text{vert}} + \varphi^{m}_{\varepsilon,\varepsilon'}\, g_{\text{hor}}$, $m = \dim V$, and keep $g_{\varepsilon,\varepsilon'} = g$ outside U_ε. It is obvious that

$$\text{sys}_k(V, g_{\varepsilon,\varepsilon'}) \leq \text{sys}_k(V, g) \quad \text{for} \quad k > m - i = \text{codim}\, M_0 \ .$$

In fact, the $g_{\varepsilon,\varepsilon'}$-volume of every k-dimensional submanifold M in V for $k > m - i = \text{rank}\, T_{\text{vert}}$ is smaller than the g-volume as the contraction effect by φ^m outweighs the expansion by φ^{-1}.

Now let us look at what happens to sys_k for $k = m - i = \text{codim}\, M_0$. For this we need

Another trivial lemma. *Let the connected components of M_0 generate $H_i(V;\mathbb{R})$ and let M_0 have trivial normal bundle. Then*

$$\text{sys}_{m-i}(V, g_{\varepsilon,\varepsilon'}) \to \infty$$

for every fixed $\varepsilon > 0$ and $\varepsilon' \to 0$, provided (U_ε, g) isometrically splits, i.e. $U_\varepsilon = M_0 \times B$, and g_{vert} is induced by the projection from some metric on (the ε-ball) B.

In order to use this lemma we recall from topology the following classical

Pontryagin-Serre-Thom theorem. *Let V be a compact n-dimensional manifold and take $i < m/2$. Then there exists a closed oriented i-dimensional submanifold $M_0 \subset V$ with trivial normal bundle whose connected components generate $H_i(M;\mathbb{R})$.*

A simple corollary of the above discussion is a possibility to construct a metric g on a closed n-dimensional manifold V with *prescribed* (positive) systoles $\text{sys}_k(V, g)$ for all $k > m/2$ for which $H_k(V;\mathbb{R})$ does not vanish. What remains unclear, however, is the existence of *isosystolic inequalities* of the form

$$(\text{sys}_i)^{\frac{1}{i}} \leq C_k(\text{sys}_k)^{\frac{1}{k}} \quad \text{for } k \geq 2i \ ,$$

that would give universal restrictions on the string of the number $(\text{sys}_1,\ \text{sys}_2, \ldots, \text{sys}_n = \text{Vol}_n)$ under suitable topological assumptions on V. (See $[\text{Ber}]_{1,2}$ and $[\text{Gro}]_{3,8}$.)

Systols of complex algebraic hypersurfaces $\mathbf{V} \subset \mathbb{C}\mathbf{P}^{n+1}$. The above discussion could convey a wrong impression that a non-trivial variation of the values of the systoles is possible only for non-Kähler manifolds. Yet, most Kähler manifolds V have the systoles different from those of $\mathbb{C}P^n$ but then, of course, these V are *not* homeomorphic to $\mathbb{C}P^n$. To get some idea we look at a smooth hypersurface $V \subset \mathbb{C}P^{n+1}$ defined by a homogeneous polynomial Q on $\mathbb{C}^{n+2}$ of degree d. We denote by ω the standard Kähler form on $\mathbb{C}P^{n+1}$

restricted to V and recall Lefschetz theorem which describes the (co)homology of V apart from the middle dimension.

If i is odd and not equal to n, then $H_i(V;\mathbb{R}) = 0$. If i is even and $< n$, then the groups $H^i(V;\mathbb{Z})$ and $H_i(V;\mathbb{Z})$ are infinite cyclic and $H^i(V;\mathbb{Z})$ is generated by $[\omega^{i/2}]$.

If i is even and lies above n, then again $H^i(V;\mathbb{Z})$ and $H_i(V;\mathbb{Z})$ are infinite cyclic, but now the class $[\omega^{i/2}] \in H^i(V;\mathbb{Z})$ is divisible by d and $d^{-1}[\omega^{i/2}]$ is a generator in $H^i(V;\mathbb{Z})$. Furthermore, the homology $H_i(V;\mathbb{Z})$ is generated by the fundamental class of the intersection of V with a linear subspace $\mathbb{C}P^j \subset \mathbb{C}P^{n+1}$, $j = i/2+1$, in general position.

Now we see as earlier that

$$\text{(1)} \qquad \operatorname{Vol}_{2n} V = d(n!)^{-1} ,$$

moreover

$$\operatorname{sys}_{2k}(V) = d(k!)^{-1} \quad \textit{for} \quad 2k > n .$$

$$\text{(2)} \qquad \operatorname{sys}_{2k} \leq d(k!)^{-1} \quad \textit{for all} \quad k = 1, \ldots, n .$$

$$\text{(3)} \qquad \operatorname{sys}_{2k} \geq (k!)^{-1} \quad \textit{for all} \quad k = 1, \ldots, n .$$

Remarks. The inequalities (2) and (3) do not allow us to determine sys_{2k} for $2k \leq n$, since the intersection $\mathbb{C}P^j \cap V$, $j = k+1$, gives us the d-multiple of a generator in $H_i(V;\mathbb{Z})$. (This intersection has volume $d(k!)^{-1}$ which implies the inequality (2)). On the other hand the lower bound (3) becomes sharp only if V contains a k-dimensional linear subspace (i.e. some $\mathbb{C}P^k \subset \mathbb{C}P^{n+1}$). One knows, this is always the case if d and k are small compared to n. For example, if $d \leq n+1$, then V necessarily contains a projective line and then $\operatorname{sys}_2 V = 1$, as for $V = \mathbb{C}P^n$. On the other hand, if d is significantly larger than n, then a generic V does not contain a line and so $\operatorname{sys}_{2k} V > (k!)^{-1}$ for all k. As we deform V in $\mathbb{C}P^{n+1}$ by varying the defining polynomial Q, the systoles vary continuously and thus the values of each (function) $\operatorname{sys}_{2k} V$, $k < n$, cover some non-empty open interval $]a,b[\subset [(k!)^{-1},\ d(k!)^{-1}]$, where, in fact, $a = (k!)^{-1}$. (The ends of $]a,b[$ may, a priori, correspond to singular hypersurfaces V.)

Conjecture. For given $k = 1, 2, \ldots$, and $s \geq (k!)^{-1}$ there exists a hypersurface V (of sufficiently large degree and dimension), such that $\operatorname{sys}_k V = s$.

Morse profile (or landscape) of the function $\mathbf{M \mapsto \operatorname{Vol} M}$. The space $\mathcal{M}$ of subvarieties $M \subset V$ can be given a topology (this can be done in a variety of essentially different ways) and then the Morse theory provides a means to extract numerical invariants (Morse profile) from every function on $\mathcal{M}$ (see $[\text{Gro}]_{5,7}$). In particular, if V carries a Riemannian metric g, then the Morse profile of the function $M \mapsto \operatorname{Vol} M$ becomes an invariant of (V,g). The simplest instance of that is $\text{sys}_i(V,g)$ representing the infimum of the volume function on a certain connected component of $\mathcal{M}$.

If V is Kähler, we have the distinguished subspace $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}$ consisting of complex subvarieties on which the values of the volume become an easily computable topological invariant (depending on $[\omega] \in H^2(V;\mathbb{R})$). This imposes strong restrictions on the Morse profile of the volume function on $\mathcal{M}$ generalizing those we have observed for the systoles, but we do not discuss such generalization in this lecture.

§3. Jacobi-Albanese varieties and minimal energy. Let $\mathbb{Z}H^i \subset H^i = H^i(V;\mathbb{R})$ denote the space of integer classes (i.e. the image of $H^i(V;\mathbb{Z})$) and similarly, let $\mathbb{Z}H_i$ denote the integer part of the real homology $H_i = H_i(V;\mathbb{R})$. We denote by J^i and J_i the quotient spaces $H^i/\mathbb{Z}H^i$ and $H_i/\mathbb{Z}H_i$ respectively which are affine tori of dimensions $\ell_i = \text{rank}\, H^i = \text{rank}\, H_i$. If V comes along with a metric g, then we can give certain metrics to J^i and J_i, and then the geometric invariants of these *Jacobians* (tori) become invariants of (V,g). For example, the volume on the i-dimensional submanifolds in V gives rise to a flat *Finsler metric* on J_i, such that the length of the shortest geodesic in this J_i becomes related to $\text{sys}_i V$ (see $[\text{Gro}]_{3,8}$).

A general scheme of constructing metrics on J^i and J_i is as follows. We start with a norm on the space Ω^i of i-forms on V which is constructed in some way using g, e.g. the L_p-norm on Ω^i. This norm restricts to the subspace of closed forms, $\text{Ker}\, d_i \subset \Omega^i$, and then goes down to the *quotient norm* on

$$H^i = \text{Ker}\, d_i / \text{Im}\, d_{i-1} .$$

Then we have the corresponding (flat Finsler) metric on J^i and, by duality, on J_i.

In what follows we concentrate on the case $i = 1$ and our starting norm is L_2 on Ω^1, i.e.

$$\|\alpha\| = \|\alpha\|_{L_2} = \left(\int_V \|\alpha_v\|^2 dv\right)^{\frac{1}{2}} .$$

The corresponding L_2-Jacobians are *flat Riemannian* tori, denoted L_2J^1 and L_2J_1, which are mutually dual (by the construction of the metrics and by the duality between H^1 and H_1). In particular, $\text{Vol}\, L_2J^1 = (\text{Vol}\, L_2J_1)^{-1}$. In fact, these volumes are exactly the invariants of (V,g) we are after in this section.

Let us give an alternative description of these volumes in terms of the *minimal energy* of *Abel-Jacobi* maps. First we observe that there exists a unique (natural) homotopy class of maps $\varphi : V \to J_1$ which induce an isomorphism on H_1. Now, given some metrics g on V and h on J_1 we can speak of the *(Dirichlet) L_2-energy* of φ, called $\text{En}\,\varphi = \frac{1}{2}\int_V \|\mathcal{D}\varphi(v)\|^2$ where the norm of a linear map $\mathcal{D}$ between Euclidean spaces is defined by $\|\mathcal{D}\|^2 = \text{Trace}\, \mathcal{D}^*\mathcal{D}$. If h is Riemannian flat as in the case of L_2J_1, then φ is locally given by d-functions

$\varphi_1, \ldots, \varphi_d$, for $d = \dim J_1$, corresponding to orthonormal linear (coordinate) functions (locally defined) on (J_1, h) and

$$\|\mathcal{D}\varphi(v)\|^2 = \sum_{i=1}^{d} \|d\varphi_i(v)\|^2 .$$

Then a simple formal argument shows that

$$(\operatorname{En}\varphi)^{1/2} / \big(\operatorname{Vol}(J_1, h)\big)^{\frac{1}{d}} \geq \sqrt{\frac{d}{2}} \Big/ (\operatorname{Vol} L_2 J_1)^{\frac{1}{d}} .$$

Moreover, if h equals the metric h_0 of $L_2 J_1$, then the infimum of the energies of φ in our homotopy class equals $d/2$ but if h is different from a scalar multiple of h_0, then this infimum is strictly greater than $\frac{d}{2}/(\operatorname{Vol} L_2 J_1)^{\frac{2}{d}}$. (Notice that scaling h by λ scales $\operatorname{Vol}(J_1, h)$ by $\lambda^{d/2}$ and $\operatorname{En}\varphi$ by λ.) This can be expressed in a more concise manner with the following definition.

Minimal Jacobian energy. First take the infimum of $\operatorname{En}\varphi$ over all φ in our homotopy class of maps $(V, g) \to (J_1, h)$ for a fixed flat Riemannian metric h on J_1 and call this infimum $\operatorname{Min}\operatorname{Ja}\operatorname{En}_h(V, g)$. Then take the infimum of $\operatorname{Min}\operatorname{Ja}\operatorname{En}_h$ over all flat Riemannian metrics h normalized by $\operatorname{Vol}(J_1, h) = 1$, and call this infimum $\operatorname{Min}\operatorname{Ja}\operatorname{En}(V, g)$. Then the above discussion shows that

$$\operatorname{Min}\operatorname{Ja}\operatorname{En} = d/2(\operatorname{Vol} L_2 J_1)^{\frac{1}{d}} = d(\operatorname{Vol} L_2 J^1)^{\frac{1}{d}}/2 \qquad (*)$$

Remarks. (a) If V is compact then the minimal energy is realized by an actual smooth map $V \to J_1$, called the *harmonic* map which is characterized by the *Laplace* equation $\Delta\varphi_i = 0$ for the above coordinate functions φ_i.

(b) The definition of Min Ja En generalizes to the more general setting of an arbitrary manifold W with a Riemannian metric h (or a family of such metrics) and a homotopy class of maps $V \to W$. Particularly interesting examples are Riemannian surfaces (W, h) and higher dimensional Kähler manifolds W of *negative curvature*, especially *locally symmetric* ones, where the complex geometry ensures holomorphicity of harmonic maps (see [Siu]).

Evaluation of Min Ja En *for Kähler manifolds.* An exterior 2-form ω on a $2n$-dimensional manifold V gives rise to the following bilinear form $\Phi = \Phi_\omega$ on the space of 1-form on V,

$$\Phi_\omega(\alpha, \beta) = \int_V \alpha \wedge \beta \wedge \omega^{n-1} .$$

If ω is closed this Φ descends to $H^1(V; \mathbb{R})$ and the resulting (anti-symmetric bilinear) form $\bar{\Phi}$ on H^1 depends only on the cohomology class $[\omega] \in H^2(V; \mathbb{R})$. Then we pass to the Jacobian $J^1 = H^1/\mathbb{Z}H^1$, denote by $\bar{\Phi}_{[\omega]}$ the corresponding 2-form on J^1 and look at the "volume" of J^1, with respect to $\bar{\Phi}_{[\omega]}$, that is the total volume of the top dimensional exterior power of $\bar{\Phi}_{[\omega]}$ (which makes sense if $d = \dim J^1$ is even), namely

$$\mathrm{Vol}(J^1, \bar{\Phi}_{[\omega]}) = \int\limits_{J^1} \left(\bar{\Phi}_{[\omega]}\right)^{d/2} .$$

Proposition. *If V is a closed Kähler manifold of real dimension $2n$ then d is even and*

$$\mathrm{Vol}(J^1, \bar{\Phi}_{[\omega]}) = c_{n,d}\,\mathrm{Vol}(L_2 J^1) \qquad (**)$$

for $c_{n,d} = (\frac{d}{2})!\left((n-1)!\right)^{d/2}$.

Proof. The Hodge theory implies that the space $\mathcal{H}^1$ of *harmonic* 1-forms on V is invariant under J, i.e. if α is harmonic then so is $J\alpha$. Then one observes that pointwise on V

$$\alpha \wedge J\alpha \wedge \omega^{n-1} = (n-1)!\|\alpha\|_g^2\, d_g v$$

by elementary linear algebra and the proposition follows.

Corollary. *If V is Kähler then* Min Ja En V *is a purely topological invariant depending on* $[\omega] \in H^2(V;\mathbb{R})$.

Example. Let V be homeomorphic to the torus T^{2n}. Then

$$\mathrm{Vol}\, L_2 J_1 = (\mathrm{Vol}\, V)^{-\frac{n-1}{n}}$$

by an elementary argument using the above $(*)$ and $(**)$.

Remark. The inequality

$$\mathrm{Vol}(J^1, \bar{\Phi}_{[\omega]}) \le c_{n,d}\,\mathrm{Vol}(L_2 J^1)$$

remains valid for *almost* Kählerian manifolds as $\|\omega\|_g \le 1$ by Wirtinger inequality.

Exercise. Let V be an almost Kähler manifold homeomorphic to T^{2n}. If $\mathrm{Vol}\, L_2 J_1 = (\mathrm{Vol}\, V)^{-\frac{n-1}{n}}$ then show that V is Kähler (i.e. the underlying almost complex structure on V is integrable).

Remarks. Let us indicate another approach to the above relation $(*)$ which is better suited for generalizations where J_1 is replaced by a more general Kähler manifold W. The key point is that the energy of a *holomorphic* map $\varphi : V \to W$ has a purely topological expression in terms of the Kähler form ω of V and the pullback ω^* of the symplectic form from W to V. Namely

$$\mathrm{En}\,\varphi = \frac{1}{(n-1)!} \int\limits_V \omega^{n-1} \wedge \omega^* ,$$

as an elementary (point-wise) computation shows. This yields $(**)$ (via $(*)$) since the L_2-Jacobian $L_2 J_1$ carries a natural complex structure which turns it (together with a metric) into a Kähler manifold, such that our homotopy class of maps $V \to J_1$ contains a *holomorphic* representative.

Also notice that each integral $\int_V \omega^i \wedge (\omega *)^{n-i}$, $i = 0, 1, \ldots, n-1$, has a Riemannian interpretation which suggests a generalization of the present discussion yet in another direction.

Variation of Min Ja En and of Vol $\mathbf{L_2 J^1}$ for non-Kähler deformations of g. *Let V be a closed connected manifold of dimension $m \geq 3$ and with* $\operatorname{rank} H_1(V) = d \geq 1$. *Then it admits a Riemannian metric g' with arbitrarily prescribed* $\operatorname{Vol}(V, g')$ *and* $\operatorname{Min} \operatorname{Ja} \operatorname{En}(V, g')$.

Proof. First we indicate a modification of a given metric g which does not change the volume but makes Min Ja En $\to 0$. This is achieved by multiplying g by a positive function φ' which equals a small constant ε outside a small ε'-ball $B' \subset (V, g)$ and which is large in the concentric $\varepsilon'/2$-ball $B'' \subset B'$ in order to make $\operatorname{Vol}(V, \varphi' g) = \operatorname{Vol}(V, g)$. The scaling g by ε scales the energy down by ε^{m-2}, $m = \dim V$, while the "large part" of φ' does not significantly contribute to the energy being located in the small ball. (We suggest the reader would go through the details by him-/herself.)

Next, we want to enlarge the energy without changing the volume. To do this we take a closed 1-dimensional submanifold $M_0 \subset V$ whose connected components generate $H_1(V; \mathbb{R})$. (This is obviously possible as $\dim V \geq 3$). We split the metric g on a tubular ε-neighbourhood $U_\varepsilon(M_0)$ into $g_{\text{hor}} + g_{\text{vert}}$ (see §2), squeeze the horizontal ("parallel" to M_0) part of g and expand in the vertical direction (normal to M_0) by

$$g'_\varepsilon | U_\varepsilon = \varepsilon \, g_{\text{hor}} + \varepsilon^{-(n-1)} g_{\text{vert}} .$$

This does not change the volume of $U_\varepsilon = U_\varepsilon(M_0)$ and so this g'_ε can be extended to all of V with $\operatorname{Vol}(V, g'_\varepsilon) = \operatorname{Vol}(V, g)$. But now Min Ja En $\to \infty$ for $\varepsilon \to 0$ as seen by looking at our maps $\varphi : V \to J_1$ restricted to U_ε. (The computation simplifies if (U_ε, g) isometrically splits, $U_\varepsilon = M_0 \times B_\varepsilon$, which we can assume as g is completely at our disposal.)

§4. Spectrum of V. *The first eigenvalue* λ_1 of a compact Riemannian manifold (V, g) is defined as the infimum of the ratio

$$\|df\|^2_{L_2} / \|f\|^2_{L_2}$$

over all Lipschitz functions $f : V \to \mathbb{R}$ satisfying $\int_V f dv = 0$. Notice that the metric g enters the definition via the two L_2-norms in different ways. The L_2-norm of a function

$$\|f\|_{L_2} = \left(\int_V f^2(v) \, dv \right)^{1/2}$$

uses only the Riemannian volume element $dv = d_g v$ whilst the norm of the differential also uses the point-wise norm on the cotangent bundle $T^*(V)$ defined by g,

$$\|df\|_{L_2} = \int_V \|df(v)\|_g^2 dv \ .$$

More generally, the *k-th eigenvalue* λ_k is defined as the infimum of those $\lambda \geq 0$ for which there exist non-zero Lipschitz functions $f_0, f_1, \ldots, f_k$ on V such that

(i) $$\int_V f_i(v) f_j(v) dv = 0 \quad \text{for} \quad 0 \leq i < j \leq k \ .$$

(ii) $$\|df_i\|_{L_2} \leq \lambda \|f_i\|_{L_2}, \quad i = 0, \ldots, k \ .$$

Remarks. One may characterize the *Laplace operator* Δ on functions on V by $\langle \Delta f, f \rangle = \langle df, df \rangle = \|f\|_{L_2}^2$ and then the set of the above eigenvalues $\{0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots\}$ can be identified with the *spectrum of* Δ.

There is a more geometric way of looking on λ_k. Namely we regard the correspondence

$$f \mapsto \|df\|_{L_2}^2 / \|f\|_{L_2}^2$$

as a function $\delta = \delta(f)$ on the infinite dimensional *projective* space P of functions on V. Then the spectrum emerges via the Morse profile (landscape) of this function δ on P (see $[\mathrm{Gro}]_{1,5}$).

Example. The unit sphere $S_1^2 \subset \mathbb{R}^3$ (of area 4π) has $\lambda_1 = \lambda_2 = \lambda_3 = 2$.

Explanation. Let $f_i : S_1^2 \to \mathbb{R}$, $i = 1, \ldots, 3$ be the linear coordinates of $\mathbb{R}^3$ restricted to S_1^2. Clearly, the norms $\|f_i\|_{L_2}$ and $\|df_i\|_{L_2}$ on S_1^2 do not depend on $i = 1, 2, 3$. Furthermore, since $\sum_{i=1}^{3} f_i^2(s) = 1$ for $s \in S_1^2$, we have

$$\sum_{i=1}^{3} \|f_i\|_{L_2}^2 = \sum_{i=1}^{3} \int_{S_1^2} \|f_i(s)\|^2 ds = \text{Area } S_1^2 = 4\pi \ .$$

Then we observe that

$$\sum_{i=1}^{3} \|df_i(s)\|^2 = 2, \quad s \in S_1^2 \ ,$$

and, consequently, $\sum_{i=1}^{3} \|df_i\|_{L_2}^2 = 8\pi$, since the map

$$(f_1, f_2, f_3) : S_1^2 \to \mathbb{R}^3$$

is *isometric* for our (standard) metric g_0 on S_1^2, i.e.

$$\sum_{i=1}^{3}(df_i)^2 = g_0 .$$

(If linear forms $\ell_1,\dots,\ell_k$ on $\mathbb{R}^n$ satisfy $\sum\limits_{i=1}^{k}\ell_i^2 = g_0$ for the (quadratic) structure form g_0 on $\mathbb{R}^n$, then $\sum\limits_{i=1}^{k}\|\ell_i\|_{g_0}^2 = n$. For example, the coordinate forms x_i, $i=1,\dots,n$, satisfy $\sum\limits_{i=1}^{n}x_i^2 = g_0$ and $\|x_i\|^2 = 1$ for $i=1,\dots,n$.) The above relations for f_i and df_i imply that

$$\|df_i\|^2/\|f_i\|^2 = 2, \quad i=1,\dots 3 ,$$

and an obvious symmetry consideration shows that

$$\int_{S_1^2} f_i(s)ds = 0 , \quad i=1,2,3$$

and

$$\int_{S_1^2} f_i(s)\,f_j(s) = 0 , \quad i \neq j$$

Thus $\lambda_i(S_1^2) \leq 2$, $i=1,2,3$, by our definition and one knows that, in fact, $\lambda_i = 2$ for $i=1,2,3$, because *every* function f on S_1^2 with $\int\limits_{2_1^2} f = 0$ satisfies the following inequality

$$\int_{S_1^2} f^2 \leq \frac{1}{2}\int_{S_1^2}\|df\|^2 .$$

(See [B-L-M].)

Hersch Theorem (see [Her]). *Let g be a Riemannian metric on S^2 normalized by the condition* $\text{Area}(S^2,g)=4\pi$. *Then the first eigenvalue of (S^2,g) satisfies*

$$\lambda_1(S^2,g) \leq 2 = \lambda_1(S_1^2) .$$

Proof. The inequality $\lambda_1 \leq 2$ is obtained by exhibiting a suitable *test function* f on S^2 for which $\|f\|_{L_2} \geq \frac{1}{2}\|df\|_{L_2}$ with our metric g. This is done by using a conformal (i.e. holomorphic) map $\varphi : (S^2,g) \to S_1^2$ and pulling back the above (coordinate) functions f_i, $i=1,2,3$, to (S^2,g). Since the test function f on (S^2,g) should have $\int\limits_{S^2} f = 0$, (see the definition of λ_1), we want our φ to pull back all f_i, $i=1,2,3$, to functions with *zero* integrals over (S^2,g). This can be expressed in terms of the *push-forward* $\varphi_*(\mu_g)$ of the

Riemannian measure μ of (S^2, g) to the (standard) unit sphere $S_1^2 \subset \mathbb{R}^3$. This measure $\varphi_*(\mu_g)$ can be thought of as a measure on $\mathbb{R}^3$ as S_1^2 lies in $\mathbb{R}^3$ and then we can speak of the *center of gravity* of this measure, which (obviously) equals the integral of the map $I \circ \varphi : (S^2, g) \to \mathbb{R}^3$ where I denotes the inclusion of the unit sphere into $\mathbb{R}^3$. Since the coordinates of the maps $I \circ \varphi$ are our pull-backs $f_i^* = f_i \circ \varphi$, the above vanishing requirement $\int_{S^2} f^*(s) ds_g = 0$ for $f = f_i$, $i = 1, 2, 3$, amounts to the center of gravity of $\varphi_*(\mu_g)$ being located at the origin.

Hersch Lemma (see [Her], [Y-Y]). *There exists a conformal map $\varphi : (S^2, g) \to S_1^2 \subset \mathbb{R}^3$ such that the center of gravity of $\varphi_*(\mu_g)$ is located at the origin in $\mathbb{R}^3$.*

Before proving the lemma we explain how it yields Hersch theorem.

Since the map φ is conformal it preserves the L_2-norm on 1-forms, as the area element $\|\ell(s)\|^2 ds^2$ on S^2 is (obviously) conformally invariant for every 1-form ℓ. Thus for every C^1-function f on $S_1^2 \subset \mathbb{R}^3$ the pull-back $f^* = f \circ \varphi$ on (S^2, g) satisfies

$$\|df^*\|_{L_2} = \|df\|_{L_2} .$$

Now we observe that the space of linear functions on $\mathbb{R}^3$ pulled back to (S^2, g) by the map $I \circ \varphi$ carries two scalar products: one induced from $\mathbb{R}^3$ and the second coming from the L_2-structure on $(S^2, d_g s = \mu_g)$. These scalar products (quadratic forms) can be simultaneously diagonalized. This amounts to finding orthonormal coordinates (frame) in $\mathbb{R}^3$ such that the pull-backs of these new coordinates to (S^2, g) are μ_g-orthogonal and so we may assume that the original coordinate functions f_i have this property. Thus

$$\int_{S^2} f_i^* f_j^* d_g s = 0, \quad 1 \le i < j \le 3 ,$$

and since the center of gravity of $\varphi_*(\mu_g)$ is zero, these f_i^* have zero integrals over S^2. Therefore each f_i may serve as a test function for λ_1.

Now, since $\sum_{i=1}^{3} \|f_i^*(s)\|^2 = 1$ on our sphere (S^2, g), we have as earlier

$$\sum_{i=1}^{3} \|f_i^*\|_{L_2}^2 = 4\pi ,$$

but now (for $(S^2, g) \neq S_1^2$) we cannot claim that $\|f_i^*\|^2$ is independent of $i = 1, 2, 3$. Yet it follows, that for some $i_0 = 1, 2, 3$, this squared norm is *at least* $4\pi/3$, while the norm $\|df_{i_0}^*\|_{L_2}$ equals $\|df_{i_0}\|_{L_2} = 8_\pi/3$ by the conformality of φ. Hence, the function $f_{i_0}^*$ has

$$\|f_{i_0}^*\|_{L_2}^2 \ge \frac{1}{2} \|df_{i_0}^*\|_{L_2} \quad \text{on } (S^2, g) ,$$

while

$$\int_{S^2} f_{i_0}^*(s) d_g s = 0 ,$$

which yields the required inequality $\lambda_1(S^2, g) \leq 2$.

Remark. The case of a general metric with Area $\neq 4\pi$ reduces to the general case as the eigenvalues of surfaces scale by

$$\lambda_i(S^2, \alpha g) = \alpha^{-1} \lambda_i(S^2, g) .$$

For example, the round Euclidean sphere $S^2_{1/2}$ of radius $1/2$ has $\lambda_1 = 4$.

Proof of Hersch Lemma. Denote by $C(\mu) \in \mathbb{R}^3$ the center of gravity of a measure μ on S^2_1 and observe that $C(\mu)$ lies in the *open* unit ball $B^3_1 \subset \mathbb{R}^3$ unless μ is a point measure. Then we consider the natural action of the group A of conformal transformations of S^2_1 on the space of measures on S^2_1 and apply C to an orbit of a measure. Namely, we take some μ and map $A \to \mathbb{R}^3$ by

$$a \mapsto C(a\,\mu) .$$

Observation. *If the support of μ contains at least two distinct points* (e.g. *if μ has continuous density function) then the resulting map of A to the open ball B^3_1 is proper as well as continuous.*

Recall, that a continuous map $p : A \to B^3_1$ is *proper* if for every *divergent* sequence $a_i \in A$ the images $p(a_i) \in B^3_1$ go to the boundary of the ball B^3_1 i.e. $\|p(a_i)\|_{\mathbb{R}^3} \to 1$, for $i \to \infty$. This properness is ensured in our case by the following important simple feature of the conformal action on S^2 called *strong Furstenberg boundary property*: for every μ and every *divergent* sequence $a_i \in A$ there is a subsequence a_{i_j}, such that the measures $a_{i_j}(\mu)$ weakly converge to a *point measure* on S^2_1.

Important example. Let μ_0 be the standard Euclidean measure on S^2_1 invariant under the group $H = SO(3)$. Then the orbit $A\mu_0$ can be identified with the homogeneous space A/H which equals the three dimensional hyperbolic space. The above map $A \to B^3_1$, for $a \mapsto C(a\mu_0)$, becomes a composition of the quotient map $A \to A/H$ and a *homeomorphism* of the hyperbolic space A/H onto the open ball B^3_1. Thus $A \to B^3_1$ is a fibration with the fiber H.

Now we invoke the following

Elementary topological lemma. *Let ψ_0 and ψ be continuous proper maps between manifolds, say $\psi_0,\ \psi : A \to B$, such that ψ_0 is a fibration and ψ is homotopic ψ_0 by a homotopy of proper maps. Then if ψ_0 is onto then ψ is also onto.*

We apply this lemma to the maps $\psi_0 : a \mapsto C(a\mu_0)$ and $\psi : a \mapsto C(a\mu)$, where the homotopy between ψ_0 and ψ is induced by a linear homotopy between the measures μ_0 and μ, i.e. $\psi_t : a \mapsto C\big(a(\mu t + \mu(1-t))\big)$. Then we see that the map $\psi : a \mapsto C(a\mu)$ maps A *onto* the ball B^3_1. In particular, there exists a conformal transformation $a \in A$ of S^2_1 which moves given measure μ' to $\mu = a\mu'$ where the center of gravity of μ is zero.

Conclusion of the proof of Hersch Lemma. By the Riemann mapping theorem there exists some conformal map $\varphi' : (S^2, g) \to S^2_1$. We compose φ with a conformal map $a : S^2_1 \to S^2_1$ which makes the center of gravity of the push-forward measure $\varphi'_*(\mu_g)$ zero and thus obtain our *conformal* map $\varphi = a \circ \varphi' : S^2 \to S_1^2$ for which the center of gravity of the measure $\varphi_*(\mu_g)$ on $S_1^2 \subset \mathbb{R}^3$ equals the origin $0 \in \mathbb{R}^3$.

Generalization of the proof of Hersch Lemma. The above "center of gravity" discussion applies to certain actions of Lie groups generalizing the conformal action of our A on S_1^2. An instance of this needed for the Kähler story is as follows.

Denote by V the (real vector) space of Hermitian forms on $\mathbb{C}^{n+1}$ and let $P_1^n \subset \mathbb{C}^{n+1}$ be the *Veronese manifold* consisting of the forms of rank one and trace one. Every form $p \in P_1^n$ equals $Z\bar{Z}$, where Z is a unit covector on $\mathbb{C}^{n+1}$ and so we have a map of the unit sphere S_1^{2n+1} in $\mathbb{C}^{n+1}$ (which we identify with the dual space of the covectors in $\mathbb{C}^{n+1}$) onto P_1^n. This map obviously factors through the Hopf map $S^{2n+1} \to \mathbb{C}P^n$ and so P_1^n can be identified with $\mathbb{C}P^n$. We also observe that $P_1^n \subset V$, being the image of the (Veronese) map $Z \mapsto Z\bar{Z},\ Z \in S^{2n+1}$, is contained in the hyperplane $V' \subset V$ given by the equation $\sum_{i=0}^{n} Z_i \bar{Z}_i = 1$. For example, if $n = 1$ and $\mathbb{C}P^1 = S^2$, the Veronese manifold can be identified with the unit sphere $S_1^2 \subset \mathbb{R}^3 = V'$.

As we identify P_1^n with $\mathbb{C}P^n$ we have an action of the group $A = SL_{n+1}\mathbb{C}$ on P_1^n and so A also acts on measures μ on $P_1^n \subset V'$.

Convex hull lemma. (See [B-L-Y].) *Let μ be a measure (with finite total mass) whose support is not contained in a hyperplane in $P_1^n = \mathbb{C}P^n$. Then the image of the map $A \to V'$ for $a \mapsto C(a\mu)$ equals the interior of the convex hull of P_1^n in V' (where C denotes the center of gravity of a measure on $V' \supset P_1^n$).*

The proof is obtained by generalizing the previous argument for $S_1^2 = P_1^1 \subset V' = \mathbb{R}^3$.

The spectrum of $\mathbb{C}\mathbf{P}^n$. Start with the standard $SU(n+1)$ invariant metric on $\mathbb{C}P^n$ normalized by the condition Area$(\mathbb{C}P^1) = \pi$ (as $\mathbb{C}P^1 = S_1^3/S^1 = S_{1/2}^2$). Then

$$\lambda_1(\mathbb{C}P^n) = 4(n+1)\ .$$

To see that we identify $\mathbb{C}P^n$ with P_1^n, or, more precisely, we use an $SU(n+1)$-equivariant map $\varphi : \mathbb{C}P^n \to P_1^n$ for the natural action of $SU(n+1) \subset GL_{n+1}\mathbb{C}$ on $P_1^n \supset V' \supset V$. This action has a unique fixed point in V', which we take for the origin and so we may speak of linear functions f on $V' \supset P_1^n$. These will be used as test functions for λ_1 but before doing this we observe that there exists a natural $SU(n+1)$-invariant positive definite scalar product on V' corresponding to the bilinear form Trace XY on Hermitian operators on $\mathbb{C}^{n+1}$ which are identified with Hermitian forms in the usual way. (In fact, this scalar product is uniquely characterized, up to a normalizing constant, by the $SU(n+1)$-invariance.) We renormalize this scalar product in order to make the map $\varphi : \mathbb{C}P^n \to P_1^n$ an isometry (this is possible as the $SU(n+1)$-invariance makes the metric in $\mathbb{C}P^n$ unique up to scale) and observe (this is a two line computation) that for this scalar product each $p \in P_1^n \subset V'$ has $\|p\|^2 = \langle p, p\rangle = n/2(n+1)$. Thus an orthonormal basis of linear functions $f_1, \ldots, f_\nu$ on $P_1^n = \mathbb{C}P^n$ satisfies

$$\sum_{i=1}^{\nu} f_i^2 = n/2(n+1) \tag{$*$}$$

while the differentials of df_i satisfy

$$\sum_{i=1}^{\nu} \|df_i\|^2 = 2n = 2 \dim_{\mathbb{R}} P_1^n . \qquad (**)$$

(Notice that $\nu = n(n+2)$.) It follows, as in the case $P_1^2 = S_1^2$, that each f_i has

$$\|df_i\|_{L_2}^2 = 4(n+1)\|f_i\|_{L_2}^2$$

and so $\lambda_1(\mathbb{C}P^n = P_1^n) \leq 4(n+1)$. Again, as in the case of S_1^2, one needs an extra argument to show that $\lambda_1 \geq 4(n+1)$ but we do not truly need this for our purposes. (See [B-L-M].)

Upper bound for λ_1 of $(\mathbb{C}\mathbf{P}^n, \mathbf{g})$. Let g be an arbitrary *Kähler* metric on $\mathbb{C}P^n$ normalized by the condition $\operatorname{Area} \mathbb{C}P^1 = \pi$ which is equivalent to $\operatorname{Vol}(\mathbb{C}P^n, g) = \operatorname{Vol} P_1^n = \pi^n/n!$.

Generalized Hersch Theorem. (See [B-L-Y].) *The first eigenvalue is bounded from above by*

$$\lambda_1(\mathbb{C}P^n, g) \leq \lambda_1(P_1^n) = 4(n+1) .$$

Proof. We start with some holomorphic map $\varphi_i : (\mathbb{C}P^n, g) \to P_1^n$ and then compare it with some (complex) projective transformation a which makes $C(a\varphi_*\mu_g) = 0$. So again we can use the φ-pull-backs of f_i as test functions on $(\mathbb{C}P^n, g)$ as they have $\int\limits_{\mathbb{C}P^n} f_i \, d\mu_g = 0$. These functions still satisfy the identity $(*)$ while the pointwise equality $(**)$ may, in general, fail. Yet, the integrated equality $(**)$ remains valid, i.e.

$$\sum_{i=1}^{\nu} \|d f_i\|_{L_2}^2 = 2n \operatorname{Vol} P_1^n \qquad (***)$$

since the energy of our (holomorphic !) map $\varphi : (\mathbb{C}P^n, g) \to P_1^n$ equals that of the identity map $P_1^n \to P_1^n$. It follows from the integrated $(*)$ and $(***)$ that for some i_0 the function f_{i_0} has

$$\|d f_{i_0}\|_{L_2}^2 \geq 4(n+1)\|f_{i_0}\|_{L_2}^2$$

which implies the desired inequality $\lambda_1 \leq 4(n+1)$. Q.E.D.

Remarks. (a) Bourguignon, Li and Yau establish (by their version of Hersch method) an upper bound on $\lambda_1(V)$ for Kähler manifolds V of (real) dimension $2n$ which admit holomorphic immersions $\varphi : V \to \mathbb{C}P^N$. Namely, they prove that

$$\lambda_1(V) \leq \frac{4n(N'+1)}{N'} d(\varphi),$$

where $N' \leq N$ denotes the (complex) dimension of the minimal projective subspace $(= \mathbb{C}P^{N'})$ in $\mathbb{C}P^N$ containing $\varphi(V)$ and $d(\varphi)$ is the following homological invariant of φ

$$d(\varphi) = \int\limits_V \varphi^*(\omega_0) \wedge \omega^{n-1} \Big/ \int\limits_V \omega^n,$$

where ω is the Kähler form of V and ω_0 is that of $\mathbb{C}P^N$. For example, if φ is *isometric* or, more generally, $\varphi^*(\omega_0)$ is cohomologous to ω, then $d(\varphi) = 1$. (See [B-L-Y].)

(b) An earlier estimate for $\lambda_1(V)$ (which gives, for example, $\lambda_1(\mathbb{C}P^n) \leq 8n$) was established by Li and Yau with a use of meromorphic maps $V \to \mathbb{C}P^1$ (compare Remark (c) following Nilpotency Theorem).

(c) The notion of the spectrum extends to singular subvarieties in $\mathbb{C}P^N$ (see $[\text{Gro}]_{5,9}$) and the estimates for λ_i remain valid in the singular case (compare [Yosh]).

(d) Lower bounds on $\lambda_i(V)$ for $V \subset \mathbb{C}P^N$ are indicated in $[\text{Gro}]_9$ but these are firmly established only for $\dim_{\mathbb{R}} V = 2$. (In general, the complex case we study here must be easier than the real case discussed in $[\text{Gro}]_9$, since the complex subvarieties in $\mathbb{C}P^N$ are *minimal* and thus satisfy certain isoperimetric inequalities.)

Upper bound on λ_k for surfaces. We start with an appropriate version of *Besikovič covering lemma.* We call a function f on a metric space *standard* if it has the support in an ε-ball B and equals in this ball to

$$f(x) = \varepsilon^{-1}(\varepsilon - \text{dist}(x, \text{ center}(B))) .$$

Lemma.* *Let V be a compact Riemannian manifold and μ a measure on V of total mass one. Then for each $k = 1, 2, \ldots$ there exist $k+1$ standard functions $f_0, \ldots, f_k$ with disjoint supports, such that*

$$\int_V f_k(x) d\mu \geq \text{const}_V^+ / k, \quad \text{for } \text{const}_V^+ > 0 .$$

This is standard. Notice that the constant const_V^+ only depends on $\dim V$, and the lower bound on the sectional curvature of V and $\text{Diam}\, V$. (Compare $[\text{Gro}]_5$ where there is a stronger unjustified claim concerning the Ricci curvature.)

Theorem. *Let V be a compact Riemann surface which admits a holomorphic map $\varphi : V \to S_1^2$ of topological degree d. Then the k-th eigenvalue of V satisfies*

$$\lambda_k / k \leq \text{const}(\text{Area}\, V)^{-1} d ,$$

for some universal const ≤ 100 .

Proof. We may assume $\text{Area}\, V = 1$ and apply the above lemma to the push-forward μ_* of the Riemannian measure μ of V to the unit sphere S_1^2. Then we pull-back to V the $(k+1)$ standard functions f_i on S_1^2 provided by the Lemma and observe that

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* N. Korevaar pointed out to me that this lemma, with our notion of "standard function" is false. A correct refinement of Besikovic lemma needed for the upper bound on the spectrum of surfaces was discovered by Korevaar, see N. Korevaar, Upper bounds for eigenvalues of conformal metrics, Journ. of Diff. Geom. 37 : 1 (1993), pp. 73–94. On the other hand, our second approach to the upper bound on the spectrum uses Kato inequality rather than a covering argument (see below).

(i) These pull-backs, say f_i^*, are mutually orthogonal on V, as the functions f_i on S_1^2 have disjoint supports.

(ii) $\int_V f_i^* \geq \text{const}^+ / k$, as follows from the assertion of the Lemma.

(iii) $\int_V \|df_i^*\|^2 \leq \text{const}' \, d$, because the standard functions f on S_1^2 have $\int_{S_1^2} \|df\|^2 \approx 1$ and the density $\|df\|^2 ds$ is (locally) conformally invariant.

It follows that the functions f_i^* may serve as test functions for λ_k in our definition of λ_k. Q.E.D.

Remarks. (a) The above argument is quite general and can be used for *non-linear spectra* of V in the sense of [Gro]$_5$. On the other hand this argument never gives sharp bounds on λ_i, not even for λ_1, where Hersch theorem is sharp.

(b) The bound on λ_1 in terms of d is due to Yang and Yau who use Hersch argument. (See [Y-Y].)

Bound on deg V. Define *the degree* of a Riemann surface V, denoted $\deg V$, as the minimum of degrees of holomorphic maps $V \to S^2$. This $\deg V$ depends only on the complex structure of V (e.g. it equals one for genus $V = 0$ and it equals two in the hyperelliptic case).

Now, let $V_j \to V$ be finite j-sheeted coverings of V which converge to the universal covering of V for $j \to \infty$ (where our j's form some *subsequence* of $1, 2, 3 \ldots$).

Surface Coverings Theorem. *If genus* $V \geq 2$ *then* $\deg V_j \to \infty$ for $j \to \infty$.

Proof. Let $N_\varepsilon(V_j)$ denote the number of the eigenvalues of V_i which are $\leq \varepsilon$. Since the spectrum of the universal covering of V does not contain zero (as genus ≥ 2), there exists $\varepsilon > 0$, such that $N_\varepsilon(V_j)/j \to 0$ for $j \to \infty$. On the other hand,

$$\lambda_k(V_j)/k \leq \text{const}_V \deg V_j / j$$

as $\text{Area}\, V_j = j\, \text{Area}\, V$. Therefore, an universal bound $\deg V_j \leq d$ would make $\lambda_k \leq \varepsilon$ for $k \approx \varepsilon j$. This is equivalent to $N_\varepsilon(V_j) \gtrsim \varepsilon j$ for $j \to \infty$ and a fixed $\varepsilon > 0$, which is incompatible with the above relation $N_\varepsilon(V_j)/j \to 0$.

Remark. One knows (this is non-trivial) that there are certain infinite sequences of coverings V_j (associated to congruence subgroups) which have $\lambda_1(V_j) \geq \varepsilon > 0$ for $j \to \infty$ and for these one necessarily has $\deg V_j \geq \varepsilon' j$ for $\varepsilon' > 0$ and all j.

Question. What is the actual behavior of $\deg V_j$ for the coverings V_j of a fixed Riemann surface V?

Bounds on λ_k for $\dim V > 2$. The above elementary "covering of S^2" argument can be applied to Kähler manifolds V of dimension $2n > 2$ via meromorphic maps $\varphi : V \to S^2$ (see remark (c) following Nilpotency Theorem) but this gives us a "wrong" asymptotic bound, namely $\lambda_k \gtrsim k$ instead of $\lambda_k \gtrsim k^n$. Sharpening the covering techniques for $2n > 2$ is a tricky matter and so we suggest here a less elementary and more analytic approach which appeals to the *Kac-Feynman formula* (or *Kato's inequality*) for some heat equation and which leads to the following estimate of some average of λ_i (which is weaker than the individual estimate).

Theorem. *Let V be a closed $2n$-dimensional Kähler manifold which admits a holomorphic map φ to $\mathbb{C}P^n$ such that the cohomology class of the Kähler form ω of V satisfies $[\omega] = \delta[\omega_0^*]$, where $\delta > 0$ is a constant and ω_0^* is induced by φ from the standard symplectic form ω_0 on $\mathbb{C}P^n$. Then the eigenvalues λ_i of V satisfy for all $t > 0$*

$$\sum_{i=0}^{\infty} \exp -t\lambda_i \geq \mathrm{const}_n(\delta/t)^n, \qquad (+)$$

where $\mathrm{const}_n > 0$ *is a universal constant.*

Proof. We may assume (by scaling ω) that $\delta = 1$ and we denote by L the holomorphic line bundle over V induced from the standard bundle over $\mathbb{C}P^n$ having curvature ω_0. It is classical ($\partial\bar{\partial}$-Lemma) that L admits a metric whose curvature equals ω and it is clear from the construction that the (k-th tensorial power) bundle L^k has at least $k^n/n!$ independent holomorphic sections (coming from $\mathbb{C}P^n$). Denote by Δ_{L^k} the (rough) Laplace operator on sections of L^k and observe that $\Delta_{L^k}(\psi) = k\psi$ for all holomorphic sections ψ of L_k. (In fact, the classical Kodaira formula says that $\Delta_L = 2\bar{\partial}_L^*\bar{\partial}_L + \mathrm{Trace}\,\omega_L$, where L is an arbitrary Hermitian line bundle with curvature ω_L and where the trace is taken against the Kähler form of V, see $[\mathrm{Dem}]_{1,2}$ for instance.) It follows that the number k appears as an eigenvalue of Δ_{L^k} with multiplicity $\approx k^n$.

Now we recall the following classical Kato inequality (or Kac-Feynman formula) which relates the eigenvalues λ_i of V with the eigenvalues $\widetilde{\lambda}_i$ of a (rough) Laplace operator on a 2-dimension bundle,

$$\sum_{i=0}^{\infty} \exp -\widetilde{\lambda}_i t \leq 2 \sum_{i=0}^{\infty} \exp -\lambda_i t, \; t > 0 \;.$$

We apply this inequality to each bundle L^k, where we have $\approx k^n$ eigenvalues of size k, which makes

$$2\sum_{i=0}^{\infty} \exp -\lambda_i t \geq k^n \exp -tk$$

for all $k = 1, 2, \ldots$, and $t > 0$, and which implies (+) by a straightforward computation. Q.E.D.

Remark. One can exclude the (standard) space $\mathbb{C}P^n$ from the statement of the theorem by requiring the class $\delta^{-1}[\omega]$ to be integral and the line bundle corresponding to $\delta^{-1}[\omega]$ to be "sufficiently ample".

Estimating δ for coverings $\mathbf{V_j \to V}$. Let $V_j \to V$ be a sequence of finite j-sheeted (unramified) coverings which converge to an infinite regular (e.g. universal) covering $V_\infty \to$

V with Galois group Γ. (Here $j \to \infty$ but we do not insist on each integer appearing among our j's.) Denote by $\delta_j = \delta(V_j)$ the maximal integer for which V_j satisfies the assumption of the above theorem. Namely, V_j admits a holomorphic map $\varphi_j : V_j \to \mathbb{C}P^n$, such that the form ω_j on V_j induced from the original form ω on V satisfies

$$[\omega_j] = \delta_j[\omega_0^*] \ ,$$

where ω_0^* is the form on V_j induced by φ_j.

Nilpotency Theorem. *If $\delta_j \gtrsim j^{\frac{1}{n}}$, i.e. $j\,\delta_j^{-n}$ remains bounded for $j \to \infty$, then the group Γ contains a nilpotent subgroup of finite index.*

Proof. Let $E_j(t)$ be the heat operator (kernel) on V_j that is $E_j(t) = \exp -t\Delta_j$ where Δ_j is the Laplace operator on V_j. Then

$$\operatorname{Trace} E_j(t) = \sum_{i=0}^{\infty} \exp -t\lambda_i(V_j) \ . \tag{$*$}$$

The sequence $E_j(t)$ converges, for each fixed t, to the heat operator $E_\infty(t)$ on V_∞ which is given by a function $E_\infty(t,x,y)$ for $x,y \in V_\infty$. For every fixed $x \in V_\infty$ the function $E_\infty(t,x,x)$ decays for $t \to \infty$ (since Γ is infinite) and the rate of the decay is essentially independent of x (since $V = V_\infty/\Gamma$ is compact). This rate is related to the growth of Γ by the following

Theorem of Varopoulos (see $[\mathrm{Var}]_{1,2}$). *If $E_\infty(t,x,x) \geq \varepsilon t^m$ for fixed m and $\varepsilon > 0$ and large $t \to \infty$, then Γ has polynomial growth of degree $2m$.*

This result combines with the polynomial growth theorem (see $[\mathrm{Gro}]_2$) and yields

Corollary. *If $E_\infty(t) \geq \varepsilon t^m$, then Γ admits a nilpotent subgroup of finite index.*

We bring into the game the heat kernel $E_j(t,x,y)$ on V_j and recall that

$$\operatorname{Trace} E_j(t) = \int_{V_j} E(t,x,x)dx \ .$$

Since $E_j(t) \to E_\infty(t)$ we can bound this trace for large $j \to \infty$ and fixed t by

$$\operatorname{Trace} E_j(t) \lesssim (\operatorname{Vol} V_j) E_\infty(t,x,x) \ . \tag{$**$}$$

On the other hand, the inequality $(+)$ gives us an upper bound on this trace, namely

$$\operatorname{Trace} E_j(t) = \sum_{i=0}^{\infty} \exp -t\lambda_i(V_j) \geq \varepsilon' j t^{-n}$$

for $\varepsilon' > 0$ since by our assumption δ^n grows as fast as j.

Finally we observe that $\mathrm{Vol}\, V_j \approx j$ and conclude that $E_\infty(t,x,x)$ decays no faster than t^n. Now the Corollary to Varopoulos theorem applies. Q.E.D.

Remarks. (a) Since V is Kähler, it seems hard for Γ to be (almost) nilpotent without being (almost) Abelian. The only feasible possibility comes from (holomorphic maps of V onto) Riemann surfaces. In particular, if $\Gamma = \pi_1(V)$ one expects "nilpotent $\Rightarrow$ Abelian" and the proof seems within reach. Granted that, one should be able to classify coverings $V_j \to V$ with $\delta_j \approx j^{\frac{1}{n}}$, where the basic example is V "sufficiently trivially" holomorphically fibered over a complex torus.

(b) For certain groups Γ one exercises an additional control over the spectrum of V_j and/or over their Picard numbers. For example, if Γ (or $\pi_1(V)$) is *Kazhdan T*, then $\lambda_1(V_j)$ remains $\geq \varepsilon > 0$ for $j \to \infty$. (The same is valid for congruence coverings V_j for arithmetic non-T groups Γ.) Also, certain Γ and V, e.g. irreducible locally symmetric spaces V_j of higher rank, have Picard number one and then every two Kähler forms are cohomologically proportional (where one of the forms may be singular, e.g. induced by a general holomorphic map). In such cases one can significantly strengthen the nilpotency theorem.

(b′) **Example.** *Suppose Γ is Kazhdan T and $\mathrm{Pic}(V_j) = 1$. Then the minimal possible (topological) degree $d_j = d(V_j)$ of a surjective holomorphic map $V_j \to \mathbb{C}P^n$ satisfies*

$$d_j \geq \mathrm{const}\, j \ ,$$

where, recall, j is the number of sheets of V_j over V.

(c) We have already mentioned that an upper bound on λ_k can be obtained with meromorphic maps into $\mathbb{C}P^1$ (which can be represented by linear systems on the manifold in question). Let us spell this out. Define $\deg\varphi$ for a meromorphic map $\varphi: V \to \mathbb{C}P^1$ as the normalized volume of the pull-back $W = \varphi^{-1}(x)$ for a generic $x \in \mathbb{C}P^1$, that is

$$\deg\varphi = (n-1)!\,\mathrm{Vol}\, W = \int_W \omega^{n-1} \ .$$

This can be expressed homologically by $\langle [\omega]^{n-1},\ [W]\rangle$ or by $\langle [\omega]^{n-1}\, v[\omega_0^*\},\ [V]\rangle$ (compare [L-Y]).

Now we can bound the eigenvalues of V by using pull-backs of functions $f: \mathbb{C}P^1 \to \mathbb{R}$ as test functions on V, since such pull-backs $f^* = \varphi \circ f$ (obviously) satisfy

$$\|df^*\|^2_{L_2} = (\mathrm{Vol}\, W)\|df\|^2_{L_2} \ .$$

Thus we see that *if V admits a meromorphic map $\varphi: V \to \mathbb{C}P^1$ of degree $\deg\varphi = d$ then*

$$\lambda_k(V) \geq \mathrm{const} \bullet (kd)/\big((n-1)!\,\mathrm{Vol}\, V\big)$$

for some universal const > 0. (This result for λ_1 is due to Li and Yau, [L-Y].)

(c′) **Example.** If $V_j \to V$ are our coverings, where Γ is Kazhdan T, and $W_p(j) \subset V_j$ are non-trivial linear systems of divisors parametrized by $p \in \mathbb{C}P^1$, then $\mathrm{Vol}\, V / \mathrm{Vol}\, W_p(j)$ remains bounded as $j \to \infty$.

Notice that V_j may have individual (rigid) divisors $W_j \subset V_j$ with $\mathrm{Vol}\, V_j / \mathrm{Vol}\, W_j \to \infty$ for $i \to \infty$.

(c″) The version of the nilpotency theorem using maps to $\mathbb{C}P^1$ rather than to $\mathbb{C}P^n$ needs the bound $\mathrm{Vol}\, W_j(p) \leq$ const for $j \to \infty$. This is a very strong condition implying that our coverings are unduced from certain coverings of surfaces and everything reduces to Surface Coverings Theorem.

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