ON THE NUMBER OF SIMPLEXES OF SUBDIVISIONS OF FINITE COMPLEXES

M. L. Gromov

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Combinatorial invariants of a finite simplicial complex K are considered that are functions of the number $\alpha_i(K)$ of simplexes of dimension i of this complex. The main result is Theorem 2, which gives the necessary and sufficient condition for two complexes K and L to have subdivisions K' and L' such that $\alpha_i(K') = \alpha_i(L')$ for $0 \le i < \infty$. The theorem yields a corollary: if the polyhedra |K| and |L| are homeomorphic, then there exist subdivisions K' and L' such that $\alpha_i(K') = \alpha_i(L')$ for $i \ge 0$.

INTRODUCTION

Let K be a finite simplicial complex, and let $\alpha_i(K)$ be the number of i-dimensional simplexes of K. The problem is which functions of α_i are combinatorial invariants of K. It is well known that the Euler characteristic $\chi(K)$ and the dimension dim K are such functions.

The problem of invariance of functions of the numbers $\alpha_i(K)$ with respect to subdivisions has two aspects. Firstly, for an individual complex K it is possible to study functions of the numbers $\alpha_i(K)$ that are invariant under subdivisions of K. Secondly, it is of interest to ascertain the cases in which we can establish with the aid of such functions that two complexes K and L are not combinatorially equivalent. The latter question can be formulated as follows: When do two complexes K and L have subdivisions K' and L' such that $\alpha_i(K) = \alpha_i(L)$ for all $i \ge 0$?

The first aspect of the problem is examined in [2], which gives for every complex K all the linear functions of $\alpha_i(K)$ that are invariant under subdivisions of K.

To the second part of the problem belongs the following result of Bing [1]: Any two closed triangulated three-dimensional manifolds M_1 and M_2 have subdivisions M_1' and M_2' such that $\alpha_i(M_1') = \alpha_i(M_2')$ for $i \ge 0$.

In the present paper we obtain the necessary and sufficient condition for two complexes K and L to have subdivisions K' and L' such that $\alpha_i(K') = \alpha_i(L')$ for all $i \ge 0$.

In Appendix 1 we formulate an analogous result for the case of more than two complexes.

In Appendix 2 we show how it is possible to obtain by our methods the fundamental theorem of [2].

§ 1. Notations and Formulation of Results

Let K be a finite simplicial complex and |K| the corresponding polyhedron. By s_i we shall denote the i-dimensional simplexes of K, as well as the corresponding open simplexes of |K|. By C(K) we shall denote a space of functions whose arguments are simplexes of K, whereas the values assumed by them belong to the group Z of integers. In the following these functions will be called co-chains. A support supp(c) of a co-chain $c \in C(K)$ is defined as a minimal subcomplex of K such that if $s \overline{\in} \operatorname{supp}(c)$, then $c(s) = 0$. The dimension dim (c) of a co-chain c is defined as the dimension of its support.

Let K' be a subdivision of K. We shall define a natural homomorphism ρ: $C(K) \to C(K')$ as follows: If $s \in K$, $s' \in K'$, and in the polyhedron |K| we have the inclusion $s' \subset s$, then we shall write $\rho c(s') = c(s)$.

If A is a subset of simplexes of K, then ψ_A will denote the characteristic function of the set A.

For every integer i we shall define a homomorphism Δ_i: $C(K) \to C(K)$ by the following conditions: $(\Delta_i(c)) \subset \operatorname{supp}(c)$, and if $s \prec s_p$, then

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$$\Delta_i\psi_{s_p}(s) = (-1)^{p+i+1} + \psi_{s_p}(s).$$

Remark 1. Let us consider the boundary $Bd(s_q)$ of the complement of the simplex s_q in the complex K (see [3]). From the definition of the homorphism Δ_i we obtain by a trivial calculation the formula

$$\Delta_i\psi_K(s_q) = 1 + (-1)^{i+q+1} + (-1)^{i+q}\chi(\mathrm{Bd}(s_q)). \tag{1}$$

In particular, if M is an n-dimensional closed h-manifold, then $\Delta_n\psi_M = 0$.

Remark 2. Any co-chain $c \in C(K)$ can be regarded as a function f(k) in the space $|K|$ by setting $f_2(k) = c(s)$, if $k \in s \subset K$. If we have two triangulations K^1 and K^2 of the space $|K|$, then the co-chains $\Delta_i\psi_{K^1}$ and $\Delta_i\psi_{K^2}$ coincide as being functions in the space $|K|$.

For the proof we must use formula (1) and the invariance of local homology groups.

Let us define the homomorphism χ: $C(K) \to Z$ by the condition $\chi(\psi s_q) = (-1)^q$. It is clear that $\chi(\psi_K) = \chi(K)$. Let us denote by P the space of polynomials of the form $\frac{1}{2}a_0 + \sum_{i=1}^{\infty} a_i x^i$, where the α_i are integers, and let us define the homomorphism χ_0: $C(K) \to P$ by the condition $\chi_0(\psi s_q; x) = x^{q+1}$. It is clear that

$$\chi_0(c; -1) = -\chi(c) \text{ and } \chi_0(\psi_K; x) = \sum_{i=0}^{\infty} \alpha_i(K)\cdot x^{i+1}.$$

In the group C(K) let us define a subset $C^*(K)$ as follows. The zero co-chain belongs to $C^*(K)$, whereas the co-chain c of dimension $n \geq 0$ belongs to the set $C^*(K)$ if and only if there exist simplexes $s_n^1, s_n^2 \in K$ such that $c(s_n^1) > 0$, and $c(s_n^2) < 0$.

Remark 3. Suppose we have two triangulations K^1 and K^2 of the space $|K|$ and two co-chains $c^1 \in C(K^1)$ and $c^2 \in C(K^2)$ that coincide as functions in $|K|$, and let $c^1 \in C^*(K^1)$. Then $c^2 \in C^*(K^2)$.

For the proof we must use the topologic invariance of dimension.

The principal result of the present paper is

THEOREM 1. Let K be a finite complex and let $c \in C(K)$. For a complex K to have a subdivision K' such that $\chi_0(\rho(c)) = 0$, where ρ: $C(K) \to C(K')$, it is necessary and sufficient that the following conditions hold:

$$\left.\begin{array}{c} c \in C^*(K),\ \chi(c) = 0 \\ \text{и если } n = \dim c, \text{ то } \Delta_n(c) \in C^*(K). \end{array}\right\} \tag{*}$$

From Theorem 1 follows

THEOREM 2. For two finite complexes K and L to have subdivisions K' and L' such that $\alpha_i(K') = \alpha_i(L')$ for all i, it is necessary and sufficient that the following conditions hold:

$$\left.\begin{array}{c} \dim K = \dim L = n,\ \chi(K) = \chi(L) \\ \text{and in the space } C(K \cup L) \text{ we have the inclusion} \\ (\Delta_n\psi_K - \Delta_n\psi_L) \in C^*(K \cup L) \end{array}\right\} \tag{**}$$

Proof. Let us apply Theorem 1 to the polyhedron $K \cup L$ and the co-chain $c = (\psi_K - \psi_L)$ $(K \cup L) \in C$. With respect to the co-chain c the conditions (**) are equivalent to the conditions (*). On the other hand,

$$\chi_0(c) = \sum_{i=0}^{\infty} x^{i+1}(\alpha_i(K) - \alpha_i(L))$$

which completes the proof of Theorem 2.

COROLLARY 1. Let us consider complexes K, L, and K_1, L_1 such that the polyhedron $|K|$ is homeomorphic to the polyhedron $|K_1|$, whereas the polyhedron $|L|$ is homeomorphic to the polyhedron $|L_1|$, and let $\alpha_i(K) = \alpha_i(L)$ for all $i \geq 0$. Then the complexes K_1 and L_1 will have subdivisions K_1' and L_1' such that $\alpha_i(K_1') = \alpha_i(L_1')$ for all $i \geq 0$.

For the proof we must use the remarks 2 and 3.

COROLLARY 2. Let us consider two n-dimensional closed h-manifolds M_1 and M_2 such that $\chi(M_1) = \chi(M_2)$. Then they have subdivisions M_1' and M_2' such that $\alpha_i(M_1') = \alpha_i(M_2')$ for all $i \geq 0$.

For the proof it is sufficient to use Remark 1.

COROLLARY 3. Let us consider a complex K of dimension n such that $\Delta_n(\psi_K) \in C^*(K)$ and let L be a complex such that L = dim K and $\chi(L) = \chi(K)$. Then there exist subdivisions K' and L' such that $\alpha_i(K') = \alpha_i(L')$ for all $i \geq 0$.

The proof is evident.

Remark. For any integers q and n such that $n \geq 0$, and if $n = 0$, then $q > 0$, there exists a complex K such that $\chi(K) = q$, dim K = n, and $\Delta_n(\psi_K) \in C^*(K)$.

Proof. If $n = 0$, then we must take K in the form of a complex consisting of q vertices. Let $n > 0$. By K_n we shall denote a complex consisting of three n-dimensional (closed) simplexes, linked along the (n-1)-dimensional boundary. Let L_q be a one-dimensional complex such that $\chi(L_q) = q - 1$. The complex K will be taken in the form of the complex $K_n \cup L_q$. The proof that the complex $K = K_n \cup L_q$ has the desired property is trivial.

The latter remark and Corollary 3 show that (in a certain sense) there do not exist combinatorial invariants, apart from dimension and Euler's characteristic, that are functions of the numbers $\alpha_i(K)$.

§ 2. Lemmas about Subdivisions

For every integral k let us define a homomorphism d_k: P → P by the formula

$$d_k p(x) = p(x) + (-1)^k p(-1 - x).$$

It can be directly verified that

$$d_{k+1} \cdot d_k = 0, \quad d_{k+2} = d_k. \tag{2}$$

In the group P let us define a subgroup P^n by the following condition: $p(x) \in P^n$ if and only if deg $p(x) \leq n$, $d_{n+1} p(x) = 0$, and the value $p(0)$ is an integer.

LEMMA 1. If $p(x) \in P^n$ and $p(0) = 0$, then $p(x) = x(x+1) \cdot q(x)$, where $q(x) \in P^{n-2}$

Proof. Since $d_{n+1} p(x) = 0$, it follows that $p(-1) = p(0) = 0$, and therefore $(x(x+1))^{-1} p(x) \in P$. In this case

$$d_{n-1}((x(x+1))^{-1} p(x)) = (x(x+1))^{-1} p(x) + (-1)^{n+1} (x(x + 1))^{-1} p(-1 - x) = (x(x+1))^{-1} d_{n+1} p(x) = 0.$$

This completes the proof of the lemma.

Let K be a finite complex. By $\chi_1(K)$ we shall denote the polynomial $1 + \sum_{i=0}^{\infty} x^{i+1} \alpha_i(K)$. Let us note that $\chi_1(K; x) = 1 + \chi_0(\psi K; x)$. We shall consider the union $K \circ L$ of two complexes. By a direct calculation we can verify the formula

$$\chi_1(K \circ L) = \chi_1(K) \cdot \chi_1(L). \tag{3}$$

LEMMA 2. Let us consider a complex K, a simplex $s_1 \in K$, a co-chain $c \in C(K)$, and a number a such that for any simplex $s \in \mathrm{st}(s_1)$ we have $c(s) = a$. Let K' be an elementary subdivision (see [3]) of the complex K with respect to the simplex s_1, and suppose that the simplex $s_1' \in K'$ lies in the simplex s_1. Let ρ: C(K) → C(K'). Then

$$\mathrm{Bd}\,(s_1) = \mathrm{Bd}\,(s_1')$$

and

$$\chi_0(\rho(c);\ x) = \chi_0(c; x) + ax(x+1)\,\chi_1(\mathrm{Bd}\,(s_1); x).$$

Proof. The first of the above formulas is evident, whereas the second formula follows from (3).

Let us define the homomorphism χ^*: $C(K) \to P$ by the formula $\chi^*(c;\ x) = \chi^0(c;\ x) + \frac{1}{2}\chi(c)$.

LEMMA 3. Let K' be a subdivision of K, let ρ: $C(K) \to C(K')$ and let $c \in C(K)$. Then:

a) $d_k \chi^*(c) = \chi^* \Delta_k(c)$;

b) $\Delta_{k+1}\Delta_k(c) = 0$, $\Delta_k(c) = \Delta_{k+2}(c)$;

c) $\chi(c) = \chi(\rho(c))$;

d) $\Delta_k \rho(c) = \rho \Delta_k(c)$;

e) if $\rho(c) \in C^*(K')$, then also $c \in C^*(K)$.

Proof. It is sufficient to verify the formulas (a), (b), (c) and (d) for co-chains $c \in C(K)$ of the form $c = \psi_s$, where $s \in K$. It is evident that

$$\chi_0(\Delta_k(\psi_{s_n});\ x) = (-1)^{n+k+1}\sum_{i=0}^{n} c_{n+1}^{i+1} x^{i+1} + x^{n+1} = (-1)^{n+k+1}(1+x)^{n+1} + x^{n+1} + (-1)^{k+n}.$$

Hence

$$\chi^*(\Delta_k(\psi_{s_n});\ x) = x^{n+1} + \frac{(-1)^n}{2} + (-1)^{k+n+1}(1+x)^{n+1} + \frac{(-1)^{k+n}}{2} = d_k\left(x^{n+1} + \frac{(-1)^n}{2}\right) = d_k\chi^*(\psi_{s_n};\ x)$$

and thus we have proved formula (a).

From (a) and (2) follows that $\chi^*\Delta_{k+1}\Delta_k(\psi_s) = 0$, but, on the other hand, if $s_i^1 \prec s$ and $s_i^2 \prec s$, then $\Delta_{k+1}\Delta_k\,\psi_s(s_i^1) = \Delta_{k+1}\Delta_k\,\psi_s(s_i^2)$; therefore we obtain for any $s_i \in K$ the relation $\Delta_{k+1}\Delta_k\,\psi_s(s_i) = 0$. Since the formula $\Delta_k \stackrel{i}{=} \Delta_{k+2}$ is evident, we thus proved (b).

Formula (c) is a consequence of the invariance of the Euler characteristic of complexes under subdivisions.

Let us denote by $|D_n|$ the closure of the simplex s_n in the polyhedron $|K|$, and let D_n' be a subdivision of the complex D_n. By $\beta_i(s)$ we shall denote the number of simplexes $s_i \in D_n'$ such that $s_i \succ s \in D_n'$ and $s_i \overline{\in}\, \partial(D_n')$. Since the polyhedron $|D_n'|$ is an n-dimensional manifold with a boundary $|\partial(D_n')|$ we have

$$\sum_{i=0}^{\infty}(-1)^i\beta_i(s) = (-1)^n.$$

By applying this formula to the case that the subdivision D_n' is induced by a subdivision K' of K, we obtain

$$\Delta_k\rho\psi_{s_n}(s) = \sum_{i=0}^{\infty}(-1)^{k+i+1}\beta_i(s) + \rho\psi_{s_n}(s) = (-1)^{n+k+1} + \rho\psi_{s_n}(s) = \rho\Delta_k\psi_{s_n}(s),$$

thus proving formula (d). For the proof of (e) it is sufficient to refer to Remark 3.

Remark. From formula (a) of Lemma 3 and Remark 1 follows that if M is a closed n-dimensional h-manifold, then $2\chi^*(\psi_M; x) \in P^{n+1}$, whereas by denoting with S^n the n-dimensional sphere and writing $\chi(M) = \chi(S^n)$, we obtain $\chi_1(M) \in P^{n+1}$.

LEMMA 4. Let K_n be a triangulation of the sphere S^n. Then there exist in the complex K_n subdivisions $K_n^1, \ldots, K_n^l$, where $l = \left[\frac{n+3}{2}\right]$, such that the polynomials $\chi_1(K_n^i)$ generate for $1 \le i \le l$ the group P^{n+1}.

Proof. For n = 0, 1 the assertion of the lemma is trivial. Now let $n \geq 2$ and let $K^1_{n-2}, \ldots, K^l_{n-2}$ be subdivisions of the sphere S^{n-2} such that the polynomials $\chi_1(K^i_{n-2})$ generate the group P^{n-1}. It is evident that the complex K has a subdivision K'_n with the following properties: There exist one-dimensional simplexes $s_1^2, s_1^3, \ldots, s_1^{l+1} \in K'_n$ such that for any i = 1, 2, . . . , l the complex $Bd(s_1^{i+1})$ is isomorphic to the complex K^i_{n-2}, and, moreover, $st(s_1^i) \cap st(s_1^j) = \phi$ for $i \neq j$. For i > 1, the complex K^i_n will be taken by us in the form of an elementary subdivision of the complex K'_n with respect to the simplex s^i_1. For i > 1 we obtain by virtue of Lemma 2

$$\chi_1(K^i_n; x) = \chi_1(K'_n; x) + x(x+1)\chi_1(K^{i-1}_{n-2}; x).$$

Since $\chi_1(K'_n; 0) = 1$ and by virtue of Lemma 1 the polynomials $\chi_1(K^i_n; x)$ generate for $1 \leq i \leq l+1$ the group P^{n+1}, we thus proved Lemma 4.

LEMMA 5. Let K be a finite complex and let $c \in C(K)$ with dim c = n. Let

$$\chi(c) = 0, \quad d_n\chi^*(c) = 0. \tag{4}$$

We shall moreover assume that there exist simplexes $s^\varepsilon_n \in K$, ε = 1, 2, such that $c(s^1_n) = v_1 > 0$ and $c(s^2_n) = v_2 < 0$, and that all the coefficients of the polynomial $\chi_0(c; x)$ are divisible by the greatest common divisor v of the numbers v_1 and v_2. Then there exists a subdivision K' of K such that $\chi_0(\rho(c); x) = \chi^*(\rho(c); x) = 0$, where $\rho: C(K) \to C(K')$.

Proof. From Lemma 4 follows that there exists a subdivision K^0 of K with the following properties:

All the "new" simplexes of the complex supp $\rho_0(c)$, where $\rho_0: C(K) \to C(K^0)$, lie in the union $s^1_n \cup s^2_n$;

there exist one-dimensional simplexes $s_1^{i,\varepsilon} \in K^0$, where $1 \leq i \leq l$ and ε = 1, 2 such that $s_1^{i,\varepsilon} \subset e^\varepsilon_h$, with $st(s_1^{i,\varepsilon}) \cap st(s_2^{j,\varepsilon}) = \phi$ for $i \neq j$;

If $B^cd(s)$ denotes the boundary of the complement of the simplex s in the complex $(\rho^0(c))$, then $\chi_1 \cdot (B^cd(s_1^{i,1}); x) = \chi_1(B^cd(s_1^{i,2}); x)$ and the polynomials $\chi_1(Bds_1^{i,1}; x)$ for $1 \leq i \leq l$ generate the group P^{n-1}.

It is clear that $\Delta_n c(s^1_n) = \Delta_n c(s^2_n) = 0$, and hence

$$\chi_0(\rho_0\Delta_n(c); x) = \chi_0(\Delta_n(c); x). \tag{5}$$

By a trivial calculation with the use of formulas (4) and (5) and Assertions (a), (c), and (d) of Lemma 3 we obtain the formula

$$d_n\chi_0(\rho_0(c); x) = 0. \tag{6}$$

It is also clear that all the coefficients of the polynomial $\chi_0(\rho_0(c); x)$ are divisible by v. By virtue of (6) we obtain

$$v^{-1}\chi_0(\rho_0(c); x) \in P^{n+1}. \tag{7}$$

Let us note that $\chi_0(\rho_0(c); 0) = 0$ and let us write $p_i(x) = \chi_1(B^cd(s_1^{i,1}); x) = \chi_1(B^cd(s_1^{i,2}); x)$. From (7) and Lemma 1 follows that there exist nonnegative integers a_i and b_i where $1 \leq i \leq l$, such that

$$\chi_0(\rho_0(c); x) = -\sum_{i=1}^{l}(a_iv_1 + b_iv_2)x(x+1)p_i(x). \tag{8}$$

If L is a finite complex and $s^1, s^2, \ldots, s^k$ are one-dimensional simplexes of L such that $st(s^i) \cap st(s^j) = \phi$ for $i \neq j$, and if $p_1, p_2, \ldots, p_k$ are nonnegative integers, then we shall denote by $L(s^1, p_1, s^2, p_2; \ldots, s_k p_k)$ a subdivision of L, defined by the following conditions: $L(s^1, 0; \ldots; s^k, 0) = L$, and the complex $L(s^1, q_1; \ldots; s, q_i; \ldots; s^k, q_k)$ is obtained by an elementary subdivision of the complex $L(s^1, q_1; \ldots; s^i, q_i; \ldots; s^k, q^k)$ with respect to any one-dimensional simplex contained in the simplex s^i.

Let us now write

$$K' = K^0(s_1^{1,1}, a_1; \ldots; s_1^{l,1}, a_l; s_1^{1,2}, b_1; \ldots; s_1^{l,2}, b_l).$$

From Lemma 2 follows

$$\chi_0(\rho(c); x) = \chi_0(\rho_0(c); x) + \sum_{i=1}^{l} (a_i v_1 + b_i v_2) x (x+1) p_i(x), \tag{9}$$

where $\rho: C(K) \to C(K')$. From formulas (8) and (9) follows that $\chi_0(\rho(c); x) = 0$, but $\chi(\rho(c)) = \chi(c) = 0$, and hence $\chi_0(\rho(c); x) = \chi^*(\rho(c); x)$. This completes the proof of Lemma 5.

LEMMA 6. For any natural number v there exists a homomorphism r_v: $P \to P$ with the following properties:

a) $d_i r_v = r_v d_i$; $\deg r_v p(x) = \deg p(x)$, and if $\deg p(x) > 1$, then the coefficient of the leading term of the polynomial $r_v p(x)$ will be divisible by v;

b) for any finite complex K there exists a subdivision K(v) with a homomorphism ρ_v: $C(K) \to C(K(v))$, such that $\chi_0 \rho_v = r_v \chi_0$.

Proof. Let us define the homomorphism r_v by the following formulas:

$$r_v(a_0 + a_1 x + a_2 x^2) = a_0 + (a_1 + (v-1)a_2)x + v a_2 x^2,$$

and if $n > 2$, then $r_v(x^n) = x r_v((x+1)^n - x^n)$.

If dim K = 0, then we set K(v) = K. Let dim K > 0. We take a one-dimensional skeleton of K and divide every one-dimensional simplex of K into v parts. Then we continue this subdivision over the entire complex K by induction on the skeletons in the following way: The subdivision of the l-dimensional simplex s_l is taken in the form of a cone over a subdivision of the boundary of the simplex s_l with vertex at the center of this simplex [let us note that K(1) = K and that K(2) is a barycentric subdivision].

It is possible to directly verify that the homomorphism r_v and the subdivision K(v), constructed by us, have the desired properties.

COROLLARY. For any natural number v and any complex K there exists a subdivision K_v of K such that the homomorphism $\rho: C(K) \to C(K_v)$ has the following properties: For any co-chain $c \in C(K)$ such that $\chi(c) = 0$ and $d_i \chi^*(c) = 0$ for some i, all the coefficients of the polynomial $\chi_0(\rho(c); x)$ will be divisible by v and $d_i \chi^*(\rho(c); x) = 0$.

Proof. Let us construct the sequence of subdivisions $K^1 = K(v)$, and $K^{i+1} = K^i(v)$. The quantity K_v is taken in the form of a subdivision K^n, where n = dim K. If $\rho: C(K) \to C(K_v)$, it follows from formula (b) of Lemma 6 that $\chi_0 \rho = (r_v)^n \chi_0$, but since $\deg \chi_0(\rho(c); x) \leq n$ and $\chi_0(\rho(c); -1) = \chi\rho(c) = \chi(c) = 0$, the desired property of the polynomial $\chi_0(\rho(c); x) = (r_v)^n \chi_0(c; x)$ will trivially follow from Assertion (a) of Lemma 6.

§ 3. Proof of Theorem 1

If $\chi_0(\rho(c); x) = 0$, then $\rho(c) \in C^*(K')$ and $\chi(\rho(c)) = 0$, so that the necessity of the conditions (*) follows from (c), (d) and (e) of Lemma 3.

Suppose that the co-chain c, which belongs to the group C(K), satisfies the conditions (*), and let dim c = n and dim $\Delta_n(c)$ = m. By setting $b = \Delta_n(c)$ and taking into account that

$$\chi^*(c;\ 0) = -\chi^*(c; -1) = \frac{1}{2}\chi(c) = 0,$$

we obtain

$$\chi(b) = 2\chi^*(b; 0) = 2d_n \chi^*(c; 0) = 0. \tag{10}$$

From formula (2) and (a) and (b) of Lemma 3 follows that

$$d_m \chi^*(b) = 0. \tag{11}$$

Since $b \in C^*(K)$, there exist simplexes s_m^1 and s_m^2 such that $b(s_m^1) = v_1 > 0$ and $b(s_m^2) = v_2 > 0$. Let us denote by v the greatest common divisor of the numbers v_1 and v_2. To the complex K and the number v we shall apply the corollary of Lemma 6 and obtain a subdivision $K_1 = K_V$ with a homomorphism ρ_1: $C(K) \to C(K_1)$. By virtue of (10) and (11) we can apply Lemma 5 to the complex K_1 and the co-chain $\rho_1(b)$; hence there exists a subdivision K_2 with a homomorphism ρ_2: $C(K) \to C(K_2)$ such that

$$\chi^{\cdot}\rho_2(b) = 0.$$

From (a) and (d) of Lemma 3 and from formula (12) we obtain

$$d_n \chi^{\cdot}\rho_2(c) = 0.$$

By using for the co-chain $\rho_2(c)$ the same reasoning as we used above for the co-chain b, we obtain a subdivision K' with the desired property.

APPENDIX 1

THEOREM 2a. Let $K_1, K_2, \ldots, K_p$ be finite complexes. Let $L = K_1 \cup K_2 \cup \ldots \cup K_p$, and suppose that every co-chain of the form $\psi_{K_i} - \psi_{K_j} \in L$ satisfies condition (*). Then there exist subdivisions K'_j of the complexes K_j such that $\alpha_i(K'_{j_1}) = \alpha_i(K'_{j_2})$ for all i and $1 \le j_1, j_2 \le p$.

The proof of this theorem is similar to the proof of Theorem 1.

APPENDIX 2

Let us consider a subgroup $P^{n,m} \subset P$, consisting of polynomials such that deg $p(x) \le n$, whereas deg $d_{n+1}p(x) \le m$ and $p(0) = p(-1) = 0$. Let K be a finite complex and let $c \in C(K)$. Let us consider all possible subdivisions K^i of K and homomorphisms ρ_i: $C(K) \to C(K^i)$; we shall define the subset $P(c) \subset P$ as the set of all polynomials of the form $\chi_0(\rho_i(c); x)$.

THEOREM 1a. Let K be a finite complex and suppose that the co-chain $c \in C(K)$ satisfies the conditions (*), with dim c = n and dim $\Delta_n(c) = m$. Then $P(c) \subset P^{n+1, m+1}$ and in the group $P^{n+1, m+1}$ there exists a subgroup $P^{n+1, m+1}$ of finite index and such that $P^{n+1, m+1} \subset P(c)$.

The proof of this theorem can be obtained by a simple reasoning from Theorem 1 and Lemma 4.

Let us denote by Π a linear space of polynomials with real coefficients and a zero free term. Let K be a finite complex. All the linear functions of the numbers $\alpha_i(K)$, invariant under subdivisions of K, form a subspace $\Pi^*(K)$ of Π^*, where Π^* is the space conjugate to Π. The space $\Pi^*(K)$ is an annihilator of the subspace $\Pi(K) \subset \Pi$ generated by polynomials of the form $p_1(x) - p_2(x)$, where $p_1(x), p_2(x) \in P(\psi_K)$.

In our terminology, the principal result of [2] is formulated as follows.

Let K be a finite complex of dimension n, and let dim $\Delta_n\psi_K = m$. Then the space $\Pi(K)$ will coincide with the linear hull, spanned over the polynomials p(x) belonging to the group $P^{n+1, m+1}$.

This theorem can be proved by applying Theorem 1a to the complex representing the union of two specimens of the complex K, and to a co-chain, equal to 1 on simplexes of one specimen of the complex, and to −1 on simplexes of the other specimen.

LITERATURE CITED

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