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SINGULARITIES, EXPANDERS AND TOPOLOGY OF MAPS. PART 2: FROM COMBINATORICS TO TOPOLOGY VIA ALGEBRAIC ISOPERIMETRY

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Abstract. We find lower bounds on the topology of the fibers $F^{-1}(y) \subset X$ of continuous maps $F : X \to Y$ in terms of combinatorial invariants of certain polyhedra and/or of the cohomology algebras $H^*(X)$. Our exposition is conceptually related to but essentially independent of Part 1 of the paper.

Contents

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1 Definitions, Problems and Selected Inequalities

The simplest measure of the fiber-wise complexity of a continuous map between equidimensional topological spaces, $F : X \to Y$, is the maximal cardinality of a fiber, $\sup_{y\in Y} |F^{-1}(y)|$. The problem we address is that of evaluating the *cardinality of* X *over* Y, defined as

$$\inf_{F\in\mathcal{F}} \sup_{y\in Y} \left|F^{-1}(y)\right| \quad \text{for a given class } \mathcal{F} \text{ of maps } F : X \to Y\,.$$

When X is a polyhedron, it is better to deal with the maximal number of closed cells in X intersected by a fiber, since the latter may only *increase* under uniform limits of maps.

Next, we extend the setting to maps from k-dimensional X to $(k+l)$-dimensional Y with a distinguished family of l-dimensional subspaces $A \subset Y$ where we search for an A_0 which is intersected by a maximal number of k-cells of X mapped to Y.

Finally, we establish lower bounds on the (co)homologies of fibers of continuous maps rather than on their cardinalities.

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1.1 $\boldsymbol{\Delta}$-Inequalities for the multiplicities of maps of the $\boldsymbol{n}$-skeleton of the $\boldsymbol{N}$-simplex to $\mathbb{R}^n$. Let X be a finite simplicial polyhedron, where the set of the k-faces (simplices) Δ^k in X is denoted by $\{\Delta^k\} = \{\Delta^k\}(X)$ and $|\{\Delta^k\}|$ denotes the cardinality of this set.

Let $F : X \to \mathbb{R}^n$ be a continuous (e.g. face-wise affine) map and let A be an affine subspace in $\mathbb{R}^n$. Denote by

$$\left|A \pitchfork_F \{\Delta^k\}\right|$$

the number of the closed k-faces Δ^k of X such that the image $F(\Delta^k) \subset \mathbb{R}^n$ intersect A.

We want to identify and/or construct polyhedra X of "sufficient combinatorial complexity", for which

> *every continuous (or, at least, every face-wise affine) map F from X to $\mathbb{R}^n$ admits an $(n-k)$-dimensional affine subspace $A_0 = A_0^{n-k} \subset \mathbb{R}^n$ (which depends on F), such that the ratio*
>
> $$\frac{|A_0 \pitchfork_F \{\Delta^k\}|}{|\{\Delta^k\}|}$$
>
> *is "reasonably large".*

A particular case of interest is where $k = n$, where $A = a$ are points in $\mathbb{R}^n$ and where the first result (I am aware of) of this kind (apart from Tverberg's theorem, see 2.8), due to Imre Barany [Ba], concerns *affine* maps of *the N-simplex $X = \Delta^N$* to $\mathbb{R}^n$, where, observe, the number of the n-faces is $|\{\Delta^n\}| = \binom{N+1}{n+1}$.

Barany's affine $\boldsymbol{\Delta}$-inequality.

> *Let $F : \Delta^N \to \mathbb{R}^n$ be an affine map. Then there exists a point $a_0 = a_0(F) \in \mathbb{R}^n$ such that the number M_0 of the closed n-faces Δ^n of the simplex Δ^N for which the image $F(\Delta^n) \subset \mathbb{R}^n$ contains a_0 is bounded from below by*
>
> $$M_0 = \left|a_0 \pitchfork_F \{\Delta^n\}\right| \geq b_{\mathrm{aff}}(n) \cdot \left|\{\Delta^n\}\right| \cdot \big(1 - O(1/N)\big),$$
>
> *for*
>
> $$b_{\mathrm{aff}}(n) \geq 1/(n+1)^{n+1}. \qquad [\Delta \to \mathbb{R}^n]_{\mathrm{aff}}$$

Equivalently,

> *given a probability measure μ on $\mathbb{R}^n$, there exists a point $a_0 \in \mathbb{R}^n$ such that the convex hull of μ-randomly chosen $n+1$ points in $\mathbb{R}^n$ contains a_0 with probability $\geq 1/(n+1)^{n+1}$.*

The true value of b_{aff} is known only for $n = 1$, where it is, obviously, $1/2$, and for $n = 2$, where the lower bound $b_{\mathrm{aff}}(2) \geq 2/9$ is due to Boros and Furedi [BorF], while the examples constructed in [BuMN] show that $b_{\mathrm{aff}}(2) \leq 2/9$; moreover it is shown in [BuMN] that $b_{\mathrm{aff}}(n) \lesssim e^{-n}$ for large n.

Barany's bound $b_{\mathrm{aff}}(n) \geq 1/(n+1)^{n+1}$ was improved by a polynomial factor in [W] (also see [DeHST]), but an exponential lower bound, $b_{\mathrm{aff}}(n) \geq \beta^n$ for some $\beta > 0$, remains problematic. (I am indebted to Janoch Pach who introduced me to the combinatorial results around the Barany inequality.)

Topological Δ-inequality.

Let $F : \Delta^N \to \mathbb{R}^n$ be a continuous map. Then there exists a point $a_0 \in \mathbb{R}^n$, such that the number M_0 of the closed n-faces Δ^n of the simplex Δ^N for which the image $F(\Delta^n) \subset \mathbb{R}^n$ contains a_0 is bounded from below by

$$M_0 = \big|a_0 \pitchfork_F \{\Delta^n\}\big| \geq b_{\mathrm{top}}(n) \cdot \big|\{\Delta^n\}\big| \cdot \big(1 - O(1/N)\big)\,,$$

for

$$b_{\mathrm{top}}(n) \geq \frac{2n}{(n+1)(n+1)!} \sim e^n/(n+1)^{n+1}. \qquad [\Delta \to \mathbb{R}^n]_{\mathrm{top}}$$

(Strangely, this is stronger than the present day bound on b_{aff} obtained by the traditional combinatorial techniques. A priori, the constant b_{top} which serves *all continuous* maps may be only *smaller* than the constant b_{aff} responsible only for the *face-wise affine* maps. The two constants happen to be equal for $n = 1, 2$, but it is unknown if $b_{\mathrm{top}}(n) < b_{\mathrm{aff}}(n)$ for any $n \geq 3$.)

The proof of the topological Δ-inequality depends on "combinatorial filling" in the *semisimplicial spaces of cycles of measurable chain complexes* (see 2.1–2.6) which is similar to, but formally independent of, the filling techniques in the Riemannian geometry.

Interestingly enough this quite formal "combinatorial filling" yields lower bounds on the *waists* of Riemannian and (some sub-Riemannian) spaces under less restrictive assumptions than those required by the known geometric arguments (see 1.3, 2.7).

The (more or less standard) geometric ideas motivating our general constructions are presented in section 3 which includes:

- A separate proof of the inequality $[\Delta \to \mathbb{R}^2]_{\mathrm{aff}}$, which contains in a nutshell the idea of our general argument;
- Basic definitions of the Riemannian geometry and a brief overview of the isoperimetric/filling techniques aimed at non-experts;
- Basics on isoperimetry in graphs and cardinalities of graphs over graphs.

1.2 $(n-k)$-Planes crossing many k-simplices in $\mathbb{R}^n$. Denote by $b_{\mathrm{aff}}(n, k)$, where $n \geq k$, the maximal number, such that

every probability measure μ in $\mathbb{R}^n$, admits an $(n-k)$-dimensional affine subspace $A_0 = A_0^{n-k}(\mu)$ such that the μ-probability of the convex hull of $k+1$ points in $\mathbb{R}^n$ to meet A_0 is bounded from below by

$$\mu^{\otimes(k+1)}\big(A_0 \pitchfork \{\Delta^k\}\big) \geq b_{\mathrm{aff}}(n, k)\,.$$

It is obvious that $b_{\mathrm{aff}}(n, 1) = 1/2$ for all n, that $b_{\mathrm{aff}}(n, k)$ is *monotone increasing* in n and that $b_{\mathrm{aff}}(n, n)$ equals the (optimal) Barany constant $b_{\mathrm{aff}}(n)$.

It is also not hard to see that if μ is *round*, i.e. spherically symmetric, e.g. the Gaussian measure $dy\big(\exp\|y\|^{-2}/\int \exp\|y\|^{-2}dy\big)$, and $\mu(\{0\}) = 0$ (no atom at the origin), then

$$\mu^{\otimes(k+1)}\big(A \pitchfork \{\Delta^k\}\big) \leq 2^{-k}$$

for all $A^{n-k} \subset \mathbb{R}^n$. (In fact, *every* μ with a continuous density function satisfies this inequality for all A, where the equality holds if and only if the radial projection of μ from some point $a \in A$ is symmetric in a [WW].)

Thus,

$$b_{\rm aff}(n,k) \le 2^{-k} \quad \text{for all } n \ge k\,.$$

On the other hand, it is shown by Bukh, Matoušek and Nivasch [BuMN] that

if $k=2$, then

$$b_{\rm aff}(n,k) \ge \frac{1}{4}\left(1-\frac{1}{(2n-1)^2}\right).$$

Our result in this regard applies to all n and k but it is not so sharp.

$[k \pitchfork n-k]$-Inequalities.

There exists a universal positive constant $B_{\rm aff} \le 2$, such that if $n \ge \lambda \cdot (k+1)^2$ for some $\lambda \ge 1$, then

$$b_{\rm aff}(n,k) \ge \frac{1}{2^k}\left(1-\frac{B_{\rm aff}}{\lambda}\right). \qquad [n \gg k^2]_{\rm aff}$$

There exists a strictly positive function, $\beta_{\rm aff}(\varepsilon)>0$ for $\varepsilon>0$, such that if $n \ge (1+\varepsilon)\cdot k$, then

$$b_{\rm aff}(n,k) \ge \beta_{\rm aff}(\varepsilon)^{k+1}. \qquad [n \gg k]_{\rm aff}$$

In other words, $b_{\rm aff}((1+\varepsilon)k,k)$ may decay *at most exponentially* in k for every given $\varepsilon>0$ (the behavior of the exponent for $\varepsilon \to 0$ remains unclear) and $b_{\rm aff}(\lambda k^2,k)/2^{-k-1}$ converges to 1 for $\lambda \to \infty$ (which is, essentially, sharp).

In fact, more precise inequalities for $n \ge 2k-1$ easily follow from *the Radon theorem* and for $k+1 \le n \le 2k-2$ from *the Tverberg theorem* (see 2.8).

Remarks and Questions. (a) The inequality $[n \gg k^2]_{\rm aff}$ says that an affine projection of μ onto a k-dimensional subspace is similar in some respect to a round measure. In fact, a recent *theorem by Klartag* [Kl] says (among other things) that

Given a probability measure μ with a continuous density function on $\mathbb{R}^n$ and a number $k \ll n$, there exists a surjective affine map $P=P_\mu : \mathbb{R}^n \to \mathbb{R}^k$ such that the pushforward measure $P_*(\mu)$ is ε-round (in the natural sense) where $\varepsilon \to 0$ for $n \to \infty$.

This provides an alternative proof of a version of $[n \gg k^2]_{\rm aff}$.

(b) Is there a topological version of, say, $[n \gg k^2]_{\rm aff}$?

It is even unclear if the constant $b_{\rm top}(n,k)$ corresponding to intersections of affine planes $A^{n-k} \subset \mathbb{R}^n$ with *curve-linear* simplicial spans of $(k+1)$-tuples of points in $(\mathbb{R}^n,\mu)$ becomes any greater than $b_{\rm top}(k,k)=b_{\rm top}(k)$ no matter how large n is compared to k.

(c) Is there a "good" asymptotic lower bound on $2^{-k-1}-b_{\rm aff}(n,k)$ for $n \to \infty$? (Some bound of this kind follows from [WW].)

1.3 Riemannian and sub-Riemannian waist inequalities.

Let $S^m \subset \mathbb{R}^{m+1}$ be the round Euclidean sphere and let $\Delta(S^m)$ be the simplex on the vertex set S^m, i.e. the i-faces of $\Delta(S^m)$ are represented by $(i+1)$-tuples of points in S^m.

Take the barycenters in $\mathbb{R}^{m+1} \supset S^m$ of $(n+1)$-tuples of points in S^m and then radially project these back to S^m. Thus, we obtain, for each $n \le m$, a map from (the geometric realization of) the n-skeleton of $\Delta(S^m)$, into S^m, say

$B_n : \bigcup_{\Delta^n}(\Delta(S^m)) \to S^m$, which is defined almost everywhere for the spherical measure ds on S^m (or, rather for the Cartesian product of the power measure $ds^{\otimes(n+1)}$ times the measure on the simplex Δ^n).

Denote by $\mathrm{subvol}^{\pitchfork}_{\Delta^n}(W)$, $W \subset S^m$, the probability that the convex geodesic n-simplex Δ^n spanned by an $(n+1)$-tuple of points in S^m (i,e, the B_n-image of an n-face of the simplex $\Delta(S^m)$) intersect W, where the probability refers to the normalized spherical measure.

Let vol_i denote the i-dimensional Hausdorff measure in S^m and observe that $\mathrm{subvol}^{\pitchfork}_{\Delta^n}(W) \le 2^{n+1}\,\mathrm{vol}_{m-n}(W)/\,\mathrm{vol}_{m-n}(S^n)$ for all closed subsets $W \subset S^m$, with the equality for subsets which meet every Δ^n at (at most) one point by the *Crofton–Wendel formula* (see 2.7).

The inequality $[\Delta \to \mathbb{R}^n]_{\mathrm{top}}$ (trivially) implies such inequality for the n-skeleton of the simplex spanned by S^m. For instance, if we apply $[\Delta \to \mathbb{R}^n]_{\mathrm{top}}$ to the composition of B_n with a continuous map $f : S^m \to \mathbb{R}^n$, we conclude to the following

Spherical waist inequality.

Let $f : S^m \to \mathbb{R}^n$ be a continuous map. Then there exists a point $a_0 \in \mathbb{R}^n$, such that

$$\mathrm{subvol}^{\pitchfork}_{\Delta^n}\left(f^{-1}(a_0)\right) \ge s_\triangle(n) \ge b_{\mathrm{top}}(n) \ge 2n/(n+1)(n+1)!\,. \qquad [\mathrm{subvol}^{\pitchfork}]$$

Therefore (see 2.7)

the $(m-n)$-dimensional Hausdorff measure of $f^{-1}(a_0)$ is bounded from below by

$$\mathrm{vol}_{m-n}\left(f^{-1}(a_0)\right)/\,\mathrm{vol}(S^{m-n}) \ge s_{\mathrm{Hau}}(n,m)$$

for

$$s_{\mathrm{Hau}}(n,m) \ge 2^{n+1}s_\triangle(n) \ge 2^{n+2}n/(n+1)(n+1)!\,. \qquad [S^m \to \mathbb{R}^n]$$

Remark. It is shown in [Gr6] that $s_{\mathrm{Min}}(n,m) = 1$ where the Hausdorff measure is substituted by the *Minkowski* volume [Gr6].

Also $\sup_a \mathrm{vol}_{m-n}(f^{-1}(a))/\,\mathrm{vol}(S^{m-n}) \ge 1$ for the *Hausdorff* vol_{m-n} if one assumes that the fibers $f^{-1}(a) \subset S^m$ of f are *rectifiable $\mathbb{Z}_2$-cycles continuous in $a \in \mathbb{R}^n$ with respect to the flat topology* (e.g. if f is a *generic* smooth map) by an old (unpublished) theorem of Almgren (see [Pi]).

It seems plausible that Almgren's method, combined with the geometric considerations in 3.4, and with the argument in 5.7 of [Gr7] for the special case $m-n=1$, [Gr6] would yield a sharp estimate of $s_{\triangle_\varepsilon}(n)$ where one counts intersections of W with ε-simplices for a small, yet positive ε. This would imply that $s_{\mathrm{Hau}}(n,m) = 1$ (compare with the last remark in 3.4).

On the other hand the true value of $s_\triangle(n)$ (for $n, m-n \ne 1$) remains unclear even hypothetically.

Contact waist inequality. Let $m = 2l+1$ and let the sphere $S^m \subset \mathbb{C}^{l+1}$ be endowed with the standard contact structure, i.e. the tangent $(m-1)$-plane field H (sub-bundle of the tangent bundle of the sphere) which is normal to the Hopfian circles (the orbits of the multiplications by complex numbers z with $\|z\| = 1$), and let the *CC-distance* be defined by the minimal length of curves between pairs of points where the curves are required to be *tangent to* H.

> *Let* $n \le l$ *and let* $f : S^m \to \mathbb{R}^n$ *be a continuous map. Then there exists a point* $a_0 \in \mathbb{R}^n$*, such that*
> $$\mathrm{vol}^{CC}_{m-n+1}\left(f^{-1}(a_0)\right) \ge \varepsilon(m) > 0\,, \qquad [CC]$$
> *where* $\mathrm{vol}^{CC}_{m-n+1}$ *denotes the* $(m-n+1)$*-dimensional Hausdorff measure for the CC-distance in* S^m.

In fact, the above proof for the Riemannian S^m extends to a class of *Carnot–Caratheodory* spaces which includes the contact S^{2l+1} (see 2.7).

Remarks. (a) The inequality $[CC]$ is qualitatively stronger than its Riemannian counterpart for maps $S^m \to \mathbb{R}^n$. For instance, the bound $\mathrm{vol}^{CC}_{m-n+1}(W) \ge 2\sqrt{\varepsilon}$ for a *smooth* $(m-n)$-dimensional submanifold $W \subset S^m$ with $\mathrm{vol}_{m-n}(W) \le 1/\sqrt{\varepsilon}$ implies that W *cannot* be approximately ε-tangent to H *outside* a subset $W_\varepsilon \subset W$ with $\mathrm{vol}_{m-n}(W_\varepsilon) < \sqrt{\varepsilon}$.

(b) If $n = 1$, then $[CC]$ follows from *the Pansu–Varopoulos isoperimetric inequality* and if f is a *generic smooth* map, then it follows for all n and m from *Robert Young's filling inequality* (see [Y]).

(c) A version of $[CC]$ applies to balls in *the Heisenberg group* H^{2l+1} and provides a lower bound on the *growth of lattices* $\Gamma \subset H^{2l+1}$ *over* $\mathbb{Z}^n$ (see 2.7, where this relative growth characterizes the growth of pullbacks of points in $\mathbb{Z}^n$ under Lipschitz maps $\Gamma \to \mathbb{Z}^n$).

1.4 Locally bounded 2-polyhedra with large cardinalities over $\mathbb{R}^2$. We shall prove (see 2.11–2.14) counterparts to $[\Delta \to \mathbb{R}^n]_{\mathrm{top}}$ for several classes of n-polyhedra smaller than the full n-skeleton of the N-simplex, but all of them have *local combinatorial degrees* growing with N.

Below is the only instance known to me of a (restricted version) of such inequality where $\deg_{\mathrm{loc}}(X) \le \mathrm{const}$.

There exists, for every positive integer N, a 2-dimensional simplicial polyhedron X_N with N vertices, where the local degrees, i.e. the numbers of faces adjacent to each vertex in every X_N, are bounded by a constant, say ≤ 1000, such that the following holds.

> *Let* $F : X_N \to \mathbb{R}^2$ *be a continuous map which is at most* k*-to-*1 *on each face and where the image of every* 1*-simplex is nowhere dense in* $\mathbb{R}^2$ (this is, probably, redundant). *Then there exists a point* $a_0 \in \mathbb{R}^2$*, which is contained in the* F*-images of at least* M *simplices of* X *for*
> $$M = \left|a_0 \pitchfork_F \{\Delta^2\}\right| \ge 0.001k^{-2}N\,. \qquad [X_N \to \mathbb{R}^2]$$
> *Moreover, one can make these* X_N *simply connected for infinitely many* N.

This inequality for the 2-skeleta of co-compact quotients Q of certain Bruhat–Tits buildings is derived (see 2.10) from *Garland's vanishing theorem* [G]; then the simply connected X_N come the same way as in 1.6(d) (see 2.10, 4.3).

It remains unclear if such an inequality ever holds for locally bounded 2-polyhedra with a constant independent of k and/or if there are similar families of n-dimensional polyhedra for $n > 2$.

On the other hand, it is shown in [Fox et al] that *randomly iterated* (see 2.14) and related n-dimensional polyhedra X_N with N vertices and $N_n \geq \mathrm{const}(n)N$ faces of dimension n satisfy

$$\sup_{a\in\mathbb{R}^n} \left|a \pitchfork_F \{\Delta^n\}\right| \geq \varepsilon N$$

for all *face-wise affine* maps $X_N \to \mathbb{R}^n$, where $\varepsilon = \varepsilon(n) > 0$ and where the lower bound on this ε may be as good as that for the full simplex Δ^{N-1}.

(This inequality fails to be true for *face-wise injective continuous* maps of random polyhedra $X(N)$ to $\mathbb{R}^n$; moreover, a significant amount of $X(N)$ admit *face-wise injective continuous* maps F where $\sup_{a\in\mathbb{R}^n} |a \pitchfork_F| \leq \mathrm{const} = \mathrm{const}(n, N_n/N)$.)

Conclude by noticing that n-polyhedra with large "cardinality over $\mathbb{R}^n$" need to have "significant local topological complexity". For example, we shall see in 2.9 that

> *every smooth n-dimensional manifold X admits a smooth generic map $F : X \to \mathbb{R}^n$, where*
> $$\sup_{y\in\mathbb{R}^n} \left|F^{-1}(y)\right| \leq 4n\,.$$

1.5 Separation inequalities in the N-torus for the homological μ_A-mass. When it comes to maps $F : X \to Y$ with $\dim(Y) < \dim(X)$ we measure the "topological size" of the fibers $F^{-1}(y) \subset X$ by their (Čech) (co)homology with coefficients in a (finite) field $\mathbb{F}$ as follows.

Given a subset in a topological space, $X_1 \subset X$, denote by

$$\mathrm{rest}^*_{/X_1} : H^*(X) \to H^*(X_1)$$

the restriction cohomology homomorphism. If $A \subset H^*(X)$ is a linear subspace, then

$$A|X_1 \text{ denotes the image } \mathrm{rest}^*_{/X_0}(A)$$

and

$$\mu_A(X_1) =_{\mathrm{def}} A \cap \mathrm{Ker}(\mathrm{rest}^*_{/(X\setminus X_0)})\,,$$

that is the kernel of the restriction of A to the complement $X \setminus X_0$.

The ranks of the $\mathbb{F}$-linear spaces $A|X_1$ and $\mu_A(X_1)$, denoted $|A|X_1|_{\mathbb{F}}$ and $|\mu_A(X_1)|_{\mathbb{F}}$, are, obviously, monotone under inclusions of subsets,

$$X_2 \supset X_1 \Rightarrow |A|X_2|_{\mathbb{F}} \geq |A|X_1|_{\mathbb{F}} \quad \text{and} \quad \left|\mu_A(X_2)\right|_{\mathbb{F}} \geq \left|\mu_A(X_1)\right|_{\mathbb{F}}\,, \tag{1}$$

and are thought of as "cohomology masses of subsets in X measured with A".

If $A = H^n = H^n(X) \subset H^*(X)$ for some $n = 1, 2, \ldots$, then every open subset $X_1 \subset X$ with boundary denoted ∂X_1 satisfies

$$|A|X_1|_{\mathbb{F}} \leq \left|\mu_A(X_1)\right|_{\mathbb{F}} + |A|\partial X_1|_{\mathbb{F}}\,. \tag{2}$$

Indeed, by the *excision property* of H^n (or by the additivity relation (3) in 4.1), the span of the kernels $K_1, K_2 \subset H^n$ of the restriction homomorphisms of H^n to X_1 and to $X_2 = X \setminus X_1$ equals the kernel K_∂ of the restriction to $\partial X_1 = \partial X_2$,

$$K_1 + K_2 = K_\partial\,;$$

therefore,

$$|A|X_1|_{\mathbb{F}} = |H^n/K_1|_{\mathbb{F}} \leq |H^n/K_\partial|_{\mathbb{F}} + |K_2|_{\mathbb{F}} = |A|\partial X_1|_{\mathbb{F}} + \left|\mu_A(X_1)\right|_{\mathbb{F}}\,.$$

(This inequality was erroneously stated for all A in the first draft of this paper as was pointed out to me by the referee.)

If two open subsets $X_1, X_2 \subset X$ cover X, then, obviously,

$$|A|X_1|_{\mathbb{F}} + \big|\mu_A(X_2)\big|_{\mathbb{F}} \geq |A|X|_{\mathbb{F}} = |A|_{\mathbb{F}} =_{\mathrm{def}} \mathrm{rank}_{\mathbb{F}}(A)\,. \tag{3}$$

Let $A = H^n$ and let $W = X \setminus (X_1 \cup X_2)$ be the complementary region ("wall") between two disjoint open subsets $X_1, X_2 \subset X$. Then (1)–(3) imply that

$$|A|_{\mathbb{F}} \leq |A|W|_{\mathbb{F}} + \big|\mu_A(X_1)\big|_{\mathbb{F}} + \big|\mu_A(X_2)\big|_{\mathbb{F}}\,. \tag{4}$$

Notice, finally (we shall not use it anywhere), that if $A \subset H^n(X;\mathbb{F})$, then

$$\big|\mu_A(X_1)\big|_{\mathbb{F}} \leq |A|X_1|_{\mathbb{F}} + \big|H^{n-1}(\partial X_1;\mathbb{F})\big|_{\mathbb{F}} \tag{5}$$

by the exactness of the cohomology sequence of the pair $(X_1, \partial X_1)$.

Multiplicative separation inequality in the N-torus.

Let $X_1, X_2 \subset \mathbb{T}^N$ be non-intersecting (closed or open) subsets and let $A_1 = H^{n_1}(\mathbb{T}^N;\mathbb{F})$, $A_2 = H^{n_2}(\mathbb{T}^n;\mathbb{F})$ for $n_i \leq N/2$, $i = 1,2$, and some field $\mathbb{F}$. Then

$$\big|\mu_{A_1}(X_1)\big|_{\mathbb{F}} \cdot \big|\mu_{A_2}(X_2)\big|_{\mathbb{F}} \leq c \cdot |A_1|_{\mathbb{F}} \cdot |A_2|_{\mathbb{F}}\,, \tag*{[$\times$]}$$

for $c = n_1 n_2/N^2$, where, observe, $|A_i = \wedge^{n_i}\mathbb{F}|_{\mathbb{F}} = \binom{N}{n_i}$, $i = 1,2$.

This is shown in 4.8 by reducing [×], by the standard ordering argument in the Grassmann algebra $\wedge^*\mathbb{F} = H^*(\mathbb{T}^N;\mathbb{F})$ (see 4.6) to the special case of X_1 and X_2 being *monomial subsets*, i.e. unions of coordinate subtori in $\mathbb{T}^N$ where [×] amounts to a combinatorial inequality due to Matsumoto and Tokushige [MatT1]. (I would not have been able to trace such a result in the literature if not for the landmark – *Kruskal–Katona theorem* that was pointed out to me by Noga Alon.)

In fact, most (all?) inequalities for the extremal set systems (see [Fr]) can be equivalently reformulated in terms of monomial subsets in $\mathbb{T}^N$. We shall see in 4.8–4.9 that some of them extend to *all* (non-monomial) subsets in $\mathbb{T}^N$ (i.e. these inequalities are invariant under *all automorphisms* of $\mathbb{T}^N$, not only under permutations of coordinates); but such extension remains problematic for the majority of these combinatorial inequalities.

Let $A_1 = A_2 = A = H^n(\mathbb{T}^N)$ and let $W = \mathbb{T}^N \setminus (X_1 \cup X_2)$ be the complementary region ("wall") between two disjoint open subsets $X_1, X_2 \subset T^N$. Let $a_i = |\mu_A(X_i)|_{\mathbb{F}}/|A|_{\mathbb{F}}$, $i = 1,2$, and $w = |A|W|_{\mathbb{F}}/|A|_{\mathbb{F}}$.

Then, [×] reduces to

$$a_1 a_2 \leq c = n^2/N^2$$

and (4) above implies that

$$w \geq 1 - a_1 - a_2\,.$$

Thus, we obtain the following:

Cohomology equipartition inequality in the N-torus.

Let $X_1, X_2 \subset \mathbb{T}^N$ be disjoint open subsets and $W = \mathbb{T}^N \setminus (X_1 \cup X_2)$. If X_1 and X_2 have equal $|\mu_{H^n}|_{\mathbb{F}}$-masses for some $n < N/2$, i.e.

$$\big|\mu_A(X_1)\big|_{\mathbb{F}} = \big|\mu_A(X_2)\big|_{\mathbb{F}}$$

for $A = H^n(\mathbb{T}^N) = H^n(\mathbb{T}^N;\mathbb{F})$, *then*

$$\big|H^n(\mathbb{T}^N)|W\big|_{\mathbb{F}} \geq \left(1-\frac{2n}{N}\right)\binom{N}{n}.$$

Consequently (see 4.8, 4.9), the maximum of the $|H^n|_{\mathbb{F}}$*-masses of the fibers of every continuous function* $F:\mathbb{T}^N\to\mathbb{R}$ *is bounded from below by*

$$\sup_{y\in\mathbb{R}}\big|H^n(\mathbb{T}^N)|F^{-1}(y)\big|_{\mathbb{F}} \geq \left(1-\frac{2n}{N}\right)\binom{N}{n}$$

for every $n < N/2$.

Remark. This is stronger than *the maximal fiber inequality* for maps $\mathbb{T}^N\to\mathbb{R}$ from [Gr8] (where the N-torus is treated as the product of smaller tori); but it remains unclear how far this improvement is from the sharp inequality. Also the corresponding strengthening of the maximal fiber inequality for maps $\mathbb{T}^N\to\mathbb{R}^k$ for $k>1$ remains problematic.

About the proof. The reader who is interested exclusively in the above results need look only into 4.1, 4.5, 4.6, 4.8 and 4.9.

1.6 $|A|_{\mathbb{F}}$**-Isoperimetry and homology expanders.** Let X be a locally compact topological space and $A\subset H^*(X;\mathbb{F})$ a linear subspace. We are concerned with bounds on the A-masses of compact subsets by A-masses of their boundary by analogy with the geometric isoperimetric inequalities; moreover, we wish to have spaces X with "strong cohomological isoperimetry" and with "simplest possible" local and global geometry/topology. Here is an instance of a family of such spaces X. (See 4.10 for the construction of these X.)

Simply connected 6-manifolds cohomology expanders.

There exists an infinite family $\{X\}$ *of smooth closed* 6*-dimensional simply connected submanifolds* $X = X^6\subset\mathbb{R}^7$, *with distinguished subspaces* $A\subset H^2(X) = H^2(X;\mathbb{F})$ *for a given field* $\mathbb{F}$ *with the ranks* $|A|_{\mathbb{F}}\to\infty$ *and such that*

(A) *Each* X *in the family has* $\|\operatorname{curv}(X)\|\leq 1$ *and*

$$\operatorname{vol}(X)\leq \operatorname{const}\cdot|A|_{\mathbb{F}} \quad \textit{for some } \operatorname{const}\leq 10^{20},$$

where $\|\operatorname{curv}(X)\|$ *stands for the norm of the second fundamental (curvature) form (shape operator) of* $X\subset\mathbb{R}^7$*;*

(B) *The* A*-mass of every open subset* $X_0\subset X$ *with smooth boundary in each manifold* X *in the family is bounded by the minimum of the masses of* X_0 *and its complement by*

$$|A|\partial X_0|_{\mathbb{F}}\geq\lambda\min\big(|A|X_0|_{\mathbb{F}}, |A|X\setminus X_0|_{\mathbb{F}}\big) \quad \textit{for some } \lambda\geq 10^{-10}.$$

Remarks and Questions. (a) The essential feature of these X which ensures (B) is a high non-degeneracy of the $\smile$-product (pairing) on $A\subset H^2(X)$ that is implemented using (graph) expanders associated to arithmetic groups (see 4.3). (It remains unclear, in general, which algebras A can be realized by cohomologies of n-dimensional Riemannian manifolds of locally 1-bounded geometries and vol $\leq$ const $\cdot$ rank(A).)

(b) The property (B) implies that generic smooth maps of X to open surfaces Y have "deep" critical sets $\Sigma \subset Y$ (see [Gr8]).

This contrasts with the 5-dimensional case: the *Smale–Barden theorem* implies that every simply connected 5-manifold admits a generic map to $\mathbb{R}^2$ where every point in $\mathbb{R}^2$ can be moved to infinity by a path that meets Σ at ≤ 100 points. (The case of simply connected 4-manifolds remains unresolved.)

(c) It is plausible in view of [Su] (see the last Remark in 4.10) that there are families satisfying (A) and (B) with $A = H^2(X)$.

(d) Are there families satisfying (A) and (B) where the rank $|A|_{\mathbb{F}}$ assumes *all* positive integer values? (Our construction depends on the *Margulis theorem* on cofiniteness of normal subgroups in lattices in semisimple Lie groups; this, being ineffective, delivers a family $\{(X, A)\}$ with an infinite but, a priori, very rare set of ranks $|A|_{\mathbb{F}}$.)

(e) Are there similar families of k-connected n-manifold X for $k \geq 2$ and $n \geq 2k + 2$, with $A \subset H^{k+1}(X)$?

About the Proof. The construction of the above 6-manifolds, which is presented in 4.10, depends on the $S_\star$*-construction* in 2.1 and on the construction of *simply connected expanders* in 4.3, while the proof of (A) and (B) uses the reduction of the topological isoperimetry to that in (cohomology) algebras (see 4.8) accompanied by a derivation of the algebraic isoperimetry in *graph algebras* from the combinatorial one in graphs (see 4.1, 4.5, 4.7).

2 Filling Profiles, Random Cones in Spaces of Cycles and Lower Bounds on Multiplicities of Maps

This section contains the proofs of the topological Barany inequality and of related results stated in 1.1–1.4.

2.1 Semisimplicial spaces and $S_\star$-construction. Let $\Delta(V)$ denote the simplex on the vertex set V and observe that the correspondence $V \rightsquigarrow \Delta(V)$ establishes an equivalence between the category $\mathcal{F}$ of maps between finite sets V and the category of simplicial maps between simplices Δ.

A *semi-simplicial structure* S on a topological space X is given by distinguishing, for every finite set V, a set $S(V)$ of continuous maps σ of $\Delta = \Delta(V)$ to X, which are called *(singular) simplices in* X *and/or semisimplicial maps* $\Delta \to X$, where the following two conditions are satisfied.

Functoriality. If a map $\sigma : \Delta_1 \to X$ is a singular simplex and $s : \Delta_2 \to \Delta_1$ is a simplicial map, then the composed map $\sigma \circ s : \Delta_2 \to X$ is also a singular simplex.

Cellularity. The images of the interiors of the singular simplices make a cellular decomposition of X. More precisely, there exists, for every $x \in X$, a singular *cell-simplex* $\sigma_x : \Delta_x \to X$, such that σ_x is a *topological embedding on the interior* $\mathrm{int}(\Delta_x) = \Delta_x \setminus \partial\Delta_x$ and $x \in \sigma_x(\mathrm{int}(\Delta_x))$. Furthermore, every singular simplex $\sigma : \Delta \to X$ with $\sigma(\mathrm{int}(\Delta)) \ni x$ decomposes, $\sigma = \sigma_x \circ s$ for some simplicial map

$s : \Delta \to \Delta_x$. In particular every singular cell-simplex σ_x is unique up to a simplicial automorphisms of Δ_x (which correspond to permutations of the vertices of Δ_x).

> *Thus, every $X = (X, S)$ defines a contravariant functor $\mathcal{S} = \mathcal{S}(X)$ from the category $\mathcal{D}$ of simplices, or, equivalently, from the category $\mathcal{F}$ of finite sets, to the category of sets* (where the cellularity can be expressed in terms of some "sheaf-like" properties of this functor).

***$X(\mathcal{D}, \mathcal{S})$*-space.** Conversely (and more generally [EZ], [Mi]), let $\mathcal{D}$ be a small category of topological spaces (e.g. of simplices $\Delta = \Delta\{0, 1, \dots, i\}$ and simplicial maps between them), let $\mathcal{S}$ be a contravariant functor from $\mathcal{D}$ to the category of sets. Define a topological space $X = X(\mathcal{D}, \mathcal{S})$ 'glued of $\Delta \in \mathcal{D}$ according to $\mathcal{S}$" as follows.

Call $s \in \mathcal{S}(\Delta)$ *degenerate* if there exists a non-injective map φ of some Δ onto Δ' such that $s \in \mathcal{S}(\varphi)$ and let $S_\Delta \subset \mathcal{S}(\Delta)$ be the set of "cells", i.e. of non-degenerate s.

Let X^* be the union of these "cells", i.e. the disjoint union of the copies of spaces Δ in $\mathcal{D}$ indexed by points in S_Δ;

$$\coprod_{\Delta \in \mathcal{D},\, s \in S_\Delta} \Delta_s \,.$$

Join two points in X^* with an arrow,

$$X^* \supset \Delta^1_{s_1} \ni x_1 \mapsto x_2 \in \Delta^2_{s_2} \subset X^*$$

if there exist

> a space $\Delta \in \mathcal{D}$ and points $x \in \Delta$ and $s \in \mathcal{S}(\Delta)$;
> an injective (face) morphism $F : \Delta \to \Delta^1$ and a surjective morphism (projection) $P : \Delta \to \Delta^2$ in $\mathcal{D}$, such that $F(x) = x_1$, $P(x) = x_2$, $\mathcal{T}(F)(s_1) = s$ and $\mathcal{S}(P)(s_2) = s$.

Let $X = X(\mathcal{D}, \mathcal{S})$ be the quotient space of X^*

If $\mathcal{D}$ is a small category of simplices and simplicial maps and $\mathcal{M}$ is the category of sets, then the resulting "new object" X is a semisimplicial complex, where the functor $\mathcal{S}' = \mathcal{S}(X)$ may be *non-equal* $\mathcal{S}$ but X' associated to $\mathcal{S}'$ is canonically isomorphic to X.

Encouragement. Category theory is intimidating for many mathematicians including the present author: the category theoretic definitions and constructions have a flavor of magic incantations with an obscure meaning. But the magic of category theory works amazingly well for you, if you *first* generate "syntactically acceptable" sentences in the category-theoretic language and *then* decipher their meaning.

A particular construction we use is that of a new mathematical object NMO as a contravariant functor $\mathcal{S}$ from a small simple category $\mathcal{D}$, such as the category of finite sets, to another "sufficiently soft", i.e. allowing many morphisms, category $\mathcal{M}$, e.g. the category of sets, of measure spaces or of linear spaces.

One may think of an object D in $\mathcal{D}$ as a "measuring rod", where $\mathcal{S}(D)$ represents a measurement/observation of NMO by means of D and where the *functoriality* rule ensures that different measurements are mutually *coherent*. Thus, NMO emerges as the totality of the results of coherent measurements/observations. (This is, apparently, how your brain forms concepts of "objects" in the external world.)

In what follows, we introduce further categorical constructions of topological spaces which will not be used until section 4.10.

Start by observing that since the category of simplices is equivalent to the category $\mathcal{F}$ of finite sets, the "gluing pattern" defined via $\mathcal{S}$ can be described in the language of $\mathcal{F}$.

Indeed, let R be a commutative topological semigroup with zero. Then the Cartesian power $V \rightsquigarrow R^V$ is a *covariant* functor from the category $\mathcal{F}$ of finite sets V to the category of topological spaces (semigroups), where every contravariant functor $\mathcal{S}$ from $\mathcal{F}$ to the category of sets defines a contravariant functor, say $\mathcal{S}_R$, from the category $\mathcal{R}$ of Cartesian powers R^V, and continuous maps (homomorphisms) induced by maps $V_1 \to V_2$; thus we get

$X(\mathcal{S},R)$*-space.* This is a topological space $X = X(\mathcal{S},R) = X(\mathcal{R},\mathcal{S}_R)$ with a distinguished point $0 \in X$ associated to each functor $\mathcal{S}$ from $\mathcal{F}$ to the category of sets.

Since this X is functorial in R, every automorphism group G of R acts on X and we "projectivize" by letting $P_{/G}(X) = P_{/G}X(\mathcal{S},R) = (X \setminus \{0\})/G$. In particular, if we take $R = \mathbb{R}_+$, $G = \mathbb{R}_+^\times$ and identify the quotients $(\mathbb{R}_+^{i+1} \setminus \{0\})/\mathbb{R}_+^\times$ with the i-simplices, we obtain

$X(\mathcal{S})$*-space:* a semisimplicial space $X(\mathcal{S}) = P_{/\mathbb{R}_+^\times}X(\mathcal{S},\mathbb{R}_+)$ associated to each functor $\mathcal{S}$ from the category $\mathcal{F}$ of finite sets to the category of sets.

If R is a topological space *with a marked point* $r_0 \in R$ (rather than a semigroup), then the Cartesian power $V \rightsquigarrow R^V$ is *covariantly* functorial on *injections* in $\mathcal{F}$. Thus we arrive at the following constructions of spaces (which will be used in 4.8).

***S*-construction.** Let $R = (R, r_0)$ be a marked topological space. Then every semisimplicial complex S, where all cell-simplices $\Delta \to S$ are injective on the faces of Δ (e.g. a simplicial complex), canonically (covariantly functorially in R) defines (via the functor $\mathcal{S}$ associated to S) a marked topological space $X = S(R)$.

$S_\star$-construction. Take the simplex $\Delta = \Delta(V)$ on the vertex set V and radially project points from the barycenters of the faces of Δ to the boundaries of these faces. Eventually, each $s \in \Delta$ ends up at the barycenter of some face and we denote by $V(s) \subset V$ the set of the vertices of this face. For example $V(barycenter) = V$ and $V(v) = v$ for the vertices v of Δ.

Let R be a marked topological space, let $R^{V(s)} \subset R^V$, $s \in \Delta$, be the coordinate subspace corresponding to $V(s) \subset V$ and observe that

$$\mathcal{D}_R : V \rightsquigarrow \Delta(V) \star R = \bigcup_{s \in \Delta(V)} s \times R^{V(s)} \subset \Delta(V) \times R^V$$

makes a covariant functor from the category of finite sets V and injective maps to the category of topological spaces. Then the S-construction applies to $\Delta(V) \star R$ instead of R and delivers *a topological space $X_\star = S_\star(R)$ associated to each simplicial space S and every marked space R.*

In simple words, if W denotes the vertex set of S, then $X_\star \subset S \times R^W$ is defined by the condition

$$X_\star \cap \big(\Delta(V) \times R^W\big) = \bigcup_{s\in\Delta(V)} s \times R^{V(s)} \subset \Delta(V) \times R^V \subset \Delta(V) \times R^W$$

for all $V \subset W$ corresponding to the faces $\Delta(V) \subset S$.

Notice that

(a) There is a natural embedding $e : S \subset X_\star$ defined by the marking in R.

(b) There is a natural map $\rho : X_\star \to S$, defined by the projections $\Delta \times R^V \supset \Delta \star R \to \Delta$, such that $\rho \circ e = id$, and where each fiber $\rho^{-1}(s) \subset X_\star$ is a Cartesian power $R^{d(s)} = s \times R^{d(s)} \subset S \times R^{d(s)}$ for $d(s) \le \dim(S) + 1$.

(c) There is a natural map $\varphi : X_\star \to X = S(R)$, defined by the projections $\Delta \times R^V \supset \Delta \star R \to R^V$, which sends $e(S) \subset X_\star = S_\star(R)$ to the marked point in X.

(d) If $\dim(S) = 1$ and $\dim(R) > 0$ then
$$\dim(X_\star) = \dim(X) = 2\dim(R)\,,$$
the map φ is one-to-one away from S and the induced cohomology homomorphism φ^* is bijective on H^i for $i \ge 2$.

(e) If R is a triangulated space, there is a triangulation of $X_\star$, where the degrees $\deg_{\rm loc}(X_\star)$ of $X_\star$ at the vertices (unlike the degrees of vertices in $X = S(R)$) are bounded by
$$\big(\dim(S) + \dim(R)\big)!\big(1 + \deg_{\rm loc}(S)\big) \cdot \big(1 + \deg_{\rm loc}(R)\big)^{1+\dim(S)},$$
and for which ρ becomes a simplicial map, where the numbers $N_\Delta(X_\star, s)$ of simplices in $X_\star$ intersected by the ρ-pull backs of the points $s \in S$ are bounded by the number of simplices in R by
$$N_\Delta(X_\star, s) \le \big(\dim(S) + \dim(R)\big)! \cdot N_\Delta(R)\,.$$
Therefore, the total number of simplices in the triangulated $X_\star$ is bounded by
$$N_\Delta(X_\star) \le \big(\dim(S) + \dim(R)\big)! N_\Delta(S) \cdot N_\Delta(R)\,.$$

2.2 Spaces cl^n_{sms} of n-cycles in chain complexes, quasitransversality and the intersection homomorphism $F^* \circ \pitchfork_n$. Let C^* be a complex of Abelian groups, $\partial^i : C^i \to C^{i+1}$, $i = 0, 1, 2, \ldots$, where $C^i = 0$ for $i < 0$, and recall the definition of the semisimplicial space of n-cycles in C^* (these will be cocycles in the topological applications).

Let $D^*(k, n) = \{\delta^i : D^i(k, n) \to D^{i+1}(k, n)\}$ be the the chain (*not* cochain) complex $C_*(\Delta^k)$ of the standard k-simplex over the integers, where the usual (decreasing) grading is reversed and shifted, such that $D^i(k, n) = C_{n-i}(\Delta^k)$ with $\delta^i = \partial_{n-i}$ and where the complex $D^*(k, n)$ is infinitely extended for $i < n - k$ and $i > n$ by zeros.

The *space of n-(co)cycles in C^** denoted $cl^n_{sms} = cl^n_{sms}(C^*)$, is defined as the semisimplicial space corresponding to the functor $\mathcal{S} : \Delta^k \to {\rm Hom}(D^*(k, n), C^*)$.

In simple words, the vertices in cl^n_{sms} are (co)cycles $c \in cl^n = \ker \partial^n$, the edges in cl^n_{sms} are $(n-1)$-chains c_{12} "joining" pairs of (co)homologous cocycles c_1 and c_2

in cl^n, which means $\partial^{n-1}(c_{12}) = c_1 - c_2$, the 2-simplices in cl^n_{sms} are "filling triangles of (semi)-triangulated 1-spheres", i.e. $(n-2)$-chains c_{123} such that $\partial^{n-2}c_{123} = \pm c_{12} \pm c_{13} \pm c_{23}$ with certain $\pm$ signs depending on orientations, etc.

EXAMPLE. If $C^i = 0$ for $i \neq 0$ then cl^n_{sms} equals the usual semisimplicial presentation of Eilenberg–MacLane space $K(H^0, n)$, since the (co)homology group $H^0 = H^0(C^*)$ equals C^* in this case.

More generally one has the following algebraic version of the

Dold–Thom–Almgren Theorem (see [A1]). *The space $cl^n_{sms}(C^*)$ is canonically (and functorially) homotopy equivalent to the Cartesian product of the Eilenberg–MacLane spaces associated with the (co)homology $H^* = \ker \partial^* / \operatorname{im} \partial^{*-1}$ of C^*,*

$$cl^n_{sms}(C^*) \simeq \times_j K(H^j, n-j)\,.$$

Thus, the connected components of cl^n_{sms} correspond to elements $g \in H^n = K(H^n, 0)$ and they are all canonically homotopy equivalent.

The subproduct $K(H^0, n) \times K(H^n, 0)$ is called the *fundamental cofactor* $fnd^n \subset cl^n_{sms}$, where cl^n_{sms} canonically retract on fnd^n.

The connected g-components of fnd^n, $g \in H^n = K(H^n, 0)$ are copies of $K(H^0, n)$; the n-dimensional homology of $K(H^0, n)$ is canonically isomorphic to H^0.

If C^* is a complex of $\mathbb{F}$-moduli over a unitary ring (e.g. a field) $\mathbb{F}$ and $H_0 = \mathbb{F}$, then an $h \in H_n(fnd^n) \subset H_n(cl^n_{sms})$ is called the *g fundamental class*, $g \in H^n$, if its restriction to the connected g-component of fnd^n equals $1 \in \mathbb{F} = H_n(K(H^0, n); \mathbb{F})$.

If a chain homomorphism between chain complexes, $\phi : C^*_1 \to C^*_2$, is an isomorphism on H^0, then it sends the fundamental g-class of $cl^n_{sms}(C^*_1) \supset fnd^n$, $g \in H^n$, to the $\phi_*(g)$-fundamental class of $cl^n_{sms}(C^*_2)$ by the functoriality of the Dold–Thom–Almgren isomorphism.

If X is a topological space, the above applies to the singular (co)chain complex of X with coefficients in some $\mathbb{F}$, where the corresponding spaces of cycles and cocycles are denoted $cl_*(X)$ and $cl^*(X)$. If X comes as a cell complex (e.g. as a simplicial complex), then (co)cycles are understood in the cellular (co)chain complex of X.

If Y is an oriented $\mathbb{F}$-homology manifold of dimension n (e.g. just a manifold) then the Poincaré duality assigns, to each i-dimensional $\mathbb{F}$-cycle c in Y, an $(n-i)$-dimensional $\mathbb{F}$-cocycle, denoted $\pitchfork_n c$. (This equally applies to the singular (co)homology and to the homology/cohomology associated to triangulations and the dual cell partitions of Y.)

The image $\pitchfork_n [Y]_\circ \in H_n(cl^n(Y)$ of the fundamental homology class $[Y]_\circ \in H_n(Y; \mathbb{F})$ equals the $[Y]^\circ$-fundamental class $\in H_n(fnd^n(Y); \mathbb{F}) \subset H_n(cl^n(Y); \mathbb{F})$, for $[Y]^\circ$ denoting the fundamental cohomology class of Y that is the Poincaré dual to $[Y]_\circ$. This implies

Non-vanishing of $\pitchfork$.

If $F : X \to Y$ is a continuous map, where X is a non-empty topological space, then the image $F^*(\pitchfork_n [Y]_\circ) \in H_n(cl^n(X))$ of $\pitchfork_n [Y]_\circ$ is non-zero.

Let X be a cellular (e.g. simplicial) space (complex) and let us recall Poincaré's description of the composed homomorphism $F^* \circ \pitchfork_n$. for continuous maps $F : X \to Y$.

A continuous map of an i-simplex to Y, say $s : \Delta^i \to Y$, is called *quasitransversal to* $F : X \to Y$ if every closed j-face Δ^j of Δ^i (mapped by s to Y) and every $(n-j)$-cell σ_{n-j} of X (mapped by F to Y) intersect in Y (if at all) *only* at their interior points. If Δ^j and σ^{n-j} are oriented, there is a well-defined $\mathbb{F}$-valued intersection number, denoted $\Delta^j \pitchfork \sigma_{n-j} \in \mathbb{F}$.

Quasitransversal $\mathbb{F}$-intersections of cells in X with the faces of i-simplices making singular i-cycles c in Y define i-cycles in $cl^{n-i}_{sms}(C^*(X))$;

> *the resulting "intersection homomorphism" $H_i(Y) \to H_i(cl^{n-i}_{sms}(X))$ equals $F^* \circ \pitchfork_n$. In particular this "intersection homomorphism" does not vanish on the fundamental homology class of Y.*

Observe that if Y is a smooth manifold and the maps F and s are face-wise smooth and face-wise transversal, then they are quasitransversal, and if $\mathbb{F}$ is a metric ring with $\|1\| = 1$, then

> *the $\mathbb{F}$-intersection "number" is bounded by the actual number of the intersection points,*
> $$\|\Delta^j \pitchfork \sigma_{n-j}\|_{\mathbb{F}} \leq \left| s(\Delta^j) \cap F(\sigma_{n-j}) \right| .$$

2.3 Filling norms $\|\partial^{-1}\|_{\mathrm{fil}}$ in metric and measurable chain complexes. *A norm* in an Abelian group A is a function $a \mapsto \|a\| = \|a\|_A$ with the values $0 \leq \|a\| \leq= +\infty$ such that $\|a\| = \|-a\|$ and where $\|a - a'\|$ satisfies the triangle inequality. (Thus, the norms correspond to invariant metrics on A where the values 0 and $+\infty$ are allowed.)

The norm of a homomorphism between normed groups, $\partial : A \to B$, is the following *function in* $\alpha \in [0, \infty)$,

$$\|\partial\|(\alpha) =_{\mathrm{def}} \sup_{\|a\|_A = \alpha} \|\partial(a)\|_B / \|a\|_A .$$

The filling norm of a $b \in B$ is

$$\|b\|_{\mathrm{fil}} =_{\mathrm{def}} \inf_{a \in \partial^{-1}(b)} \|a\|_A .$$

Thus, $\|b\|_{\mathrm{fil}} < \infty$ if and only if $b \in \partial(A) \subset B$.

If $A' \subset A$ is a subgroup in a normed group A and $C \subset A$ is an A'-coset in A, then *the (minimal) norm* $\|[a/A']\|$ *of an* $a \in C$ *modulo* A' or *the quotient norm* $\|C\|$ is defined as the infimum of the A-norms of all $a' \in C$.

Metric complexes and their systoles. A metric complex $C^* = \{\partial^i : C^i \to C^{i+1}\}$ is a complex of Abelian groups with norms. This norm passes to the quotient norms on (co)homology $H^i = \ker \partial^i / \operatorname{im} \partial_{i-1}$, where we define the i-systole of C^* by

$$\mathrm{syst}_i = \inf_{h \neq 0} \|h\| \quad \text{for } h \in H^i .$$

Observe that $\|b\|_{\mathrm{fil}} < \infty$ for all i-cocycles $b \in \ker \partial^i$ with $\|b\| < \mathrm{syst}_i$.

Define the $(i+1)$-*filling profile of* C^* also called *the inverse filling norm of* ∂^i, as the following function in $\beta \geq 0$,

$$\left\|(\partial^i)^{-1}\right\| = \left\|(\partial^i)^{-1}\right\|_{\mathrm{fil}}(\beta) = \sup_{\|b\| = \beta} \|b\|_{\mathrm{fil}} / \|b\| \quad \text{for } b \in \operatorname{im} \partial^i \subset C^{i+1} .$$

Measurable complexes. Let X be a cell (e.g. simplicial) complex where each i-cell σ is assigned a weight denoted $\|\sigma\| = \|\sigma\|_i$ and where the set Σ_i of i-cells is endowed with a measure (e.g. these may be *unitary (atomic)* measures, where each sigma has measure 1 or probability measures where the total mass of all i-cells equals 1).

Then the group $C^i = C^*(X;\mathbb{F})$, where $\mathbb{F}$ is a normed Abelian group, of measurable $\mathbb{F}$-valued i-cochains c, that are measurable $\mathbb{F}$-valued functions on the (oriented) i-cells σ in X, is given the L_1-*norm*,

$$\|c\|_i = \int_{\Sigma_i} \|c(\sigma)\|_{\mathbb{F}} \cdot \|\sigma\|_i d\sigma .$$

An X with measures $|\dots|_i$ on all Σ_i is called a *measurable cell complex* if the coboundary homomorphisms $\partial^i : C^i \to C^{i+1}$ send measurable cochains (with arbitrary coefficients) to measurable ones and a(lmost) e(verywhere) vanishing of a c implies a.e. vanishing of $\partial^i c$.

The notation $\|\dots\|_{\mathrm{fil}}$ applied to X refers in this context to $C^*(X;\mathbb{F})$, and $\mathrm{syst}^i(X;\mathbb{F})$ signifies $\mathrm{syst}_i(C^*)$. Observe that

$$\mathrm{syst}^0(X;\mathbb{F}) = \inf_{0\neq f\in\mathbb{F}} \|f\|_{\mathbb{F}} ,$$

e.g. $\mathrm{syst}^0(X;\mathbb{Z}) = 1$, while $\mathrm{syst}^0(X;\mathbb{R}) = 0$.

***Measurable simplex* $\mathbf{\Delta = \Delta(V)}$.** Every contravariant functor $\mathcal{T}$ from the category of finite sets to the category of measure spaces defines *a measurable semi-simplicial complex* $X(\mathcal{T})$.

For instance, the Cartesian power functor $F \rightsquigarrow V^F$ for a given probability space V defines "measurable semi-simplex" on the vertex set V, where the measurable simplex $\Delta = \Delta(V)$ is obtained by removing the diagonals from V^F (which does not change anything if V contains no atom.)

In other words, the set $\Sigma_i = \sigma_i(\Delta) = \Sigma_i(V)$ of *i-faces* of Δ equals the Cartesian power of V minus the diagonals divided by the permutation group,

$$\Sigma_i = (V^{i+1} \setminus \mathrm{Diag})/\Pi(i+1)$$

and Σ_i is endowed with the probability measure denoted $d\sigma = d\sigma_i$ on Σ_i that is induced from the normalized measure $(dv)^{i+1}$ on $V^{i+1} \setminus \mathrm{Diag}$ (where removing the diagonals is unnecessary if V has no atoms).

$\mathbb{F}$-valued i-cochains on Δ can be represented by measurable functions $c : V^{i+1} \to \mathbb{F}$ that are *antisymmetric* under permutations of coordinates. Such a representation is "almost unique", it depends on a choice of an *orientation* on the $(i+1)$-element set $I = I_{i+1}$ indexing the coordinates. An orientation on a finite set I is an order on I up to an *even* permutation from $\mathrm{Aut}(I)$. Changing orientation switches $c \leftrightarrow -c$.

The coboundary $\partial^i(c) : V^{i+2} \to \mathbb{F}$ of a $c : V^{i+1} \to \mathbb{F}$ is defined with the lifts $\tilde{c}_j$ of c to V^{i+2} by the $i+2$ coordinate projections $V^{i+2} \to V^{i+1}$ as the sum

$$\partial^i(c) = \sum_{j=1,\dots,i+2} \pm\tilde{c}_j \qquad (\pm)$$

where the $\pm$-signs are taken according to orientations. It follows, in particular, that

$$\left\|\partial^i(c)\right\| \leq (i+2)\|c\| .$$

Riemannian* Δ_ε*-complex. Every measurable simplicial complex X with the vertex set V can be realized as a subcomplex in $\Delta(V)$ but the measures on the sets $X^i \subset \Sigma_i(V)$ of i-simplices for $i > 0$ do not necessarily come from the (Cartesian power) measures on $\Sigma_i(V)$. Here is a representative example.

Let $\tilde{V}$ be a complete simply connected Riemannian manifold of constant curvature and $\Delta^i_\varepsilon(\tilde{V}) \subset \Sigma_i(\tilde{V})$ be the set of regular convex i-simplices in $\tilde{V}$ with the edge length ε. This set carries a unique, up to scaling, measure invariant under the isometry group $\mathrm{iso}(\tilde{V})$ but it is infinite unless the curvature of $\tilde{V}$ is positive, i.e. $\tilde{V}$ is a round sphere. But if we divide $\Delta^i_\varepsilon(\tilde{V})$ by a discrete subgroup $\Gamma \subset \mathrm{iso}(\tilde{V})$ of finite covolume the measure becomes finite and can be normalized to a probability measure. Thus we obtain a measurable semi-simplicial (simplicial if the action of Γ on $\tilde{V}$ is free) complex, with the vertex set $V = \tilde{V}/\Gamma$, where the set of i-simplices equals $\Delta^i_\varepsilon(\tilde{V})/\Gamma$.

Our objective is evaluation of the filling profiles, i.e. of norms of the inverse to the (co)boundary operators in measurable complexes, and related invariants, where we follow the lead (unfortunately, not far) of geometric measure theory (see 3.3–3.5).

On cocycles in metric complexes. If C^* is a metric complex then the set $cl^n = \ker\partial^n \subset C^n$ of n-cocycles inherits the norm topology from the space C^n of n-cochains, where, observe, cl^n serves as the set of vertices for cl^n_{sms}.

Let the spaces of cochains C^i be contractible and locally contractible for all $i \le n$. Then the identity map from cl^n to itself continuously extends to a map from cl^n_{sms} to cl^n, and if $\|(\partial^i)^{-1}\|_{\mathrm{fil}}(\beta)$ is bounded for all $i \le n$, then this map (obviously) is a weak homotopy equivalence.

EXAMPLE. Let $C^* = C^*(\Delta(V); \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z})$ for the measurable simplex $\Delta(V)$. If the probability space V has no atoms, then the above applies and the space of n cocycles with the norm topology (as well as cl^n_{sms}) is weakly homotopy equivalent to $K(\mathbb{Z}_p, n)$.

2.4 Compounded filling profiles Φ^n_i of metric complexes and contractions in spaces of cycles. Let $(C^*, \partial) = (\partial^i : C^i \to C^{i+1})$, $i = -1, 0, 1, \dots$, where $C^{-1} = 0$, be a (co)chain complex with a given a norm/metric structure. If $\sigma \subset cl^n_{sms}$ is a k-cell-simplex represented by a non-zero homomorphism $\sigma : D^*(k,n) \to C^*$, let $\|\sigma\|$ denote the norm on the σ-image of the generator in $D^{n-k}(k) = C_k(\Delta^k) = \mathbb{Z}$.

Let

$$fl_i(C^*;\beta) = fl_i(\beta) = \sup_{\|c\|=\beta} \|c\|_{\mathrm{fil}} \quad \text{for } c \in \mathrm{im}\,\partial^i.$$

Thus, $fl_i(\beta) = \beta \cdot \|(\partial^i)^{-1}\|_{\mathrm{fil}}(\beta)$ for $\beta_{i+1} < \mathrm{syst}_{i+1}(C^*)$, $i = -1, 0, 1, \dots$, and $fl_i(\beta) = \infty$ for $\beta \ge \mathrm{syst}_{i+1}(C^*)$, where $\mathrm{syst}_{i+1}(C^*)$ is the infimum of the norms of non-zero (co)homology classes in $H^{i+1} = \ker\partial^{i+1}/\,\mathrm{im}\,\partial_i$. Observe that $fl_{-1}(\beta) = 0$ if $\beta < \mathrm{syst}_0 = \inf_{c\neq 0}\|c\|$, $c \in C^0$.

Define, for every $n = 0, 1, \dots$, the following (non-linear) operator from $(i+1)$-tuples of (one variable) functions $f_{n-j-1}(\beta_{n-j})$, $j = 0, 1, \dots, i \le n$, to functions Φ_i in the $i+1$ variables, $\beta_n, \dots, \beta_{n-i}$,

$$\big\{f_{n-1}(\beta_n), \dots, f_{n-i-1}(\beta_{n-i})\big\} \mapsto \Phi_i(\beta_n, \dots, \beta_{n-i}),$$

where we start with $\Phi_0(\beta_n) =_{\rm def} f_{n-1}(\beta_n)$ and then, by induction on i, let

$$\Phi_i(\beta_n,\dots,\beta_{n-i}) = f_{n-i-1}\big(\beta_{n-i}+(i+1)\Phi_{i-1}(\beta_n,\dots,\beta_{n-i-1})\big) \quad \text{for } i\geq 0\,.$$

EXAMPLE. If $f_{n-i-1}(\beta_{n-i}) = c_{n-i}\cdot\beta_{n-i}$, then

$$\Phi_i(\beta_n,0,\dots,0) = (i+1)!\prod_{j=0,\dots,i} c_{n-j}\,.$$

Let $\Phi_i^n(C^*;\beta_n,\dots,\beta_{n-i})$ denote the function obtained with the "filling functions" $f_{n-j-1} = fl_{n-j-1}(C^*;\beta_{n-j})$.

Filling/contraction inequality.

Let $B = B^i \subset cl^n_{sms}(C^*)$ be an i-dimensional closed semi-simplicial subset that is contained in a single connected component of cl^n_{sms}. Let β_{n-j} denote the suprema of $\|\sigma^j\|$ over all j-cells σ^j in B and let β_n° be the infimum of $\|\sigma^0\|$ over all vertices in the connected component of cl^n_{sms} containing B. If the function $\Phi_i = \Phi_i^n(C^*;\beta_{n-j})$ satisfies

$$\Phi_i(\beta_n+\beta_n^\circ,\beta_{n-1},\dots,\beta_{n-i})<\infty\,,$$

or, equivalently, if

$$(i+1)\Phi_{i-1}(\beta_n+\beta_n^\circ,\beta_{n-1},\dots,\beta_{n-i+1})+\beta_{n-i} < \mathrm{syst}_{n-i}(C^*)\,,$$

then B is contractible in X.

Proof. Contract B in cl^n_{sms} to (the cocycle in $cl^n\subset cl^n_{sms}$ corresponding to) a vertex $\sigma^0_\circ$ with $\|\sigma^0_\circ\| = \beta^\circ_n$ (we may assume such vertex exists) by mapping a cone over B to cl^n_{sms} by the usual induction on skeleta $B^j\subset B$, $j\leq i$, as follows.

Step 0. Make $\mathrm{cone}(B^0)\to cl^n_{sms}$ by joining all points (vertices) $\sigma_1^0,\sigma_2^0,\sigma_3^0,\dots$ in B^0, that are n-cocycles in C^*, with $\sigma^0_\circ$ by edges $\sigma_1^1,\sigma_2^1,\sigma_3^1,\dots\in cl^n_{sms}$ that are $(n-1)$-cochains in C^*, such that $\partial^{i-1}(\sigma^1_l) = \sigma^0_l-\sigma^0_\circ$, $l=1,2,3,\dots$, and such that these cochains σ^1_l are norm minimizing, i.e.

$$\|\sigma^1_l\| = \|\sigma^0_l-\sigma^0_\circ\|_{\rm fil}\leq fl_{n-1}\big(\|\sigma^0_l-\sigma^0_\circ\|\big)\leq\Phi_0(\beta_n+\beta_n^\circ)\,,$$

where such ε-minimizing $(n-1)$-cochains exist by the definition of $\|\dots\|_{\rm fil}$ and where we eventually send $\varepsilon\to 0$ and pretend that $\|\sigma^1_l\|$ are minimizing to start with.

Step 1. Extend the above cone/map to $\mathrm{cone}(B^1)\supset B^1\cup\mathrm{cone}(B^0)$ by filling-in all triangles. Every such triangle σ^2 may have (at most) one "old" edge σ^1 in B^1 of norm $\leq b_{n-1}$ and *two* σ^1_l-edges issuing from the apex β°_n of $\mathrm{cone}(B^0)$ that have norms $\leq\Phi_0(C^*;\beta_n+\beta_n^\circ)$. We take every such σ^2 (i.e. a cochain in C^{n-2} that has the coboundary of the form $\partial^{n-2}(\sigma^2) = \pm\sigma^1\pm\sigma^1_{l_1}\mp\sigma^1_{l_2}$) with minimal possible norm (up to $\varepsilon\to 0$), i.e. (ignoring the ε) with the norms

$$\|\sigma^2\|\leq fl_{n-2}\big(\beta_{n-1}+2\Phi_0(\beta_n+\beta_n^\circ)\big)\leq\Phi_1(\beta_n+\beta_n^\circ,b_{n-1})$$

for all these σ_2.

Step j. Extension of the cone/map to $\mathrm{cone}(B^j)\supset B^j\cup\mathrm{cone}(B^{j-1})$ needs filling-in (semi)triangulated i-"spheres" by $j+1$-simplices σ^{j+1}. Every such "sphere" has (at most) one "old" face σ^j in B^j and $j+1$ faces σ^j_l in $\mathrm{cone}(B^{j-1})$; thus, the minimal such filling σ^{j+1}, regarded as a cochain in C^{n-j-1}, satisfies

$$\partial(\sigma^{j+1}) = \pm\sigma^j\mp\sum_{l=1,\dots,j+1}\pm\sigma^j_l$$

and, by minimality, has

$$\begin{aligned}\|\sigma^{j+1}\| &\le fl_{n-j-1}\Big(\|\sigma^i\| + \sum_{l=1,\dots,j+1} \|\sigma^j_l\|\Big)\\ &\le fl_{n-j-1}\big(\beta_{n-j} + (j+1)\Phi_{j-1}(\beta_n + \beta^\circ_n, \dots, \beta_{n-j+1})\big)\\ &= \Phi_j(\beta_n + \beta^\circ_n, \dots, \beta_{n-j})\,.\end{aligned}$$

Such σ^{j+1} exists, provided $fl_{n-j-1}\big(\big\| \pm \sigma^i \pm \sum_{l=1,\dots,j+1} \pm\sigma^j_l\big\|\big) < \infty$. If $\Phi_i < \infty$, this is the case for all $j \le i$ and the construction *does deliver* the required cone(B^i). □

2.5 Lower bounds on multiplicities of maps of polyhedra to manifolds. Let X be a finite (or, more generally, measurable) cell complex, $\mathbb{F}$ be a unitary metric ring, and C^* a $\mathbb{F}$-cochain complex of X with the the norm corresponding to that of $\mathbb{F}$. Let Y be an oriented n-dimensional $\mathbb{F}$-homology manifold and $F : X \to Y$ be a continuous map. Let c be a singular cycle representing the fundamental class $[Y]_\circ \in H_n(Y;\mathbb{F})$ that is quasitransversal to F, i.e. the F-images of i-cells do not intersect (the images of) the $(n-i-1)$-faces of the (singular) simplices constituting c.

Let $m^\circ_n = \|F^*([Y]^\circ)\|$ for the fundamental cohomology class $[Y]^\circ \in H^n(Y;\mathbb{F})$ and the quotient norm on the homology of C^* and let

$$m_i = m_i(F,c) = \sup_{\Delta^{n-i}} \int_{\Sigma_i} \|\sigma^i \pitchfork \Delta^{n-i}\|_{\mathbb{F}} d\sigma^i,$$

where the integral is taken over the space Σ_i of i-cells in X and supremum is taken over all $(n-i)$-faces Δ^{n-i} of the singular n-simplices constituting c.

Then

$$\Phi_n(m_n + m^\circ_n, m_{n-1}, \dots, m_0) = \infty\,;$$

equivalently,

$$\Phi_{n-1}(m_n + m^\circ_n, m_{n-1}, \dots, m_1) + m_0 \ge \mathrm{syst}^0(X;\mathbb{F})\,.$$

Indeed, the homology class $F^*(\pitchfork_n [Y]_\circ) \in H_n(cl^n_{sms}(C^*);\mathbb{F})$ is non-zero by non-vanishing of $\pitchfork$ (see 2.2) and the above filling/contraction inequality applies.

Let us specialize to smooth manifolds Y and a generic piecewise smooth map F of an n-dimensional measurable simplicial complex X to Y. If c is represented by a sufficiently fine generic smooth triangulation of Y, then

$$m_n \le M^\pitchfork_n = M^\pitchfork_n(F) =_{\mathrm{def}} \sup_y \int_{\Sigma_n} \big|F^{-1}(y) \cap \sigma\big| d\sigma\,,$$

where Σ_n is the (measure) space of n cells σ in X, where the supremum is taken over "generic" points in Y, i.e. away from the image of the $(n-1)$-skeleton of X and where F is transversal to y. Furthermore, m_i for $i < n$ are bounded by the "normalized compounded degrees" D_i of the i-faces at j-faces in X, $j < i$, defined as follows.

Denote by $d_i(\Delta^j)$, $j \le i$, the the measure of the i-faces adjacent to a given j-face Δ^j in X and let $d_{ij} = \sup_{\Delta^j} d_i(\Delta^j)$. (If the measures on the sets of j-simplices have no atoms then $d_{ij} = 0$.) Let D_i be the supremum of those sums $d_{ij_1} + d_{ij_2} + \dots + d_{ij_k}$, where $(n-j_1) + (n-j_2) + \dots + (n-j_k) \le n$.

Then $m_i \le D_i$ for $i = n-1, \ldots, 1$ and we conclude with the

Generic syst^0-inequality.

$$(n+1)\Phi_{n-1}(M_n^{\pitchfork} + m_n^{\circ}, D_{n-1}, \ldots, D_1) + m_0 \ge \mathrm{syst}^0(X; \mathbb{F})$$

where this m_0 equals the supremum of the masses of atoms (if there are any) in the 0*-skeleton of X.*

Observe that $m_n^{\circ} = 0$ if Y is an *open* manifold, e.g. $Y = \mathbb{R}^n$ (where homology is taken with infinite support and the cohomology with compact supports) and that D_i are usually much smaller than $M_n^{\pitchfork}$; thus, a bound on m from below would follow from an upper bound on $\|(\partial^i)^{-1}\|_{\mathrm{fil}}$ and on syst_i in C^*.

2.6 Random cones, bound $\|\partial^{-1}\|_{\mathrm{rand}} \le 1$ in $\Delta(V)$ and the proof of the topological Δ-inequality. Let V be a probability space and $\Delta(V)$ denote the (measurable) simplex $\Delta(V)$ on the vertex set V. The cone from a vertex $v \in V$ over an $(i+1)$-cocycle b, say with $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients, makes an i-cochain c_v with $\partial^i(c) = b$, where, obviously, $\int_V \|c_v\| dv \le 1$ if V has no atoms or if it consists of finitely many equal atoms.

This shows that

$\|(\partial^i)^{-1}\|_{\mathrm{fil}} \le 1$*, which concludes the proof of the topological Δ-inequality* $[\Delta \to \mathbb{R}^n]_{\mathrm{top}}$ *from 1.1.* (See the end of this section for details.)

Let us proceed more formally and spell out the definitions that are also useful for similar constructions in more general polyhedra.

Given a homomorphism $\partial : A \to B$ between normed Abelian groups (see 2.3), define *a contraction or a cone* as a homomorphism $\delta : B \to A$, such that $\partial \circ \delta(b) = b$ for all $b \in \partial(A) \subset B$.

A random contraction (cone) in B is a family of contractions parametrized by a probability space, say δ_p, $p \in P$, where the norm, denoted $\|\delta_p\|(\beta)$, is defined, at each $\beta \ge 0$, as the expectation of $\|\delta_p\|(\beta)$, that is $\int_P \|\delta_p\|(\beta) dp$ (where $\|\delta\|(\beta)$ is defined according to 2.3).

The randomized contraction profile of ∂ is

$$\|\partial^{-1}\|_{\mathrm{rand}}(\beta) =_{\mathrm{def}} \inf_{\delta_p} \|\delta_p\|(\beta)\,,$$

where the infimum is taken over all random contractions on B. Clearly, this bounds the filling profile of ∂,

$$\|\partial^{-1}\|_{\mathrm{rand}}(\beta) \ge \|\partial^{-1}\|_{\mathrm{fil}}(\beta)\,.$$

Represent $\mathbb{F}$-valued i-cochains on Δ by measurable functions $c : V^{i+1} \to \mathbb{F}$ which are *antisymmetric* under permutations of coordinates and where the coboundary $\partial^i(c) : V^{i+2} \to \mathbb{F}$ of $c : V^{i+1} \to \mathbb{F}$ is defined with the lifts $\tilde{c}_j$ of c to V^{i+2} by the $i+2$ coordinate projections $V^{i+2} \to V^{i+1}$ as the sum

$$\partial^i(c) = \sum_{j=1,\ldots,i+2} \pm \tilde{c}_j \qquad (\pm)$$

where the $\pm$-signs are taken according to orientations.

Notice that if $\mathbb{F} = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, then one does not need orientation: $\mathbb{Z}_2$-cochains (unambiguously) are *symmetric* (i.e. $\Pi(i+1)$-invariant) functions $c : V^{i+1} \to \mathbb{Z}_2$.

Every such c equals the indicator (characteristic) that is a $\Pi(i+1)$-*invariant* subset in V^{i+1} of measure $|S| = \|c\|$.

The coboundary operator for $\mathbb{F} = \mathbb{Z}_2$ can be be described in terms of the pullbacks $\tilde{S}_j \subset V^{i+2}$, $j = 1, 2, \ldots, i+1$, of the support $S = \mathrm{supp}(c) \subset V^{i+1}$ under the coordinate projections $V^{i+2} \to V^{i+1}$: the support $\mathrm{supp}(\partial^i(c)) \subset V^{i+2}$ equals the subset $\tilde{S}_{odd} \subset \cup_j \tilde{S}_j$ of the points covered by *odd* number m of $\tilde{S}_j$. In other words,

$$\partial^i(c)(\sigma) = \tfrac{1}{2}\big(1 - (-1)^{m(\sigma)}\big)\,,$$

where $m(\sigma)$, $\sigma \in V^{i+2}$, is the multiplicity (function) of the family $\{\tilde{S}_j\}$, that is the number of $\tilde{S}_j$ containing σ, and where, observe,

$$\int_{V^{i+2}} m(\sigma)d\sigma = (i+2)|S| = (i+2)\|c\|\,.$$

Interior product and random cones. Given two non-intersecting oriented faces σ and σ' in $\Delta = \Delta(V)$ of dimensions $k < i$ and $k' = i - k - 1$, denote by $\sigma \vee \sigma'$ the i-face spanned by the two of them. Then define $\sigma \wedge c \in C^{i-k-1}(\Delta;\mathbb{F})$ for all $c \in C^i(\Delta;\mathbb{F})$ by $(\sigma \wedge c)(\sigma') = c(\sigma \vee \sigma')$ with the agreement $(\sigma \wedge c)(\sigma') = 0$ if σ' intersect σ.

Observe that

$$\int_{\Sigma_k} \|\sigma \wedge c\| d\sigma = \|c\| \qquad (\| \wedge_{\mathrm{rand}} \| = 1)$$

for all c, provided V has no atoms.

If $k = 0$ and $\sigma = v \in V$, then the homomorphism $c \mapsto v \wedge c$ is, obviously, a cone, that is

$$\partial^{i-1}(v \wedge c) = c \quad \text{for all cocycles } c \in \ker \partial^i \subset C^i(\Delta;\mathbb{F})\,,$$

and the above formula shows that the corresponding random cone has norm 1. It follows that if V has no atoms, then every cocycle (coboundary) b satisfies

$$\|b\|_{\mathrm{fil}} \le \inf_{v \in V} \|v \wedge b\| \le \int_V \|\sigma \wedge b\| dv = 1\,.$$

In other words,

> *the norms of the random and, hence, of the filling inversions of ∂^i are bounded by* 1*:*
> $$\big\|(\partial^i)^{-1}\big\|_{\mathrm{fil}} \le \|(\partial^i)^{-1}\|_{\mathrm{rand}} \le 1$$
> *for all V without atoms.*

This can be slightly sharpened for the finite probability spaces V made of N atoms of equal weights ($= 1/N$). Indeed, since $(v \wedge c)(\sigma') = 0$ whenever one of the i vertices of σ' equals v,

> *the norm of the random cone over every c equals $\frac{(N-i)}{N}\|c\|$; therefore*
> $$\big\|(\partial^{i-1})^{-1}_{\mathrm{fil}}\big\| \le (N-i)/N$$
> *for all $i = 0, 1, 2, \ldots$.*

Sharp bound on $\|(\partial^0)^{-1}\|_{\mathrm{fil}}$ over $\mathbb{Z}_2$. Exact 1-cochains b with $\mathbb{Z}_2$-coefficients on $\Delta(V)$ correspond to partitions of V into two subsets, say $V = V_+ \cup V_-$ where b equals the coboundary of either of the two characteristic functions, of V_+ and/or of V_-.

If V has no atoms, then $\|b\| = 2|V_+| \cdot |V_-| \le 1/2$;, in general, if there are atoms, $\|b\| \ge 2|V_+| \cdot |V_-|$. In any case,

$$\|b\|_{\mathrm{fil}} = \min\big(|V_+|, |V_-|\big) \le \frac{1}{2} \quad \text{and} \quad \|b\|_{\mathrm{fil}} \le \frac{1-\sqrt{1-2\|b\|}}{2} \quad \text{for } \|b\| \le \frac{1}{2}\,.$$

In other words,

$$\big\|(\partial^0)^{-1}\big\|_{\mathrm{fil}}(\beta) \le \frac{1-\sqrt{\max(0,1-2\beta)}}{2\beta}\,.$$

Thus, $\|(\partial^0)^{-1}\|_{\mathrm{fil}}(\beta) < 1$ for $\beta < 1/2$ and if V has no atoms, $\|(\partial^0)^{-1}\|_{\mathrm{fil}}(\beta) \to 1/2$ for $\beta \to 0$.

If $i \ge 0$, evaluation of $\|(\partial^i)^{-1}\|_{\mathrm{fil}}(\beta)$ remains problematic for most $\beta \in [0,1]$. It is not even clear at which β the function $\|(\partial^i)^{-1}\|_{\mathrm{fil}}(\beta)$ equals one. (We show in 3.7 that $\|(\partial^1)^{-1}\|_{\mathrm{fil}}(\beta) < 1$ for $\beta < 1/81$.)

Proof of $[\Delta \to \mathbb{R}^n]_{\mathrm{top}}$ *from 1.1.* Since the simplex $\Delta(V)$ has $\mathrm{syst}_i = \infty$ for $i > 0$, the above bounds imply that the complex $C^* = C^*(\Delta(V);\mathbb{Z}_2)$ has $fl_i(\beta) \le \beta$ for all $i > 0$ and $fl_0(\beta) \le \big(1-\sqrt{\max(0,1-2\beta)}\big)/2$ by the definition of fl_i in 2.4. Then an obvious computation for $\Phi_i = \Phi_i^n(C^*;\beta_n,\dots,\beta_{n-i})$ (see 2.4) shows that

$$(n+1)\Phi_{n-1}(\beta_n,0,\dots,0) < 1 = \mathrm{syst}^0\big(\Delta(V);\mathbb{Z}_2\big) \quad \text{for } \beta_n < \frac{2n}{(n+1)(n+1)!}$$

(with equal weights assigned to all $v \in V$ for finite V) and these bounds on fl_i and Φ obviously pass to the n-skeleton X of $\Delta(V)$.

If V has no atoms, the "compounded degrees" D_i are zero (see 2.4). If $|V|{=}N{<}\infty$, the normalized degrees of the i-faces of X at the j-faces are $\binom{N-j-1}{i-j}\binom{N}{i+1}^{-1} = O(N^{-j-1})$; hence, the "normalized compounded degrees" D_i (see 2.5) are $O(N^{-1})$ for $i < n$. Therefore,

$$(n+1)\Phi_{n-1}(\beta_n, D_{n-1},\dots,D_1) < 1$$

if

$$\beta_n \le \frac{2n}{(n+1)(n+1)!} - \varepsilon(N) \quad \text{for some } \varepsilon(N) = O(N^{-1})\,.$$

Then the "generic syst^0-relation" in 2.5 implies that generic piecewise smooth maps $F : \Delta^N = \Delta(V) \to \mathbb{R}^n$ satisfy

$$M_n^{\pitchfork}(F) \ge \frac{2n}{(n+1)(n+1)!} - O(N^{-1})\,,$$

where, clearly,

$$\max_{y\in\mathbb{R}^n} \big|F^{-1}(y)\big| \ge M_n^{\pitchfork}(F)\binom{N+1}{n+1} \ge \frac{2n}{(n+1)(n+1)!}\binom{N+1}{n+1}\big(1-O(N^{-1})\big)\,. \quad \square$$

Finally, returning to the notation in 1.1, we conclude

Let V be a probability space without atoms, $\cup_{\Delta^n}(V)$ be the geometric realization of the n-skeleton of the $\Delta(V)$ on the vertex set V and $F : \cup_{\Delta^n}(V) \to \mathbb{R}^n$ a measurable map which is continuous on each n-face $\Delta^n \subset \{\Delta_n\}(V)$. Then there exists a point $a_0 \in \mathbb{R}^n$ for which the probability that the n-face of $\cup_{\Delta^n}(V)$ contains a_0 is bounded from below by

$$\big|a_0 \pitchfork_F \{\Delta^n\}\big| \ge \frac{2n}{(n+1)(n+1)!}\,. \qquad [a_0 \pitchfork \{\Delta^n\}]$$

Questions. (a) Can $2n/(n+1)(n+1)!$ be replaced by const^{-n}, at least for face-wise affine maps F? (An optimist's suggestion would be const $\sim e = 2.71\ldots$ for large n.)

(b) Is there an LG-invariant (see 4.13.A) version of the above inequalities with some algebraic rank(c) instead of $\|c\|$ for cochains $c \in C^*(\Delta(V)) = \bigwedge^*(C^0(V))$?

2.7 Crofton–Wendel formula, waist inequalities and relative growth of infinite groups. Given a (polish) topological space V with a Borel probability measure μ, define a $\{\Delta^n\}$-*structure* on V as a Borel map from the geometric realization of the n-skeleton of $\Delta(V)$ to V, denoted $B_n : \cup_{\Delta^n}(V) \to V$, such that B_n is continuous on almost every $\Delta^n \subset \cup_{\Delta^n}(V)$ and equals the identity map on $V \subset \cup_{\Delta^n}(V)$.

Given a closed subset $W \subset V$, let $|W \pitchfork_{B_n} \Delta^n|$ denote the cardinality of the intersection of $B_n^{-1}(W) \subset \cup_{\Delta^n}(V)$ with a face $\Delta^n \subset \cup_{\Delta^n}(V)$, set

$$\mathrm{vol}^{\pitchfork}_{\Delta^n}(W) = \int_{V^{n+1}} |W \pitchfork_{B_n} \Delta^n| d\mu^{\otimes(n+1)}$$

and let

$$\mathrm{subvol}^{\pitchfork}_{\Delta^n}(W) = \big|W \pitchfork_{B_n} \{\Delta^n\}\big| = \int_{V^{n+1}} \min\big(1, |W \pitchfork_{B_n} \Delta^n|\big) d\mu^{\otimes(n+1)}$$

be the probability of an n-face in $\cup_{\Delta^n}(V)$ meeting $B_n^{-1}(W)$, where, clearly,

$$\mathrm{subvol}^{\pitchfork}_{\Delta^n}(W) \le \mathrm{vol}^{\pitchfork}_{\Delta^n}(W)\,.$$

Examples. **(a)** Let V be the unit m-sphere $S^m \subset \mathbb{R}^{n+1}$ and B_n the barycenter map (see 1.3). Then, by the Fubini–Crofton formula,

$$\mathrm{vol}_{\Delta^n}(W) \le \delta_n(m)\,\mathrm{vol}_{m-n}(W)/\,\mathrm{vol}(S^{m-n})\,,$$

where vol_{m-n} denotes the $(m-n)$-dimensional Hausdorff measure and where $\delta_n(m) = \mathrm{vol}_{\Delta^n}(S^{m-n})$. (If W is an $(m-n)$-dimensional *rectifiable* set, then, clearly, $\mathrm{vol}_{\Delta^n}(W) = \delta_n(m)\,\mathrm{vol}_{m-n}(W)/\,\mathrm{vol}(S^{m-n})$. (There are counterexamples for non-rectifiable sets going back to Besicovitch which were pointed out to me by Larry Guth, see 13.2.5 in [BurZ] and 3.3 in [F].)

Since the spherical measure is $\pm$-symmetric, the convex hulls of all 2^{n+1} tuples $(\pm s_i,\ i = 0,\ldots,n)$ have equal expectations of their volumes, and since $\sum_\pm \mathrm{vol}_n(\mathrm{conv}(\pm s_i)) = \mathrm{vol}(S^n)$, the expected volume δ_n^{n+1} of each $\mathrm{conv}(\pm s_i)$ equals $2^{-n-1}\,\mathrm{vol}(S^n)$; thus, $\delta_n(m) = 2^{-n}$, because a generic equatorial sphere S^n in S^m meets S^{n-m} at two points.

This and the above $[a_0 \pitchfork \{\Delta^n\}]$ imply the spherical waist inequality from 1.3.

Every continuous map $f : S^m \to \mathbb{R}^n$ admits a point $a_0 \in \mathbb{R}^n$, such that

$$\mathrm{vol}_{m-n}\left(f^{-1}(a_0)\right) \ge \frac{2^{n+2}\cdot n \cdot \mathrm{vol}(S^{n-m})}{(n+1)(n+1)!}.$$

Remark. The relation $\delta_n^{n+1} = 2^{-n-1}\,\mathrm{vol}(S^n)$ (which smells three hundred years old) is called *Wendel's formula* with the reference to [We], where the author, who attributes this to R.E. Machol and L.J. Savage, adds the following observation.

Let a $\pm$-symmetric probability measure (say, with measurable density) on S^n be given.

Then the probability $P(N)$ of $\mathrm{conv}(s_0,\dots,s_N)$ for $N \geq n$ being *contained in a hemisphere* in S^n equals $PD_n(N+1)/2^{N+1}$ for the *Pascale–Descartes number* $PD_n(N+1)$, i.e. the number of the complementary components to $N+1$ equators in general position in S^n.

Indeed, $P(N)$ equals the probability of the hemispheres $\mathrm{hem}(s_i) \subset S^n$ with centers $s_0,\dots,s_N \in S^n$ having a *non-empty intersection*. Since, generically, among 2^{N+1} intersections $\bigcap_{i=0,\dots,N} \mathrm{hem}(\pm s_i)$ there are (exactly) $PD_n(N+1)$ non-empty ones, the proof follows.

(Notice that $PD_1(N) = 2N$, $PD_n(N) = 2^{N+1}$ for $N \leq n+1$, and $PD_n(N+1) = PD_n(N) + PD_{n-1}(N)$; thus $PD_n(N+1) = 2\sum_{i=0,\dots,n} \binom{N}{i}$ for $N \geq n$.)

(b) Let V be an m-dimensional Riemannian manifold and $\Lambda = (\lambda_1,\dots,\lambda_k)$, $1 \geq k \geq m-1$, be a k-tuple of smooth 1-forms on V. Denote by $H \subset T(V)$ the kernel of Λ and by $\Omega : \bigwedge^2(H) \to H^\perp =_{\mathrm{def}} T(V)/H$ the differential $d\Lambda = (d\lambda_i)$ restricted to H and factored to $H^\perp$.

Say that a linear subspace $S = S_v \subset H_v \subset T_v(V)$, is Ω-*regular* if the linear forms $\lambda_i(v) : T_v(V) \to \mathbb{R}$ are linearly independent and the linear map $\Omega_S : H_v \to \mathrm{Hom}(S, H_v^\perp)$ defined by Ω is surjective.

If Λ is generic, then the forms λ_i are linearly independent away from a $(k-1)$-dimensional subset in $\Sigma_0 \subset V$ and if, furthermore $n \leq (m-k)/(k+1)$ then Ω-regularity fails for n-dimensional subspaces $S = S_v \subset H_v$, $v \in V \setminus \Sigma_0$ only away from a codimension 1 subset in the space of all $S = S_v$. Moreover (and this is what we need for the present purpose),

> *every generic* Λ *admits an* Ω*-regular and* Ω*-isotropic* n*-dimensional subspace* $S_0 \subset H_{v_0}$ *at some point* $v_0 \in V$*, provided* $n \leq (m-k)/(k+1)$, where Ω-isotropic signifies that Ω vanishes on $\bigwedge^2(S_0)$.

Denote by $|\dots|_{CC}$ the metric defined via the lengths of shortest paths $P : [0,1] \to V$ between $v_0 = P(O)$, $v_1 = P(1) \in V$ which are tangent to H, i.e. such that $\Lambda \circ D(P) = 0$.

If $m-k \geq 2$, this is a true metric for *generic* Λ and if $k \leq (m-k)(m-k-1)/2$ then the Hausdorff dimension of $(V, |\dots|_{CC})$ equals $m+k$ (see [Gr4] and references therein).

If, furthermore, $n \leq (m-k)/(k+1)$ and V is $(n-1)$-connected, then, by *local (folded) h-principle* (see [Gr4]) there exists a Λ-*adapted* $\{\Delta^n\}$-structure, $B_n : \cup_{\Delta^n}(V) \to V$, i.e. where the map B_n is smooth on almost all (for the Riemannian measure on V) n-faces $\Delta^n \subset \cup_{\Delta^n}(V)$, tangent to H on almost all Δ^n and such that

$$\mathrm{vol}^{\pitchfork}_{\Delta^n}(W) \leq \mathrm{const}(V,\Lambda)\, \mathrm{vol}^{CC}_{m-n+k}(W)\,,$$

where $\mathrm{vol}^{CC}_{m-n+k}$ denotes the $(m-n+k)$-dimensional Hausdorff measure associated to the metric $|\dots|_{CC}$.

In fact, we need B_n below only on a small neighborhood $U \subset V$, and the existence of such B_n in a neighborhood of a point v_0, where there is an Ω regular $S_0 \subset H_{v_0}$. This follows (as in [Gr4]) by combining the *Poenaru pleating lemma* (see [Gr2]) and the *microflexibility of sheaves of solutions of infinitesimally invertible differential equations* (see (a) below).

Thus, we conclude,

Let Λ *be a smooth* k*-tuple of* 1*-forms on an* m*-dimensional Riemannian manifold* V *which admits an* Ω*-isotropic and* Ω*-regular* n*-dimensional subspace* $S_0 \subset H_{v_0}$ *(e.g.* Λ *is generic and* $n \leq (nm-k)/(k+1)$*). Then every continuous map* $f : V \to \mathbb{R}^n$ *for* $n \leq l$ *admits a point* $a_0 \in \mathbb{R}^n$*, such that*

$$\mathrm{vol}^{CC}_{m-n+k}\left(f^{-1}(a_0)\right) \geq \varepsilon(V,\Lambda) > 0\,. \qquad [CC]_{m-n+k}$$

This applies, in particular, to the standard contact form on $V = S^{m=2l+1}$ (where $k=1$), and yields the bound

$$\mathrm{vol}^{CC}_{m-n+1}\left(f^{-1}(a_0)\right) \geq \varepsilon(m) > 0$$

claimed in 1.3.

Remarks. (a) Infinitesimal invertibility in the present case amounts to an *algebraic* non-degeneracy condition on (partial derivatives of) Λ which generalizes the Ω-regularity; this condition is satisfied by many non-Ω-regular Λ, but its verification may require a lengthy (albeit algorithmic) computation in certain cases.

The implication *infinitesimal invertibility* $\Rightarrow$ *microflexibility* is based on a localized version of the *Nash implicit function theorem* [Gr2] with a heavy analytic proof. Possibly, our application of this to lower bounds on CC-measures can be obtained with the *formal* (approximate) implicit function theorem with a purely algebraic (and trivial) proof.

(b) Take an adapted $\{\Delta^n\}$-structure B^0_n on the sphere S^{2n+1} with the standard $U(n+1)$-invariant contact structure; assume for the moment B^0_n is $U(n+1)$-equivariant and embed $S^{2n+1} = S^{2l+1} \cap \mathbb{C}^{n+1} \subset S^{2l+1}$ for some $l \geq n$. Then B^0_n uniquely extends to an $U(l+1)$-equivariant structure B_n on $S^{2l+1} \supset S^{2n+1}$ adapted to the $U(l+1)$-invariant contact structure on S^{2l+1}.

An $U(n+1)$-equivariant structure B^0_n on S^{2n+1} is not hard to construct; but even if B^0_n, is not equivariant, a $U(l+1)$-equivariant extension makes sense as a "random $\{\Delta^n\}$-structure", i.e. a family of $\{\Delta^n\}$-structures parametrized by a probability space – the group $U(l+1)$ with the normalized Haar measure in the present case.

If we use such an "induced from S^{2n+1}" structure B_n on S^{2l+1} ("random" is OK), then the above constant $\varepsilon(m)$ can be expressed as $\varepsilon(n)c(m-n)$ for a "standard" constant $c(m-n)$, similar to $\mathrm{vol}(S^{m-n})$ in the Riemannian case. But this still remains far from the unknown sharp constant.

(c) The local h-principle provides a similar inequality in a wider range of k and l, which does not, however, cover what can be expected by the obvious estimate for the CC-Hausdorff dimensions of *smooth submanifolds* $W \subset V$ (see [Gr4] for the related discussion).

(d) Question. Can the space cycles in a Riemannian (or sub-Riemannian) manifold V be approximated by the space of cocycles in some measurable complex C^*_ε constructed with suitable ε-small simplices in V?

We want such an approximation to be *sharp* (unlike the one we used above), such that the full measure geometric portrait of V, including the Plateau problem, would emerge from C^*_ε in the limit for $\varepsilon \to 0$. In particular, we would like to obtain *sharp* filling/waist inequalities in V by an argument similar to that in the above Examples **(a)** and **(b)**.

Relative growth of discrete metric spaces. Let X and Y be metric spaces where X is discrete and let $\phi : X \to Y$ be an L-Lipschitz map. Consider all subsets $B \subset X$, such that $\mathrm{diam}_X(B) \leq R$ and $\mathrm{diam}_Y(\phi(B)) \leq r$ and let $|Y \backslash_\phi X; r, R|$ denote the supremum of the cardinalities of all these $B \subset X$. Set

$$|Y \backslash X; L, rR| = \inf_\phi |Y \backslash_\phi X; r, R| \,,$$

where the infimum is taken over all L-Lipschitz maps ϕ.

> *Let $X_\circ$ be a finitely generated nilpotent group with a word metric, such that its tangent cone at infinity, which is a nilpotent Lie group with a CC-metric, say X_∞, has dimension m and which satisfies $[CC]_{m-n+k}$. Then there exist positive constants C_1 and C_2 depending on $X_\circ$, such that*
> $$|\mathbb{R}^n \backslash X; L, r, R) \geq C_2 R^{m-n+k} \quad \textit{for all } r \geq C_1 L \,.$$

Sketch of the Proof. Assume for simplicity's sake that $X_\circ$ admits a dilation and, thus, cocompactly embeds into X_∞. Extend a given Lipschitz map $\phi_\circ : X_\circ \to \mathbb{R}^n$ to a piecewise affine map $\phi : X_\infty \to \mathbb{R}^n$ (for some $X_\circ$-invariant triangulation of X_∞) and let $\phi_R : X_\infty \to \mathbb{R}^n$ equal the composition of ϕ with the self-homothety (for the CC-metric) of X_∞ which scales the R-ball $B(R) \subset X_\infty$ around the origin to the unit ball.

Then $[CC]_{m-n+k}$, applied to ϕ_R on the unit ball, provides the following lower bound on the growth of the Riemannian $(m-n)$-volumes (for a left-invariant Riemannian metric on X_∞) of the intersections of the fibers of ϕ with the Riemannian R-balls $B(R) \subset X_\infty$,

$$\sup_{y \in \mathbb{R}^n} \mathrm{vol}_{m-n}\big(\phi^{-1}(y) \cap B(R)\big) \geq \mathrm{const}\, R^{m-n+k} \quad \text{for large } R \,,$$

which trivially implies the corresponding lower bound on $|\mathbb{R}^n \backslash X_\circ; L, r, R|$.

Example. If $X_\circ \subset X_\infty = H^{2l+1}$ is a lattice in the Heisenberg group, then $|\mathbb{R}^n \backslash X; L, r, R) \sim R^s$, where $s = 2l+1-n$ for $n > l$ and $s = 2l+2-n$ for $n \leq l$.

Remarks. (a) The above scaling/discretization argument is standard, and filling in the details is straightforward.

(b) Since we need here only piecewise affine maps, we can prove it with Robert Young's filling inequality instead of $[\Delta \to \mathbb{R}^n]$. (This is definitely so for the Heisenberg and similar groups, but I am not certain if the general conditions in [Y] are identical to ours.)

(c) The geometry of $\mathbb{R}^n$ enters (essentially) only via the bound on the *asymptotic dimension* (see [Gr3]) of $\mathbb{R}^n$ by n, but the role of the geometry of Y (e.g. of $\mathbb{R}^n$) becomes prominent if we look at the asymptotic of $|Y \backslash X; L, r, R|$, when r also tends to infinity.

(d) Let X be a discrete cocompact group acting on symmetric space of dimension m of non-positive curvature. Then the hyperspherical waist inequality (see 3.5) implies that $|\mathbb{R}^n \backslash X_\circ; L, r, R) \sim R^{m-n}$ for $n \geq m - \operatorname{rank}(X)$ and $|\mathbb{R}^n \backslash X_\circ; L, r, R) \sim (1+\varepsilon)^R$ for $n < m - \operatorname{rank}(X)$.

(e) QUESTIONS:

(1) Can $|\mathbb{R} \backslash X_\circ; L, r, R|$ be evaluated in terms of the Følner function of the group $X_\circ$?

(2) Do the groups of exponential growth have $|\mathbb{R} \backslash X_\circ; L, r, R|$ exponentially growing in R?

(3) Does (d) extend to polycyclic groups?

The role of the rank for an ambient solvable Lie group X may possibly be replaced by distortion features of Abelian and/or nilpotent subgroups in X. For example, the presence of undistorted Abelian subgroup of rank p makes $|\mathbb{R}^n \backslash X_\circ; L, r, R|$ $\precsim R^p$ for $n = \dim(X) - p$; one wonders if all upper bounds on $|\mathbb{R}^n \backslash X_\circ; L, r, R|$ are of similar origin.

What can be said of pairs of groups X, Y with slowly growing $|Y \backslash X; L, r, R|$ for $R \to \infty$?

2.8 Randomized Radon–Tverberg theorem. Recall (see 1.2) that

$$b_{\text{aff}}(n,k) =_{\text{def}} \inf_{\mu} \sup_{A^{n-k}} \mu^{\otimes(k+1)}\big(A^{n-k} \pitchfork \{\Delta^k\}\big)$$

where μ runs over all probability measures on $\mathbb{R}^n$ and A^{n-k} over all $(n-k)$-dimensional affine subspaces in $\mathbb{R}^n$ and where $\mu^{\otimes(k+1)}(A \pitchfork \{\Delta^k\})$ denotes the probability that the convex span of a $(k+1)$-tuple of points in $\mathbb{R}^n$ intersects A.

Take $q \geq 2$, let $N = N_q = (n+1)(q-1)+1$, let P be a decomposition (partition) of $N-k-1$ into the sum

$$N - k - 1 = m_1 + n_2 + n_3 \cdots + n_q\,,$$

let $N_i = n_1 + n_2 + \cdots + n_i$ for $n_1 = k+1+m_1$ and denote by $\pitchfork_P(\mu)$ the $\mu^{\otimes N}$ measure of N-tuples $(y_1, \dots, y_N) \in (\mathbb{R}^n)^N$ such that the convex hull $\Delta = \Delta(y_1, \dots, y_{k+1}) \in \mathbb{R}^n$ intersects the affine span

$$A = A(y_{k+2}, \dots, y_{n_1}) + \bigcap_{i=1,\dots,q-1} A(y_{N_i+1}, \dots, y_{N_{i+1}})\,,$$

where $A(\dots) \subset \mathbb{R}^n$ denotes the affine span of points $(\dots)$ in $\mathbb{R}^n$.

Denote by $\pitchfork_q(\mu)$ the sum of $\pitchfork_P(\mu)$ over all decompositions (partitions) P of the number $N-k-1$ into q summands, $N-k-1 = m_1 + \sum_{i=2,\dots,q} n_i$ and observe that, for all $q = 2, 3, \dots$,

$$b_{\text{aff}}(n,k) \geq r(n,k,q) =_{\text{def}} \frac{1}{|\{P\}_q|} \inf_{\mu} \pitchfork_q(\mu)\,,$$

where $|\{P\}_q|$ for $q \geq 3$ denotes the number of partitions P and where we agree that $|\{P\}_2| = 1$, which is admissible since different partitions $n+2 = N_2 = m_1 + n_2$ amount to the same $A = A(y_{k+2}, \dots, y_{N_2})$.

At this point we recall *Tverberg's theorem* [Ma1,2],

every N_q-tuple Y of points in $\mathbb{R}^n$ for $N_q=(n+1)(q-1)+1$ in $\mathbb{R}^n$ can be partitioned into q disjoint subtuples of cardinalities n_i, for $i=1,2,\dots,q$, and $\sum n_i = N_q$, such that their convex hulls have a common point in $\mathbb{R}^n$.

Now, let us order the points in Y, say $Y=\{y_1,\dots,y_{N_q}\}$, such that the Tverberg partition agrees with this order, i.e. our subtuples are $(y_1,\dots,y_{n_1}),\dots,(y_{N_{q-1}+1},\dots,y_{N_q})$. for $N_i=n_1+\dots+n_i$, and let us suppose that the first subtuple contains at least $k+1$ points, i.e. $n_1=k+1+m_1$ for $m_1\geq 0$. Then the convex hull $\Delta\in\mathbb{R}^n$ of the first $k+1$ points $y_1,\dots,y_{k+1}$ intersects the above affine space A which is the sum of the affine span of m_1-subtuple $(y_{k+2},\dots,y_{n_1})$ for $n_1=k+1+m_1$ with the intersection of affine spans of the n_i-subtuples $(y_{N_{i-1}+1},\dots .y_{N_i})$ for $N_i=n_1+n_2+\dots+n_i$ and $i=2,\dots,q$.

It follows by averaging over all ordered N_q-tuples of points in $\mathbb{R}^n$ that

$$\frac{\pitchfork_q(\mu)}{|\{P\}_q|}\geq r(n,k,q)\geq r_0(n,k,q)=_{\rm def}\frac{1}{|\{P\}|}\min_{P\in\{P\}_q}\frac{\sum_i\binom{n_i}{k+1}}{\binom{N_q}{k+1}}\,,$$

where $\{P\}_q$ for $q\geq 3$ denotes the set of partitions P of N_q-k-1 into a sum $N_q-k-1=m_1+n_2+\dots+n_q$ and where we agree that $|\{P\}_2|=1$ and that $\binom{a}{b}=0$ for $a<b$.

It remains to estimate $r_0(n,k,q)$ from below, where a non-trivial bound is possible if and only if one of n_i is necessarily $\geq k+1$, i.e. if $N/q\geq k+1$ for $N=N_q=(n+1)(q-1)+1$, for which it is sufficient to have

$$q\geq\frac{n+1}{n-k}\,.$$

Furthermore, if $n_i\geq k+1$, then

$$\frac{\binom{n_i}{k+1}}{\binom{N}{k+1}}\geq\left(\frac{n_i-k-1}{N-k-1}\right)^{k+1}\,,\quad\text{for all } N\geq n_i\,,$$

which we combine with the Hölder inequality,

$$\frac{1}{N^{k+1}}\sum_i n_i^{k+1}\geq\frac{1}{q^k}\,,\quad\text{for } N=\sum_i n_i\,,$$

and where we recall that $|\{P\}_2|=1$ and observe that $|\{P\}_q|\leq(N_q-k-1)^{q-1}$ for $q\geq 3$.

If $q=2$, where Tverberg's theorem goes back to Radon, the above implies that, for each k,

$$b_{\rm aff}(n,k)\geq r_0(n,k,2)\,,\quad\text{where } 2^k r_0(n,k,2)\to 1 \text{ for } n\to\infty\,,$$

and where the rate of convergence, which was specified in 1.2, follows by a trivial computation.

If $(n-k)/(n+1)\geq\varepsilon_0>0$ we take $q_0=[\varepsilon_0^{-1}]+1$, where $[\dots]$ denotes the entire part of a number. A rough evaluation shows that, for each $\varepsilon_0>0$ (being kept fixed) and large $n\to\infty$,

$$b_{\rm aff}(n,k)\geq r_0(n,k,q_0)\gtrsim 1/q_0^{\gamma_0 k},$$

where $\gamma_0\approx\varepsilon_0^{-1}$ will do. This implies the bound stated in 1.2. (We do not attempt to be more precise as our $r_0(n,k,q)$ are far from the true lower bounds on $r(n,k,q)$ anyway.)

QUESTIONS. Let Φ be a formula involving convex hulls, affine spans and intersections of subsets and let us apply Φ to a system S of subsets Y_s, $s \in S$, in an N-tuple Y of points in $\mathbb{R}^n$ where the result is denoted $\Phi_S(Y_s) \subset \mathbb{R}^n$. When can one meaningfully evaluate the range the expectation E of the cardinality $|\Phi_S(Y_s)|$ with respect to the measure $\mu^{\otimes N}$, where μ is a probability measure on $\mathbb{R}^n$?

Is the range $E_\Phi \subset \mathbb{R}_+$ of possible values E for all probability measures μ on $\mathbb{R}^n$ effectively computable?

(The example of $r(n,k,q)$ indicates that it is worthwhile considering several Φ simultaneously, e.g. by looking at sums $\sum_S |\Phi_S(Y_s)|$ over suitable systems S of subsets in Y.)

2.9 Open questions on multiple points of maps. **a.** Can one significantly improve the above lower bounds on $b_{\text{aff}}(n,k)$ for $(1-\varepsilon)n \le k \le n$ in order to make them stronger (rather than weaker) than the present-day bounds on $b_{\text{aff}}(n,n)$?

Something can be done modulo *the Sierksma conjecture* according to which the number $T_{\text{aff}}(q,n)$ of Tverberg's partitions is bounded from below by

$$T_{\text{aff}}(q,n) = \big((q-1)!\big)^n,$$

but this barely brings the bound to Barany's k^{-k} for $n = k$.

Notice that the established lower bounds on the Tverberg numbers $T_{\text{aff}}(q,n)$ go via their topological counterparts $T_{\text{top}}(q,n)$ which concern intersections of images of faces of an $(N-1)$-simplex Δ^{N-1} *continuously* (rather than affinely as for T_{aff}) mapped to $\mathbb{R}^n$, i.e. where the convex hulls of n_i-tuples of points $y_1, \dots, y_{n_i} \in \mathbb{R}^n$ are replaced by (images of) continuous maps of $(n_i - 1)$-simplices to $\mathbb{R}^n$ (which are faces of Δ^{N-1}) with their vertices sent to $y_1, \dots, y_{n_i}$.

For example, it is shown in [Hel] (where references to earlier papers can be found) that if q is a power of a prime number, $q = p^r$, then

$$T_{\text{aff}} \ge T_{\text{top}}(q,n) \ge \frac{1}{(q-1)!}\left(\frac{q}{r+1}\right)^{\left[\frac{N-1}{2}\right]}.$$

b. The (known) proofs of the non-vanishing T_{top} and of lower bounds T_{top} are based on the *Borsuk–Ulam theorem* for free Z_q-actions (see[BaSS], [S]). For example, (see [BB])

> *every continuous map f of a convex body $X \subset \mathbb{R}^{n+1}$ to $\mathbb{R}^n$ admits a pair of non-intersecting (i.e. non-equal parallel) supporting hyper-planes $T_1, T_2 \subset \mathbb{R}^{n+1}$ such that the images $f(T_1 \cap X)$ and $f(T_2 \cap X)$ intersect.*

(If ∂X is smooth *strictly* convex, this is the usual Borsuk–Ulam and the general case follows by approximation.)

This, applied to the convex simplex $X_\Delta^{n+1} \subset \mathbb{R}^{n+1}$, yields the topological Radon (Bajmóczy–Barany) theorem: $T_{\text{top}}(2,n) \ge 1$.

c. The topological Tverberg theorem, whenever available, implies the *van Kampen–Floris theorem* (classical for $q = 2$):

Let $n = qk/(q-1)$ be an integer, let $N \ge N_{nq} = (n+2)(q-1)$ and let f be a continuous map of the k-skeleton $X = \Delta^{N,k}$ of the N-simplex to $\mathbb{R}^n$. Then there *are at least*

$$M \geq m(q,n)(N+1)!(q!)^{-1}\big((k+1)!\big)^{-q}$$

disjoint q*-tuples of points* $x_{ij} \in X$, $i = 1,\dots,q$, $j = 1,\dots,M$, *such that* $f(x_{ij}) = f(x_{i'j})$ *for all* $i, i' = 1,\dots,q$ *and* $j = 1,\dots,M$, *where*

$$m(q,n) \geq t(q,n) = \frac{T_{\mathrm{top}}(q,n)}{(N_{qn}+1)!(q!)^{-1}((k+1)!)^{-q}} \qquad [VKF]_q$$

(and thus $M \sim N^{q(k+1)}$ *for* $N \to \infty$*).*

Indeed, extend f to a map $F : \Delta^N \to \mathbb{R}^{n+1} \supset \mathbb{R}^n$, where $F^{-1}(\mathbb{R}^n) = \Delta^{N,k}$, and observe that every q-tuple of disjoint faces in Δ^N which meet in $\mathbb{R}^{n+1}$ under F yields a q-tuple of disjoint k-faces meeting in $\mathbb{R}^n$.

Sometimes, e.g. for $q = 2$ where $T_{\mathrm{top}}(2,n) = 1$, one has a strict inequality $m(q,n) > t(q,n)$, as follows from *Ramsey theorem*, but it is unclear what happens in general.

It also seems unknown, for any $q = 6, 10, \dots$ which *is not* a prime power, whether every compact k-dimensional topological space X, where $n = qk/(q-1)$ is an integer, admits a $(q-1)$-to-1 map to $\mathbb{R}^n$.

d. What are relations between the Barany numbers $b_{\mathrm{aff}}(n,k)$?

It is obvious that $b_{\mathrm{aff}}(n+1,k) \geq b_{\mathrm{aff}}(n,k)$ and $b_{\mathrm{aff}}(n+1,k+1) \leq b_{\mathrm{aff}}(n,k)$. Also

$$b_{\mathrm{aff}}(n, k_1+k_2) \geq b_{\mathrm{aff}}(n,k_1) b_{\mathrm{aff}}(n-k_1,k_2) \binom{(k_1+1)(k_2+1)}{k_1+k_2+1}^{-1},$$

since if k_2+1 convex simplices $\Delta_i^{k_1} \subset \mathbb{R}^n$, $i = 0,\dots,k_2$, meet $A^{n-k_1} \subset \mathbb{R}^n$ at some points a_i and if the convex hull $\Delta^{k_2} \subset A^{n-k_1}$ of a_i meets $A^{n-k_1-k_2} \subset A^{n-k_1}$, then the convex hull $\Delta^{k_1+k_2} \subset \mathbb{R}^n$ of k_1+k_2+1 (out of total $(k_1+1)(k_2+1)$) vertices of some $\Delta_i^{k_1}$ also meets $A^{n-k_1-k_2}$ by the *Caratheodory theorem*.

But this seems very weak. For example, starting from $b_{\mathrm{aff}}(n,1) = 1/2$ this yields by induction the mere $b_{\mathrm{aff}}(n,n) \gtrsim 2^{-n^2}$, and if we depart from the above (derived from Radon's theorem) inequality $b_{\mathrm{aff}}(n,k) \geq 2^{-n-1}$, for $n \geq 2k-1$, we arrive at something like $b_{\mathrm{aff}}(n,n) \leq (10n)^{-n}$ or $(10n)^{-10n}$. Probably, there should be stronger inequalities between $b_{\mathrm{aff}}(n,k)$ that would at least match the (known) lower bounds for $b_{\mathrm{aff}}(n) = b_{\mathrm{aff}}(n,n)$.

e. Are there smaller k-polyhedra X than the full k-skeleton $\Delta^{N,k}$ that have many q-multiple points under continuous (or, at least affine) maps to $\mathbb{R}^n$ with $q(n-k) \leq n$?

One would like to have families of such polyhedra $X(N)$ where the number $|\Sigma_k(X(N))|$ of k faces $\to \infty$ such that the number M_q of q-multiple points satisfies a lower bound similar to $[VKF]_q$, i.e. where $M_q \geq \mathrm{const} \cdot |\Sigma_k(X(N))|^q$, where const > 0 is independent of N or, if it decays, then slowly with $|\Sigma_k(X(N))|$, and, ideally, one wishes to have all $X(N)$ of uniformly bounded local degrees.

One knows, that expanders X^1 on N-vertices do have $\sim N^2$ crossings in the plane.

Also, one knows that every λ-expander X_N^1 on N-vertices contains "many" (randomly chosen) topological copies of the full bipartite $(3+3)$-graph X_{33}, where, moreover, each has about $(\log(N))^c$ edges in X^1 and these copies approximately uniformly, up-to $(\log(N))^c$-factor, cover X^1.

This can be combined, via the standard averaging argument with *Skopenkov's theorem* [Sk] saying that

every continuous map $(X_{33})^n \to \mathbb{R}^{2n}$ *has a double point.*

But what comes out of it for maps $(X^1_N)^n \to \mathbb{R}^{2n}$ is only $M_2 \geq (\log(N))^{-C})N^2$ (if not less, I did not check it carefully, but the referee pointed out that $M_2 \geq (\log(N))^{-C})N^n$), rather than $\sim N^{2n}$ double points.

It seems not hard to prove qualitatively optimal lower bounds on q-self-intersections of spherical buildings for $q = p^r$. But can one do this for "thinner" polyhedra than spherical buildings? For example, let X be the k-skeleton of a compact quotient of a Bruhat–Tits building or of a symmetric space of non-compact type with $\mathrm{rank}_{\mathbb{R}} \geq k+1$ (as in the concluding examples in 2.10). Do maps of such X to $\mathbb{R}^{2k}$ have "many" double points?

f. Apparently, the existence of many double points for maps $X^k \to \mathbb{R}^{2k}$ as well of highly multiple points for maps $X^k \to \mathbb{R}^k$ is due to the presence of many connected components in $X \setminus \Sigma(X)$, where $\Sigma(X)$ is the set of (singular) points where X fails to be a manifold. For example, if X^k is a manifold, then it embeds into $\mathbb{R}^{2k}$ by the Whitney theorem and

Every smooth k-manifold X admits a smooth nap $F : X \to \mathbb{R}^k$, where $\sup_{y\in\mathbb{R}^k} |F^{-1}(y)| \leq 4k$.

Sketch of the proof. Let $F' : X \to \mathbb{R}^{k-1}$ be a generic smooth map and let us construct a function $f_0 : X \to \mathbb{R}$, such that $F = F' \oplus f_0 : X \to \mathbb{R}^k = \mathbb{R}^{k-1} \oplus \mathbb{R}$ has the required property.

Shrink each connected component of every fiber of F' to a point, where, observe, each fiber $(F')^{-1}(y)$, $y \in \mathbb{R}^{k-1}$, is a possibly disconnected graph. Thus, we obtain a (stratified $(k-1)$-dimensional) space $\underline{X}$ and a factorization of the map F' into $F'_1 : X \to \underline{X}$ and $F'_2 : \underline{X} \to \mathbb{R}^{k-1}$, where F'_1 has connected fibers, i.e. where the F'_1-pullbacks of the points $\underline{x} \in \underline{X}$ are connected (graphs), while F'_2 has finite fibers.

Let $f_0 : \underline{X} \to \mathbb{R}$ be a generic function, let $f_1 = f_2 \circ F'_1 : X \to \mathbb{R}$ and slightly perturb f_1 to the desired $f_0 : X \to \mathbb{R}$ as follows.

The fibers $X_{\underline{x}} = (F'_1)^{-1}(\underline{x})$ are connected graphs with loops, where each $X_{\underline{x}}$ can be subdivided such that the resulting graph has no loops and contains at most $l = l(\underline{x}) \leq 4k$ edges. We design f_0 such that every graph $X_{\underline{x}}$ is sent by f_0 into a small neighborhood of $f_2(\underline{x}) \in \mathbb{R}$ with multiplicity $\leq l$. The images of different components of the fibers of F' do not meet in $\mathbb{R}$ unless the points $f_2(\underline{x})$ do and a counting of parameters (+ the usual general position argument) shows that these $f_2(\underline{x}_i)$ do not meet in $\mathbb{R}$ unless $\sum_i l(\underline{x}_i) \leq 4k$; then f_0 serves our purpose.

There are (in principle) computable obstructions (e.g. characteristic classes) for the existence of low multiplicity smooth generic maps; for example, complex projective spaces of even complex dimension k, probably, admit no maps into $\mathbb{R}^{2k}$ of multiplicity less than $4k$ or something like that. (A similar but apparently easier question is obstructing maps with a bound on the ranks of their local rings expressing the multiplicity of the analytic continuations of real analytic F.) But if X is stably parallelizable, then there is no apparent obstruction for maps of multiplicities 4.

If X_0 admits a map of multiplicity m and X is obtained from X_0 by a surgery adding handles of dimensions $\leq n-l$, then X admits a map of multiplicity $\leq m+n/(n-l)$ which opens a way for constructing maps of low multiplicity with some cobordism theory.

Finally, does every *closed* manifold of dimension k admit a map of multiplicity ≤ 4 into an open k-manifold?

2.10 l_2-Filling bounds and maps of 2-polyhedra to $\mathbb{R}^2$. Let

$$\|c\|_{l_2} = \left(\int_{\Sigma_i} \|c\|^2_{\mathbb{F}} d\sigma\right)^{1/2}$$

for measurable $\mathbb{F}$-cochains c on the set Σ_n of i-cells of a measurable complex X and let $\|c\|^{l_2}_{\text{fil}}$ and $\|(\partial^{i-1})^{-1}\|^{l_2}_{\text{fil}}$ be defined with this norm as was done in 2.3 with the l_1-norm $\|c\| = \|c\|_{l_1} = \int_{\Sigma_i} \|c\|_{\mathbb{F}} d\sigma$, where $\|(\partial^i)^{-1}\|^{l_2}_{\text{fil}}(\beta)$ as well as $\|(\partial^i)^{-1}\|_{\text{fil}}(\beta)$ are true norms independent of $\beta > 0$ if $\mathbb{F} = \mathbb{R}$.

Let X be a measurable cell complex where the spaces of cells $(\Sigma_i, d\sigma_i)$ are probability spaces and let $\mathbb{F} = \mathbb{R}$. Let Y be an oriented surface and $F : X \to Y$ be a continuous map. Let c_2 be a singular cycle which represents the fundamental class $[Y]_\circ \in H_2(Y;\mathbb{R})$ and which is quasitransversal (see 2.2) to F. Recall the supremum of the integrated intersection numbers of 1-cells in X with 1-faces of singular 2-simplices of c_2,

$$m_1 = m_1(F,c) = \sup_{\Delta^1} \int_{\Sigma_1} \|\sigma_1 \pitchfork \Delta^1\| d\sigma^1,$$

let

$$m_2^{l_2} = m_2^{l_2}(F,c) = \sup_{\Delta^0} \left(\int_{\Sigma_2} \|\sigma^2 \pitchfork \Delta^0\|^2 d\sigma^2\right)^{1/2},$$

where the supremum is taken over all vertices Δ^0 of the singular 2-simplices constituting c and let

$$M_2 = m_2^{l_2} + \|F^*[Y]^\circ\|_{l_2}$$

for the fundamental cohomology class $[Y]^\circ \in H^2(Y;\mathbb{R})$ of Y.

> *Let $H^1(X;\mathbb{R}) = 0$ and set*
> $$M_1 = m_1 + 2M_2 \cdot \left\|(\partial^1)^{-1}\right\|^{l_2}_{\text{fil}}.$$
> *Then*
> $$3M_1 \cdot \left\|(\partial^0_{\mathbb{Z}})^{-1}\right\|_{\text{fil}}(M_1) \geq 1\,,$$
> *where $\|(\partial^0_{\mathbb{Z}})^{-1}\|_{\text{fil}}$ denotes the filling norm in $C^*(X;\mathbb{Z})$).*

Proof. Observe that

> *the filling norm of every 2-coboundary $b \in \partial^1(C^1(X;\mathbb{Z}))$, call it $\|b\|^{\mathbb{Z}}_{\text{fil}}$, equals the filling norm $\|b\|^{\mathbb{R}}_{\text{fil}}$ of this b in the ambient $C^2(X;\mathbb{R}) \supset C^2(X;\mathbb{Z})$. In simple words you can find a $\mathbb{Z}$-(co)filling of b with the l_1-norm just as small as for any (co)filling in $C^1(X;\mathbb{R})$.*

Indeed, let $b = \partial^1(c)$ for some $c \in C^1(X;\mathbb{R})$ and let $\square_c \subset C^1(X;\mathbb{R})$ consist of the cochains $c' \in c + \partial^0(C^0(X;\mathbb{R}))$, i.e. $d = c - c' \in \partial^0(C^0(X;\mathbb{R}))$, such that

1. If c takes an integer value at a 1-cell σ^1, i.e. $c(\sigma_1) \in \mathbb{Z}$, then

$$d(\sigma^1) = c'(\sigma^1) - c(\sigma^1) = 0\,.$$

2. If $c(\sigma^1)$ is *not* an integer, then $c'(\sigma_1) \in (n, n+1)$ for the minimal integer interval $(n, n+1) \subset \mathbb{R}$ containing $c(\sigma^1) \in \mathbb{R}$. Equivalently,

$$\alpha_- < d(\sigma^1) < \alpha_+ \,,$$

where

$$-1 < \alpha_- = n - c(\sigma^1) < 0 \quad \text{and} \quad 0 < \alpha_+ = 1 + \alpha_- = n + 1 - c(\sigma^1) < 1 \,.$$

Notice that $\square_c$ is a *convex* subset in $C^1(X; \mathbb{R})$, since **1** and **2** are convex inequalities imposed on coboundaries d of real 0-cochains (where the coboundary operator ∂^0, being linear, preserves convexity). Furthermore, if $\partial^1 c \in C^2(X; \mathbb{Z})$, then

all extremal points c_{extr} of $\square_c$ are integer, i.e. $c_{\mathrm{extr}}(\sigma^1) \in \mathbb{Z}$ for all σ^1.

Indeed, if $\partial^1 c$ is an integer, then there exists a map, say ϕ from the 1-skeleton of X to the circle $S^1 = \mathbb{R}/\mathbb{Z}$ with the following properties:

- The map ϕ is locally affine on each 1-cell σ^1.
- •• The value $c(\sigma^1)$ equals the integral $\int_{\sigma^1} \phi^*(ds)$ for all 1-cells σ^1 in X, where ds is the canonical 1-form on the circle. Consequently, the length of $\phi(\sigma^1)$ counted with multiplicity equals $|c(\sigma^1)|$.

Moreover, this $\phi = \phi_c$ is unique up to rotations of S^1 and adding exact 1-cochains $d = \partial^0 f$ to c corresponds to homotopies of ϕ issuing from moving the points $\phi(\sigma^0) \in S^1$ for the 0-cells σ^0 of X.

If c is non-integer (modulo constants), then there are (at least) two 0-cells in X which are sent by ϕ to different points in S^1; this allows an obvious homotopy bringing these points together, thus, showing that such a c *cannot* be an extremal point in $\square_c$.

It follows that the infimum of the l_1 norm of c' on $\square_c$ equals that on $\square_c \cap C^1(X; \mathbb{Z})$; hence, every *real* cochain c with $\partial^1 c \in C^2(X; \mathbb{Z})$ can be replaced by an *integer* cochain c' with the same coboundary and the same (up to an arbitrarily small error) l_1-norm. Thus, as we claimed,

the complexes $C^*(X; \mathbb{Z})$ and $C^*(X; \mathbb{R}) \subset C^*(X; \mathbb{Z})$ have equal norms $\|b\|_{\mathrm{fil}}$ for all $b \in \partial^1(C^1(X; \mathbb{Z})) \subset C^2(X; \mathbb{Z}) \subset C^2(X; \mathbb{R})$.

Finally, since our norms are defined with a *probability* measure, the l_1-norm in $C^1(X; \mathbb{Z}) \subset C^1(X; \mathbb{R})$ is bounded by the l_2-norm, and 2.5 applies (with the use of only the first two steps in the definition of the compounded profile in 2.4).

Corollary. *Let X be a finite 2-dimensional complex where $\|(\partial^i)^{-1}\|_{\mathrm{fil}}^{l_2} \le C_i$ for $i = 0, 1$, let Y be an open surface and $F : X \to Y$ a generic piecewise smooth generic map, which is (at most) k-to-one on every 2-cell in X.*

Then there exists a point $y \in Y$ such that the number $|F^{-1}(y) \pitchfork \Sigma_2|$ of open 2-cells σ^2 in X such that $F(\sigma^2) \ni y$ is bounded from below by the number N_2 of all 2-cells in X for large N_2 as follows:

$$\left|F^{-1}(y) \pitchfork \Sigma_2\right| / N_2 \ge \left(4k(3C_0 + 6C_0C_1)\right)^{-2} + o(1) \,,$$

provided the numbers of 1- and 2-cells in X adjacent to the vertices (i.e. the degrees d of the vertices) are uniformly $O(1)$.

Indeed, take a sufficiently fine generic triangulation of Y and observe that if $|F^{-1}(y) \pitchfork \Sigma_2|/N_2 < \varepsilon$, then $m_2^{l_2} \le k\varepsilon^{1/2}$, while $\|F^*[Y]^\circ\|_{l_2}$ and $m_1 = o(n_1)$. Then the inequality $3M_1 \cdot \|(\partial^0_{\mathbb{Z}})^{-1}\|_{\mathrm{fil}}(M_1) \ge 1$ applies, since $\|(\partial^0_{\mathbb{Z}})^{-1}\|_{\mathrm{fil}}$ is estimated by $\|(\partial^0)^{-1}\|_{\mathrm{fil}}^{l_2}$ according to the

Mazia–Cheeger inequality.

$$\sup_{\beta\ge 0} \big\|(\partial^0_{\mathbb{Z}})^{-1}\big\|_{\mathrm{fil}}(\beta) \le 2 \sup_{0\le\beta\le 1} \big\|(\partial^0_{\mathbb{Z}_2})^{-1}\big\|_{\mathrm{fil}}(\beta) \le 4\big(\|(\partial^0)^{-1}\|_{\mathrm{fil}}^{l_2}\big)^2 .$$

Indeed, the l_1-norm of a function $c : \Sigma_0 \to \mathbb{Z}$, (i.e. $c \in C^0(X;\mathbb{Z})$) with given $\|\partial^0(c)\|$ can be bounded by adding up the $\mathbb{Z}_2$-bounds for the characteristic functions of the levels $c^{-1}(-\infty, n]$ and $c^{-1}[n, \infty)$ for those c, where the $d\sigma^0$ measures of $c^{-1}(-\infty, -1]$ and of $c^{-1}[1, \infty)$ are $\le 1/2$.

The second inequality, $\|(\partial^0_{\mathbb{Z}_2})^{-1}\|_{\mathrm{fil}} \le 2(\|(\partial^0)^{-1}\|_{\mathrm{fil}}^{l_2})^2$, is (trivially) obtained by applying the l_2-estimate to functions $c(1 - \|c\|) + (c-1)\|c\|$, where 0-cochains $c \in C^0(X, \mathbb{Z}_2)$ are regarded as functions $c : \Sigma_0 \to \{0, 1\} \subset \mathbb{R}$.

Concluding examples of polyhedra with l_2-bounds on ∂^{-1}. The bounds on $\|(\partial^i)^{-1}\|_{\mathrm{fil}}^{l_2}$ are available for many polyhedra X, which may have a given number N of vertices and all local degrees d bounded by a constant independent of N. This suffices for the inequality $[X_N \to \mathbb{R}^2]$ from 1.4. modulo a specification of C_0, C_1 and d, where the known examples of such X are obtained as follows.

Let G be a locally compact group such that $H^i(G; H) = 0$ for $0 < i < n$ for *all* Hilbert (space) G-modules H and let $\Gamma \subset G$ be a lattice. Let Γ act freely and discretely on a contractible n-dimensional locally finite simplicial polyhedron. Then the quotient space $X = \tilde{X}/\Gamma$ has $H^i(X;\mathbb{R}) = 0$ and $\|(\partial^{i-1})\|_{\mathrm{fil}}^{l_2} \le C_i(\tilde{X}) < \infty$ for $0 < i < \dim X$.

The simplest instances of such G are *(semi)simple p-adic Lie groups G acting on Bruhat–Tits buildings* $\tilde{X}$ according to the *Garland vanishing theorem* (see [G], [Bo]). The *congruence subgroups* Γ_i *of p-adic arithmetic groups* ([G], [BoH]) provide $X_i = X/\Gamma_i$ with $p^{ni}\,\mathrm{const}(\tilde{X})$ vertices. (I owe the reference [BoH]) to Akshay Venkatesh.) Moreover, Garland's argument delivers specific values of the constants $C_i(\tilde{X})$ and then one can trivially arrange X_N with any number N of vertices and slightly greater C_i.

Remark. One does not know if there are comparable l_1-bounds on $(\partial^i)^{-1}$ for $i \ge 1$, nor do such bounds seem in sight for complexes with $\mathbb{Z}_2$-coefficients. On the other hand, some l_p-estimates with $p < 2$ seem to be available. This, for $i = 1$, would allow an improvement of the above lower bound on $|F^{-1}(y) \pitchfork \Sigma_2|$ by $|F^{-1}(y) \pitchfork \Sigma_2|/N_2 \gtrapprox k^{-2+\varepsilon}$ with $\varepsilon > 0$.

Finally, to make some X_N simply connected (as was promised in 1.4), we attache discs to X_N along non-contractible curves and observe with Margulis' normal subgroup theorem (see [M2] and 4.3) that the universal coverings of the resulting spaces are compact (as well as simply connected).

2.11 Isoperimetry in cubes. The random cone construction applies to many measurable polyhedra besides the simplex $\Delta(V)$, where an individual filling (e.g. a contraction) can be averaged, e.g. over a compact group of symmetries of X. Thus, one obtains lower bounds on the maximal cardinalities of fibers of maps of such X to Euclidean spaces as in the case of $\Delta(V)$.

EXAMPLE: PRODUCT INEQUALITY FOR $\|\partial^{-1}\|_{\mathrm{fil}}$. Let X_0 be a measurable cellular space and let $X = X_0 \times \Delta(V)$ for the simplex $\Delta(V)$ spanned by a probability space V without atoms or made of finitely many equal atoms.

Each cocycle $b \in C^{i+1}(X)$ splits into $b = b_0 + b_*$, where b_0 equals the restriction of b to $\Sigma_i(X_0) \times V$, where $\Sigma_i(X_0)$ denotes the set of i-faces in X_0, and $b_* =_{\mathrm{def}} b - b_0$.

The contractions of $\Delta(V)$ to the vertices $v \in V$ define cochains $c_v \in C^i(X)$ such that $\partial(c_v) = b - b_v$ for the restrictions $b_v = |(X_0 \times v) \in C^{i+1}(X_0 = X_0 \times v)$, where

$$\int_V \|b_v\| dv = \|b_0\|$$

and

$$\int_V c_v dv \leq \|b_*\| \,.$$

by the averaging in $\Delta(V)$ (see 2.6).

On the other hand

$$\|b\|_{\mathrm{fil}} \leq \inf_{v\in V} \big(\|b_v\|_{\mathrm{fil}} + \|c_v\|\big) \leq \int_V \big(\|b_v\|_{\mathrm{fil}} + \|c_v\|\big) dv \,;$$

therefore,

$$\|b\|_{\mathrm{fil}} \leq \|b_*\| + \int_V \|b_v\|_{\mathrm{fil}} dv \leq \|b\| \cdot \max\big(1, \|(\partial^i_{X_0})^{-1}\|_{\mathrm{fil}}\big) \,.$$

EXAMPLE: ISOPERIMETRY IN THE CUBE. Let $X = [-1,1]^N$ be the N-cube with the uniform probability measures on the sets of its faces. Then

$$\big\|(\partial^i_X)^{-1}\big\|_{\mathrm{fil}} \leq \frac{\|\Sigma_i(X)\|}{2\|\Sigma_{i-1}(X)\|} = \frac{\binom{N}{i}}{2\binom{N}{i-1}} = \frac{N-i+1}{2i} \,.$$

Proof. This follows from the above, since the number $\|\Sigma_i(X)\|$ of i-faces in X equals $\binom{N}{i}$, where the additional 1/2 factor is due to the equality $\|(\partial^1)^{-1}\|_{\mathrm{rand}} = 1/2$ for the 1-simplex $[-1,1]$.

COROLLARY. *Let $F : X \to \mathbb{R}^n$ be a continuous map and let $|y \pitchfork_F \Sigma_n|$, $y \in \mathbb{R}^n$, denote the number the n-faces σ of X, such that $F(\sigma) \ni y$. Then*

$$\max_{y\in\mathbb{R}^n} |y \pitchfork_F \Sigma_n| \geq 2^{N-n}\binom{N}{n}\bigg((n+1)! \prod_{i=1,\dots,n} \frac{N-i+1}{2i}\bigg)^{-1} = 2^N/(n+1)!$$

where, observe, the number $|\Sigma_n(X)|$ of all n-faces in the N-cube equals $2^{N-n}\binom{N}{n}$.

Let us improve this to

$$\max_{y\in\mathbb{R}^n} |y \pitchfork_F \Sigma_n| \geq 2^{N-n} - \binom{N}{n}$$

as follows.

Finely triangulate the $(N-1)$-sphere $S = S^{N-1} = \partial([-1,1]^N)$, denote the triangulated sphere by by $S_\triangle$ and approximate $F|S = S_\triangle$ by a generic facewise affine map, say $f : S_\triangle \to \mathbb{R}^n$. It is clear that

(a) $\max_y |y \pitchfork_f \Sigma_n| \leq \max_y |y \pitchfork_F \Sigma_n|$;
(b) the fibers $f^{-1}(y) \subset S_\triangle$, $y \in \mathbb{R}^n$, are $(N-1-n)$-dimensional pseudomanifolds;
(c) the intersection of each fiber $f^{-1}(y)$ in S with the $(n-1)$-skeleton of the cubical decomposition of $S = S_\square = (\partial[-1,1]^N)$ is a finite set of cardinality $\leq n$.

Take the (diamond) decomposition (triangulation) $S_\diamond$ of S into 2^N simplices of dimension $N-1$ corresponding to the 2^N coordinate "octants" of $\mathbb{R}^N \supset [-1,1]^N$ and observe that $S_\diamond$ is the combinatorial dual to the cubical decomposition $S_\square$.

Approximate the identity map $S_\triangle \to S_\diamond$ by a simplicial map, say $a : S_\triangle \to S_\diamond$ (where each vertex x of $S_\triangle$ goes to the vertex $s = s(x)$ of $S_\diamond$ such that the star of s in the barycentric subdivision of $S_\diamond$ contains x) and observe that the images $c(y) = a(f^{-1}(y)) \subset S_\diamond$, $y \in \mathbb{R}^n$ make a family of $(N-n-1)$-dimensional piecewise linear $\mathbb{Z}_2$-cycles in $S = S_\diamond$, where we identify $\mathbb{Z}_2$-cycles with their supports.

Take a point $x \in f^{-1}(y)$, let Δ_x be a closed simplex of $S_\triangle$ containing x and consider the following possibilities.

(0) The simplex Δ_x does not intersect the n-skeleton $S^n_\square \subset S_\square$. Then $a(\Delta_x)$ is a simplex in $S_\diamond$ of dimension $< N-n-1$.
(1) Δ_x does not intersect the $(n-1)$-skeleton $S^{n-1}_\square$ and it intersects a *single* n-cube $\square^n$ which is also intersected by the fiber $f^{-1}(y)$. Then $a(\Delta_x)$ is a face of the $(N-n-1)$-simplex in $S_\diamond$ which is dual to $\square^n$.
(2) Δ_x is neither (0) nor (1). Then the dimension of the simplex $a(\Delta_x)$ may be $\geq N-n-1$.

In this latter case Δ_x must lie close to a cubical cell $\square = \square^m$, $m < n$, of $S_\square$ with $a(\Delta_x)$ being a face of the dual $(N-m-1)$-simplex $\Delta = \Delta^{N-m-1}$ of $S_\diamond$. If there are k such $\Delta = \Delta^{N-m_i-1}$, then there is a fiber $f^{-1}(y')$ for some $y' \in \mathbb{R}^n$ close to y, and points $x_i \in f^{-1}(y')$, such that $a(x_i)$ are contained in the interiors of the cubical cells $\square^{m_i}$ of $S_\square$; thus, $k \leq n-1$. Moreover, the genericity of f implies that the dimensions m_i satisfy
$$\sum_{i=1,\dots,k} (n - m_i) \leq n\,.$$

Therefore, one has the following (1) + (2)-relation:

Every cycle $c(y) = a(f^{-1}(y)) \subset S_\diamond$ consists of at most $M(F;y)$ simplices of $S_\diamond$ of dimension $N-n-1$ of type (1) *and of k-intersections of $(N-n-1)$-equators of the sphere $S = S_\diamond$ with the simplices Δ^{N-m_i-1} from* (2).

The spherical volumes of these equatorial (inter)sections are bounded by
$$\mathrm{vol}_{N-n-1}(S^{N-n-1} \cap \Delta^{N-m_i-1}) \leq (n - m_i + 1)^{-1} 2^{-(N-n)} \binom{N-m_i}{N-n}$$
with the normalization $\mathrm{vol}(S^{N-n-1}) = 1$. In fact, the volumes of the l-codimensional sections Q of any convex spherical polyhedron $P \subset S^L$ are bounded by $(l+1)^{-1} \mathrm{vol}_{L-l} P^{L-l}$, for P^{L-l} denoting the $(L-l)$-skeleton of P, since a generic l-equator intersecting P meets P^{L-l} at at least $l+1$ points while meeting Q at at most one point. (This bound is sharp as far as general P are concerned but it is very crude for regular simplices, where a better inequality must be known.)

It follows that the spherical $(N-n-1)$-volumes of $c(y)$ are bounded by

$$\mathrm{vol}_{N-n-1}(c(y)) \le 2^{-(N-n)} M(F;y) + \sum_{i=1,\dots,k} (n-m_i+1)^{-1} 2^{-(N-n)} \binom{N-m_i}{N-n},$$

which implies with the above bound on $n-m_i$ that

$$\mathrm{vol}(c(y)) \le 2^{-(N-n)} \left(M(F;y) + \binom{N}{N-n} \right).$$

On the other hand, by Almgren's waist inequality (see 3.3, 3.4)

$$\max_{y\in\mathbb{R}^n} \mathrm{vol}(c(y)) \ge 1$$

and the proof follows.

REMARKS. (a) The bound $\|(\partial_X^1)^{-1}\|_{\mathrm{fil}}(\beta) \le N/2$ is non-sharp except for $\beta = 1/2N$. The sharp bound is provided by the *the Harper inequality* which is a special case of the *Shannon inequality* for the entropies of measures on product spaces (see [Gr7] and references therein).

(b) If n is close to $N/2$, then the inequality $\max_{y\in\mathbb{R}^n} M(F;y) \ge 2^{N-n} - \binom{N}{n}$ is far from being sharp; here a better result is desirable.

(c) The *(1)+(2)*-relation remains valid for arbitrary families of $(N-n-1)$-cycles $c(y)$ in S parametrized by m-dimensional spaces Y (instead of $\mathbb{R}^n$), where $S(=S_\diamond)$ is a triangulated (Riemannian) $(N-1)$-space. One is concerned here with a lower bound on the maximal cardinality $M_n^{\pitchfork}(c(y))$ of the intersection of $c(y)$ with the n-skeleton of the combinatorial dual $S^\perp$ (instead of $S_\square$). The *(2)*-term in this case is about $m\binom{N}{n}$ and the issuing lower bound on $M_n^{\pitchfork}(c(y))$ in terms of $\max_y \mathrm{vol}(c'(y))$ is similar to the above. (This, with a rougher constant, is proven in [Gu1].)

QUESTION. When is a family of $\mathbb{F}$-cycles $c(y)$ with $\max_y \mathrm{vol}(c(y)) \le V_0$ homotopic to family $c'(y)$ with $M_n^{\pitchfork}(c'(y)) \le M_0 = M_0(V_0)$ for a given function $M_0(V)$? This is (almost) fully resolved in [Gu2] for $\mathbb{F} = \mathbb{Z}_2$ (where an essential point is to have the function $M_0(V)$ independent of the dimension of $Y \ni y$).

(d) The simplicial approximation c' of an individual generic cycle c has no *(2)*-term in it. In particular, if $F : [-1,1]^N \to \mathbb{R}^n$ is a smooth generic $\pm$-*symmetric* (i.e. $F(-x) = -F(x)$) map, then the symmetric simplicial approximation c' of $c = F^{-1}(0)$ is a cycle with $\mathrm{vol}_{N-n}(c') \le 2^{N-n}|0 \pitchfork \Sigma_n|$ in the $(N-1)$-sphere (where $|0 \pitchfork_F \Sigma_n|$ stands for the cardinality of the intersection of $F^{-1}(0)$ with the n-skeleton of the sphere $\partial[-1,1]^N$). Since the corresponding cycles $c/\pm$ and $c'/\pm \sim c/\pm$ in the projective space $\mathbb{R}P^{N-1} = S^{N-1}/\pm$ are non-homologous to zero, every n-equator $S^n \subset S^{N-n-1}$ meets c' and $\mathrm{vol}_{N-n}(c') \ge 1$ by Crofton's formula.

This yields

The Barany–Lovasz inequality. The cardinality $|0 \pitchfork_F \{\square^n\}|$ of the set of the closed n-faces of the N-cube $[-1,1]^N$ which intersect the zero set of a $\pm$-symmetric continuous (e.g. linear) map $F : [-1,1]^N \to \mathbb{R}^n$ satisfies

$$\left|0 \pitchfork_F \{\square^n\}\right| \ge 2^{N-n}.$$

(See [BaL] for further variations and applications of this argument.)

This inequality can also be derived (see [BaL]) from the following three combinatorial properties of the n-skeleton $\square^{N,n}$ of the cubical complex $[-1,1]^N$ and the $\pm$-involution on $\square^{N,n}$.

1. Σ^n is made of $K = 2^{N-n}\binom{N}{n}$ cells of dimension n and the involution has no fixed point in $\square^{N,n}$.
2. $\square^{N,n}$ contains an $(n-1)$-*acyclic* $\pm$-invariant subcomplex S with (at most) $L = 2\binom{N}{n}$ cells of dimension n, where "$(n-1)$-acyclic" means that $H^i(S;\mathbb{Z}_2) = 0$ for $i = 1,\dots,n-1$.

An example of $S \subset \square^{N,n}$ is the "folding locus" of a generic linear projection $P : [-1,1]^N \to \mathbb{R}^{n+1}$. In fact, such a P is one-to-one over the boundary of its image, say $S_P = \partial(P([-1,1])) \subset \mathbb{R}^{n+1}$, and S is *defined* as $P^{-1}(S_P)$. Clearly, S_P is a convex polyhedral n-sphere with a *single* pair of $\pm$-opposite n-faces for each sub-product of n out of N segments $[-1,1]$ in $[-1,1]^N$. This makes $\binom{N}{n}$ such pairs of faces in S_P, and, hence, in S.

3. The group G of cellular automorphisms g of Σ^n which commute with $\pm$ is *transitive* on the set of n-cells of $\square^{N,n}$.

It follows from 2 and 3 that

$\square^{N,n}$ is covered by $\pm$-symmetric $(n-1)$-acyclic subcomplexes $S_g = g(S)$, each having (at most) L cells, such that the number of S_g which contain an n-cell of $\square^{N,n}$ is independent of this cell.

Every generic $\pm$-symmetric map $F : \square^{N,n} \to \mathbb{R}^n$ has zeros in the interiors of at least 2 cells of each S_g by the Borsuk–Ulam theorem; hence $F^{-1}(0)$ meets at least $2K/L = 2^{N-n}$ cells in $\square^{N,n}$.

2.12 Filling with subdivided cones. Let X be a measurable cell complex, where the sets $\Sigma_n = \Sigma_n(X)$ of n-cells σ in X are *probability* spaces, where all *closed* cells are *embedded* into X and where the face maps $\Sigma_n \to \Sigma_{n-1}$ are measure preserving.

Let $Z = X^{n-1} \times [0,1]$, where $X^{n-1} \subset X$ denotes the $(n-1)$-skeleton of X, and let Z' be a measurable cellular subdivision of Z. Denote by $\Sigma'_n = \Sigma_n(Z')$ the set of n cells in Z' and observe that every cell $\sigma' \in \Sigma'_n$ is contained in $\sigma \times [0,1]$ for a unique $\sigma = \sigma(\sigma') \in \Sigma_{n-1} = \Sigma_{n-1}(X)$. This gives us a map, say $r : \Sigma'_n \to \Sigma_{n-1} = \Sigma_{n-1}(X)$ with finite $r^{-1}(\sigma) \subset \Sigma'$. Let $d\sigma'$ be the measure on Σ' defined by

$$\int_{\Sigma'_n} \phi(\sigma')d\sigma' = \int_{\Sigma_{n-1}} \Big(\sum_{\sigma' \in r^{-1}(\sigma)} \phi(\sigma') \Big) d\sigma\,.$$

Let P be a probability space and $R = \{R_p\} : Z' \times P \to X^n \subset X$, be a cellular map such that the corresponding map on the sets of n-cells, denoted $R^n : \Sigma_{n-1} \times P \to \Sigma_n$, is measurable.

Given $b \in C^n(X;\mathbb{F})$ for a given field $\mathbb{F}$, define

$$c_p(\sigma) = \sum_{\sigma' \subset \sigma \times [0,1]} b\big(A(\sigma')\big) \in C^{n-1}(X;\mathbb{F})\,,$$

and observe that the family δ of homomorphisms $\delta_p : b \mapsto c_p$, $p \in P$, makes a random contraction (cone) for $\partial^{n-1} : C^{n-1}(X) \to C^n(X)$ (see 2.6) if

- the map R_p sends every cell of $Z = Z \times 0 \times p$ onto itself with $\mathbb{F}$-degree 1 for almost all $p \in P$;

•• the homology homomorphism induced by $R_p(Z \times 1 \times p) \subset X$ on the boundary $\partial(\sigma) \subset X^{n-1}$ of each n-cell σ in X vanishes on the fundamental class $[\partial(\sigma)] \in H_{n-1}(\partial(\sigma);\mathbb{F})$ for almost all $p \in P$ (e.g. $H_{n-1}(R_p(Z \times 1 \times p);\mathbb{F}) = 0$ for (almost) all $p \in P$).

Denote by $|dR^n|(\sigma)$ the the *Radon–Nikodym derivative of* R^n, which is defined by the condition

$$\int_{\Sigma'_n \times P} \varphi\big(R^n(\sigma')\big) d\sigma' dp = \int_{\Sigma_n} |dR^n|(\sigma)\varphi(\sigma) d\sigma$$

for all functions $\varphi : \Sigma_n \to \mathbb{R}$, and observe that

$$\int_P \|c_p\| dp \le \int_{\Sigma_n} |dR^n|(\sigma)\|b(\sigma)\|_{\mathbb{F}} d\sigma \le \|b\| \sup_{\sigma \in \Sigma_n} |dR^n|(\sigma)\,.$$

Therefore (see 2.6)

$$\big\|(\partial^{n-1})^{-1}\big\|_{\text{fil}} \le \big\|(\partial^{n-1})^{-1}\big\|_{\text{rand}} \le \|\delta\| \le \sup_{\sigma \in \Sigma_n} |dR^n|(\sigma)\,.$$

Let us exclude $|dR^n|$ from the bound on $\|(\partial^{n-1})^{-1}\|_{\text{fil}}$ in a presence of symmetry of X as follows.

Let G be a group of measurable automorphisms of X with the finite set O of ergodic components $o \subset \Sigma_n$.

Denote by $m'_o(\sigma \times [0,1])$, for $o \in O$ and $\sigma \in \Sigma_{n-1}$, the number of n-cells σ' in $\|Z'\|$, such that $h(\sigma') \in o$, and let

$$m'_o = \sup_{\sigma \in \Sigma_{n-1}} m'_o\big(\sigma \times [0,1]\big)\,.$$

If the function $|dR^n|$ *is in* $L_1(\Sigma_n)$ *(i.e.* $\int_{\Sigma_n} |dR^n|(\sigma)d\sigma < \infty$*) and the coefficient field* $\mathbb{F}$ *is finite, then*

$$\big\|(\partial^{n-1})^{-1}\big\|_{\text{fil}} \le M = \sup_{o \in O} |o|^{-1} m'_o\,.$$

Proof. It suffices to show that

$$\inf_{g \in G} \int_{\Sigma_n} |dR^n|(g(\sigma))\|b(\sigma)\|_{\mathbb{F}} d\sigma \le M\,,$$

or, equivalently, that

$$\inf_{g \in G} \int_{\Sigma_n} |dR^n|(\sigma)\big\|b(g(\sigma))\big\|_{\mathbb{F}} d\sigma \le M$$

where, recall, $\|b\| = \int_{\Sigma_n} \|b(\sigma)\|_{\mathbb{F}} d\sigma$.

Let $B \subset L_2(\Sigma_n)$ be the closed convex hull of the orbit $G(b) \subset L_2(\Sigma_n)$ and let $b_0 \in B$ be the (unique!) vector minimizing the L_2-norm on B. Clearly, b_0 is G-invariant, i.e. $b_0(\sigma) = b_0(o)$ for $o \ni \sigma$,

$$\|b\| = \|b_0\| = \sum_{o \in O} b_0(o)|o|$$

and

$$\inf_{g \in G} \int_{\Sigma_n} |dR^n|(\sigma)\big\|b(g(\sigma))\big\|_{\mathbb{F}} d\sigma \le \int_{\Sigma_n} |dR^n|(\sigma) b_o(\sigma) d\sigma = \sum_{o \in O} b_0(o) \int_o |dR^n|(\sigma) d\sigma\,.$$

Finally,

$$\sum_{o\in O} b_0(o)\int_o |dR^n|(\sigma)d\sigma \le \sum_{o\in O} b_0(o)m'_o \le \Big(\sup_{o\in O}|o|^{-1}m'_o\Big)\sum_{o\in O} b_0(o)|o| = M\|b_0\|\,,$$

and the proof follows.

2.13 Colored polyhedra and spherical buildings. Let V be a probability space, let P be a finite measurable partition of V, which may be regarded as the quotient map $P : V \to V/P$ called a *V/P-coloring of* V. Let $P^{\triangle} : \Delta(V) \to \Delta(V/P)$ be the simplicial map induced by this map.

Let $X = \Delta^n_{\mathrm{col}}(V/P) \subset \Delta(V)$ be the union of the n-faces of the simplex $\Delta(V)$ on which the map P is injective. This X is called *the full V/P-colored polyhedron*; it is a measurable subcomplex in $\Delta(V)$ and we normalize the induced measures on the sets of faces of X to probability measures.

If $\mathrm{card}(V/P) \ge n+1$ then X is nonempty and it is $(n-1)$ connected. Indeed, a full $(\{1,\dots,i+1\}$-colored space $X(i+1)$ equals the union of the cones from the $(i+1)$-colored vertices over the full $(\{1,\dots,i\}$-colored $X(i) \subset X(i+1)$. Since these cones intersect over $X(i)$, the $(i-1)$-connectedness of $X(i)$ implies i-connectedness of $X(i+1)$ by (the trivial part of) Freudenthal's suspension theorem.

Thus, the cone Z over the $(n-1)$-skeleton $X^{n-1} = \Delta^{n-1}_{\mathrm{col}}(V/P) \subset X$ admits a map $R_0 : Z \to X$ which equals the identity on the base of the cone, that is $X^{n-1} \subset Z$, and which is simplicial with respect to some subdivision Z' of Z.

Such a contraction can be implemented by induction on skeletons of X^{n-1} where one extends maps from subdivided boundaries of simplices Δ^i, $i = 1,\dots,n$, to further subdivided Δ^i. Since every finite set V_0 of vertices of X is contained in a full colored subpolyhedron $X_0 = X_0(V_0)$ with at most $|V_0|+n-1$ vertices, and since all these X_0 are all $(n-1)$-connected, such extensions take place within subpolyhedra in X of uniformly bounded size. It follows that each n-simplex in Z contains at most κ_n simplices of Z' for some universal constant κ_n (which is $\precsim n^n$ by a rough estimate).

If V is a finite set, then such R_0 is measurable and if both spaces V and V/P have all atoms of equal weight, then the automorphism group G of X is transitive on the n-faces of X and we conclude (see 2.6) that the filling norm of $(\partial^{n-1})^{-1}$ in the complex $C^*(X;\mathbb{Z}_2)$ is bounded by κ_n. Then an obvious arrangement of a measurable family of contractions deliverers the same bound for infinite V if all atoms in V/P have equal weights. It follows that

> *every measurable map $F : X \to \mathbb{R}^n$ (where $X = \Delta^n_{\mathrm{col}}(V/P)$ is a full colored n-dimensional simplicial space) which is continuous on the n-simplices, admits a point $y \in \mathbb{R}^n$, such that the (probability) measure m of the set of the n-faces in X which intersect $F^{-1}(y) \subset X$ satisfies*
>
> $$m \ge c(n) \ge \Big(\prod_{i=1,\dots,n} \kappa_i(i+1)\Big)^{-1}.$$

Remarks. (a) If V is a finite space with N equal atoms, one has an additional $-O(N^{-1})$ term as in estimates for Φ_{n-1} in 2.5.

(b) If the number of colors is significantly greater than n, then the $[\Delta \to \mathbb{R}^n]$-inequality (see 1.1) gives a better lower bound on m. Also, the contraction/averaging argument delivers a bound on m without direct use of automorphisms of V/P (which allows a similar bound for $X = \Delta_{\rm col}(V/P)$ where V/P may have atoms of different weights), but the resulting lower bound on $c(n)$ remains poor.

(c) The $(n-1)$-connectedness of X is (essentially) equivalent to the color extension property proven in [I].

Let $\{S_\bullet\}$ be a collection of connected simplicial complexes (polyhedra) $S_\bullet$.

A (measurable) $\{S_\bullet\}$-*connected space* X is a (measurable) simplicial complex with a distinguished family of subcomplexes S, called *apartments* in X, such that every S is isomorphic to some $S_\bullet \in \{S_\bullet\}$, and such that every two simplices of X lie in a common apartment.

Let X be an $\{S_\bullet\}$-connected space, such that

all $S_\bullet \in \{S_\bullet\}$ (and, hence, all apartments in X) are $(n-1)$-connected;

if apartments $S_i \in X$ have an n-simplex in X in common, then the intersection $\cap_i S_i$ is $(n-2)$-connected;

every simplex of X is a face of an n-simplex in X.

Then, clearly, X is $(n-1)$-connected: the cone Z over the $(n-1)$-skeleton $X^{n-1} \subset X$ admits a map $R_0 : Z \to X$ which equals the identity on the base of the cone and which is simplicial with respect to some subdivision Z' of Z. Moreover, as in the full colored case, each n-simplex in Z contains at most $\kappa(N_\bullet)$ simplices of Z' for $N_\bullet$ denoting the supremum of the numbers of simplices in $S_\bullet \in \{S_\bullet\}$ and some universal function $\kappa(N_\bullet)$ which is $< \infty$ for $N_\bullet < \infty$.

The classical examples are *spherical Tits' buildings X over locally compacts fields* $\mathbb{K}$, where there is a single $S_\bullet$ which is a triangulated n-sphere, where the intersection of $k \geq 2$ different apartments with an n-simplex in common is contractible and where every X admits a compact automorphism group which is transitive on the set of n-simplices of X.

For instance, *the complex of flags of projective subspaces in* $\mathbb{K}P^{n+1}$ makes such a building, where each S is the subcomplex of flags of intersections of $n+1$ hyperplanes in $\mathbb{K}P^{n+1}$ in general position. (The corresponding $S_\bullet$ is isomorphic to the barycentric subdivision of the boundary of the $(n+1)$-simplex.)

The inverse filling norms of $C^*(X;\mathbb{Z}_2)$ for classical n-dimensional buildings X are bounded, as in the full colored case, by

$$\left\|(\partial^i)^{-1}\right\|_{\rm fil} \leq \kappa_\bullet(n)$$

for some universal function $\kappa_\bullet$, since $N_\bullet$ is bounded by the order of a reflection (Weyl) group W with $|W| \leq {\rm const}_n$. Consequently,

every measurable map $F : X \to \mathbb{R}^n$ which is continuous on the n-simplices admits a point $y \in \mathbb{R}^n$, such that the (probability) measure m of the set of the n-faces in X which intersect $F^{-1}(y) \subset X$ satisfies

$$m \geq {\rm const}(n) \geq \left(\kappa_\bullet(n)^{n+1}(n+1)!\right)^{-1}.$$

L_2-REMARK. There are sharp bounds on the inverse filling L_2-*norms* in the L_2-cochain complexes of classical n-dimensional buildings X for all n. These bounds for $n=2$ provide better estimates than the multiplicities m of maps F of 2-dimensional buildings X to $\mathbb{R}^2$ under the assumption of injectivity of F on the 2-simplices of X (as in 2.10).

On measurable algebraic polyhedra. Classical spherical buildings are instances of *simplicial algebraic polyhedra* defined as follows.

Start with an algebraic variety V (defined over some field or ring, e.g. over $\mathbb{Z}$) and take an n-dimensional subcomplex X in the simplex $\Delta(V)$, such that the sets of ordered i-faces $\Sigma_i^{\mathrm{ord}}(X) \subset \Sigma_i^{\mathrm{ord}}(\Delta(V)) = X^{i+1}$ are algebraic subvarieties.

If the codimensions of $\Sigma_i^{\mathrm{ord}}(X) \subset X^{i+1}$ are small compared to $\dim(V)$ and n, then the set of points of X over a "sufficiently large" field $\mathbb{K}$ is likely to be $(n-1)$-connected (possibly, with a mild genericity assumption on $\Sigma_i^{\mathrm{ord}}(X)$). If $\mathbb{K}$ is locally compact, then X can be given a rather canonical measure structure (e.g. by an embedding of V to a projective space); then the inverse filling norms are, probably, bounded by the above argument.

An example of such X, closely related to the flag complex, is where V is the Euclidean N-sphere, and $\Sigma_i(X)$ consists of the $(i+1)$-tuples of mutually orthogonal vectors in this $V = S^N \subset \mathbb{R}^{N+1}$.

If $n=1$, i.e. X is an algebraic graph, then the most attractive case is where the dimension of the edge set $\Sigma_1(X) \subset V \times V$ equals the dimension of V. Here the connected components of X make a partition of the space of $X(\mathbb{K})$ of $\mathbb{K}$-points of V, which is, typically, non-measurable for infinite locally compact fields $\mathbb{K}$ with apparently interesting dynamical/arithmetical properties.

For instance, if $\mathbb{K}$ is a finite field, and $\Sigma_1(X)$ has *several* (more than 2) irreducible components, then, typically, $X(\mathbb{K})$ is an expander graph. In general, the foliation must have some Kazhdan-T-like properties.

QUESTION. Is there a similar class of "small" n-dimensional algebraic polyhedra for $n \geq 2$, incorporating quotients of Bruhat–Tits buildings by arithmetic groups (as in 2.10)?

2.14 Random polyhedra. Let X be an n-dimensional cell complex and let $X_{\mathrm{ran}}(N)$ denote the union of N randomly chosen closed n-cells in X.

Assume that the numbers of cells of dimensions $n-1$ and n in X satisfy

$$\left|\Sigma_{n-1}(X)\right| = N_{n-1} \leq N \leq N_n/2 = \left|\Sigma_n(X)\right|/2\,.$$

Then the number of cochains in $C^{n-1}(X;\mathbb{Z}_2)$ is much smaller than the number of N-tuples of cells in $\Sigma_n(X)$,

$$2^{N_{n-1}} \ll \binom{N_n}{N}\,.$$

It follows that the norms of the coboundaries of $\mathbb{Z}_2$-cochains $c \in C^{n-1}(X;\mathbb{Z}_2)$ with respect to the uniform probability measure on the sets $\Sigma_{n-1}(X)$ and $\Sigma_n(X)$ do not change much for large N as we pass from X to $X_{\mathrm{ran}}(N)$, for a fixed $\theta = 1 - N_{n-1}/N > 0$ and $N \to \infty$. Namely, they satisfy the following *combinatorial sampling inequality* (probably, several hundred years old).

There exists a universal strictly positive monotone increasing function $\alpha(t)$, $t > 0$, *such that* $\alpha(t) \to 1$ *for* $t \to \infty$ *and such that the coboundaries* $b = \partial^{n-1}(c)$ *of all cochains* $c \in C^{n-1}(X_{\mathrm{ran}}; \mathbb{Z}_2) = C^{n-1}(X; \mathbb{Z}_2)$ *for* $X_{\mathrm{ran}} = X_{\mathrm{ran}}(N = (1+\theta)N_{n-1})$ *satisfy*

$$\|b\|_{X_{\mathrm{ran}}} \geq \alpha\big(\theta \cdot \|b\|_X\big)\|b\|_X$$

with probability $P(\theta)$, *where*

$$1 - P(\theta) \leq 1/\big(1+\alpha(\theta)\big)^{N_{n-1}} . \qquad [CSI]$$

Therefore, there exists a universal continuous function $\gamma(\beta)$, $\beta \geq 0$, such that $\gamma(\beta) \to 0$ for $\beta \to 0$ and such that

the filling norms $\|b\|^{\mathrm{ran}}_{\mathrm{fil}}$ *of all cocycles* b *in* $C^n(X_{\mathrm{ran}}; \mathbb{Z}_2)$ *are bounded, with probability* $P(\theta) \geq 1 - 1/(1+\alpha(\theta))^{N_{n-1}}$, *in terms of the inverse filling norm in* X, *that is*

$$\big\|(\partial_X^{n-1})^{-1}\big\|_{\mathrm{fil}} = \sup_{b \in \partial^{n-1}C^{n-1}(X);\mathbb{Z}_2)} \|b\|_{\mathrm{fil}}/\|b\| ,$$

as follows:

$$\|b\|^{\mathrm{ran}}_{\mathrm{fil}} \leq \gamma(\|b\|)\big\|(\partial_X^{n-1})^{-1}\big\|_{\mathrm{fil}} .$$

Remarks. (a) This inequality shows that the maps $F : X_{\mathrm{ran}} \to \mathbb{R}^n$ for polyhedra X from sections 2.11–2.13, satisfy roughly the same lower bonds on their multiplicities $m = \max_{y \in \mathbb{R}^n} |F^{-1}(y)|$ as X themselves.

What remains unclear, however, is if this remains true for all (non-random) subpolyhedra $X' \subset X$ with "many" cells, i.e. where $|\Sigma_i(X')|/|\Sigma_i(X)| \geq \text{const} > 0$ for $|\Sigma_0(X| \to \infty$. (This is known for a face-wise affine map under the name of the "second selection lemma", see [Ma1]. Such X' may have large inverse filling norms (e.g. they may be disconnected), but apparently, they decompose into "few" clusters with high connectivity and "good" filling properties.

(b) A sharper version of [CSI] with a control on $\alpha(\theta)$ for small θ and on local degrees is used in the Kolmogorov–Brazdin–Pinsker theorem [P], [KoB] on the existence of *graph expanders* (see 4.3).

(c) How small an n-polyhedron on N_0 vertices having all inverse filling norms bounded by a constant independent of N_0 as $N_0 \to \infty$ may be is unclear. For example, one has the following:

n-Expander question. Are there simplicial n-polyhedra on N_0-vertices, for arbitrary large N_0 with $\|(\partial^i_X)^{-1}\|_{\mathrm{fil}} \leq \text{const}$, $i = 0, 1, 2, \ldots, n-1$, and with the *n-degrees of the vertices*, i.e. the numbers of n faces adjacent to all vertices also bounded by a constant or, at least, with the numbers of n-simplices bounded by $\text{const} \cdot N$?

The difficulty in constructing random n-polyhedra for $n \geq 2$ is due to the fact that the space of n-polyhedra X on N_0-vertices conditioned by the inequality $\deg_n(X) \leq \text{const}$ at all vertices does not seem to have, for $n \geq 2$, a "good parametrization/approximation" by a product probability space (in the spirit of the approximation of the microcanonical measure of a free particle system by the Gauss–Gibbs measure). The absence of a "good parametrization/approximation" makes evaluation of the expectations of invariants of such random polyhedra unapproachable by the straightforward large deviation analysis (but no lower bound of any specific invariant measuring the efficiency of such parametrizations/approximations of subsets in binary spaces seems to be known). Probably, one can precisely state (and eventually prove) non-existence of a "good" theory of "random n-dimensional polyhedra of bounded degrees" for $n \geq 2$ (or, at least, for $n \geq 5$).

The presence of a "good parametrization/approximation" becomes, apparently, less likely (and the problem becomes more intriguing) with the additional conditioning on the topology of X, e.g. by requiring that fundamental group of X (or even $H_1(X)$) be trivial, since the triviality of a finitely presented group (the fundamental group $\pi_1(X)$ in the present case) is algorithmically unsolvable. Yet, this does not (?) rule out a rough evaluation of the asymptotics, for $N \to \infty$, of the number of *connected simply connected* 2-polyhedra on N-vertices with all vertices with 2-degree $\leq$ const or with the number of the 2-faces $\leq \mathrm{const} \cdot N$. (See [Gr5].)

The π_1-problem seems more approachable with a bound on the (normalized) *Dehn function* $\delta(r)$, of X_{ran} that is the minimal function, such that every minimal (area) disk D in X of area $\leq r$ has

$$\mathrm{area}(D) \leq \delta(r)\,\mathrm{length}(\partial D)\,.$$

The inequality $\delta(r) \leq$ const, say in the range $r \lessapprox \log N$, makes π_1 infinite hyperbolic in many/most (?) cases (see [O] and references therein). Alternative conditions are $\delta(r) \leq \mathrm{const} \cdot r$ for $r \leq R(N)$, say for $R(N) = \mathrm{const} \cdot N$ or bounds on the minimal Lipschitz constants of maps of the unit disk into X bounding curves of length $\leq r$. But (approximate) counting numbers of polyhedra satisfying such conditions remains an open problem.

(d) One may try to diminish the n-degree of the n-skeleton of a polyhedron by "blowing up" the $(n-1)$ skeleton, i.e. removing a small neighborhood of this skeleton and gluing in something of smaller degree by elaborating on the construction of iterated cubical graphs [Gr1] and/or on the *zig-zag product* in [HoLW]. But, this does not seem work well for $n \geq 2$.

Alternatively, one may try an induction on n, where a "random n-polyhedron" is obtained by attaching cones to randomly chosen $(n-1)$-cycles in a random $(n-1)$-polyhedron. But, again, it seems too hard to reconcile "small degree" with a bound on the inverse filling norm, except, possibly, for $n = 2$.

Randomly iterated polyhedra. One can obtain a variety of classes of random polyhedra by successively adding simplices of various dimensions. Here is an instance of such a class mentioned in 1.4.

Given a polyhedron X on a vertex set V of cardinality N, choose R_1 pairs of disjoint faces in X and let $X^+(N; R_1)$ be the polyhedron, where the simplices correspond to the unions of the vertices in these pairs.

Iterate this (+)-construction n times starting with $X = V$ and denote by $X^n(N; R_1, R_2, \dots, R_n)$ the n-skeleton of the resulting polyhedron (of dimension $2^n - 1$). These $X^n(N; R_i)$ are called *graph-iterated n-polyhedra*, since each step amounts to taking a graph on the barycenters of the simplices of the preceding polyhedron.

If $R_1 \geq \mathrm{const}_1(n)N$ and $R_i \geq \mathrm{const}_i(n)R_{i-1}$, for $i = 2, \ldots, n$, then, according to Naor and Pach (see 1.4), these graphs are expanders and the proportion of the iterated n-polyhedra $X_N = X^n(N; R_i)$ which fail to satisfy the inequality

$$\sup_{a\in\mathbb{R}^n} \left| a \pitchfork_F \{\Delta^n\} \right| \geq \varepsilon N$$

for all *face-wise affine* maps $X_N \to \mathbb{R}^n$, tends to zero for $N \to \infty$; moreover, this remains true for random iterated polyhedra with uniformly (in N) bounded degrees.

3 Appendices to Section 2

The material presented in this section is, formally speaking, unnecessary for our main results.

Section 3.1 contains notation, standard definitions and elementary properties of finite graphs which pertain to our main topics.

In section 3.2 we prove the Barany–Boros–Furedi inequality $b_{\mathrm{aff}} \geq 2/9$ in the plane by specializing our general argument to this case.

Sections 3.3–3.5 present basics on Plateau's problem with an emphasis on isoperimetric/filling inequalities, combinatorial and topological versions of which are established in other sections of our paper.

Sections 3.6–3.8 contain a few remarks concerning the norms of coboundary operators which lead to some combinatorial problems which are left untouched in the main body of the paper.

3.1 Isoperimetry with $\vec{\partial}$-boundary and cardinality of graphs over graphs.

A. ***Cardinality*** $|X/_{\mathcal{F}}Y|$**.** Consider some class $\mathcal{F}$ of continuous maps $F : X \to Y$ and recall the *$\mathcal{F}$-cardinality of X over Y*, now denoted

$$|X/_{\mathcal{F}}Y| = \inf_{F\in\mathcal{F}} \sup_{y\in Y} \left|F^{-1}(y)\right| .$$

If $\mathcal{F}$ equals the class of all proper continuous maps we write $|X/Y| = |X/_{cont}Y|$.

Let X and Y be graphs, regarded as 1-dimensional topological spaces with the natural combinatorial structure where all vertices have valencies $\neq 2$. Then every continuous map $X \to Y$ can be straightened on the edges (except for loops) without increasing the cardinality of the fibers; the cardinality $|X/Y| = |X/_{cont}Y|$ equals this cardinality for the class consisting of the maps that are locally one-to-one on the edges (and two-to-one on the loops). In particular, $|X/\mathbb{R}|$ can be evaluated within the class of functions $F : X \to \mathbb{R}$ that are affine on the edges of X with an exception for the loops that may be present in X.

Thus one sees that $|X/Y| \leq N_{\mathrm{edg}}(X)$ if there are no loops; if there are loops (not counted as edges), then $|X/Y| \leq N_{\mathrm{edg}} + 2N_{\mathrm{loop}}$.

This bound can be improved roughly by the factor of 1/2 by interchanging the values of functions F at the pairs of vertices x_1 and x_2 in so far as this diminishes $\max_y |F^{-1}(y)|$.

The N-*cliques*, i.e. the full graphs $X = X(N)$ on N vertices, (the 1-skeleta of the $(N-1)$-simplices), provide examples, where this bound is sharp, since, obviously, $|X(N)\mathbb{R}|$ equals $\frac{1}{2}N_{\mathrm{edg}}\big(1 + \frac{1}{N-1}\big)$ for even N and $\frac{1}{2}N_{\mathrm{edg}}\big(1 + \frac{1}{N}\big)$ for odd N.

Furthermore (exercise (a) on page 215 in [Ma1]),

if a graph X on N vertices has $N_{\mathrm{edg}}(X) \geq \varepsilon\binom{N}{2}$, then $|X/\mathbb{R}| \geq \frac{1}{2}\varepsilon^2\binom{N}{2} - O(\varepsilon^3)$ for $\varepsilon \to 0$.

Indeed, think of our graph mapped to $\mathbb{R}$ as a subset of measure $\varepsilon/2$ in the (right lower) triangle under the diagonal of the unit square, $Z \subset \underline{\Delta}^2 \subset [0,1] \times [0,1]$. Consider the rectangulars $\square_y = [y,1] \times [0,y] \subset \underline{\Delta}^2$, observe that

$$\max_y \operatorname{area}(Z \cap \square_y) \geq \int_0^1 \operatorname{area}(Z \cap \square_y)dy \geq \varepsilon^2 - O(\varepsilon^3)\,,$$

and discretize this back to combinatorial language.

The extremal set $Z = Z(\varepsilon)$ for this inequality in $\underline{\Delta}^2$ is, approximately (but not sharply) equals the $\sqrt{2}\varepsilon$-band around the diagonal side of $\underline{\Delta}$. (Probably, the extremal $Z(\varepsilon)$ can be explicitly determined.) It follows, that

if the N-clique is covered by subgraphs $X_1, X_2, \ldots, X_k$, then

$$\max_i |X_i/\mathbb{R}| \geq \frac{1+\delta(k)}{2k^2}\binom{N}{2} - O(1/k^2)\,,$$

for some strictly positive $\delta(k)$ (whose value, I guess, is unknown).

B. *Edge isoperimetric profile and the* max*-cardinality of the fibers.* Given a subset X_0 of vertices in X, denote by $\overrightarrow{\partial} X_0$ the set of edges issuing from (the vertices in) X_0 and landing outside X_0, i.e. $\overrightarrow{\partial}$ equals the set of the edges $[x_0, x_1]$, where $x_0 \in X_0$ and $x_1 \in \operatorname{vert}(X) \setminus X_0$.

Assign probability measures with atoms of equal weights to the sets of vertices and edges of X, denote them $|\ldots|_{vr}$, and $|\ldots|_{ed}$, and observe that each subset X_0 in the vertex set of X, which has X_0 the smallest measure among all vertex sets X_0' with $\partial X_0' = \partial X_0$ (if X is connected, there are a most two such subsets), satisfies

$$|X_0|_{vr}/|\overrightarrow{\partial} X_0|_{ed} \leq \big\|(\partial^0)^{-1}\big\|_{\mathrm{fil}}\big(|\overrightarrow{\partial} X_0|_{ed}\big)$$

for the $\mathbb{Z}_2$-filling norm defined in 2.3.

Profile $\|\overrightarrow{\partial}\|_{e/v}(r)$ for graphs. It is often convenient to encode $\|(\partial^0)^{-1}\|_{\mathrm{fil}}$ into a function of $r = |X_0|_{vr}$ (rather than of $|\overrightarrow{\partial} X_0)|_{ed}$). We do this by introducing $\|\overrightarrow{\partial}\|_{e/v}(r)$ which is the maximal function, such that all X_0 satisfy

$$|\overrightarrow{\partial} X_0|_{ed} \geq |X_0|_{vr}\|\overrightarrow{\partial}\|_{e/v}\big(|X_0|_{vr}\big)\,.$$

Observe that the (trivial) bound on $\|(\partial^0)^{-1}\|_{\mathrm{fil}}$ for $\Delta(V)$ from 2.6 translates to the (equally obvious) inequality $\|\overrightarrow{\partial}\|_{e/v}(r) \geq 2(1-r)$ for *cliques*, i.e. the full graphs on their vertex sets.

Similarly, *full colored graphs* have $\|\overrightarrow{\partial}\|_{e/v}(r) \geq 2(1-r)$ where a graph is called *full colored* if the the vertex set is divided into several subsets with the edges between *all pairs* of vertices in *different* subsets.

Lower bounds on $|X/Y|$ with $\|\overrightarrow{\partial}\|_{e/v}$. If $F : X \to \mathbb{R}_+ \subset \mathbb{R}$ is a proper continuous map, which is *injective* on the vertex set of X, then the number of vertices in the sublevels $F^{-1}(-\infty, y)$, $y \in \mathbb{R}$ assumes all values $0, 1, 2, \ldots, N_{vr}(X)$; hence,

$$\sup_{y\in\mathbb{R}}\big|F^{-1}(y)\big| \geq N_{ed}(X) \sup_{0\leq r\leq 1} r \cdot \|\overrightarrow{\partial}\|_{e/v}(r)\,.$$

To remove "injective", let $|F^{-1}(y)|_{ed}$ denote the number of edges in X that intersect the level $F^{-1}(y) \subset X$ and observe that this number (unlike $|F^{-1}(y)|$) is semicontinuous: if $F_i \to F$, and $y_i \to y$, then $\limsup |F_i^{-1}(y_i)|_{ed} \leq |F^{-1}(y)|_{ed}$. Clearly, $\sup_y |F^{-1}(y)| \geq N_{ed} \sup_r r \cdot \| \overrightarrow{\partial} \|_{e/v}(r)$ for all continuous maps F.

This generalizes to maps $F : X \to Y$ for all trees Y as follows.

An open subset in a topological tree, $Y_0 \subset Y$, is called a *multibranch at* $y \in Y$ if y is the *only* boundary point of Y_0. Connected mutibranches are called *branches at* y.

$\left[\frac{1}{3}, \frac{2}{3}\right]$-Inequality.

Every continuous map F of a finite graph X with N_{ed} edges to a locally finite tree Y satisfies

$$N_{ed}^{-1} \sup_{y \in \mathbb{R}} \left|F^{-1}(y)\right|_{ed} \geq \inf_{1/3 \leq r \leq 2/3} r \cdot \| \overrightarrow{\partial} \|_{e/v}(r)\,.$$

Proof. Because of semicontinuity, one may assume that F is injective on the vertex set and that no vertex of X lands at a vertex of Y (provided $N_{vr}(Y) \geq 2$). Then it suffices to find a $\frac{1}{3}$-*centrum point* $y_c \in Y$ such that the number of vertices from X landing in some multibranch $Y_c \subset Y$ at y_c satisfies

$$\tfrac{1}{3} \leq N_{vr}\big(F^{-1}(Y_c)\big)\big/N_{vr}(X) \leq \tfrac{2}{3}\,. \qquad \left[\tfrac{1}{3}, \tfrac{2}{3}\right]$$

To obtain y_c, let $y_0 \in Y$ be an arbitrary point that is not an image of a vertex from X and such that no multibranch at y_0 satisfies $\left[\frac{1}{3}, \frac{2}{3}\right]$. Then there exists a branch at y_0, say $B_0 \subset Y$, such that $N_{vr}(F^{-1}(B_0)) > \frac{2}{3} N_{vr}(X)$, since every finite set I of numbers $0 \leq m_i \leq 2/3$, $i \in I$, with $\sum_i m_i \geq 1$ contains a subset $J \subset I$ such that $\sum_{j \in J} m_j \in \left[\frac{1}{3}, \frac{2}{3}\right]$.

Let $E = E(y) \subset Y$ be the edge of B_0 adjacent to y and let $y_1 \in E$ be either a non-vertex point in B_0, such that the interval (y_0, y_1) contains a *single* F-image of an X-vertex, or y_1 be the vertex of Y with no F-images of X-vertices in $(y_0, y_1]$. If $y_1 \in B_0 \subset Y$ still violates $\left[\frac{1}{3}, \frac{2}{3}\right]$, go to y_2, etc., and thus, arrive at the desired $y = y_c$ (provided $N_{vr}(X) \geq 2$).

Remarks. (a) A similar argument shows that

$$N_{ed}^{-1} \sup_{y \in \mathbb{R}} \left|F^{-1}(y)\right|_{ed} \geq \sup_{\varepsilon \leq \alpha \leq 1} \inf_{\frac{\alpha}{3} \leq r \leq \frac{\alpha}{3}} r \cdot \| \overrightarrow{\partial} \|_{e/v}(r)\,, \quad \text{for } \varepsilon = 2/N_{vr}(X)\,.$$

(b) The difference between $|F^{-1}(y)|$ and $|F^{-1}(y)|_{ed}$ is seen in the example of the clique $X(N)$ mapped onto the tree Y_M made of $M = \binom{N}{2}$ copies of $[0,1]$ joined at 0, such that the $F^{-1}(0)$ equals the vertex set of $X(N)$ and where each edge of $X(N)$ folds in an edge of Y_M. Here, $|F^{-1}(0)| = N$ and $|F^{-1}(y)| \leq 2$ for the other points $y \in Y_M$, while $|F^{-1}(0)|_{ed} = \binom{N}{2}$.

C. *Cardinalities of trees over* $\mathbb{R}$. A binary rooted tree X_d of depth d has $|X_d/\mathbb{R}| = d$ for $d = 1, 2$ and $|X_{d+2}/\mathbb{R}| = |X_d/\mathbb{R}| + 1$ for $d + 2 \geq 3$.

To see that $|X_{d+2}/\mathbb{R}| \geq |X_d/\mathbb{R}| + 1$, take a simple path in X mapping onto the full image of f and observe that the complement of every path in X_d contains a subtree of depth $d - 2$. Conversely, the forest complementary to a longest path P in X_d maps to P with cardinality $|X_{d-1}/\mathbb{R}|$.

A similar estimate holds for trees over trees.

The cardinality of the binary tree $X = X_{d+\delta}$ *over* $Y = X_d$ *satisfies* $|X/Y| \geq \frac{1}{2}\delta - \log_2(\delta)$.

Indeed, if a graph Y has (at most) N vertices of degrees ≥ 2 and if an X contains M disjoint subgraphs X_i with $X_i/\mathbb{R} \geq \delta_0$, then $|X/Y| \geq \min(M/N, \delta_0)$.

This applies to $X = X_{d+\delta}$ which contains $M \geq 2^{d_0}$ disjoint binary trees of depth $d + \delta - d_0$ for every $d_0 \leq d + \delta$, e.g. for $d_0 = d + \log_2(\delta)$, while the number N of 3-valent vertices in $Y = X_d$ is $\leq 2^d$.

Problem. Determine $|X/Y|$ for given trees X and Y.

An upper bound on the cardinality of graphs over trees. The $|X/trees|$-and $d|X/trees|$-inequalities provide lower bounds on the cardinality of a graph X over trees by the isoperimetric profile of X and thus, by the supremum of the isoperimetric profiles of all subgraphs $X' \subset X$. Here is an opposite (standard and obvious) inequality.

Let the vertex set V' of every subgraph X' of X be partitioned into two subsets V'_1 and V'_2 of cardinalities N'_1 and $N'_2 = N' - N'_1$ for $N' = \mathrm{card}(V')$, such that $N'^1 \geq N'_2 \geq C \cdot N'$ for some constant C and such that the number of edges between V'^1 and V'^2 is bounded by $J(N')$ for some real function J vanishing for $N' < 1$. Then there exists a map f of X onto subtree Y in a binary tree Y_d of depth $d \leq C \cdot \log_2(N)$, such that the vertices of X go to the leaves of Y and the f-pullbacks of all points $y \subset Y$ have cardinalities $\leq C \cdot kN \cdot \sum_{i=1,2,\dots} J\big(1 - \frac{1}{C}\big)^i$, where N denotes the number of vertices in X.

Remarks and Questions. It seems little is known about the cardinality of graphs over graphs, either for distinguished examples, e.g. for cliques over trees, or for classes of graphs, e.g. for expanders over expanders and/or for random graphs with given numbers of edges and vertices.

For example, let X and Y be graphs on N vertices, all having the valences (degrees) of all vertices bounded by d (say, by $d = 3$), where Y is a λ-*expander*, i.e. $\|(\partial^0)^{-1}\|_{\mathrm{fil}} \leq 1/\lambda$ (say with $\lambda = 0.1$).

Is then $|X/Y| \leq \mathrm{const}(\log N)^\alpha$? Does an opposite inequality hold true for random graphs X in the above class?

D. ***Graphs in the circle.*** Let X_0 be an N-point set in the circle S^1, take two points $y_1, y_2 \in S^1 \setminus X_0$, consider the two segments in S^1 joining these points, take the segment which contains $M \leq N/2$ points from X_0 and let $|y_1 - y_2|_{X_0} = M/N$. It is obvious that

if every pair of points in $y_1, y_2 \in Y \subset S^1 \setminus X_0$ satisfies $|y_1 - y_2|_{X_0} < 1/3$ then Y is contained in open segment $S = S_Y \subset S^1$ with less than $N/3$ points from X_0 in this S.

Let $X_0 \subset S^1$ serve as the vertex set of a graph X, where every edge e is implemented by one of the two segments in S^1 between the end vertices of e in S^1. Let $N_e(y)$, $y \in S^1 \setminus X_0$, be the number of the segments corresponding to the edges of

X which contain y, and let $\nu(y) = N_e(y)/N_{ed}$, for N_{ed} denoting the number of all edges in X.

Take two points $y_1, y_2 \in Y \subset S^1 \setminus X_0$, let S_1 and S_2 be the two segments between them in S^1 and observe that the number of edges in X between the vertex sets $X_0 \cap S_1$ and $X_0 \cap S_2$ equals $\nu(y_1) + \nu(y_2)$. Then, according to the definition of the profile $\|\overrightarrow{\partial}\|_{e/v}(r)$ of X in **B**,

$$\|\overrightarrow{\partial}\|_{e/v}\big(|y_1 - y_2|_{X_0}\big) \leq \frac{\nu(y_1) + \nu(y_2)}{|y_1 - y_2|_{X_0}} .i$$

COROLLARY. *If*

$$\varepsilon < \tfrac{r}{2}\|\overrightarrow{\partial}\|_{e/v}(r) \quad \textit{for all } \tfrac{1}{3} \leq r \leq \tfrac{1}{2}\,,$$

then the set of the points $y \in S^1 \setminus X_0$ for which $\nu(y) \leq \varepsilon$ is contained in an open segment $S \subset S^1$ with $< N/3$ of the points from X_0 in S.

EXAMPLE. Let X be the full graph on N vertices (the N-clique). Since $\|\overrightarrow{\partial}\|_{e/v}(r) = 2(1-r)$ for this X, the set of the points $y \in S^1 \setminus X_0$ which are covered by $< \frac{2}{9}\binom{N}{2}$ edges is contained in a segment in S^1 with less than one third of the points from X_0 in it.

3.2 Barany–Boros–Furedi inequality for cliques in the plane. The abstract proof of the $[\Delta \to \mathbb{R}^n]$-inequality, when applied to an affine map of the $(N-1)$-simplex to the plane, simplifies as follows. (See [Fox et al] for a more straightforward argument.)

Let X be a simplicial 2-complex with the vertex set X^0 of cardinality N_{vr} in the plane, where no three points lie on a line, where the edges making the 1-skeleton X^1 of X are realized by straight segments between these vertices, and the 2-simplices of X are the Euclidean triangles. We write $X^0 \subset \mathbb{R}^2$ and denote by $\underline{X}^1 \subset \mathbb{R}^2$ the union of these segments corresponding to the edges in X (where some segments may intersect in $\mathbb{R}^2$). Let $\|\overrightarrow{\partial}\|_{e/v}(r)$, $0 \leq r \leq 1$, be the isoperimetric profile of the graph X^1 defined in **B** of the previous section.

> **1.** *If, for a given m, every point $z \in \mathbb{R}^2 \setminus \underline{X}^1$ admits a straight ray $R_y \subset \mathbb{R}^2$ issuing from z and intersecting $< m$ segments in $\underline{X}^1$, then*
>
> $$\inf_{\frac{1}{3} \leq r \leq \frac{1}{2}} r \cdot \|\overrightarrow{\partial}\|_{e/v}(r) \leq \big(m + \deg(X^1)\big)/N_{ed}(X)\,, \qquad \left(\tfrac{1}{3}\right)$$
>
> *where* $\deg(X^1)$ *denotes the maximum of degrees (valences) of the vertices in the* 1*-skeleton of X.*

Proof. Radially project X^1 from a point z to a large round circle $S^1 \subset \mathbb{R}^2$ surrounding $\underline{X}^1$ in the plane, assume $\left(\frac{1}{3}\right)$ *does not* hold for this z and apply the corollary from **D** above. Thus we obtain a unique *non-empty* segment $S = S(z) \subset S^1$ containing all points covered by less than m edges of X projected to S^1.

Let D be the disk in the plane bounded by S^1 and let $\Gamma_F \subset D \times S^1$ be the closure of the set of the pairs $(z \in D \setminus \underline{X}^1, s \in S(z))$ (that we regard as the graph of the interval valued map $D \to S^1 = \partial D$. Observe that this graph is a piecewise linear subset in $D \times S^1$).

The presence of the extra deg-summand in $\binom{1}{3}$ guarantees that the intersection of Γ_F with the circle $z \times S^1$ for *every* $z \in D$ is a non-empty segment that *is not* equal to all of $z \times S^1$. Thus the projection $\Gamma_F \to D$ is a homotopy equivalence, hence Γ_F is contractible.

On the other hand, the map $z \mapsto (z, s = z)$ for $y \in \partial D = S^1$ sends S^1 to Γ_F, where the composition of this map with the projection $\Gamma_F \to S^1$ is the identity map. Since the circle is non-contractible, this is impossible and so $\binom{1}{3}$ *does* hold. (This is similar to the reduction of the Brower fixed point theorem in D to non-contractibility of S^1.)

2. Let a point $z \in \mathbb{R}^2 \setminus \underline{X}^1$ be contained in at most cN_{Δ^2} triangles corresponding to the 2-simplices of X. Then, by an obvious counting/averaging argument, there is a point $x_0 = x_0(z) \in X^0 \subset \mathbb{R}^2$ such that y is contained in at most $cN_{\delta_2}(x_0)$ triangles having x_0 as a vertex, where $N_{\delta_2}(x_0)$ denotes the number of the 2-simplices in X adjacent to x_0.

3. If $X = X^2(N)$ is made of *all* $\binom{N}{3}$ simplices spanned by the vertices in X^0, then

($\star$) *the ray issuing from z to ∞ that continues the segment $[x_0, z] \subset \mathbb{R}^2$ meets as many segments as there are triangles containing z and having x_0 as one of its vertices.*

By combining **1–3** with the (obvious) inequality $\|\overrightarrow{\partial}\|_{e/v}(r) \geq 2(1-r)$ for the N-cliques, we conclude with the

Boros–Furedi inequality.

Given a set $X^0 \subset \mathbb{R}^2$ of N points in general position, there exists a point $y \in \mathbb{R}^2$ that is contained in at least $\frac{2}{9}\binom{N}{3} - O(N^2)$ open triangles convexly spanned by triples of points in X^0.

REMARKS. (a) Our argument gives a poorer evaluation of the $O(N)$-term than the original proof by Boros–Furedi.

(b) The seemingly trivial property ($\star$) creates problems when it comes to similar inequalities for general polyhedra X, since the natural counterparts to ($\star$) are hard to verify and/or achieve in most cases.

3.3 Filling-in geometric cycles via the local-to-global variational principle.

Let X be a metric space and

$$C_* = \big(C_*(X, \mathbb{F}, \partial_*)\big) = \big(\{\partial_i : C_i \to C_{i-1}\}_{i=0,1,\dots,n=\dim(X)}\big)$$

be the complex of *Lipschitz singular chains*: an i-chain is a finite sum $c = \sum_j f_j \sigma_j$, where every $\sigma_j : \Delta^i \to X$ is a *Lipschitz map* of the standard i-simplex to X and $f_i \in \mathbb{F}$. (A map between metric spaces, say $f : A \to B$, is called Lipschitz if $\mathrm{dist}_B(f(a_1, f(a_2))) \leq \mathrm{const} \cdot \mathrm{dist}_A(a_1, a_2)$ for all $a_1, a_2 \in A$.)

The union of the images of the maps σ_j is called the *support* of c. We shall often make no distinction between chains and their supports regarding chains as subsets in X.

If $\mathbb{F}$ comes with a norm (e.g. $\mathbb{F}$ equals $\mathbb{R}$, $\mathbb{Z}$, or $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$), then each chain $c \in C^i$ is given the *i-volume norm*, $\mathrm{vol}_i(c) = \|c\|_{\mathrm{vol}_i} = \sum_i \|f_j\| \,\mathrm{vol}_i(\sigma_j)$ where this volume is understood as the i-dimensional Hausdorff measure of Δ^i in the metric induced by the map (σ_j) from X. (If the map is not one-to-one, this metric may vanish at some pairs of points, but the Hausdorff measure is well defined all the same.)

The corresponding minimal norm on the homology $H_i(X) = \mathrm{Ker}\,\partial_i / \mathrm{Im}\,\partial_{i+1}$, that is the infimum of the volumes of the cycles $c \in \mathrm{Ker}\,\partial_{i+1}$ representing an $h \in H_i(X)$, is called *the volume or the mass norm*,

$$\mathrm{vol}_i(h) = \inf_{[c]=h} \mathrm{vol}_i(c)\,.$$

One can show (see [Wen]) that if X is a compact locally contractible space, then this norm does not vanish: $\inf_{h\neq 0} \mathrm{vol}_i(h) > 0$, for $h \neq 0$.

The i-th $\mathbb{F}$-systole of X is

$$\mathrm{syst}_i(X) = \inf_{h\neq 0} \mathrm{vol}_i(h)\,, \quad \text{where } h \in H_i(X)\,.$$

Define the *filling volume* of an i-cycle b in X, denoted $\mathrm{filvol}_{i+1}(b)$, as the infimum of the $(i+1)$-volumes of the chains $c \in C_{i+1}$ "filling in" b, i.e. with $\partial_{i+1}(c) = b$.

Observe that the supremum of those β for which the filling volume is finite for all b with $\mathrm{vol}_i(b) \leq \beta$ equals $\mathrm{syst}_i(X)$ and that the filling norm from 2.3 is related to the filling volume by

$$\|\partial_{i+1}^{-1}\|_{\mathrm{fil}}(\beta) = \beta^{-1} \sup_{\|b\|=\beta} \mathrm{filvol}_{i+1}(b)\,.$$

Evaluation of $\|\partial_{i+1}^{-1}\|_{\mathrm{fil}}$ via $\|\partial_{i+1}^{-1}\|_{\mathrm{rand}}$ for round spheres. Let X be the unit Euclidean sphere, $X = S^n \subset \mathbb{R}^{n+1}$ with the $O(n+1)$-invariant i-volumes normalized so that the equatorial spheres $S^i \subset S^n$ have volume 1 for all $i = 0, 1, 2, \dots, n$.

Let b be an i-cycle in S^n and $s \in S^n$ a point such that $-s$ is not in (the support of) b. Then there is a unique distance minimizing geodesic (ark of a great circle) in S^n between s and every point in b and *the geodesic cone $c = c_s(b)$ from s over b* made of all these geodesics fills in b.

If, for instance, b equals an equatorial i-sphere $S^i \subset S^n$, then such a cone c_s equals an $(i+1)$-hemisphere with the boundary equal to our S^i, where, observe, $\mathrm{vol}_{i+1}(c_s) = \frac{1}{2}\mathrm{vol}(b)$. Consequently, the average of $\mathrm{vol}_{i+1}(c_s)$ over S^n equals $\frac{1}{2}\mathrm{vol}(b)$ as well.

Since the orthogonal group $O(n+1)$ is transitive on the set of tangent i-planes in S^n, the average of the $(i+1)$-volumes of the s-cones $c_s(b)$ over *every i-chain b* also equals $\frac{1}{2}\mathrm{vol}(b)$. Thus,

$$\|\partial_{i+1}^{-1}\|_{\mathrm{rand}}(\beta) = 1/2 \quad \text{for all } i = 1, 2, \dots, n-1 \text{ and } \beta \geq 0\,.$$

This averaging argument is similar to the one used in 2.6 for simplices and it suffers the same drawback of not being sharp: the inequality $\|\partial_{i+1}^{-1}\|_{\mathrm{fil}}(\beta) \leq 1/2$ in the range $0 < \beta \leq 1$ (i.e. for cycles of volume ≤ 1) is sharp only for $\beta = 1$. (It would be interesting to link the two averagings by the moment map $S^{2n+1} \to \Delta^n$.)

But unlike the combinatorial case, the sharp bound in S^n is available for all $\beta \leq 1$. In fact, as everybody would guess, *if X is either $\mathbb{R}^n$ or S^n, then the round (umbilical) i-spheres of volume β (filled in by flat $(i+1)$-discs) have maximal filling volumes among all i-cycles b in X, with* $\mathrm{vol}(b) = \beta$.

REMARKS. (a) The idea of "filling" goes back to Federer and Fleming, [F], [FF], [BurZ], who used among other things an averaging (similar to the above case of S^n) combined with "cutting and pasting" also used in related contexts, [Gr1], [K], [Wen], [Y].

The spherical case was reduced to $\mathbb{R}^{n+1} \supset S^n$ by Bruce Kleiner (private communication). In fact, a slight modification of Almgren's argument applies to manifolds of non-negative curvature and related spaces.

(b) $X = S^n$, Kleiner's result leaves open the sharp bound on $\|\partial_{i+1}^{-1}\|_{\mathrm{fil}}(\beta)$ for $\beta > 1$ that may depend on n (if $n > i+1$) and on the coefficient field $\mathbb{F}$ in a rather complicated manner. Also, the "filling extremality" of the round i-spheres remains unproven in the hyperbolic n-spaces for $2 \leq i \leq n-2$.

***Local-to-global variational principle:* $\|\dots\|_{\mathbf{fil}} \leq \|\dots\|_{\mathbf{fil}}^{\mathbf{loc}}$.** Let (C_*, ∂_*) be a normed chain complex, $h \in H_i(C_*)$ a homology class and let $B = B_i(h) \subset C_i$ be the space of i-cycles in the class of h with the filling metric $\mathrm{dist}_B(b_1, b_2) = \|b_1 - b_2\|_{\mathrm{fil}}$. Define the supremum norm of the "downstream gradient" of the function $b \mapsto \|b\|$ on B as follows,

$$\|\downarrow b\|_{\sup} = \limsup_{\|c\| \to 0} \frac{\|b\| - \|b + \partial_{i+1}(c)\|}{\|c\|} \quad \text{for } c \in C_{i+1} \setminus \{0\}\,.$$

Observe that this norm on smooth submanifolds Y representing cycles b in Riemannian manifolds X equals the supremum of the norm of the *mean curvatures* of Y, denoted $\|M\|(Y) = \sup_y \|M_y(Y)\|$. (We define M at the end of this section.)

Let $m(\beta) = \inf_{\|b\|=\beta} \|\downarrow b\|_{\sup}$, and suppose that there exists, for some β, an *extremal* cycle b supported on a *smooth* submanifold Y_β with $\mathrm{vol}_i(Y) = \beta$ and such that $\mathrm{filvol}_{i+1}(b) = \beta \|\partial_{i+1}^{-1}\|_{\mathrm{fil}}(\beta)$.

Then, for infinitesimally small $\varepsilon > 0$,

$$\mathrm{filvol}(\beta_0) \leq \mathrm{filvol}(\beta) - \varepsilon \|M\|(Y_\beta) + \varepsilon + o(\varepsilon)\,,$$

where $\mathrm{filvol}(\beta)$ stands for $\beta \|\partial_{i+1}^{-1}\|_{\mathrm{fil}}(\beta)$. Therefore,

if, for each $0 \leq \beta \leq \beta_0$ there is a unique smooth extremal cycle Y_β and if we have a lower bound on $\|M\|(Y_\beta)$ in terms of β, then we can bound $\mathrm{filvol}(\beta)$ *from above.*

Comparison of Y with a model space. To see the explicit relation between filvol and $\|M\|$ geometrically, look at a "model" $X_{\mathrm{mod}} = S \times (0, R)$, $R \in (0, \infty)$, where S is an i-dimensional Riemannian manifold and where X_{mod} is given a Riemannian metric which is uniquely defined by the following conditions.

- The segments $x \times (0, R)$ in X_{mod} are geodesics *isometrically* parametrized by $r \in (0, R)$ and which are normal to $S(r) = Y \times r \subset X_{\mathrm{mod}}$ for all $r \in (0, R)$.
- The induced metric on each $S(r) = Y_{\mathrm{mod}} \times r$ equals the original metric on S times a constant $c(r)$ which is monotone increasing in r and tends to zero with $r \to 0$.

An archetypical example of this is the punctured Euclidean space $\mathbb{R}^{i+1} \setminus 0$ represented in the polar coordinates as $S^i \times (0,\infty)$. Also a round sphere punctured at two opposite points is such a model space. Conversely, if the isometry group is *transitive* on the (one point) completion of a model space, then it is isometric to a sphere, to a Euclidean space or to a *hyperbolic space* (of constant curvature).

Now suppose that the norms of the mean curvatures of our extremal submanifolds $Y_\beta \subset X$ satisfy

$$\|M\|(Y_\beta) \geq m_{\mathrm{mod}}(r) =_{\mathrm{def}} \|M\|(S(r)), \qquad [M \geq M_{\mathrm{mod}}]$$

where $r = r(\beta) < R$ satisfies $\mathrm{vol}_i(S(r)) = \beta$.

Then, obviously,

if $r(\beta) < R$, *then every* i*-cycle* b *in* X *with* $\mathrm{vol}_i(b) \leq \beta$ *satisfies*

$$\mathrm{filvol}_{i+1}(b) \leq f(r) = \mathrm{vol}_{i+1}\left(S \times [0,r]\right), \qquad [\mathrm{fil} \leq \mathrm{fil}_{\mathrm{mod}}]$$

where, observe, $\mathrm{vol}_{i+1}(S \times [0,r]) = \mathrm{filvol}_{i+1}(S(r))$ if we add the $R = 0$ point to X_{mod}.

Of course, one cannot, in general, guarantee the existence and uniqueness of smooth extremal Y_β, but the above inequality holds just the same by the following theorem of Almgren [A2].

Let $Y \subset X$ be a closed subset and $U_\varepsilon(Y) \subset X$ be the ε-neighbourhood of Y, i.e. the set of points within distance $\leq \varepsilon$ from Y, where $\mathrm{dist}(x,Y) =_{\mathrm{def}} \inf_{y\in Y} \mathrm{dist}(x,y)$.

Given a subset $Z \subset Y$, let $U_\varepsilon^{\perp_Y}(Z) \subset U_\varepsilon(Y)$ denote the set of the points $x \in U_\varepsilon(Y)$ such that $\mathrm{dist}_X(x,Z) = \mathrm{dist}_X(x,Y)$.

Let $B_y(Y,\delta) \subset Y$ denote the δ-ball in Y around x, i.e. the intersection of Y with $B_y(X,\delta) \subset X$. Say that Y is i*-coregular in* Y at a point $y_0 \subset Y$ if y_0 admits a neighborhood $Y_0 \subset Y$ such that all $y \in Y_0$ satisfy

$$\varepsilon^{-(n-i)}\delta^{-i}\,\mathrm{vol}_n\left(U_\varepsilon^{\perp_Y}(B_y(Y,\delta))\right) \leq \mathrm{const}_0 \quad \text{for } 0 < \delta \leq \varepsilon \leq \varepsilon_0\,,$$

where $\varepsilon_0 > 0$, and $\mathrm{const}_0 < \infty$ are constants depending on y_0 but not on $y \in Y_0$, ε and δ.

For example, if $Y \subset X$ is a smooth i-submanifold with a boundary, then all interior points of Y are i-coregular, while the boundary points of Y are not coregular in Y. However, all boundary points are $(i-1)$-coregular in Y (as well as in $\partial(Y)$).

A closed i-dimensional subset Y in X is called *quasiregular*, if it is a C^2-smooth submanifold in the complement of a closed subset of zero i-dimensional Hausdorff measure and if *all* points of Y are i-coregular in Y.

Denote by $\|M\|(Y)$ the supremum of the norm of the mean curvature over the subset $\mathrm{reg}(Y) \subset Y$ of all C^2-regular points of Y.

Almgren Filling Theorem. *Let* X *be a closed Riemannian manifold and* $X_{\mathrm{mod}} = S \times \mathbb{R}$ *be a model manifold, such that all compact quasiregular subsets* $Y \subset X$ *of volume* $\beta \leq \beta_0$ *satisfy the above* $[M \geq M_{\mathrm{mod}}]$ *for a given* β_0. *Then every* i*-cycle* b *in* X *with* $\mathrm{vol}_i(b) \leq \beta$ *satisfies* $[\mathrm{fil} \leq \mathrm{fil}_{\mathrm{mod}}]$.

Idea of the proof. Almgren shows [A2] that, for every β, there is a compact quasiregular subset $Y = Y_\beta \subset X$ which is *extremal* in the following sense. There exists, for every $\varepsilon \geq 0$, a cycle b_ε, in X, such that

- $\mathrm{vol}_i(b_\varepsilon) \to \beta$ for $\varepsilon \to 0$.
- $\mathrm{filvol}_{i+1}(b_\varepsilon) \to \mathrm{filvol}_{i+1}(\beta) = \beta \|\partial^{-1}\|_{\mathrm{fil}}(\beta)$.
- The part of b_ε which does not belong to $\mathrm{reg}(Y)$ has the i-volume $\le \varepsilon$. (In fact, one can make B_ε with the support entirely contained in the ε-neighborhood $U_\varepsilon(Y)$.)
- The complement to the support of b_ε in $\mathrm{reg}(Y)$ has $\mathrm{vol}_i(\mathrm{Reg}(Y) \setminus \mathrm{supp}(b_\varepsilon)) \le \varepsilon$. (This is a minor technicality.)

If $\mathrm{vol}_i(Y) = \beta$, then the above "model argument" directly (and obviously) applies; otherwise, if $\mathrm{vol}(Y) < \beta$, one may additionally assume that the mean curvature function of the model, i.e. $\|M\|(S(r))$, is monotone increasing, which is the case in our examples.

If $\|M\|(S(r))$ is non-monotone, one has to "unfold" b_ε by a slight perturbation in order to bring the measure of its support close to β. (This is needed, for example, if Y is a manifold where the cycle is represented by a finite covering of Y.)

REMARKS. (a) The condition $[M \ge M_{\mathrm{mod}}]$ is needed only for the extremal subsets Y, and these have many additional properties, e.g. if $i = n-1$ they have constant mean curvature on $\mathrm{reg}(Y)$.

Moreover, one does not even need $[M \ge M_{\mathrm{mod}}]$ for all extremal Y but only where, a priori, the above sequence b_ε violates the filling inequality $[\mathrm{fil} \le \mathrm{fil}_{\mathrm{mod}}]$, i.e. $\limsup_{\varepsilon \to 0} \mathrm{filvol}(b_\varepsilon) > f(r) = \mathrm{filvol}_{i+1}(S(r)) = \mathrm{vol}_{i+1}(S \times [0,r])$, which eventually is proven to hold anyway.

(b) The proof of regularity in Almgren's theorem is purely local (albeit quite difficult) and does not need the compactness of Y, but the existence of extremal Y does.

However, it is not hard to see that the filling conclusion holds for complete non-compact manifolds with a mild condition at infinity which is satisfied for most examples; moreover, this is also true for some non-complete manifolds if they are not very pathological at the completion points.

For example, let $\Sigma \subset X$ be a k-dimensional submanifold. If one slightly expands or contracts the metric near Σ in the directions *normal* to Σ, the resulting metric on X becomes singular at Σ but the Almgren theorem still delivers extremal cycles in X, with the quasi-irregularity condition satisfied at the points in $X \setminus \Sigma$.

(c) Almgren's local-to-global principle originates from "isoperimetric ideas" of Max Dehn in combinatorial group theory and of Paul Levy in convexity. It is frequently used but rarely stated explicitly; yet, see [A2] and references therein.

Definitions of Φ, of M and of the sectional curvature. Let Y be a smooth submanifold in a Riemannian manifold X, where the Riemannian metric, that is a quadratic form on the tangent bundle $T(X)$, is denoted by $g = g_x(\tau)$, $\tau \in T_x(X)$. Let $T^\perp(Y) \subset T(X)|Y$ denote the normal bundle of Y in X and let $\nu : Y \to T^\perp(Y)$ be a normal vector field along Y.

Extend ν to a vector field $\tilde\nu$ in a neighborhood of Y in X and take the (Lie) $\tilde\nu$-derivative $\tilde\nu(g)$.

This $\tilde{\nu}(g)$ is a quadratic form on the tangent bundle of (a neighborhood in) X. What is significant, albeit obvious, is that the restriction of $\tilde{\nu}(g)$ to $T(Y) \subset T(X)$, which is a quadratic form on $T(Y)$, does *not* depend on all of $\tilde{\nu}$, but only on ν itself. We denote this restriction by Φ_ν.

It easy to see that the map $\nu \mapsto \Phi_\nu$ is linear:

$$\Phi_{\nu_1+\nu_2} = \Phi_{\nu_1} + \Phi_{\nu_2} ,$$

and that Φ_ν, on each tangent space $T_x(Y)$, $x \in Y$, depends only on the vector $\nu(x) \in T_x^\perp(Y)$.

The resulting field on Y of linear maps from $T_y^\perp(Y)$, $y \in Y$, to the space of quadratic forms on $T_y(Y)$ given by $\nu \to \Phi_\nu$ is called the *second fundamental form* or *the extrinsic curvature (form)* of Y in X, denoted $\Phi_\nu = \Phi_\nu(\tau) = \Phi_\nu(\tau|Y \subset X)$.

Two i-submanifolds in X with a common tangent space $T_x^i \subset T_x(X)$ at some point $x \in X$ have equal second fundamental forms on T_x^i for all vectors $\nu_x \in T_x(X)$ normal to T_x^i, if and only if these submanifolds are *second-order tangent* at x, where, observe, the order of tangency does not depend on the Riemannian metric on X. Thus, Φ fully and faithfully encodes the second-order infinitesimal information on Y in X at each point $x \in Y$. It follows, that Φ is determined by its restrictions to curves Z in Y.

More generally, let Z be a submanifold in $Y \subset X$, and ν a vector normal to Z in X which is canonically decomposed as $\nu = \nu_1 + \nu_2$, where ν_1 is normal to Y and ν_2 is tangent to Y and normal to Z in Y. Then the second fundamental form $\Phi_\nu(\tau|Z \subset X)$ of Z in X satisfies

$$\Phi_\nu(\tau|Z \subset X) = \Phi_{\nu_1}(\tau|Y \subset X) + \Phi_{\nu_2}(\tau|Z \subset Y)$$

for all vectors τ tangent to Z. In particular $\Phi(\tau, |Y \subset X)$, can be reconstructed from $\Phi(\tau|Z \subset X)$ for all geodesic lines Z in Y as these have $\Phi(\tau|Z \subset Y) = 0$.

The mean curvature $M = M(\nu)$ is the linear function (covector field) on the normal bundle $T^\perp(Y)$ defined by $M_y(\nu) = \mathrm{trace}_g(\Phi_\nu)$. It is easy to see (and this was used in the local-to-global principle) that

> *the derivative of the i-volume of Y under the flow in X generated by $\tilde{\nu}$ equals $\int_Y M_y(\nu)dy$. Conversely, if such equality is satisfied by some covector field M' and all ν, then $M' = M$.*

Finally, define the *sectional curvature* of X at a tangent 2-plane $T^2 \subset T_x(X)$ as follows.

Take a hypersurface $Y = Y^{n-1} \subset X$, such that the tangent space $T_x(Y) \subset T_x(X)$ contains a unit vector $\tau \in T^2$ and such that the normal unit vector $\nu \in T^2$ is also normal to $T_x(Y)$.

Denote by $\tilde{\nu}$ the gradient field of the function $x \mapsto \mathrm{dist}_X(x, Y)$ defined locally in the small half-ball in X on the side of ν. Let $\tilde{g}''$ be the second (Lie) derivative of the Riemannian quadratic form g of X under $\tilde{\nu}$.

If the second fundamental form of Y (that is the first derivative $\tilde{g}'$) vanishes at x – and one always can choose such a Y – then the number $-\tilde{g}''(\tau)$ depends only on T^2 and it is called the *sectional curvature* of X at T^2 (being independent of τ and Y; this is a miracle, but verifying it is rather trivial), where the minus sign is taken in agreement with the convention of the unit 2-sphere having sectional curvature +1: a normal equidistant push of an equatorial arc $Y \subset S^2$, *shortens* the length of Y.

3.4 Moves, tubes and filling inequalities. The following process of filling-in cycles by pushing them with families of moving hypersurfaces was suggested by Max Dehn in combinatorial group theory about hundred years ago.

Let X admit a family of properly immersed cooriented smooth hypersurfaces S_r, $r > 0$, such that

(a) *The i-mean curvatures* $M_{i-1}(S_r)$ of all S_r, i.e. the traces of the restrictions of the *second fundamental form* Φ of S_r to all tangent i-planes to S_r, are bounded from below by a positive constant m_0.

(b) There exists a locally compact space $\tilde{X}$, a proper continuous map $p : \tilde{X} \to X$ and a continuous function $f : \tilde{X} \to \mathbb{R}_+$, such that

  (b_1) $p^{-1}(S_r) = f^{-1}(r)$ for all $r > 0$;

  (b_2) The map p properly embeds the 0-level $f^{-1}(0) \subset \tilde{X}$ to X, where the image is a rectifiable set and where either $\dim(f^{-1}(0)) \le i - 1$ (e.g. $f^{-1}(0)$ is empty) or $f^{-1}(0)$ is contractible of dimension i.

Then every quasiregular Y*, that is not contained in* $p(f^{-1}(0))$*, (obviously) has* $\sup_y \|M(Y, y)\| \ge m_0$*; hence,*

$$\|\partial_{i+1}^{-1}\|_{\mathrm{fil}}(\beta) \le m_0^{-1}$$

for all $\beta \ge 0$.

Sub-example. The concentric r-spheres S_r in the hyperbolic n-space X (and in every complete simply connected manifold of curvature ≤ -1 for this matter) have $M_i(S_r) \ge i$; thus, $\|\partial_{i+1}^{-1}\|_{\mathrm{fil}}(\beta) \le i^{-1}$ for all $i \ge 1$ in these X. Dehn's argument also applies to the symmetric spaces X of non-compact types with $\mathrm{rank}_{\mathbb{R}} \le i$ and, in a suitable form, to the Bruhat–Tits buildings.

Remark. The Dehn inequality is never sharp (at least in the natural examples) and the true value of $\|\partial_i^{-1}\|_{\mathrm{fil}}$ remains unknown in most cases, even in the hyperbolic n-space for $3 \le i \le n - 1$.

Tube volume estimates and filling inequalities. The local-to-global inequality delivers a lower bound on filling volumes of i-cycles, and thus, on $\|\partial_{i+1}^{-1}\|_{\mathrm{fil}}(\beta)$, in a Riemannian manifold X, provided one can bound from below the i-volumes of all quasiregular subvarieties Y in X with $\|M\|(Y) \le \mathrm{const}$. Such a bound can be achieved if

- on one hand, one can bound from above the n-volume of the *R-neighborhood* $U_R(Y) \subset X$ (i.e. the set of all points in X within distance $\le R$ from Y), which is also called *the R-tube* if R is not assumed small, of every quasiregular $Y \subset X$ with $\|M\|(Y) = \sup_{y \in Y} \|M_y\| \le m$ by
$$\mathrm{vol}_n\left(U_R(Y)\right) \le \gamma(m) \cdot \mathrm{vol}_i(Y)$$
for a suitable function $\gamma = \gamma_X(m)$;

•• on the other hand, one has some lower bound on the volumes of such R-neighborhoods, e.g. with an a priori bound on the volumes of all R-balls in X for a large R. Below are instances of where it works or does not quite work.

1. *Curves in the Plane.* Let Y be a *normally oriented* smooth curve in the plane $\mathbb{R}^2$ and let the map $\exp_\perp : Y \times \mathbb{R} \to \mathbb{R}^2$ send (y, r) to the second end of the straight segment normal to Y at y, where the choice of the normal direction is determined by the normal unit vector ν_y given by the normal orientation.

The Jacobian of this map at a point (y, r), obviously, equals $1 + r \cdot M_y(\nu)$, in so far as the segment $[y, \exp_\perp(y)] \subset \mathbb{R}^2$ minimizes the distance from $\exp_\perp(y) \in \mathbb{R}^2$ to Y. Notice that such a segment *fails* to be minimizing for $1 + r \cdot M_y(\nu) < 0$.

On the other hand, if Y is a *closed* curve then the area of the R-neighborhood of Y in $\mathbb{R}^2$ equals the integral of this Jacobian over the subset $\tilde{U} \subset Y \times \mathbb{R}$ of points y where the segment $[y, \exp_\perp(y)]$ is minimizing.

It easily follows that the area of the R-neighborhood of Y is bounded by the length of Y at least as efficiently as happens to a circle S^1_m of (constant) curvature $m = \sup_y \|M_y\|$,

$$\mathrm{vol}_2\left(U_R(Y)\right) / \mathrm{vol}_1(Y) \le \mathrm{vol}_2\left(U_R(S^1_m)\right) / \mathrm{vol}_1(S^1_m) .$$

Let $R \to \infty$ and observe that $\mathrm{vol}_2(U_R(Y))/vol_2(U_R(S^1_m)) \to 1$; hence $\mathrm{vol}_1(Y) \le \mathrm{vol}_1(S^1_m)$. Therefore, among all closed curves with (mean) curvatures bounded by m, the circles of curvature $= m$ have minimal lengths. Finally, we invoke the local-to-global principle and arrive at the isoperimetric inequality in the plane:

> *among all closed curves of given length the circles have the maximal filling areas* (which happen to be the areas bounded by these curves if they have no self-intersections).

2. *The Euclidean tube formula.* Let $Y = Y^i$ be a smooth submanifold in the Euclidean space $\mathbb{R}^n$, let $T^\perp(Y)$ be the normal bundle of Y and $\exp_\perp : T^\perp(Y) \to \mathbb{R}^n$ be the normal exponential map that sends each point $(y, \nu_y) \in T^\perp(Y)$ to the second end of the straight segment in $\mathbb{R}^n$, which issues from y, which is directed by the normal vector ν_y and which has the length equal the norm $\|\nu_y\|$.

It is easy to see that the Jacobian of $\exp_\perp$ at a point (y, ν_y), or, rather, the n-form on $T^\perp(Y)$ induced by $\exp_\perp$ from $\mathbb{R}^n$, depends only on $\|\nu_y\|$ and on the second fundamental form Φ of Y at y. In other words, if two submanifolds are second order tangent at y, then they have equal Jacobians of their respective $\exp_\perp$ along every straight line normal to them at y.

In fact, by looking closer, one easily sees that this Jacobian is determined by the symmetric operators Φ^*_ν on the tangent spaces of Y associated to Φ_ν via the Euclidean scalar product on the tangent bundle $T(Y)$, i.e. $\langle \Phi^*(\tau_1), \tau_2 \rangle = \Phi(\tau_1, \tau_2)$, (where the quadratic form Φ is regarded as a bilinear form) by the following:

The Hermann Weyl tube formula.

$$\mathrm{Jac}(\nu) = \det\left(1 + \|\nu\| \cdot \Phi^*_{\nu'}\right) \quad \text{for } \nu' = \nu / \|\nu\| .$$

Then an elementary estimate of the determinant on the right-hand side shows that, in-so-far as it remains positive, it is majorized by the norm of the mean curvature covector $M_y(\nu)$. Therefore,

the volume of the R-neighborhood $U_R(Y) \subset \mathbb{R}^n$ of an i-submanifold $Y \subset \mathbb{R}^n$ with the mean curvature M is majorized by $\mathrm{vol}_n(U_R(S^i_m))$ *for the round i-sphere $S^i_m \subset R^n$ with the mean curvature* $m = \sup_{y\in Y} \|M_y\|$.
We call the resulting bound *Euclidean Weyl's tube inequality.*

If Y is quasiregular, we argue as in the case of curves in the plane by applying the local-to-global principle and by sending $R \to \infty$. Thus, we arrive at Almgren's theorem cited in the previous section.

Among all (singular Lipschitz) i-cycles of given i-volume the round i-spheres have the maximal filling volumes.

Let X be a complete n-dimensional Riemannian manifold and $Y = Y^i \subset X$ a smooth submanifold. The *normal exponential map* $\exp_\perp : T^\perp(Y) \to X$ is defined as earlier by sending geodesics in X directed by the normal vectors $\nu \in T^\perp(Y)$. There are instances, most of the known ones concern (possibly) singular spaces with *curvatures bounded from below* (in contrast with Dehn's argument which relies on *upper* bound on the curvatures), where one can bound the Jacobian of this map [Buj], [HeK], and thus obtain estimates on the filling volumes of cycles in X which are sharp in certain cases. Below are a few concrete examples.

A. *Kleiner's filling inequality in the sphere.* The Weyl tube inequality extends to the unit sphere S^n with $\det(1 + \|\nu\| \cdot M^*_\nu)$ replaced by $\det(1 + \gamma_1(\|\nu\|) \cdot M^*_{\nu'})$, where the function γ_1 is seen by looking at round i-spheres in S^n for Y. Thus, one obtains a sharp volume bound on the R-neighborhoods of $Y^i \subset S^n$ with $\sup_y \|M_y\| \leq m$, and then Kleiner's theorem on the filling extremality of round spheres cited earlier follows by looking at $U_R(Y)$ for $R = \pi$.

B. *Tubes in hyperbolic spaces.* There is a similar formula with $\det(1 + \gamma_{-1}(\|\nu\|) \cdot M^*_{\nu'})$ in the hyperbolic space $X = H^n$ of constant curvature -1 which provides the bound on the volume of $U_R(Y)$ by the volume of the R-tube $U(\beta) \subset X$ around the round i-sphere $S^i \subset H^n$ of i-volume $\beta = \mathrm{vol}_i(Y)$. However, this gives a poor filling inequality, especially for large β, since there is no satisfactory lower bound on the volume of $U_R(Y) \subset H^n$, albeit such a bound is plausible in terms of filvol(Y). (Such a bound for $i = n - 1$ is obtainable by the *Schwartz symmetrization.*)

C. *Tube inequalities in symmetric spaces.* There are "explicit" tube formulas in all *symmetric spaces* X, since the curvature tensor is parallel and, thus, Jacoby fields along geodesics γ in X satisfy linear ODE-systems with *constant* coefficients, but the issuing bounds on $\mathrm{vol}_n(U_R(Y))$ are rarely sharp.

For instance, such a formula provides a bound on the $2k$-volumes of tubes around i-submanifolds in $X = S^k \times S^k$ for $i = k + 1, \dots, 2k - 1$, but the issuing filling inequality on the volumes of minimal $(i+1)$-chains filling in i-cycles remains unsharp. If $i = n - 1$ one can "in principle" obtain the sharp bound by the Schwartz symmetrization, but the case $i \leq n - 2$ seems difficult, except, maybe for $i = 1$.

A rare exception, besides spaces of constant curvature, is the complex projective space $X = \mathbb{C}P^k$ where the tube volume bound is sharp for subvarieties Y with zero mean curvature. It follows that *every $2j$-dimensional quasiregular subset $Y \subset \mathbb{C}P^k$ with mean curvature vanishing on the regular locus* $\mathrm{reg}(Y)$ *has* $\mathrm{vol}_{2j}(Y) \geq \mathrm{vol}_{2j}(\mathbb{C}P^j)$.

However, the corresponding sharp bound is unknown for odd-dimensional Y. For instance, if $Y \subset \mathbb{C}P^k$ is a hypersurface, $\dim(Y) = 2k-1$, one expects that its volume is bounded from below by the volume of some *homogeneous* $Y_0 \subset \mathbb{C}P^k$ with $\|M(Y_0)\| = \sup_y \|M(Y)|_y$, where "homogeneous" means that the isometry group of $\mathbb{C}P^k$ preserving Y_0 is transitive on Y_0.

On the other hand, if, for instance, $M(Y) = 0$, then the extremal Y_β that would perfectly match the tube volume bound needs to be a totally geodesic submanifold. But there is no i-dimensional totally geodesic $Y_\beta \subset \mathbb{C}P^k$ of volume $\beta > 0$ for odd i, except for $i = 1$.

D. *Tubes and Filling for Positive Curvature.* The volume tube bounds from [Buj] and [HeK] allow an extension of the above proof of the Almgren and Kleiner inequalities to manifolds X with the sectional curvatures $\geq \kappa \geq 0$ (and sometimes with $\kappa < 0$).

If $\mathrm{curv}(X) > 0$ the constant in the resulting filling inequality depends on $\mathrm{vol}(X)$; if $\mathrm{curv}(X) \geq 0$ and X is complete non-compact, then the constant depends on the rate of growth of the volumes of concentric balls $B(R) \subset X$, $R \to \infty$, where the latter inequality is sharp for cones and the former for spherical suspensions.

Questions. (a) Can one "hybridize" Dehn's and Almgren's inequalities, e.g. for Cartesian products of manifolds of positive and of negative curvatures and/or for spherical buildings? More generally, is there a "Künneth formula" relating filling invariants of $X = X_1 \times X_2$ to those of X_1 and X_2?

One can show, for instance, that if $X = S^n \times \mathbb{R}^m$ and $i \geq n+1$ then

the extremal i-cycles in X of $\mathrm{vol}_i \geq \mathrm{const}_{m,n}$ *are products of S^n with round $(i-n)$-spheres in $\mathbb{R}^m$,*

but the full filling profile of this X is more complicated.

(b) Is there a meaningful version of Weyl's formula in infinite-dimensional spaces of constant curvature, say in the Hilbert space, such that all infinities cancel one another?

Alternatively, one may search for a geometric inequality directly comparing the exponential image $\exp(T^\perp(Y^i))$ with $\exp(T^\perp(Y_0^i))$ for a suitable umbilical $Y_0^i \subset X$.

Notice that Almgren's proof of the local-to-global principle applies to compact Y^i, $i < \infty$, in infinite-dimensional spaces X.

On the other hand, if $X = X^\infty$ is a infinite-dimensional Riemannian manifold which densely and isometrically contains an increasing union of finite-dimensional submanifolds, $X^\infty \supset \cdots \supset X^{n+N} \supset \cdots \supset X^n$, such that all X^{n+N}, $N = 1, 2, \dots$, have $\|\partial_i^{-1}\|_{\mathrm{fil}}(\beta_0) \leq \delta_0$, for some $i(< \infty)$, then, obviously, X^∞ also has $\|\partial_i^{-1}\|_{\mathrm{fil}}(\beta_0) \leq \delta_0$. This applies, for example, to the Hilbert space $\mathbb{R}^\infty$, to the Hilbertian sphere $S^\infty \subset \mathbb{R}^{\infty+1}$ and to other infinite-dimensional symmetric spaces of "compact type", where the argument depends on the N-asymptotic of the $(n+N)$-volumes of X^{n+N}.

(c) Is there a dimension-free proof applicable to more general X^∞ e.g. to S^∞ divided by an infinite discrete isometry group Γ? (Notice that the simplicial volume-like invariants of Γ can be defined as the minimal volumes of certain homology classes in such spaces.)

(d) Is there a (sufficiently) *sharp* generalization of Almgren's filling bound in $\mathbb{R}^n$ to non-Euclidean Banach–Minkowski spaces X^n in the spirit of the Brunn–Minkowski inequality (corresponding to $i = n-1$)? Are there such inequalities in the metric spheres in Minkowski spaces and similar (e.g. Grassmann spaces) spaces S?

More realistically, one expects coarse filling inequalities in such S associated to *uniformly convex* Minkowski spaces in terms of the modulus of convexity.

(e) Does the variational method apply to $\Delta(V)$ and similar measurable complexes, and improve the bound $\|(\partial^i)^{-1}\|_{\mathrm{fil}} \leq 1$? (The complex of ε-simplices in the round sphere looks promising.)

(f) Is there an algebraic/topological version of the local-to-global principle and of $\|\dots\|_{\mathrm{fil}}^{\mathrm{loc}}$ in the context of our section 4?

(g) If $i = n-1$, then the filling volume of $Y \subset \mathbb{R}^n$ (which equals the volume of the domain encompassed by Y in this case) can be bounded by the measure of the set of straight lines in $\mathbb{R}^n$ meeting Y, see [Gr7].

It is conceivable that (similar to the remark following the spherical waist inequality in 1.3) the isoperimetric inequality stated at the end of section 5.7 of [Gr7] for $i = n-1$ generalizes to all i, where instead of the volumes of i-dimensional submanifolds in $\mathbb{R}^n$ and of their $(i+1)$-fillings one uses their "projection volumes" that are the measures of the sets of the $(n-i)$-planes and $(n-i-1)$-planes meeting these submanifolds and their fillings respectively. (One needs the "projection volumes" for the fillings as well as for the i-submanifolds themselves as was pointed out to me by Anton Petrunin.)

This may, possibly, work for submanifolds in the n-sphere but the hyperbolic case seems more difficult.

3.5 Lower bounds on volumes of minimal varieties and waist inequalities. There is an extension of the local-to-global variational principle to families of cycles, called *The Almgren–Morse theory* [Pi]. This implies, for instance, a lower bound on the *i-waist* of an X, provided there is a bound on the volumes of tubes of minimal (i.e. with zero mean curvature) i-subvarieties $Y \subset X$.

Recall that the *i-waist* of X is the infimum of the numbers w, such that the fundamental $\mathbb{Z}_2$-homology class in the space of i-cycles in X, say $[X]_{-i} \in H_{n-i}(cl_i;\mathbb{Z}_2)$, $n = \dim X$, can be represented by an $(n-i)$-family of i-cycles c in X with $\mathrm{vol}_i(c) \leq w$.

Thus, the Weyl volume tube formula in the sphere S^n implies the

Almgren's spherical waist inequality.

Every generic smooth map $F : S^n \to \mathbb{R}^{n-i}$, admits a point $y \in \mathbb{R}^{n-i}$, such that

$$\mathrm{vol}_i\big(F^{-1}(y)\big) \geq \mathrm{vol}_i(S^i)\,.$$

Remarks. (a) As we mentioned in 1.3, the smoothness/genericity condition is apparently redundant but the details have not been checked.

(b) The most frequently used lower bound on the volumes of minimal subvarieties $Y = Y^i \subset X$, called *monotonicity (property)*, is obtained by estimating the rate of growth of the volumes $A(R)$ of intersections of Y with R-balls $B(R, y_0)$ in X centered at some point $y_0 \in Y$ by integrating over Y the divergence of a suitable radial vector field in X projected to Y.

For example, if $X = \mathbb{R}^n$ and $Y \ni y_0 = 0$, then

$$\mathrm{vol}\big(Y \cap B(R,0)\big) \geq \mathrm{vol}\big(\mathbb{R}^i \cap B(R,0)\big)\,, \quad \text{for all } R \geq 0\,,$$

and similar estimates are available in other cases [Fo].

For instance the monotonicity estimate is sharp in S^n, and this was used by Almgren in his original proof of the waist theorem.

(c) The monotonicity argument usually (e.g. for $\mathbb{R}^n$, S^n and the hyperbolic space H^n) exploits an *upper* bound on the sectional curvature of X (which is zero for $\mathbb{R}^n$, it is $+1$ for S^n and -1 in H^n) and on the injectivity of the exponential maps $\exp_x : T_x(X) \to X$, $x \in X$, on the R-balls in some range $0 \leq R < R_0$ (where $R_0 = \infty$ for $\mathbb{R}^n$ and H^n, while S^n has $R_0 = \pi$).

But the tube argument relies on the *lower* bound on the curvature where it applies to lower bounds on waists in conjunction with a lower bound on $\mathrm{vol}(X)$ similarly to how it was indicated in D from the previous section for the filling problem.

(If X is complete non-compact with a lower bound on the rate of growth of volumes of concentric balls $B(R) \subset X$ for $R \to \infty$, then X satisfies *asymptotic waist inequalities*, defined similarly to the relative growth in 2.7.)

Apparently, the monotonicity works better for symmetric spaces, but the tube formula leads to sharper (sometimes even sharp) lower bounds of waists for many non-symmetric spaces of positive curvature.

(d) Even if the lower volume bound on minimal Y is sharp, this does not guarantee the sharpness of the corresponding waist inequality.

For instance, let $n = 2k$ and $X = \mathbb{C}P^k$. The lower bound on the volumes, obtainable by either of the two methods, is sharp for minimal subvarieties $Y^i \subset \mathbb{C}P^k$ for even $i = 2j$, with the equality for $\mathbb{C}P^j \subset \mathbb{C}P^k$.

However the corresponding waist inequality,

$$\sup_y \mathrm{vol}_i\big(F^{-1}(y)\big) \geq \mathrm{vol}_i(\mathbb{C}P^j)\,,$$

remains non-sharp, since the fibers $Y = F^{-1}(y) \subset \mathbb{C}P^k$ are homologous to zero and must have $\mathrm{vol}_i(Y) \geq \mathrm{vol}_i(\mathbb{C}P^j) + \varepsilon$, where the exact value of this $\varepsilon = \varepsilon(k) > 0$ seems hard (?) to guess.

(e) Sometimes positive and negative curvatures go along. For instance, if B^n is the unit Euclidean or hyperbolic ball, then

> *every generic smooth (apparently, continuous will do) map $F : B^n \to \mathbb{R}^{n-i}$, admits a point $y \in \mathbb{R}^{n-i}$, such that* $\mathrm{vol}_i(F^{-1}(y)) \geq \mathrm{vol}_i(B^i)$ *for the corresponding (Euclidean or hyperbolic) unit i-ball B^i.*

This trivially follows from the spherical waist inequality with an obvious (radial) i-volume contracting map from B^n to the sphere $S^n(R)$ of radius R which is chosen such that $\mathrm{vol}_i(S^n(R)) = \mathrm{vol}_i(B^i)$.

(f) A monotonicity property also holds for submanifolds with upper bounds on their mean curvatures, but I am not certain in which cases the *sharp* inequalities are available.

Analysis and geometry play complementary parts in the filling/waist inequalities.

The analytic existence/regularity statements are quite general and intuitively obvious (probably due to the limitations of our imagination), but the proofs are hard. The geometric lower volume bounds on minimal $Y \subset X$, which depend on a particular geometry of an X, may look striking, but once you have the idea, the proofs are elementary and straightforward.

The sparkle comes when the two collide.

3.6 Crofton's formulas and lower bounds on $\|\partial^{-1}\|_{\mathrm{fil}}$ by calibrations. *Crofton calibration in projective spaces and spheres.* Let $P^n = S^n/\pm$ be the real projective n-space with the usual metric of constant curvature and with the normalized i-volumes, such that projective i-subspaces have i-volumes 1.

Let dg denote the $O(n+1)$-invariant probability measure on the Grassmannian $G = G_{n-i}(P^n)$ of $(n-i)$-subspaces g in P^n and recall that every smooth i-submanifold V (and every rectifiable subset for this matter) satisfies *Crofton's formula*:

$$\mathrm{vol}_i(V) = \int_G |g \cap Y| dg\,.$$

It follows, that every i-cycle c in P^N, which is non-homologous to zero, has $\mathrm{vol}_i(c) \geq 1$, since it intersect each $g \in G$. Hence

$$\mathrm{syst}_i\big((P^n);\mathbb{Z}_2\big) = 1\,, \quad \text{for all } i = 0, 1, \dots, n\,.$$

Similarly, let $U \subset S^i \subset S^n$ be an open subset with smooth boundary in the i-sphere, which contains *no pair of opposite points* $(s, -s)$ in it. Then almost every equatorial $(n-i)$-sphere intersects U at a single point if at all; hence, the filling norm of the boundary $b = \partial U$ of U (which is an $i-1$-cycle) has

$$\|b\|_{\mathrm{fil}} = \mathrm{vol}_i(U)\,,$$

by the Crofton formula in S^n.

In other words, U itself provides the minimal filling of b in the ambient sphere $S^n \subset S^i \supset U$.

Remarks and Questions. (a) Both statements remain valid for $n = \infty$, but no "direct infinite-dimensional" proof seems to be known.

(b) Does the equality $\|b\|_{\mathrm{fil}} = \mathrm{vol}_i(U)$ hold for all U with $\mathrm{vol}_i(U) \leq \mathrm{vol}(S^i)/2$?

The Crofton formula for intersections of $(i+1)$-dimensional subvarieties $V \subset S^n$ with $(n-i+1)$-dimensional (rather than with $(n-i)$-dimensional) equatorial spheres implies this, provided the intersection of U with every equatorial circle $S^1 \subset S^i$ has $\mathrm{length}(S^1 \cap U) \leq \mathrm{length}(S^1)/2$.

Furthermore, a symmetrization argument apparently reduces the general case to that of $n = i+1$. Then, by the *Morse variational lemma*, the minimal filling of b

must be invariant under the isometry group Iso(b) of S^i preserving b, and if Iso(b) is transitive on b (e.g. if U equals a ρ-neighborhood of a j-equator in S^i for some $j < i$), then the problem reduces to an ODE-computation which seems manageable.

(c) Intersections of i-chains with a family G of $(n-i)$-cycles in $X = X^n$ parametrized by a probability space defines an i-cocycle in X. If $\mathbb{F} = \mathbb{R}$, a similar role is played by closed differential i-forms ω in X which can be used for lower bounds on norms of homology classes $h \in H_i(X;\mathbb{R})$, since, on the one hand, the integral of a closed form over an i-cycle c depends only the class $h = [c] \in H_i(X;\mathbb{R})$, and, on the other hand

$$\|c\| \geq \frac{\int_c \omega}{\sup_{x\in X} \|\omega\|_x} .$$

When such a bound is sharp one says, following Harvey and Lawson, that ω *calibrates* h.

EXAMPLE. If X is a Kähler manifold of complex dimension n, then the i-th power of the Kähler 2-form, $\omega = \omega^i_{\text{Kahl}}$, thought of as the function on the tangent real $2i$-planes in X, assumes its maximum on the complex i-planes according to *the Wirtinger inequality.* Therefore (see [F]),

every closed complex subvariety $V \subset X$ of complex dimension i is volume minimizing in its homology class $[V] \in H_{2i}(X;\mathbb{R})$.

For instance the projective subspaces $\mathbb{C}P^i \subset \mathbb{C}P^n$ are $\mathbb{Z}$-minimizing.

REMARKS AND QUESTIONS. (a) A use of the Crofton formula for the Grassmannian G of complex projective subspaces $\mathbb{C}P^{n-i} \subset \mathbb{C}P^n$ provides a lower bound on the norms of Z_p-cycles, but to make such bounds sharp one needs a stronger version of the Wirtinger inequality. (This is easy for $n = 2$ and $i = 1$.)

(b) One can get (apparently sharp) lower bounds on volumes of (at least even-dimensional) minimal subvarieties V in $\mathbb{C}P^n$ applying the *monotonicity estimate* that is a lower bound on the volumes vol($V(R)$) of intersections of V with the R-balls in $\mathbb{C}P^n$ with the center $v_0 \in V$.

If, for instance, V represent a $\mathbb{Z}_p$-cycle which is *not* a $\mathbb{Z}$-cycle, where p is a prime number, then V must have a point v_0 of density $p/2$ which makes $\mathrm{vol}(V(R)) \geq (p/2)\,\mathrm{vl}(R)$, for a universal function $\mathrm{vl}(R, i = \dim V)$.

(c) There are, besides Kähler, other remarkable forms which allow sharp lower bounds on volumes of (minimal) subvarieties. These were discovered by Harvey and Lawson [HL] where they called *calibrating forms.*

Let us reformulate the concept of calibration in the context of general measurable *chain* complexes C_* (where we prefer "chains" over "cochains" following the geometric picture).

Recall the quotient norm $\|h\|$ for $h \in H_i = H_i(C_*) = \ker(\partial_i)/\operatorname{im}(\partial_{i-1})$ as the infimum of the norms $\|c\|$ over all cycles $c \in \ker(\partial_i)$ representing h, and let

$$\mathrm{syst}_i(C_*) =_{\mathrm{def}} \inf_{0\neq c\in H_i} \|c\| .$$

In other words, this systole equals the infimum of the norms of the i-cycles c with $\|c\|_{\mathrm{fil}} = \infty$.

Define *i-cocycles* as homomorphisms g of C_i into a normed Abelian group, such that g vanishes on $\partial_{i-1}(C_{i-1}) \subset C_i$ and let

$$\|g\| =_{\rm def} \sup_{0\neq c\in C_i} \|g(c)\|/\|c\| .$$

In other words, this is the minimal norm for which $\|g(c)\| \leq \|g\| \cdot \|c\|$ for all i-chains $c \in C_i$.

A *random i-cocycle* is a family $G = \{g_p\}$ of i-cocycles g_p parametrized by a probability space P, where $\|G\| =_{\rm def} \int_P \|g_p\| dp$ (and where we do not exclude g_p with the cohomology class depending on p).

Obviously, the expectation $\int_P \|g_p(c)\| dp$ for a cycle $c \in \ker(\partial_i)/\operatorname{im}(\partial_{i-1})$ and a given G depends only on the homology class $h = [c] \in H_i(C_*)$ and $\int_P \|g_p(c)\| dp \leq \|G\| \cdot \|c\|$.

Define the *calibrated norm* of a homology class $h = [c]$ by

$$\|h\|_{\rm cal} = \sup_r \sup_{\|G\|\leq r} r^{-1} \int_P \big\|g_p(c)\big\| dp$$

where the supremum is taken over all random i-cocycles $G = \{g_p\}$ and all $r > 0$ and introduce the *the calibrated systole* by

$$({\rm syst}_i)_{\rm cal}(C_*) =_{\rm def} \inf_{h\neq 0} \|h\|_{\rm cal} .$$

In other words the calibrated systole expresses the best possible lower bound on the true systole available with random cocycles.

A cycle $c \in \ker(\partial_i) \subset C_i$ is called *calibrated* by a random i-cocycle $G = \{g_p\}$ if $\|g_p(c)\| = \|G\| \cdot \|c\|$ for almost all $p \in P$. Clearly, as in the Riemannian case (see Harvey–Lawson [HL])

every calibrated cycle c is norm minimizing in its homology class: $\|c\| = \|[c]\|$.

Similarly, define *the relative calibrated norm* $\|[c|\partial(c)]\|_{\rm cal}$, for $c \in C_i$, by

$$\big\|[c|\partial(c)]\big\|_{\rm cal} = \sup_r \sup_{\|g\|\leq r} r^{-1} \int_P \big\|g_p(c)\big\| dp ,$$

and observe that

$$\big\|[c|\partial(c)]\big\|_{\rm cal} \leq \|c|\partial c\| \quad \text{for all } c \in C_i ,$$

where $\|c|\partial c\| =_{\rm def} \|[c/\partial_{i+1}(C_{i+1})]\|$ is the quotient norm in $C_i/\partial_{i+1}(C_i)$.

If $H_i(C_*) = 0$, then

$$\partial_i(c') = \partial_i(c) \quad \text{implies } \big\|[c'|\partial(c)']\big\|_{\rm cal} = \big\|[c|\partial(c)]\big\|_{\rm cal} ;$$

thus the filling norm of every $b = \partial_i(c) \in C_{i-1}$ is bounded from below by

$$\|b\|_{\rm fil} \geq \big\|[c|\partial(c)]\big\|_{\rm cal} .$$

In other words, the filling profile satisfies

$$\big\|(\partial_i)^{-1}\big\|_{\rm fil}(\beta) \geq \beta^{-1} \sup_{\|\partial_i(c)\|=\beta} \big\|[c|\partial(c)]\big\|_{\rm cal} .$$

Say that a chain $c \in C_i$ is *calibrated (relative to* $\partial_i c$*)* by a random i-cocycle $G = \{g_p\}$ if $g_p(c) = \|G\| \cdot \|c\|$ for almost all $p \in P$ and observe that

if $H_i(C_*) = 0$, *then every calibrated i-chain c is minimizing: all i-chains* c' *with* $\partial_i c' = \partial_i c$ *have* $\|c'\| \geq \|c\|$.

About Examples. Calibrations provide a quite general method for obtaining lower bounds on the norms of cycles (as well as cocycles), but such estimates, apart from calibrations by differential form and of product spaces, are known to be sharp only in the presence of symmetries.

Sometimes, calibrations survive a break of symmetry. For example the "$\square$-discretization" of the Crofton calibration in P^n is again a calibration which yields the Barany–Lovasz inequality. (See, 3.3, where the subcomplex S in the second proof carries a cycle, such that the orbit $G(S)$ calibrates the cocycle supported by $0 \pitchfork_F \{\square^n\}$.)

Similarly, the "Δ-discretization" of Crofton's calibration in S^n implies that

the cochain complex $C^* = C^*(\Delta(V);\mathbb{Z}_2)$ of the simplex $\Delta(V)$ on the probability space V without atoms satisfies

$$\big\|(\partial^i)^{-1}_{\mathrm{fil}}\big\|(\beta) = 1 \quad \textit{for } \beta = 2^{-(i+1)}.$$

Proof. Recall that $\|(\partial^i)^{-1}_{\mathrm{fil}}\|(\beta) \le 1$ for all β (see 2.6) and, in order to prove the opposite inequality, we exhibit the following i-cocycle b with $\|b\| = \|b\|_{\mathrm{fil}} = 2^{-(i+1)}$.

Take V to be the round sphere, $V = S^{i-1} \subset \mathbb{R}^i$, and let $b \in C^i = C^i(V;\mathbb{Z}_2)$ be the cocycle with the support consisting of the i-simplices $\Delta^i \subset \mathbb{R}^i$ which contain the origin $0 \in \mathbb{R}^n$; observe that $\|b\| = 2^{-(i+1)}$ by Wendel's formula.

Take an $(i-1)$-face (simplex) $\Delta^{i-1} \subset \Delta(V)$ and observe that its orbit under the $\pm$-involutions of its vertices makes an $(i-1)$-cycle $c_{i-1} = \partial_i c_i$, where $c_i \in C_i(\Delta(V)\mathbb{Z}_2)$ is the cone over c_{i-1} from a point in V.

The i-chain c_i is determined by $i+1$ points in V: the vertices of Δ^{i-1} and the vertex of the cone, where the full family of all these c_i (parametrized by V^{i+1}) calibrates b, again by Wendel's formula (since $\langle c^{i-1}, c_{i-1}\rangle = \langle b, c_i\rangle = 1$ and $\|\langle c^{i-1}, c_{i-1}\| \le \|c^{i-1}\| \cdot \|c^{i-1}\|$). Thus every $c^{i-1} \in C^{i-1}$ with $\partial^{i-1}c^{i-1} = b$ satisfies $\|c^{i-1}\| \ge 2^{-(i+1)}$.

Remark. There is another value of β where $\|(\partial^i)^{-1}_{\mathrm{fil}}\|(\beta)$ is known to equal 1, namely $\beta = (i+2)!/(i+2)^{i+2} \sim e^{-i}$.

Indeed, partition V into $i+2$ measurable subsets X_j and let $b \in C^{i+1}(\Delta(V);\mathbb{Z}_2)$ be the $(i+1)$-cochain supported on the simplices with vertices $v_j \in X_j$, $j = 1, 2, \dots, i+2$. Clearly

$$\|b\| = (i+2)! \prod_{j=1,\dots,i+2} |X_j|\,.$$

Observe that $b = \partial^i(c)$, where c is an i-cochain that is supported on the simplices with vertices $v_j \in X_j$, $j = 1, 2, \dots, i+1$, and that, if the set X_{i+2} has the maximal measure among all X_j, i.e. $|X_{i+2}| = \max_j |X_j|$, then every c' with $\partial(c') = b$ has

$$\|c'\| \ge \|c\| = (i+1)! \prod_{j=1,\dots,i+1} |X_j|\,.$$

Thus,

$$\|b\|_{\mathrm{fil}}/\|b\| = (i+2)^{-1}|X_{i+2}|^{-1}$$

that equals 1 if all X_j have measures $1/(i+2)$. $\square$

QUESTIONS. (a) Are there further instances of calibrated combinatorial (co)cycles?

(b) Partition V into $N+1$ subsets of, say, equal measure and let $f:\Delta(V)\to\Delta^N=\Delta\{0,1,\dots,N\}$ be the induced simplicial map. When can one (effectively) evaluate the filling norms of cocycles $c^*=f^*(c)\in C^*(\Delta(V))$ for $c\in C^*(\Delta^N)$ in terms of (combinatorics of) c?

3.7 Bound $\|(\partial^1)^{-1}\|_{\mathrm{fil}}(\beta)<1$ for small β. Strengthening $\|(\partial^i)^{-1}\|_{\mathrm{fil}}(\beta)\le 1$ to $\|(\partial^i)^{-1}\|_{\mathrm{fil}}(\beta)\le 1-\varepsilon_i$ at least for small (but not too small) $\beta\le\beta(i)$ (e.g. for $\beta(i)=10^{-i}$ and all large i) would improve the constant b_{top} in the $[\Delta\to\mathbb{R}^n]$-inequality from 1.1

Here is such a bound I was able to prove by a baby version of the local to global principle, but, unfortunately, only for very small β precluding applications to lower bound on b_{top}.

Let V be a probability space without atoms and let $\sigma'\prec\sigma$ denote the $(i-2)$-faces of an $(i-1)$-face σ in $\Delta=\Delta(V)$. Given $c\in C^{i-1}(\Delta;\mathbb{F})$, let

$$\|\Sigma\|(c,\sigma)=\sum_{\sigma'\prec\sigma}\|\sigma'\wedge c\|=\sum_{\sigma'\prec\sigma}\int_V\big\|(\sigma'\wedge c)(v)\big\|dv\,,$$

and observe that

$$\big\|\sigma\wedge\partial^{i-1}(c)\big\|=\int_V\big(\sigma\wedge\partial^{i-1}(c)\big)(v)dv\ge(i+1)\big(\|c(\sigma)\|-\|\Sigma\|(c,\sigma)\big)\,,$$

since the contribution $(i+1)\|c(\sigma)\|$ of $c(\sigma)$ to $\|\sigma\wedge\partial^{i-1}(c)\|$ at the i-faces $\tilde\sigma\succ\sigma$ can be cancelled by $c(\sigma'')$ for other $(i-1)$-faces $\sigma''\prec\tilde\sigma$ (that are adjacent to the $(i-2)$-faces $\sigma'\prec\sigma$ and where $c(\sigma'')$ enters with a $\pm$-sign according to the orientation of σ'') by an amount not exceeding $(i+1)\|\Sigma\|(c,\sigma)$. Consequently,

$$\big\|\partial^{i-1}(c)\big\|=\int_{\Sigma_{i-1}}\big\|\sigma\wedge\partial^{i-1}(c)\big\|d\sigma\ge(i+1)\int_{\Sigma_{i-1}}\max\big(0,\|c(\sigma)\|-\|\Sigma\|(c,\sigma)\big)d\sigma\,.$$

Let

$$\|\Pi\|(c,\sigma)=\prod_{\sigma'\prec\sigma}\|\sigma'\wedge c\|=\prod_{\sigma'\prec\sigma}\int_V\big\|(\sigma'\wedge c)(v)\big\|dv\,,$$

and observe that

$$\int_{\Sigma_{i-1}}\big(\|\Pi\|(c,\sigma)\big)^{\frac1i}d\sigma\le\|c\|^{\frac{i+1}{i}}$$

by the Loomis–Whitney inequality (see [Gr7] and references therein; we use below only the case of $i=1$ where, obviously, $\int_{\Sigma_1}\|\Pi\|(c,\sigma)d\sigma=\|c\|^2$).

Now, let $\mathbb{F}=\mathbb{Z}_2$ with the standard norm and assume that

$$\|\sigma'\wedge c\|\le\tfrac1i\quad\text{for all } (i-2)\text{-faces } \sigma'.$$

Then

$$\|\Sigma\|(c,\sigma)\le\frac{i-1}{i}+\big(\|\Pi\|(c,\sigma)\big)^{\frac1i}\qquad\text{for all } (i-1)\text{-faces } \sigma\,;$$

therefore,

$$\begin{aligned}\big\|\partial^{i-1}(c)\big\|&\ge(i+1)\int_{\mathrm{supp}(c)}\left(1-\frac{i-1}{i}-\|\Pi\|(c,\sigma)^{\frac1i}\right)d\sigma\\&\ge\frac{i+1}{i}\|c\|\big(1-(i+1)\|c\|^{\frac1i}\big)\,.\end{aligned}$$

Furthermore, if we assume $\|\sigma' \wedge c\| \leq \frac{1}{i} + \varepsilon$ instead of $\|\sigma' \wedge c\| \leq \frac{1}{i}$, we obtain

$$\big\|\partial^{i-1}(c)\big\| \geq \frac{i+1}{i}\|c\|\big(1-(i+1)\|c\|^{\frac{1}{i}}\big) - \varepsilon', \quad \text{where } \varepsilon' = \varepsilon'_i(\varepsilon) \to 0 \text{ for } \varepsilon \to 0\,.$$

Conclusion: $\frac{2}{3}$-bound on $\|(\partial^{i-1})^{-1}_{\mathrm{fil}}\|$ for $i = 2$.

If V has no atom then the filling norm of the 2-cocycles $b \in C^2(\Delta(V);\mathbb{Z}_2)$ satisfies.

$$\|b\|_{\mathrm{fil}} - \tfrac{2}{3}\|b\| \leq 3\|b\|^{1/2}_{\mathrm{fil}}.$$

Consequently, since $\|b\|_{\mathrm{fil}} \leq \|b\|$,

$$\big\|(\partial^1)^{-1}_{\mathrm{fil}}\big\|(\beta) \leq \frac{2}{3(1-3\beta^{1/2})}\,. \qquad \left(\tfrac{2}{3}\right)$$

Proof. Pretend for the moment that there exists a *minimal* 1-cochain c with $\partial(c) = b$, i.e. where $\|c\| = \|b\|_{\mathrm{fil}}$, and observe that such a c has the (essential) supremum of $\|v' \wedge c\|$, $v' \in V = \Sigma_0$, bounded by 1/2. Indeed, if $\|v' \wedge c\| < 1/2$ on a subset $V' \subset V$ of positive measure, one could diminish the norm of c by subtracting the coboundary of a 0-cochain supported on a (possibly) smaller subset $V'' \subset V'$. Then the above inequality (without ε), specialized to $i = 2$, yields the required relation $\|c\| - \frac{2}{3}\|b\| \leq 3\|c\|^{1/2}$.

Finally, to avoid the minimality problem, use δ-*minimizing* cochains c_δ with $\delta \to 0$ where $\|c\| \leq \|b\|_{\mathrm{fil}} + \delta$, and apply the above inequality with $\varepsilon \to 0$.

Remarks. One may expect, by analogy with Almgren's isoperimetric inequality (see 3.3, 3.4), that $\|(\partial^i)^{-1}_{\mathrm{fil}}\|(\beta) \leq Ci$ with some constant C (say, $C = 4$) for all $i = 1, 2, \dots$ and $\beta \leq (\beta_0)^i$ where $\beta_0 > 0$ is another universal constant (say, 1/4). However, it is not even clear whether $\liminf_{\beta\to 0} \|(\partial^i)^{-1}_{\mathrm{fil}}\|(\beta) < 1$ for $i \geq 2$. Also, it is unclear if there are "small" cocycles b (i.e. with small norm $\|b\|$) that *locally* (for the $\|b_1 - b_2\|$-metric) minimize $\|b\|$, or even if all (small?) cocycles b are decomposable into sums of smaller cocycles.

3.8 Bounds on $\|\partial^i\|$ and Turan's graphs. *Norm of* $\partial^0 : C^0 = C^0(\Delta(V);\mathbb{Z}_2) \to C^1 = C^1(\Delta(V);\mathbb{Z}_2)$. Every 0-cochain c equals the ($\mathbb{Z}_2$-valued) characteristic function of its support $\mathrm{supp}(c) \subset V$, where $\mu^1(\mathrm{supp}(c)) = \|c\|$ and ∂c is supported on the symmetrized set $\mathrm{supp}(c) \times (V \setminus \mathrm{supp}(c)) \subset V \times V$; thus

$$\big\|\partial^0(c)\big\| = 2\|c\|\big(1 - \|c\|\big)$$

and

$$\|\partial^0\|(\alpha) = 2 - 2\alpha\,.$$

Evaluation of the norm of $\partial^1 : C^1 = C^1(\Delta(V);\mathbb{Z}_2) \to C^2 = C^2(\Delta(V);\mathbb{Z}_2)$.

If V has no atoms then

$$\|\partial^1\|(\alpha) = 1/\alpha \quad \textit{for } 1 \geq \alpha \geq \tfrac{1}{2}\,,$$

$$\|\partial^1\|(\alpha) = 3 - 2\alpha \quad \textit{for } \alpha = \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots,$$

$$\|\partial^1\|(\alpha) < 3 - 2\alpha \quad \textit{for } \alpha \neq 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\,.$$

Furthermore, the functions $\|\partial^i\|(\alpha)$ are strictly monotone decreasing in $\alpha \in [0,1]$, for all $i = 0, 1, 2, \dots$.

Proof. Denote by $\mathbf{1}^i \in C^i$ the i-cochain that equals 1 on all i-faces of $\Delta(V)$. Clearly, $\|\mathbf{1}^i\| = 1$ and

$$\big\|\partial^1(\mathbf{1}^i)\big\| = 1 \quad \text{for odd } i \quad \text{and} \quad \|\partial^1(\mathbf{1}^i)\| = 0 \quad \text{for even } i\,;$$

thus $\|\partial^i\|(1) = 1$ for odd i, since $\|\partial^i\|(\alpha) \le 1/\alpha$ for all $i = 0,1,2,\dots$ and $0 \le \alpha \le 1$.

Next observe that

$$\|\mathbf{1}^i - c\| = 1 - \|c\| \quad \text{and} \quad \big\|\partial(\mathbf{1}^i - c)\big\| = 1 - \|\partial c\|$$

for all odd i and all $c \in C^i$. It follows that

$$\big\|\partial^i(\mathbf{1}^i - \partial b)\big\| = 1 \quad \text{for all } b \in C^{i-1}.$$

Hence,

$$\|\partial^i\|(\alpha) = 1/\alpha$$

for

$$\alpha \ge 1 - \sup_{b\in C^{i-1}} \|\partial^{i-1} b\| = \sup_\beta \beta \|\partial^{i-1}\|(\beta).$$

Since, $\|\partial^0\|(1/2) = 1$ the equality $\|\partial^1\|(\alpha) = 1/\alpha$ for $\alpha \ge 1/2$ follows.

Given a countable set S, probability spaces $V_s = (V_s,\mu_s)$, $s \in S$, and positive numbers p_s with $\sum_s p_s = 1$ let $V = \coprod p_s V_s$ be the probability space decomposed into the disjoint union of $p_s V_s =_{\text{def}} (V_s, p_s\mu_s)$. Then define the corresponding weighted sum of i-cochains $c_s \in C^i(\Delta(V_s);\mathbb{Z}_2)$, denoted $\coprod_s p_s \blacktriangle c_s \in C^i = C^i(\Delta(V);\mathbb{Z}_2)$.

Clearly, $\|\coprod_s p_s \blacktriangle c_s\| = \sum_s p_s^{i+1}\|c_s\|$ and

$$\Big\|\partial^i\Big(\coprod_s p_s \blacktriangle c_s\Big)\Big\| = \sum_s p_s^{i+2}\|\partial^i c_s\| + (i+2)\sum_s (1-p_s)p_s^{i+1}\|c_s\|\,.$$

For example,

$$\Big\|p\blacktriangle c \coprod (1-p)\blacktriangle \mathbf{0}\Big\| = p^{i+1}\|c\|$$

and

$$\Big\|\partial^i\Big(p\blacktriangle c \coprod (1-p)\blacktriangle \mathbf{0}\Big)\Big\| = p^{i+2}\|\partial^i(c)\| + (i+2)(1-p)p^{i+1}\|c\|$$

for $\mathbf{0}$ denoting the identically zero cochain. Since, $\|\partial^i(\alpha)\| < i+2$ for $\alpha > 0$, the ratio $\|c\|/\|\partial^i(c)\|$ is strictly and definitely greater than $1/(i+2)$, for $\|c\| = \alpha > 0$; thus

$$\Big\|p\blacktriangle c \coprod (1-p)\blacktriangle \mathbf{0}\Big\| / \|c\| < \Big\|\partial^i\Big(p\blacktriangle c \coprod (1-p)\blacktriangle \mathbf{0}\Big)\Big\| / \|\partial^i(c)\|$$

for $\|c\| = \alpha > 0$ and $p < 1$. Consequently,

> *the norm $\|\partial^i\|(\alpha)$ is a strictly monotone decreasing (and, obviously, continuous) function in α.*

Next, let $c_s = \mathbf{1}^i = \mathbf{1}^i_s \in C^i(\Delta(V_s))$ and observe that the cochain $c(p_s) = \coprod_s p_s \blacktriangle \mathbf{1}^i \in C^i(\Delta(V);\mathbb{Z}^2)$ satisfies

$$\big\|c(p_s)\big\| = \sum_s p_s^{i+1} \quad \text{and} \quad \|\partial^i c\| = \chi_i \sum_s p_s^{i+2}\|\partial c_s\| + (i+2)\sum_s (1-p_s)p_s^{i+1}$$

where $\chi_i = 0$ for even i and $\chi_i = 1$ for odd i.

This, with $c(1/k, 1/k, \dots, 1/k)$, i.e. where $p_s = 1/k$, $s = 1,2,\dots,k$, shows that

$$\|\partial^i\|\left(\frac{1}{k^i}\right) \ge \frac{\chi_i}{k} + (i+2)\left(1 - \frac{1}{k}\right).$$

In particular, $\|\partial^1\|(\alpha) \ge 3 - 2\alpha$ for $\alpha = 1/k$, $k = 1,2,\dots$.

Let us refine the upper bound $\|\partial^i\| \leq i+2$ where we need the following notion.

Density of cochains. Given a positive function φ on the set $\Sigma_i = V^{i+1}/\Pi(i+1)$ of the i-faces of $\Delta(V)$ (if V has no atoms the diagonals in V^{i+1} do not matter) define its j-*density* (*degree*) $d_j = \mathrm{den}_j(\varphi) : \Sigma_j \to \mathbb{R}_+$ via the push-forwards of the measure $\varphi\mu^{i+1}$ by the coordinate projection $V^{i+1} \to V^{j+1}$. Clearly $d_{j'}(d_j(\varphi)) = d_{j'}(\varphi)$ for $j' < j$ and $\|d_j\| =_{\mathrm{def}} \int_{\Sigma_j} d_j\mu^{j+1} = \int_{\Sigma_i} \varphi\mu^{i+1} = \mathrm{def}\|\varphi\|$.

If $\varphi(\sigma = \|c(\sigma)\|$ for an i-cochain on $\Delta(V)$ then $\mathrm{den}_j(\varphi)(\sigma') = \|\sigma' \wedge c\|$ that is also denoted $d_j(c)(\sigma')$.

Denote by $\partial^i_m(c) \in C^{i+1}(\Delta(V);\mathbb{Z}_2)$ for $c \in C^i(\Delta(V);\mathbb{Z}_2)$, the cochain supported on those $(i+1)$-faces that contain at least m faces in the support of c and observe that

$$\|\partial^i(c)\| = \sum_m \chi_m\big(\|\partial^i_m(c)\| - \|\partial^i_{m+1}(c)\|\big) \leq (i+2)\|c\| - 2\big\|\partial^i_2(c)\big\|\,,$$

where the equality holds if and only if $\partial^i_3(c) = \partial^i_2(c)$ and $\partial^i_m(c) = 0$ for $m > 3$.

The $(i-1)$-density of $\partial^i_2(c)$ is related to that of c by

$$\int_{\Sigma_{i-1}} \mathrm{den}_{i-1}\big(\partial^i_2(c)\big) \geq \int_{\Sigma_{i-1}} \big(\mathrm{den}_{i-1}(c)\big)^2;$$

hence

$$\big\|\mathrm{den}_{i-1}(\partial^i_2(c))\big\| \geq \|c\|^2$$

by the Cauchy–Schwartz inequality. Therefore,

$$\|\partial^i\|(\alpha) \leq (i+2) - 2\alpha\,.$$

Finally, let us explain why this upper bound for $i = 1$ is *not* sharp, unless $\alpha = 1, 1/2, 1/3, 1/4, \dots$.

If $\|\partial^1(c)\|/\|c\| = 3 - 2\alpha$, then c must have *constant* density $\mathrm{den}_0(c) = \alpha = \|c\|$ (when "Cauchy–Schwartz" becomes an equality) and the support $S \subset X \times X$ of c must be an *equivalence relation*:

$$(x_1,x_2),(x_2,x_3) \in S \Rightarrow (x_1,x_3) \in S$$

for almost all x_2 and almost all pairs (x_1,x_3) in the "slice" $S \cap x_2 \times X$ where the latter "almost all" refers to the canonical (Fubini) measure in this slice. Since the density is constant, the equivalence classes must have equal measures, all equal α, and so α must be an inverse integer.

Thus, $\|\partial(c)\|/\|c\| < 3 - 2\alpha$ for $\|c\| = \alpha \neq 1, 1/2, 1/3, 1/4, \dots$, but it does not imply the required bound for $\|\partial^1\|(\alpha) = \sup_{\|c\|=\alpha} \|\partial(c)\|/\|c\|$ since the supremum does not, a priori, have to be attained. However, by looking closely at the above argument, one sees that, for every $0 < \alpha < 1$, the support S of every chain c with $\|c\| = \alpha$ and $\|\partial^1(c)\|/\|c\| \geq 3-2\alpha-\varepsilon$ defines "a ε'-equivalence relation": a collection of disjoint subsets of measures ε'-close to α and with the union of measure $\geq 1-\varepsilon'$, where $\varepsilon' \to 0$ for $\varepsilon \to 0$. Hence, the equality $\|\partial^1\|(\alpha) = 3 - 2\alpha$ does imply that α is an inverse integer.

QUESTIONS. (a) Can one describe the set $A(C^i) \subset \mathbb{R}^\infty$ of norms of integral cochains and their boundaries modulo the prime numbers,

$$A(C^i) = \big\{\|c/\mathbb{Z}_p\|, \|\partial^i(c/\mathbb{Z}_p)\|\big\} \quad \text{for all } c \in C^i(\Delta(V);\mathbb{Z})\,,$$

where $c/\mathbb{Z}_p$ denotes the reduction modulo p for all primes p?

(b) Do the cochains $c = \coprod_s p_s \blacktriangle \mathbf{1}^1$ for $s = 1, 2, 3 \ldots, k$, with $p_1 \leq p_2 = p_3 = \cdots = p_k = p$ maximize $\|\partial^1(c)\|$ among all $\mathbb{Z}_2$-cochains with given norm $\|c\| = \alpha = p_1^2 + (k-1)p^2$?

Remarks. (a) Recently Rasborov and Nikiforov proved that these $\coprod_s p_s \blacktriangle \mathbf{1}^1$ maximize $\|\partial_1^1(c)\|$ or, equivalently, that $\mathbf{1} - \coprod_s p_s \blacktriangle \mathbf{1}^1$ minimize ∂_3^1 (see [R], [Ni], I owe these references to Noga Alon):

among all measurable graphs $S \subset V \times V$ with α edges, the Turan graphs, that are the supports of $\mathbf{1} - \coprod_s p_s \blacktriangle \mathbf{1}^1$, have the minimal "numbers" of triangles,

(b) If V has no atoms and $\mathbb{F} = \mathbb{Z}_2$, then

$$\|\partial^i\|(\alpha) \geq (i+2)\alpha(1-\alpha^{i+1}) \quad \text{for even } i\,,$$

and

$$\|\partial^i\|(\alpha) \geq \alpha^{\frac{i+2}{i+1}} + (i+2)\alpha(1-\alpha^{i+1}) \quad \text{for odd } i\,.$$

In fact, this lower bound on $\|\partial^i\|(\alpha)$ is seen on the i-cochains c_μ that are equal 1 on all i-simplices in the sub-simplex $\Delta(V_\mu)$ spanned by a subset $V_\mu \subset V$ of measure $\mu \in 0, 1$, and that vanish on the rest of the i-simplices in $\Delta(V)$. Clearly $\|c_\mu\| = \mu^{i+1}$, while

$$\big\|\partial^i(c_\mu)\big\| = (i+2)\mu^{i+2}(1-\mu) \quad \text{for even } i\,,$$

and

$$\big\|\partial^i(c_\mu)\big\| = \mu^{i+2} + (i+2)\mu^{i+2}(1-\mu) \quad \text{for odd } i\,.$$

4 Homological Isoperimetry

We start this section with formalizing and slightly refining the lower bound from [Gr8] on the cohomologies of the fibers of continuous maps $F : X \to Y$, and then study in greater detail the case $\dim(Y) = 1$ with the help of *isoperimetric/separation inequalities* for the algebra $H^*(X)$.

The subsections 4.3 and 4.7 are needed for 4.10 where we construct homology expanders promised in 1.6, while the result of 4.6 is used in 4.8 and 4.9, where we prove the separation inequality for the N-tori stated in 1.5 and its generalizations.

4.1 Ideal valued measures and cohomological widths. Let $A = \oplus_n A(n)$ be a graded (anti)commutative algebra over some field $\mathbb{F}$ where the product in A is denoted by "$\smile$", let $\mathcal{I} = \mathcal{I}(A)$ be the set of the graded ideals $I \subset A$ and let Y be a topological space.

An $\mathcal{I}$-mass μ on Y is an assignment $\mu = \mu_A : U \mapsto \mu(U) \in \mathcal{I}(A)$, for all open $U \subset Y$ which satisfy the following five conditions:

(0) *Normalization.* $\mu(\emptyset) = 0$.
(1) *Monotonicity.* $U_1 \subset U_2 \Rightarrow \mu(U_1) \subset \mu(U_2)$.
(2) *Continuity.* If U equals the union of an increasing sequence of open subsets $U_1 \subset U_2 \subset \cdots \subset U$, then

$$\mu(U) = \mu(U_1) \cup \mu(U_2) \cup \ldots\ .$$

(3) *Additivity.* If subsets $U_i \subset Y$ pairwise do not intersect, then
$$\mu(\cup_i U_i) = +_i \mu(U_i)\,,$$
where "+" denotes the span/sum of linear subspaces in the vector space A.

(4) *Multiplicativity.* $\mu(U_1 \cap U_2) \supset \mu(U_1) \smile \mu(U_2)$.

We distinguish masses with two additional properties.

(5) *Intersection property.* Let $W_i \subset Y$ be *disjoint closed subsets* and denote $\mathbf{0}_A(W_i) = \mu(X \setminus W_i)$. Then
$$\mathbf{0}_A\big(\cup_i W_i)\big) = \cap_i \mathbf{0}_A(W_i)\,.$$

(6) *Fullness.* $\mu(Y) = A$.

These properties are obvious for $A = H^n(X)$ and $\mu_A(U)$ equal the kernel of the restriction homomorphism of A to $X \setminus U$ (see 1.5), where, maybe, the inclusion part $\mu(\cup_i U_i) \subset +_i \mu(U_i)$ of (3) needs an explanation.

To show this, observe that every cohomology class h from $\mu(\cup_i U_i)$ can be represented by a cocycle c with the support in $\cup_i U_i$. Since the sets U_i are disjoint, this c decomposes into the sum of cochains c_i with supports in U_i and such that $c_i|U_i = c|U_i$. Clearly, these cochains are cocycles and $\sum_i [c_i] = [c] = h$.

Push-forward and restriction. If $F : X \to Y$ is a continuous map and μ is an $\mathcal{I}(A)$-mass on X, then the push-forward $\mu_\diamond = F_\diamond(\mu)$ of an $\mathcal{I}$-mass μ on X to Y, defined by $\mu_\diamond(U) = \mu(F^{-1}(U))$ for $U \subset Y$, is an $\mathcal{I}$-mass on X, where the intersection property and fullness are preserved under this push-forward.

The restriction of an $\mathcal{I}(A)$-mass on X to an open subset $Y \subset X$ is an $\mathcal{I}(A)$-mass on Y, but the intersection property and fullness are not necessarily preserved under restriction.

A-Covariant functoriality. Let $\phi : A \to B$ be a surjective homomorphism. Then the map $\mathcal{I}(A) \ni I \mapsto \phi(I) \in \mathcal{I}(B)$ sends every $\mathcal{I}(A)$-mass μ on Y to a $\mathcal{I}(B)$-mass. This preserves fullness but not necessarily the intersection property since, in general, $\phi(I_1 \cap I_2) \neq \phi(I_1) \cap \phi(I_2)$.

H^*- AND $F_\diamond$-EXAMPLES. Let X be compact, $\mathbb{F}$ finite and A equals the Čech cohomology $H^*(X) = H^*(X;\mathbb{F})$. Then the kernels $\ker(\mathrm{rest}^*_{X\setminus U})$ of the restriction homomorphisms $\mathrm{rest}^*_{X\setminus U} : H^*(X) \to H^*(X \setminus U)$ define a full mass with the intersection property, $U \mapsto \ker(\mathrm{rest}^*_{X\setminus U})$ on X called *the $H^*(X)$-mass* and denoted $\mu = \mu_{H^*(X)}$.

REMARK. One needs Čech cohomology with finite coefficients in order to guarantee continuity, which may, in general fail. However, if the sets U in question have non-pathological boundaries, then any cohomology there will do. For example the mass $\mu_{H^*(X)}$ is defined and has all of the above properties on semialgebraic subsets in simplicial polyhedra.

The push-forward $F_\diamond(\mu)$ of the mass $\mu_{H^*(X)}$ to Y under a continuous map $F : X \to Y$ is called *the $F_\diamond$-mass $\mu_{F_\diamond}$ on Y.*

If the induced cohomology homomorphism $F^*: H^*(Y) \to H^*(X)$ is *surjective* then

$$F_\diamond\big(\mu_{H^*(X)}\big) = F^*_{\mathcal{I}}\big(\mu_{H^*(Y)}\big)\,,$$

where $F^*_{\mathcal{I}} : (H^*(Y)) \to \mathcal{I}(H^*(X))$ is the F^*-pullback map on the set of ideals in $H^*(Y)$.

Given an arbitrary $\mathcal{I}(A)$-mass μ on Y, we write $\mathbf{0}_A(Y_1) = \mathbf{0}^\mu_A(Y_1) \in \mathcal{I}(A)$ for the value of μ on the complementary subset in X, i.e. the ideal $I = \mu(Y \setminus Y_1)$ and let $A|Y_1 = A|_\mu Y_1 = A/\mathbf{0}_A(Y_1)$ for all $Y_1 \subset Y$. Observe that $\mu(Y_1) \smile \mathbf{0}_A(Y_1) = 0$ for all open subsets $Y_1 \subset Y$ by the multiplicativity of the mass.

We are particularly interested in the maxima of the $\mathbb{F}$-ranks of the algebras $A|y$ and of their n-th grades, denoted $|A(n)|y|_{\mathbb{F}}$, at the points $y \in Y$, especially for the push-forward masses $\mu_{F_\diamond}$ on Y for continuous maps $F : X \to Y$, where $A(n) = H^n(X;\mathbb{F})$.

The infimum of these maxima over all full $\mathcal{I}(A)$-masses μ on Y (where we may or may not require the intersection property), called the *width of A over Y*, denoted

$$\mathrm{width}_*(A/Y) = \inf_\mu \sup_{y\in Y} |A|_\mu y|_{\mathbb{F}}\,,$$

and

$$\mathrm{width}_n(A/Y) = \inf_\mu \sup_{y\in Y} |A(n)|_\mu y|_{\mathbb{F}}\,.$$

These widths are evaluated for certain A and Y in the following sections and applied for lower bounds on the *cohomological width*, denoted $\mathrm{width}^*(X/Y)$, of topological spaces X over Y, defined with $A = H^*(X)$, where the infimum taken over the $\mu_{F_\diamond}$-masses on Y for all continuous maps $F : X \to Y$.

This constitutes a (small) step toward the solution of the following general

$\mathcal{I}\mu$-Problem. Given a class $\{Y\}$ of topological spaces (e.g. of Y with $\dim(Y) \le d$ for some d) and a class $\{B\}$ of graded algebras (e.g. of B with $\mathrm{rank}_{\mathbb{F}}(B) \le r$ for some r), describe the class $\{A\} = \{A\}(\{Y\};\{B\})$ of graded algebras such that each $A \in \{A\}$ admits a full (where one my insist on the intersection property) $\mathcal{I}(A)$-mass over some $Y \in \{Y\}$, where $A|y \in \{B\}$ for all $y \in Y$.

This problem can usually be reduced to its combinatorial counterpart where $\{Y\}$ is a class of simplicial complexes Y which is closed under subdivisions of complexes, and where the condition $A|y \in \{B\}$ is replaced by $A|\Delta \in \{B\}$ for all simplices $\Delta \in Y$.

However, even in the simplest case where $\{Y\}$ is the class of all finite graphs and $\{B\}$ is the class of algebras with $\mathrm{rank}_{\mathbb{F}}(B) \le r$ for some r, our results provide only limited information on $\{A\} = \{A\}(graphs; r)$.

The $\mathcal{I}\mu$-problem is motivated by the corresponding topological problem where we are a given, instead of $\{B\}$, a class $\{Z\}$ of (homotopy classes of) topological spaces (e.g. of spaces decomposable into r cells for some r) and we want to decide whether a given space X admits a continuous map F to some $Y \in \{Y\}$, such that $F^{-1}(y) \in \{Z\}$ for all $y \in Y$.

A particular case of this is the *restricted $\mathcal{I}\mu$-problem*, where we allow only those $\mathcal{I}(A)$-masses μ on Y of the form $\mu_{F_\diamond}$, where $A = H^*(X)$ for some X and where $F : X \to Y$ is a contionous map.

It seems unclear by how much this restriction affects the answer. If for example, $\mathbb{F} = \mathbb{Q}$, and $\{B\}$ is defined by $\mathrm{rank}(B(n)) \leq r$, then the "restricted class" $\{A\}_{\mathrm{res}}(\{Y\}; (n, r))$ is closed under taking quotients of A, at least for (truncated) free algebras A; but this, probably, does not hold for the corresponding non-restricted class $\{A\}$.

Apparently, if $\mathbb{F} = \mathbb{Q}$, then some extension of the $\mu_{F_\diamond}$-masses from cohomology to the free differential algebras representing minimal model of spaces in question may fully reflect the topological picture; and, if $\mathbb{F} = \mathbb{F}_p$, then $\mu_{F_\diamond}$ can probably be enhanced by incorporating the action of the Steenrod algebra. Also, other cohomology theories, e.g. K-theory with the Adams operations, may prove useful.

Finally, observe that the $\mu_{F_\diamond}$-mass represents a small part of the information contained in the Leray–Grothendieck sheaf of the map F, but I have not worked out any example of a lower bound on the cohomological width of X over Y with the use of the (multiplicative) Leray spectral sequence.

QUESTION. Is there some "integration theory" associated to μ_A?

4.2 Maximal fiber inequality revisited. Let A be a (anti)commutative graded algebra over some field $\mathbb{F}$ with the product in A written as "$\smile$" and let $A^{/r} \subset A$ denote (differently from that in [Gr8]) the intersection of the graded ideals $I \subset A$ with $\mathrm{rank}(A/I) < r$.

Recall that $\mathrm{rank}^{\smile}_d(A)$ is defined (see [Gr8]) as the maximal number r, such that the d-multiple cup-product map $(A)^{\otimes d} \to A$ is not identically zero on $A^{/r} \subset A$.

For example, if $A = H^*(X; \mathbb{F})$, where X is the Cartesian product of k closed connected manifolds X_i (orientable, unless $\mathbb{F} = \mathbb{Z}_2$), then

$$\mathrm{rank}^{\smile}_k(A) \geq \min_i \mathrm{rank}_{\mathbb{F}}\big(H^*(X_i)\big)\,.$$

The proof of the cohomological maximal fiber inequality in [Gr8] shows that

$$\mathrm{width}_*(A/Y) \geq \mathrm{rank}^{\smile}_{d+1}(A)$$

for all compact topological spaces Y with $\dim(Y) \leq d$, where width_* is defined (see 4.1) with the $\mathcal{I}(A)$-masses on Y which satisfy the intersection property.

This inequality is non-vacuous only if $\mathrm{length}_{\smile}(A) \geq 2d+2$ for $d = \dim(Y)$, i.e. if A contains $2d+2$ elements of positive degrees with non-zero $\smile$-product (e.g. if $A = H^*(X_0^{d+1})$ where X_0 is a closed surface of positive genus).

Let us describe a class of algebras A with $\mathrm{length}_{\smile}(A) = 2d+1$ which may have arbitrary large $\mathrm{width}_*(A/Y)$ for all compact d-dimensional spaces Y.

Let $a^{\perp} \subset A$ for $a \in A$ denote the $\smile$-orthogonal complement to a, i.e. the kernel of the operator $\smile a : A \to A$ of the right multiplication by a, let $a + a^{\perp}$ be the linear span of a and $a^{\perp}$ and define

$$\mathrm{rank}_{\perp}(\smile a) = \big|A/(a + a^{\perp})\big|_{\mathbb{F}} =_{\mathrm{def}} \mathrm{rank}_{\mathbb{F}}\big(A/(a + a^{\perp})\big)\,.$$

Notice that if the cup-product by a has finite rank, $\mathrm{rank}_{\mathbb{F}}(\smile a) < \infty$, then either $\mathrm{rank}_{\perp}(\smile a) = \mathrm{rank}(\smile a)$ or $\mathrm{rank}_{\perp}(\smile a) = \mathrm{rank}(\smile a) + 1$ depending on whether $a_{\perp}$ contains a.

Let $I_{\perp}(A, r) \subset A$ be the graded ideal generated by all a with $\mathrm{rank}_{\perp}(\smile a) < r$ and let $\mathrm{rank}'^{\smile}_d(A)$ be the maximal number r for which the image $(A^{/r})^{\smile d} \subset A$ of the d-multiple $\smile$-product map $(A^{/r})^{\otimes d} \to A$ is *not contained* in $I_{\perp}(A, r)$.

Example. Let $A_0 = A_0(0) \oplus \dots \oplus A_0(p)$ and $A_1 = A_1(0) \oplus \dots \oplus A_1(p+q)$ be graded algebras of finite $\mathbb{F}$-ranks, such that $\operatorname{rank}_k^{\smile}(A_0) \geq r_1$ for some k and r_1 and such that the operators $\smile a : A_1(0) \oplus \dots \oplus A_1(q) \to A_1$ have $\operatorname{rank}(\smile a) \geq r_2$ for all $0 \neq a \in A_1(1) \oplus \dots \oplus A_0(p) \subset A_1$. Then the truncated tensor product algebra

$$A = \oplus_{i+j \leq p+q} A_0(i) \otimes A_1(j)$$

has $\operatorname{rank}_k'^{\smile}(A) \geq \min(r_1, r_2)$, since

$$A^{/r_1} \supset A_0 \otimes \mathbf{1} \cap A^{/r_1} \supset (A_0 \otimes \mathbf{1})^{/r_1} = A_0^{/r_1}$$

and

$$I_\perp(A, r_2) \subset A(p+1) \oplus \dots \oplus A(p+q)\,.$$

For instance, if X is the product of a Cartesian power of a closed orientable surface and the N-torus, $X = X_0^d \times \mathbb{T}^N$, then the algebra $A = H^0(X) \oplus \dots \oplus H^{2d+1}(X)$ for $H^* = H^*(\cdots; \mathbb{F})$ has

$$\operatorname{rank}_d'^{\smile}(A) \geq \min\bigl(N, \operatorname{rank}(H^*(X_0))\bigr)\,.$$

Consequently, the cohomology of the $(2d+1)$-skeleton of every cell decomposition of X also has $\operatorname{rank}_d'^{\smile} \geq \min(N, \operatorname{rank}(H^*(X_0)))$.

$(2d+1)$-Width inequality.

Let Y be a compact space of dimension d and $\mu = \mu_A$ be an $\mathcal{I}$-mass on Y as in 4.1 with values in the set $\mathcal{I} = \mathcal{I}(A)$ of graded ideals $I \subset A$. Then

$$\operatorname{width}_*(A/Y) \geq \operatorname{rank}_d'^{\smile}(A)\,.$$

Proof. By the multiplicativity of μ (see 4.1), every open subset $U_0 \subset Y$ satisfies

$$|A|U_0|_{\mathbb{F}} \leq s = \sup_{a \in \mu(U_0)} \operatorname{rank}_\perp(a)\,,$$

where, recall, $A|U_0 = A/\mu(Y \setminus U_0)$ (where $\mu_A(U_0)$ corresponds to the kernel of the restriction cohomology homomorphism of A to the complement $X \setminus U_0$ in the topological context) and $|\dots|_{\mathbb{F}}$ denotes $\operatorname{rank}_{\mathbb{F}}(\dots)$.

Cover Y by sufficiently small open subsets, $Y = \cup_{ij} U_{ij}$, $i = 0, \dots, d$, $j = 1, 2, \dots$, such that U_{ij} is disjoint from $U_{ij'}$ for all i and $j \neq j'$ and

$$|A|U_{ij}|_{\mathbb{F}} \leq s = \sup_{y \in Y} |A|y|_{\mathbb{F}}$$

for all U_{ij}, which is possible due to the continuity of the mass (compare the proof of the maximal fiber inequality in [Gr8]). Let $U_i = \cup_j U_{ij}$ and $U = \cup_{i>0} U_i$. Then, by the additivity and monotonicity of the mass

$$\mathbf{0}_A(U) \subset \mu(U_0) \subset I_\perp(A, s)\,.$$

On the other hand, since $\mathbf{0}_A(U_i) = \cap_j \mathbf{0}_A(U_{ij})$ by the intersection property and since $|A|U_{ij}|_{\mathbb{F}} \leq s$, the ideals $\mathbf{0}_A(U_i) \subset A$, $i = 1, \dots, d$, contain $A^{/s} \subset A$, while the product of these ideals is contained in $\mathbf{0}_A(U)$ by the multiplicativity of the mass. Therefore,

$$(A^{/s})^{\smile d} \subset I_\perp(A, s)$$

and the proof follows.

Questions. Are there algebras A with $\operatorname{length}_\smile(A) = \delta < 2d-1$, for $d \geq 2$, with arbitrary large width (the intersection property is required) over all compact d-dimensional spaces Y. (An encouraging sign comes from the lower bound on the fibers of maps of hyperbolic $(d+2)$-dimensional spaces X to Y with $\dim(Y) = d$, since our argument in [Gr8] is cohomological in nature.)

Does it help to limit to $Y = \mathbb{R}^d$? Does anything of the topology of Y besides $d = \dim(Y)$ matter at all? (We shall see in 4.12 that this is the case for $d = 1$.)

Does the cohomological width of spaces X, such as product of spheres and/or of unions of some coordinate sub-products $X_0 \subset X$ over, say, $\mathbb{R}^d$ equal the widths of their respective cohomology (Stanley–Reisner) algebras? (Notice that the latter may, apparently, only diminish under passing to quotient algebras.)

Can one fully describe algebras of width $\leq r$ over a given graph Y and/or over a given class of graphs? (See the following sections for partial results in this direction.)

Define by induction the "higher-order cohomological widths", where $\mathrm{width}_1^* = \mathrm{width}^*$ and where, e.g. the "second Euclidean width" is

$$\mathrm{width}_2^*(X/\mathbb{R}^{n_1}/\mathbb{R}^{n_2}) = \inf_F \sup_{y \in \mathbb{R}^{n_2}} \mathrm{width}_1^*\left(F^{-1}(y)/\mathbb{R}^{n_1}\right)$$

for F running over all continuous maps $X \to \mathbb{R}^{n_2}$.

Can one evaluate these with suitable invariants of the cohomology algebra of X?

4.3 Construction of simply connected expanders. Recall (see 3.1) that the edge boundary of a vertex subset V_0 in the vertex set of a graph (V, E), denoted $\overrightarrow{\partial}(V_0) \subset E$, is the set of edges issuing from the vertices in V_0 and terminating in the complement $V \setminus V_0$.

A locally bounded expander is, by definition, a family $\{V\} = \{(V, E)\}$ of finite connected graphs with degrees $\deg(V) \leq d_0 < \infty$ where $|V| \to \infty$ and such that the isoperimetric profiles of $\|\overrightarrow{\partial}\|_{e/v}$ of these graphs (see 3.1) are bounded from below by a *strictly* positive constant λ. More precisely, every finite subset $V_0 \subset V$ in each graph $V \in \{V\}$ satisfies

$$\left|\overrightarrow{\partial}(V_0)\right| \geq \lambda \cdot |V_0|, \qquad [\overrightarrow{\partial} \succ \lambda]$$

in so far as the cardinality of V_0 satisfies $|V_0| \leq |V|/2$, where the key point is that λ is bounded away from zero for all graphs $V \in \{V\}$ regardless of their cardinality.

If you try to construct an expander from scratch you may be justified in concluding they do not exist at all; yet they do. Moreover,

random d-valent graphs for $d \geq 3$ are expanders

according to Lemma 1 in [KoB], which is combined with the following proposition $(\star)$. (The main result in [KoB, Th. 1] consists in a construction of "least volume consuming" topological embeddings of graphs into $\mathbb{R}^3$; expanders in $(\star)$ serve as examples which show that the construction is essentially optimal.)

$(\star)$ *Let $U \subset \mathbb{R}^3$ be an open subset and $V \subset U$ be a topologically embedded λ-expanding (sub)graph with N-vertices, all of degree $\leq d$, such that U homotopy retracts to V and such that* $\mathrm{dist}_{\mathbb{R}^3}(V, \partial U) \geq 1$. *Then*

$$\mathrm{Vol}(U) \geq \mathrm{const}(\lambda, d) N\sqrt{N}\,.$$

(The 1967 paper [KoB] by Kolmogorov–Brazdin was pointed out to me by Larry Guth; the most frequently cited paper [P] appeared in 1973, where an equivalent *superconcentration* property of random graphs was proven.)

Another (non-random) source of expanders was discovered by Margulis who related λ to the first non-zero eigenvalue λ_1 of the combinatorial Laplace operator on V (i.e. the norm of $\|(\partial^0)^{-1}\|^{l_2}_{\mathrm{fil}}$, see 2.10) and proved the following:

Let G be a Kazhdan T group and $\tilde{X}$ be a connected simplicial polyhedron. Then the 1-skeleta of the compact quotients $X = \tilde{X}/\Gamma$ for lattices $\Gamma \subset G$ which freely and discretely acting on $\tilde{X}$ are expanders, [M1].

(D. Kazhdan defined T in [Ka] by requiring that the *trivial* representation of G was *not* a weak *limit of non-trivial* irreducible unitary representations and he proved that the simple Lee groups G of $\mathbb{R}$-ranks ≥ 2 and their lattices are T.

Deciphering Kazhdan's definition leads to the required universal bound on $\|(\partial^0)^{-1}\|^{l_2}_{\mathrm{fil}}$. Notice that this bound for p-adic G, which is sufficient for our applications, is also a corollary of the *Garland vanishing theorem* [G] but this had not been observed till several years afterwards.)

A family $\{X\}$ of simplicial 2-polyhedra is called a (2-dimensional) *locally bounded (edge-wise) λ-expander* if the degrees of all X (i.e. the numbers of the faces attached to all vertices) are bounded by a constant $d_0 < \infty$ and the 1-skeleta of X make a λ-expander. Such a family (expander) is called *simply connected* if all X in the family are simply connected.

Construction of locally bounded simply connected expanders. Take a symmetric space $\tilde{X}$ with non-positive curvature of $\mathrm{rank}_{\mathbb{R}} \geq 2$ (or a Bruhat–Tits building), let Γ_0 be a discrete isometry group, freely and co-compactly acting on $\tilde{X}$, and let $\{\Gamma\}$ be an infinite family of subgroups $\Gamma \subset \Gamma_0$ of finite index in Γ_0.

Since Γ_0 is Kazhdan T (see [Ka], [M2]), the family $\{X^2\} = \{X^2(\Gamma)\}$ of lifts of the 2-skeleton X_0^2 of a triangulation of $\tilde{X}/\Gamma_0$ to $\tilde{X}/\Gamma$ make a locally bounded λ-expander for some $\lambda > 0$. (This is nearly obvious, modulo Margulis' spectral reformulation of the expander property, for the canonical triangulations of Bruhat–Tits buildings; while the Riemannian case needs a minor adjustment.)

Take the shortest non-contractible closed curve of edges in each $X^2 = X^2(\Gamma)$ and attach a disk D by its boundary to X^2 along this curve C. Denote by $\Gamma[C] \subset \Gamma$ the normal subgroup generated by the homotopy class of C and observe that the fundamental group $\Pi_C = \Gamma/\Gamma[C]$ of the resulting space, say X_D^2, is *finite*, according to

Margulis' Theorem. Γ *contains no non-trivial normal subgroup of infinite index (see section 4.4 [M2]).*

Subdivide the attached disk without introducing extra vertices to the triangulation and adding no more than three 2-simplices at each vertex, and let $\hat{X}_D^2$ be the (finite!) universal covering of the so subdivided space X_D^2.

This $\hat{X}_D^2$ is, in fact, obtained by attaching $|\Pi_C|$ discs to $X^2(\Gamma[C])$ that are lifts of $D \subset X_D^2$, where, observe, the family $\{X^2(\Gamma[C])\} \supset \{X^2(\Gamma)\}$ for all Γ and all C is an expander for the same reason that the original family is. It follows that the family $\hat{X}_D^2$ is also an expander which is, obviously, locally bounded.

Remarks. (a) The set of the above expanding graphs $G = \hat{G}_\gamma \in \mathcal{G}$, albeit infinite, can be, a priori, very rare, since the Margulis theorem provides no effective estimate on the order of the group Π_C. Probably, a reasonable lower bound on the density of this set can be extracted from Margulis' proof. (A last resort could be *cut elimination* as in [Gi].)

(b) Margulis' theorem does not apply to locally symmetric spaces with *strictly* negative curvature: if, say $K \leq -1$, then the total length of geodesics needed to kill π_1 is about $\log(\mathrm{Inj}\,.\,\mathrm{Rad}) \cdot \mathrm{vol}$, and so they will crowd at some point, bringing logarithmic local complexity.

4.4 Coarse geometry versus topology. In this subsection, we address general issues of relations between geometry and topology of metric spaces which pertain to the above construction, but which are not used in the sequel.

Let $\mathcal{X}(n,k,d,N)$ denote the class of n-dimensional k-connected simplicial polyhedra made of N simplices and having local degrees (i.e. the numbers of simplices adjacent to the vertices) bounded by d and where each X is endowed with the length metric corresponding to the standard Euclidean metrics in the regular unit simplices.

How much does the "coarse geometry" of an $X \in \mathcal{X}(n,k,d,N)$ depend on n,k,d and N?

For example, let n and d be fixed. How much does the asymptotic behaviour of the geometry of $X = X(N) \in \mathcal{X}(n_0,k,d_0,N \to \infty)$ depend on k, say for $n_0 = 2$, $d_0 = 1000$, and $k = 0,1,2$?

More specifically, let $L \cdot \mathcal{X}(n_0,k,d_0,N)$, $L \geq 1$, denote the class of the metric spaces which are L-bi-Lipschitz equivalent to the above metric spaces (where $1 \cdot \mathcal{X}(n,k,d,N) = \mathcal{X}(n,k,d,N)$).

What is the Hausdorff distance between a given metric space X_0 and $\mathcal{X} = L \cdot \mathcal{X}(n,k,d,N)$, for

$$\mathrm{dist}_{\mathrm{Hau}}(X_0,\mathcal{X}) =_{\mathrm{def}} \inf_{X \in \mathcal{X}} \mathrm{dist}_{\mathrm{Hau}}(X_0,X)\,?$$

For example, what is the supremum of this distance over all $X_0 \in \mathcal{X}\ (n_0, k_0 < k, d_0, N)$? Can this supremum, denoted

$$D(n,n_0,k,k_0,d,d_0,N,L) = \sup_{X_0} \mathrm{dist}_{\mathrm{Hau}}(X_0,\mathcal{X})\,,$$

be bounded for $N \to \infty$?

Is every $X_0 \in \mathcal{X}(n_0,k_0,d_0,\infty)$ quasi-isometric to some $X \in \mathcal{X}\ (n = n_0,\ k > k_0,$ $d \gg_0, \infty)$?

Quasi-isometry does not account for all of the "coarse geometry" as it leaves the following questions untouched.

Given an $X_0 \in \mathcal{X}(n,k,d,N)$, what is the bound in terms of n,k,d and N, on the minimal numbers $L_k = L_k(X_0)$ and $V_k = V_k(X_0)$, such that the $(k+1)$-dimensional homotopy group of X can be generated by classes of L_k-Lipschitz maps $S^{k+1} \to X$ or of maps of volumes$\leq V_k$? (See [NR] for some general results about this.)

What is (the bound on) *the (filling) Dehn functions* $D_k(l)$ and the *Lipschitz Dehn function* $D_k^{\mathrm{Lip}}(l)$ of X_0? (Recall that $D_k(l)$ equals the minimal number a such that every map $S^k \to X$ of volume $\leq l$ extends to a map of ball of volume $\leq a$, and

$D_k^{\mathrm{Lip}}(l)$ is a similar function where l and a stand for the Lipschitz constants of the maps.)

In particular, what are the above invariants for $X_0 = X[C]$ from the previous section and of other spaces similarly associated to curves C for $[C]$ running over all conjugacy classes in Γ?

Are there 2-dimensional *contractible* locally bounded edge expanders?

Is there an effective bound on the Dehn functions of locally bounded contractible 2-polyhedra?

It is unclear, in general, if every simply connected locally bounded polyhedron X_0 is quasi-isometric to a locally bounded contractible n-polyhedron for a given $n \geq 2$. In fact, the positive answer to the Andrews–Curtis conjecture on balanced group presentations with an effective bound on the number of Nilson moves would imply that the simply connected 2-polyhedra X_0 with large Dehn functions, e.g. with no recursive bound by N, cannot be quasi-isometrically approximated by contractible polyhedra. Probably, the class of quasi-isometries of contractible 2-polyhedra is large enough to contain a counterexample to the Andrews–Curtis conjecture but it is smaller than the class of all locally bounded simply connected polyhedra.

If X is a closed simply connected Riemannian 3-manifold with locally 1-bounded geometry (i.e. with $\mathrm{curv}(X) \leq 1$ and $\mathrm{injrad}(X) \geq 1$), then the Perelman theorem on Hamilton's flow along with Rubenstein's algorithm on recognition of 3-spheres probably provides an effective bound on the Dehn function $D_1(l, v)$ of X: every closed curve in X of length l bounds a disk of area $a \leq a(l, v = \mathrm{vol}(X))$ for a computable function a. How large can this a be? (This may be rather difficult to determine, as Bruce Kleiner pointed out to me.)

CONSTRUCTIONS. One can pass from contractible k-dimensional polyhedra P to n-spheres X quasi-isometric to P, provided $n \geq \max(2k-1, k+2)$: immerse P into $\mathbb{R}^{2k}$ and take the boundary of the tubular neighborhood of this immersion. (This also applies to $k = 2$, provided P is *collapsible*, as defined below.)

It remains unclear if the quasi-isometry class of contractible (or of p-connected) locally bounded k-dimensional polyhedra stabilizes for large k, but we shall see below some spaces where one can reduce dimensions without changing their homotopy and quasi-isometry classes (much) and keeping (or only slightly increasing) the local bounds on geometries.

Let X be a compact n-dimensional Riemannian manifold X with 1-bounded local geometry and $S \subset X$ a simple curve of length L, where $L \approx \mathrm{vol}(X)$ in the relevant cases.

If X is a closed manifold, then the complement $X \setminus S$ is homotopy equivalent to X minus a point; if X has boundary and S meets the boundary at an endpoint, then $X \setminus S$ is homotopy equivalent to X.

If $\dim(X) \geq 3$, then the induced *length metric* on $X \setminus S$ equals dist_X restricted to $X \setminus S$, but the k-th Dehn functions, $k = n - 2$, of X and $X \setminus S$ may be quite different: (arbitrarily) small k-spheres in $X \setminus S$ positioned close to the center of S need a homotopy of size $\approx L/2$ in order to be contracted in $X \setminus S$.

Now assume that

- the curvature of S is bounded by $\approx 1/2$;

•• S is ε-dense in X, say, for $\varepsilon \approx 0.1$;

••• the *normal* δ-neighborhood $W = W_\delta(S)$ for $\delta \approx 0.02$, *smoothly embeds* to X, i.e. every point in X within distance δ from X has a *unique* nearest point in S. We call such W an *$\varepsilon\delta$-worm of length L* in X.

The cut locus $X_0 \subset X$ of X with respect to X is homotopy equivalent and ε-quasi-isometric to S, and, under the above assumptions, it can be regularized by a small perturbation that would bound its local geometry (by something of order ε/δ).

Thus, for example, every closed Riemannian n-manifold, $n \geq 3$, of bounded geometry contains a locally bounded subpolyhedron P of dimension $n-1$ which is homotopy equivalent to X minus a point and such that P contains the 1-skeleton of some locally bounded triangulation of X, thus being quasi-isometric to X.

Here is another use of "worms". Let X be the round 3-sphere S^3 of (large) radius R and take two "worms" W_1, W_2 both of the length $\approx R^n$ in X. Then there is an (almost) canonical diffeomorphism between their boundaries established with the normal (Frenet) frame along their axial curves. Glue the complements of the worms by this diffeomorphism of their boundaries and observe that the resulting manifold say X_1 is still a topological sphere with a metric of locally bounded geometry (with bounds independent of R). But the global geometry of X_1 may be rather far from that of X_0, since the gluing can bring far away points in $X \setminus W_1$ to nearby points in $X \setminus W_2$.

What is the first non-zero eigenvalue of the Laplace operator on X_1 for two "random" worms?

What happens if we iterate this construction, where X_2 is obtained with two worms in X_1, etc?

Let X be a graph on N vertices and take two random simple paths of edges $S_1, S_2 \subset X$ of equal length L slightly less than the number of vertices in X. What can be said about the first non-zero eigenvalue (or the isoperimetric profile) of the graph X_1 obtained by gluing the two graphs along these paths?

What are the geometric possibilities of such a construction with "p-dimensional worms", i.e. small neighborhoods of topological p-disks similarly embedded into Riemannian n-manifolds?

Below, we shall use higher-dimensional "worms" associated to collapsible polyhedra of dimensions≥ 2 (similar to spanning trees of the adjacency graphs of the top-dimensional cells) but it is less clear what to do with more complicated contractible subpolyhedra in $P \subset X$, e.g. spanning trees of the adjacency graphs of the n-simplices of locally bounded triangulations of X?

The Dehn function $D_1(l)$ admits no recursive bound in terms of l and N on the set of locally bounded contractible 3-dimensional polyhedra (and on Riemannian 5-spheres with locally bounded geometries) by a theorem by Novikov. But the underlying (central extension) construction (see, e.g. [N] provides a very rare set of examples (somewhat similar to what happens to simply connected expanders), and most quasi-isometry questions for the class $\mathcal{X}(3,2,d,N)$ remain open. (Yet, some estimates of this kind are established in [NW].)

Let X be a simplicial polyhedron which admits an *L-exhaustion with respect to a subpolyhedron* $X_0 \subset X$, say, $X_0 \subset X_1 \subset \dots . \subset X$, which means that

every sub-polyhedron X_{i+1} is adjacent to X_i, $i = 1,2,\dots,$ i.e. every vertex of X_{i+1} is joint by an edge with some vertex in X_i;
every X_i, $i = 0,1,\dots,$ is $(1,L)$-locally connected: if two vertices in X_i are joint by an edge in X, then they are joined by a path of at most L edges in X_i.

Move every vertex v in $X_{i+1} \setminus X_i$ to a nearest vertex in X_i, say $p(v) \in X_i$ and, whenever two vertices are joined by an edge in X_{i+1}, join them by a shortest edge path in X_i, say, $p[v_1,v_2] \subset X_{i+1} \setminus X_i$. Then attach a 2-cell to every closed path of edges of the form $[v_1,v_2] \circ [v_2,p(v_2)] \circ p[v_2,v_1] \circ [p(v_1),v_1]$.

The resulting cell complex $P = P(X,X_0) = P(X,X_0,p) \supset X_0$ satisfies the following obvious properties:

(1) X_0 is a homotopy retract in $P \supset X_0$; moreover, P *collapses* to X_0: there is a retraction decomposable into homotopy retraction of 1- and 2-cells into their boundaries.
(2) The 1-skeleton of P equals that of X.
(3) $\dim(P) \leq \max(2, \dim(X_0))$.
(4) The boundary of each 2-cell σ in $P \setminus X_0$ is a simple closed path in the 1-skeleton $X^1 = P^1$ of length$\leq L+3$.
(5) Each $\sigma \subset P \setminus X_0$ can be subdivided into $L+1$ triangles while keeping the 0-skeleton unchanged; thus, P becomes a simplicial complex, say P' with the vertex set equal to that of X,
(6) The metric on the 0-skeleton of X induced from P' satisfies $\mathrm{dist}_X \geq \mathrm{dist}_{P'} \geq L^{-1}\,\mathrm{dist}_X$.
(7) The local degrees of P‘ are bounded by $\deg(P') \leq \deg(X)^{L+3}$.
(8) If every $(L+3)$-Lipschitz map of the unit circle to X extends to an L_1-Lipschitz map of the unit disk bounded by this circle, then the identity map on the $X_0 \cup P_1$ extends to a $10L_1$-Lipschitz map $P' \to X$.

Examples. (a) Every triangulation of S^n into *convex* simplices (for the standard projective structure on S^n) admits a $10n$-exhaustion with respect to a vertex for all $n \geq 2$. This shows, in particular, that the metrics associated to such triangulations (where each simplex given the unit geometry) are rather special. For example the triangulated Novikov spheres are far from this class.

On the other hand, the topological 2-disk with an arbitrary triangulation admits a 10-exhaustion with respect to its boundary.

Consequently, if the 2-skeleton of a polyhedron X is obtained from a given X_0 by consecutive attaching triangulated 2-disks to (closed or non-closed) paths of edges, then X admits a 10-exhaustion.

(b) Let the cube $[-N,N]^n$ be subdivided into $(2N)^n$ unit subcubes and then every unit cube is subdivided into affine simplices without introducing new vertices. Then the triangulated boundary sphere $S^{n-1}(N)=\partial[0,N]^n$ admits an L-exhaustion with respect to each vertex with $L\le(10n)^{10n}$, provided $n\ge 3$.

(c) Let $X_0\subset\mathbb{R}^n$ be the union of the above spheres $S^{n-1}(N)$ of radii $N=1,2,4,8,\dots$ and of the straight ray $r_0\subset\mathbb{R}^n$ issuing from the origin in the direction of the first coordinate. Then $\mathbb{R}^n$ admits an L-exhaustion with respect to X_0 with $L\le(10n)^{10n}$.

(d) Take $P(j)=S^{n-1}(2^j)\subset\mathbb{R}^n$, $j=1,2,\dots$, and let $X_0\subset\mathbb{R}^n$ be the union of these sphere and the ray r_0. Let $n\ge 2$ and $X_1\supset\mathbb{R}^n$ be obtained by replacing each sphere $S^{n-1}(2^j)$ by the polyhedron $P(j)=P(S^{n-1}(2^j))$, $r_0\cap S^{n-1}(2^j)$, where each $P(j)$ is attached to $\mathbb{R}^n$ at its 1-skeleton (which equals that of $S^{n-1}(2^j)\subset\mathbb{R}^n$).

Then the 2-dimensional polyhedron $Q=Q(\mathbb{R}^n)=P(\mathbb{R}^n,X_0)$ has the following properties:

- • The 0-skeleton of Q equals that of $\mathbb{R}^n$, i.e. the set of the integer points in $\mathbb{R}^n$.
- •• The distances between the vertices in Q and in $\mathbb{R}^n$ satisfy $\mathrm{dist}_Q\le\mathrm{dist}_{\mathbb{R}^n}\le L^2\,\mathrm{dist}_Q$ for $L\le(10n)^{10n}$; thus Q is quasi-isometric to $\mathbb{R}^n$.
- ••• The polyhedron Q is collapsible, in particular Q is contractible; moreover, Q is exhausted by compact collapsible subpolyhedra $Q_j=Q([-2^j,2^j]^n)\subset Q$ with collapsible boundaries $\partial Q_j=P(S^{n-1}(2^j))$.

It follows that the boundary of an $(m+1)$-dimensional manifold thickening of Q for each $m\ge 3$ is

> *a complete Riemannian manifold X of bounded geometry which is diffeomorphic to $\mathbb{R}^m$ and is, at the same time, quasi-isometric to $\mathbb{R}^n$, where m and n are arbitrarily given numbers satisfying $m,n\ge 3$.*

(This answers a question put to me by Itai Benjamini some time ago. Probably, the construction extends to complete metrics on $\mathbb{R}^n$ of *non-positive curvatures* on one hand and of *positive curvatures* instead of the Euclidean metric on the other hand, with a use of convex *non-radial* exhaustions in the curv >0 case.)

Remarks. (A) No metric on $\mathbb{R}^2$ is quasi-isometric to $\mathbb{R}^n$ for $n>2$, since a *subquadratic* isoperimetric profile implies *hyperbolicity* for simply connected surfaces.

(B) If a manifold X is quasi-isometric to a surface with a complete metric (e.g. to $\mathbb{R}^2$), then X is *non-simply connected at infinity.* In particular, no metric on $\mathbb{R}^n$, $n\ge 3$, is quasi-isometric to $\mathbb{R}^2$ (nor, obviously, to $\mathbb{R}^1$).

(C) The Dehn function of a collapsible 2-polyhedron X is bounded by the number of 2-cells in X. It follows that the quasi-isometry class of such polyhedra is strictly smaller than the class of the simply connected polyhedra.

Indeed, if $f_0:X_1^0\to X_2^0$ is an L-bi-Lipschitz bijective map between the 0-skeleta of two polyhedra, then the first Lipschitz Dehn function of X_2 satisfies $D_1^{\mathrm{Lip}}(X_2,l)\le D_1^{\mathrm{Lip}}(X_2,10L^2)D_1^{\mathrm{Lip}}(X_1,L\cdot l)$ for all $l\ge 0$.

It follows that the 2-polyhedra with large Dehn functions for large l, e.g. those (re)presenting trivial groups with unsolvable isomorphism problem, cannot be quasi-isometrically approximated by collapsible polyhedra. (This severely limits applicability of the above construction $X \rightsquigarrow P(X)$.)

RELATED QUESTIONS. Consider a sequence P_i of simplicial polyhedra (e.g. graphs) of a fixed dimension k, built of i simplices where all P_i have their local degrees (i.e. the maximal number of simplices adjacent to each vertex) bounded independently of i. Under what kind of conditions does there exist another sequence of polyhedra, say Q_i, also of bounded local degrees which satisfy the following conditions?

(1) There is a combinatorial embedding $P_i \subset Q_i$ for all $i = 1, 2, \dots$;
(2) The number of simplices making Q_j is bounded by $C \cdot i$ with a constant C independent of i;
(3) All spaces Q_i are combinatorially equivalent to an n-ball for a fixed $n \geq k$.
(3′) One may require some weaker conditions on Q_i, such as all Q_i being all mutually homotopy equivalent manifolds, for example with or without asking for the bound on their local complexity.

Notice that a fixed triangulation of an n-ball B^n with $i_+ \leq \text{const} \cdot i$ simplices contains at most 2^{i_+} subcomplexes while the number of graphs with i edges is about $i!$. Thus one needs a lot of different triangulations of B^n (which must be far removed from the "obvious" triangulations according to ($\star$) in 4.3) in order to accommodate all graphs and one does not know if this "lot" exists. For example, it is unknown if the number $M_n(i)$ of triangulations of B^n into at most i simplices grows at most exponentially in i.

On the other hand, one can always find Q_i satisfying (1) and (3) and having about $i \cdot \log(i)$ simplices.

A closely related question is as follows. Can a connected graph G be embedded into the 1-skeleton of a *simply connected* 2-dimensional simplicial complex G_+, such that the local complexity of G_+ is bounded by that of G (i.e. the number of simplices in G_+ at each vertex is bounded by some universal function of the maximal valency of the vertices in G), and the number M_+ of the simplices in G_+ is linearly bounded by the number M of the edges in G, i.e. $M_+ \leq \text{const} \cdot M$, where "const" must be independent of M (but it may depend on the maximal valency of vertices in G)?

It is easy to make such an embedding with M_+ roughly bounded by $M \log(M)$ for all graphs G but it is unclear what happens with the linear bound in general.

4.5 Separation and isoperimetric profiles of graded semigroups and algebras.

Isoperimetry in semigroups. Let $G_\circ$ be a commutative semigroup with 0, where the product is denoted by "$\smile$" and where, by definition, $g \smile 0 = 0$ for all g in G and where $G \subset G_\circ$ denotes the set of non-zero elements, i.e. $G = G_\circ \setminus \{0\}$. For instance, a commutative ring makes such a semigroup under multiplication.

Two elements g_0 and g_1 in G are called $\smile$*-orthogonal* if $g_0 \smile g_1 = 0$. Accordingly, subsets in G_0 and G_1 in G are called $\smile$*-orthogonal (or separated)* if every g_0 in G_0 is $\smile$-orthogonal to all g_1 in G_1.

The $\smile$-orthogonal complement $G_0^\perp \subset G$ is, by definition, the set of all g_1 which are $\smile$-orthogonal to all g_0 in G_0.

The $\partial_\perp$-boundary of a subset $G_0 \subset G$ is defined as the complement to G_0 and $G_0^\perp$ in G

$$\partial_\perp(G_0) = G \setminus (G_0 \cup G_0^\perp).$$

GRAPH EXAMPLE. Let (V,E) be a graph without multiple edges, G be the disjoint union of V and E and let $G_\circ$ equal G augmented by en extra element called 0. Define the product in $G_\circ$ as follows.

If $g_1 = v_1$ and $g_2 = v_2$, where v_1 and v_2 are vertices in V joined by an edge $e \in E$, then $v_1 \smile v_2 = e$; otherwise the product $g_1 \smile g_2 = 0$.

Two subsets in the vertex set $V \subset G$ are *$\smile$-orthogonal* if and only if they are *edge separated*, i.e. if there is no edges between these subsets.

The $\perp$-boundary $\partial_\perp(V_0)$ of a subset $V_0 \subset V$ is contained in V, where it equals the set of all vertices in the complement $V \setminus V_0$ which are joined by edges with vertices in V_0.

Also observe that the intersection $V_0 \cap V_0^\perp$ equals the set of *isolated* vertices v_0 in V_0, i.e. where there is no edge between v_0 and any other vertex in V_0.

Finally, notice the cardinality of the $\perp$-boundary is related to that of the edge boundary $\overrightarrow{\partial}$ (see 3.1 and 4.3) as follows:

$$\tfrac{1}{d}\big|\overrightarrow{\partial}(V_0)\big| \le \big|\partial_\perp(V_0)\big| \le \big|\overrightarrow{\partial}(V_0)\big|,$$

where d denotes the maximum of the degrees (valences) of the vertices in V, and where, recall, $\overrightarrow{\partial}(V_0) \subset E$ is the set of edges between V_0 and its complement $V \setminus V_0$.

Granted a concept of boundary, one may speak of the *isoperimetric profile* of $G_\circ$ defined as the set $\mathcal{M} = \mathcal{M}(G_\circ)$ of pairs of numbers, say (M_0, M_1), for which G admits a subset V_0 where the cardinality of V_0 equals M_0 and the cardinality of its boundary is $\le M_1$, and where one is mainly concerned with *isoperimetric inequalities* that are *lower bounds* on M_1 in terms of M_0.

Let us generalize the above to *graded* semigroups $G_\circ$, where a grading means a decomposition of G into disjoint union of subsets $G(n)$, $n = 1, 2, \ldots$, such that

$$\text{if } g \in G(i) \text{ and } g' \in G(j), \text{ then } g \smile g' \in G(i+j), \text{ unless } g \smile g' = 0.$$

We want to bound from below the cardinalities of products of "large" subsets V_i in $G = G_\circ \setminus \{0\}$, where this bound may (or may not) be specified in each grade. In other words, we are concerned with "graded cardinalities" of subsets V in G, denoted $|V|(*)$, where $* = 1, 2, \ldots$, and $|V|(n) =_{\rm def} |V \cap G(n)|$

Besides commutative semigroups (and algebras later on), we also allow *anti-commutative* ones with respect to a given involution in $G_\circ$, called a *$\pm$-involution*, which must have the usual properties of such involution in multiplicative semigroups of rings.

EXAMPLE: STANLEY–REISNER SEMIGROUPS. Every simplicial complex G defines $G_\circ$ where the (only) non-zero products are $\Delta^{n_1+n_2+1} = \Delta^{n_1} \cdot \Delta^{n_2}$ for pairs of faces in the simplices $\Delta^{n_1+n_2+1}$ in G which span $\Delta^{n_1+n_2+1}$.

DEFINITIONS. Let V_i, $i = 1, 2, \ldots, k$, be subsets in G such that

$$\big|V_{i_1}(n_1) \smile \cdots \smile V_{i_l}(n_l)\big| \leq M_{i_1,\ldots,i_l}(n_1, \ldots, n_l)\,, \qquad [\smile_G \leq M^\star]$$

where $M_{i_1,\ldots,i_l}(n_1, \ldots, n_l)$ are given numbers, some of which may be taken $+\infty$, and where $V_{i_1}(n_1) \smile \cdots \smile V_{i_l}(n_l)$ denotes the set of all *non-zero* products of $g_l \in V_{i_l}(n_l)$.

Let $M^\star$ denote the totality of numbers $M_{i_1,\ldots,i_l}(n_1, \ldots, n_l)$ and define the *separation profile* $\mathcal{M}^*$ of $G_\circ$,

$$\mathcal{M}^*(G_\circ; M^\star) = \big\{M_i(n)\big\}(M^\star)\,, \quad n = 1, 2, \ldots\,,$$

as the set of graded cardinalities $M_i(*) = M^*(V_i) = |V_i|(*)$ for *all* V_i in G which satisfy $[\smile_G \leq M^\star]$.

The set $\mathcal{M}^*$ is insufficient for reconstruction of the $\partial_\perp$-isoperimetric profile in the graph example. To compensate for this, we augment the above set $\mathcal{M}^*(M^\star)$ by the graded cardinalities $M_\cup(V_i)(*) = |\cup_i V_i|(*)$, $* = 1, 2, \ldots, n, \ldots$, and denote the resulting set by $\mathcal{M}^*_\cup(G_\circ; M^\star)$.

Thus, the *separation $\cup$-profile* $\mathcal{M}^*_\cup(M^\star)$ is the set of pairs

$$\mathcal{M}^*_\cup(G; M^\star) = \big\{M_\cup(V_i), M^*(V_i)\big\}$$

for all k-tuples of subsets V_i in G which satisfy $[\smile_G \leq M^\star]$.

Clearly, this $\mathcal{M}^*_\cup$ encodes sufficient information for reconstruction of the $\partial_\perp$-isoperimetric profile but this information, as we shall see, is rather unstable under linearization, i.e. when we pass from semigroups to algebras.

Profiles of algebras. Let us extend the above definitions to graded commutative or anti-commutative algebras $A = \oplus_n A(n)$, $n = 1, 2, \ldots$, over a field $\mathbb{F}$, where the product is denoted by "$\smile$".

Denote by $|\ldots| = |\ldots|_\mathbb{F}$ the ranks of linear subspaces in A and let $|\ldots|(*) = |\ldots|_\mathbb{F}(*)$, $* = 0, 1, \ldots, n, \ldots$, denote the graded ranks of graded subspaces in A. This notation extends to arbitrary subsets in A by passing to the linear spans of subsets.

We are concerned with possible values of ranks of graded subspaces $A_i = \oplus_n A_i(n) \subset A$, $i = 1, \ldots, k$, where the ranks of the linear spans of the products $A_{i_1}(n_1) \smile \cdots \smile A_{i_l}(n_l)$ are bounded by certain numbers, some of which may be equal $+\infty$ (i.e. the corresponding inequality is vacuous),

$$\big|A_{i_1}(n_1) \smile \cdots \smile A_{i_l}(n_l)\big| \leq M_{i_1,\ldots,i_l}(n_1, \ldots, n_l)\,, \qquad [\smile_A \leq M^\star]$$

where $M^\star$ denotes the totality of these numbers M.

The separation profile of A is the set

$$\mathcal{M}^*(A; M^\star) = \big\{M_i(n)\big\}(M^\star)$$

of $M^*(A_i)$ for *all* A_i in A which satisfy $[\smile_A \leq M^\star]$.

Besides the ranks of A_i, we are interested, as in the $G_\circ$-case, in upper bounds on the ranks of their spans, denoted $+_i A_i$. (These can be replaced by the ranks of the corresponding graded quotient spaces $A -_i A_i = A / +_i A_i$ which is more appropriate for infinite-dimensional $A(n)$.)

Accordingly, the above set $\mathcal{M}^*(M^\star)$ is augmented by the graded rank, called $M_+(A_i)(*) = |+_i A_i|(*)$, $* = 1, 2, \ldots, n \ldots$, and the result, called the *separation $+$-profile of* A, is denoted $\mathcal{M}^*_+(A; M^\star)$.

Thus, $\mathcal{M}^*_-(M^\star)$ is the set of pairs

$$\mathcal{M}^*_+(A;M^\star) = \big\{M_+(A_i), M^*(A_i)\big\}$$

for all k-tuples of graded subspaces A_i in A which satisfy $[\smile_A \leq M^\star]$.

Notice that the M_+-component of every M^*_+ in $\mathcal{M}^*_-$ is constrained by the M^*-component via the obvious inequality $|+_i A_i|(*) \leq \sum_i |A_i|(*)$. This allows us, in some cases, to reduce bounds on M_+-numbers to bounds on M^*-numbers.

Let us look closely at the case where there are only two subspaces and $M^\star = 0$, i.e. where the $\smile$-product between these subspaces is zero.

Define the *$\smile$-orthogonal complement* $A_0^\perp \subset A$ to a graded linear subspace $A_0 \subset A$ as the maximal (necessarily graded) subspace such that $a \smile a^\perp = 0$ for all $a \in A_0$ and $a^\perp \in A_0^\perp$ and let *the $\smile$-boundary* $\partial_\smile(A_0)$ be the the quotient space $B = A/(A_0 + A_0^\perp)$.

The graded isoperimetric $\partial_\smile$-profile of A is the set of pairs (M_0, M_1) of sequences $M_0 = M_0(n)$, $M_1 = M_1(n)$, $n = 1, 2, \ldots$, for which A admits a graded subspace A_0, such that $|M_0|_{\mathbb{F}}(n) = M_0(n)$ and $|\partial_\smile(A)|_{\mathbb{F}}(n) \leq M_1(n)$. An isoperimetric inequality for A is a lower bound on M_1 in terms of M_0.

The isoperimetric $\partial_\smile$-profile of A is obviously expressible in terms of $\mathcal{M}^*_+(0)$, but not in terms of $\mathcal{M}^*(0)$; yet, sufficiently strong bounds on $\mathcal{M}^*(0)$ may suffice for meaningful lower bounds on $|\partial_\smile|$, as happens in some cases of interest.

Bilinear form example. If A has only two non-zero grades, say $A = A(1) \oplus A(2)$, then the product is given by a symmetric or antisymmetric $A(2)$-valued bilinear form $\smile$ on $A(1)$ and everything is expressed in terms of $\smile$-orthogonal subspaces in A(1).

For instance, let $\mathbb{F} = \mathbb{R}$, $|A(2)|_{\mathbb{R}} = 2$, and $\smile$ be represented by a pair of positive quadratic forms. Then $|\partial|_\smile(M) = 0$ for all M. But generic *triples* of forms have $M(1)^{-1}|\partial|_\smile(M)(1) \geq \lambda_0 > 0$ for all $M(1) \leq \frac{1}{2}|A(1)|$ (similar to 3-valent graphs).

Profiles of monomial algebras. Let

$$G_\circ = \{0\} \cup_n G(n)\,, \quad n = 1, 2, \ldots\,,$$

be a *graded commutative or anti-commutative semigroup with zero*, let $\mathbb{F}^G$ be the space of $\mathbb{F}$-valued functions a on G, where $a(-g) = -a(g)$ for the $\pm$-involution in the anti-commutative case. and let $\mathbb{F}[G_\circ] \subset \mathbb{F}^G$ be the space of functions with finite supports.

The semigroup $G_\circ$ naturally embeds into $\mathbb{F}[G_\circ]$ (with $0 \mapsto 0$) and the product in $G_\circ$ extends to a bilinear (product) map $\mathbb{F}[G_\circ] \otimes \mathbb{F}[G_\circ] \to \mathbb{F}[G_\circ]$, also denoted "$\smile$"; moreover, if every $g \in G = G_\circ \setminus \{0\}$ admits *at most finally many* decompositions $g = g_1 g_2$, this "$\smile$" extends to a product $\mathbb{F}^G \otimes \mathbb{F}^G \to \mathbb{F}^G$.

More generally, given a class $\mathcal{G}$ of subsets in $G = G_\circ \setminus \{0\}$, such that the product map $G_1 \times G_2$, for $(g_1, g_2) \mapsto g_1 \cdot g_2$, is finite-to-one and has its image in $\mathcal{G}$, the space $\mathbb{F}\{G_\circ\}_{\mathcal{G}} \subset \mathbb{F}^G$ makes an algebra. These are called *$G_\circ$-algebras or monomial algebras A*, since G makes what is called the *monomial basis* in $\mathbb{F}[G_\circ]$.

The sets (separation profiles) $\mathcal{M}^*(G_\circ; M^\star)$ and $\mathcal{M}^*_\cup(G_\circ; M^\star)$ for semigroups can be expressed in terms of $\mathbb{F}[G_\circ]$ by using the subspaces A_i, satisfying the inequalities $[\smile_A \geq M^\star]$, such that each $A_i(n)$ equals the set of *all* functions on its support in $G(n)$. Therefore,

$$\mathcal{M}^*(G_\circ; M^\star) \subset \mathcal{M}^*\big(\mathbb{F}[G_\circ]; M^\star\big) \quad \text{and} \quad \mathcal{M}^*_\cup(G_\circ; M^\star) \subset \mathcal{M}^*_+\big(\mathbb{F}[G_\circ]; M^\star\big)\,.$$

4.6 Combinatorial reduction of separation inequalities in orderable monomial algebras. Our objective is to establish opposite relations, possibly non-sharp ones, that would reduce isoperimetric/separation inequalities in monomial algebras to those in semigroups. (This is similar to the combinatorial approach to *group algebras* in [Gr7], where, however, the isoperimetry is defined differently; also see [D] for isoperimetric inequalities in general algebras.)

The two properties of $G_\circ$ we need are *faithfulness* and *order*.

Faithful $G_\circ$**.** This means that the equality $g_1 \smile g = g_2 \smile g \neq 0$, implies $g_1 = g_2$. Notice that the Stanley–Reisner semigroup $G_\circ$ associated to a simplicial complex G is faithful.

Ordered $G_\circ$**.** This signifies an order relation on each $G(n) \subset G = G_\circ \setminus \{0\}$, denoted $g_1 \preceq g_2$, such that $g_1 \preceq g_2$ and $g'_1 \preceq g'_2$ implies $g_1 \smile g'_1 \preceq g_2 \smile g'_2$, unless one of the two products equals zero.

Example. Every order on the vertex set V of a simplicial complex G lexicographically extends to an order on G that makes the associated semigroup $G_\circ$ ordered.

If a faithful semigroup $G_\circ$ admits an order, then

$$\mathcal{M}^*\big(\mathbb{F}[G_\circ]; M^\star\big) = \mathcal{M}^*(G_\circ; M^\star)\,. \qquad [A \sim_{\mathcal{M}^\star} G]$$

Proof (Compare 3.2 in [Gr7]). Given a non-zero function $a : G \to \mathbb{F}$, with the support admitting a maximal element $g \in G$, denote this g by $g = [a]_{\max}$ and extend this notation to linear subspaces $A_0 \subset \mathbb{F}[G_\circ](n_0) \subset \mathbb{F}[G_\circ]$ by

$$[A_0]_{\max} = \bigcup_{0\neq a\in A_0} [a]_{\max} \subset G(n_0)\,.$$

Observe that

$$\big|[A_0]_{\max}\big| = |A_0|_{\mathbb{F}}\,, \tag{1}$$

$$\text{either} \quad [a_0 \smile a_1]_{\max} = [a_0]_{\max} \smile [a_1]_{\max} \quad \text{or} \quad [a_0]_{\max} \smile [a_1]_{\max} = 0\,, \tag{2}$$

for all homogeneous a_i, say $a_i \in \mathbb{F}[G_\circ](n_i), i = 0, 1$; therefore

$$[A_0 \smile A_1]_{\max} \supset [A_0]_{\max} \smile [A_1]_{\max} \quad \text{and, consequently}\,,$$

$$\big|[A_0 \smile A_1]_{\max}\big| \geq \big|[A_0]_{\max} \smile [A_1]_{\max}\big| \tag{3}$$

for all homogeneous A_i, say $A_i \subset \mathbb{F}[G_\circ](n_i)$, $i = 0, 1$.

Thus, the $[\smile_A \leq M^\star]$ inequalities for graded subspaces A_i in A imply the $[\smile_G \leq M^\star]$ inequalities for the subsets $[A_0]_{\max} \subset G$ with the *same* $M^\star$-numbers; hence,

$$\mathcal{M}^*(G_\circ; M^\star) \supset \mathcal{M}^*\big(\mathbb{F}[G_\circ]; M^\star\big)\,,$$

and the proof follows.

Idealization of $\mathcal{M}$-profiles. Define $\mathcal{IM}^*(M^\star)$ as well as $\mathcal{IM}^*_\cup(M^\star)$ and $\mathcal{IM}^*_+(M^\star)$ for semigroups and algebras by limiting the $[\smile_G \leq M^\star]$ and $[\smile_A \leq M^\star]$ inequalities to those subsets V_i in G and graded subspaces A_i in A which make ideals in $G_\circ$ and A correspondingly.

The relations **(2)** and **(3)** show that if a graded subspace $A_0 \subset A = \mathbb{F}[G_\circ]$ is an ideal, then the graded subset $[A_0]_{\max} \subset G_\circ$ is a semigroup ideal, which implies the idealized version of $[A \sim_{\mathcal{M}^*} G]$,

$$\mathcal{IM}^*(G_\circ; M^\star) = \mathcal{IM}^*\big(\mathbb{F}[G_\circ]; M^\star\big)\,, \qquad [A \sim_{\mathcal{IM}^*} G]$$

for all graded faithful orderable semigroups $G_\circ$ *with* 0.

Remarks. (a) The use of ordering is standard in the study of ideals in exterior algebras, e.g. see the survey [MoS]. I learned this idea from Dima Grigoriev in the context of polynomial algebras.

(b) The original draft of this paper contained a "proof" of the equality $\mathcal{M}^*_\cup(G_\circ; M^\star) = \mathcal{M}^*_+(\mathbb{F}[G_\circ]; M^\star)$. The mistake in the argument was pointed out by the referee who has also indicated how one could circumvent this equality in the topological context of 4.8–4.10.

4.7 $\partial_\smile$-isoperimetry in graph algebras. As indicated by the referee the equality $\mathcal{M}^*_\cup(G_\circ; M^\star) = \mathcal{M}^*_+(\mathbb{F}[G_\circ]; M^\star)$ may fail in general. Below is a counterexample which, albeit seemingly trivial, points to a true relation between the combinatorial and algebraic isoperimetric profile for the *graph algebras* (defined below).

Let A be the algebra associated to the single edge graph, $[v, v']$, i.e. A is $\mathbb{F}$-linearly spanned by v, v' and $e = [v, v']$, where $v \smile v' = v' \smile v = e$ and the other products are zero.

The "bad" subspaces are A_1 and A_2 generated by $v+v'$ and by $v-v'$ respectively: these two subspaces are $\smile$-orthogonal and they generate A, unless the characteristic of the field $\mathbb{F}$ equals 2, while no two edge separated subsets in $V = \{v_1, v_2\}$ cover all V.

More generally, let (V, E) be a graph, let $V_1, V_2 \subset V$ be two vertex sets and $E_\cap \subset E$ be a set of edges, such that

The end vertices of all edges from $E_\cap$ are contained in the intersection $V_1 \cap V_2$.
No two edges from $E_\cap$ have a common vertex.
Every two vertices v_1 from V_1 and $v_2 \in V_2$ are either edge separated or they are joined by an edge from $E_\cap$.

Let A_i, for an $i = 1, 2$, be the linear space of $\mathbb{F}$-valued functions $a_i(v)$ on V such that

1. The functions $a_i(v)$ vanish outside V_i, i.e. the *support of* A_i, denoted $[A_i]_{sp}$, is contained in V_i.
2. Every $a_i \in A_i$ satisfies $a_i(v) = c_i(e) a_i(v')$ for the pairs of the vertices of all edges $e \in E_\cap$, where $c_i : E_\cap \to \mathbb{F}$, $i = 1, 2$, is a non-vanishing function.
3. The functions c_1 and c_2 are related by the equality $c_1(e) = -c_2(e)$ for all edges in $E_\cap$.

Clearly, these spaces are $\smile$*-orthogonal* in the corresponding *commutative graph algebra* that is the commutative Stanley–Reisner algebra A associated to the graph (V, E) via the semigroup $G_\circ = 0 \cup V \cup E$.

Furthermore, the dimensions of these spaces are

$$|A_i|_{\mathbb{F}} = |V_i| - |E_\cap|\,,$$

while the dimension of their span satisfies

$$|A_1 + A_2|_{\mathbb{F}} = |V_1 \cup V_2|\,, \quad \text{if } \operatorname{char}\mathbb{F} \neq 2\,,$$

and

$$|A_1 + A_2|_{\mathbb{F}} = |V_1 \cup V_2| - |E_\cap|\,, \quad \text{if } \operatorname{char}\mathbb{F} = 2\,.$$

Notice that the equality $|A_1+A_2|_{\mathbb{F}} = |V_1 \cup V_2| - |E_\cap|$ is also valid for *anti*commutative graph algebras over the fields of characteristics $\neq 2$, since two $\smile$-orthogonal lines in a 2-dimensional space necessarily coincide if "$\smile$" is antisymmetric, i.e. $v \smile v' = -v' \smile v$.

Let us show how an *arbitrary pair* of $\smile$-orthogonal linear subspaces A_1 and A_2 in A can be reduced to the above example.

General definitions. Let A_0 be a linear space of functions on a set G. An *injective reduction* A'_0 of A_0 to a subset $V'_0 \subset G$ is the projection of A_0 to the space of functions vanishing outside V'_0, i.e to the space $A'_0 = \mathbf{1}_{V'_0}(A_0)$ for the indicator function $\mathbf{1}_{V'_0}$, such that the projection operator, i.e. multiplication by $\mathbf{1}_{V'_0}$, is injective on A_0.

Notice that such a reduction is practical only if V'_0 is contained in the *support* of A_0, denoted $[A_0]_{sp}$. On the other hand every A_0 admits an injective reduction to a subset $V'_0 \subset [A_0]_{sp}$ with $|V'_0| = |A_0|_{\mathbb{F}}$.

For example, the reduction of A_0 to the subset $[A_0]_{\max} \subset [A_0]_{sp}$ for some ordering of the support $[A_0]_{sp}$ is injective.

Given a k-tuple of linear spaces A_i, $i = 1, 2, \dots, k$, a $+$-*injective reduction* is a k-tuple of injective reductions A'_i, such that the resulting linear map between the spans, $+_i A_i \to +_i A'_i$, is injective.

An elementary linear algebraic argument shows that

- *an arbitrary k-tuple $\{A_i\}$ of linear spaces A_i of functions with finite supports on a set V admits a $+$-injective reduction $\{A'_i\}$, where the corresponding subsets $V'_i \subset [A_i]_{sp}$ satisfy*
$$|V'_i| = |A'_i|_{\mathbb{F}} \quad \textit{for } i = 1, \dots, k \quad \textit{and} \quad |\cup_i V'_i| = |+_i A'_i|_{\mathbb{F}}\,.$$

Finally, if the functions in question come with a $\smile$-product, a reduction $\{A'_i\}$ of $\{A_i\}$ is called $[\smile_A \leq M^\star]$-*admissible* if the (reduced) spaces A'_i satisfy the inequalities $[\smile_A \leq M^\star]$ from 4.5.

Notice that our proof of $[A \sim_{\mathcal{M}^*} G]$ amounts to showing that the collection of (injective) reductions of A_i to the subsets $[A_i]_{\max} \subset [A_i]_{sp}$ is $[\smile_A \leq M^\star]$-admissible. (This reduction is *not*, in general, $+$-injective as was pointed out by the referee.)

Let us return to graph algebras A and concentrate on the case $M^\star = 0$, where we look at $+$-*irreducible* k-tuples of mutually $\smile$-orthogonal subspaces A_i of $\mathbb{F}$-functions on the vertex set V of a graph (V, E), where $+$-irreducible means that the k-tuple $\{A_i\}$ admits *no non-trivial* $[\smile_A \leq 0]$-admissible (i.e. preserving orthogonality) $+$-injective reduction and where "trivial" signifies $A'_i = A_i$.

Combinatorial reduction of $+$-separation inequalities in graph algebras. Let (V, E) be a graph, $\sigma(e)$ be a ± 1-valued function on its edges and let A be the algebra made of finite $\mathbb{F}$-linear combinations of vertices v in V and edges e

in E, where the product $v \smile v'$ is non-zero if and only if v and v' are joined by an edge e where $v \smile v' = \sigma(e)v' \smile v = \pm e$.

Let $\{A_i\}$ be an $+$-*irreducible* k-tuple of mutually $\smile$-orthogonal subspaces $A_i \subset A$ of $\mathbb{F}$-functions on V with finite supports denoted $V_i = [A_i]_{sp}$ and let $E_\cap \subset E$ be the set of all edges $e = [v, v']$ in E such that v is contained in some V_i and $v' \in V_j$ with $j \neq i$.

(a) *If a vertex v of an edge e from $E_\cap$ is contained in some V_i then the second vertex v' of e is also contained in V_i.*
(b) *Every pair of edges from $E_\cap$ is disjoint, i.e. there is at most one edge $e \in E_\cap$ issuing from each vertex v in V.*
(c) *If $e \in E_\cap$, then $\sigma(e) = 1$; in particular, if $\sigma(e) = -1$ for all edges $e \in E$, then the subsets V_i are mutually edge separated.*
(d) *The cardinalities of V_i and of their union satisfy*
$|V_i| = |A_i|_\mathbb{F} + |E_{\cap_i}|$, *for* $i = 1, 2, \dots, k$, *and* $|\cup_i V_i| = |+_i A_i|_\mathbb{F} + |E_\cap|$,
where $E_{\cap i} \subset E_\cap$ denotes the set of edges from $E_\cap$ with the vertices in V_i.

In particular, if $\sigma(e) = -1$ for all $e \in E$, then
$$|V_i| = |A_i|_\mathbb{F} \quad \textit{for } i = 1, 2, \dots, k, \quad \textit{and} \quad |\cup_i V_i| = |+_i A_i|_\mathbb{F}.$$
Thus, if the $\smile$-product in a graph semigroup $G_\circ = 0 \cup V \cup E$ is anti-commutative, e.g. commutative and $\operatorname{char}\mathbb{F} = 2$, *then the graph algebra $A = \mathbb{F}[G_\circ]$ has the same isoperimetric profile as $G_\circ$; moreover,*
$$\mathcal{M}^*_+\big(\mathbb{F}[G_\circ]; 0\big) = \mathcal{M}^*_\cup(G_\circ; 0). \qquad [A_+ \sim_{0^\star} G_\cup]$$

Proof. The property (a) does not depend on irreducibility. In fact the orthogonality relation $(a_i \smile a_j)(e) = 0$ for $a_i \in A_i$ and $a_j \in A_j$ at an edge $e = [v_1, v_2]$ from E with the vertices $v_i \in [A_i]_{sp}$ and $v_j \in [A_j]_{sp}$, $j \neq i$, is possible only if both vertices are contained in the intersection of the supports $[A_i]_{sp}$ and $[A_j]_{sp}$.

(b) The restriction of each linear space A_i to the pair of vertices (v, v') of any edge from $E_\cap$ has $|A_i|\{v, v'\}|_\mathbb{F} \leq 1$. Moreover, if V' is the set of vertices of a *connected* subgraph of (V, E) with the edges from $E_\cap$, then also $|A_i|V'|_\mathbb{F} \leq 1$. It follows, that if there are two edges issuing from a v with the second ends denoted v' and v'' then simultaneous reduction of all A_i to the complement $V \setminus \{v''\}$ is $+$-injective and 0-admissible.

(c) If an edge $e = [v, v']$ from $E_\cap$ has $\sigma(e) = -1$, then one may reduce $\{A_i\}$ to the subset $V \setminus \{v'\}$.

(d) Granted (a), (b) and (c), this (d) follows from the above •.

Lower bound on $\partial_\smile$ in graph algebras by the combinatorial $\overrightarrow{\partial}$. The isoperimetric inequalities in graphs are usually stated (and proven) as low bounds on the cardinalities of the *edge boundaries* $\overrightarrow{\partial}(V_0)$ of subsets $V_0 \subset V$ of the vertex set of a graph (V, E), where the edge boundary $\overrightarrow{\partial}(V_0)$ is the set of edges issuing from all $v \in V_0$ and terminating in the complement $V \setminus V_0$ (see 3.1 **B**).

On the other hand, when it comes to graph algebras A, we want to bound from below the ranks of the quotient spaces $B = \partial_\smile(A_0) = A/(A_0 + A_0^\perp)$ for a linear subspaces $A_0 \subset A$ and their $\smile$-orthogonal complements $A_0^\perp$.

The following shows how the latter bound can be reduced to the former.

Let (V, E) be a finite or infinite graph where every connected component contains at least two edges, and where the degrees (valencies) of all vertices are bounded by $d < \infty$. Let $A(1)$ be the space of linear $\mathbb{F}$-valued functions on V and let $\smile$ be a bilinear map, denoted "$\smile$", from $A(1)$ to $A(2)$ which is the space of functions on E, where $v \smile v' = 0$ if there is no edge between v and v' and $v \smile v' = \pm e$ if v and v' are joined by an edge e in the graph.

Let A_0 be a linear space of functions on V with finite support $[A_0]_{sp} \subset V$. Then there exists a subset $V_0' \subset [A_0]_{sp}$ of cardinality $|V_0'| = |A_0|_\mathbb{F}$, such that the rank of $\partial_\smile$-boundary $\partial_\smile(A_0) = A/(A_0 + A_0^\perp)$ of $A_0 \subset A(1)$ is bounded from below by the cardinality of the edge boundary of V_0' as follows:

$$\big|A/(A_0 + A_0^\perp)\big|_\mathbb{F} \geq \tfrac{1}{d}\big|\overrightarrow{\partial}(V_0')\big|. \qquad [\partial_\smile \succ \overrightarrow{\partial}]$$

Proof. Denote by W_0 the complement $W_0 = V \setminus ([A_0]_{sp} \cup [A_0^\perp]_{sp})$ and let $W_0' \subset V$ be the set of points in the complement $V \setminus V_0$, for $V_0 = [A_0]_{sp}$, which can be joined by an edge with V_0. Observe that

$$W_0' \subset W_0 \quad \text{and that } \tfrac{1}{d}\big|\overrightarrow{\partial}(V_0)\big| \leq |W_0'| \leq |W_0| \leq \big|A/(A_0 + A_0^\perp)\big|_\mathbb{F}.$$

Assume without loss of generality that the pair of spaces $\{A_0, A_0^\perp\}$ is irreducible and let $E_\cap \subset E$ be the set of edges with the above properties (a)–(d).

Let $V_0' \subset V_0 = [A_0]_{sp}$ be obtained from V_0 by removing one vertex from each edge $e \in E_\cap$, where we choose and remove from among the two ends of e, the vertex which has degree > 1 in the ambient graph (V, E).

(d) above implies that $|V_0'| = |A_0|_\mathbb{F}$, while it follows from (b) that the cardinality of the edge boundary of V_0' satisfies

$$\big|\overrightarrow{\partial}(V_0')\big| \leq \big|\overrightarrow{\partial}(V_0)\big|.$$

Indeed,

$$\overrightarrow{\partial}(V_0') \setminus \overrightarrow{\partial}(V_0) = E_\cap, \quad \text{while} \quad \big|\overrightarrow{\partial}(V_0) \setminus \overrightarrow{\partial}(V_0')\big| \geq |E_\cap|,$$

since every vertex $v \in V_0 \setminus V_0'$ coming from an edge $e \in E_\cap$ admits another edge $e' \neq e$ issuing from v, where, observe, the second vertex v' of e' necessarily lies in the complement $V \setminus V_0$. Thus,

$$\big|A/(A_0 + A_0^\perp)\big|_\mathbb{F} \geq \tfrac{1}{d}\big|\overrightarrow{\partial}(V_0)\big| \geq \tfrac{1}{d}\big|\overrightarrow{\partial}(V_0')\big|. \qquad \square$$

QUESTION. Is there a generalization of the above to non-graph algebras and/or to where $M^\star \neq 0$?

4.8 $\partial_\smile$-control of homological isoperimetry and topological translation of extremal set systems inequalities.

Graded homological isoperimetry and $[\partial_A \geq \partial_\smile]$-inequality. Let X be a topological space and $A \subset H^* = H^*(X; \mathbb{F})$ a linear subspace. Recall (see 1.5) that *the restriction* $A|X_0$, denotes the quotient space,

$$A|X_0 = A/\mathbf{0}_A(X_0) \quad \text{where} \quad \mathbf{0}_A(X_0) = A \cap \ker(rest^*_{/X_0}) \subset A,$$

where $\mathrm{rest}^*_{/X_0} : H^*(X) \to H^*(X_0)$ is the restriction cohomology homomorphism and that the A-*mass* $\mu_A(X_0) \subset A$ is defined via the complement of X_0 in X, by

$$\mu_A(X_0) = \mathbf{0}_A(X \setminus X_0)\,.$$

We are concerned with bounds on this "mass" in term of the A-*boundary* of X_0, that is

$$\partial_A(X_0) = A|\partial(X_0)\,,$$

where $\partial(X_0) \subset X$ is the topological boundary of X_0 and where we encode such bounds in terms of inequalities between the graded ranks of the linear spaces $\mu_A(X_0)$ and $\partial_A(X_0)$.

If $A = H^*$, then such inequalities can be derived from the corresponding isoperimetric inequalities for the $\partial_\smile$-boundary in this A (see 4.5, 4.7).

In fact, the additivity and multiplicativity of the mass μ_{H^*} (see 4.1) imply that there is a *surjective* homomorphism from $A|\partial X_0$ onto the space $\partial_\smile(A_0) = A/(A_0 + A_0^\perp)$ for $A_0 = 0_A(X \setminus X_0) = \mu_A(X_0)$; thus, the rank of $\partial_A(X_0)$ is bounded in each grade $n = 1, 2, \dots,$ by

$$\big|\partial_A(X_0)\big|_{\mathbb{F}}(n) \geq \big|\partial_\smile(\mu_A(X_0))\big|_{\mathbb{F}}(n) \qquad [\partial_A \geq \partial_\smile]$$

for A being the Čeck cohomology algebra $H^*(X,\mathbb{F})$ with coefficients in a (preferentially finite) field $\mathbb{F}$.

★ $[\partial_A \geq \partial_\smile]$-*Inequality for* $\smile$-*retracts and coannulators.* The above proof of this inequality remains valid for $\smile$-*retracts* $A \subset H^*$, i.e. the subspaces which admit graded projectors $P : H^* \to H^*$ (projector means $P^2 = P$), with $P(A) = A$, which are $\smile$-algebra homomorphisms in the grades $i+j$, where $A(i) \oplus A(j) \neq 0$.

The simplest examples of $\smile$-retracts are *homogeneous coannulators*, i.e. subspaces $A \subset H^i$ which admit complementary subspaces $B \subset H^i$ such that $B \smile H^i = 0$.

More generally, let $A = \oplus A(n) \subset H^*(X\ \mathbb{F})$ be a graded subalgebra, $X_i \subset X$, $i = 0, 1, \dots, k$, be open subsets and let $X_I \subset X$, $I \subset \{0, 1, \dots, k\}$ denote the intersections $\cap_{i \in I} X_i$.

Since the $\smile$-products of the graded ideals $A_i = \mu_A(X_i) \subset A$ satisfy

$$\smile_{i\in I} A_i = \smile_{i \in I} \mu_A(X_i) \subset A_I =_{\mathrm{def}} \mu_A(X_I)\,,$$

the set $M^*(X_i)$ of the graded $\mathbb{F}$-ranks of $\mu_A(X_i)$ is contained in such a set for A_i, where the ranks of A_I are bounded by $\mu(A(X_I))$, i.e.

$$M^*(X_i) \in \mathcal{M}^*(A; M^\star) \quad \text{for} \quad M^\star = \big\{|\mu_A(X_I)|_{\mathbb{F}}(n)\big\}_I\,,$$

where $\mathcal{M}^*$ denotes the separation profile of A defined in 4.5. This is especially useful if

A is a semigroup algebra, $A = \mathbb{F}[G_\circ]$ for a faithful ordered semigroup $G_\circ$ with zero,

since $\mathcal{M}^*$ for such an A equals the corresponding profile of $G_\circ$ which, in turn, can be often evaluated by combinatorial means as we shall see presently.

Homological isoperimetry in the torus. Let X be the N-torus. Then $A = H^*(\mathbb{T}^N;\mathbb{F})$ is isomorphic to the exterior algebra $\wedge^*\mathbb{F}^N = \mathbb{F}[\Delta_\circ^{N-1}]$ for the graded semigroup $G_\circ = \Delta_\circ^{N-1}$ associated to the simplex Δ^{N-1} on N-vertices (see 2.1).

Namely, $G_\circ$ equals $2^{\{1,\dots,N\}}$ that is the set of subsets g in $\{1,\dots,N\}$, where $G(n) \subset G_\circ$ consists of all subsets of cardinality n and where the product $g_i \smile g_2$ for $g_i \subset \{1,\dots N\}$ is defined as follows:

If g_1 intersects g_2, then $g_1 \smile g_2 = 0$; otherwise, $g_1 \smile g_2 = \pm g_1 \cup g_2$.

(If $\operatorname{char}\mathbb{F} = 2$ one does not have to bother with the specification of the $\pm$ sign.)

Thus,

bounds on cardinalities of subsets $G_i \subset 2^{\{1,\dots,N\}}$ established in extremal set theory in terms of the numbers of non-intersecting members $g_i \in 2^{\{1,\dots,N\}}$ regarded as subsets $g_i \subset \{1,\dots,N\}$ imply corresponding inequalities between the cohomology masses of subsets $X_i \subset \mathbb{T}^N$ and of their intersections.

Example: Matsumoto–Tokushige inequality [MatT1].

Let $G_i \subset G(n_i) \subset G_\circ = 2^{\{1,\dots,N\}}$, $i = 0,1$, be subsets such that the intersections $g_0 \cap g_1$ in $\{1,\dots,N\}$ are non-empty for all $g_0 \in G_0$ and $g_1 \in G_1$. If $n_0, n_1 \le N/2$, then the cardinalities of these sets satisfy

$$|G_0| \cdot |G_1| \le \binom{N-1}{n_0-1}\binom{N-1}{n_1-1}.$$

This implies *the following homological separation inequality* for pairs of disjoint subsets $X_0, X_1 \subset \mathbb{T}^N$ (stated in slightly different notation in 1.5), i.e. an upper bound on the ranks of the n_0- and n_1-grades of their cohomology masses,

If $n_0, n_1 \le N/2$, then

$$\big|(\mu_{H^*}(X_0)(n_0)\big|_{\mathbb{F}} \cdot \big|(\mu_{H^*}(X_1)(n_1)\big|_{\mathbb{F}} \le \binom{N-1}{n_0-1}\binom{N-1}{n_1-1}.$$

In particular,

if X_1 can be obtained from X_0 by a homeomorphism of $\mathbb{T}^N$ homotopic to the identity, then

$$\big|(\mu_{H^*}(X_0)(n)\big|_{\mathbb{F}} = \big|(\mu_{H^*}X_1)(n)\big|_{\mathbb{F}} \le \binom{N-1}{n-1}.$$

(This corresponds to the *Erdös–Ko–Rado theorem* (see [Fr]).)

***t*-Disjointness and *t*-intersection.** Subsets $X_0, X_1 \subset \mathbb{T}^N$ are called t-disjoint, $t = 1,2,\dots$, if the homotopy class of every coordinate projection $P : \mathbb{T}^N \to \mathbb{T}^{N-t+1}$ has a continuous representative $P' : \mathbb{T}^n \to \mathbb{T}^{N-t+1}$, such that the images $P'(X_0)$ and $P'(X_1)$ are disjoint in $\mathbb{T}^{N-t+1}$. This corresponds to the t-intersection in the extremal set theory (see [Fr]); thus, the t-version of the Erdös–Ko–Rado theorem [MatT2] implies that

if the above $X_0, X_1 \subset \mathbb{T}^N$ are t-disjoint, then

$$\big|(\mu_{H^*}(X_0)(n)\big|_{\mathbb{F}} = \big|(\mu_{H^*}(X_1)(n)\big|_{\mathbb{F}} \le \frac{n}{N}|H^n|_{\mathbb{F}} = \binom{N-1}{n-1} \le \binom{N-t}{n-t}$$

for all sufficiently large $N \ge N_0(n,t)$.

Remarks. (a) The intersection (e.g. t-disjointness in $\mathbb{T}^N$) pattern of subsets $X_i \subset X$, $i = 1, \dots, k$, can be expressed in terms of intersections of $\times_i X_i \subset X^k$ with diagonals in X^k, which suggests a more general version of separation inequalities.

(b) If X is a Cartesian product, say of spheres and projective spaces, then its cohomology is a semigroup algebra. Probably, the combinatorial inequalities cited above extend to this context which would yield the corresponding cohomological separation inequalities in X. (Products of *odd-dimensional* spheres have the same cohomological separation inequalities as tori; but the products of *even-dimensional* spheres, where the cohomology algebras are truncated polynomial rings, need combinatorial inequalities for systems of *sets with multiplicities.*)

(c) The multiplicativity property of the $\mathcal{I}$-mass carries over to algebras representing minimal models of spaces X. Probably, it leads to *homotopy* separation inequalities refining, in certain cases, the homological inequalities. An example one may start with is a compact nil-manifold X with the free nilpotent fundamental group of nilpotency degree δ on N generators.

4.9 Parametric separation inequalities. Let X be a smooth manifold, $F : X \to \mathbb{R}_+$ a proper generic smooth (Morse) function and $A \subset H^* = H^*(X;\mathbb{F})$ a graded linear subspace. Then the rank of the restriction of A to a y-sublevel of F is a ± 1-*continuous* function in $y \in \mathbb{R}_+$, i.e.

$$1 \geq \lim_{y \to y_0} \left|A|F^{-1}[0,y]\right|_{\mathbb{F}} - \left|A/F^{-1}[0,y_0]\right|_{\mathbb{F}} \geq -1$$

for all graded subspaces $A \subset H^* = H^*(X;\mathbb{F})$, all coefficient fields $\mathbb{F}$;

therefore, the function $|A|F^{-1}[0,y)|_{\mathbb{F}}$, $y \in \mathbb{R}_+$, assumes all integer values between 0 and $|A|_{\mathbb{F}}$.

It follows, that if X is compact, there is a $y_0 = y_0(A) \in \mathbb{R}_+$ such that

$$2 \geq \left|A|F^{-1}[0,y_0)\right|_{\mathbb{F}} - \left|A|F^{-1}(y_0,\infty)\right|_{\mathbb{F}} \geq -2$$

where, moreover, one can replace "2" by "0" if X is a closed N-manifold and $A \subset H^* < N/2$.

In particular, as explained in 1.5, "the H^n-mass" of the F-level of such $y_0 = y_0(A = H^n)$ for the torus $X = \mathbb{T}^N$ satisfies

$$\left|H^n|F^{-1}(y_0)\right|_{\mathbb{F}} \geq (1 - 2n/N)|H^n|_{\mathbb{F}} = (1-2n/N)\binom{N}{n}$$

for every $n < N/2$ and since every continuous function can be approximated by smooth generic ones, such a y_0 exists for all continuous functions on $\mathbb{T}^N$, as was claimed in 1.5.

Remarks. (a) The above argument depends on the multiplicativity and additivity of the μ_{H^*}-mass but does not need the intersection property.

(b) A continuous function F on an arbitrary compact space X can be approximated by generic smooth maps F_ε of smooth thickening of nerves of finite open covers of X where the A-mass $|A(k)|F_\varepsilon^{-1}[0,y]|_{\mathbb{F}}$ assumes all values $m = 0, 1, \dots, |A(k)|$.

Then the maximum of the (Čech) cohomology (with finite coefficients), restricted to the levels of F, i.e.

$$\sup_{y \in \mathbb{R}_+} \left|H^*|F^{-1}(y)\right|_{\mathbb{F}},$$

can be bounded from below in terms of the separation profile of the algebra $H^* = H^*(X;\mathbb{F})$.

Let $F_s : X \to \mathbb{R}$, $s \in S$, be a smooth generic family of proper smooth maps parameterized by a k-dimensional family S. Then the map $(s,y) \mapsto |A|F^{-1}(-\infty,y)|_{\mathbb{F}}$ is $\pm k$ continuous for every finite-dimensional linear subspace $A \subset H^*(X)$.

For example, let X be a compact manifold, $F : X \to \mathbb{R}^k$ be a smooth generic map and $A_i \subset H^*(X) = H^*(X;\mathbb{F})$, $i = 1,\dots,k$ be linear subspaces. Then the *Borsuk–Ulam theorem* shows that

> *there exists an affine hyperplane $P \subset \mathbb{R}^k$, such that $-2k \leq |A_i|F^{-1}(P_+)|_{\mathbb{F}} - |A_i|F^{-1}(P_-)|_{\mathbb{F}} \leq 2k$, $i = 1,2,\dots,k$, where $P_\pm \subset \mathbb{R}^k$ are the two half spaces in the complement to H.*

For example, let $X = \mathbb{T}^N$ and $n_1,\dots,n_k$ be given positive integers.

> *Then there exists a hyperplane $P \subset \mathbb{R}^k$ such that*
> $$\big|H^{n_i}|F^{-1}(P)\big|_{\mathbb{F}} \geq (1 - 2n_i/N)\binom{N}{n_i} - 4k\,, \quad i = 1,\dots,k\,.$$

(In fact, one can replace $4k$ by $2k$.)

Similarly, let X be the connected sum of k copies of $\mathbb{T}^N$ and let $n \leq N/2$.

> *Then there exists a hyperplane $P = P(F,n) \subset \mathbb{R}^k$, such that*
> $$\big|H^n|F^{-1}(P)\big|_{\mathbb{F}} \geq k(1 - 2n/N)\binom{N}{n} - 4k\,.$$

Remarks and Questions. (a) The above extends to arbitrary compact spaces X where suitable bounds on the $\smile$-profiles of their cohomology algebras are available.

(b) What happens if "hyperplane" is replaced by an "affine subspace of codimension m" for $m \geq 2$? This, for $X = \mathbb{T}^N$, is reminicent of sections of an N-simplex by affine subspaces (see 1.1), since the cohomology of $\mathbb{T}^N$ equals the cochain complex of δ^{N-1}. Is there a true connection here?

(c) There is another possible link between affine section of simplices and topology (or rather geometry of toric varieties). Consider the standard action of the $\mathbb{C}$-torus $(C^\times)^N$ on $\mathbb{C}P^N$ and recall that $\mathbb{C}P^N/\mathbb{T}^N = \Delta^N$ for the subgroup $\mathbb{T}^N \subset (C^\times)^N$.

Let $G \subset (C^\times)^N$ be a connected subgroup, let $G(x) \subset \mathbb{C}P^N$ be an orbit of a point $x \in \mathbb{C}P^N$ and let $\pitchfork_k(G,x)$ denote the number of k-dimensional orbits of $(C^\times)^N$ in $\mathbb{C}P^N$ which intersect $G(x)$. Can one bound $\sup_x \pitchfork_k(G,x)$ from below for suitable classes of subgroup G and use this for lower bounds on Barany's constants $b_{\mathrm{aff}}(n,k)$?

4.10 Homological realization of monomial algebras by spaces with locally bounded geometries. A (typically infinite) family $\{X\}$ of (e.g. finite) simplicial polyhedra X is said to have *locally bounded geometry* if the degrees of all vertices x in all X in the family (i.e. the numbers of simplices in X adjacent to x) are bounded by a constant $d = d\{X\} < \infty$ (which is independent of X and x).

One can also express this by saying that the set $L = (\{X\}$ of isomorphism classes of links of all x in all X is finite and observe that $|L| \leq d^d$; however, if $\dim(X) \geq 2$, there is *no* bound of d in terms of the cardinality $|L|$.

Next, given a family $\{X\}$ of metric spaces (e.g. of simplicial polyhedra with the standard metrics), denote by $N_{\mathrm{Lip}}(\{X\};\rho,\lambda)$ the minimal number N for which there exists a set $\{Y\}$ of N metric spaces Y, such that every ρ- ball in every $X \in \{X\}$ is contained in a neighborhood $U \subset X$ which is $(1+\lambda)$-bi-Lipschitz equivalent to some $Y \in \{Y\}$. (One may say in this case that the family $\{X\}$ has its *ρ-local λ-Lipschitz geometry is bounded by N*. This agrees with the above "bounded geometry" for polyhedra, where, in fact, one could equally use the Hausdorff distance $\mathrm{dist}_{\mathrm{Hau}}(U,Y)$ instead of $\mathrm{dist}_{\mathrm{Lip}}$.)

For example, if $\{X\}$ is the class of all complete n-dimensional Riemannian manifolds with sectional curvatures bounded by $|\,\mathrm{curv}\,| \leq \kappa^2$ and with injrad $\geq 1/r$, then, by Cheeger's compactness theorem, $N_{\mathrm{Lip}}(\{X\};\rho,\lambda) \leq C(n,\rho,\lambda,\kappa,r) < \infty$ for all $\rho,\lambda,\kappa,r > 0$; moreover, this C is bounded by something like $\exp(\exp(n\rho\kappa r/\varepsilon))$.

We say, in this case, that the local (Riemannian) geometries of the spaces X are bounded by κ. For instance, if a smooth closed submanifold $X \subset \mathbb{R}^N$ has the norm of second fundamental form bounded by 1, then the induced Riemannian metric in X has its local geometry bounded by something like $10n^2$.

Given a simplicial complex S and a marked topological space $R = (R, r_0)$ there are topological spaces $X = S(R)$ and $X_\star = S_\star(R)$ canonically (covariantly functorially in R) associated to these S and R, which are, recall, glued of Cartesian powers R^i for $i = 1,2,\dots,\dim(S)+1$ according to the combinatorial pattern encoded by S. (See 2.1 and (b) at the end of this section for a geometric counterpart of the functorial construction in 2.1.)

If the cohomology $H^*(R) = H^*(R;\mathbb{F})$ is *faithful*, which means it is (isomorphic to) the monomial algebra associated to a faithful ordered semigroup $G_\circ$ with 0 (see 4.6), then $H^*(X)$ is also faithful being associated to the obviously defined semigroup $H_\circ = S(G_\circ)$; hence the homological separation profile M^* of X equals that of $H_\circ$ (see 4.6).

For example, if R is the homology i-sphere, then $H_\circ = 0 \cup_{n=0,i,2i,\dots} H(n)$ equals the Stanley–Reisner semigroup associated to S and the cohomology $H^*(X) = \oplus_n H^{in}$ is the (monomial) Stanley–Reisner algebra $\mathbb{F}\{H_\circ\}$ associated to S (see 4.5, 4.6).

Therefore, if $\dim(S) = 1$ and either i is odd or the coefficient field $\mathbb{F}$ has characteristic 2, then the cohomological isoperimetric profile of X equals the $\perp$-profile of S; in any case, for all i and $\mathbb{F}$, the profile of X is bounded from below by the ordinary $\overrightarrow{\partial}$-profile of S up to $1/\deg(S)$-factor, unless some connected component of S consists of a single edge (see 4.7).

The space $X = S(R)$ is locally homeomorphic at the marked point to the cone over something which is, roughly, as complicated as S. Since we look for infinite families of spaces with uniformly bounded local geometries, we turn to the spaces $X_\star = S_\star(R)$ that do have locally bounded geometry if the polyhedra S do. This creates, however, a (minor in the present context) problem, since the homology of $X_\star$ is more complicated than that of X as some part of it comes from S. Yet this can be either controlled or modified to our liking as we shall see below.

Questions. Can one achieve bounded geometry *without* changing the homotopy type of $X = S(R)$ or, at least, its cohomology algebra? (An alternative to $X_\star$ could be removing a small ball around the "bad" marked point in X and then doubling the resulting space across the boundary of this ball.)

More generally, given a family $\{X\}$ of finite n-dimensional simplicial polyhedra, can one find another family of polyhedra $\{\tilde{X}\} = \{\tilde{X}(X)\}$ of a given dimension $\tilde{n} \geq n$, such that every $\tilde{X}$ is homotopy equivalent to the respective X and such that the *local combinatorial degrees of all $\tilde{X}$ are bounded by a constant independent of $\tilde{X}$*, while the numbers of simplices in $\tilde{X}$ are bounded by $N_\Delta(\tilde{X}) \leq CN_\Delta(X)$ for a constant C?

Presumably, the answer is in the negative for many "natural" families, but, it is apparently unsettled even for the family of *all* finite polyhedra, allowing *non-simply connected* ones; then one wonders what is the asymptotic of functions $C(N)$ such that $N_\Delta(\tilde{X}) \leq C(N_\Delta(X))N_\Delta(X)$ can be, where one expects $C(N) \sim (\log N)^{\alpha(n)}$ for "typical" families, e.g. for families of random polyhedra.

1. 2-Dimensional homology expanders. (a) Let S be a *simply connected* 2-dimensional simplicial polyhedron and $R = \mathbb{T}$ be the circle with a marked point. Let $X^2_\star = S_1(\mathbb{T}) \cup S \subset X_\star = S_\star(\mathbb{T})$, where $S_1 \subset S$ is the 1-skeleton of S and where, recall (see 2.1) S is naturally embedded to $X_\star$. Denote by $\Pi : X^2_\star \to X$ the restriction of the map $X_\star \to X$ (see 2.1) to $X^2_\star$ and observe that Π induces an isomorphism $\Pi^* : H^1(X;\mathbb{F}) \to H^1(X^2_\star;\mathbb{F})$ for all fields $\mathbb{F}$ and that the homomorphism $\Pi^* : H^2(X;\mathbb{F}) \to H^2(X^2_\star;\mathbb{F})$ is injective.

It follows that

> the isoperimetric (separation) $\smile$-profile of $H^1(X^2_\star)$ is equal to that of $H^1(X)$; hence, (see 4.5) the isoperimetric H^1-profile of $X^2_\star$ can be estimated in terms of the $\overrightarrow{\partial}$-profile of the graph S_1 according to $[\partial_\smile \succ \overrightarrow{\partial}]$ in 4.6 as follows:
>
> *The rank of the cohomology boundary $\partial_{H^1}(X_0) = H^1|\partial(X_0)$ of every open subset $X_0 \subset X^2_\star$ is bounded from below in terms of (the rank of) its cohomology mass $|\mu_{H^1}(X_0)|_\mathbb{F}$ as follows:*
> $$\left|\partial_{H^1}(X_0)\right|_\mathbb{F} \geq \tfrac{1}{d}\left|\overrightarrow{\partial}(V_0)\right|,$$
> *where $V_0 = V_0(X_0)$ is a set of vertices in S_1 of cardinality*
> $$|V_0| = \left|\mu_{H^1}(X_0)\right|_\mathbb{F}$$
> *where d is the maximum of the degrees of vertices in S_1 and where $\overrightarrow{\partial}(V_0)$ the set of edges between V_0 and its complement in the full vertex set of S_1.*

One can "simplify" this $X^2_\star$ further, by adding the 2-handles corresponding to the disc D attached to the circle $\mathbb{T}$ by a map $\partial(D) \to \mathbb{T}$ of degree p. The resulting space, say $X^2_p \supset X^2_\star$ has a finite Abelian fundamental group and the same $\mathbb{F}_p$-cohomology as $X^2_\star$.

If we take such $X^2_p = X^2_p(S)$ with a family $\{S\}$ of 2-dimensional simply connected uniformly locally bounded polyhedra S (see 2.1) that make an edge-wise expander (see 4.3) we obtain a family $\{X^2_p\}$ of finite *uniformly* locally bounded 2-dimensional polyhedra X with *finite Abelian* fundamental groups and with $|H^1(X^2_p;\mathbb{F}_p)|_{\mathbb{F}_p} \to \infty$, such that, *every compact space X that is homotopy equivalent to some member of the family $\{X_p^2\}$ is an H^1-expander, for $H^1 = H^1(X;\mathbb{F}_p)$, i.e. all closed subsets $X_0 \subset X$ with $|H^1|X_0|_{\mathbb{F}_p} \leq \frac{1}{2}|H^1|_{\mathbb{F}_p}$, satisfy*

$$\big|H^1|\partial(X_0)\big|_{\mathbb{F}_p} \geq \lambda \cdot |H^1|X_0|_{\mathbb{F}_p} \quad \textit{for} \quad \lambda = \lambda\{S\} > 0\,.$$

2. Simply connected 4-dimensional homology expanders. Proceed as above with R being the 2-sphere instead of the circle and denote the resulting space by $X_\star^4 = S(R = S^2)$. Observe (see 2.1) that if S is simply connected then $X_\star^4$ is also simply connected.

Since the complement $X_\star^4 \setminus S_1(R) \subset S$ equals the union of disjoint open 2-simplices, the kernel $K \subset H^2(X_\star^4)$ of the inclusion homomorphism $H^*(X_*^4) \to H^*(S_1(R)$ is a $\smile$-annulator; thus the $H^4(X_\star^4)$-valued $\smile$-product on every subspace $A_\star \subset H^2(X_\star^4)$ complementary to K, that is $\smile$: $A_\star \oplus A_\star \to H^4(X_\star^4)$, equals the $H^4(S_1(R))$-valued $\smile$-product on $H^2(S_1(R)) = A = H^2(X_\star^4)/K$ for $H^4(S_1(R) = H^4(X_\star^4)$.

Since $A_\star$ is a coannulator in the sense of 4.8, the cohomological $A_\star$-mass profile of $X_\star^4$, concerning the ranks of the linear spaces $\mu_{A_\star}(X_0) \subset A_\star(2)$ and $\partial_{A_\star}(X_0) = A_\star|\partial(X_0)$ for $X_0 \subset X_\star^4$, is bounded from below by the $\smile$ profile, of $A_\star$; therefore, the $A_\star$-mass profile of $X_\star^4$ is bounded from below via $[\partial\smile \succ \overrightarrow{\partial}]$ from 4.6 by the $\overrightarrow{\partial}$-profile of the 1-skeleton S_1 of S upto the $1/d$ factor where d is maximum of degrees of vertices in S_1. That is

for every subset $X_0 \subset X_\star^4$, there exists a subset V_0 of vertices in the graph S_1, such that $|V_0| = |\mu_{A_\star}|_{\mathbb{F}}$ and $|\overrightarrow{\partial}(V_0)| \leq d \cdot |\partial_{A_\star}(X_0)|_{\mathbb{F}}$.

3. Manifolds homology expanders. Let us turn the above X_p^2 and $X_\star^4$ into smooth manifolds as follows. Observe that X_p^2 and $X_\star^4$ can be immersed into $\mathbb{R}^4$ and $\mathbb{R}^6$ respectively and thus they are homotopy equivalent to compact 4- and 6-manifolds with boundaries.

Let $M_p^4 = M_p^4(S)$ and $M_\star^6 = M_\star^6(S)$ be the doubles of these. Clearly, M_p^4 and $M_\star^6$ are closed manifolds which admit Riemannian metrics with bounds on their local geometries depending (only!) on that of S (and on p for M_p^4) and with volumes bounded by the numbers of simplices in S (and p for M_p^4). Also observe that X_p^2 and $X_\star^4$ embed as retracts into M_p^4 and $M_\star^6$ correspondingly and that the manifolds $M_\star^6$ are simply connected (for simply connected S) while M_p^4 have finite Abelian fundamental groups.

Let $A_1 \subset H^1(M_p^4;\mathbb{F}_p)$ and $A_2 \subset H^2(M_\star^6;\mathbb{F})$ (for any $\mathbb{F}$) be the images of $H^1(X_p^2;\mathbb{F}_p)$ and of the above $A_\star \subset H^2(X_\star^4)$ under the retractions $M_p^4 \to X_p^2$ and $M_\star^6 \to X_\star^4$. The isoperimetric homology and $\smile$-profiles for A_1 and A_2 are, obviously, equal to those of $H^1(X_p^2;\mathbb{F}_p)$ and of $A_\star$; hence, by 4.7 and ★ in 4.8,

The families of closed manifolds $M_p^4\{S\}$ and $M_\star^6\{S\}$ are, respectively, A_1- and A_2-expanders, whenever $\{S\}$, the family of 2-polyhedra S, is an (edge-wise) expander.

3-Manifold homology expanders. The polyhedron $X = S_1(\mathbb{T})$ can be (obviously) immersed into $\mathbb{R}^3$, then thickened to a manifold with boundary and doubled to a closed 3-manifold $M^3 = M^3(S)$ (which is a graph manifold in the usual sense). The homological profile of this M^3 with respect to the image A_1 of $H^1(X;\mathbb{F})$ equals that of X, and M^3 enjoys all of the above properties except for having infinite $H^1(M^3;\mathbb{Z})$ that we render zero as follows.

Take a system of disjoint simple closed curves in M^3 making a basis in $H_1(M^3;\mathbb{Q})$, and perform surgery along the slightly perturbed p-multiples of these curves with some frames. The resulting manifold, say M^3_p, has $H^1(M^3_p;Z) = 0$ while the cohomology algebra with $\mathbb{F}_p$ coefficients splits in the grades 1 and 2,

$$H^{1,2}(M^3_p : \mathbb{F}_p) = H^{1,2}(M^3;\mathbb{F}_p) \oplus B\,,$$

where the image of the coordinate embedding of $H^{1,2}(M^3;\mathbb{F}) \to H^{1,2}(M^3_p)$ is a $\smile$-retract (as in ★ in 4.8). Indeed, a surgery along a p-multiple curve adds one $\mathbb{F}$ summand to H^1 which is $\smile$-orthogonal to H^1 and one to H^2.

Thus we obtain

> *closed Riemannian 3-manifolds $M^3_p = M^3_p(S)$ with volumes and local geometries bounded by those of S, with $H^1(M^3_p;\mathbb{Z}) = 0$ and with $A = H^1(X;\mathbb{F}_p)$ contained in $H^1(M^3_p;\mathbb{F}_p)$ as $\smile$-retracts, where the latter makes homological A-isoperimetry of M^3_p equal that of S.*

REMARKS. (a) The Margulis theorem on normal subgroups is not needed in this case, vanishing of $H^1(\Gamma;\mathbb{Z})$ suffices.

(b) The above construction of manifolds with controlled $\smile$-products can be seen in a (more general) geometric light as follows. Let X_i, $i \in I$, be closed smooth manifolds of dimension n and $Y_{ij} \subset X_i$, $j \in J_i$, be smooth closed codimension k submanifolds with a given set $\{D\}$ of diffeomorphisms between some pairs of Y_{ij}.

For example, one may have two manifolds $X_i \supset Y_i$, $i = 1,2$, with a single diffeomorphism $Y_1 \to Y_2$, or a single $X \supset Y$, with a finite group of diffeomorphisms of Y. (Also one may include immersed rather than embedded hypersurfaces.)

Glue X_i by $\{D\}$, denote the resulting n-dimensional space by $X_{\{D\}}$ and suppose that $X_{\{D\}}$ is locally homeomorphic at each point to the union of linear subspaces of codimension k in $\mathbb{R}^{n+k}$ in general position. (One may allow non-general position as well.)

Then, modulo an obvious obstruction, $X_{\{D\}}$ can be thickened to an $(n+k)$-manifold M with boundary, where the intersection ring generated by $[X_i] \in H_n(M)$ can be expressed in terms of $\{D\}$ and in the intersections of $[Y_{ij}] \in H_{n-k}(X_i)$.

For instance, if X_i are 2-tori and Y_{ij}, $j = 1,2$, are the coordinate circles, we recapture the above 3-manifold thickening of $S(\mathbb{T})$ and get more examples by gluing surfaces X_i along systems of closed curves $Y_{ij} \subset X_i$.

(c) According to [Su], every antisymmetric 3-form can be realized by intersections of 2-cycles in a 3-manifold X which leads to examples of 3-manifolds homology expanders with $A = H^1(X)$.

4.11 Maps to trees and folds in $\mathbb{R}^2$. Let $F : X \to Y$ be a continuous map of a compact space X with finite-dimensional cohomology $H^k = H^k(X;\mathbb{F})$ for given $k = 1,2,\dots$, and $\mathbb{F}$ to a locally finite tree Y and denote the normalized cohomology mass of open subsets $Z \subset Y$ by

$$m(Z) = \big|\mu_{H^k}(Z)\big|_{\mathbb{F}}/|H^k|_{\mathbb{F}}\,.$$

The set function m is *monotone*: $Z_1 \subset Z_2 \Rightarrow m(Z_1) \leq m(Z_2)$, *(semi)continuous*: $m(Z) = \lim m(Z_i)$ for open exhaustions $Z_1 \subset Z_2 \subset \cdots \subset Z$ and subadditive $m(Z_1 \cup Z_2) \leq m(Z_1) + m(Z_2)$ for all pairs of disjoint open subsets (see 4.5). It follows that

there exists a point $y_c \in Y$ such that either

$$m\big(Y \setminus \{y_c\}\big) \leq 3/4\,,$$

or there is a multibranch $Z_c \subset Y$ at y_c (see 3.1), such that the masses of Z_c and of the complementary multibranch $Z_c^\perp =_{\text{def}} Y \setminus (Z_c \cup \{y_0\})$ satisfy

$$\tfrac{1}{2}m(Z_c) \leq m(Z_c^\perp) \leq m(Z_c)\,. \qquad \big[\tfrac{1}{2}\cdot m\big]\,.$$

Proof. If a point $y_0 \in Y$ violates $\big[\frac{1}{2}\cdot m\big]$, then there is a branch B_0 at y_0, such that

$$m(B_0) > 2m(B_0^\perp)\,, \qquad [>]$$

since every subadditive measure m on a finite set I (of branches at y_0), where $m(i) \leq 2m(I \setminus \{i\})$ for all $i \in I$ admits a subset $J \subset I$, such that $m(J) \leq m(I \setminus J) \leq 2m(J)$ (as in the proof of $\big[\frac{1}{3},\frac{2}{3}\big]$ in 3.1).

If there exists a point $y_\varepsilon \in B_0$ arbitrarily close to y_0, such that the branch $B_\varepsilon \subset B_0$ of Y at y_ε has

$$m(B_\varepsilon) \leq \tfrac{1}{2}m(B_\varepsilon^\perp)\,,$$

then $[>]$ implies that

$$m(B_0) + m(B_0^\perp) < \tfrac{3}{4}m(B_\varepsilon^\perp) + \tfrac{1}{2}\big(m(B_0) - m(B_\varepsilon)\big)\,.$$

Since $m(B_\varepsilon^\perp) \leq 1$ and since $m(B_0) - m(B_\varepsilon) \to 0$ for $\varepsilon \to 0$ (in fact, $m(B_0) - m(B_\varepsilon) = 0$ in the present case) we conclude that

$$m\big(Y \setminus \{y_c\}\big) \leq m(B_0) + m(B_0^\perp) \leq \tfrac{3}{4}\,.$$

If there is a point y_1 in the open edge E of B_0 adjacent to y_0, such that $m(B_1) \leq \frac{1}{2}m(B_1^\perp)$, while all points y_ε that lie closer to y_0 have $m(B_\varepsilon) \geq \frac{1}{2}m(B_\varepsilon^\perp)$, then such y_1 which is the farthest from y_0 serves for the required y_c. Otherwise take the vertex of Y in E for $y_1 = y_1(y_0)$, then pass to $y_2 = y_2(y_1)$, $y_3 = y_3(y_2)$, etc., until the process terminates at y_c.

COROLLARY: Lower bounds on the cohomology mass over trees by the isoperimetric profile. *Let $|\partial_k|(m)$ be a function (cohomological isoperimetric profile of X) such that all open subsets $X_0 \subset X$ satisfy*

$$\big|H^k|\partial(X_0)\big|_{\mathbb{F}} \geq |\partial_k|\big|\mu(X_0)\big|\mathbb{F} \quad \textit{for } H^k = H^k(X;\mathbb{F})$$

and let m_1, m_2 be integers such that $m_1 < \frac{1}{3}|H^k|_{\mathbb{F}}$ and $m_2 > \frac{2}{3}|H^k|_{\mathbb{F}}$.

Then every continuous map $F : X \to Y$, where Y is a tree, admits a point $y \in Y$, such that

$$\big|H^k|F^{-1}(y)\big|_{\mathbb{F}} \geq \min\Big(\tfrac{1}{4}|H^k|, \inf_{m_1 \leq m \leq m_2} \|\partial_k|_{\text{int}}(m)\Big).$$

Furthermore, this inequality also holds for maps to arbitrary graphs Y, provided $H^1(X;\mathbb{Z}) = 0$ (e.g. if $H_1(X)$ is pure torsion), since every map to $X \to Y$ lifts to a map $X \to \tilde{Y}$ for the the universal covering $\tilde{Y}$ of Y.

We apply this to the families of polyhedra $\{X\} = \{S\}(\mathbb{R})$, associated to "simply connected" expanders $\{S\}$ (see 4.3) and where R is a sphere (of dimension 1 or 2 for the present purpose) and conclude that maps of these to trees have

$$\sup_{y\in Y} \big|H^k|F^{-1}(y)\big|_{\mathbb{F}} \geq \lambda |H^k|_{\mathbb{F}} \quad \text{for } \lambda = \lambda\{S\} > 0\,.$$

Then we recall the corresponding families of manifolds M from 4.8 of dimensions 3, 4 and 6, which all have $H_1(M) = 0$, and where M^6 are simply connected, while M^4 have finite fundamental groups. Every such M has a corresponding X embedded into it such that the restriction cohomology homomorphisms are surjective and their kernels K are bounded by $|K| \leq 10|H^*(X)|$. Thus the above inequality remains valid for M as well.

Finally, we invoke the depth inequality (proof) from [Gr8] and conclude with the following:

Deep Fold Examples.

There exist three families $\{M\}$ of closed Riemannian manifolds:

(1) *of dimension* 6*, which have trivial* π_1*;*
(2) *of dimension* 4*, which have finite* π_1*;*
(3) *of dimension* 3*;*

where all these M have 1-bounded local geometries and volumes tending to infinity, and such that a generic map F of every member M of such family to an open surface Y admits a point $y \in Y$, where every path from y to ∞ crosses the folding locus of F at least N times for $N \geq \lambda \cdot \mathrm{vol}(M)$ for $\lambda = \lambda\{M\} > 10^{-100} > 0$.

4.12 Tree-like algebras over graphs. There are spaces X, including simply connected 6-manifolds, which admit maps to "deep" binary trees with "homologically small" fibers but where every map F of X to a "shallow" tree Y (e.g. to $Y = \mathbb{R}$) necessarily has a "large" fiber $F^{-1}(y)$ for some $y \in Y$, namely where the A-mass $|A|F^{-1}(y)|_{\mathbb{F}}$ (see 1,5) is rather large. This follows from the properties of the kernels of restriction homomorphisms $H^*(X) \to H^*(F^{-1}(Z))$, $Z \subset Y$ which are enumerated in the following definition.

Let $A = A(S, \mathbb{F})$ be the Stanley-Reisner algebra associated to a connected graph S, i.e. $A = H^{*>0}(X; \mathbb{F})$ for $X = X_\star = S_\star(R)$, where R is the k-sphere and $\mathbb{F}$ is a field. Recall that $A^k = H^k(X; \mathbb{F})$ equals the space of $\mathbb{F}$-valued function a on the vertex set $\mathrm{vert}(S)$.

Given a linear subspace $I^k \subset A^k$, let $\mathrm{supp}(I^k) \subset \mathrm{vert}(S)$ denote the support of I^k i.e. the set of those $s \in \mathrm{vert}(S)$, for which $a(s) \neq 0$ for some function $a \in I^k$ and let $\overline{\mathrm{supp}}(I^k)$ be the subgraph in S spanned by $\mathrm{supp}(I^k)$.

Let $I_1^k, I_2^k \subset A^k$ be $\smile$-orthogonal, i.e. $I_1^k \smile I_2^k = 0$. Then, clearly,

$$\big|\mathrm{vert}(S) \setminus (\mathrm{supp}(I_1^k) \cup \mathrm{supp}(I_2^k))\big| \leq \mathrm{corank}(I_1^k + I_2^k) =_{\mathrm{def}} \mathrm{rank}\big(A^k/(I_1^k + I_2^k)\big)\,;$$

$\mathrm{supp}(I_2^k)$ *does not intersect the exterior boundary* $\partial_{\mathrm{ext}}(\mathrm{supp}(I_1^k) \subset \mathrm{vert}(S)$*;*

the intersection $\overline{\mathrm{supp}}(I_1^k) \cap \overline{\mathrm{supp}}(I_2^k)$ consists of the union of disjoint subgraphs in S which simultaneously serve as connected components of $\overline{\mathrm{supp}}(I_1^k)$ and of $\overline{\mathrm{supp}}(I_2^k)$;

if S_0 is a connected subgraph in $\overline{\mathrm{supp}}(I_1^k)$, then the rank of the restriction of (the linear space of functions $a : \mathrm{vert}(S) \to \mathbb{F}$ from) I_2^k to the vertex set $\mathrm{vert}(S_0) \subset \mathrm{vert}(S)$ is at most 1; moreover, if the rank of the restriction of I_1^k to $\mathrm{vert}(S_0)$ is ≥ 2, then the restriction of I_2^k to $\mathrm{vert}(S_0)$ is zero.

It follows that

The rank of the restriction of I_2^k to the union $\mathrm{supp}(I_1^k) \cup \partial_{\mathrm{ext}}(\mathrm{supp}(I_1^k))$ is bounded by the number of the connected components of $\mathrm{supp}(I_1^k)$.

The rank of the restriction of $I_1^k + I_2^k$ to the vertex set of a union of l connected subgraphs in $\overline{\mathrm{supp}}(I_1^k) \cap \overline{\mathrm{supp}}(I_2^k)$ is $\leq 2l$.

Let S have at least 3 vertices and denote $S_{12} =_{\mathrm{def}} \mathrm{supp}(I_1) \cap \mathrm{supp}(I_2)$. Then the rank R_{12} of the restriction of $I_1^k + I_2^k$ to $S_{12} \cup \partial_{\mathrm{ext}}(S_{12})$ is bounded by

$$R_{12} \leq \left|S_{12} \cup \partial_{\mathrm{ext}}(S_{12})\right| - \frac{|S_{12}|}{3\deg(S)} \qquad [3\deg]$$

where $\deg(S)$ denotes the maximum of degrees (valences) at the vertices of S.

Take an $\mathcal{I}(A)$-valued mass μ on $Y = \mathbb{R}$ for the above $A = A(S;\mathbb{F})$ and denote by $S_-(y)$ and $S_+(y)$, $y \in \mathbb{R}$, the supports of the ideals $I_-(y) =_{\mathrm{def}} \mu(-\infty, y) \subset A$ and $I_+(y) =_{\mathrm{def}} \mu(y, +\infty)$, where, observe,

$$I_-(y) \smile I_+(y) = 0 \quad \text{for all } y \in \mathbb{R}$$

and notice that

$$S_-(+\infty) =_{\mathrm{def}} \bigcup_{y\in\mathbb{R}} S_-(y) = S_+(-\infty) =_{\mathrm{def}} \bigcup_{y\in\mathbb{R}} S_+(y) = \mathrm{supp}(I(\mathbb{R})) \subset \mathrm{vert}(S)\,.$$

Let

$$I_-(y] = \bigcap_{\varepsilon>0} I_-(y+\varepsilon)\,, \quad S_-(y] = \mathrm{supp}\big(I_-(y]\big) \subset \mathrm{vert}(S)$$

and

$$S_-(y^{\uparrow}] = S_-(y] \setminus S_-(y) \subset \mathrm{vert}(S)\,.$$

Denote

$$m(y) = \left|\mathrm{supp}(I_{\perp}^k(\mathbb{R}))\right| - \mathrm{rank}\big(I_-(y) + I_+(y)\big)$$

and observe that

$$m(y+\varepsilon) = m(y) \quad \text{for small } \varepsilon > 0$$

by the continuity of the $\mathcal{I}$-mass (see the above (3)).

Let us bound the cardinality $|S_-(y^{\uparrow}]|$ by $m(y)$ as follows. Write

$$S_-(y^{\uparrow}] = \big(S_-(y^{\uparrow}] \cap S_+(y)\big) \cup \big(S_-(y^{\uparrow}] \setminus S_+(y)\big) \subset \big(S_-(y] \cap S_-(y)\big) \cup \big(S_-(y^{\uparrow}] \setminus S_+(y)\big)$$

and observe that if the graph S has at least 3 vertices, then the above [3 deg] applies to $I_1 = I_-(y+\varepsilon)$ and $I_2 = I_+(y+\varepsilon)$ and yields, in the limit for $\varepsilon \to 0$, the inequality

$$\left|S_-(y] \cap S_-(y)\right| \leq 3\deg(s) m(y)\,.$$

Since, obviously,

$$\left|S_-(y^{\uparrow}] \setminus S_+(y)\right| \leq m(y)\,,$$

we conclude with the inequality

$$\left|S_-(y^{\uparrow}]\right| \leq m(y)\big(3\deg(S) + 1\big)\,.$$

3. Map the subgraph $\overline{\text{supp}}(\mu(\mathbb{R})) \subset S$ to $[-\infty, +\infty) = \mathbb{R} \cup \{-\infty\}$ by sending each vertex $s \in \text{supp}(\mu(\mathbb{R})) = \text{vert}(\overline{\text{supp}}(\mu(\mathbb{R})))$ to $f(s) \in [-\infty, +\infty)$ equal to the *supremum* of the points $y \in \mathbb{R}$, such that s *is not contained* in the support of the subspace $I_-(y) =_{\text{def}} I^i_\perp(-\infty, y) \subset A^i$ and where f is extended to continuous monotone maps on the edges of $\overline{\text{supp}}(\mu(\mathbb{R}))$.

Let us bound the number $N(y)$ of the edges $[s_-, s_+]$ in $\overline{\text{supp}}(\mu(\mathbb{R}))$, such that $f[s_-, s_+] \ni y$, i.e. of the edges issuing from vertices in $S_-(y]$ and terminating in $\text{supp}(\mu(\mathbb{R})) \setminus S_-(y)$.

There are *at most* $m(y)$ among the s_+-ends of these edges which are *not contained in* $S_+(y)$; thus there are at least $N(y) - \deg(S)m(y)$ edges with the s_+-ends in $S_+(y)$ whose s_--ends necessarily lie in $S_-(y^\uparrow]$ since $I_-(y) \smile I_+(y) = 0$. It follows that

$$N(y)/m(y) \le 3\deg(S)\big(\deg(S) + 1\big)\,. \qquad [3\deg^2]$$

4. Mapping to graphs corollary. *Let $A = A^k \oplus A^{2k}$ be the Stanley–Reisner algebra associated to a connected graph S of degree (at most) d, let X be a compact topological space with $H^{*>0}(X;\mathbb{F}) = A$ and F be a continuous map of X to a finite graph Y with $N_{vr} = N_{vr}(Y)$ vertices of degrees ≥ 3.*

If the A^k-mass of every fiber $F^{-1}(y) \subset X$ (i.e. the rank of the restriction homomorphism $H^k(X) \to H^k(F^{-1}(y))$) is at most m, then there exists a subset $V_0 \subset \text{vert}(S)$ with $|V_0| \le m \cdot N_{vr}$ such that the graph $S_0 \subset S$ spanned by the complement $\text{vert}(S) \setminus V_0$ admits a continuous map $f : S_0 \to \mathbb{R}$ where $|f^{-1}(y)| \le 3md(d+1)$ for all $y \in \mathbb{R}$.

Proof. Let $I_0 \subset A^k$ be the kernel of the restriction homomorphism from $H^k(X)$ to the F-pullbacks of the set of vertices in Y of degrees ≥ 3 and let $S_0 = \overline{\text{supp}}(I_0)$. Then every connected component of S_0 serves as a component of the support of $\mu((y, y')) \subset A^k$ for some open edge $(y, y)'$ in Y, where, recall, $\mu((y, y'))$ denotes the kernel of the restriction homomorphism $H^k(X) \to F^{-1}(Y \setminus (y, y'))$.

Since (y, y') is homeomorphic to $\mathbb{R}$, the above **3.** applies to the $\mathcal{I}(A)$-valued mass $U \mapsto H^*(F^{-1}(U))$, $U \subset (y, y')$, and the proof follows.

5. Example: Mapping tree-like spaces to trees. Let S be a simplicial binary rooted tree $X_{d+\delta}$ of depth $d + \delta$ and let R be a sphere S^k and $X_\star = S_\star(R)$, be as in 4.8. We know (see 2.1) that $X_\star$ comes with a map $\rho : X_\star \to S$ where all fibers are either S^k or $S^k \times S^k$. On the other hand, the above shows that

> *every continuous map F of $X_\star$ to the binary tree Y of depth $\le d$ has a fiber $F^{-1}(y) \subset X_\star$, $y \in Y$, such that the A-mass of $F^{-1}(y)$ for $A = H^k(X_\star;\mathbb{F})$ satisfies,*
>
> $$\big|A|F^{-1}(y)\big|_{\mathbb{F}} \ge \text{const} \cdot \delta - 1 \quad \textit{for some } \text{const} \ge 0.1\,.$$

Finally, every such $X_\star$ can be turned into a manifold, as earlier, and in particular, one obtains, for each $k = 1, 2, \dots$ and every $n \ge 2k + 1$,

> *closed Riemannian manifolds of dimension n, say $M(d) = M^n_\star(d)$, $d = 1, 2, \dots,$ of volumes $N = 2^d$ with uniformly bounded local geometries, such that every $M(d)$ admits a map into a binary tree Y_d of depth d with the diameters of the pullbacks of all points $y \in Y_d$ bounded by a*

constant independent of d, and, yet, every continuous map $F : M(d) \to Y_{d-\delta}$ *admits a point* $y_0 \in Y_{d-\delta}$*, such that the rank of the restriction homomorphism* $H^k(M;\mathbb{F}) \to H^k(F^{-1}(y_0);\mathbb{F})$ *is bounded from below by* $\mathrm{const}\cdot\delta - 1$ *for* $\mathrm{const} \geq 0.1$ *and every coefficient field* $\mathbb{F}$*. Moreover, if* $k \geq 2$ *and* $n \geq 2k+2$*, there are such simply connected* $M(d)$*.*

4.13 Perspectives and problems.

A. On *LG*-invariance of ∂^i. Let $\mathbb{F}$ be a ring with a unit (e.g. a field), C a module (vector space) over $\mathbb{F}$ with a distinguished non-zero element (vector) denoted $\mathbf{1} \in C$ and $C^i = \wedge_{\mathbb{F}}^{i+1} C$ be the $(i+1)$-th exterior power of C. Then the linear map $\partial^i_{\mathbf{1}} : C^i \to C^{i+1}$ for $c \mapsto c \wedge \mathbf{1}$ satisfies $(\partial^i_{\mathbf{1}})^2 = 0$ and if $C = C^0$ is the space of measurable $\mathbb{F}$-valued function on V with $\mathbf{1} \in C^0$ being the constant unit function, then $\partial^i_{\mathbf{1}}$ equals the coboundary operator $\partial^i : C^i(\Delta(V);\mathbb{F}) \to C^{i+1}(\Delta(V);\mathbb{F})$ on the $\mathbb{F}$-cochain complex of the simplex $\Delta(V)$ on the vertex set V, where, recall V is a measure space (see 2.3).

Thus, the complex $(C^*(\Delta(V), \partial^*))$ (but not the norms $\|\dots\|$ on $C^i(\Delta(V))$ is invariant not only under the group G of measurable transformations of V but also under the (much larger) group LG of invertible "measurable linear operators" of C fixing $\mathbf{1}$: a measurable linear operator $A : C \to C$ is, by definition, given by the measure space W with an $\mathbb{F}$-valued function K on W (the "integral kernel" of A) and a pair of measurable maps $M_i : W \to V$, $i = 1,2$, where M_2 is finite to one. Then every function $c : V \to \mathbb{F}$ is first pulled back to W with M_1, then multiplied by K and finally pushed forward to V with M_2. (If V is a finite set and $\mathbb{F}$ is a field, then LG equals the group of linear transformations of $C = \mathbb{F}^V$ fixing $\mathbf{1}$.)

Is there an LG-Invariant version of the homological isoperimetry, e.g. for the N-torus?

B. On measurable functoriality. Let X and Y be measurable cell complexes, let $\{Y \to X\}$ denote the space of cellular maps and let $[Y \to X]$ be the set of connected components in $\{Y \to X\}$ (i.e. of homotopy classes of cellular maps). For example, if X and Y are simplicial complexes, then $[Y \to X]$ equals the set of simplicial maps $Y \to X$.

The correspondence $Y \rightsquigarrow [Y \to X]$ is a set valued contravariant functor from the category $\mathcal{Y}$ of cellular complexes Y with $\mathrm{Hom}(Y_1, Y_2) = [Y_1 \to Y_2]$. This functor on the category $\mathcal{Y}$ of finite complexes Y essentially recovers X. One would like to have a similar definition of a measurable cell complex (e.g. of a measurable simplicial complex) as a contravariant functor from $\mathcal{Y}$ to the category of measure spaces.

In fact, in most examples, say of measurable simplicial complexes X and for all finite simplicial complexes Y, one can introduce measure structures on the sets $[Y \to X]$; however, the restriction maps $R_0 : [Y \to X] \to [Y_0 \to X]$ for subcomplexes $Y_0 \subset Y$ may be non-measurable: the R_0-pullbacks of null-sets in $[Y_0 \to X]$ may have positive measure in $[Y_0 \to X]$. This happens when "most" simplicial maps $s_0 : Y_0 \to X$ do not extend to $Y \supset Y_0$ and the best one can do is to require the existence of "many" extensions of s_0 to a simplicial subdivision Y' of Y, where the restriction map $R'_0 : [Y' \to X] \to [Y_0 \to X]$ becomes measurable.

To make this consistent one needs the following "coherence" property of the measure spaces $[Y \to X]$. Let $Y_1, Y_2 \supset Y_0$ be complexes for which the restriction maps $R_{10} : [Y_1 \to X] \to [Y_0 \to X]$ and $R_{20} : [Y_2 \to X] \to [Y_0 \to X]$ are measurable and let Y be obtained by gluing Y_1 and Y_2 along Y_0. Then the measure space $[Y \to X]$ is equivalent to the fiber product of $[Y_1 \to X]$ and $[Y_2 \to X]$ over $[Y_0 \to X]$.

C. Cohomological width of spaces X over coverings of graphs Y. Let X be a compact topological space and let $F : X \to Y$ be a continuous map such that the F_*-image of the fundamental group $\pi_1(X)$ in the (free) group $\pi_1(Y)$ has at most l-generators. (For instance, $l = 0$ if $H^1(X : \mathbb{Z}) = 0$ and $l \le 1$ if $\pi_1(X)$ is Abelian.) Then F lifts to a map $\tilde{F} : X \to \tilde{Y}$ where $\tilde{Y}$ is a graph (covering Y) which has at most l independent cycles; thus, $\tilde{Y}$ admits a map to a tree, say $\varphi : \tilde{Y} \to \overline{Y}$, where $|\varphi^{-1}(\overline{y})| \le 2l$ for all $\overline{y} \in \overline{Y}$ and then X goes to $\overline{Y}$ by $\overline{F} = \varphi \circ \tilde{F} : X \to \overline{Y}$.

Since $\overline{Y}$ is a tree, the maximum $\overline{m}$ of the H^k-masses $|H^k(X;\mathbb{F})/\overline{F}^{-1}(\overline{y})|$ can be bounded from below by the isoperimetric profile of $A = H^*(X;\mathbb{F})$; then a lower bound on $m = \max_y |H^k(X;\mathbb{F})/F^{-1}(y)|$ follows, for $m \ge \overline{m}/2l$. (This applies, in particular, to X where $H^k(X;\mathbb{F}) \oplus H^{2k}(X;\mathbb{F}) = A^k \oplus A^{2k}$ for $A = A(S^1(N);R)$ and R being a k-sphere.)

In general, with no assumption on $\pi_1(X)$, one has to deal with the map $\tilde{F} : \tilde{X} \to \tilde{Y}$, where $\tilde{Y}$ is the universal covering of Y and $\tilde{X}$ is the induced covering of X with the (free) Galois group $= \pi_1(Y)$.

Is there a non-trivial lower bound on $\max_{\tilde{y}} |H^k(\tilde{X})/\tilde{F}^{-1}(\tilde{y})|$ in terms of $H^*(X)$?

What is the condition on $H^*(X)$ such that every proper map $\tilde{F}$ of every regular covering $\tilde{X}$ of X with a free Galois group to a tree, say $\tilde{F} : \tilde{X} \to \tilde{Y}$ (where $\tilde{Y}$ is not necessarily associated to any Y), satisfies a non-trivial lower bound on $\max_{\tilde{y}} |H^k(\tilde{X})/\tilde{F}^{-1}(\tilde{y})|$?

Can one control the isoperimetric profile of the algebra $H^*(\tilde{X})$ by an appropriate profile of $H^*(X)$ itself?

D. Homological filling. The simplest (co)homological model of an n-cycle c in a space X with the cohomology algebra $A = H^*(X;\mathbb{F})$ is given by a graded algebra $C = C(c)$, a graded homomorphism $h = h(c) : A \to C$ and a linear map $l = l(c) : C \to \mathbb{F}$.

Denote,

$$|c|_{\mathbb{F}} = |C|_{\mathbb{F}}, \quad |A/c|_{\mathbb{F}} = \mathrm{rank}_{\mathbb{F}}(h) \quad \text{and} \quad [c] = l \circ h : A \to \mathbb{F}.$$

We think of c as a representative of the class $[c]$ and introduce the following "norms":

$$|[c]|_{\mathbb{F}} = \inf_{c \in [c]} |c|_{\mathbb{F}} = |C(c)|_{\mathbb{F}} \quad \text{and} \quad |A/[c]|_{\mathbb{F}} = \inf_{c \in [c]} |A/c|_{\mathbb{F}}.$$

The first (apparently easy) question is the evaluation of these "norms"(ranks) on linear maps $[c] : A \to \mathbb{F}$ for particular algebras A, e.g. for the cohomology algebras A of products of Eilenberg–MacLane spaces.

Next, if a "cycle" c has $[c] = 0$, we define a "filling" b of c as an algebra B and a decomposition of $h : A \to C$ into homomorphisms $A \underset{g_1}{\to} B \underset{g_2}{\to} C$. Then we set

$$\|c\|_{\mathrm{fil}} = \inf_b |b|_{\mathbb{F}} = |B|_{\mathbb{F}} \quad \text{and} \quad \|A/c\|_{\mathrm{fil}} = \inf_b |A/b|_{\mathbb{F}} = \mathrm{rank}_{\mathbb{F}}(g_1)$$

where the infima are taken over all "fillings" b of c.

What are the the "filling inequalities" between $|[c]|_{\mathbb{F}}$ and $|A/[c]|_{\mathbb{F}}$ on the one hand and their filling counterparts $\|c\|_{\text{fil}}$ and $\|A/c\|_{\text{fil}}$ on the other for particular algebras A (e.g. for free anticommutative algebras)?

Eventually, we want to find lower bounds on the cohomological width$^*(X/Y)$, say for $Y = \mathbb{R}^m$, by a filling argument similar to that in 2.4, but one needs for this, besides filling inequalities, an appropriate semisimplicial structure in the space of cycles in $A = H^*(X)$. It seems unlikely, however, that this structure can be constructed while remaining within $H^*(X)$, since "gluing fillings across common boundaries" involves (the multiplicative structure on) the relative cohomology that is not contained in the restriction homomorphism alone. In any case, a realistic evaluation of the cohomological width$^n(X/\mathbb{R}^m)$ remains open even for such X as the product of Eilenberg–MacLane spaces.

If $\mathbb{F} = \mathbb{F}_p$, then A comes with an action of the Steenrod algebra and one may insist on C and h being compatible with this action and if $\mathbb{F} = \mathbb{Q}$ one may use the full minimal model of X instead of the cohomology.

If one wants to be "homotopically comprehensive", one takes the set $\mathcal{H}_*(X)$ of the isomorphism classes of objects of the *homotopy category over* X, where the objects are the homotopy classes of (say, polyhedral) spaces P along with homotopy classes of maps $f : P \to X$, where the morphism are the homotopy classes of maps $p_{12} : P_1 \to P_2$, such that the triangular diagrams over X are (homotopically) commutative, and where "$*$" refers to the filtration by $\dim(U) \leq i$. Besides this filtration, $\mathcal{H}_*(X)$ carries a variety of combinatorial (and algebraic) structures (e.g. a partial order) and the corresponding "homotopy masses" take values in (the set of subsets in) $\mathcal{H}_*(X)$ compatible with these structures.

Finally observe that the main difficulty of the higher-codimensional filling is, apparently, due to *non-monotonicity* (see **F.** below) of the filling invariants.

E. Homological moment problem. Given a continuous map $F : X \to Y$ let $X^N_{/Y}$ denotes the N-th fibered product power of X over Y and $(E^N/Y)^* : H^*(X^N) \to H^*(X^N_{/Y})$ the restriction cohomology homomorphism for the embedding E^N/Y of $X^N_{/Y}$ to the Cartesian power X^N for $H^*(\dots) = H^*(\dots;\mathbb{F})$ as usual. Define the Poincaré series in the formal variables t and u by

$$P(X/Y;t,u) = \sum_{n,N} t^n u^N \big|H^n(X^N_{/Y})\big|_{\mathbb{F}}$$

and

$$P(E/Y;t,u) = \sum_{n,N} t^n U^N \operatorname{rank}_{\mathbb{F}}(E^N/Y)^{*=n}.$$

What are recursion relations between the coefficients of these series? Do they admit analytic continuations to $\mathbb{C}^2$ with mild singularities (poles?) for "reasonable" maps F, e.g. for simplicial maps, smooth generic maps, complex analytic maps? When are these series rational functions? Are there inequalities on their coefficients and/or on the values at particular (t,u) refining the inequalities on the cohomological width of X over Y?

This is unclear even for *fibrations* over the figure ∞ if the action of the (free) fundamental group on the cohomology of the fibers is not reductive. On the other hand, the reductive case (where the Zariski closure of the monodromy is reductive) seems easier. For example, the ranks of the spaces of fixed vectors in tensorial N-th powers of representations of reductive groups (which corresponds to H^0 with local systems for coefficients) are well controlled by the Hermann Weyl character formula as was explained to me by Sasha Kirillov.

F. "Quasi-isometry" invariants of graded algebras. Let $A = A_1 \oplus A_2$ by the Stanley–Reisner algebra associated to an infinite connected graph S of degree (at most) d. (This A equals the cohomology of $X_\star(S)$ *with compact supports*.) Which quasi-isometry invariants of S can be expressed in terms of A?

Since S can be reconstructed from the lattice of the graded ideals in A, *every* invariant of S can be read from A; but we want to be more specific than that and to define invariants in the spirit of the isoperimetric profile of A.

For example, given an $A = A_1 \oplus A_2$, consider a graph S with the vertex set $V \subset A_1$ which makes a linear basis in A_1 (or just linearly spans A_1) and where two vertices $v_1, v_2 \in V$ are joined by an edge if and only if $v_1 \smile v_2 \neq 0$. Then, given a *monotone* invariant Inv on graphs (which may only increase if an edge is added), extend Inv to A by taking the infimum of $\mathrm{Inv}(S)$ over all above S.

This allows one, for instance, to define the *growth function* of A and to show that it agrees with the combinatorial one for the Stanley–Reisner algebras. Similarly, one can define and evaluate the *relative growth* (see 2.7), say $|\mathbb{R}\backslash A|$ of Stanley–Reisner algebras A associated to the Cayley graphs of nilpotent groups.

However, most invariants (e.g. Dehn functions) are non-monotone and it remains unclear how much of the (coarse) asymptotic geometry can be extended to algebras. Is there, for example, a working notion of hyperbolicity for A?

G. On equivariant cohomology measures. The notion of the ideal valued mass μ_{H^*} from 4.1 admits an equivariant counterpart where an (infinite-dimensional) X is acted upon by an (infinite) amenable group Γ (e.g. where X is a Cartesian Γ-power, $X = \underline{X}^\Gamma$), where the subsets $U \subset X$ must be Γ-invariant and where the ranks of cohomology groups are taken in a suitable category of Γ-moduli as in [L], [Gr9], [BeG]. (Since Γ is amenable, this may even make sense for not invariant $U \subset X$.) Possibly, some (all?) results of (both parts of) the present paper admit such Γ-generalization.

Another potentially interesting possibility is that of using, instead of cohomology, the Chow ring of cycles in an algebraic variety X or in a symbolic algebraic variety, e.g. in $\underline{X}^\Gamma$, where $\underline{X}$ is algebraic as in [Gr10].

H. On smooth maps. If X is a smooth manifold, the above (co)-homological arguments can be applied to the subalgebra in $H^*(X)$ generated by the characteristic classes, thus providing non-vanishing results for certain characteristic classes of some smooth fibers of generic smooth maps.

If $\dim(X) = 4$ one expects much finer lower bounds on the topology (e.g. maximal genera) of the fibers of generic smooth maps of X to surfaces that would depend on the smooth structures in X, apply to simply connected manifolds (as well as non-simply connected ones) and would be related to the Donaldson–Seiberg–Witten invariants. (Lower bounds in [Gr8] for hyperbolic manifolds rely on $\pi_1(X)$ and do not incorporate the smooth structure, thus, being non-specific for dimension 4.)

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