Number of Questions.

Misha Gromov

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Contents

Abstract

Nothing is ever settled until it is settled right.
RUDYARD KIPLING

I collect in this paper questions that occupied me during my mathematical life and that have remained unsettled.

Diverse questions are presented in different sections; I have made every effort to make "local reading" everywhere possible: whenever necessary, I repeatedly, almost on every page, remind all relevant definitions and notations rather than referring to previous sections. Also we often discuss the same problem in different contexts in different sections.

Although standing unsolved, many problems we present here have evolved and changed in the course of years, but some old problems still linger before my eyes in the light of my first impression upon coming across at them[1]and my perception of them remained childishly naive.

I do not attempt to be comprehensive – I apologise to those who, unlike myself, seriously and successfully worked on some of these questions.

Motivation, Terminology, References, Conjectures. Most of you find here is an exposition of *known* mathematical phenomena, where our aim is to show how they are immersed into the often invisible ocean of *unknown*.

We try to articulate theorems in *maximally general* terms that would make transparent "free parameters" within these theorems. This allows us to *automatically generate questions* by extending and modifying these "parameters".

What we write is intended as a message from a non-expert author to a non-expert reader. We explain many simple "well known" properties of our mathematical objects starting from the level zero and we try to be as much self-explanatory in our terminology and notation as we can.

For the sake of references, we also reproduce traditional terminology, with concepts and theorems often named after their (presumed) discoverers that are largely unknown to the outsiders of the respective fields.

The manner we refer to our sources is to make easy finding them on the web; only exceptionally we refer to non-freely accessible items.

We formulate certain questions as "conjectures" not out of a deep belief in their validity but because they sound better stated definitely.

The present text will be continuously updated and posted. At the moment, it contains about 15% of the intended material. (Only the sections 1.1 -1.12 and 2.1-2.7 are anywhere close to the final stage.)

I invite readers to communicate their comments to me that I will be happy to put on my web page.[2]

[1]We all know that the feeling of faithful recollection of the thoughts we had in the past is illusory.

[2]Such a comment may be included as a separate item if it comes in pdf and it may be incorporated to the present text with the reference to the author if in latex.

1 Schur, Ramsey, Borsuk-Ulam, Grothendieck, Dvoretzky, Milman.

Issai Schur conjectured[3] in 1916 and Van der Waerden proved in 1927 the following

MONOCHROMATIC PROGRESSION THEOREM.

An arbitrary finite coloring of the set of integers admits arbitrary long ***monochromatic*** *arithmetic progressions.*

(Here and below *finite coloring* of a set S means *partition* of S into finitely many subsets and *monochromatic* stands for *"contained in single one of these subsets"*.)

This was popularised in Russia by Khinchin in his 1947 book *Three Gems of Number Theory* and it was often suggested as a problem*** [4] to school children in "math kruzhoks" as well to the first year undergraduates in the Lenigrad and Moskow Universities.[5]

BABY RAMSEY.

A similar but much easier problem commonly suggested at Leningrad's Mathematics Olimpiades was as follows:

every group of six people always contains three members such that

either *every two are mutually acquainted,*
or *no two of the three are acquainted.*

If you are familiar with the concept of *graph*, you are likely to reformulate this question in terms of 2-colorings of the edges of the *full graph*[6] with six vertices, call them $v_0, v_1, ..., v_5$.

Denote the colors by $\circ$ and $\bullet$ and take three *monochromatic* edges issuing from some vertex, say from v_0.

Since there are *five* edges issuing from v_0, (at least) *three* among them ought to be monochromatic; you may assume, by renaming them if necessary, that these go from v_0 to v_1, v_2 and v_3.

Forget about v_4 and v_5, assume that the common colour of the edges $[v_0, v_1]$, $[v_0, v_2]$ and $[v_0, v_3]$ equals $\circ$ and observe that there are two possibilities of colour patterns of the three edges in the triangle $\triangle(v_1, v_2, v_3)$, both allowing a monochromatic triangle.

$\blacktriangle$ Either *the edges of the triangle* $\triangle(v_1, v_2, v_3)$ *are all* $\bullet$*-coloured,*

$\triangle$ or *one of the three triangles:* $\triangle(v_0, v_1, v_2)$, $\triangle(v_0, v_1, v_3)$, $\triangle(v_0, v_2, v_3)$, *has all its edges* $\circ$*-colored.*

[3] See the *Ramsey Theory* book (2011) edited by Alexander Soifer, Volume 285 of Progress in Mathematics - Birkhuser.

[4] These *** indicate the top level of difficulty. Only one or two of mathematically inclined youngsters in a group of 20-30 were expected to solve such a problem.

[5] Khinchin writes in the introduction that reading his book should stimulate development of young mathematicians. It is hard to say whom he had in mind – I have never met anybody who was able to fight his/her way through Khinchin's notation. Yet, almost all my mathematicians friends who had attended math classes (kruzhki) in Leningrad for high school children worked out, in desperation, the proof of Schur's conjecture by themselves.

[6] *"Full"* means that every two vertices are joined by an edge.

This kind of proof by *focusing different colors*[7] on a single point or on a smallish group of points reappears in many general Ramsey-type settings, starting from the following.

RAMSEY MONOCHROMATIC SUB-SIMPLEX THEOREM.

Let $G = G(V)$ be the full graph on a vertex set V, say, with even number of points, $card(V) = 2N$. If the edges of G are colored into two colors, say our $\circ$ and $\bullet$, then every vertex $v_0 \in V$, has at least N monochromatic edges issuing from it.

Therefore, the sum of the numbers $n_\circ(G)$ and $n_\bullet(G)$ of vertices in the *maximal full monochromatic* subgraphs in G is *greater* (at least) by one than this sum for the full graph on the set of the second ends of N monochromatic edges issuing from v_0. Numerically,

$$n_\circ(2N) + n_\bullet(2N) \geq n_\circ(N) + n_\bullet(N) + 1$$

and consequently,

an edge 2-colored full graph on 2^{2n} vertices contains a full monochromatic subgraph on n vertices.

Then the general (finite) case of Ramsey theorem follows by the same argument:

$[\triangle]_d$ *If the set of d-dimensional faces of the N-simplex Δ^N is k-colored i.e. partitioned into k subsets, then Δ^N contains a* ***monochromatic*** *n-face $\triangle^n \subset \Delta^N$ (i.e all d faces of which lie in one of these monochromatic subsets), provided N is sufficiently large compared to d, k and n.*

Proof. Let $\Delta^M \subset \Delta^N$ be the largest size subsimplex that *contains a given vertex* v of Δ^N and such that all d-faces in this Δ^M that contain v are monochromatic. Observe that, validity of $[\triangle]_{d-1}$ (applied to the set of $(d-1)$-faces in Δ^M that *do not contain* v) implies that $M \to \infty$ for $N \to \infty$.

Denote by $n_i(M,d)$, $i = 1, 2, ...k$, the numbers of vertices in the monochromatic subsimplices in Δ^M of maximal sizes with their d-faces of the i-th colour and observe that

$$n_1(N,d) + n_2(N,d) + ... + n_k(N,d) \geq n_1(M,d) + n_2(M,d) + ... + n_k(M,d) + 1.$$

Since, as we know, $M \to \infty$ for $N \to \infty$ by induction on d, the proof follows by induction on n and delivers

a monochromatic face $\triangle^n$ in the simplex Δ^N with a k-colored set of its d-faces, whenever

$$N \geq \underbrace{f(f(f...(f(nk))...))}_{d} \text{ for } f(x) = x^x.$$

Lower Bound on the Hypergraph Ramsey Numbers. There is multiexponential discrepancy between the *known lower* (like the above) bounds and *upper bounds* on the *maximal* N, such that *some k-colouring* of the d-faces of the simplex Δ^N admits *no monochromatic* Δ^n, where the basic *upper bound*

$$\binom{N+1}{d+1} < k^{\binom{n+1}{d+1}-1},$$

[7] This rather recently introduced "colourful terminology" greatly aids the readers in many current expositions of Ramsey theory.

was obtained in 1947 by Erdös, who pointed out (this is obvious once being stated) that

a randomly chosen k-colouring of the set of d-faces of Δ^N where N satisfies the above inequality admits no monochromatic face $\triangle^n$.[8]

Explanation(?). The essential reason for the gap between the lower and upper bounds on N is, probably, due to the fact that (the probability distribution in) the "random argument" is filly symmetric under the permutation group of the sets of d-faces of simplices Δ^N, while the above construction of monochromatic faces $\triangle^n$ of Δ^N fundamentally depends on the break of this symmetry.

A partition Π of a set S into subsets S_i, $i \in I$, can be seen via the corresponding (quotient) map $f : S \to I = S/\Pi$. Conversely, an arbitrary function $f(s)$ on S defines a partitions of S into the *levels* $S_c = f^{-1}(c) \subset S$, where $f(s) = const = c$; thus, *monochromaticity* of subsets $T \subset S$ translates to *constancy* of f on such T.

Now the following starts looking very much as the Ramsey theorem.

MONOCHROMATIC ORTHONORMAL FRAME THEOREM:
(Kakutani 1942, Yamabe–Yujobo 1950).

Let $f(s)$ be a continuous function on the unit sphere $S^n \subset \mathbb{R}^n$. Then there exists a ***full orthonormal*** *frame $\{s_0, s_1, s_2, ..., s_n\}$ of unit vectors $s_i \in S^n$, such that the function f is* ***constant*** *on this frame:*

$$f(s_0) = f(s_1) = f(s_2) = ... = f(s_n).$$

This goes along with the following purely topological (and obvious by the modern standards)

BORSUK-ULAM MONOCHROMATIC $\mathbb{Z}_p$-ORBIT THEOREM.
(Borsuk 1933, Lusternik-Schnirelmann 1930, Bourgin 1955, Yang 1955.)

Let the cyclic group $\mathbb{Z}_p$ continuously act on an m-connected[9] manifold S, e.g on the n-dimensional sphere S^n with $n > m$, (where, observe, n must be *odd* for $p \neq 2$ if we insist on freedom of this action) and let $f : S \to \mathbb{R}^k$ be a continuous map. Then in the following two cases

the map f is constant on some orbit of this action, i.e. f sends all points[10] *of such an orbit to the same point in $\mathbb{R}^k$.*

(i) $p = 2$ *and* $k \leq m - 1$ (Borsuk 1933, Lusternik-Schnirelmann 1930);

(ii) p *is an odd prime and* $2k(p-1) \leq m$ (Bourgin 1955, Yang 1955.).

Examples. (a) If $S = S^n$ and $k = n$, the above (i) is the original Borsuk-Ulam theorem saying that every continous map $S^n \to \mathbb{R}^n$ brings together a pair of

[8]Noga Alon pointed out to me that a much better lower bound is achieved with a use of the *Erdos and Hajnal stepping up lemma* and its generalisation, see *Hypergraph Ramsey numbers* by David Conlon, Jacob Fox and Benny Sudakov, http://people.maths.ox.ac.uk/ conlond/offdiagonal-hypergraph.pdf

[9]A topological space S is called m-connected if the continuous maps of all m-dimensional polyhedra into it are contractible, where 0-connected = connected and simply connected = 1-connected.

[10]We may assume there are p-points in every orbit, i.e. the action is *free*; otherwise, what we about to say becomes trivial,

opposite points.[11]

(b) Let $St_2(\mathbb{R}^{N+1})$ be *the Stiefel manifold* that is the space of (equatorial) isometric maps from the unit circle S^1 to the sphere $S^N \subset \mathbb{R}^{N+1}$. This manifold is naturally acted by S^1 (that is the group of complex numbers of with one) and; hence, by all cyclic groups $\mathbb{Z}_p$.

Since $St_2(\mathbb{R}^{N+1})$ is $(N-2)$-connected (being a vibration with S^{N-1}-fibers over S^N) the above (ii) implies that every continuous function is constant on some orbit of such a Z_p-action if $2(p-1) \leq N-2$.

In fact, since the Euler class χ of the canonical S^1-fibration of the Stiefel manifold over the Grassmanian of the 2-planes, $St_2(\mathbb{R}^{N+1}) \to Gr_2(\mathbb{R}^{N+1})$, satisfies $\chi^{N-1} \neq 0$, the inequality $p \leq N+1$ suffices for this $St_2(\mathbb{R}^{N+1})$. This implies, in particular, the following

$\pentagon_p$-*Coloring Theorem.* Given a continuous function $f : S^N \to \mathbb{R}$, and a prime number $p \leq N+1$ there exists a regular p-gon $\pentagon_p$ inscribed in some equatorial circle $S^1 \subset S^N$ such that f is constant on the vertex set of this p-gon.[12] .

The logic of the proofs of the topological monochromaticity theorems is well illustrated by the following simple but instructive exampe.

Simplices Monochromatized by Euclidean Translations. Let

$$T = \{s_0, s_1, ..., s_n\} \subset S = \mathbb{R}^n$$

be an $(n+1)$-tuple of points where the n difference vectors vectors $r_i = s_i - s_0 \in \mathbb{R}^n$, $i = 1, 2, ..., n$, are *linearly independent* and let a continuos real function $f : \mathbb{R}^n \to \mathbb{R}$ be asymptotic to the Euclidean norm, that is

$$f(s) - ||s|| \to 0 \text{ for } s \to \infty.$$

Then there exists a vector (parallel translation) $r \in \mathbb{R}^n$ *such that*

$$f(s_0 + r) = f(s_1 + r) = f(s_2 + r) = ... = f(s_n + r).$$

Proof. Let $H_i(f) \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, $i = 0, 1, ..., n+1$, be the r_i-translates of the graph $H_0(f) \subset \mathbb{R}^{n+1}$ of the function f by the above vectors $r_i = s_i - s_0$. (The vectors $r_i \in \mathbb{R}^n$ act on $\mathbb{R}^n \times \mathbb{R}$ by $(s, r) \mapsto (s + r_i, r)$ where we take $r_0 = 0$.) Let us show that

all these graphs meet at some point in $\mathbb{R}^{n+1}$.

This is clear for $f(s) = ||s||$; in fact, the hypersurfaces $H_i(||...||)$ *intersect at a single point*, since the (non-degenerate!) n-simplex with the vertices s_i admits a *unique circumscribed sphere around it.*

It is also clear that this intersection is *transversal*; hence,

the (algebraic) index of intersection between (infinite n-cycles represented by) $H_i(||...||)$ in $\mathbb{R}^{n+1}$ *does not vanish.*

[11]Similar result is proven for maps $S^n \to R$ for all *open* n-dimensional target manifolds R by Akopyan, Karasev and Volovikov in *Borsuk-Ulam type theorems for metric spaces.* The authors also discuss in this paper *Hopf's theorem*: *Every continuous map f of a closed Riemannian n-manifold $X \to \mathbb{R}^n$ admits a geodesic segment in X of a given length l, the two ends of which are brought together by f.*

[12]This theorem, that is obvious by the modern standards, can be seen in 1955 papers by Bourgin's and by Yang, but, likely, this was known before 1955. On the other hand, I am not certain this is known (true?) for all non-prime numbers p.

Since this *index is the same* for all hypersurfaces that are *asymptotic to* $H_i(||...||)$,

arbitrary $n+1$ *hypersyrfaces* $H_i \subset \mathbb{R}^{n+1}$ *asymptotic to* $H_i(||...||)$, $i = 0, 1, ..., n$, *intersect.*

If these hypersurfaces H_i happen to be our graphs $H_i(f)$ their intersection points are pairs $(s, r) \in \mathbb{R}^n \times \mathbb{R}$ such that $f(s + r_i) = r$. Since $s + r_i = s_i + s - s_0$, the translation by $r = s - s_0$ monochromatizes the subset $T = \{s_i\} \subset \mathbb{R}^n$ and the proof is concluded.

Another famous result strikingly similar to the Ramsey simplex coloring theorem came as

Dworetzky's solution of a problem posed by Grothendieck:

VIRTUALLY ROUND SECTION THEOREM

Every infinite dimensional Banach (normed) space X admits a sequence of subspaces (sections) $Y_n \subset X$, $n = 1, 2, 3, ...,$ of given finite dimensions (e.g. all of a given dimension d or of dimensions $d_n = n$) with ***virtually Euclidean*** *geometries for large $n \to \infty$.*[13]

The latter means that there exist Euclidean metrics $dist_{Eucl_n}$ on the spaces Y_n, such that the metrics $dist_{Ban_n}$ on Y_n induced from the Banach metric of X multiplicatevly converge to these Euclidean $dist_{Eucl_n}$:

$$\frac{dist_{Ban_n}(y_1, y_2)}{dist_{Eucl_n}(y_1, y_2)} \to 1 \text{ for all } y_1 \text{ and } y_2 \neq y_1 \text{ in } Y_n \text{ and } n \to \infty,$$

where this convergence is *uniform* on the pairs of points $y_1 \neq y_2 \in Y_n$, i.e. the suprema and the infima of these ratios on (the distinct pairs of points in) Y_n converge to one.

(The existence of almost round 2-dimensional sections trivially follows from the $\pentagon_p$-Coloring Theorem[14] but if $n \geq 3$ there is no apparent topological solution of the Grothendieck problem.[15])

The virtually round section theorem comes especially close to the following geometric corollary to the Ramsey theorem:

Let $\Delta^N \subset \mathbb{R}^N$ be an affine simplex with the length all its edges in the interval $[a, \lambda a]$ and let numbers $\varepsilon > 0$ and $n = 1, 2, 3...$ be given. If N is sufficiently large compared to λ, ε^{-1} and n, then Δ^N contains an ε*-regular* face Δ^n, i.e. where the edges have their lengths in an interval $[b, (1 + \varepsilon)b]$.

Universal Questions.[16]

Is there something deep behind the apparent similarity between the Ramsey, Kakutani and Dworetzky theorems?

[13]If $X = l_p$ with the norm $(\sum_i |x_i|^p)^{1/p}$, this result follows from *Hilbert's symmetrisation lemma* used in his solution of the *Waring-Hurwitz problem*, as we shall explain in section 1.12.

[14]It was pointed out by Milman in *"A few observations on the connections..."* that the approximate roundness of the 2D-sections implied by $\pentagon_p$ is better than what is delivered by other proofs of the Dvoretzky theorem and that such roundness estimates for higher dimensional sections remain problematic.

[15]See section 1.8 and *Topological aspects of the Dvoretzky Theorem* by Burago and Ivanov.

[16]Everyting you meet on the pages of our text, not only Ramsey type results, must be accessed in the light of such questions.

What are ultimate generalizations[17] *of these theorems?*

What are constructions/operations producing new Ramsey-like structures from given ones?

Is there a Grothendieck-style framework embracing these theorems?

Are there "natural extensions" of these theorems that would have non-trivial overlaps between their proofs?

Is there a meaningful classification/clusterization of Ramsey phenomena based on deep invisible structures inherent in them?

Are there "dictionaries" for "transition/translation rules" of Ramsey type properties from combinatorics to geometry, to topology and back?

I have been pondering about these questions ever since I came across the Russian 1964 translation of the Dworetzky article (originally published in 1961). I have not gone anywhere myself and, later on, I had hard time trying to absorb the overwhelming flow of results in the convex geometry around the Grothendieck – Dworetzky's theorem and in Ramsey's style combinatorics projected to other domains, including ergodic theory, model theory and non-linear Fourier analysis.

But despite enormous developments in these domains for the last fifty years, there has been a limited exchange of ideas between different fields[18] that makes a Grothendieck satisfactory solution of these problems more and more problematic. Yet, some hope remains and to allow a balanced judgement we present in the following sections 1.1 – 1.12 a (superficial) overview of Ramsey phenomena in combinatorics, geometry and topology.

1.1 Monochromaticity in Combinatorial Categories, Diagrams and their Actions on Sets.

In order to understand the message carried by the above cited monochromaticity theorems we need a *simple language*,[19] that would define a proper context for these theorems where they could make what we may call "sense". Below is an obvious (naive?) candidate for such a language adapted to the present situation.

Combinatorial Diagrams of Sets and Maps. Such a diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ is understood as a class $\mathcal{S}$ of sets S and maps $\sigma : S_1 \to S_2$ between them, while "combinatorial" or "locally finite" means that these sets S are *finite*.

Binary Diagrams. Such a digram is, by definition, given by maps between two sets, say from E to V; thus for instance, a binary diagram with a *two point set* for E can be visualised as a *directed graph* on the vertex set V.

In most (all?) examples of Ramsey theory these diagrams $(\mathcal{S}, \Sigma)$ are, in fact, *categories of sets*, i.e. the classes Σ of our maps σ contain the *identity maps*

[17]Such a generalisation, beside enlarging classes of objects in question, might allow a significant widening of the concepts of "partition" (colouring) and/or of "monochromaticity".

[18]Many accounts of the Ramsey theory by combinatorialists do not even mention the Kakutani and Grothendieck – Dworetzky theorems. But the conceptual settings of Furstenberg and Katsenelson in their 1989 paper "*Idempotents in compact semigroups and Ramsey theory*" and even more so of Pestov in "*Dynamics of infinite-dimensional groups and Ramsey-type phenomena*" (2005) emphasise a unified view on Ramsey's and the Dworetzky – Milman style theorems.

[19]"Simple" means for us abstract and general, and also free of the venerable load of accumulated tradition.

$id_S : S \to S$ for all $S \in \mathcal{S}$ and also Σ are closed under compositions of maps: $\sigma_2 \circ \sigma_1 : S_1 \to S_3$ for all $S_1 \overset{\sigma_1}{\to} S_2 \overset{\sigma_2}{\to} S_3$.

Actions of Categories on Sets. It is often (always?) convenient[20] to think of σ as of morphisms in a certain abstract category that *act on sets* $S \in \mathcal{S}$, by being *represented by maps* between these sets.

For instance, morphisms σ from a category $\mathcal{S}$ act on sets of morphisms ϕ in $\mathcal{S}$ by left and by right compositions:

$$\sigma_{left} : \phi \mapsto \sigma \circ \phi \text{ and } \sigma_{right} : \phi \mapsto \phi \circ \sigma.$$

Homogeneous Spaces. Ramsey phenomena often (but not always) appear in a highly *symmetric* environment where the diagram $(\mathcal{S}, \Sigma)$ is associated with an action of a set (e.g. a group or a semigroup) G of transformations of another set R, say by maps $g : R \to R$, that together act *transitively* on R. Then one takes the set of all finite subsets $S \subset R$ for $\mathcal{S}$ with the maps σ between them obtained by restricting all g to these subsets. Relevant instances of this are:

• the group $G = aut(F)$ of *all bijective* self mapping of a set F that acts on the set $F_{[d+1]}$ of $(d+1)$*-tuples* of points in F (Ramsey);

• the group $G = aff(\mathbb{R})$ of *affine transformations* of the real line $\mathbb{R}$. (Schur-Van der Waerden);

• the *linear group* $G = GL_n$ acting on the *Grassmannian of linear d-subspaces* in a linear n-space (Rota-Graham-Rothschild).

If a set S is *coloured* i.e. partitioned into some subsets called *monochromatic parts* of S, then a map $\sigma : T \to S$ is *monochromatizing* if its image is *monochromatic*, i.e. it is contained in a single monochromatic part of S.

A (partial) *colouring of a class* $\mathcal{S}$ is defined by colouring of (some) sets $S \in \mathcal{S}$; such a colouring is called *finite* if the numbers of monochromatic parts in all (coloured) S are bounded by some $k < \infty$.

Remark. All sets $S \in \mathcal{S}$ in all (?) diagrams $\mathcal{D}$ of Ramsey theory come on an equal footing and "a colouring of $\mathcal{S}$" refers to *colourings of all* $S \in \mathcal{S}$, unless otherwise stated.

On the other hand, if we speak of a colouring of a *hypergraph* , e.g. of a graph that is represented by a binary diagram $\mathcal{D}$ of edge-maps from a two point set $\{\cdot\cdot\}$ to the graph vertex set V, then V is colored while monochromatization applies to $\{\cdot\cdot\}$.

Ramsey Diagrams and Actions.

Ramsey Σ-Monochromatizing Property. A coloring of a sets & maps diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ is said to satisfy this property if

every set $S \in \mathcal{S}$, *can be monochromitized by a map* $\sigma \in \Sigma$ *from* S *to some set* $S' \in \mathcal{S}$.

It is convenient in some cases to admit *infinite* set $S \in \mathcal{S}$ but require the above only for *finite* subsets in S. Then we speak of Ramsey Σ-monochromatizing property *on finite sets* meaning that *all finite subsets* $S_0 \subset S$, for all $S \in \mathcal{S}$, can be monochromitized by restrictions of maps $\sigma : S \to S'$ to $S_0 \subset S$.

The action of Σ on $\mathcal{S}$ according to a diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ and/or the diagram $\mathcal{D}$ itself are called *Ramsey monochromatizing*, if *all finite colorings* of $\mathcal{S}$ satisfy

[20]Here we follow S. Solecki's *Abstract approach to Ramsey theory and a self-dual Ramsey theorem.*

Ramsey Σ-monochromatizing property and, accordingly, we define the Ramsey monochromatizing property of $\mathcal{D}$ *on finite sets*.

If $\mathcal{S}$ is a *set* rather than a class (often it is a a countable set) then one may take the disjoint union of all $S \in \mathcal{S}$, denoted

$$\mathcal{S}^{\cup} = \bigcup_{S \in \mathcal{S}} S$$

and define the set $\Sigma^{\cup}$ of maps $\mathcal{S}^{\cup} \to \mathcal{S}^{\cup}$ as the corresponding disjoint union of the maps $\sigma \in \Sigma$.

Granted such a set $\mathcal{S}^{\cup}$ with a given collection (usually a semigroup) $\Sigma^{\cup}$ of self maps, the *Ramsey property* for colourings of the set $\mathcal{S}^{\cup}$ can be expressed by requiring the existence of a single monochromatizing map $\sigma : \mathcal{S}^{\cup} \to \mathcal{S}^{\cup}$, $\sigma \in \Sigma^{\cup}$.

Dynamical Interpretation. Colourings κ of $\mathcal{S}^{\cup}$ can be treated as maps $\kappa = \kappa(s^{\cup})$ from $\mathcal{S}^{\cup}$ to a *set of colours*, call it K, where, accordingly, the *space of colouring* κ is denoted $K^{\mathcal{S}^{\cup}}$.

The action of $\Sigma^{\cup}$ on $\mathcal{S}^{\cup}$ induces an obvious (shift) action on $K^{\mathcal{S}^{\cup}}$, where monochromatization of a colouring κ_0, by a map $\sigma : \mathcal{S}^{\cup} \to \mathcal{S}^{\cup}$ translates to $\sigma(\kappa_0) \in K^{\mathcal{S}^{\cup}}$ being *a fixed point* of the action of $\Sigma^{\cup}$ on $K^{\mathcal{S}^{\cup}}$.

And if the sets $S \in \mathcal{S}$ are *finite* and the action of $\Sigma^{\cup}$ on $\mathcal{S}^{\cup}$ is *transitive* (this is a necessary condition for Ramsey), then the existence of a monocromatizing map $\mathcal{S}^{\cup} \to \mathcal{S}^{\cup}$ is equivalent to the existence of such a *fixed point in the closure* of the $\Sigma^{\cup}$-orbit $\Sigma^{\cup}(\kappa_0) \subset K^{\mathcal{S}^{\cup}}$, where "closure" refers to the pointwise stabilisation topology for which the space $K^{\mathcal{S}^{\cup}}$ is *compact*, assuming (which we do) K is a *finite* set.

Five Famous Examples.

1. Injective maps between sets, $\sigma_{1,2} : F_1 \to F_2$, naturally acts on sets $S = S_{[d]}$ of subsets of cardinalities $d+1$ in sets F, for $\sigma_{1,2}$ sending $S_{[d]}(F_1) \to S_{[d]}(F_2)$. Thus, we may say that

the category $\mathcal{F}_{inj}$ of finite sets F and injective maps σ (naturally) acts on the class $\mathcal{S}_{[d]}$ of the sets $S = S_{[d]} = S_{[d]}(F)$.

The Ramsey theorem from the previous section now reads as follow:

Ramsey Monochromatic Sub-Simplex Theorem (1930). *This action of $\mathcal{F}_{inj}$ on $\mathcal{S}_{[d]}$ is Ramsey monochromatizing for all $d = 1,2,3...$.*

This means that

given a finite colouring of $\mathcal{S}_{[d]}$ (i.e. all sets $S \in \mathcal{S}_{[d]}$ are k-colored for some $k < \infty$), every set $S \in \mathcal{S}_{[d]}$, can be monochromatized by a map representing some morphism $\sigma \in \mathcal{F}_{inj}$.

Since injections between sets extend to bijections between larger ones, one can "pack" all of $\mathcal{F}_{inj}$ into a single infinite set Q and reformulate the above as the Ramsey property of the natural action of the group G of bijective transformations of Q on the set $R = Q_{[d+1]}$ of $(d+1)$-tuples of points in Q.

Given a a finite colouring of R, every finite subset $S \subset R$, can be made monochromatic by some bijective transformation $g : Q \to Q$ naturally acting on $R = Q_{[d+1]}$.

Then one visualises $(d+1)$-tuples of points in Q as *d-faces* of the *simplex* $\Delta(Q)$ on the vertex set Q and state the Ramsey theorem geometrically as follows.[21]

if Q is infinite then all finite colourings of d faces of $\Delta(Q)$ admit monochromatic n-faces $\Delta^n \subset \Delta(Q)$ for all (arbitrarily large!) $n \geq d$.

Here, Δ^n is called *monochromatic* if all d-faces in Δ^n are of the same colour.

Infinite Countable Ramsey. A slight modification of the *colour focusing argument* used in the proof of $[\triangle]_d$ from the previous section delivers an *infinite dimensional* monochromatic face $\Delta^\infty \subset \Delta(Q)$, and, thus, shows that

the action of the category $\mathcal{C}_{inj}$ of countable sets C and injective maps σ between them on the class $\mathcal{S}_{[d]}$ of the sets $S = S_{[d]} = S_{[d]}(C)$ of d-tuples of points in sets $C \in \mathcal{C}$ is Ramsey monocromatizing.

For instance, let $d = 1$ and let the edges of the simplex $\Delta(Q)$ on an infinite vertex set Q be finitely coloured. Take a vertex, say $q_1 \in Q_1 = Q$ and observe that the complementary set $Q \setminus \{q_1\}$ contains an infinite subset, denoted Q_2, such that all edges from q_1 to *all* $q \in Q_2$ are of *the same colour*. Apply the same procedure to the simplex $\Delta(Q_2)$ and some vertex $q_2 \in Q_2$, thus, arrive at $Q_3 \subset Q_2 \setminus \{q_2\}$ and continue indefinitely.

This gives you a "*telescope*" of *infinite* strictly descending sequence of *infinite* subsets

$$Q = Q_1 \supset Q_2 \supset ... \supset Q_i \supset ...$$

and a sequence of points $q_i \subset Q_i \setminus Q_{i+1}$ such that the sets E_i of edges from q_i to Q_{i+1} are *monochromatic for all* $i = 1, 2, 3, ...$

Since the number of colours is finite, one of them is present for infinitely many i, say for i_j, $j = 1, 2, 3, ...$, which means that all edge sets E_{i_j} are *of the same colour*. It follows that the simplex $\Delta^\infty \subset \Delta(Q)$ with the vertex set $\{q_{i_j}\}_{j=1,2,3,...}$ is monochromatic since all its edges are contained in the union $\cup_j E_{i_j}$.

Pessimistic Remark. One feels uneasy with this monochromatic $\Delta^\infty \subset \Delta(Q)$, since the above monochromatizing map $\Delta^\star_{[d+1]} \to \Delta^\star_{[d+1]}$ is not defined in the semigroup theoretic language of self mappings $Q \to Q$; this feeling is substantiated by the *Paris–Harrington theorem*:

the following obvious finitery corollary $[!]_{d,k,N}$ of the infinite Ramsey admits no proof in the Peano arithmetic.

Let the vertices of an N-simplex Δ^N be linearly ordered, e.g. let $\Delta^N = \Delta(\{0, 1, 2, ..., N\})$ and let us call a face $\Delta \subset \Delta^N$ *huge* if the *number of vertices* in this Δ is *greater* then the *minimal vertex* $v \in \{1, 2, ..., N\}$ of Δ^N.

$[!]_{d,k,N}$. *If N is sufficiently large compared to given integers $d, k > 0$, then every k-coloring of the set of d-faces of Δ^N admits a huge monochromatic face.*

Another annoying (for some of us) related fact is that the infinite (unlike finite) Ramsey fails, in general, to be true for colorings applied to the set of *all* finite dimensional faces of $\Delta(Q)$, i.e. where the faces of *all* dimensions $d = 0, 1, 2, ...$, are simultaneously colored, and where one seeks (preferably large) faces $\Delta \subset \Delta(Q)$ with monochromatic sets of d-faces in Δ for all d (yet, allowing different colours for different d).

[21] The geometric formulation, besides an obvious terminological superiority of "*d-face*" over "*subset of cardinality* $d + 1$" or "$(d + 1)$-*tuple of points*", may direct you toward something similar for other natural polyhedra.

(I guess, the existence of such an infinite *poly-monochromatic* face Δ for sets Q of large cardinalities in the Zermelo-Fraenkel set theory is known to set theoretic combinatorialists but an intrinsically mathematical formate for this seems unclear.)

2. Let S be an affine space over the field $\mathbb{R}$ of reals (in fact any field of zero characteristic[22] will do) and let H be the set of *homotheties* $h : S \to S$ that are translations $s \mapsto s + s_0$ followed by scalings $s \mapsto \lambda s$ with respect to some point taken for zero in L and for all $\lambda \in \Lambda \subset \mathbb{R}$, where Λ is a *subsemigroup* of the *additive group*[23] of real numbers, $\Lambda \subset \mathbb{R}$, e.g. $\Lambda = \mathbb{N} = \{1, 2, 3, ...\}$.

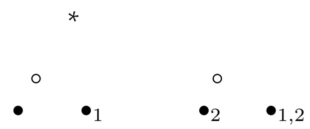
Monochromatizing Homotheties Theorem (Van der Waerden, 1927).[24] *The action of H on S has finite Ramsey monochromatizing property:*

Every finite colouring of $\mathcal{S}$, satisfies H-monochromatizing property on finite subsets in S.

In simple words,

given a finite partition of an affine space S, e.g. of the plane $S = \mathbb{R}^2$, and a finite subset $S_0 \subset S$, there exists a homothetic transformation $h : L \to L$ for which the image $h(S_0) \subset L$ is monochromatic.

$\left(\circ^{*}\ \bullet\right)_4$ *Focusing on "∗".* To grasp the idea of the proof of the existence of monochromatic triples of points in a 2-colored plane $\mathbb{R}^2$, look at the following 7-point configuration with four mutually homothetic triangles in it where the colors are denoted $\circ$ and $\bullet$.



No matter what is the colour at $*$, be it $\circ$ or $\bullet$, one of the triangles will be monochrmatic, since *both colors come into focus at the point* $*$.

We shall explain in the next section how one achieves monochromaticity of the four point "$\bullet$-colored base" in this picture (this is essentially, obvious) and how this colour focusing yields the general case of the theorem, where, observe, the above geometric formulation trivially implies the more traditional form of the Schur – Van der Waerden monochromatic progression theorem where the role of L is played by the group $\mathbb{Z}$.

3. Cartesian Diagrams. A Diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ is called *Cartesian* if

• the class $\mathcal{S}$ is closed under *Cartesian products* of sets: if $S_1, S_2 \in \mathcal{S}$ then also $S_1 \times S_2 \in \mathcal{S}$,

and

• the class Σ is closed under *diagonals* and *Cartesian products* of maps: if $\sigma_1 : T \to S_1$ and $\sigma_2 : T \to S_2$ are in Σ then their (obviously defined) *diagonal* $(\sigma_1, \sigma_2) : T \to S_1 \times S_2$ and the Cartesian product map $\sigma_1 \times \sigma_2 : T \times T \to S_1 \times S_2$ are also in Σ.

[22]If $\mathbb{F}$ has characteristic $p \neq 0$, then what we are about to say will become vacuous.

[23]The *monochromatizing homotheties theorem* stated below fails for *multiplicative* sub-semiproups, e.g. for $\Lambda \subset \mathbb{N}$ equal the multiplicative semigroup of *odd* numbers.

[24]This formulation is sometimes attributed to T. Gallai.

(Usually, one asks for Cartesian products of maps $T_i \to S_i$, $i = 1, 2$ where $T_1 \neq T_2$ and applies the Cartesian condition to categories.)

Affine Example. The category of *linear spaces* L over some field $\mathbb{F}$ and *affine* maps between these sets is an instance of a Cartesian diagram.

Hales-Jewett Block-Diagonal Monochromaticity Theorem (1963). Let $\mathcal{D} = (\mathcal{S}, \Sigma)$ be a locally finite (all sets S are finite) Cartesian diagram that contains the identity maps $id_S : S \to S$ for all $S \in \mathcal{S}$ and the constant maps $T \to s \in S$ for all $S, T \in \mathcal{S}$ and all $s \in S$.

Then the subdiagam $\mathcal{D}_{inj} = (\mathcal{S}, \Sigma_{inj})$ of injective maps in $\mathcal{D}$ is Ramsey: every finite colouring of $(\mathcal{S}, \Sigma_{inj})$ satisfies Ramsey monochromatizing property.

(This means, we recall, that every $S \in \mathcal{S}$ admits a monochromatizing map $\sigma : S \to S'$ for some $S' \in \mathcal{S}$ and an injective $\sigma \in \Sigma$.)

Minimal Examples. Every set F_0 is contained in a (small) category $\mathcal{D}(F_0)$ that is *generated* as a Cartesian category $\mathcal{D} = \mathcal{D}(F_0)$ by this F_0 and the semi-group Σ_0 of all constant maps $F_0 \to F_0$ and the identity map: the sets in the corresponding $\mathcal{S} = \mathcal{S}(F_0)$ are Cartesian powers F_0^X, that are sets of functions $X \to S$ for finite sets X, and where the morphisms are whatever comes from those in Σ_0 by taking diagonals and Cartesian products of maps.[25]

Combinatorial Lines and Block-diagonals. The images of F_0 under non-constant (hence, injective in the present case) maps $F_0 \to F_0^X$ in this category are called *diagonals* or *combinatorial lines* in F_0^X and the images of maps $F_0^Y \to F_0^X$ are sometimes called *block-diagonals.*

The the Hales-Jewett Theorem as it is commonly formulated in these terms applies to

F_0^X *partitioned into* k (monochromatic) *parts*

as follows.

Monochromatic Combinatorial Line Theorem. *If the number $N = card(X)$ is very large compared to k and to the cardinality $card(F_0)$, then F_0^X contains a monochromatic combinatorial line.*

This trivially implies the existence of monochromatizing homotheties from **2**, since suitable finite sets of "diagonal" maps can be realised by homotheties in H.

To be concrete, let $F_0 = \{0, 1, 2, ..., 9\}$. Then, for all $N = 1, 2, ...$, the decimal representation of integers identifies *the N-th Cartesian power* $F_0^N = F_0^{\{1,2,...,N\}}$ (of N-strings in ten digits that may start with zeros, e.g. as 000322096= 322096 for $N = 9$) with *the subset* $\{0, 1, 2, ..., 10^N - 1\} \subset \mathbb{Z}_+$, that is

$$\{0, 1, 2, ..., 9\}^{\{1,2,...,N\}} = \{0, 1, 2, ..., 10^N - 1\} \text{ in writing },$$

where the combinatorial lines in $F_0^{\{1,2,...,N\}}$ become 10-term arithmetic progressions in this subset of integers.

(In fact *all "direct elementary"* proofs of the Van der Waerden theorem are derived, often implicitly, from the combinatorial line theorem.)

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[25]The block-diagonal monochromaticity property in the *"cubes"* $F_0^N = F_0^{\{1,...,N\}}$ for the *two point* set $F_0 = \{0, 1\}$ was already observed by Hilbert in 1892. (See *The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of its Creators* by Alexander Soifer for the history of the Ramsey theory.)

4. Let $\mathbb{F}$ be a field and Σ the category of linear spaces over $\mathbb{F}$ and injective linear maps. This Σ naturally acts on the class $\mathcal{G}_{[d]}$ of Grassman spaces $Gr_d(L)$ of d-dimensional subspaces in linear spaces L.

Grassmanian Monochromaticity Theorem. (Conjectured by Gian-Carlo Rota in 1967$-\varepsilon$ and proved by Graham and Rothschild in 1971.)

The action of Σ on $\mathcal{G}_{[d]}$ satisfies the finite Ramsey monochromatizing property for all $d = 1, 2, ...$

An essential case of this is where the underlying field $\mathbb{F}$ is *finite* and where this theorem says, in effect, that

an arbitrarily finitely colored Grasmanian $Gr_d(\overset{\infty}{L})$ for an infinite dimensional linear space $\overset{\infty}{L}$, contain monochromatic sub-Grassmaninans $Gr_d(L_i) \subset Gr_d(\overset{\infty}{L})$ for some linear subsubspaces $L_i \subset \overset{\infty}{L}$ of all dimensions $i = 1, 2, 3, ...$

(If the field $\mathbb{F}$ is *infinite* this monochromaticity theorem trivially follows from Ramsey property for block-diagonals applied to the category of *affine*, rather than projective, spaces.)

The first novel case that does not automatically reduce to the block diagonal monochromaticity for affine spaces is where we look for a monochromatic projective line l in a k-colored projective space P^N over $\mathbb{F}$ where N is very large compared to k.

By the affine monochromaticity, the complement to a hyperplane, $P^N \smallsetminus P^{N-1} = \mathbb{F}^N$ contain an affine monochromatic subspace $A = \mathbb{F}^M$ of $rank = M \to \infty$ for $N \to \infty$. If no projective line l in the projective completion $P^M \supset A$ that meets $P^{M-1} = P^M \supset A$ at single point is monochromatic, then P^{M-1} is coloured by $k - 1$ colores and the obvious induction in k applies.

This kind of colour focusing in *Schubert's decompositions* scales up to the proof of the general case of Rota's conjecture for Grassmanians, similarly to how it works for the Ramsey's monochromatic faces in simplices in $[\triangle]_d$ in the previous section. (I learned this in a conversation with Joel Spencer around 1977.)

QUESTION. What are other linear (classical?) groups G besides $GL(\infty)$ (as it acts on on $Gr_d(\overset{\infty}{L})$) and their actions on homogeneous spaces S that monichromitize finite subsets in S? For example, is it true for the groups of orthogonal and/or symplectic transformations when they act on the spaces of isotropic subspaces, or such as $O(\infty, 1)$ that acts on the space of timelines in the case $\mathbb{F} = \mathbb{R}$?

Flag Coloring Problem. Ramsey monochromatization, as it stands, fails for spaces of *flags* of linear subspaces and it is unclear what the correct formulation (that would not trivially follow from Ramsey for Grassmannians) should be.

5. A semigroup S is called *idempotant* if $s \cdot s = s$ for all $s \in S$. Such an S is called called *free commutative*, if it is isomorphic to the semigroup of finite subsets of some set with the union of subsets taken for the product.

Observe that subsemigroups generated by collections of disjoint subsets in the full semigroup of of subsets are free.

Hindman's Monochromatic Finite Sums Theorem (1974). *The category of free commutative idempotent semigroups and their monomorphisms is Ramsey. Moreover, an arbitrarily k-colored infinite countable free commuta-*

tive idempotent semigroup admits a self-monomorphism with a monocromatic image.

The natural question, that has been raised many times but has hardly been fully answered, reads:

What are other categories of semigroups and/or of other algebraic objects with operations that have Ramsey (or similar to Ramsey) properties?

1.2 Products, Colors, Focuses, Telescopes, Selfsimilarity.

Let us spell down in words the message conveyed by the above 7-point configuration $\left(\circ^{*}\ \bullet\right)_4$ with four triangles "focused" on one point $*$ and thus prove the Hales-Jewett combinatorial line and the block-diagonal monochromaticity theorems.

We proceed in two steps (see **1** and **2** below) that mimic those in the 1927 proof by Van der Waerden in his solution of Schur monochromatic progression conjecture and that are present in most later expositions.

(The two steps logic of the original 1927 paper is highlighted by Van der Waerden in his account of the history of the solution of the Schur conjecture[26] that came up in conversations he had with E. Artin and O. Schreier.[27] Not until Shelah's 1988 paper a novel direct elementary argument became available.)

Ramsey Diagrams. Recall that a diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ of sets $S \in \mathcal{S}$ and maps $\sigma : S_1 \to S_2$ is *Ramsey* if all finite colouring of $\mathcal{S}$, that are colouring of the sets $S \in \mathcal{S}$ into $k < \infty$ colours, have the Ramsey Σ-monochromatizing property, i.e. every set $S \in \mathcal{S}$, can be monochromotized by some map $\sigma : S \to T$, $\sigma \in \Sigma$.

Cartesian terminology.

The Cartesian product $\mathcal{S}_1 \times \mathcal{S}_2$ of two classes of sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is the class of the Cartesian product $S_1 \times S_2$ for all constituent sets $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$.

The Cartesian product $\mathcal{D}_1 \times \mathcal{D}_2 = (\mathcal{S}_1, \Sigma_1) \times (\mathcal{S}_2, \Sigma_2)$ *of two sets & maps diagrams* $(\mathcal{S}_1, \Sigma_1)$ and $(\mathcal{S}_2, \Sigma_2)$ consists of the Cartesian products of their constituent sets along with Cartesian products of maps, say $\sigma_1 \times \sigma_2 : S_1 \times S_2 \to T_1 \times T_2$, for all $\sigma_1 \in \Sigma_1$, and and $\sigma_2 \in \Sigma_2$.

1. Ramsey Product Property.[28]

Cartesian Products of locally finite Ramsey sets & maps diagrams,

$$\mathcal{D}_1 \times \mathcal{D}_2 = (\mathcal{S}_1, \Sigma_1) \times (\mathcal{S}_2, \Sigma_2) = (\mathcal{S}_1 \times \mathcal{S}_2, \Sigma_1 \times \Sigma_2)$$

are Ramsey.

(Recall that an $\mathcal{S}$ is "locally finite" if all sets $S \in \mathcal{S}$ are finite.)

Proof. If a *finite* set S_2 from an arbitrarily k-colored class $\mathcal{S}_2$ for $k < \infty$ can be monochromotized by some map $\sigma_2 \in \Sigma_2$ to another *finite* set $T_2 \in \mathcal{S}_2$ that a priori depends on the colouring, then, by an obvious "assume otherwise" argument, there exists a *single* finite set $T_2^* = T_2(S_2) \in \mathcal{S}_2$ such that *every* k-coloring of the class $\mathcal{S}_2$ admits a monocromatizing map $\sigma_2^* \in \Sigma_2$ from S_2 to T_2^*.

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[26] According to Van der Waerden he learnt this conjecture from P. J. H. Baudet.

[27] This is reproduced in *Mathematical Coloring Book* by A. Soifer, 2009.

[28] This property accounts for the existence of the monochromatic bottom $\bullet\ \ \bullet_1\ \ \bullet_2\ \ \bullet_{1.2}$ in $\left(\circ^{*}\ \bullet\right)_4$ from section 1. 1.

Therefore, every finite coloring of the class

$$\mathcal{S}_1^* =_{def} \mathcal{S}_1 \times T_2^* \subset \mathcal{S}_1 \times \mathcal{S}_2$$

of the sets $S_1^* = S_1 \times T_2^*$, $S_1 \in \mathcal{S}_1$, say into k colors, defines an *also finite* colouring of $\mathcal{S}_1 = \mathcal{S}_1^*$, namely, into $K^* = k^{card(T_2^*)}$ colors, where colorings of the "T_2^*-slices" $T_2^* = s_1 \times T_2^* \subset S_1 \times T_2^*$ (that are the fibers of the coordinate projections $S = S_1 \times T_2^* \to S_1$) serve as the colours of the underlying points $s_1 \in S_1$.

Thus, the Cartesian products $\sigma_1 \times \sigma_2^*$ of monocromatizations $\sigma_1 \in \Sigma_1$ of sets in the K^*-coloured class $\mathcal{S}$ with the above σ_2^* provide monocromatizations of the product sets $S_1 \times S_2 \in \mathcal{S}_1 \times \mathcal{S}_2$.

Remarks. The above argument apples to all, possibly infinite, sets S_2; thus the Ramsey property for Cartesian product of two diagrams needs local finiteness of only one of the two factors.

Furthermore, in view of the *Löwenhei-Skolem-De Bruij-Erdös compactness theorem*[29], this argument also applies if the *receiving sets* $T_2 \in \mathcal{S}$ for monochromatizing maps of (finite!) $S_2 \in \mathcal{S}$ are *infinite.* This shows that

(Mubayi&Rödl, 2004). The *chromatic number* of the Cartesian products of *hypergraphs* with *infinite* chromatic numbers is *infinite.*

(See http://www.math.cmu.edu/ mubayi/papers/bergesimon.pdf.)

The asymmetry of the two factors in the proof of the product property seems to be related to asymmetry of the tensorial products of ultrafilters (appearing as $u_1 \otimes u_2$ on $S_1 \times S_2$ in section 1.3). This asymmetry and lost of precision seem, in general, unavoidable; yet, the product property can be probably proven symmetrically and quantitatively effectively in specific examples.

2. Self Similar Focal Decompositions.[30] Start with the following.

Motivating Example. Let S be a projective space over some field, let $S' \subset S$ be a hyperplane and $S^- = S \setminus S'$ be the complementary affine space. Then all points in S' serve as "terminals" or *focal points* of affine lines from S^-; or one may say that affine lines from S^- *focus* on all points in S'.

We use this as a model for the following general definition applied to a diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ of sets S and maps σ between them as follows.

• Let $\mathcal{D}'$ and $\mathcal{D}^-$ be two *subdiagams* in $\mathcal{D}$.

Here, "subdiagram", say $\mathcal{E} = (\mathcal{T}, \Omega)$ in $\mathcal{D}$ is, by definition, a collection (class) of subsets $T \subset S \in \mathcal{S}$ and maps ω between them such that

[·] each $S \in \mathcal{S}$ contains at most one (possibly none) $T \in \mathcal{T}$;

[··] every map $\omega : T_1 \to T_2$, $T_i \subset S_i$, $i = 1, 2$, $\omega \in \Omega$, comes from $\mathcal{D}$, i.e. it equals the *restriction* of some map $\sigma : S_1 \to S_2$ from Σ to the subset T_1, such that $\sigma(T_1) \subset T_2$.

• Let $S_0 \in \mathcal{S}$ be a set decomposed into two non-empty subsets,

$$S_0 = S_0^- \cup S_0' \text{ for } S^- \in \mathcal{S}- \text{ and } S_0' \in \mathcal{S}'.$$

[29] This, essentially obvious, theorem says that every first order language expressible property of relations on a *well-ordered* set S is already seen on finite subsets of S. This is commonly applied to *all* sets as these can be well ordered in the *Zermel-Fraenkel set theory*.

[30] Such decompositions formalise the colour focusing on $*$ property in $\begin{pmatrix} * & \\ \circ & \bullet \end{pmatrix}_4$ from section 1. 1.

Focusing and Self Similarity. Say that the subdiagram

$\mathcal{D}'$ *lies fully in the* S_0*-focus of* $\mathcal{D}^-$ and/or that $\mathcal{D}^-$ *focus on all of* $\mathcal{D}'$,

if all maps $\sigma' : S'_0 \to S'$, for all $S' \in \mathcal{S}'$, extend to maps $\sigma : S_0 \to S \supset S'$, that send the subset $S_0^- \subset S$ to $S^- \subset S$.

Call a pair of subdiagrams $\mathcal{D}^-$ and $\mathcal{D}'$ in $\mathcal{D}$

a self similar S_0*-focal decomposition of* $\mathcal{D}$

if

the diagram $\mathcal{D}'$ is *isomorphic to* $\mathcal{D}$ *and it lies fully in the* S_0*-focus of* $\mathcal{D}^-$,

where, for the terminological simplicity sake, we assume at this point that $\mathcal{S}$ is a *set* (rather than a class) of sets that makes the expression *isomorphism between* $\mathcal{D}$ *and* $\mathcal{D}'$ acceptable.

Also notice, that despite the use of the word "decomposition" we *do not* require that $S = S' \cup S^-$.

Below are three standard examples of such decompositions, where the self similarity and the focality properties are seen immediately.

Example 1. *Self similar focal decomposition of the category of projective spaces* S *over a field* $\mathbb{F}$ *and projective embeddings* $\sigma : S_1 \to S_2$ *between these spaces.*

Decompose each S as $S^- \cup S'$ for a hyperplane $S' \subset S$ and $S^- = S \setminus S'$, take some so decomposed projective spaces for S_0 and let the corresponding $\mathcal{D}^-$ and $\mathcal{D}'$ via projective embeddings $S'_1 \to S'_2$ and affine maps $S_1^- \to S_2^-$.

Example 2. *Self similar focal decomposition decomposition of the category the sets* $S = \Delta(Q)_{[d]}$ *of* $(d-1)$*-faces of simplices* $\Delta = \Delta(Q)$ *on vertex sets* Q *with the maps* $\sigma : S_1 \to S_2$ *induced by injective maps* $Q_1 \to Q_2$.

Take a point $q \in Q$ in each set Q, decompose every S into the $S^- \subset S$ of those $(d-1)$-faces that *contain* q and $S' \subset S$ being the set of faces that *do not contain* q, take some of so decomposed sets of faces for S_0 and make the diagramms of these sets by restricting to them the maps $\sigma : S_1 \to S_2$

Example 3. *Self similar focal decomposition decomposition of Cartesian categories.*

Let S_0 be a set and let $\mathcal{D} = \mathcal{D}(S_0)$ be the Cartesian category generated by the semigroups of the constant maps $S_0 \to s_0 \in S_0$ and of the identity map $S_0 \to S_0$.

This means that $\mathcal{D} = (\mathcal{S}, \Sigma)$ where $\mathcal{S}$ is the class of the (Cartesian powers) sets $S = S_0^X$, for all sets X, and Σ be the class of maps $S_1 \to S_2$ that is block diagonally generated by S_0, that is the minimal class of maps that contains Σ_0 and that is closed under the diagonal maps $S \to S^X$ and the Cartesian product of maps.

(Recall, that S_0^X is the space of functions $X \to S_0$ and that the diagonal map $S \to S^X$ sends the points $s_0 \in S_0$ to the *constant* functions $s(x) = s_0$.)

Let $S'_0 \subset S_0$ consist of a single point, say $S'_0 = \{s_*\}$, $s_* \in S_0$, let S^- equal the complement $S_0 \setminus S'_0$ and let $\mathcal{S}^-$ consist of the Cartesian powers $S^- = (S_0^-)^X \subset S = S_0^X$.

Finally, to define, $\mathcal{S}'$, choose points $x_* \in X$ – a single point in each set X – take their complements $X' = X \setminus \{x_*\}$ and let $S' = S_0^{X'}$, where every such S' is embedded into $S = S_0^X$ by extending functions $s(x)$ from X' to $X \supset X'$ by

$s(x_*) = s_*$; conclude the description of $\mathcal{D}'$ and $\mathcal{D}^-$ in $\mathcal{D} = (\mathcal{S}, \Sigma)$ by defining the maps in these subdiagrams as restrictions of the maps $\sigma \in \Sigma$.

Let us formulate and proof a simple property of our decompositions that, when applied to the above Example 3, delivers the (standard) proof of the Hales-Jewett line and block-diagonal monochromaticity theorems.

Telescoping Decompositions. Let $(\mathcal{D}', \mathcal{D}^-)$ be a self similar S_0-focal decomposition of a diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ and let $\Psi = \{\Psi_S : S \to S\}_{S \in \mathcal{S}}$, be an endomorphism of $\mathcal{D}$ that implements an isomorphism $\mathcal{D} \to \mathcal{D}'$. Thus, these Ψ_S injectively map the sets S onto $S' \subset S$ for all $S \in \mathcal{S}$.

Define a descending sequenece of subsets in $S \in \mathcal{S}$,

$$S^{(0)} = S \supset S^{(1)} \supset S^{(2)}, ..., \supset S^{(i)}, ...$$

and subsets

$$S^{-i} \subset S^{(i)}$$

by letting

$$S^{(1)} = S' = \Psi_S(S),\ S^{(2)} = S'' = \Psi_S(S'),\ \dots\ , S^{(i)} = \Psi_S\left(S^{(i-1)}\right)$$

and, similarly, let

$$S^{-0} = S^- \subset S = S^{(0)},\ S^{-1} = \Psi_S\left(S^{-0}\right) \subset S^{(1)},\ \dots\ , S^{-i} = \Psi_S(S^{-i-1}) \subset S^{(i)}.$$

Clearly, the corresponding subdiagrams $\mathcal{D}^{-j} = \Psi^{\circ j}(\mathcal{D}^-)$ in $\mathcal{D}^{(j)}$, for $\Psi^{\circ j}$ denoting the j-th iterate $\Psi \circ \Psi \circ ... \circ \Psi$, are *$S_0$-focused on all of $\mathcal{D}^{(i)}$ for all i and $j = 0, 1, ..., i-1$.* In particular,

$\mathcal{D}^{-i}$ fully lies in the S_0-focus of $\mathcal{D}^{-j}$ for all $j < i$.

Now let $\mathcal{S}$ be finitely colored and let the sets $\mathcal{S}^{-j}$ can be made monochromatic by applying to them some maps $\sigma \in \Sigma$, where the essential example is where $\mathcal{D}^-$ *is Ramsey.*

To save notation, we assume that these sets are monochromatic to start with, and if the number of them is greater than the number of colours– this can be always achieved for finite coloringds– then two of these sets, say $\mathcal{S}^{-j}$ and $\mathcal{S}^{-i}$, $i > j$ will be of the same colour.

It follows that if the subset $F_0' \subset F_0$ admits a map $\sigma' : F_0' \to \mathcal{S}^{-i}$, $\sigma' \in \Sigma'$, then, because of the focusing property, this σ' extends to a map $\sigma \in \Sigma$ from F_0 to the union $\mathcal{S}^{-j} \cup \mathcal{S}^{-i} \subset S^{(i)} \in \mathcal{S}$, where this σ is *monochromatizing* since $\mathcal{S}^{-j}$ $\mathcal{S}^{-i}$ are monochromatic with same colour.

This leads to the following

Lemma. Let $(\mathcal{D}', \mathcal{D}^-)$ be a self similar S_0-focal decomposition of a diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ where $\mathcal{D}^-$ is Ramsey.

If the subset $S_0' \subset S_0 \in \mathcal{S}$ consists of a single point, call it $s_* \in S_0$, and if all points $s' \in S'$ come from s_* by maps $\sigma' \in \Sigma'$, for all $S' \in \mathcal{S}'$, i.e. $s' = \sigma'(s_*)$ for all $s' \in S'$ and some σ' depending on s', then F_0 can be monochomatized by some map $\sigma \in \Sigma$.

Indeed the above map $\sigma' : F_0' \to \mathcal{S}^{-i}$ trivially exits in tho case.

Let us apply this to Cartesian categories "generated" by sets S_0, denoted $\mathcal{D} = \mathcal{D}(S_0) = (\mathcal{S} = \{S_0^X\}, \Sigma)$ as in the above Example 3 and conclude:

The Ramsey property for the *Cartesian category* $\mathcal{D}^- = \mathcal{D}(F_0^-)$ for $F_0^- = F_0 \smallsetminus \{s_*\}$, $s_* \in S_0$ – that is the validity of Hales-Jewett *block-diagonal* monochromaticity theorem for $\mathcal{D}^-$ – implies *the existence of monochromatic combinatorial lines* for all finite colouring of $\mathcal{S}$ that is the validity of the monochromatic *combinatorial line* theorem in $\mathcal{D} = \mathcal{D}(S_0)$.

On the other hand, if the set F_0 is finite, then *Ramsey* for $\mathcal{D}(F_0)$ follows from the existence of monochromatic combinatorial lines by the *Ramsey product property* (see the above **1**). Then the Hales-Jewett block-diagonal monochromaticity theorem for $\mathcal{D}(F_0)$ follows by induction on the cardinality $card(F_0)$.

Recollection and Discussion. Around 1961-1962, three of us, then undergraduates at the Leningrad University – Emmik Gerlovin[31], Yura Ionin.[32] and myself – were competing for finding the simplest possible proof of the Van der Waerden theorem.

To our surprise, our seemingly different arguments converged to the above $\left(\circ^{*}\ \bullet\right)_4$ picture, that translates to

Cartesian products + telescoping focusing of colors

in words.

We were frustrated as none of us could find a proof that would retain the *inherent symmetry* of the Schur problem, e.g. by a *computational density* argument.[33]

We were further discouraged by number theorists around Yuri Vladimirovich Linnik who were not impressed by the following "Diophantine approximation" rendition of the Van der Waerden theorem.

$(*)$ *Let $U \subset \mathbb{R}^n$ be a Borel subset and $S_0 \subset \mathbb{R}^n$ a finite set. If the intersections of U with all unit Eucildean balls in $\mathbb{R}^n$ have their Lebesgue measures$\geq \varepsilon > 0$, than S_0 can be moved to U by an integer homothety of the Euclidean space $\mathbb{R}^n$: there exists an integer $N > 0$ and a vector $t \in \mathbb{R}^n$, such that $N \cdot S_0 + t \subset U$.*

We were instructed that such a result is not number theoretically acceptable unless it is accompanied by a good effective estimate on $N = N(\varepsilon)$ comparable to that for the classical Dirichlet's $N \sim 1/\varepsilon$ for $U = U_\varepsilon \subset \mathbb{R}$ that is the set of points $\{i + \delta\}$ for all integer i and $0 \leq \delta \leq \varepsilon$ and where S_0 is a two point set, say $S_0 = \{0, \alpha\}$ for an irrational number α.

But we realised to our dismay that the logic of proof of the Ramsey product property (see the above **1**) precludes anything that can be called effective.

An effective bound on N was found only in 1988 by Saharon Shelah who

[31]Emmanuel Gerlovin, worked on the development of computer aided design products, including Pro/ENGINEER. He served as Vice President of Geometry Software at Exa and a Senior Vice President of Advanced Design at Parametric Technology Corporation. He died on August 16, 2012.

[32]Yury J. Ionin spent 25 years of his life teaching mathematically gifted youngsters in Leningrad. From about 1990 until 2007 he held a position of a professor at Central Michigan University. Together with Mohan Shrikhande they wrote the book *Combinatorics of Symmetric Designs*.

[33]The existence of *arbitrary long arithmetic progressions in subsets of positive density* in integers was eventually proven by E. Szeremédi in 1975 following partial results by K. Rot for *triple progressions* obtained by *Fourier-Hardy-Littlewood analysis* (1954) and for progressions of length four by Szemerédi (1969) and Rot (1972). The corresponding density version of the Hales-Jewett combinatorial line theorem was proven by Furstenberg and Katznelson in 1991 by an *ergodic-theoretic method* and a combinatorial proof of this was furnished by D.H.J. Polymath in 2010.

shifted the center of gravity of the above argument from products toward the colour focusing. In fact he established a primitive recursive, but still horrendous, bound on $N = N(card(S_0), k)$, such that

every k-colouring of the Cartesian power $S_0^{\{1,...,N\}}$ contains a monochromatic combinatorial line.

Question. Can one translate Shelah's argument to our language similarly to how it is done by Solecki in his "*Abstract approach to finite Ramsey theory....*"?

The Shelah bound was greatly improved in the Schur - Van der Waerden case, that is for $U \subset \mathbb{Z}$ and $F_0 = \{1, 2, ..., n\}$, by Timothy Gowers, who using his Fourier theoretic method, established in 2001 the estimate $N \sim \exp\exp C$ with C reasonably bounded in terms $1/\varepsilon$ and $n = card(F_0)$.

In fact, Gowers proved this for all subsets $U \subset \mathbb{Z}$ with *asymptotic densities*[34] at least ε, thus delivering an effective sharpening of *Szeremédi's theorem* on the existence of

arbitrary long arithmetic progressions in subsets of integers with positive densities.

1.3 Diagram Construction and Block Schubert Decomposition.

A variety of Ramsey and Ramsey-like structures as well as of general constructions of such structures appear in:

Ramsey's Theorem for a Class of Categories, Graham, Leeb and Rothschild, (1972),

Some unifying principles in Ramsey theory, Carlson (1988),

Idempotents in compact semigroups and Ramsey theory, Furstenberg and Katznelson (1989),

Dynamics of infinite-dimensional groups and Ramsey-type phenomena, Pestov (2005),

Introduction to Ramsey Spaces, Todorcevic (2010),

Ultrafilters, IP sets, Dynamics, and Combinatorial Number Theory, Bergelson (2010),

Density Hales-Jewett Theorem for matroids, Geelen and Nelson (2012),

Abstract approach to finite Ramsey theory and a self-dual Ramsey theorem, Solecki (2013),

Some recent results in Ramsey Theory, Dodos (2013),

....

....

Below we present some of these constructions in a purely categorical language.

Actions on Arrows. Every category $\mathcal{C}$ acts on sets of its own morphisms by the left and right compositions, where the two action commute. Thus, the right action of $\mathcal{C}$ on the sets $Hom(C_0 \to C)$ (that is composition of morphisms $C_0 \to C$ from $Hom(C_0 \to C)$ with morphisms $C \to C'$) factored by the group

[34] Gowers proof is tersely written on ≈100 pages, while a detailed proof of seemingly similar (∗) needs a single page.

$aut(C_0)$ of automorphism of C_0 defines an action on

$$\text{the class } \mathcal{S}_{\uparrow C_0} \text{ of sets } Hom(C_0 \to C)/aut(C_0) \text{ for all } C \in \mathcal{C}.$$

Similarly, $\mathcal{C}$ acts from the left on

$$\text{the class } \mathcal{S}_{C_0\downarrow} \text{ of sets } Hom(C \to C_0)/aut(C_0).$$

EXAMPLES. The standard Ramsey-type theorems can be reformulated in this language as follows.

A: *Classical Finite Ramsey.* Let $\mathcal{C}$ equal the category of finite sets and *injective* maps. Then

the action of $\mathcal{C}$ on $\mathcal{S}_{\uparrow C_0}$ satisfies the Ramsey monochromatizing property for all $C_0 \in \mathcal{C}$.

A*: *Dual Ramsey* (Graham-Rothschild)[35]. Let $\mathcal{C}$ equal the category of finite sets and *surjective* maps. Then

the action of $\mathcal{C}$ on $\mathcal{S}_{C_0\downarrow}$ satisfies Ramsey monochromatizing property for all $C_0 \in \mathcal{C}$.

B: *Graham-Rothschild Grassmanian Monochromaticity Theorem.* Let $\mathcal{C}$ equal the category of finite dimensional linear spaces over a finite field and injective linear maps. Then

the action of $\mathcal{C}$ on $\mathcal{S}_{\uparrow C_0}$ satisfies the Ramsey monochromatizing property for all $C_0 \in \mathcal{C}$.

C: *Graham-Rothschild Monochromaticity Theorem for Block Diagonal Grassmannians.*[36] Let $\mathcal{D}$ be the Cartesian category generated by the constant maps and by the identity map of a finite set and let $\mathcal{C}$ be the subcategory of injective maps in $\mathcal{D}$.

Then

the action of $\mathcal{C}$ on $\mathcal{S}_{\uparrow C_0}$ satisfies the Ramsey monochromatizing property for all $C_0 \in \mathcal{C}$.

Questions. The above "arrow construction" generalises to graphical diagarms also called *quivers* Γ of arrows in $\mathcal{C}$, where there are particular natural subcategories in such $\mathcal{A}$ distinguished by properties of morphisms and possible commutation relations between them.

What is the behaviour of "Ramsey properties" under such construction(s)?

Some answers can be seen in the above cited papers by Graham-Leeb-Rothschild and/or Solecki.

For instance, Solecki proves *Self-dual Ramsey Theorem* for two arrow quivers $C_0 \to C \to C_0$ in the category of finite sets where these pairs of arrows satisfy some additional properties.

Can one express these properties in general categorical terms?[37]

Can one translate the description of other Solecki's constructions to our diagram-categorical languagea and render similarly the multitude of Ramsey theorems he proves?

[35] This appears in their 1971 paper *Ramsey theorem for n-parameter sets* that, along with Milmans's 1971 *A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies*, set a new stage for the Ramsey theory.

[36] In the literature, this goes under the heading *"Parameter Sets"*.

[37] The composed maps $C_0 \to C_0$ equal *identity* but there are other conditions as well.

Notice in this regard that the essential ingredient of the standard proof of the Ramsey monochromatic subsimplex theorem can be seen in terms of the telescopic decompositions (see previous section) of diagrams $\mathcal{D} = (\mathcal{S}, \Sigma)$ with endomorphisms $\Psi = \{\Psi_S : S \to S\}_{S \in \mathcal{S}}$, that are given, recall, by descending sequeneces of subsets in $S \in \mathcal{S}$,

$$S^{(0)} = S \supset S^{(1)} \supset S^{(2)}, ..., \supset S^{(i)}, ... \text{ and subsets } S^{-i} \subset S^{(i)}$$

where $S^{(i)} = \Psi_S\left(S^{(i-1)}\right)$ and $S^{-i} = \Psi_S(S^{-i-1}) \subset S^{(i)}$

The relevant property of such a decomposition needed for the Ramsey theorem is

$$\bigcup_{i=0,1,...} S^{-i} = S \text{ for all } S \in \mathcal{S}.$$

This may be seen as a (quite primitive) counterpart of *Schubert decomposition*, where the true Schubert decomposition of Grassmaninas, when formulated in diagrammatic terms, yields the proof of Rota's conjecture for Grassmannias (Spenser 1979). Probably, a suitable diarmmartic version of such decomposition for the block-Grassmannias would deliver, by induction, the parametric sets theorem of Graham-Rothschild as well, where the combinatorial arrangements of such "Schubert decompositions" serve as "templates" for the logical schemes of the induction.

Cartesian Powers of Categories etc. We have observed earlier that the Ramsey monochromatization property was preserved by Cartesian product of diagrams[38], that our argument was non-symmetric with respect to the two factors, that the resulting estimates were non-effective and that one could probably improve the matters in specific examples, e.g. for the Cartesian powers $\mathcal{D}^X$ for "nice and simple" diagrams $\mathcal{D}$ and all finite (infinite?) sets X.

What, in general, is the "Ramsay behaviour" of *Cartesian powers of categories*, say $\mathcal{D}^{\mathcal{C}}$ where $\mathcal{D}$ admits direct (Cartesian like) products and $\mathcal{C}$ is a category of sets, where the basic example one wants to emulate is the construction of the Hales-Jewett Cartesian category (diagram) of Cartesian powers $\mathcal{D}(F_0) = (F_0^X, \Sigma_0^X)$?

Is there something of Ramsey in *Markovian symbolic categories*[39] $\mathcal{M}$ where the objects are $\mathcal{D}^V$ for *multicategories* $\mathcal{D}$ and sets V?

Here, for instance, V may be the set of vertices of a directed graph with the edge set $E \subset V \times V$ and morphisms in $\mathcal{D}^V$ say $D^V \to D^E$ that are defined via bimorphisms, say $(D_{v_1}, D_{v_2}) \to D_e$, $e = (v_1, v_2) \in E$, that are combined in more general case with morphisms between graphs.

Semigroups and Ultrafilters. Many Ramsey results, such as *Hindman's monochromatic finite sums theorem*, depend on the *semigroup* and/or *ultrafilter* methods around the following

Ellis Idempotency Theorem. *Every compact (often unmetrizable and typically without unity) left topological semigroup G (e.g. a finite one) has an idempotent, that is a $g \in G$ such that $g \cdot g = g$.*

[38]See **1** in section 1.2 entitled **Ramsey Product Property**, where we consider Cartesian products $\mathcal{D}_1 \times \mathcal{D}_2$ of combinatorial diagrams.

[39]These are discussed in section?? and also in my articles: *Manifolds: Where Do We Come From?...*(section 11) *Spaces and Questions* and *Symbolic Algebraic Varieties.*

This, in particular, applies to *semigroups of ultrafilters* where, recall, an *ultrafilter*[40] u on a countable set S is a $\{0,1\}$-valued *finitely additive* measure $u(T)$ (with the Boolean addition $1+1=1$) defined on *all* subsets $T \subset S$, where the "interesting" u called *non-principal* ultrafilters are those where $u(T)=0$ for all *finite* sets T.

One also can define the space of ultrafilters along with its topology as the *Stone-Čech compactification* of S that is the *maximal compact* space $U(S)$ such that S is dense in $U(S)$.

Here, maximal means that every map from S to an arbitrary compact space X admits an extension to a continuous map $U(S) \to X$, where the subspace of *non-principal ultrafilters* naturally identifies with the *Stone-Čech ideal boundary* $\partial(S) = U(S) \setminus S$.

Ultrafilters behave in many respects similarly to ordinary measures, e.g. maps $f : S_1 \to S_2$ send ultrafilters u_1 on S_1 to $u_2 = f_*(u_1)$ on S_2. Also the Cartesian product of sets say $S_1 \times S_2$ admits a *canonical* ultrafilter $v = u_1 \otimes u_2$, such that $v(T)=1$, $T \subset S_1 \times S_2$, if and only if $u_2(s_1 \times S_2) = 1$ for u_1-almost all $s_1 \in S_1$.

The essential property of this $u_1 \otimes u_2$ (that is not hard to prove) is associativity $(u_1 \otimes u_2) \otimes u_3 = u_1 \otimes (u_2 \otimes u_3)$.

This property, and, of course, *the existence of nonprinciplal ultrafilters* that is proven with a standard use of the *Zorn Lemma*, allows one to rephrase the telescopic argument in the proof of the *Infinite Ramsey* from section 1.2. in familiar "measure theoretic" terms as follows.

$\left[\otimes\right]$ Let Q be a countable set and u an ultrafilter (thought of as a measure) on it. If u is non principal, then $u^{\otimes d}$*-almost all* points $(q_1, ..., q_i, ..., q_j, ..., q_d)$ in the Cartesian power Q^d have *no* $q_i = q_j$.

Break the symmetry by endowing Q with an order structure isomorphic to the usual order on the set $\mathbb{N}$ of natural numbers and identify points in $Q_{[d]}$ with d-tuples of points in Q with ordered sequences $(q_1 < ... < q_i < ... < q_j < ... < q_d) \in Q^d$.

If C is a finite colouring of $Q_{[d]}$, then $u^{[d]}$*-almost all* $q_{[d]} \in Q_{[d]}$ are of the same colour c and $u^{[n]}$*-almost all* n-tuples $q_{[n]} \in Q_{[n]}$, $n \geq d$, have all their d-subtuples $q_{[n]} \subset q_{[d]}$ also of the same colour c.

Finally, the coherence of these "$u^{[n]}$-almost all" for all $n = d, d+1, d+2, ...$ implies the existence of an infinite subset in Q with all its d-tuples being of colour c. QED

If S has a structure of a semigroup then the pushfoward of ultrafilters under the product map $S \times S \to S$ defines a convolution product on ultrafilters denoted $u_1 * u_2$ that endows the set $U = U(S)$ of ultrafilters on S with a semigroup structure where the left[41]multiplication is continuous for the Stone-Čech topology in U; thus, Ellis' idempotency theorem applies.

The resulting class of idempotent ultrafilters u with $u * u = u$ is used for variety of Ramsey results,[42] such as (Sanders-Folkman-)Hindman's monochromatic finite sums theorem that is customary stated as follows. *An arbitrarily*

[40]This concept goes back to *Filtres et ultrafiltres* (1937) by Henri Cartan.

[41]There is no – and there can not be – any *mathematical* definition of "left"; but inhabitants of Universes where CP-symmetry is violated can tell "left" from "right".

[42]See Bergelson's survey *Ultrafilters, IP sets, Dynamics, and Combinatorial Number Theory.*

finitely partitioned (coloured) set $\mathbb{N} = \{1, 2, 3, ...\}$ *of natural numbers admits a monochromatic* IP *subset,*[43] *i.e. a set which contains all finite sums of an infinite subset in* $\mathbb{N}$.

Proof. Let u be an idempotent ultrafilter on $\mathbb{N}$. Then one of the part of our partition (i.e. a maximal monochromatic subset), say $A \subset \mathbb{N}$ is *u-prevalent*, i.e. has $u(A) = 1$ and, by deciphering the meaning of the relation $u * u = u$, one sees that this prevalence makes the sums of the numbers from u^d-almost all sequences of numbers $n_1 < n_2 < ... < n_d$ contained in A.

This yields the finite version of Hindman theorem and the existence of such an infinite sequence follows similarly to how it goes in the above $[\otimes]$ with the ultrafilter proof of the classical Ramsey theorem.

A different way of of using semigroup structures and Ellis' theorem is suggested by Furstenberg and Katznelson in *Idempotents in compact semigroups and Ramsey theory* where, in particular, they give a short semigroup theoretic proof of the existence of *infinite* monochromatic (Hales-Jewett) block-diagonals in F_0^X for finite F_0 and infinite X theorem and also they obtain the following general finite result.

Let Γ be a countable semigroup, let $G \subset \Gamma^F$ be a subsemigroup in the Cartesian power Γ^F for a finite set F such that G contains the diagonal $\Gamma_\Delta \in \Gamma^F$.

Then, for any finite coloring of Γ*, every two-sided ideal*[44] H *in* G *contains a monochromatic element* $h = \{h_\phi\}_{\phi \in F}$*, i.e. where all coordinates* $h_\phi \in \Gamma$ *carry the same colour.*

This translates to the Hales-Jewett monochromatic combinatorial line theorem for the Cartesian category $\mathcal{D}(F_0)$ "generated" by a finite set F_0 in the case where:

• the free semigroup of words in the letters that are the elements from the set F_0 is taken for Γ;

• the subsemigroup in Γ^{F_0} generated by the diagonal $\Gamma_\Delta \in \Gamma^{F_0}$ and the product of all elements/letters in F_0 in some order is taken for G;

• The complement $G \smallsetminus \Gamma_\Delta$ is taken for H.

Questions. Does the Furstenberg-Katznelson proof of the monochromatic combinatorial line theorem rely on the same combinatorial input as the original one of Van der Waerden-Hales-Jewett?

Does the Ellis idempotency theorem properly reflect self similarity seen in most (all) Ramsey problems?

Is there something of Ramsey in other self similar combinatorial structures, such as *Grigorchuk's groups* for instance?

Are there higher Cartesian power Ramsey counterparts to the constructions of *infinitely iterated wreath products* of groups and semigroups?

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[43]The existence of monochromatic triples $(x, y, z = x + y)$ was proven by Schur in *Über die Kongruenz* $x^m + y^m = z^m$ *(mod p)*, 1916.

[44]A subset H in a semigroup G is a *two sided ideal* if it is invariant under left and rift multiplication by all $g \in G$. For instance , the numbers $r \geq r_0 \geq 0$ make an ideal in the senigroup of positive numbers.

1.4 Symmetry Ruined by Order and Measure.

What, in general, is the *mathematical* role of ultrafilters, Ellis-type theorems etc. in the proofs of Ramsey-like theorems?

Do they reveal combinatorial structures that are invisible by elementary/finitery means or they serve as logical bookkeeping devices?

The following observation[45] suggests the latter.

Let $\mathcal{P}$ be a class of subsets in an infinite countable set S that are distingushed by some property Π. Then the following [A] and [B] are equivalent

[A] Some part of every finite partition of S satisfies property Π (e. g. contains arbitrary long arithmetic progression as in the van Der Waerden case where $S = \mathbb{N}$).

[B] There exists an ultrafilter (thought of as a $\{0,1\}$-measure) u on S, such that all *u-prevalent* subsets $P \subset S$ satisfy Π, i.e.

$$u(P) = 1 \Rightarrow P \in \mathcal{P}.$$

In a similar vein, Jean-Yves Girard presents a logical dissection[46] of the Furstenberg-Weiss proof[47] of the existence of monochromatic arithmetic progressions in the framework of topological dynamics with an essential use of existence of *non-empty minimal* closed invariant subsets for group actions on *compact* spaces.

Namely, Girard shows that Furstenberg-Weiss proof can be transformed to the elementary Van der Waerden's one by some *universal logical procedure*, mainly by *cut elimination*.

Apparently, the topological dynamics vocabulary serves to "condense" the mathematical induction steps in the proof of the **Ramsey Product Property** in 2.2 (in the case of arithmetic progressions) to a single existence statement of minimal sets that is possible due to "logical homogeneity" of the ensemble of these steps; the cut elimination reverses this "condensation" by returning "individuality" to these steps by assigning names to them.

A Girard-style analysis was applied to many other proofs of Ramsey-type theorems (I must admit I have not looked into the relevant papers) but we want to bring to light the following *mathematical* (rather than logical) problem related to all these proofs

In many (all?) cases Ramsey structure arise in a symmetric – be it group theoretic or categorical – environment. However,

ALL AVAILABLE PROOFS OF COMBINATORIAL RAMSEY-TYPE THEOREMS FAIL TO EXPLOIT THIS SYMMETRY.

Paradoxically (regretfully?) all proves depend on a radical break of symmetries.

This becomes most apparent when the Ramsey theorems are formulated as symmetrically as they come; then one can see what is broken by their proofs.

For instance, the construction of a monochromatic n-face in a simpex $\Delta^N = \Delta(Q)$ via an "infinite Ramsey telescope" $Q_1 \supset Q_2 \supset ... \supset Q_i \supset ...$ in 1.1 is acived

[45] The standard reference is to Hindman's *Ultrafilters and combinatorial number theory* 1979, but this might have been known to model theorists prior to 1979.

[46] *Proof Theory and Logical Complexity*, Volume I of Studies in Proof Theory, 1987.

[47] *Topological dynamics and combinatorial number theory,* 1978.

with *arbitrary choices* of points $q_i \subset Q_i \setminus Q_{i+1}$, with no analysis of the totality of such choices and/or with no attempt of an optimisation of such a choice.

Non-surprisingly, most (all) non-elementary combinatorial Ramsey proofs depends on the Zermelo axiom of choice[48] that is often apears in the form of *Zorn's Lemma* or *Cantor's well ordering theorem*. Indeed, this axiom is indispensibale for systematic symmetry breaking as has been known since the misadventure of Buridan's ass.[49]

In particular, ultrafilters, unlike classical measures,[50] admit no non-trivial symmetries. Namely, if an ultrafilter u on a set S is invariant under a transformation $g : S \to S$, then value of u on the support of g, i.e. on the subset of those $s \in S$ for which $g(s) \neq s$, equals 0.

Thus, for instance, one there is no symmetric product of ulterfilters, $u_1 \otimes u_2$ on the Cartesian product $S_1 \times S_2$ can not be equal to $u_2 \otimes u_1$, since no ulterfilter v on $S \times S$ can be symmetric under the permutation of the two factors unless v is supported on the diagonal.

This kind of break of symmetry also seen in the very formulation of the monochromatic combinatorial line theorem by Furstenberg and Katznelson, where they represent the Cartesian power F_0^X by the set of N-strings in $\{\phi_i\}$, $i = 1, 2, ..., card(F_0)$, $N = card(X)$. This automatically forces an ordering on the sets F_0 and X and, thus, removes the automorphisms (permutation) groups of these sets from the game. (The symmetry inherent in the semigroup structure on ϕ_i-words is much weaker than that on the Cartesian/block diagonal maps $F_0^X \to F_0^X$.)

Besides ordering, the proofs of some Ramsey theorems, especially around the Grothendieck-Dworetzky-Milman virtually round section theorem, benefit from rigidification of your objects by endowing them with auxiliary (sometimes rather arbitrary) measures or measure like structures such as ultrafilters.

But breaking symmetries by hand makes elementary proofs awkwardly long; this invites a use of a symmetry breaking logical machinery.

(Combinatorialists have their own means for condensing their proofs. They work exclusively with ordered sets that are automatically identified with segments of integers and are denoted by $[n]$ for $n = card[n]$. This kind of notations compensate for arbitrariness of choices in some combinatorial constructions. Thus, for example, Graham and Rothschild manage to write *A Short Proof of Van Der Waerden's Theorem on Arithmetic progressions* on a single page.[51])

Will the Ramsey theory ever achieve a deep understanding of relevant combinatorial structures, similarly to how such structures were revealed in topology and algebraic geometry, or it will be dominated by constellations of clever tricks designed for avoiding this understanding?

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[48] *Martin axiom* sometimes enters as well, e.g. see *Combinatorial Set Theory: With a Gentle Introduction to Forcing* 2011 by L. J. Halbeisen.

[49] A logician may smile at the naivety of the ass who was unable to choose an item out of collection of two, but modern mathematicians side with the ass rather than with Zermelo: realisation of impossibility of such a choice being consistent brought to life *Galois' theory* and also the *algebraic topology of fiber bundles* along with *gauge theories*. In fact the Borsuk-Ulam theorem on colouring spaces with involutions can be seen as the direct vindication of the hesitancy of the ass in choosing a pile of hay.

[50] Most (all?) interesting measures in mathematics and mathematical physics are derived from *Haar measures*.

[51] The *shortest* proof of this theorem, that, according to *Kolmogorov's complexity theory*, would be indistinguishable from a random string of symbols, still awaits its turn to be written.

Notice at this point, that non-elementary *Fourier theoretic* proofs by Roth and by Gowers, that had been worked out so far only in the *arithmetic environment*, strive on (some aspects of the relevant) symmetries of the objects (arithmetic professions) they apply to and, eventually, will lead to understanding of deeper level of structure. Unsurprisingly, these methods yield results unachievable by any other available means.

However, this does not exclude a (sad?) possibility, that the apparent symmetry in *purely combinatorial* Ramsey structures is illusory and the asymmetry of the proofs is unavoidable.

Further Questions. (a) What is a "fully asymmetric" Ramsey theory in something like the category of *well order sets*?

(b) Monochromataizing maps in a coloured diagram/category $\mathcal{D}$ of *sets* make a "right ideal" in $\mathcal{D}$. Can one one extract essential properties of such "ideals" applicable to *"non-set"* categories?

The colour focusing argument and alike applied to hypergraphs G arising in the Ramsey theory deliver witness sub-hypergraphs H with large colouring numbers. and with some bound on the size of H.

Can one effectively describe (classes of) such "witnesses" H without direct reference to G?

Given a (large) number R, what are "concrete and effective" (very large) hypergraphs G that have large colouring numbers $N_{col}(G)$ but where all sub-hypergraphs H with $size(H) \leq R$ (kind of R-balls in G) have their colouring numbers much smaller than $N_{col}(G)$?[52]

1.5 ε-Monochromaticity, Virtual Equalization and Virtual Constancy of Functions .

ε-Monocromatization and ε-Equalization. Recall the definition of *the ε-neighbourhood* of a subset T in a *metric space* S, denoted $U_\varepsilon(T) \subset S$, sometimes called the *$\varepsilon$-thickening* of T, that is the set of points in S within distance at most ε from T, i. e. the union of all metric ε-balls in S, let them be open, with their centres in T.

Observe that

$$U_\varepsilon(T_1 \cup T_2) = U_\varepsilon(T_1) \cup U_\varepsilon(T_2)$$

and

$$U_\varepsilon(T_1 \cap T_2) \subset U_\varepsilon(T_1) \cap U_\varepsilon(T_2)$$

where, in general, the intersection $U_\varepsilon(T_1) \cap U_\varepsilon(T_2)$ may be quite large, e.g. equal all of S, while $T_1 \cap T_2$, and hence, $U_\varepsilon(T_1 \cap T_2)$ are empty.

Given a colouring, that is a partition, or more generally a covering of S, say $S = \cup_i U_i$, a subset $T \subset S$ is called *ε-monochromatic* if it is contained in the *ε-neighbourhood* of some monochromati subset $U_i \subset S$.

A map $\sigma : S_0 \to S$ is called *ε-monochromatizing* if it sends S_0 into the ε-neighbourhood of a monochromatic subset in S.

◇ *Discretization* If the spaces $S \in \mathcal{S}$ are compact and the maps $\sigma \in \Sigma$ are simultaneously uniformly continuous (e.g being all λ-Lipschitz for some λ) then

[52]Ramanujan graphs G have this property that suggests parallels between Ramsey and "spectrality" of hypergraphs.

one can approximate $\mathcal{D}$ by combinatorial (locally finite) diagrams $\mathcal{D}_\varepsilon$ for all $\varepsilon > 0$ by taking finite ε'-nets S_ε in all $S \in \mathcal{S}$, say with $\varepsilon' = \varepsilon/10$, and, accordingly, by approximating maps $\sigma : S_1 \to S_2$ by $\sigma_\varepsilon : S_{1,\varepsilon} \to S_{2,\varepsilon}$.

Thus, the notion of *ε-monochromatization* for topological diagrams reduces to the ordinary monochromatization for combinatorial diagrams.

Notice, however, that even if $\mathcal{D}$ is a *category none of* $\mathcal{D}_\varepsilon$ *makes a category*, since the composition of ε-maps, $S_{1,\varepsilon} \to S_{2,\varepsilon} \to S_{3,\varepsilon}$, say $\sigma_{23,\varepsilon} \circ \sigma_{12,\varepsilon}$, is not, a priori, an ε-map but rather a 2ε-map $\sigma_{13,2\varepsilon}$.

The concept of *ε-monochromatization* may be applied to continuous R-colorings associated to continuous maps $f : S \to R$ (these are partitions of S into the f-pullbacks of points $r \in R$), where, in the case of metric spaces R, there is the following concept of *ε-equalization.*

A map $\sigma : S_0 \to S$ is said to *ε-equalize* f if the image of the composed map $f \circ \sigma : S_0 \to R$ has diameter at most ε.

Observe, this obvious, that if the map $f : S \to R$ is 1-Lipschitz, that is distance decreasing, then

all maps $\sigma : S_0 \to S$ *that ε-monochromatize the continuos R-coloring associated with f, are 2ε-equalizing for f, with this 2 replaced by 2λ for λ-Lipschitz maps f.*

A *colouring* of a diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ (e.g. a category) of metric spaces S and maps $\sigma : S_1 \to S_2$, i.e. where all $S \in \mathcal{S}$ are coloured, is called *virtually monochromatizable by* Σ if

given a set $S_0 \in \mathcal{S}$ *and* $\varepsilon > 0$, *there exist another set* $S \in \mathcal{S}$ *and an ε-monochromatizing map* $\sigma : S_0 \to S$, $\sigma \in \Sigma$.

An $\mathcal{F}$*-coloring of a diagram* $\mathcal{D} = (\mathcal{S}, \Sigma)$ *over a class* $\mathcal{R}$ *of metric spaces* R , is given, by definition, by maps $f : S \to R$ for all (sometimes for some) $S \in \mathcal{S}$. Such a colouring, is called *virtually equalizable* by Σ if

given a set $S_0 \in \mathcal{S}$ *and* $\varepsilon > 0$, *there exist another set* $S \in \mathcal{S}$ *and a map* $\sigma : S_0 \to S$ $\sigma \in \Sigma$ *that ε-equalizes the map* $f : S \to R$, $f \in \mathcal{F}$.

($\mathcal{R}$ may consist of a single space R but even then the values of nearly constant functions $f \circ \sigma$ on S_0 may strongly vary for different equalisers σ.)

Uniform Continuity and Lipschitz. An essential role in possibility of virtual equalization is often played (we shall see this below) by *uniform continouty of families* $\mathcal{F}$, i.e. where all maps $f \in \mathcal{F}$ are simultaneously uniformly continuous, e.g. all being λ-Lipschitz for some $\lambda < \infty$.

Lipschitz maps may seem special but... .

If S is a Riemannian manifold or, more generally a length space, then every uniformly continuous function $f : S \to \mathbb{R}$ *admits a uniform ε-approximation by λ-Lipschitz functions* f_ε, *where* $|f - f_\varepsilon| \le \varepsilon$ *and where λ depends only on the modulus of continuity of f and on ε.*

Proof. Let $T \subset S$ be a maximal δ-separated subset in S and let

$$d_t(s) = \lambda \cdot dist(t, s) + f(t) \text{ and } f_\varepsilon(s) = \inf_{t \in T} d_t(s).$$

If λ is large depending on the value of the continuity modulus of f at ε and $\delta \le \varepsilon/10\lambda$, then the functions $f_\varepsilon(s)$ provide the required approximation of f.

⊞ *From $(k+1)$-Colorings to $\mathbb{R}^k$-Valued Lipschitz $\mathcal{F}$-colorings.* Given a covering of a metric space S by $k+1$ subsets,

$$S = \bigcup_{i=0,1,...,k} U_i,$$

one associate to it a Lipschitz map $f : S \to \mathbb{R}^k = \mathbb{R}^{k+1}/diagonal$ that is given by the distance functions $s \mapsto dist(s, U_i)$, $i = 0, 1, ..., k$, defined up to a common additive constant.

It is obvious that if f is *ε-equalized* by some map $\sigma : S_0 \to S$, then the image of this map lies ε-close to some U_i; thus, σ serves to *ε-monochromatize* this covering seen as a colouring.

Most (all?) diagrams in what we call *the Ramsey-Milman theory* (presented below) are of homogeneous origin where one has (usually an infinite dimensional) coloured metric space S_* acted upon by (usually isometry) groups G, where monochromatization and equalisation refer to transformation $g \in G$ applied to compact subsets in S_*.

Sometimes one requires such a property only of *finite subsets* $S_0 \subset S_*$ and then derives (this is often possible as well as trivial) the corresponding property for *all compact* subsets in S_*.

The main purpose of the above definitions of virtual monocromatization & equalization is to embed the following theorem into a general context.

Virtually Constant Slices of Functions on the Sphere S^∞. (Milman 1969, 1971).[53] *Uniformly continuous functions $f : S^\infty \to \mathbb{R}$, where S^∞ is the unit sphere in infinite dimensional Euclidean (e.g. Hilbert) space $\mathbb{R}^\infty$, are virtually equalizable on compact subsets $S_0 \subset S^\infty$ by the (group of the) isometries σ of this sphere:*

the (diameters of the) images of the composed maps $f(\sigma(S_0)) \subset \mathbb{R}$ can be made arbitrarily small with suitable choices of isometries $\sigma : S^\infty \to S^\infty$.

Equivalently,

Every uniformly continuous $\mathbb{R}$-valued $\mathcal{F}$-coloring of the category of unit Euclidean spheres $S^n \subset \mathbb{R}^{n+1}$, $n = 1, 2, 3, ...,$, and of (necessarily equatorial) isometric embeddings $\sigma : S^n \to S^N$ is virtually equalizable.

In fact, Milman has established a *quantitative version* of the corresponding $\varepsilon_{n,N}$-equalizaton needed for the following version of

Almost Round Section Theorem for Convex Bodies. Let $||x|| = ||x||_N$ be some (Minkowski-Banach) norms (that are convex homogeneous functions) on the Euclidean spaces $\mathbb{R}^{N+1}$ and let $\mathcal{F}_{conv}$ be the following family of functions $f = F_N$ on the N-spheres,

$$f(s) = \log ||s||, \; s \in S^N \subset \mathbb{R}^{N+1}.$$

Then this $\mathcal{F}_{conv}$ is is virtually equalizable by isometric embeddings (maps) between these spheres.

In other words,

[53]Originally, Milman derived such results from Dvoretzky's theorem, but, due to the doubts expressed by some experts on the completeness of Dvoretzky's argument, he suggested in 1971 his own proof (based on *concentration*) of the Grothendieck conjecture and his proof eventually became standard. (Myself, I have carefully read only Milman's proof.)

Given $n = 1,2,...$ and $\varepsilon > 0$, then, for all sufficiently large $N \geq N_0(n,\varepsilon)$, the restriction of any given norm $||...||$ from $\mathbb{R}^{N+1}$ to some equatorial sphere $S^n \subset S^N \subset \mathbb{R}^{N+1}$ satisfies

$$\frac{\sup_{s\in S^n} ||s||}{\inf_{s\in S^n} ||s||} \leq 1+\varepsilon.$$

Observe that "convex" families $\mathcal{F}_{conv}$ are *not*, in general, uniformly continuous and an application of the virtually constance slices theorem, even in its quantitative form, is by no means automatic; yet, Milman's proof guarantees $N_0 \leq \exp const(\varepsilon)N$.

Two Proofs of "the Virtually Constant Slices" Theorem. The simplest proof of this theorem (corresponding, I guess, to what is suggested by Dworetzky in his 1961 paper) follows by integration over spaces of smooth maps between manifolds, e.g. $S^n \to S^N$ with an essential use of the *Pythagorean theorem* applied to squared norms of the differentials of these maps. The second proof suggested by Milman in 1971 is derived from the *spherical isoperimetric inequality* with *Paul Lévy's concentration* as an intermediate.

We present these proofs below and then turn to their formalised versions that, albeit being longer, open avenues for generalisations.

1. *Pythagorean Proof.* Let ds, dS and $d\sigma$ be the normalised (i.e. of the total masses one) *Haar measures* on the spheres S^n and S^N and on the space $\Sigma = \Sigma_{n,N}$ of isometric (equatorial) imbedding $S^n \to S^N$, where "Haar" means invariant under the respective isometry groups, that are $O(n+1)$ for S^n, $O(N+1)$ for S^N and $O(n+1)\times O(N+1)$ for Σ.

Then the integral of the square norm of the differential Df of an arbitrary λ-Lipschitz (e.g. C^1-differentiable with $|Df| \leq \lambda$) function $f(S) = f_N(S)$ on S^N, that is $f: S^N \to \mathbb{R}$, satisfies

$$\int_{S^N} ||Df(S)||^2 dS = \frac{N}{n}\int_{\Sigma} d\sigma \int_{S^n} |Df(\sigma(s))|^2 ds$$

by the Pythagorean theorem applied to the vectors in the tangent spaces $T_S(S^N) = \mathbb{R}^N$ and $T_s(S^n) = \mathbb{R}^n$.

It follows that if a family of functions $f = f_N$ on S^N, $N = 1,2,...$, is uniformly continuous, hence, approximable by λ-Lipshitz functions with (uniformly bounded) λ *independent of* N, then the integral $\int_{S^n} |Df(\sigma(s))|^2 ds$ becomes arbitrarily small for some (in fact, for most) $\sigma = \sigma(N) \in \Sigma = \Sigma_{N,n}$ as $N \to \infty$.

Finally, since the family of the functions $f = f_N(S)$ is uniformly continuous, the family of composed functions $f_N(\sigma(s))$ on S^n is also uniformly continuous and bounds on their integrals of these functions yield comparable bounds on their total oscillations,

$$\int_{S^n} |Df_N(\sigma(s))|^2 ds \underset{N\to\infty}{\to} 0 \Rightarrow diam(f_N(\sigma(S^n)) \underset{N\to\infty}{\to} 0.$$

QED.

2. *Concentration Proof.* If $f: S^N \to \mathbb{R}$ is a continuous function then some level $S_r = f^{-1}(r) \subset S^N$, $r \in \mathbb{R}$, serves as *the Lévy mean* of f, i.e. it divides the spherical volume of S^N into "essentially equal" halves; more precisely,

$$vol_N(f^{-1}(-\infty,r]) \geq \frac{1}{2}\mathrm{vol}(S^N) \text{ as well as } vol(f^{-1}(r,+\infty]) \geq \frac{1}{2}\mathrm{vol}(S^N).$$

(Of course, except for an obvious "pathology", one has $vol_N(f^{-1}(-\infty, r]) = vol(f^{-1}(r, +\infty]) = vol(S^N)/2$ for this r.)

It follows by *the spherical isometric inequality* – and this is the only technically non-trivial ingredient of the proof – that the ε-neighbourhoods of $U_\varepsilon(S_r) \subset S^N$ have their volumes greater than such neighbourhoods of the equators $S^{N-1} \subset S^N$.

Hence, – this was pointed out by Paul Lévy – these neighbourhoods contain almost all volumes of the spheres S^N for $N \to \infty$,

$$\frac{vol_N(U_\varepsilon(S_r))}{vol_N(S^N \setminus U_\varepsilon(S_r))} \underset{N\to\infty}{\to} \infty \text{ for all } \varepsilon > 0,$$

as it follows from the corresponding fundamental, albeit obvious. (at least this was obvious to Maxwell if not to Bernoulli?) property of the equatorial bands $U_\varepsilon(S^{N-1}) \subset S^N$.

(The volumes

Then, according to *the Buffon-Crofton formula*, almost all n-volume of the subsphere $S^n = \sigma(S^n) \subset S^N$ is contained in $U_\varepsilon(S_r)$ for most $\sigma \in \Sigma = \Sigma_{N,n}, N \to \infty$, that is

$$\frac{vol_n(\sigma(S^n) \cap (U_\varepsilon(S_r))}{vol_n(\sigma(S^n) \setminus U_\varepsilon(S_r))} \underset{N\to\infty}{\to} \infty \text{ for all } \varepsilon > 0,$$

Since n is kept fixed as $N \to \infty$,

$$diam(f_N(\sigma(S^n)) \underset{N\to\infty}{\to} 0$$

by the uniform continuity of $f = f_N$. QED.

1.6 Prevalently Constant Functions on Pythagorean Diagrams.

Let us make better visible what makes the above **1** work.

A *measure* $\mathcal{M}$ on a diagram $\mathcal{D} = (\mathcal{S}, \Sigma)$ is given, by definition, by measures $\mu = \mu_{12} \in \mathcal{M}$ on the sets Σ_{12} of maps $\sigma : S_1 \to S_2$, $S_1, S_2 \in \mathcal{S}$, where we do not necessarily insist, as it is common in the category theoretic context, that the composition maps $\Sigma_{12} \times \Sigma_{23} \to \Sigma_{13}$ are measure preserving.

Prevalent Constancy. An $\mathcal{F}$-coloring of $\mathcal{D}$ over $\mathcal{R}$, (that is, recall, a family of maps f from sets $S \in \mathcal{S}$ to *metric* spaces $R \in \mathcal{R}$) is called *(virtually) $\mathcal{M}$-prevalently constant* for a measure $\mathcal{M}$ on $\mathcal{D} = (\mathcal{S}, \Sigma)$, if

the composed maps $f \circ \sigma : S_0 \to R$ are virtually constant on S_0 for most maps $\sigma : S_0 \to S$.

That is,

given $S_0 \in \mathcal{S}$ and numbers $\varepsilon, \epsilon > 0$, there exists an $S \in \mathcal{S}$ such that the relative measure of those maps $\sigma : S_0 \to S$ for which $diam(\sigma(S_0)) \geq \varepsilon$ is at most ϵ.

In other words, the function $d(\sigma) = diam(\sigma(S_0))$ on the space of maps Σ_0 from S_0 to S with our the measure μ concentrates at 0:

$$\frac{d^{-1}[\varepsilon, \infty)}{d^{-1}[0, \varepsilon]} \leq \epsilon.$$

The proofs in **1** and **2** both yield the following improvement of the "the virtually constant slices" theorem.

[$\star$] **Prevalent Constancy Theorem**. *Every uniformly continuous $\mathbb{R}$-valued $\mathcal{F}$-coloring of the category $\mathcal{D}$ of unit Euclidean spheres $S^n \subset \mathbb{R}^{n+1}$, $n = 1, 2, 3, ...,$, and of isometric (equatorial) embeddings $\sigma : S^n \to S^N$ is $\mathcal{M}$-prevalently constant for the family $\mathcal{M}$ of Borel (Haar) probability measures on the spaces if isometric maps between spheres that are invariant under the isometry groups of these spheres.*[54]

In simple words, if $\mathcal{F} = \{f_N : S^N \to \mathbb{R}\}_{N=1,2,...}$ is a uniformly continuos family of functions, then

for every n and most isometric embeddings $\sigma = \sigma_N : S^n \to S^N$, where $N \to \infty$, the total oscillations of the composed functions $f_N(\sigma_N(s))$ on S^n tends to 0.

Let us extract the essential properties spaces and maps from the above Pythagorean proof of [$\star$].

Pythagorean Measures. A Borel measure ν also written as $\nu(x)$ on the Euclidean space $\mathbb{R}^N$ is called *Pythagorean* if one of the two (obviously) equivalent conditions are satisfied.

(i) The ν-integral of the square of every linear function $l : \mathbb{R}^N \to \mathbb{R}$ satisfies

$$\int_{\mathbb{R}^N} l^2(x) d\nu(x) = \frac{\nu(\mathbb{R}^N)}{N} \cdot \sup_{||x||=1} l^2(x).$$

(ii) The ν-integral of every quadratic function (form) Q equals $const \cdot trace(Q)$ for some $const = const(\nu)$ independent of Q.

Thus, Pythagorean measures constitute a convex cone of codimension $\frac{N(N+1)}{2}$ in the space of all measures on $\mathbb{R}^N$.

Examples. (a) Equidistributed measures on orthonormal frames of vectors $\{x_1, ..., x_N \in \mathbb{R}^N\}$ are Pythagorean by the Pythagorean theorem.

(b) A measure invariant under the action by an orthogonal group G acting on $\mathbb{R}^N$ is (obviously) Pythagorean if this G admits *no non-trivial invariant subspace* in $\mathbb{R}^N$.

A measure $\nu = \nu(g)$ on the Grassmann space $Gr_n(\mathbb{R}^N)$ of linear n-subspaces in $\mathbb{R}^N$ is *Pythagorean* if the traces of the restrictions of an arbitrary quadratic form Q from $\mathbb{R}^N$ to linear n-spaces $\mathbb{R}^n_g \subset \mathbb{R}^N$, $g \in Gr_n(\mathbb{R}^N)$, satisfy

$$\int_{Gr_n(\mathbb{R}^N)} trace(Q|\mathbb{R}^n_g) d\nu(g) = \frac{n \cdot \nu(Gr_n(\mathbb{R}^N))}{N} trace(Q).$$

A measure ν on the the Grassmann manifold $Gr_n(S)$ of tangent n-planes T^n in an N-dimensioanal Riemannian manifold S called *Pythagorean* if its "restrictions" (conditionings) to ν-almost all Grassmanian fibres $Gr_n(T_s(S) = \mathbb{R}^N)$ are Pythagorean.

Now, the essential *Pythagorean property* of families $\Sigma = \Sigma_N$ of maps between Riemannian manifolds, say $\sigma : S_0 \to S = S_N$ where $dim(S_0) = n$ and $dim(S_N) = N$, can be seen as follows.

[54] Unlike to how it works in the combinatorial environment, it seems virtually impossible, locate "*virtual something*" in a compact homogeneous space (e.g. in the space of isometric maps $S^n \to S^N$) unless this something is virtually prevalent, such as proving the existence of maps σ with a certain property, without proving *virtual prevalency* of the maps with this property.

There are measures ν_N on S_0 and μ_N on Σ_N, such that the differentials $D\sigma : T(S_0) \to T(S_N)$ of almost all maps $\sigma : S_0 \to S_N$ are injective almost everywhere on S_0 and the *push forward measures* ν_* of the measures $\nu_N \otimes \mu_N$ from the $S_0 \times \Sigma_N$ to $Gr_n(S_N)$ by the map

$$(s, \sigma) \mapsto T^n = D\sigma(T_s(S)) \subset T(S)$$

are *Pythagorean.*

Restrictions and Generalizations.

(A) *On Uniform Positivity of Measures* ν_n *on* S_0. The Pythagoriam property implies by itself that

$$\int_{S_0} \|Df_N(\sigma_N(s))\|^2 d\nu_0(s) \underset{N\to\infty}{\to} 0$$

for uniformly continuous families of Lipschitz functions f_N on S_n and maps σ_N. But derivation of the prevalent constancy of the composed functions $f_N(\sigma_N(s))$ needs certain assumptions on the measures ν_N. For instance, if S_0 is compact, then it is sufficient to require *uniform positivity* of these measures on non-empty open subsets $U_0 \subset S_0$, that is a bound $\nu_N(U_0) \geq \delta(U_0) > 0$ for all N and all U_0.

(B) On Maps $F = F_N : S_N \to R$ for $m = dim(R) \geq 2$. If, for instance, $R = \mathbb{R}^m$ and $F = (f_1, ... f_m)$, then prevalent constancy of all m functions $f_1, ..., f_m$ individually (obviously) implies prevalent constancy of F.

One could do even better using directly the Pythagorus theorem that implies that, prevalently, i.e. for most σ_N, the convergence

$$\int_{S_0} \|DF_N(\sigma_N(s))\|^2 d\nu_0(s) \underset{N\to\infty}{\to} 0$$

remain valid for λ-Lipschitz maps $F = F_N : S_N \to R$, provided the ranks of the differentials of these maps $F = F_N$ almost everywhere satisfy

$$\frac{rank(DF)}{dim(S)} \underset{N\to\infty}{\to} 0,$$

where $R = R_N$ may be arbitrary Riemannian manifolds and where the Lipschitz constants λ of these maps must be independent of N.

(C) *Prevalent Constancy for Amenable Spaces with Infinite Measures.* The Pythagorean proof of the relation

$$\int_{S_0} \|Df_N(\sigma_N(s))\|^2 d\nu_0(s) = o(N)$$

depends not so much on *integration* over spaces S_N but on taking averages. The latter are available in some cases when the Pythagorean measure is question, that is ν_* on the Grassmann manifold $Gr_n(S_N)$ is infinite, where the relevant condition is *Følner amenability* of the push forward measure $\nu_\star$ of ν_* from $Gr_n(S_N)$ to $S = S_N$ reads:

There is an exhaustion of S by relatively compact domains, say $V_1 \subset V_2 \subset ..., \subset V_i \subset ... \subset S$, such that the measures of their ρ-neighbourhoods satisfy

$$\nu_\star(U_\rho(V_i))/\nu_\star(V_i) \underset{i\to\infty}{\to} 1 \text{ for all } \rho > 0.$$

[◯] *Euclidean Example.* Let $\mathbb{R}^\infty$ be an infinite dimensional Hilbert space, let R be a finite dimensional locally compact metric space, e.g. a Riemannian manifold.

Let $f : \mathbb{R}^\infty \to R$ be a uniformly continous (e.g. λ-Lipschits) map and $S_0 \subset \mathbb{R}^\infty$ be a compact subset. Since S_0 can be approximated by its intersections with Euclidean subspaces $\mathbb{R}^N \subset \mathbb{R}^\infty$ and since Euclidean spaces are amenable,

there exists a sequence of isometric transformations $\sigma_N : \mathbb{R}^\infty \to \mathbb{R}^\infty$ such that the diameters of the f-images of the transformed S_0 satisfy

$$\lim_{N\to\infty} diam(f(\sigma_N(S_0))) = 0.$$

Categorical Remark. This can be reformulated in terms of the diagram $\mathcal{D}_{r,N}$ of isometric maps between Euclidean balls $B^N(r) \subset \mathbb{R}^N$ (for all radii, especially for $r \to \infty$ and $N \to \infty$), where this $\mathcal{D}_{r,N}$ is, in fact, *a category.*

Question? Can one make a full list of all categorical Pythagorian Diagrams similar to these Euclidean $\mathcal{D}_{r,N}$?

[⋈] *Amenable Hyperbolic Example.*[55] Let H^∞ be a increasing union of hyperbolic spaces H^N, $N \to \infty$, of negative curvatures -1, that is $H^\infty = \cup_n H^n$, let $G = \cup_n G_n$ be the (increasing) union of *co-amenable* isometry groups of H^N and let $\Gamma = \cup_N \Gamma_N$ be a discrete group isometrically acting on H^∞, where "coamenable" means tha the quotient spaces H^N/Γ_N are amenable.

Let $f : H^\infty \to \mathbb{R}^m$ be a *uniformly continuos* map that is *invariant under* Γ and let $S_0 \subset H^\infty$ be a compact subset.

Then there exists a sequence of isometric transformations $\sigma_N : H^\infty \to H^\infty$ such that the diameters of the f-images of the transformed S_0 satisfy

$$\lim_{N\to\infty} diam(f(\sigma_N(S_0))) = 0.$$

Making [⋈] Quasi-Categorical. This example is inherently non-categorical, since H^N and S_0 are objects of different nature, where the spaces H^N can not be replaced by large balls in them, since the presence of Γ is indispensable: the hyperbolic distance function to a fixed point, $f(x) = dist(x, x_0)$, $x, x_0 \in H^\infty$, sends all unit geodesic segments from H^∞ to segments of length > 0.1 in $\mathbb{R}$.

On the other hand, hyperbolic (quotient) manifolds $S_N = H^N/\Gamma_N$ for certain groups Γ_N without torsion (this restriction is needed to avoid terminology of orbifolds) satisfy:

Given λ-Lipshitz maps $f_N : S_N \to \mathbb{R}^m$ and a number $p \geq 1$, there exist hyperbolic n-dimensional submanifolds $S_{n,N} \subset S_N$ such that the averages of the L_p-norms of their differentials satisfy

$$\frac{\int_{S_{n,N}} ||df_N(s)||^p ds}{vol_n(S_{n,N})} \leq c_{n_N}\lambda^p \textit{ where } c_{n_N} \underset{N\to\infty}{\to} 0.$$

In fact, such $S_{n,N}$ exist in many cases of arithmetic groups Γ_N, e.g for $\Gamma_N = O(n,1;\mathbb{Z})$, due to uniformity of distributions of arithmetic n-submanifolds in these $S_N = H^N/\Gamma_N$.

[55] See section 9.3 in my *Filling Riemannian Manifolds* for a similar result for other symmetric spaces.

(D) On Singularities and on Trees. All of the above applies to piecewise smooth spaces S_0 and S_N, where, in particular, on has the following.

Pythagorean Liouville Measure on Spaces of Maps from Trees to Riemannian Manifolds. Let $T \subset S_0$ be a *geometric subtree of finite length* , that is a closed contractible set that equals a finite or countable union of geodesic segments with finite total length. A map $\sigma : T \to S$ is called *nearly isometric* if σ locally isometrically sends each edge in T to a geodesic segment in S, such that the angles between these segments equal the corresponding angles in S_0.

Then the space Σ of nearly isometric maps $\sigma : T \to S$ (where the dimension of this space may grow with the combinatorial size of T) carries a natural measure μ that, when restricted to all edges of T becomes the Liouville Measure. Clearly, the measure $dt \otimes \mu$ is Pythagorean.

This may be used for proving the equalization property for maps $S_0 \to S$ via maps $T \to S$ with a suitably "dense" subtrees in $T \subset S_0$.

Uniform Continuity and Convexity. The above does not directly yield Dvoretzky's almost round section theorem since restrictions of (logarithms of) convex functions (norms) from $\mathbb{R}^{N+1}$ to $S^N \subset \mathbb{R}^{N+1}$ are not uniformly continuous. This brings forth the following questions.

Is there a natural class of functions $f(s) = f_N(s)$ on spheres S^N that would include both uniformly continuous ones as well as convex ones (coming from homogeneous convex functions on $\mathbb{R}^{N+1} \supset S^N$) for which the virtual constancy of f would hold true?

(Recall that a family of maps f_N from the N-spheres S^N to a metric space R is called virtually constant, if, for every given n, there exit isometric embeddings $\sigma : S^n \to S^N$ such that $diam_R(f \circ \sigma(S^n)) \to 0$ for $N \to \infty$.)

One may start by defining some class $\mathcal{C}_n$ of functions on S^n for a fixed n and then to look at the function on S^N the restrictions of which to all n-dimensianl equators belong to $\mathcal{C}_n$. Notice that both convex and uniformly continuous, say λ-Lipschitz for a given λ, classes of functions on S^N come this way with $n = 1$.

Also notice that the class of convex norm-functions on S^n is invariant under *linear transformations of* $\mathbb{R}^{n+1} \supset S^n$, where functions on the sphere S^n are regarded as homogeneous functions of degree one on the ambient Euclidean space $\mathbb{R}^{n+1} \supset S^n$.

What happens if you replace "convex" by "λ-Lipschitz"?

Namely, let $\mathcal{C}_{n,\lambda}$ be the minimal class of functions on S^n that contains *all* λ-Lipschitz function for some λ and that is invariant under linear transformations of $\mathbb{R}^{n+1} \supset S^n$ and let $\mathcal{C}$ be the class of functions f on S^N such that the restrictions of f to all n-dimensional equators belong to $\mathcal{C}_{n,\lambda}$.

Do the families $f = f_N$ of such functions enjoy the virtual constancy property?

These questions also arise for other families of spaces S_N where the virtual constancy is known for uniformly continuous classes $\mathcal{C}_{unif}$ of functions but where one does not (?) even have counterparts to convex functions.

For instance,

are there significant extensions of the class $\mathcal{C}_{unif}$ in the Euclidean and/or in the amenable hyperbolic examples (see $[\bigcirc]$and $[\bowtie]$ above) for which the virtually constancy property remains valid?

Similarly, one wants to know how much the Pythagorean argument for virtual/prevalent constancy of composed maps $f \circ \sigma : S_0 \to R$, for Lipshitz (and more general) maps $S \to R$ extends to non-Riemannian "*locally coherently concentrated*" spaces S, e.g. to *Finsler spaces* (e.g. with uniformly convexity condition on small balls), to *Carnot-Caratheodory spaces* and, possibly, to *fractal spaces.*

Also one looks for sharpening of virtual/prevalent constancy estimates with some conditions on the target spaces R, as it has been done for *concentration of maps into $CAT(\kappa)$ spaces* (with curvatures $\kappa < 0$).

1.7 ABC of Concentration.

Recall that a *metric measure* space is a metric space with a Borel measure on it.

Say that a family of subsets in metric measure spaces, $T_N \subset S_N$, is *virtually prevalent* if the ε-neighbourhoods $U_\varepsilon(T_N) \subset S_N$ are of "eventually full measure" in S_N,

$$\frac{\mu_N(U_\varepsilon(T_N))}{\mu_N(S_N \setminus U_\varepsilon(T_N)} \underset{N\to\infty}{\to} \infty \text{ for all } \varepsilon > 0.$$

The following two (essentially obvious) examples, that carry the same message as the Law of Large numbers for normal distributions, are associated with the names of Maxwell (Euler? Bernoulli?) and Poincare.

[○] *Prevalence of Equators in S^N.* If the dimensions of equatorial subspheres $S^M \subset S^N$ satisfy

$$\liminf_{N\to\infty} \frac{M}{N} > 0$$

then these S^M are virtually prevalent in S^N.

[•] *Prevalence of Balls in the unit Spheres $S^N \subset \mathbb{R}^{n+1}$.* (a) If the radii r of metric balls $B^N = B^N(r) \subset S^N$ are such that the measures of B^N satisfy

$$\liminf_{N\to\infty} \frac{\mu_N(B^N)}{\mu_N(S^N)} > 0,$$

then these B^N are virtually prevalent in S^N.

Notice that [•] for hemispheres is essentially the same as [○] for $M = N - 1$.

Expanding. A family of metric measure spaces S_N, $N = 1, 2, 3, ...$, is called *Lévy expanding* or *a Lévy expander* if, for all families of Borel subsets $T_N \subset S_N$, the condition

$$\liminf_{N\to\infty} \frac{\mu_N(T_N)}{\mu_N(S_N)} > 0,$$

implies that the subsets T_N are virtually prevalent in S_N:

$$\frac{\mu_N(U_\varepsilon(T_N))}{\mu_N(S_N \setminus U_\varepsilon(T_N)} \underset{N\to\infty}{\to} \infty \text{ for all } \varepsilon > 0.$$

Paul Lévy proves in his book[56] that

[56]Problémes concrets d'analyse fonctionnelle,1951

the family of the spheres S^N is a Lévy expander

by observing that this follows from the above [•] via the

Spherical Isoperimetric Inequality. *Among all Borel subsets $T \subset S^N$ with a given measure μ_0, the metric balls have minimal measures of their ε-neighbourhoods $U_\varepsilon(T) \subset S^N$.*[57]

On Paul Lévy Isoperimetric Ineqiality. Lévy sketches a proof of a similar inequality for general convex hypersurfaces $S \subset \mathbb{R}^{N+1}$ that depends on solution of the minimisation problem for the measure $\mu(U_\varepsilon(T))$ over all $T \subset S$ with $\mu(T) = \mu_0$ and that uses a bound on the volume (measure) of the ε-neighbourhoods of hypersurfaces $H^{n-1} \subset S$ in terms of the mean curvatures of these H^{n-1}. This proof needs certain regularity of the boundaries H^{n-1} of extremal T that still remain problematic.

On the other hand, *Almgren-Allard regularity theorems* (unavailable to Lévy) justify a slight modification of Levy's argument that yields isoperimetric inequalities in *Riemannian manifolds with lower bounds on their Ricci curvatures.*

Monochromatic Concentration. A family of metric measure spaces S_N, $N = 1, 2, 3, ...$, is called *monochromatically concentrated* if

almost all measures of the spaces S_N coloured into k colores eventually, for $N \to \infty$, concentrate near a single colour.

In writing, if

$$S_N = \bigcup_{i=1,\dots,k} T_{N,i},$$

then there exist monochromatic subsets $T_N = T_{N,i_0=i_0(N)} \subset S_N$, that are virtually prevalent.

Continous Concentration. A family of metric measure spaces S_N, $N = 1, 2, 3, ...$, is called *continously concentrated* over a topological space R if

given continuous maps $f_N : S_N \to R$, there exist points $r_N \in \mathbb{R}$, such that the r_N-levels of these functions, that are the pullbacks $T_N = f_N^{-1}(r_n) \subset S_N$, are virtually prevalent:

$$\frac{\mu_N(U_\varepsilon(T_N))}{\mu_N(S_N \setminus U_\varepsilon(T_N)} \underset{N\to\infty}{\to} \infty \text{ for all } \varepsilon > 0.$$

It is essentially obvious to see that

Lévy's expanders are monochromatically as well as functionally concentrated over $\mathbb{R}$ (where one needs the expanding proper only of subsets of measure $1/2$) and according to Lévy this taks care of spheres S^N and, more generally, of families of closed Riemannian manifolds S_N with their Ricci curvatures bounded from below by $const_N \to \infty$.

Also observe that

continuous concentration over $\mathbb{R}$ implies the monochromatic one.

Indeed, if S is 2-colored, i.e. covered by two subsets U_1 and U_2, we do it with the distance functions to these subsets similarly how we derived monochromatization from equalization (see ⊞ in the previous section). Then the case of k-coloring trivially follows by induction on k starting from $k = 2$.

[57]The proof of this (exercise in *Schwarz symmetrisation*) appears in writing in E. Schmidt's *Beweis der isoperimetrischen Eigenschaft der Kugel im hyperbolischen und sphärischen Raum jeder Dimensionszahl.* Math. Z., 49, pp 1-109, 1943/44.

On Quantitative Prevalence and Concentration. All of the above admit obvious (nearly sharp) quantitative versions. This in conjunction with *the John-Dvoretzky-Rogers ellipsoid/cube sandwich theorem*[58] applies (not quite directly) to "kind of convex" functions f on the spheres $S^N \subset \mathbb{R}^{N+1}$, that depict unit spheres of Banach norms on $\mathbb{R}^{N+1}$; thus, Milman (1971) obtains his quantitate version of the almost round section theorem.

Continous Concentration Versus Prevalent Constancy of Functions. Continous concentration is by far stronger property than Prevalent constancy. For example, Lipschitz functions on the spaces H^N/Γ in the amenable hyperbolic example (see [⋈] in the previous section) may be prevalently constant but, apparently, never virtually concentrated.

Continuous Concentration Over $\mathbb{R}^k$*.* Continuous concentration property for maps $\mathbb{R}^k$, unlike how it is with prevalent constancy (that concerns composed maps $f \circ \sigma : S_0 \to S$ for uniformly continuous families of maps $F : S \to \mathbb{R}^k$ with measures on sets of maps $\sigma : S_0 \to S$ rather than on S per se), *does not* follow[59] but with somewhere from this property for $\mathbb{R}$-valued functions. Yet, the family of spheres S^N, $N \to \infty$, is functionally concentrated over $\mathbb{R}^k$ for all k; moreover, this is true whenever $N/k \to \infty$ by the following

Waist of the Sphere Theorem. *Every continuos map* $f : S^N \to \mathbb{R}^k$ *admits a level* $S_r = f^{-1}(r) \subset S^N$*,* $r \in \mathbb{R}^k$*, such that the volumes of all its* ε*-neighbourhoods* $U_\varepsilon(S_r) \subset S^N$ *are bounded from below by those of the equatorial spheres in* S^N *of codimension* k*.*

This follows from from a version of Borsuk-Ulam theorem along with some higher codimensional counterpart of Schwartz symmetrisation.[60]

Induced Concentration . The continuous concentration property (obviously) passes over from families of metric measure spaces S_N to families of "eventually full measure" subsets $U_N \subset S$, i.e. where the measures of these subsets satisfy

$$\frac{\nu(U_N)}{\nu(S_N \setminus U_N)} \underset{N\to\infty}{\to} \infty.$$

Then concentration "descends" from U_N to V_N under families of surjective maps $P : U_N \to V_N$, provided the maps in these families are *uniformly continuous* (e.g. all λ-Lipschitz with λ independent of N) and where the relevant measures on V_N are the push-forwards of the measures on U_N.

This can be applied to *prevalent* families of "mildly distorted" submanifolds in the N-dimensipanl spheres, say $T_N \subset S^N$, where all ε-neighbourhoods $U_\varepsilon(T_N) \subset S^N$ are of eventually full spherical measure and where the normal projections $U_\varepsilon(T_N) \to T_N$ are well defined and uniformly continuous. Thus, for instance, Milman proves the concentration property over $\mathbb{R}$ for the *Stiefel manifolds* $St_{M,N}$ that are spaces of (equatorial) isometric maps $S^{M-1} \to S^{N-1}$. These manifolds naturally embed into the sphere S^{MN-1} with codimensions $\approx M^2/2$ where they are prevalent in-so-far as $M = o(N)$ for $N \to \infty$.

In fact, this (properly formulated) continuous concentration property of $St_{M,N}$ over $\mathbb{R}$ remains valid for all $M \leq N$ by the Paul Lévy inequality but

[58]See lectures on *Measure Concentration* by Alexander Barvinok and *Lectures in Geometric Functional Analysis* by Vershynin

[59]I would try manifolds with large positive Ricci curvatures and small sectional curvatures for specific examples.

[60]See my paper on *Isoperimetry of Waists.*

there is no, at least not for the moment, alternative proof of such concentration over $\mathbb{R}^k$ for $k \geq 2$.

ε-Monochromaticity and Concentration for Product Spaces. Combinatorial approximation (see $\diamondsuit$ in 1. 5) allows an extension of Ramsey product property (see **1** in section 1.2) to ε-monochromatizable diagrams of compact spaces.

This approximation brings along asymmetry and non-effectiveness to the Ramey properties of product spaces, while stability of *prevalent constancy* under Cartesian products (e.g. of geometric Pythagorean diagrams), that effortlessly comes by a direct (and obvious) argument, is free of these shortcomings.

Concentration and isoperimetry are also effectively product stable,[61] but the mechanisms behind this stability are more elaborate.[62]

Question. Are there interesting instances of Cartesian products of continuous diagrams with combinatorial ones?

Is there any worth in the continuous counterpart to the (Hales-Jewett) monochromatic block-diagonal theorem for Cartesian diagrams/categories[63] generated by geometric diagrams such as that of isometric embeddings between spheres?

Perspectives on Concentration. An unexpectedly rich geometry of concentration of multi-parametric systems to single points (dictated by the law of large numbers in the probability theory) emerged from the work by Emil Borel, Paul Lévy and Vitali Milman. (See M. Ledoux' monograph *The concentration of measure phenomenon.*)

More general and poorer understood phenomenon of concentration of "projections" of (huge) spaces of microstates of a statistical ensembles to low dimensional "screens of macroobservables", is described in geometric terms in section $3\frac{1}{2}$ of my book *Metric Structures for Riemannian and Non-Riemannian Spaces.*

An alternative view on concentration described by *Poincare inequalities on infinite dimensional foliated spaces* is indicated in my survey article *Spaces and Questions.*

1.8 Overlaps, Disentanglements and Fourier Analysis Approach to Dvoretzky's Theorem.

A topological binary diagram $\mathcal{D} = (\{S, T\}, \Sigma)$ where S and T are topological (often metric) spaces and Σ is a space of continuous (often isometric) maps $\sigma : S_0 \to S$ is called *disentangled* if the images of the maps σ *do not overlap*, that is if $\sigma_1(T)$ and $\sigma_2(T)$ intersect in S, then $\sigma_1(T) = \sigma_2(T)$. In other words, S is *partitioned* (fibered) by these images.

Notice that one can always *disentangle* any $\mathcal{D}$ by replacing S by the set $\tilde{S}$ of pairs (V, s) for $V \subset S$ and $s \in V$ and where $V = \sigma(T)$ for some $\sigma \in \Sigma$. Then the maps $\sigma \in \Sigma$ send $T \to \tilde{S}$ by $t \mapsto (\sigma(T), \sigma(t))$.

Example: Orbit Diagrams. If S is acted by a group Γ, one associates to it *the*

[61]Concentration in product spaces had been nurtured by Bernoulli under the name the law of large numbers and it flourishes in "physical" spaces with Gibbsian quaisiproduct measures.

[62]See Talagran's *Concentration of Measure and Isoperimetric Inequalities in Product Spaces* and my *Isopermetry of Waists and Concentration of Maps.*

[63]Such a diagram (as defined in **3** of 1.1) contains Cartesian products of spaces and maps as well as all diagonal maps.

orbit diagram $\mathcal{D} = (\{S, \Gamma\}, \Sigma)$ with Σ being the space of the equivariant (orbit) maps $\Gamma \to S$.

The topological chromatic number of a topological binary diagram $\mathcal{D} = (\{S, T\}, \Sigma)$ is the *minimal number* k such that S can be covered by k open subsets with the property that *no monochromotizing map* $\sigma : T \to S$, $\sigma \in \Sigma$ exists. (If the space S is discrete this reduces to the combinatorial chromatic number of a hypergraph.)

There are no non-trivial lower bounds on chromatic numbers of *combinatorial* disentangled diagrams (i.e. for discrete S) but such bounds are *essential* in topology.

For instance, the Borsuk-Ulam monochromatic $\mathbb{Z}_p$-orbit theorem stated in section 1 has the following obvious corollary.

Let S be a topological k-connected[64] manifold acted upon by the cyclic group $\mathbb{Z}_p$. Then the chromatic number of the corresponding orbit diagram $\mathcal{D} = (\{S, \mathbb{Z}_p\}, \Sigma)$ satisfies:

$$chr_{top}(\mathcal{D}) \geq k + 2 \text{ for } p = 2 \text{ and } chr_{top}(\mathcal{D}) \geq k/2 \text{ for primes } p \neq 2.$$

This has the (obvious and well known) combinatorial interpretation.[65]

Let Σ_ε be a (arbitrarily small) neighbourhood of the above Σ in the space of (all) continuous maps $\mathbb{Z}_p \to S$.

Then the *combinatorial* chromatic numbers of the corresponding ε*-orbit diagrams* $\mathcal{D}_\varepsilon = (\{S, \mathbb{Z}_p\}, \Sigma_\varepsilon)$ are bounded from below by

$$chr_{comb}(\mathcal{D}_\varepsilon) \geq chr_{top}\mathcal{D}.$$

An amusing feature of this can be seen in the simple case of there standard ± 1-action of $\mathbb{Z}_2$ on the sphere S^N, where the corresponding diagrams are graphs, call them G_ε, on the vertex set $S = S^N$, where s_1 is joined by an edge with s_2 if $dist(s_1, -s_2) \leq \varepsilon$ and where

$$chr_{comb}(G_\varepsilon) \geq N + 1$$

by the Borsuk Ulam theorem.

It follows from general (and obvious by the to-days standards) principles[66] that G_ε, for all $\varepsilon > 0$, contains a *finite* subgraph F_ε with $chr_{comb}(F_\varepsilon) = N + 1$. In fact, one can achieve this directly in the present case by taking a δ-dense subset in S with $\delta \leq \varepsilon/n$ (of cardinality about $\exp(n\delta^{-1})$ for the vertex set of F_ε.

Also, it is clear that all (sub)graphs F on $N < 1/\varepsilon$ vertices $s \in S = S^n$ where the distances between the ends of the edges in F are all $\geq \pi - \varepsilon$ can be 2-colored.

Notice in this regard that whichever number N you take,

> *suitably modified random graphs may have arbitrary large chromatic numbers, with all N-vertex subgraphs in it being acyclic* (Erdös, 1959).

[64] This means the homotopy groups $\pi_i(S) = 0$ for $i = 0, 1, , , , k$.

[65] For more see *Using the Borsuk-Ulam theorem* by Matoušek, *Suborbits in Knaster's problem* by Bukh and Karasev and *Extensions of theorems of Rattray and Makeev* by Blagojević and Karasev.

[66] The most general such principle is called the *Löwenhei–Skolem compactness theorem* in model theory (1915), and the relevant combinatorial version of this is known as *De Bruij–Erdös theorem* (1951).

But there are few "specific natural classes" of *non-random* graphs[67] with this property, or even of those graphs the chromatic numbers of which are much greater than those of all moderate size subgraphs in them.

Disentangled diagrams, in particular, orbit diagrams are studied in the framework of algebraic topology but the following questions have some combinatorial flavour.

[I] Can one eliminate the cycles of length $\leq N$ in the above graphs G_ε by (systematically) removing some edges from them, while keeping the chromatic numbers of these graphs unchanged?

[II] Let $\Gamma = \mathbb{Z}_p \times \mathbb{Z}_q$ for primes $p \neq q$. Then, conjecturally, the chromatic numbers of the ε-orbit diagrams $\mathcal{D}_\varepsilon$ associated to a continuous actions A of Γ on compact manifold S are *universally bounded* by $const = const(\Gamma)$ for all small positive $\varepsilon \leq \varepsilon(S, A) > 0$.

(This must follow from the existence of actions of these Γ without fixed points on contractible manifolds and be extendable to all *non-Smith type* groups, see *Current Trends in Transformation Groups* by A. Bak, M. Morimoto, F Ushitaki.)

What makes [II] combinatorially attractive is that (the orbit diagram of) the Cartesian product action of $\Gamma = Z_p \times \mathbb{Z}_q$ on the product of infinite dimensional spheres, $S = S^\infty \times S^\infty$, $S^\infty \subset \mathbb{C}^\infty$, has *infinite* chromatic number by the Ramsey product property (see **1** in 1.2), while the above conjectural bound suggests "effective" counterparts to probabilistic examples of products of hypergraphs with bounded chromatic numbers.[68]

It remains unclear, in general, by how much the chromatic number may *decrease under disentanglement* of a diagram, where a significant decrease is expected for the Kakutani-Yamabe-Yujobo diagrams of (isometric maps from) orthonormal $(N+1)$-frames in $S^N \subset \mathbb{R}^{N+1}$, say for $N+1 = p \cdot q$ where $p \neq q$ are primes and where amazingly (for a topologist) the existence of monochromatic orthonormal frames does not come from the corresponding property of a disentangled diagram but is obtained by an elementary continuity argument.

But this seems an exception rather than a rule, especially in view of recent counterexamples to the *Knaster conjecture*[69]. Yet, the overall picture remains unclear.

Question. Let $S = G/H$ where G be a *connected Lie group* (or a more general topological group) that naturally (and transitively) act on S and let $T \subset S$ be a finite subset.

What are conditions on $S = G/H$ and on T, such that the topological chromatic number of the diagram $\mathcal{D} = (\{S, T\}, \Sigma = G)$ (i.e. where Σ consists of all $g \in G$ applied to $T \subset S$) does not change (or, on the contrary, significantly decreases) when $\mathcal{D}$ is disentangled?

[67] Most (all?) interesting examples of natural infinite families of finite non-random graphs are of group theoretic or of arithmetic origin, as what you see in *Elementary Number Theory, Group Theory and Ramanujan Graphs* by Davidoff, Sarnak and Valette.

[68] See Mubayi & Rödl on http://www.math.cmu.edu/ mubayi/papers/bergesimon.pdf.

[69] The full conjecture was disproved by Kashin and Szarek and it's weaker form saying that *every continuous function* $f : S^\infty \to \mathbb{R}$ *(for* S^∞ *being unit sphere in the infinite dimensional Hilbert space) can be monochromatized by an isometric motion of a given finite subset* $T \subset S^\infty$ is most likely to be false as well. (See *Suborbits in Knaster's problem* by Bukh and Karasev.)

There is also a continuous version of this question for "continuous colorings" into pullbacks of points $r \in R$ under continuous maps $S \to R$, where the dimension k of R, e.g. for $R = \mathbb{R}^k$ plays the role of the number of colors.

Disentangling Dvoretzky's Almost Round Section Theorem.

Dvoretzky's theorem concerns diagrams of isometric embedding between spheres , say $S^n \to S = S^N$ where one may take $N = \infty$, where the space $\tilde{S}$ of the corresponding disentagled diagram makes the n-sphere bundle over the Grassmann manifold $Gr_{N+1}(\mathbb{R}^N)$ where this bindle is universal for $N = \infty$ since the total space of the corresponding principle $O(n+1)$-bundle over $Gr_{n+1}(\mathbb{R}^\infty)$ is contractible.

What can be said of functions on $\tilde{S} \to \mathbb{R}$ that are continous functions on n-sphere "parametrised" by $g \in Gr_{n+1}(\mathbb{R}^\infty)$?

According to the *Peter-Weyl theorem* the space of L_2-functions on S^n decomposes into orthogonal sum

$$L_2(S^n) = \oplus_i L_i$$

where L_i are *finite dimensional* linear subspaces in $L_2(S^n)$ that are invariant under the orthogonal (isometry) group $O(n+1)$ that acts on S^n, where L_0 denotes the 1-dimensional subspace of constant functions on S^n and where there is *no non-zero invariant vector* in $\oplus_i L_i$, $i \neq 0$, that is *fixed by* $O(n+1)$.

The family of L_i over all spheres $S^n = S^n_g \subset \tilde{S}$, $g \in Gr_{n+1}(\mathbb{R}^\infty)$, define a *vector bundle*, $\mathcal{L} \to Gr_{n+1}(\mathbb{R}^\infty)$ associated to the principal $O(n+1)$-bundle bundle over $Gr_{n+1}(\mathbb{R}^\infty)$ via the action (linear representation) of $O(n+1)$ on L_i.

Since L_i has *no invariant vector*, the bundle $\tilde{L}_i$ has no "natural" non-zero sections and one is *tempted to assume* that, due to the universality of the bundle $\mathcal{L} \to Gr_{n+1}(\mathbb{R}^\infty)$, it has *no continuous non-vanishing section* at all.

The simplest criterion for this would be *non-vanishing* of the Euler class χ of $\mathcal{L}_i$ that happens to be true for the tori $\mathbb{T}^n$:

if a linear representation L of $\mathbb{T}^n$ has no invariant non-zero vector, then the corresponding vector bundle $\tilde{L}$ associated to the *universal* torus bundle has $\chi(\tilde{L}) \neq 0$ (this obvious if you know ABC of caracteristic classes); hence, *every continuous section of* $\tilde{L}$ *has zeros.*

But if such $\tilde{L}$ is associated to a universal $U(n)$-bundle ($U(n)$ is as good as $O(n+1)$ as far as applications go), then $\chi(\tilde{L}) \neq 0$ for for *analytic and anti-analytic* representations L of $U(n)$ but not for tensor products of of them.

Now, given a continuous function f on $\tilde{S}$, one orthogonally projects its restriction $f_{|S^n_g}$ to the space L_i for all $g \in Gr_{n+1}(\mathbb{R}^\infty)$ and thus gets a continuous section $\tilde{P}_i(f) : Gr_{n+1}(\mathbb{R}^\infty) \to \mathcal{L}_i$.

If one knew that such a section has zeros for all representations $L = L_i$ without invariant vectors one would immediately arrive at a *non-holonomic* (called so by Burago and Ivanov) Dvoretzky theorem, since vanishing of many initial spherical harmonics of a "convex" function on on S^n makes this function nearly constant. But this "non-holonomic Dvoretzky" was shown to be false by Burago and Ivanov in *Topological aspects of the Dvoretzky Theorem* (2009).[70]

[70] I tried to prove the vanishing property of sections of $\mathcal{L}_i$ before I realised this would be in conflict with *Floyd examples* of fixed point free group actions. I thought I could go around this by confronting several principal unitary bundles over parts of Grassmannians $Gr_{n+1}(\mathbb{R}^\infty)$

Remarks and Questions. (a) How do overlaps in the original diagram of isometric maps $S^n \to S^N$ influence the sections $\tilde{P}_i(f)$?

For instance, let L be a finite dimensional linear subspace in the space of L_2-functions on S^n that is invariant under $O(n+1)$ and that is orthogonal to the constants. Let f be a bounded continuous function on the sphere S^∞ and let a number $\varepsilon > 0$ be given.

Does there always exist an equatorial subsphere $S^n = S^n_g \subset S^\infty$, $g \in Gr_{n+1}(\mathbb{R}^\infty)$, such that the normal projection $P_g(f) \in L = L_g$ of this function f restricted to S^n_g sastisfies $||P_g(f)|| \leq \varepsilon$?

(b) The vanishing argument does apply to continous sections of associated vector bundles $\mathcal{L}$ over over $Gr_2(\mathbb{R}^\infty)$ where the relevant group is the circle $\mathbb{T}$. This delivers "homologically large" subsets H_N in the Grassmanian $Gr_2(\mathbb{R}^{N+1})$ of equatorial circles $S^1 \subset S^N$ such that a given "convex" function f on S^N is nearly constant on the circles from this H_N.

The subsets H_N can be shown to be prevalent in $Gr_2(\mathbb{R}^{N+1})$ for $N \to \infty$ which leads to a yet another proof of the prevalent constancy theorem for uniformly continuous functions,[71] but it is unclear if this helps in the convex case.

(c) Does the Fourier analysis approach to Dvoretzky's theorem has anything to do with such an approach by Roth and by Gowers to monochromatic arithmetic progressions?

1.9 Projections of Sets and of Measures and Grothendieck-Dvoretzky Problem for Crofton Metrics in Desarguesian Spaces.

By duality, Dvoretzky's theorem yields *almost round projections* along with almost round sections of high dimensional convex bodies. Since taking convex hulls commutes with projections, this implies a similar statement for families of arbitrary open bounded subsets in $U_N \subset \mathbb{R}^N$;

There exists surjective affine maps $\sigma_{n,N_n} : \mathbb{R}^N \to \mathbb{R}^n$, $n = 1,2,3,...$, such that the Hausdorff distances from the images of U to the unit balls in $\mathbb{R}^n$ satisfy

$$dist_{Hau}(\sigma_{n,N_n}(U), B^n(1)) \underset{n\to\infty}{\to} 0.$$

Now, thinking functorially, you dismiss projections of sets and look at pushfowards of measures where the natural expectation is the following.

Let μ_N, $N = 1,2,...$, be probability measures on $\mathbb{R}^N$, such that $\mu_N(H) = 0$ for all affine hyperplanes $H \subset \mathbb{R}^N$.[72] *Then there exist surjective affine maps $\sigma_{n,N_n} : \mathbb{R}^N \to \mathbb{R}^n$ such that the push forwards of these measures are ε_n-radial where $\varepsilon_n \to 0$ as $n \to \infty$.*

with variable n and even tried to explain my "proof" to Dvoretzky at the 1966 Congress of Mathematician in Moscow. Dvoretzky was not impressed and suggested to me looking for a purely affine proof of his theorem instead.

During the congress I stayed at Dima Kazhdan's place who, one evening, jumped out of his bed and started explaining to me the idea of the T-property. It took me 20 years afterwards to understand what had made him so exited about it.

[71] In fact, the unit sphere bundles in $\mathcal{L}_i$ admit no *uniformly continuous* sections as we show with Milman in *A Topological Application of the Isoperimetric Inequality.*

[72] This condition is not *truly* necessary

Here, a measure μ in $\mathbb{R}^n$ is called radial if it is $O(n)$-invariant and ε-radial means that that it is invariant in the weak topology up-to ε with respect to the *Monge-Kantorovich transportation metric* (any metric defining the weak topology in the space of measures will do) where this "up-to" is arranged in an obvious way to be invariant under the scalings $x \mapsto \lambda x$ of $\mathbb{R}^n$.

I formulated this in *"Dimension, Non-linear Spectra and Width"* appearing in 1988 issue of *Geometric Aspects of Functional Analysis* as an "obvious corollary" to Dvoretzky's theorem; this was dismissed as non-interesting due its triviality by the geometric functional analysis community.

But when 20 years later I tried to apply this to construction of $(n-k)$-*planes crossing many k-simplices in* $\mathbb{R}^n$ in the second part of *"Singularities, Expanders..."* I realised that my "trivial argument" did not work.

At that time, Bo'az Klartag was visiting Courant institute and, almost instantaneously, he furnished the proof[73] that relied on a lemma from a paper by Bourgain, Lindenstrauss and Milman in this very volume of *Geometric Aspects of Functional Analysis.*

Questions. Does Klartag's theorem extend to measures on vector spaces $\mathbb{F}^N$ for locally compact non-Archimedean fields $\mathbb{F}$ (where the role of $O(N)$ is played by maximal compact subgroups in $GL_N(\mathbb{F})$)?

What are counterparts to Klartag's theorem in combinatorics for functions $f = f(s)$ with values in commutative semigroups and push forwards of such functions under (surjective?) maps $\sigma : S \to T$ for

$$(\sigma_* f)(t) = \sum_{\sigma(s)=t} f(s)$$

in suitable diagrams of such maps.

Can one see Dvoretzky's theorem as a limiting case[74] of a generalised Klartag theorem?

Desarguesian Perspective on Measures, Metrics and Convexity. Let us bring Klartag's theorem closer to that of Dvoretzky in the following setting.

A metric $dist = dist(s_1, s_2)$ on spherical domain $U \subset S^N$ is called *Desarguesian* if simple curves in U that are contained as subsegments in equatorial semicircles in S^N are distance minimizing for this $dist$ as it is in the case for the standard spherical metric restricted to U.

Exampe. Minkowski-Banach metrics on the Euclidean space $\mathbb{R}^N$ (projectively realised by a hemisphere) are Desarguesian. They are distinguished by being *scale invariant* that, along with the triangle inequality, implies convexity of metric balls in theses spaces.

Observe that a linear combination $dist = \sum_i c_i dist_i$ of Desarguesian metrics $dist_i$ is a Desarguesian metric, provided this "$dist$" is a metric at all (i.e. positive and satisfies the triangle inequality). In particular, positive combinations of Desarguesian metrics are Desarguesian; thus, Desarguesian metrics constitute a *convex* cone, call it $\mathbb{D}es = \mathbb{D}es_N$, in the space of all metrics on the N-sphere

[73] *On nearly radial marginals of high-dimensional probability measures.*

[74] This could be a kind of *tropical limit* of *the Radon transform* similar to *Legendre transform* associated to *Fourier-Laplace transform* in *tropical geometry.* (See *"Tropical Mathematics, Idempotent Analysis..." by Litvinov.)*

S^N that is invariant under the action of the linear group $GL(N+1)$ on S^N by projective transformations.

Desarguesian metrics, were singled out by Hilbert in his fourth problem[75] by analogy with the Minkowski metrics associated to convex bodies in affine spaces. Despite significant efforts one's understanding of $\mathbb{D}es$ lags behind that of the Minkowski-(Banach) and Kähler cases.[76]

The simplest class of Desarguesian spaces is associated to measures as follows.

A Crofton geometry of codimension one on a manifold V is given by a Borel measure ν on the space $\mathcal{H}_{-1}$ of hypersurfaces $H \subset S$. Such a measure defines by *Poincare-Crofton's duality* the length function of curves $l \subset S$ by integrating the number of intersections of H with L

$$length_{\nu^\perp}(l) = \int_{\mathcal{H}_{-1}} card(l \cap H) d\nu.$$

Then the corresponding metric[77] $dist_{\nu^\perp}$ on S is defined as usual by by minimising this $\nu^\perp$-length of curves between given pairs of points in S.[78]

It is obvious but significant that if ν is supported on the subspace $S_\perp \subset \mathcal{H}_{-1}$ of (cooriented if you wish) equatorial $(N-1)$-spheres in S^N then the metric $dist_{\nu^\perp}$ is *Desarguesian.* We call these *Crofton-Desargues* metrics on S^N.

Exampe. Let $B \subset S^N$ be an open ball in S^N that is strictly contained in a hemisphere. Then the standard hyperbolic Riemannian metrics on B of constant negative curvatures are $dist_{\nu^\perp}$ for an infinite measure ν that is supported on the equators in S^N that intersect B and that is invariant under projective transformations of S^N that preserve B.

It was shown by Pogorelov that

all Desarguesian metrics on the 2*-sphere are Crofton*

but there are many non-Crofton Desarguesian metrics in dimensions ≥ 3. These *Crofton-Desargues* metrics constitute a (proper for $N \geq 3$) closed convex invariant subcone in the cone of Desarguesian metrics, say $\mathbb{CD} \subset \mathbb{D}es$, which equals (this is easy to show) the closed convex conical hull of the $GL(N+1)$-orbit of the standard spherical metric $dist_\circ$ on S^N.

Let $dist$ be a Desarguesian metric on S^N such that

$$diam(S, dist) =_{def} \sup_{s_1, s_2 \in S} dist(s_1, s_2) = \pi.$$

Conjecture. There exits an equatorial subsphere $S^n \subset S = S^N$ such that the metric space $(S^n, dist_{|S^n})$ is ε-close to the standard sphere S^n where $\varepsilon \leq \varepsilon_{n,N} \to 0$ for $N \to \infty$.

To be precise one needs to specify what ε-closeness between metric spaces signifies.

[75]See the survey *On Hilbert's fourth problem* by Papadopoulos (?2013) and references therein.

[76]Kähler metrics minimise areas of holomorphic curves in complex manifolds and, thus, can be seen as complex counterparts to Desarguesian metrics

[77]A priori, this $dist_{\nu^\perp}$ may vanish or to be infinite. This can be ruled out by imposing a mild regularity/finiteness condition on ν.

[78]The most significant Crofton geometries are given by measures on the spaces of hypersurfaces (of real codimension two) in complex algebraic manifolds that define Kähler metrics (or possibly singular closed positive $(1,1)$-current) on these manifolds.

The weakest(?) and the safest definition will be that of the *Hausdorff distance between metric spaces* S_1 and S_2 that is the minimal ε for which there exists an $(S_1 \leftrightarrow S_2)$-correspondence, that is a subset $H_\varepsilon \subset S_1 \times S_2$, that projects *onto* S_1 and S_2 and such that

$$|dist_1(s_1, s_1') - dist_2(s_2, s_2')| \le \varepsilon$$

for all pairs of pairs of H_ε-*corresponding* points $(s_1 \leftrightarrow s_2, s_1' \leftrightarrow s_2')$, where $s_1 \leftrightarrow s_2$ signifies that these points come from the same point in H_ε under the projections $H_\varepsilon \to S_1, S_2$.

The strongest(?) condition would be in terms of the standard spherical metric $dist_\circ$ on $S = S^N$. Namly one may require that $|dist - dist_\circ|$ is bounded (as a function in two variable) on S^n by ε.

Notice that this conjecture for *Crofton-Desargues* metrics is (essentially) equivalent to Klartag's theorem and its general form, probably, implies Dvoretzky's theorem.

We shall return to Crofton geometries and to Desarguesian spaces in section??? and conclude here by formulating the following quintessentially convex theoretic theorem due to Milman[79] the quantitative formulation of which carries yet undisclosed (in my view) qualitative (Ramsey theoretic?) message.

There exist a positive function $c = c(\varepsilon) > 0$, $\varepsilon > 0$, *such that every Minkowski-Banach space* S *of dimension* N *admits a subspace* $S' \subset S$ *with an* n*-dimensional quotient space* S'' *of* S' *such that* $n \ge c \cdot N$ *and such that* S'' *is* ε*-close to the Euclidean space* $\mathbb{R}^n$ *for the natural (Banach-Mazur) distance between Banach spaces.*

1.10 Banach Conjecture and Topology of Bifibrations.

Banach conjectured that if all k-dimensional subspaces for $k \ge 2$ in a Banach space X of dimension $n > k$ are mutually isometric, mutually isometric, then X is isometric to the Euclidean/Hilbertian space.

The solution in the case $k = 2$, as it is indicated by Dvoretzky in his paper,[80] was, according to Banach, obtained by S. Mazur and if $dim(X) = n = \infty$ this (trivially) follows for all $k < \infty$ from the almost round section theorem of Dvoretzky.

Mazur derived his case of the conjecture from Poincarés "hairy sphere" theorem by showing that

> a ***continuous*** *family of mutually isometric*[81] ***non-Euclidean*** *norms on the 2-planes in* $\mathbb{R}^3$ *would give rise to a* ***non-vanishing continuous*** *vector field on the sphere* S^2.

Similarly, let $Gr_k(L^n)$ denote the Grassmann manifold of oriented linear k-subspaces L^k_g, $g \in Gr_k(L^n)$, in the linear n-space $L^n = \mathbb{R}^n$, and let $||...||_g$ be a continuous family of norms in the spaces.

If the Banach spaces $(L^k_g, ||...||_g)$ are mutually isometric, say all being isometric to a Banach space $Y = (L^k, ||...||_Y$, then, obviously

[79] *Almost Euclidean quotient spaces of subspaces of a finite-dimensional normed space* (1985). Amusingly, as Milman told me, this four page article was rejected by BAMS being called "non-interesting" by a referee.

[80] I learned about this conjecture from this paper but I have not read Banach.

[81] "Isometric" here means the *existence* of linear isometries between these spaces.

the structure group of the canonical k-vector bundle over $Gr_k(\mathbb{R}^n)$ reduces to the isometry group $Iso(Y) \subset GL(n, \mathbb{R})$

Now elementary topology of fiber bundles tells you that

(a) *if k-is even and $n > k$ then the group $Iso(Y)$ is transitive on the unit sphere in Y; hence, Y and consquenly X are isometric to Euclidean spaces;*

(b) if $n \geq 3k - 2$ then $Iso(Y)$ equals the full special orthogonal group $SO(k)$; thus again, X is isometric to the Euclidean space.

Furthermore, non-parallelizibilty of spheres S^{n-1} for $n \neq 2, 4, 8$ implies the following.

(c) *If $n \neq 4, 8$ and $k = n - 1$ the Banach spaces $L^k_g, \|...\|_g$ admit non-trivial one parameter groups of linear isometries.*

Connecture. If all k-dimensionl subspaces in Banach space X of dimension $n > k$ admit non-trivial isometries then X is isometric to the Euclidean n-space.

(This together with (c) would imply Banach conjecture for $dim(X) \neq 4, 8$)

The above (a)(b)(c) do not depend on "interactions" between "overlapping" (i.e. intersecting) subspaces $L^k \subset L^n$, but these overlaps can be used for proving Banach conjecture for odd n and $k = n - 2$ as follows.

Let $C = St_4(\mathbb{R}^n)$ be the Stiefel manifold of 4-tuples called c of orthonormal vectors $a_1, a_2, b_1, b_2 \in \mathbb{R}^n$, $n \geq 4$, and define maps

$$C \to A = St_2(\mathbb{R}^n) \text{ and } C \to B = St_2(\mathbb{R}^n)$$

by

$$p_A : c = (a_1, a_2, b_1, b_2) \mapsto (a_1, a_2) \text{ and } p_B : c = (a_1, a_2, b_1, b_2) \mapsto (b_1, b_2)$$

where, observe these maps are fibrations with the fibers $D = St_2(\mathbb{R}^{n-2})$.

If n is odd, then – this is a textbook topology – the spaces A, B and D are $\mathbb{Q}$-homologically (in fact, $\mathbb{Q}$-homotopically) are equivalent to spheres,

$$A = B \sim_{\mathbb{Q}} S^M,\ M = 2n - 3, \text{ and } D \sim_{\mathbb{Q}} S^m,\ m = 2n - 7,$$

while

$$C \sim_{\mathbb{Q}} S^M \times S^m$$

with the maps $p_A : C \to A$ and $p_B : C \to B$ both being $\mathbb{Q}$-equivalent to the projection $S^M \times S^m \to S^M$.

Let $L^{n-2}_b \subset \mathbb{R}^n$, $b \in B$, denote the linear subspace normal to the frame $b = (b_1, b_2)$, $b_1, b_2 \in \mathbb{R}^n$, and let $A_b \in A$ denotes the subset of the frames a that are contained in this subspace, where observe

$$A_b = St_2(L^{n-2}_b = \mathbb{R}^{n-2}) = D \sim_{\mathbb{Q}} S^m$$

.

Lemma. Let $U \subset A$ be a *non-empty* subset such that the intersection $U \cap A_b \subset A$ is "homotopically constant" as a function of $b \in B$, which means that the restriction of the map $p_B : C \to B$ to the pull back $\tilde{X} = p_A^{-1}(X) \to B$, denoted $p_B|\tilde{U} : \tilde{U} \to B$, is a *Serre fibration.* Then

the subset $U \subset A$ is, in fact, equals all of A.

Proof. Since U is non-empty, $\tilde{U}$ contains some fiber $D \sim_{\mathbb{Q}} S^m$ of the map $p_B : C \to B$, where, recall, $C \sim_{\mathbb{Q}} S^M \times S^m$; hence, the $\mathbb{Q}$-homology inclusion

[$\neq 0$] *homomorphism* $H_m(\tilde{U};\mathbb{Q}) \to H_m(\tilde{C};\mathbb{Q}) = \mathbb{Q}$ *does not vanish.*

But if $U \neq A$ then $X \cap A_b \neq A_b$ for some $b \in B$, and (by homotopical constancy) all intersections $U \cap A_b$, $b \in B$ have $\mathbb{Q}$-homologies

$$H_m(U \cap A_b;\mathbb{Q}) = 0.$$

Since $B \sim_{\mathbb{Q}} S^M$, $M > m$,

the homology $H_i(B;\mathbb{Q})$ *vanishes for all* $i = 1, 2, ..., m$;

hence,

[$= 0$] $$H_m(\tilde{U};\mathbb{Q}) = 0$$

for $\tilde{X} \to B$ is a Serre fibration with the fibers $\tilde{X}_b = U \cap A_b$. Thus, the proof follows by contradiction.

Now Banach conjecture for normed spaces $X = (\mathbb{R}^n, ||...||_X)$ with isometric k-planes $L^k \subset X = \mathbb{R}^n$ for $k = n-2$ and odd n follows from Mazur's case of $k = 2$ as follows.

Pick up a 2-plane, say $L_0^2 \subset X$, and let $U = A = St_2(\mathbb{R}^n)$ be the set of the 2-tuples of those orthonormal vectors in X that span 2-planes $L^2 \subset X$ *isometric to* L_0^2 with the norm induced from X.

If two $(n-2)$-dimensional subspaces $L_1^{n-2}, L_2^{n-2} \subset X$ are isometric, then the intersections

$$U \cap A_{b_1} \subset A_{b_1} = St_2(L_1^{m-2}) \text{ and } U \cap A_{b_2} \subset A_{b_2} = St_2(L_2^{m-2})$$

for two frames $b_1, b_2 \in B = St_2(\mathbb{R}^n)$ normal to L_1^{n-2} and to L_2^{n-2} are mutually linearly equivalent; hence, "homotopically constant". Then the Lemma says that $U = A$, which means all 2-planes in X are isometric to L_0^2, and Mazur's theorem applies.

This leaves us with the case of n-dimensional Banach spaces X with mutually isometric $(n-1)$-dimensional subspaces where the Banach conjecture *remans unsettled for all even* $n \geq 4$.

Remarks and Questions. A potential approach to the Banach problem, say for $n = 4$, is trying first to understand the case where there exists a 2-dimensional subspace $L_0^2 \subset X = (L^4, ||...||_X)$ such that the spaces $V_0 \subset Gr_2(L^3)$ of isometric copies of this L_0^2 in all (mutually isometric!) subspaces $L^3 \subset X$ are topologically 0-*dimensional*, e.g. this set V_0 is finite. Then all of X is obtained by "affine rotations" of its 3-dimensional subspaces around these 2-dimensional ones that seems to imply this X must be "impossibly round".

In general, Banach norms $||...||_X$ on a linear space L^n define partitions $\Pi_k = \Pi_k(X)$ of the Grassmann manifold $Gr_k(L^n)$, $k < n$, into the $||...||_X$-isometry classes of k-dimensional subspaces $L_g^k \subset L^n$, $g \in Gr_k(L^n)$, where most features of $\Pi_k(X)$, even for $k = 2$, remain obscure.

For instance:

1. What are Banach spaces $X = (L^n, ||...||_X)$ where the quotient space $Gr_k(L^n)/\Pi_k(X)$ is "small" e.g. $dim_{top}(Gr_k(L^n)/\Pi_k(X)) = 1$?

2. Let $||...||_1$ and $||...||_2$ be two norms on L^n such that the Banach spaces $(L^k, ||...||_1)$ and $(L^k, ||...||_2)$ are isometric for every k-dimensional subspace $L^k \subset$

L^n (as it happens, for instance, for all pairs of Euclidean norms on L^n). Does it follow that $(L^n, ||...||_1)$ is isometric to $(L^n, ||...||_1)$ (where such an isometry must be given by some linear transformation of L^n that moves $||...||_1$ to $||...||_2$)?

3. How much of the above extends to infinite dimensional Banach spaces X with all subspaces of given finite codimension being isometric?

4. Can one reprove Bourgain's theorem on Banach spaces with approximately isometric subspaces[82] in the above topological framework?

Bifibrations. Recall that a *correspondence* between two spaces A and B, that is a subset $C \subset A \times B$, is called a (smooth) *bifibration* if the projection p_A and p_B of C to A and B are (smooth) fibrations.

One can also think of this as a familyy of subsets $A_b \in A$ paramertrized by $b \in B$ where C comes about as the set of pairs (a, b) such that $a \in A_b$.

Call a subset $U \subset A$ *topologically bifibrated* if the intersection $U \cap A_b$ is "topologically constant" as a function of $b \in B$, that is if the projection of lift $\tilde{U} = p_A^{-1} \subset C$ to B is a topological fibration.

Homogenous Bifibrations . Let G be a topological groups, let $G_1, G_2, G_3 \subset G$ be closed subgroups where $G_3 = G_1 \cap G_2$ and let

$$A = G/G_1,\ B = G/G_2 \text{ and } C = G/G_3.$$

Then the obvious maps $C \to A$ and $C \to B$ define a bifibration since they are fibrations and since they *embed* $C \to A \times B$.

Common examples of this come from classical linear groups and their classical subgroups, such as spaces $C = Fl_{k_1<k_2}(L^n)$ of (k_1, k_2)-*flags* in a linear space L^n that are pairs of linear subspecies $L^{k_1} \subset L^{k_2} \subset L^n$ naturally projected to the Grassmann manifolds $A = Gr_{k_1}(L^n)$ and $B = Gr_{k_2}(L^n)$.

Conjecture. Let $A \leftarrow C \to B$ be a classical bifibration. Then all *topologically bifibrated* subsets $U \subset A$ equal deformations of some "classical" $U_0 \subset A$ associated to some subgroup of the automorphism group of $(A \leftarrow C \to B)$

Example. If $C = Fl_{2,2m-1}(\mathbb{R}^{2m} = \mathbb{C}^m)$, then $U(m)$-invariant subspaces $U \subset A = Gr_2(\mathbb{R}^{2m})$ are instances of our "classical subbifibrations" that accompany the complex projective space $U_\circ = \mathbb{C}P^{m-1} \subset Gr_2(\mathbb{R}^{2m})$.

Spaces $X = X^n$ with constant invariants $inv(g) = inv(L^k_g, ||...||_g)$. The condition requiring all k-dimensional subspaces in $X = (L^n, ||...||_X)$ to be mutually isometric is grossly overdetermined in contrast with with similar (linear) conditions on functions in the context of *integral geometry* of bi-fibrations.

For instance if f is a continuous centrally symmetric function on S^{n-1}, then, in order to be constant, it only needs to have *mutually equal integrals over all equatorial subspheres* $S_1^{n-2}, S_2^{n-2} \subset S^{n-1}$ that is an incomparably weaker conditions than the existence of isometries $S_1^{n-2} \leftrightarrow S_2^{n-2}$ that equate $f|S_1^{n-2}$ with $f|S_2^{n-2}$.

Are there numerical invariants of Banach spaces $Y = (L^k.||...||_Y)$ comprable in their expressive power to these integrals?

The apparent difficulty in constructing "nice and natural" invariants of *isomorphism classes* of finite dimensional Banach spaces is due to the fact that these classes are defined via the action of the group $GL(k)$ on the space of

[82]See: *On finite dimensional homogeneous Banach spaces.*

norms on L^k: *one can not compensate $GL(k)$-ambiguity by integration*, since this group is non-compact and even non-amenable.[83]

Combinatorial Diagrams as Bifabrations. A binary diagram of injective maps can be seen as a diagram $A \overset{p_A}{\leftarrow} C \overset{p_B}{\rightarrow} B$ for $C \subset A \times B$, regarded as a family of subsets $A_b = p_A(p^{-1}(b)) \subset A, \in B$ where in the diagrmiic terms these subsets are injective images of a single set, say T.

One is deals here with R-colorings of A that are R-valued maps f from A to some set R where one one looks at the lifts of these maps f to C as $\tilde{f} = \tilde{f}(c) = f(p_A(c))$ on C where these $\tilde{f}(c)$ are constant on the p_A-fibers $C_a = p_A^{-1}(a) \subset C$.

Question What are possibilities of restrictions of these C to the p_B-fibers $p_B^{-1}(b) \subset C$ that correspond to subsets $A_b \in A$?

One wants to understand this for the kinds of finite diagrams we met earlier, where, observe, Ramsey and Banach problems lie at two extremal ends of the avenue of possible enquiries.

Some specific questions can be formulated (approached?) in terms of *Radon transform* that makes sense if R is an Abelian semigroup:

the Radon transform sends R-valued function $f(a)$ on A to functions $\psi(b)$ on B by

$$\psi(b) = \sum_{a \in A_b} f(a) = \sum_{c \in p_B^{-1}(b)} \tilde{f}(c) \text{ for } \tilde{f}(c) = f(p_A(c)).$$

If, for instance, R equals the cyclic group $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ for $m = card(A_b) = card(T)$, then *monochromaticity* of a subset $A_b \subset A$ amounts to *vanishing* of $\psi(b)$ at this $b \in B$. And a closely related problem of describing $\{0,1\}$-functions $f(a)$ such that its $\mathbb{Z}$-valued Radon transform $\psi(b)$ for $\mathbb{Z} \supset \{0,1\}$ is *constant on* B has some flavour of the Banach conjecture.

If our diagram/bifibarition admits a large symmetry group, as, e.g. in the case of flag spaces $C = Fl_{k_1<k_2}(L^n)$ for vector spaces L^n over a finite field $\mathbb{F}$, then the solution of such Banach-type problems is easy since a full description of the Radon transform is achievable with elementary representation theory.

It is more interesting (at least more challenging) to understand the Radon transform for the Cartesian powers $A = T^n$ of a finite set T, where A_b are *combinatorial lines* in T^n that equal the images of the maps $T \to A$, such that their projections to all coordinates, that are $T \to T_i = T$, $i = 1, 2, ..., n$, are either constant maps or equal the identity map with at least one of these projections being non-constant.

1.11 Linearized Monochromaticity.

Question. Is there some Ramsey theory in Abelian categories, e.g. in categories of vector spaces and linear maps?

Let us indicate possibilities in approaching this question by departing from a *hypergraph*, that is a collection G of subsets $A_g \subset S$ in a (vertex) set A and expressing its chromatic number that is

[83]Much of the asymptotic geometry of finite dimensional Banach spaces depends on the break of the $GL(k)$-symmetry" by choosing a maximal compact subgroup (isomorphic to $O(k)$) in $GL(k)$.

Sometimes, e.g. in the *Knothe proof of the Brun-Minkowski theorem*, one may also use the (maximal amenable) subgroup of triangular matrices in $GL(k)$.

the minimal k, such that A can be partitioned into k subsets with no monochromatic subset among $A_g \subset A$, $g \in G$.

in linear terms as foliow

Let $L = \mathbb{F}^A$ be the set of functions on A with values in a field $\mathbb{F}$ and observe that subsets $B \subset A$ correspond to *coordinate linear subspaces* $L_B = \mathbb{F}^B \subset L$ that consist of function with *supports* equal B, where, recall, the support of a function $l(a)$ is the subset of those $a \in A$ where $l(a) \neq 0$.

Denote by $Gr_*(L)$ the set of all linear subspaces in L and let us associate to our hypergraph (A, G) the subset $G^L \subset Gr_*(L)$ of the coordinate subspaces

$$L_g = L_{A_g} = \mathbb{F}^{L_g} \subset L = \mathbb{F}^A$$

that correspond to the subsets $A_g \subset A$, $g \in G$.

Define a *linear k-subcoloring* of L as a collection of k-linearly independent subspaces $M_i \subset L$, $i = 1, 2, ..., k$, and say that a linear subspace $L_0 \subset L$ is *monochromatic* if the projection of L to some quotient space

$$M_{\hat{i}} = M/(M_1 + ... + M_{i-1} + M_{i+1} + ... + M_k)$$

is injective (where "+" denotes the linear sum or span of subspaces).

Given a collection H of linear subspaces $L_g \subset L$ define the *linear chromatic number of* (L, H) as the minimal k, such that L admits a k-subcoloring with *no monochromatic* subspace among $L_h \subset A$, $h \in H$.

Observe that every partition of A into k-subset defines a k-subcoloring of L, (where, in fact, $+_i M_i = L$) where monochromaticity of a subset $A_0 \subset A$ is equivalent to monochromaticity of the corresponding coordinate linear subspace $L_0 = L_{A_0} \subset L$.

What is slightly more interesting is that

the linear chromatic number of the collection $G^L \subset Gr_*(L)$ of (coordinate!) linear subspaces $L_g \subset L$ that correspond to subset $A_g \subset A$, $g \in G$, equals the chromatic number of the hypergrpah (A, G).

$$chr_{lin}(G^L) = chr(G).$$

Proof. Every subpartition $\{M_i\}$ of the space $L = \mathbb{F}^A$ of functions $l : A \to \mathbb{F}$ can be moved to a *coordinate* subpartition $\{M'_i\}$ where all M'_i are *coordinate subspaces* that are obtained by *coordinate projections* of M_i such that these projection *invectively* map $M_i \to M'_i$ for all i.

This follows by application of *Hall's marriage Lemma*, that tells you that every set of linearly independent functions $l_i(a)$, $i = 1, 2, ...k$, on A admits k *distinct* points $a_i \in A$ such that each a_i is *contained in the support* of l_i, i.e. $l_i(a_i) \neq 0$.

Question. Are there color-wise interesting non-coordinate families $H \subset Gr_*(L)$ of supspaces L_h in linear spaces L?

Remark. There is another way to linearize a hypergraph (A, G), namely, by associating to it the following polynomial in the variables x_a, $a \in A$,

$$Q_G(x_a) = \sum_{g \in G} \prod_{a \in A_g} x_a.$$

This Q_G can be seen as a polynomial function on $L = \mathbb{Z}^A$, where *isotropic subspaces* $M \subset L$ on which Q_G *vanishes* correspond to subsets $B \subset A$ that *contain none of subsets* A_g. Thus, the chromatic number of G can be seen in terms of decompositions of L into sums of Q_G-isotropic subspaces.

This kind of polynomials can be realised within cohomology algebras of certain topological spaces W [84] that serves to show that every open covering $\cup_i U_i = W$ contains a member $U_{i_0} \subset W$ with "multiplicatively interesting" restriction cohomology homomorphism $H^*(W) \to H^*(U_{i_0})$; but it is unclear if this points toward "homotopy Ramsey theory".

What, for instance, can be seen of Q_G in the category theoretic framework in the (characteristically Ramsey) case where G equals the set of images of maps/morphisms between sets, e.g. $B \to A$ when one starts composing morphisms, say $C \to B \to A$.

1.12 Ramsey-Dvoretzky Equations for Polynomial Maps.

One can formulate the Ramsey problem in a general category $\mathcal{D}$ as follows. Let three classes of morphisms in $\mathcal{D}$ be given:

C, where morphisms $c \in C$ are "simple", e.g. constant, functions,
Σ, where $\sigma \in \Sigma$ are our "unknowns",
F, where $f \in F$ serve as "right hand sides" of equations in σ.

Let morphisms $f = f_S \in F$ issuing from the objects $S \in \mathcal{C}$ be given and let T be an object in $\mathcal{D}$.

The Ramsey property in this terms says that

the equation $f \circ \sigma \in C$, for $T \overset{\sigma}{\to} S \overset{f}{\to} R$, is solvable in σ for some object $S = S(T) \in \mathcal{D}$.

Thus, even if the original maps f can be "complicated",

an arbitrary f can be transformed to a a simple often (approximately) symmetric map c by composing f with a map σ from a given large and well structurally organised class Σ of simple maps.

Similarly, one may formulate the Banach-kind conjecture/principle:

if the composed maps $f_0 \circ \sigma : T \to R$ are "relatively simple" for all $\sigma : T \to S$, e.g. belong to a "smallish" space B of maps $T \to R$, then the map $f_0 : S \to R$ must be comparably simple to start with.

Specific Questions. Let $\mathbb{F}$ be a field and $\Sigma = \Sigma(n, N; d)$ be the space of injective polynomial maps $\sigma : \mathbb{F}^n \to \mathbb{F}^N$ of degree d, i.e. where the coordinate functions of σ are polynomials $\mathbb{F}^n \to \mathbb{F}$ of degrees d defined over $\mathbb{F}$.[85]

[1] Algebraic Banach Problem. Describe algebraic, e.g. polynomial, $\mathbb{F}^k$-valued functions f on $\mathbb{F}^N$ such that the space $f \circ \Sigma$ of the composed maps $f \circ \sigma : \mathbb{F}^n \to \mathbb{F}^k$ has dimension (as an algebraic variety) at most M for a given M that is significantly smaller than $dim(\Sigma)$?

[2] Algebraic Grothendieck-Dworetzky Problem. Let C_n and F_N be two sets of algebraic $\mathbb{F}^k$-valued functions: $c \in C_n$ on $\mathbb{F}^n$ and $f \in F_N$ on $\mathbb{F}^N$.

[84] See part 2 of my "*Singulairities,Expanders...*"

[85] To define "degree" of a polynomial in n variables one has to choose a norm on the "lattice of degrees" that is $\mathbb{Z}^n_+$. The two commonly used "degrees" are associated with the l_1-norm $\sum_i |d_i|$ and with the l_∞-norm $\sup_i d_i$. Strangely, there is no(?) applications of the l_2-norms on degrees.

Under what (natural) condition(s) the equation $f \circ \sigma = c$ is solvable in $(\sigma, c) \in \Sigma \times C_n$ for all $f \in F_N$?

If the field $\mathbb{F}$ is algebraically closed e. g. $\mathbb{F} = \mathbb{C}$ then solvability of $f \circ \sigma \in C_n$ often follows from the inequality

$$dim(\Sigma) \geq dim(F_N) - dim(C_n)$$

or from a slightly strengthened version of it.

In general, this comes as a Diophantine problem that, amazingly, can be solved in some cases by the following

HILBERT'S SYMMETRIZATION ARGUMENT.[86] Let $\phi = \phi(x)$ be a polynomial on $\mathbb{R}^m$ with rational coefficients and let

$$f_N(x_1, ..., x_k, ..., x_N) = \phi(x_1) + ... + \phi(x_k) + ... + \phi(x_N),\ x_k \in \mathbb{R}^m,$$

that is the direct sum of N copies of ϕ,

$$f = f_N = \underbrace{\phi \oplus \phi \oplus ... \oplus \phi}_{N} : \mathbb{R}^{mN} \to \mathbb{R}.$$

Then, if $N \geq N_0 = N_0(m, n, deg(\phi)$ for a given n, there exists an injective linear map $\sigma : \mathbb{R}^n \to \mathbb{R}^{mN}$ defined over *rationals* (i.e. $\mathbb{Q}^n \to \mathbb{Q}^{mN}$), such that the composed polynomial $c = f \circ \sigma$ on $\mathbb{R}^n$ equals the weighted sum of powers of sums of squares,

$$c(x_1, ..., x_n) = \sum_{j \leq deg(\phi/2)} w_j \cdot \left(x_1^2 + ... + x_n^2\right)^j.$$

In fact, let Q_ϕ denote the subset in the linear space P of polynomials on $\mathbb{R}^n$ that are induced from ϕ by linear maps $\mathbb{R}^n \to \mathbb{R}^m$. The orthogonal group $O(n)$ naturally acts on Q_ϕ and (the interior of) the convex hull of Q_ϕ in P contains $O(n)$-*invariant* polynomials c, that are integrals of $O(n)$-orbits in Q_ϕ over $O(n)$.

Since the linear span of Q_ϕ in P is finite dimensional, these integrals c are representable by *finite* linear combinations $\sum_{=1,...,N} ka_k \cdot (f \circ \sigma_k)$ with some positive coefficients $a_k \geq 0$.

Such combinations can be equated with polynomials induced from $f = f_N$ on $\mathbb{R}^{mN}$ by linear maps $\sigma : \mathbb{R}^n \to \mathbb{R}^{mN}$; on the other hand, $O(n)$-invariant polynomials c are exactly those of the form $\sum_{j \leq deg(\phi/2)} w_j \cdot \left(x_1^2 + ... + x_n^2\right)^j$.

Since rational points are dense in $O(n)$, the linear spans of $O(n)$-orbits of rational points in P are *rational* linear subspaces and one may assume that our c lies in the *interior* of the convex hull of such an orbit; then both σ_k and a_k admit rational approximation by elementary linear algebra. QED.

This averaging combined with a Borsuk-Ulam argument shows that an *arbitrary real polynomial* f on $\mathbb{R}^N$ becomes $\sum_{j \leq deg(\phi/2)} w_j \cdot \left(x_1^2 + ... + x_n^2\right)^j$ under composition with some linear $\sigma : \mathbb{R}^n \to \mathbb{R}^N$, provided N is sufficently large,[87]

[86] Possibly, my attribution of this to Hilbert rather than to Hurwitz, is erroneous – I have not read the original papers and I borrowed the references from textbooks, see, e.g. p. 76 in *Additive Number Theory The Classical Bases* by Nathanson.

[87] See *Dvoretzky type theorems for multivariate polynomials and sections of convex bodies* by Dol'nikov and Karasev.

and, probably, it is not hard to find a *rational* σ if f is a rational (i,e, with rational coefficients) polynomial.

Remark. Hilbert's symmetrisation applies to all functions $f = \phi \otimes \phi \otimes ... \otimes \phi$ where it yields approximately round composed maps $f \circ \sigma$, e.g. the Dworetzky theorem for l_p-spaces.

Questions. Can one make any use of Hilbert's symmetrisation for other locally compact fields $\mathbb{F}$ and compact subgroups in $GL_n(\mathbb{F})$?

If $\mathbb{F}$ is a finite field then all $\mathbb{F}$-valued functions on $\mathbb{F}^N$ are representable by polynomials f and Grassmanian monochromaticity results/conjectures by Rota-Graham-Rothschild (see section 1.1) can be interpreted, but hardly(?) approached, as particular solutions to the above [2] over $\mathbb{F}$.

On the other hand, if we pass to a *sufficiently large* algebraic extension $\mathbb{F}'$ of $\mathbb{F}$, then the equation $f \circ \sigma = const$ in σ extended to $\mathbb{F}'$ can be solved by algebraic geometric means in the spirit of Deligne-Lang-Weil.

What happens to the equation $f \circ \sigma = const$ over *moderate* extensions of $\mathbb{F}$?

1.13 Monochromaticity, Fixed Points, Transversal Measures and Extremely Amenable Categories.

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1.14 Gibbsian Limits and Concentrated Spaces.

1.15 Selected Books and Articles on the Ramsey-Dworetzky-Borsuk-Ulam Type Theorems and the Concentration Phenomenon.

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2 Isoperimetric, Filling and Waist Inequalities.

The *sharp isoperimetric inequality* in the Euclidean space $\mathbb{R}^n$ bounds the volume of a domain $U\subset\mathbb{R}^n$, say with a smooth boundary ∂U, by:

$$[\star]\qquad vol_n(U)\leq\beta_n vol_{n-1}(\partial U)^{\frac{n}{n-1}},$$

where β_n is the constant from the corresponding equality for the unit ball $B^n\subset\mathbb{R}^n$ and its boundary sphere,

$$\beta_n=\frac{vol_n(B^n)}{vol_{n-1}(S^{n-1})^{\frac{n}{n-1}}}.$$

Basic Question. What is the ultimate generalization/refinement of this inequality.[88]

One may approach this question by analysing the existing proofs of [$\star$] that rely on four (more?) different techniques:

- *Integral Geometry;*
- *Calculus of Variations in the framework of the geometric measure theory;*
- *Rearrangements such as symmetrisation and mass transform;*
- *Complexification and Hodge-type Inequalities.*

[88] This, along with other results and problems on isoperimetry, I learned from Yura Burago between 1965 -1970.

The integral geometry approach has been successful so far only in $\mathbb{R}^2$ (Santalo, 1953) and in $\mathbb{R}^4$ (Croke 1984) that brings forth the following

PROBLEM. *Find an integral-geometric proof of the isoperimetric inequality* [☆] *for all* n.

Possibly, such a proof does not exists.

QUESTIONS. *Can one precisely formulate and eventually rigorously prove insufficiency of the integral geometry arguments for derivation of* [☆]?

More generally,

Can one evaluate "the amount of logically infinite/transcendental" involved in a particular geometric argument, where integral-geometric proofs would be minimalistic in this respect – nearest to purely algebraic ones, while variational proofs would be regarded as those with "maximal transcendentality" in them?

2.1 Santalo's Proof of the 2D-Isoperimetric Inequality.

We present below Santalo's argument for bounded plane domains $U \subset \mathbb{R}^2$ with smooth boundaries $\Sigma = \partial U$ in full detail as it exemplifies what we expect of an ideal integral-geometric proof of the isoperimetric inequality.

Given a straight segment $[\sigma, u] \subset U$ with $\sigma \in \Sigma = \partial U$ and $u \in U$, let $\alpha_\sigma(u)$ be the derivative $[d\angle_u(\sigma)]/d\sigma$ where $[d\angle_u(\sigma)]$ is the infinitesimal visual angle of an "element of length" $d\sigma$ in Σ at $\sigma \in \Sigma$ as it is seen from u.

In other words, α equals the derivative (one-dimensional Jacobian) of the radial projection $\sigma \mapsto \angle_u(\sigma) \in S^1_u$ from the subset $\Sigma_u \subset \Sigma$ of points $\sigma \in \Sigma$ that are visible from u to the unit circle S^1_u in the tangent plane $T_u(U) = \mathbb{R}^2$, where $\Sigma_u = \Sigma$ if and only if (U, u) is *star convex*, i.e. the radial projection from Σ to $S^1_u \subset T_u(U)$ is one-to-one.

Since the map $\Sigma_u \to S^1$ is *onto* as well as *one-to-one* where $\alpha_\sigma(u) \neq 0$,

$$\int_{\Sigma_u} \alpha_\sigma(u) d\sigma = 2\pi$$

and

$$[2\pi \cdot area] \qquad \int_U du \int_{\Sigma_u} \alpha_\sigma(u) ds = 2\pi \cdot area(U).$$

Rewrite this by changing the order of integration

$$\int_U du \int_{\Sigma_u} \alpha_\sigma(u) ds = \int_\Sigma ds \int_{U_\sigma} \alpha_\sigma(u) du,$$

where $U_\sigma \subset U$ is the set of points that are visible from σ, and look at the integrals

$$\int_{U_\sigma} \alpha_\sigma(u) du \text{ for } \sigma \in \Sigma.$$

If $U = B^2 = B^2(R) \subset \mathbb{R}^2$ is a planar disc, then, the function $\alpha_\sigma(u)$ is defined an the half-plane $R^2_+(\sigma)$ bounded by the straight line tangent to Σ at the point σ, such that the segments $[s, u] \subset B^2$ are contained in this half-plane.

! Now comes the key moment where the dimension $dim(U) = 2$ enters the game:

by elementary geometry, the function $\alpha_\sigma(u)$ is *constant on the circle* $S^1(R) = \partial B^2(R)$ and it *goes down outside* B^2,[89]

$$\alpha_\sigma(u) \geq \alpha_\sigma(u) \text{ for all } u \in B^2 \text{ and all } u' \in R^2_+(\sigma) \setminus B^2.$$

It follows that

among all plane domains U with given area A the integral $\int_U \alpha_\sigma(u)du$ is maximised by balls/discs $B^2 = B^2(R) = B^2(R(A))$ of areas A (that is where $R = \sqrt{A/\pi}$) where, observe, this integrals over balls do not depend on the boundary points σ. Therefore, according to the above $[2\pi \cdot area]$,

$$2\pi{\cdot}area(U) = \int_U du \int_{\Sigma_u} \alpha_\sigma(u)ds = \int_\Sigma ds \int_{\Sigma_{U_\sigma}} \alpha_\sigma(u)du \leq length(\Sigma){\cdot}\int_{B^2(R)} \alpha(b)db.$$

This delivers a bound on $area(U)$ in terms of $length(\Sigma)$ and the function $\phi(a) = \int_{B(R)} \alpha(b)$. Since the above inequality become an *equality for the disk* $B(R)$ of area $A = area(U)$ this bound is sharp; hence, it represents the required isoperimetric inequality for *plane domains U*.

On Surfaces with Curvatures $\kappa \leq 0$ The above arguments extends to to surfaces U with *unique visibility* metrics of non-positive sectional curvature, where "unique visibility" signifies that every two points $u_1, u_2 \in U$ are joined by *at most* one geodesic segment $[u_1, u_2] \subset U$ and where all one needs is to "transplant" the inequality

$$\int_U \alpha_\sigma(u)du \leq \int_{B^2} \alpha(b)db$$

to such surfaces.

Such a "transplantation" relies on two facts.

1. *The symmetry of the Jacobians*[90] $J(u_1{\to}u_2)$ *of the exponential maps* for all geodesic segments $[u_1, u_2]$ in U:

the number $J(u_1{\to}u_2)$, that is the value of the Jacobian of the (partially defined) exponential map $\exp_{u_1}$ *from the tangent space $T_{u_1}(U)$ to U at the tangent vector that reaches u_2, i.e. at the pullback* $\exp^{-1}_{u_1}(u_2) \in T_{u_1}(U)$, *equals* $J(u_2{\to}u_1)$ *for* $\exp_{u_2} : T_{u_2}(U) \to U$.

2. The Cartan-Alexandrov-Toponogov comparison theorem for spaces with $\kappa \leq 0$ that delivers the lower bound $J(u_1{\to}u_2) \geq 1$ for all geodesic segments $[u_1, u_2] \subset U$.

Remark. There is a more general (also sharp) isoperimetric inequality for surfaces U with boundaries proved with *the t-equidistant deformation* $\Sigma_t \subset U$, $t \geq 0$, of $\Sigma = \Sigma_0 = \partial U$, where the Gauss-Bonnet theorem provides a sharp upper bound on the (typically negative) derivative $\partial(length(\Sigma_t))/dt$, but Santalo's integral geometric argument (that applies to unique visibility surfaces with curvatures $\leq \kappa_0$ for all $-\infty < \kappa_0 < +\infty$) gives a better, in a way *perfect, lower bound* on the "error" $\frac{length(\Sigma)}{length(\partial B^2)} - 1$ by the geometry of U, where B^2 is the disc with $area(B) = area(U)$.

[89] This is also the key point in Crokes 1984 proof of the isoperimetric inequality for $n = 4$ where one integrates over $\partial U \times \partial U$ instead of $U \times \partial U$.

[90] This follows from the *symmetry of the Heat operator* that is, in turn, is the (trivial) Riemannian case of the *Einstein-Onsager relation.*

For instance, if $U \subset \mathbb{R}^2$, this error is expressed in terms of "the average overlap" of U with the discs $B^2_\sigma \subset \mathbb{R}^2$, $\sigma \in \Sigma$, the centres of which lie on the normal segments to $\Sigma = \partial U$ at $\sigma \in \Sigma$ that are directed inward U and such that the boundaries $S^1_\sigma = \partial B^2_\sigma$ are tangent to $\Sigma = \partial U$ at $\sigma \in \Sigma$.

Questions. What should be a Santalo type "error estimate" for dimensions $n > 2$, at least for Euclidean domains $U \subset \mathbb{R}^n$?

Can one "interpolate" between the 2-dimensional proof by Santalo's and the 4-dimensional argument by Chris Croke (see section 2.4), such that this "interpolation" would deliver the sharp isoperimetric inequality for 3-dimensional manifolds with sectional curvatures $\kappa \leq 0$? [91]

On Algebraic Souls of Geometric Inequalities. Santalo's argument demonstrates the general principle: sharp geometric *inequalities* are rooted in algebraic/analytic *identities*. Apparently, in order to find a Santalo-type proof of the isoperimetric inequality in other dimensions and spaces, we need to learn something new (recognise something old?) about the elementary Euclidean geometry of balls and spheres.

2.2 Liouville Measure on the Space of Lines, Buffon-Cauchy-Crofton Integral Formulas, Visual Area and Boltzmann Entropy.

The space $\mathcal{L}$ of straight lines $L \subset \mathbb{R}^n$ carries a unique up to a scaling constant measure, called after *Liouville*, that is invariant under isometries of $\mathbb{R}^n$.

We choose this constant with agreement with *Cauchy Formula*, such that

the Liouville measure λ of the subset $\mathcal{L}_{\pitchfork Y_0} \subset \mathcal{L}$ of the lines intersecting a given domain $Y_0 \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$ e.g. the unit cube $[0,1]^{n-1}$ equals the $(n-1)$-volume of this domain denoted $vol_{n-1}(Y_0)$.

Then, obviously, the $(n-1)$-volumes (Hausdorff measures) of all *rectifiable*, e.g. piecewise smooth, hypersurfaces $Y \subset \mathbb{R}^n$ are expressible via cardinalities of their intersection with lines by

CAUCHY AREA FORMULA.[92]

$$\int_{\mathcal{L}} card(L \cap Y) dL = vol_{n-1}(Y).$$

Since straight lines $L \subset \mathbb{R}^n$ can be represented by pairs (g, x) where $g = g(L) \in Gr_{n-1}(\mathbb{R}^n)$ are the hyperplanes normal to L and where $x = x(L) \in \mathbb{R}^{n-1}_g$ are the intersection points of L with $\mathbb{R}^{n-1}_g$, the above integral can be expressed as follows.

Let us also use the notation $\mathbb{R}^{n-1}_g \subset \mathbb{R}^n$ for the hyperplanes (corresponding to) $g \in Gr_{n-1}(\mathbb{R}^n)$ an let $P_g : Y \to \mathbb{R}^{n-1}_g$ be the normal projections of Y to these hyperplanes. Observe that the pullbacks $P_g^{-1}(x) \subset Y$ equal the intersections of Y with the lines $L = L_{g,x} \subset \mathbb{R}^n$ corresponding to the pairs (g, x) and let

[91] The only available proof of this due to Bruce Kleiner depends on the geometric measure theory á la Almgren.

[92] This can (should?) be taken for definition of vol_{n-1}. In fact, any measure on the space of curves in $\mathbb{R}^n$ (or any *two point homogeneous* space for this matter) that is invariant under isometries satisfies a similar formula.

$mult_Y(x)$, $x \in \mathbb{R}^{n-1}_g$ be the multiplicity functions that are

$$mult_Y(x) = card(P_g^{-1}(x)) = card(Y \cap L_{g,x}),$$

where, observe, $mult_Y(x) = 0$ for $x \notin P_g(Y)$.

Thus,

$$\int_{\mathcal{L}} card(L \cap Y)dL = \int_{Gr_{n-1}(\mathbb{R}^n)} dg \int_{\mathbb{R}^{n-1}_g} mult_Y(x)dx$$

where dg is a suitably normalised $O(n)$-invariant measure on the Grassman manifold $Gr_{n-1}(\mathbb{R}^n) \ni g$. (This manifold can be identified with the projective space $\mathbb{R}P^{n-1} = S^{n-1}/\pm 1$ of the lines normal to hyperplanes $\mathbb{R}^{n-1}_g$, $g \in Gr_{n-1}(\mathbb{R}^n)$.)

Convex and non-Convex Examples. If Y serves as the boundary of a compact domain in $\mathbb{R}^n$, say $Y = \partial U$, then the lines L that intersect U also intersect $Y = \partial U$.

If U is convex, then the cardinalities of almost all non-empty intersections $L \cap \partial Y$ equal 2 and so the Liouville measure λ of the lines that intersect a convex $U \subset \mathbb{R}^n$ satisfies

$$\lambda(\mathcal{L}_{\pitchfork U}) = \lambda(\mathcal{L}_{\pitchfork Y}) = \frac{1}{2} vol_{n-1}(Y = \partial U).$$

In general, if U is non-convex, one has only the inequality

$$\lambda(\mathcal{L}_{\pitchfork U}) = \lambda(\mathcal{L}_{\pitchfork Y}) \geq \frac{1}{2} vol_{n-1}(\partial U)$$

that (this is easy to show) needs no regularity assumption on ∂U if vol_{n-1} is understood as the $(n-1)$-dimensional Hausdorff measure.

The Cauchy area formula is accompanied by the following

Buffon-Crofton Integral formula:[93]

$$\int_{\mathcal{L}} length(L \cap U)dL = c_n \cdot vol_n(U),$$

where $length =_{def} vol_1$ stands for the Lebesgue measure on the line, where vol_n denotes the Lebesgue measure in $\mathbb{R}^n$ and where this constant is determined by applying this formula to the unit n-ball

$$c_n = \frac{\int_{\mathcal{L}} length(L \cap B^n)dL}{vol(B^n)}.$$

The Euclidean Visual Area Conjecture. *Among all compact domains $U \subset \mathbb{R}^n$ with unit volume, the balls of unit volume $B^n_{vol=1} \subset \mathbb{R}^n$ minimise the Liuoville measure λ of the set of the lines intersecting U,*

[I] $$\lambda(\mathcal{L}_{\pitchfork U}) \geq \lambda(\mathcal{L}_{\pitchfork B^n_{vol=1}}) \text{ for all } U \subset \mathbb{R}^n \text{ with } vol_n(U) = 1.$$

Discussion. (a) If U is convex, this follows from the isoperimetric inequality and if dimension $n = 2$. the general case trivially reduced to the convex one.

[93]See *Geometric Probability and Stereology* by Hykšová-Kalousová-Saxl, Image Anal Stereol 31:1-16, p. 1-16, (2012).

(b) Almgren's argument relying on the geometric measure theory from his 1986 paper *Optimal Isoperimetric Inequalities* yields this conjecture for all n but only for domains with *smooth* (rectifiable will do) boundaries.[94]

It would be nice to have a more elementary (integral geometric?) proof that would need no regularity assumption on the boundary, where observe, the boundary of U does not even enter into the statement of the conjecture.

The Liouville measure $\lambda(\mathcal{L}_{\pitchfork U})$ can be represented by the integral of the volumes of the orthogonal projections P_g of U to the hyperplanes $\mathbb{R}^{n-1}_g$,

$$\lambda(\mathcal{L}_{\pitchfork U}) = \int_{Gr_{n-1}(\mathbb{R}^n)} vol_{n-1}(P_g(U)dg$$

and then the above inequality can written as

$$\int_{Gr_{n-1}(\mathbb{R}^n)} vol_{n-1}(P_g(U))dg \geq \int_{Gr_{n-1}(\mathbb{R}^n)} vol_{n-1}(P_g(B^n_{vol=1}))dg.$$

Let us normalise dg to be the *probability measure*, i.e. of total mass one, think of the integral $\int_{Gr_{n-1}(\mathbb{R}^n)} vol_{n-1}(P_g(U))dg$ as *arithmetic mean* of the volumes $vol_{n-1}(P_g(U))$ over the Grassmanian $Gr_{n-1}(\mathbb{R}^n)$. We abbreviate by denoting $G = Gr_{n-1}(\mathbb{R}^n)$ and formulate the following

Geometric Mean Conjecture.

[II] $$\int_G \log vol_{n-1}(P_g(U))dg \geq \int_G \log vol_{n-1}(P_g(B^n_{vol=1}))dg.$$

Clearly, this [II] is stronger than [I]. Also observe that a non-sharp version of this conjecture,

$$\int_G \log vol_{n-1}(P_g(U))dg \geq const_n \int_G \log vol_{n-1}(P_g(B^n_{vol=1}))dg.$$

follows from the *Loomis-Whitney inequality*.[95]

Let μ_U be the measure $\chi_U(x)dx$ on $\mathbb{R}^n$ for χ_U denoting the characteristic function of U, where this is a *probability measure* because we assumed $vol_n(U) = 1$. Let $P_{g*}(\mu_U)$ be the pushforwards of μ_U to the hyperplanes $\mathbb{R}^{n-1}_g$ by the normal projections $P_g : \mathbb{R}^n \to \mathbb{R}^{n-1}_g$ and formulate the following refinement of the geometric mean conjecture.

Entropic Isoperimetry Conjecture.

[III] $$\int_G ent\,(P_{g*}(\mu_U))\,dg \geq \int_G ent\,\big((P_{g*}(\mu_{B^n_{vol=1}})\big)\,dg,$$

where, recall, the *Boltzmann entropy* of a probability measure $\mu = f(x)dx$ on $\mathbb{R}^{n-1}$ can be computed by the integral

$$ent(\mu) = -\int_{\mathbb{R}^{n-1}} f(x)\log f(x)dx.$$

[94] This is explained at the end of section 5.7 in my paper *"Entropy and Isoperimetry..."*. I could not locate this result in the literature and I apologize to an author whose paper I failed to find.

[95] The measures μ_i, $i = 1, 2, ..., n$, of the projections of a $U \subset \mathbb{R}^n$ to the n coordinate hyperplanes satisfy $\prod_i \mu_i \geq \mu(U)^{n-1}$.

Notice that [III] implies [II], since the entropy of a probability measure $\phi(x)dx$ is smaller than the logarithm of the dx-measure of the support of ϕ. Also observe that a non-sharp version of [III] follows from the *Brascamp-Lieb inequality* that says in the present (rather spacial) case that

among all probability measures $\mu = \phi(x)dx$ on $\mathbb{R}^n$ with given entropy the minimum of $\int_G ent\,((P_{g*}(\mu))\,dg$ is achieved by a Gaussian measure $C_1 e^{-C_2\|x\|^2}dx$.[96]

2.3 Area Shrinking and Filling Volume in bi-Contracting Spaces.

A metric space X is called *(linearly geodesyclly) bi-contracting*, if it admits a family of maps $E_\lambda : X \times X \to X$, $\lambda \in [0,1]$, with the following properties.

• The maps (secretly, distance minimizing geodesics) $G_{x_1,x_2} : [0,d] \to X$ for $d = dist(x_1,x_2)$ defined by $G_{x_1,x_2}(t) = E_{t/d}(x_1,x_2)$ are *isometric* and they satisfy

$$E_{x_1,x_2}(0) = x_1 \text{ and } E_{x_1,x_2}(t) = E_{x_2,x_1}(d-t);$$

• the maps $E_\lambda(x_1,*) : X \to X$ for $x \mapsto E_\lambda(x_1,x)$ are λ-*Lipshitz*, i.e.

$$dist_X(E_\lambda(x_1,x),E_\lambda(x_1,x')) \le \lambda \cdot dist_X(x,x');$$

For instance, Banach spaces and $CAT(0)$ *spaces* are linearly bi-contracting where the latter are defined as follows.

A metric space X is called $CAT(0)$ or $CAT(\kappa \le 0)$ if every 1-*Lipshitz*, i.e. (non-strictly) *distance deceasing* map from $X_0 \subset \mathbb{R}^n$ to X extends to a 1-Lipschitz map $\mathbb{R}^n \to X$ for all $n = 1,2,3,...$, and all subsets $X_0 \subset \mathbb{R}^n$.

Remark. If one limit the above condition to $n = 1$, one obtains much larger class of metric spaces, called *length spaces* where distances between pairs of points equal the minimal lengths of paths between these points.

Examples.(a) Euclidean spaces $\mathbb{R}^N$ are $CAT(0)$, since they satisfy the Lipschitz extension property by *Kirszbraun's Theorem*; moreover, complete simply connected spaces with sectional curvatures $\kappa \le 0$ are also $CAT(0)$, where one may allow *singular spaces*, with the condition $\kappa \le 0$ *defined* as the 1-Lipshitz extension property from triples of points $\{x_1,x_2,x_3\}$ in the plane $\mathbb{R}^2$ to a fourth point $x_4 \in \mathbb{R}^2$.[97]

(b) Basic example of smooth $CAT(0)$-spaces are *Riemannian symmetric spaces of non-compact type* such as *hyperbolic spaces of constant negateive curvature.*

The simplest singular spaces are *(metric) trees*, their *Cartesian products* and, more general *polyhedral spaces* such as *the Bruhat-Tits buildings*.

Let V be a k-dimesional Riemannian manifold. Say that a Lipschitz map $f : V \to X$ is *k-volume (non-strictly) λ^k-contracting* if it can be

reparametrized by a volume preserving self map $V \to V$ to a λ-Lipschitz map $V \to X$.

Equivalently, one may requite that f decomposes as $f = h \circ g$ for $V \overset{h}{\to} V_1 \overset{g}{\to} X$ where g *is λ-Lipshitz* and h is Lipschitz *volume decreasing*, i.e. the Rieman-

[96]This, along with the the Loomis-Whitney and basics on entropy is explained in my articles *"Entropy and Isoperimetry...."* and "In a Search for a Structure".

[97]See Lang-Schroeder, *Kirszbraun's Theorem and Metric Spaces of Bounded Curvature.*

nian volumes of the h-pullbacks all open subset $U \subset V_1$ satisfy $vol_k(h^{-1}(U)) \geq vol_k(U)$.

We often say "volume contracting" instead of "volume 1-contracting" and observe that, in general, volume λ-contracting maps are contracting for $\lambda \leq 1$ but may be expanding if $\lambda > 1$.

Also observe that "1-volume λ-contracting" is equivalent to "λ-Lipschitz".

Area Shrinking Problem. Let $W = V \times [R_1, R_2]$ be a compact Riemannian n-manifold with two boundary components, $V_1 = V \times R_1$ and $V_2 = V \times R_2$, with the metric $dR^2 + R^2 dv^2$ for some Riemannian metric dv^2 on V.

The basic examples of such W are the annuli between concentric spheres in the Euclidean space, that are the complement between balls, $B^n(R_2) \setminus B^n(R_1) \subset \mathbb{R}^n$, where the unit sphere $S^{n-1} \subset \mathbb{R}^n$ plays the role of V.

What is the minimal number $\lambda_{min} \geq 0$ for which all $(n-1)$-volume contracting maps $V_2 \to X$ extend to n-volume λ^n_{min}-contracting maps $f : W \to X$ such the the restrictions of f to V_1 are $(n-1)$-volume contracting?

One knows in this regard the following.

[A] *If X equals the Euclidean space $\mathbb{R}^N$, then $\lambda_{min} = 1$.*

If $n = 2$, the proof is straightforward and, probably, goes back to Alexandrov.

In the general case of $X = \mathbb{R}^{n \geq 2}$ this was proven by Almgren in his (technically difficult) 1986 paper "*Optimal Isoperimetric Inequalities*".[98]

[B] If X is *a three dimensional Roemannian $CAT(0)$-manifold*, then the equality $\lambda_{min} = 1$ was proven in 1992 by Bruce Kleiner by elaborating on Almgren's geometric measure theory techniques combined with the Gauss-Bonnet theorem for surfaces.

[C] The sharp Chris Croke isoperimetric inequality (see the next section) implies that $\lambda_{min} = 1$ for 4-*dimensional Riemannian $CAT(0)$-manifolds and* $R_0 = 0$.

[D] The number λ_{min} is bounded by a universal (and unpleasantly large) constant C_n for *all bi-contractible* spaces X.

This follows from my *Filling of Riemannain manifolds* paper with improvements by Stefan Wenger in his *A short proof of Gromov's filling inequality*.

Sharp Shrinking/Filling Conjecture in $CAT(0)$-Spaces. *If X is $CAT(0)$ then $\lambda_{min} = 1$ for all dimensions n.*

This remains open for all $n \geq 2$ except for the cases covered by the above [A], [B], [C] and by the sharp isoperimetric inequalities that are know to hold in the Riemannain products X of manifolds of dimensions ≤ 4 with hyperbolic spaces that resolves the equidimensional (i.e. where $dim(W) = dim(X)$) *filling* case (i.e. where $R_0 = 0$) of the conjecture for these products. (We return to this this in the next section.)

Local Corollary to the Global Conjecture. If X is a smooth Riemannian manifold, $f_0 : V_0 \to X$ is a smooth immersion, then by letting $R_0 \to R_1$, one arrive with the shrinking conjecture at the following proposition.

[98] Almgren does not prescribe the topology of his shrinkings and fillings as we do, but this makes little difference due to "topological freedom for fillings" that was observed in our paper *Construction of nonsingular isoperimetric films* with Yasha Eliashberg and in *Mappings that minimize area in their homotopy classes* by Brian White.

$\left[\star\right]$ *there exists a point* $x_\star \in f_0(V_0) \subset X$*, and a unit tangent vector*[99] $\tau_\star \in T_{x_\star}(X)$*, such that the mean curvature of* $f_0(V_0)$ *in the direction* $\tau_\star$ *is greater or equal than* $(n-1)R_1^{-1}$ *that is the mean curvature of the sphere* $S^{n-1}(R_1) \subset \mathbb{R}^n$*.*

In fact, Almgren shows that the validity of $\left[\star\right]$ for all, possibly singular, $(n-1)$-subvarieties in a complete smooth Riemannian manifold X (with mild assumptions at infinity) implies the conclusion of the area shrinking conjecture for X. But the proof of $\left[\star\right]$ remains problematic even for surfaces in hyperbolic N-spaces for $N \geq 4$.[100]

Questions. Let $U \subset \mathbb{R}^n$ be a compact domain with smooth *connected* boundary ∂U. Does there always exist a point $x_\star \in \partial U$ where the mean (or a another kind of) curvature is $\geq const_n vol_n(U)^{-\frac{1}{n}}$ for some constant $const_n > 0$?

The above conjecture for $CAT(0)$ spaces extends to $CAT(\kappa \leq \kappa_0)$ spaces X for all κ_0, where the problem remains open even for real hyperbolic spaces $X = H^N$ and $R_0 = 0$:

Does every $(n-1)$*-cycle* $C \subset H^N$ *admits a filling by an n-chain* $D \subset H^N$ *("filling" means that* $\partial D = C$*), such that the n-volume of D is bounded by that of the hyperbolic ball* $B^n \subset H^n \subset H^N$ *with* $vol_{n-1}(\partial B^n) = vol_{n-1}(C)$*?*

If $n \geq 2$ this is known only in the equidimnsional case of $N = n$ where *Schwartz symmetrisation* applies.

Besides filling estimates and Schwartz symmetrisation there are other means for proving equidimensional isoperimetric inequalities. These suggest particular avenues for generalisations one of which is indicated in the next section.

Questions. What is a minimal set of invariants of Riemannan manifolds X_1 and X_2, suitable conditions on which would imply (some form of) the shrinking area conjecture for their Cartesian product $X_1 \times X_2$?

One knows, for instance, that the Schwatz' "virtual symmetrisation" inequalities delivers isoperimetric, sometimes sharp, inequalities for hypersurfaces Y in Cartesian and more general *warped products* as well as in *geometric (e.g. spherical) suspensions* and *geometric joins* in terms of such inequalities for the factors.[101] For instance, these inequalities for the Euclidean, spherical and hyperbolic spaces follow this way.

But it is unclear what should be a counterpart of this "symmetrisation" for submanifolds $Y \subset X_1 \times X_2$ of codimnsion ≥ 2.

For instance, is the Almgren's shrinking (or filling) inequality stable under Cartesian products with Euclidean spaces?

Does it hold, for example, for the Cartesian products of 2-manifolds with negative curvatures by Euclidean spaces?

Do compact domains U in minimal (but not necessarily volume minimizing) n-dimensional subvarieties $W \subset \mathbb{R}^N$ satisfy the sharp Euclidean isoperimetric inequality $vol_n(U) \leq \beta_n vol_{n-1}(\partial U)^{\frac{n}{n-1}}$?

(Recall that "sharp Euclidean" signifies that $\beta_n = \frac{vol_n(B^n)}{vol_{n-1}(\partial B^n)^{\frac{n}{n-1}}}$ for the unit ball $B^n \subset \mathbb{R}^n$.)

[99] One may be assume that $\tau_\star$ is normal to the tangent space of the image $f_0(V_0) \subset X$.

[100] The existence of $x_\star$ for $N = 3$ was proved by Kleiner for 3-manifolds X with non-positive sectional curvature.

[101] I explain this in section 9 of my "*Isoperimetry of Waists...*" but I guess it was written earlier in some textbooks somewhere.

Shrinking and Filling Problems in General bi-Contractible Spaces X. As we mentioned above the known bound on λ_{min} by a constant C_n is unsatisfactory crude.

What is the true value of this constant? What are the corresponding extremal manifolds?

A potential candidate for W_{extr} is the hemisphere S^n_+ bounded by S^{n-1}, where its conjectural extremity is twofold.

(1) Let W be a Riemannian manifold with the boundary S^{n-1} such that $\partial W = S^{n-1}$ and such that every semi-circle in this S^{n-1} is *distance minimizing in* $W \supset S^{n-1}$. Then, conjecturally, $vol_n(W) \geq vol_n(S^n_+) = \frac{1}{2} vol_n(S^n)$.

(2) Let W be a compact smooth n-manifold with boundary V, where V is endowed with a Riemannain metric g such that the volume of V with respect to g satisfies $vol_{n-1}(V) \leq vol_{n-1}(S^{n-1})$.

Then, again conjecturally, g admits extensions to Riemannian metrics h_ε on W for all $\varepsilon > 0$, such that $dist_{h_\varepsilon}(v_1, v_2) = dist_g(v_1, v_2)$ for all $v_1, v_2 \in V$ (i.e. no two points v_1, v_2 can be joint by a geodesic in W of length $< dist_g(v_1, v_2)$ and such that the h_ε-volumes of W satisfy $vol_n(W) \leq vol_(S^n_+) + \varepsilon$.

These (1) and (2) would resolve the above problem in the filling case ($R_0 = 0$) but (2) looks too good to be true.

Also it is tempting to think but hard to believe that every metric g on V with $vol_{n-1} \leq vol_{n-1}(S^{n-1})$ extends to h_ε on W with $dist_{h_\varepsilon}(v_1, v_2) = dist_g(v_1, v_2)$ and such that $dist_{h_\varepsilon}(w, V) \leq \frac{\pi}{2} + \varepsilon$ for all points $w \in W$ (as it is the case for $W = S^n_+$ with $\varepsilon = 0$).

2.4 Visual Area for the Liouville Measure and Sharp Isoperimetry for Negative Curvature.

Let us formulate another kind of generalisation of the sharp isoperimetry conjecture for spaces X with non-positive curvatures.

Let X be a (possibly non-compact and non-complete) *unique visibility* Riemannian manifold without boundary where, recall every two points are joined by *at most one* geodesic segment. Then the space $\mathcal{L} = \mathcal{L}(X)$ of (finite or infinite) geodesic segments $L \subset X$ *properly* imbedded to X, (i.e. with none of its two ends in the interior in X) also carries such a unique measure dL_X, also denoted λ_X and called *transversal Liouville mesaure*, that is characterised by the following properties.

• Let $U \subset X$ be an open subset and let us map $\mathcal{L}(U) \to \mathcal{L}(X)$ by extending proper geodesic segments from U to such segments in X. This map is countable-to-one and *it locally induces the measure dL_Y, from dL_X.* For instance, if U has piecewise smooth boundary, then almost all $L \in \mathcal{L}(Y)$ admit neighbourhoods, say $M_L \subset \mathcal{L}(U)$ where the map $M_L \to \mathcal{L}(X)$ *is one-to-one and measure preserving.*

• The scaling rule for the Liouville measure is the same as that for the $(n-1)$-volume (of hypersurfaces) in X:

dL_X transforms by the factor C^{n-1}, $n = dim(X)$, under scaling $X \rightsquigarrow C \cdot X$ where $C \cdot X$ is the same X but with the metric $dist_{\lambda X} =_{def} C \cdot dist_X$.

• Recall that *the tangent cone of X at $x \in X$*, that is a naturally defined limit of $C \cdot X$, $C \to \infty$, as it is seen from x, equals the tangent splace $T_x(X)$ that is isometric to the Euclidean space $\mathbb{R}^n$. In the course of this limit the spaces

$\mathcal{L}(C\cdot X)$ naturally converges to $\mathcal{L}=\mathcal{L}_{T_x(X)=\mathbb{R}^n}$ and *the Riemannian Liouville measures* $dL_{C\cdot X}$ *converge to the Euclidean* d_L.

• *Cauchy -Liouville formula.* All smooth $(n-1)$ submanifolds $Y\subset X$ satisfy

$$\int_{\mathcal{L}} card(L\cap Y) = vol_{n-1}(Y).$$

(This agrees with our normalisation of the Euclidean Liouville measure in section 2.2 that was characterised, *uniquely up-to a scaling constant*, by its invariance under isometries of $\mathbb{R}^n$.)

The Buffon-Santalo formula.[102] The measure dL_X is related to the Riemannin volumes of Borel subsets $U\subset X$ by

$$\int_{\mathcal{L}} length(L\cap U)dL_X = c_n vol_n(U)$$

for the constant c_n in this this equality for the balls $B^n\subset\mathbb{R}^n$ taken for U.

Example: "Visual Liouville Area". Denote by $\mathcal{L}_{\pitchfork Y}$ the subset of the lines $L\subset X$ that *intersect* a subset $Y\subset X$ and think of the Liouville measure $\lambda_X(\mathcal{L}_{\pitchfork Y})$ of it as the "visual area" of Y.

Here, similarly to the case $X=\mathbb{R}^n$, if Y serves as the boundary of a relatively compact *convex* subset $U\subset X$ i.e. such that the intersections $L\cap U$ are connected, then

$$\int_{L_X} card(L\cap Y)dL = 2\lambda_X(\mathcal{L}_{\pitchfork U}) = 2\lambda_X(\mathcal{L}_{\pitchfork Y}) = vol_{n-1}(Y).$$

But if $U\subset X$ is *non-convex*, then the the visual area is *strictly smaller* than one half of the $(n-1)$-volume of its boundary $Y=\partial U$,

$$\lambda_X(Y)=\lambda_X(U)<\frac{1}{2}vol_{n-1}(Y).$$

Visual Area Conjecture. *Let X be a unique visibility Riemannian manifold of dimension n with non-positive sectional curvatures, where "unique visibility" means that every two points in X are joined by at most one geodesic segment, and let $U\subset X$ be a relatively compact open subset with piecewise smooth boundary $Y=\partial U$. Then*

$$\int_{\mathcal{L}_X} length(L\cap U)dL_X \le \gamma_n\lambda_X(\mathcal{L}_{\pitchfork Y})^{\frac{n}{n-1}} = \gamma_n\lambda_X(\mathcal{L}_{\pitchfork U})^{\frac{n}{n-1}}$$

for the "Euclidean constant" γ_n, i.e. such that

$$\int_{\mathcal{L}} length(L\cap B^n)dL_X = \gamma_n\lambda_{\mathbb{R}^n}(\mathcal{L}_{\pitchfork B^n})^{\frac{n}{n-1}}$$

for the round balls $B^n\subset\mathbb{R}^n$.

Discussion. If the subset $U\subset X$ is convex, this conjecture is equivalent to the sharp isoperimetry conjecture for manifolds X with non-positive curvatures.

[102]This follows by the Fubini theorem, since the Liouville measure λ on the unit tangent bundle of X is invariante under the geodesic flow.

In fact, this latter conjecture makes sense also for *singular* $CAT(0)$ spaces where it is formulated as follows.

$[\star]_{\kappa\leq 0}$ *Let* X *be a geodesicly complete* $CAT(0)$ *space, i.e. where every geodesic segment, that is an isometric map* $[a,b]\to X$*, extends to a full geodesic that is an isometric map of the line* $(-\infty,\infty)\supset[ab]$ *to* X*. Then the Hausdorf measures of all compact subsets* $U\subset X$ *satisfy*

$$Hau_n(U)\leq\beta_n Hau_{n-1}(\partial U)^{\frac{n}{n-1}},$$

where $n=1,2,...,$ *and* β_n *is the constant that enters the corresponding equality for the balls* $B^n\subset X=\mathbb{R}^n$ *taken for* U*.*

This is unknown even for the 4-dimensional complex hyperbolic space and the counterpart of this conjecture for $CAT(\kappa\leq 1)$ spaces is unknown for the complex projective plane (that has real dimension 4).

As we mentioned earlier the inequality $[\star]_{\kappa\leq 0}$ for all (not necessarily convex) domains U has been established for $CAT(\kappa\leq 0)$- manifolds X of dimensions $2,3,4$. Then the *Virtual Schwartz symmetrisation argument*, yields $[\star]_{\kappa\leq 0}$ for *Riemannian products* of $CAT(0)$-manifolds of dimensions ≤ 4, trees and and spaces with constant negative curvatures.

But $[\star]_{\kappa\leq 0}$ remains conjectural for all *irreducible symmetric spaces* (and Bruhat-Tits buildings, where it seem easier) except for those of constant curvature.

The first instances of such spaces are $X=SL_3(\mathbb{R})/O(3)$ where $dim(X)=5$ and *the complex hyperbolic space* of the (real) dimension 6.

If X is an *irreducible symmetric spaces* X *of rank* 1, then, conjecturally, the boundary of each extremal domains $U\subset X$ is *transitively acted upon* by some subgroup of the isometry group of X.[103]

But if $rank(X)\geq 2$, then the orbits of relevant subgroups have *positive codimensions* in hypersirfaces in X and description of extremal domains $U\subset X$, even conjecturally, is problematic. Yet one may think that the extremal hypersurfaces ∂U have maximal possible homogeneity. For instance if $rank(X)=2$ then ∂U must be acted upon by an isometry (sub)group G of X with $dim(\partial U/G)=1$ where, probably, *the constant mean curvature equation* (that characterizes extremality) can be explicitly solved.

A sharp bound on volumes of 4-dimensional *unique visibility*[104] Riemannian compact manifolds U with boundaries in terms of the "parts of the boundary ∂U that are seen from the points $u\in\partial U$" was proven Croke in his 1984 paper. In fact, Croke shows that if $dim(U)=4$ and the sectional curvatures κ of the metric in U are ≤ 0, then the (transversal) Liouville measure of the space of geodesic segments in $[u,v]\subset U$ with their ends $u,v\in\partial U$ is *sharply bounded* in terms of suitably weighted average A of measures of subsets $\Sigma_u\subset\partial U$ that are visible from u, i.e, serve as second ends $v\in\partial U$ of the segments $[u,v]$.

This A satisfies $A\leq vol_3(\partial U)$, where the equality holds if and only if the hypersurface ∂U is convex and where Croke shows that $vol_4(U)\leq\beta_4 A^{\frac{4}{3}}$ with the Euclidean constant β_4. This implies the (refined) sharp isoperimetric bound on $vol_4(U)$ by *the Buffon-Santalo formula.*

[103]Often but not always this is the subgroup of isometries fixing a point $x\in X$.

[104]This means that every pair of points is joined by *at most one* geodesic segment.

Observe that Crokes "area" is bounded by $A \leq \inf_{u\in\partial U} vol_3(\Sigma_u)$ and so the above implies that

$$vol_4(U) \leq \beta_r \inf_{u\partial U} vol_3(\Sigma_u)^{\frac{4}{3}}$$

Questions. Does this inequality hold true for the Euclidean domains $U \subset \mathbb{R}^n$ for all n?

Can adapt Croke's proof to *singular* 4-dimensional $Cat(0)$-spaces?

(An essential step in such "adaptation" should be a substitute for the Liouville measure for the singular case with *no unique extension property* of geodesics.)

Can Kleiner's $3D$ Almgren-stile argument, be used for proving the visual area conjecture in unique visibility 3-manifolds of non-positive curvature.

(This is plausible in view of how Almgren's argument applies to the visual area conjecture in $\mathbb{R}^n$ in my "*Singularties and Expanders...*".

If one is not concerned with constants, one can be satisfied with the following

Generalized Non-sharp Visual Area Inequality.

Let X be a Riemannian manifold of dimension n with non-positive sectional curvatures where, moreover, every two points are joined by at most one geodesic segment and let $Y \subset X$ be a hypersurface. Then the integral of diameters of the intersections of Y with proper geodesics segments $L \subset X$ is bounded by the visual area of Y as follows

$$\int_{\mathcal{L}} diam(L \cap Y) dL_X \leq \gamma_n \cdot \left(const \cdot \lambda_X(\mathcal{L}_{\pitchfork Y})^{\frac{n}{n-1}}\right),$$

for the above "*Euclidean constant*" γ_n and some universal $const \leq 10$.

Proof. This is obtained by integration of the corresponding local/infinitesimal inequality for Y consisting of two hypersuface germs $Y_1, Y_2 \subset X$ at some points $y_1 \subset Y_1, y_2 \subset Y_2$, where the local inequality is obvious in the Euclidean space and if X is $CAT(\kappa \leq 0)$ it follows from the super-Euclidean rate of divergence of geodesics in X.

Remarks. (a) The extremal hypersurfaces for the above inequality are pairs of germs $Y_1 \ni y_1, Y_2 \ni y_2$ in X where the geodesic segments $[y_1, y_2] \subset X$ are *normal* to Y_1 and to Y_2, i.e. such that the angles between $[y_1, y_2]$ and the two hypersurfaces satisfy

$$\angle_{y_1} = \angle(Y_1, [y_1, y_2]) = \angle_{y_2} = \angle([y_1, y_2], Y_2) = \pi/2.$$

But it is unclear what, conjecturally, should be less trivial *sharp global* form of this inequality, that would take into account the distribution of the values of these angles $\angle_{y_1}, \angle_{y_2}$ on $Y_1 \times Y_2$ as it is accomplished in the proof of the isoperimetric inequality by Croke in $4D$ spaces.[105]

(b) Even if $Y = \partial U$ for some U, this generalised inequality does not follow from the statement of the visual area conjecture, since the diameters $diam(L \cap Y) = diam(L \cap U)$ are strictly greater than $length(L \cap U)$ for some geodesic lines $L \subset X$ if U is non-convex.

Paradox of Singularities in Geometric Inequalities. Complications in proofs of geometric inequalities often arise because of singularities, that may underly

[105] If Y equals the Euclidean sphere S^{n-1} and n is large, then most pairs of vectors $y_1, y_2 \in S^{n-1} \subset \mathbb{R}^n$ are *mutually almost normal* and the angles $\angle_{y_1} = \angle_{y_2}$ are close to $\pi/4$.

the assumptions (such in the geometry of $CAT(\kappa \le \kappa_0)$-spaces and of Alexanrov spaces with curvatures $\kappa \ge \kappa_0$) or to pop up in the course of constructions as it happens to solutions of auxiliary variation problems. However, the experience shows, that such inequalities in many (most?) cases makes the inequalities only stronger.

Can one formulate and prove a general principle of this kind that would allow one carry on proofs without bothering abut singularities?

About Counterexamples. The sharp isoperimetry conjecture for negatively curved spaces may be hard to falsify, while stronger and more technical conjectures have a better chance to be seen from an opposite angles and proven to be false.

Possibly, some of conjectures we stated in this section belongs to this "falsifiable category".

2.5 Waists of Spheres.

Let X be a "geometric space" and $\{Y_p \subset X\}$ be a family of subsets parametrized by a topological space $P \ni p$. One seeks lower bounds on the "maximal geometric size" of Y_p, $p \in P$, in terms of a (sometimes hidden) lower bound on the topology of the family $\{Y_p \subset X\}_{p \in P}$.

We restrict ourselves in this section to the case where X equals the unite sphere S^n and start with discussing the following

Spherical Waist Conjecture for the Hausdorff Measure. *Let $X = S^n \subset \mathbb{R}^n$ be the unit n-sphere and $f : S^n \to \mathbb{R}^{n-m}$, $m \ge 0$ be a continuous map. Then there exists a point $p \in \mathbb{R}^{n-m}$, such that the* ***m-dimensioanl Hausdorff measure*** *of the the p-fiber $Y_p = f^{-1}(p) \subset S^n$, is* ***bounded from below*** *by the volume $vol_{n-m}(S^{n-m})$ of an equatorial sub-sphere $S^{n-m} \subset S^n$,*

$$[HAU]_p^m \qquad Hau_m(Y_p) \ge vol_m(S^m).$$

The essential difficulty here is due to the "geometric pathologies" of the "fibers" $Y_p = f^{-1}(p) \subset S^n$ of general continuous maps as well as of certain (non-genetic) smooth maps: these fibers may be *non-rectifiable* and even when rectifiable, their dependence on p may be rather discontinous.

This problem, however, can be resolved for $n - m = 1$, where the required $Y_p \in S^n$ is furnished by *the Lévy mean of f*, that is the value p of f such that the *hypersurdace* $Y_p = f^{-1}(p) \subset S^n$, divides S^n into "equal halves", i.e. such that the subsets $f^{-1}(-\infty, p]$ and $f^{-1}[p, +\infty)$ in S^n have measures at least one half[106] of that of S^n.

The bound $[HAU]_p^m$ for this p follows from the spherical isoperimetric inequality.

The "geometric pathology" is absent if f a *generic smooth* map or an *arbitrary real analytic map*, where the lower bound on waists is proven (stated?) by means of a geometric measure theory in the long unpublished Almgren's 1965 paper *The theory of varifolds*, where he proves the following

[106]We say "at least" since we do not, a priori, exclude Y_p having non-zero measure.

MINMAX THEOREM. *Let $\{Y_p\}$, $p \in P$, be a continuous*[107] *family of rectifiable m-dimension cycles with $\mathbb{Z}_l = \mathbb{Z}/l{\cdot}\mathbb{Z}$, $\in \mathbb{Z}$, coefficients in a compact n-dimensional Riemannain manifold X such that the resulting map from P to the space of cycles induces non-zero homomorphism on homology. Then X contains a minimal subvariety (varifold) $Y_{min} \subset X$ of dimension m, such that*

$$Hau_m(Y_{min}) \le \sup_{p\in P} Hau_m(Y_p)$$

.

This applies in the present case since the genericity assumption[108] on f ensures rectifiabilty of $Y_p = f^{-1}(p)$ and their continuity in p and since minimal subvarieties in S^n have their Hausdorff measures (that are the same as their volumes) at least as large as that of S^m.

A non-shrap version of the inequality $[HAU]^m_p$ for all continuos maps f from the unit sphere S^n to $\mathbb{R}^{n-m}$, namely

$$\sup_{p\in\mathbb{R}^{n-m}} Hau_m(Y_p) \ge c_n$$

for a rather small (yet, positive!) constant c_n is proven in section 1.3 in the part 2 of my "*Singularities, Expanders....*". But one is unable to validated such an inequality with a constant $const_m > 0$.

The "geometric pathology" can be tamed with the use of the *Minkowski measure* that is defined via the top-dimensional volumes of the ε-neighbourhoods of subsets $Y \subset X$ as

$$[MINK]_\varepsilon \qquad Mink_m(Y) = \liminf_{\varepsilon\to 0} b^{-1}_{n-m}\varepsilon^{-n+m} vol_n(U_\varepsilon(Y)),$$

where b_{n-m} stands for the volume of the unit ball $B^{n-m} \subset \mathbb{R}^{n-m}$.

If Y is a "nice" m-dimensional subvariety, then $Mink_m(Y) = Hau_m(Y)$. For instance, the above Y_{min} is in this "nice" category. But in general, a (rectifiable if you wish) subset Y with finite Hausdorff measures may easily have $Mink_m(Y) = \infty$.

The lower bound on the maximal Minkowski measure of the fibers of continuous maps $S^n \to \mathbb{R}^{n-m}$ follows from *the waist of the sphere theorem* stated in section 1.7,

given a continuous map $f : S^n \to \mathbb{R}^{n-m}$ there exists a point $p \in \mathbb{R}^{n-m}$ such that the volumes of the ε-neighbourhoods of the subset $Y_p = f^{-1}(p) \subset S^n$ are bounded from below by

$$vol_n(U_\varepsilon(Y_p)) \ge vol_n(U_\varepsilon(S^m)) \textit{ for all } \varepsilon > 0.$$

This, in the limit for $\varepsilon \to 0$, yields the desired lower bound on supremum of the Minkowski measures of Y_p, $p \in P$.

Remark. The essential feature of the waist of the sphere theorem is the existence of a fiber Y_p where *all* ε-neighbourhoods *simultaneously* have large

[107] This continuity refers to the so called *flat topology* in the space of cycles.

[108] Every closed subset in X may come up as $f^{-1}(p) \subset X$ for a continuous, even smooth, map $f : X \to P$ where, moreover, these $f^{-1}(p)$ are, in general, rather discontinuous in $p \in P$.

volumes. This seems hard to achieve by variational techniques despite the fact that minimal subvarieties $Y_{min} \subset S^n$ do have $vol_n(U_\varepsilon(Y_p)) \geq vol_n(U_\varepsilon(S^m))$ *for all* $\varepsilon > 0$.

A visible disadvantage of the Minkowsli measures due to a possibility of the strict inequality $Mink_m(Y) > Hau_m(Y)$ does not seem to be serious as this can be remedied by suitable regularisations of flat continuos families Y_p of rectifiable cycles that seems not difficult.

However, this would reduce the waist corollary of Almgren's Minmax theorem to (available generalizatins of) $[MINK]_\varepsilon$ *only for* $\mathbb{Z}_2$*-cycles,* since the only known proof of $[MINK]_\varepsilon$ depend on a kind of Borsuk-Ulam theorem and the the corresponding $\mathbb{Z}_l$-waist inequality remains conjectural.

A representative special case of this conjecture can be formulated as follows.

Let Z be an n-dimensional topological pseudo-manifold, e.g. a smooth manifold and P be a topological pseudomanifold of dimension $n-m$ and let $f: Z \to P$ and $g: Z \to S^n$ be continuous maps.[109]

$\mathbb{Z}_l$-Waist Conjecture for Spheres. If the map g is homologically *non-trivial in the top dimension*, i.e. the homology homomorphism

$$g_*: H_n(Z; \mathbb{Z}_l) \to H_n(S^n; \mathbb{Z}_l) = \mathbb{Z}_l$$

does not vanish for some integer l, then there exists a point $p \in P$ such that the volumes of the ε-neighbourhoods of the g-image in S^n of the p-fiber of f are (sharply!) bounded from below by

$$vol_n(U_\varepsilon(g(f^{-1}(p)) \geq vol_n(U_\varepsilon(S^m).$$

.

Conjectural Quasi-Corollary. Let $Y \subset \mathbb{R}^n$ be a compact subset, let $S^m = S^m(R) \subset \mathbb{R}^{m+1} \subset \mathbb{R}^n$ be the sphere of radius R and let ε be a positive (possibly large) number.

Let

$$incl^i(R): H^i(U_R(Y); \mathbb{Z}_l) \to H^i(Y; \mathbb{Z}_l), \; i, l = 1, 2, 3, ...$$

denote the cohomology homomorphisms induced by the inclusion of Y into its closed R-neighbourhood $U_R(Y) \subset \mathbb{R}^n$.

If the n-volumes of the ε*-neighbourhoods of* Y *are bounded by those of* $S^m = S^m(R)$ *in* $\mathbb{R}^n$*,*

$$vol_n(U_\varepsilon(Y)) \leq vol_n\left(U_\varepsilon(S^m)\right),$$

then

$$incl^i(R) = 0 \text{ for all } l = 1, 2, 3, ... \text{ and all } i \geq m.$$

Proof Modulo a Bound on $\mathbb{Z}_l$*-Waist of* S^{n-1}*.* Assume $R = 1$ and let $incl^i(R) \neq 0$, say for $i = m$. Then there exists a $\mathbb{Z}_l$ cycle $P \subset \mathbb{R}^n$, of dimension $n - m - 1$ such that

• $dist(y, p) > 1$ for all $(y, p) \in Y \times P$

and

[109] If $m = dim(P) = n$ one should assume that f is homotopic to a map with the image of dimension $n-1$.

• the cycle P is linked with some cycle in Y, that is the map $g : Y \times P \to S^{n-1} \subset \mathbb{R}^n$ for

$$(y, p) \mapsto y/||y||$$

induces non-zero homomorphism on the $(n-1)$-dimensional $\mathbb{Z}_l$ cohomology.

The condition $dist(y, p) > 1$ implies that the maps

$$g_p : Y = Y \times p \to S^{n-1} \text{ for } g_p(y) = g(y, p)$$

are *distance decreasing* and, according to a *theorem by Bezdek and Connelly* (see below), the Euclidean ε-neighbourhoods of the images $g_p(Y) \subset S^{n-1} \subset \mathbb{R}^n$ satisfy

$$vol_n(U_\varepsilon(g_p(Y))) \le vol_n(U_\varepsilon(Y)) \text{ for all } p \in P.$$

But, since f is homologically *non-trivial*, the $\mathbb{Z}_l$-wast conjecture would imply that there exists a point $p \in P$, such that

$$vol_n(U_\varepsilon(g_p(Y))) \ge vol_n(U_\varepsilon(S^m)),$$

where both neigbotrhoods are taken in $\mathbb{R}^n$ and where a passage from spherical neighbourhood $U_\varepsilon \subset S^{n-1}$ (as in the waist conjecture) to the present Euclidean ones $U_\varepsilon \subset \mathbb{R}^n \supset S^{n-1}$ is easy.

Conclude by observing that the above is a valid proof for $l = 2$ and that the resultng bound on the minimal *filling radius* R of $Y \subset \mathbb{R}^n$ for which $incl^m(R) = 0$ follows, in the limit for $\varepsilon \to 0$, from Bombiery-Simon solution of *Gehring's linked spheres conjecture.*

The Kneser-Poulsen conjecture and Bezdek-Connelly Theorem. This conjecture claims

monotonicicty of the volumes of ε-neighbourhoods in Euclidaen spaces under *surjective distance decreasing maps*

$$\mathbb{R}^n \supset Y_1 \overset{g}{\to} Y_2 \subset \mathbb{R}^n :$$

if g is distance decreasing then

$$vol_n(U_\varepsilon(Y_2 = g(Y_1))) \le vol_n(U_\varepsilon(Y_1)).$$

Bezdek-Connelly proved (among other things) this monotonicity if there is a *dinstance decreasing homotopy* g_t of the identity map $g_0 : Y_1 \to Y_1$ to $g = g_1 : Y_1 \to Y_2$, i.e. such that the maps g_t are distance decreasing and, moreover, $g_{t_2} : Y_1 \to Y_{t_2}$ for $Y_t = g_t(Y_1)$ factor as

$$Y_1 \overset{g_{t_1}}{\to} Y_{t_1} \overset{g_{t_1,t_2}}{\to} Y_{t_2}$$

for some distance decreasing maps g_{t_1,t_2} and all $0 \le t_1 < t_2 \le 1$.

This applies to the above maps $g_p : Y \to g_p(Y) \subset S^{n-1} \subset \mathbb{R}^n$, ($g_t$ is not the same as g_p, sorry for this) since the obvious radial homotopy is distance decreasing.[110]

[110]There must be a direct proof of the monotonicty of the volumes of U_ε for these g_p but I forgot the argument.

2.6 Waists of Other Simple Spaces.

$\mathbb{Z}_2$-*Waist Conjecture in Banach Spaces.* Let X be an n-dimensional *(Mikowski)-Banach space*, where the norm is denoted by $||...||_X$, let Y be a closed Riemannian manifold of dimension m and P be a pseudomanifold of dimension $n-m-1$.

Let $g: Y \times P \to X$ be a continuous map of *norm one*, i. e. $||g(y,p)||_X = 1$, for all $y \in Y$, $p \in P$, and such that the restrictions of g to all $Y = Y \times p \subset Y$, $p \in P$, are *distance decreasing* maps $Y \to X$.

If the homomorphism on the homology of X minus the origin,

$$g_*: H_{n-m-1}(Y \times P; \mathbb{Z}_2) \to H_{n-m-1}(X \smallsetminus 0; \mathbb{Z}_2),$$

does not vanish, then

$$vol_m(Y) \geq c_m \textit{ for some constant } c_m > 0 \textit{ independent of } X.$$

This conjecture, would imply a bound on the filling radii of submanifolds Y in X similarly how it works for the spaces $\mathbb{R}^n$ with the ordinary Pythagorean norms $(\sum_{i=1,...,n} |x_i|^2)^{\frac{1}{2}}$, (see the proof of the . "Conjectural Quasi-Corollary" in the previous section) and such a bound may, conceivably, could be better than those delivered by the existing proofs.[111]

But possibly, the above conjecture is false: an absence of a *simple and general* formulation for *infinite dimensional* Banach spaces X makes one suspicious about the finite dimensional case as well.

The basic instance of a space X where the issue is undecided is L^n_∞ that is the space of real functions x on a finite set I of cardinality n with the norm

$$||x|| = \sup_{i \in I} |x(i)|.$$

The above conjecture raises the following *Parametric Lipschitz Comparison Problem* between (spheres in) Banach spaces.

Let X_1 and X_2 be finite dimensional Banach spaces and $S_i \subset X_i$, $i = 1, 2$, be the spheres of vectors of *norm one* in these spaces.

Denote by $contr(S_1 \to S_2)$ the minimal λ such that the space of λ-Lipschitz maps $S_1 \to S_2$ *does not contract to the subspace of constant maps* $S_1 \to s \in S_2$. The problem consist in evaluating this *contr* for particular Banach spaces.

For instance, let X_i be the Euclidean spaces $\mathbb{R}^{n_i}$, $i = 1, 2$. Then $contr(S_1 \to S_2) = 1$ for $n_1 \leq n_2$ and if $n_1 > n_2$, then $contr(S_1 \to S_2) > 1$; probably, $contr(S_1 \to S_2) \geq 2$.

Conjecture. Let $X_1 = L^{n_1}_\infty$ and $X_2 = \mathbb{R}^{n_2}$. Then

$$contr(S_1 \to S_2) \to \infty \text{ for } n_1, n_2 \to \infty.$$

Moreover, still conjecturally, this remains true with arbitrary *uniformly convex* spaces instead of $\mathbb{R}^{n_2}$.

Notice that a universal bound $contr(S_1 \to S_2) \leq const$ for all n_1 and all *sufficiently large* $n_2 >> n_1$, would yield the *zero in the spectrum conjecture*, and, probably, *the Novikov higher signatures conjecture* that are stated in section ???.

[111]see my *Filling of Riemannian manifolds* and *A short proof of Gromov's filling inequality* by Stefan Wenger, arXiv:math/0703889.

Waists of Arithmetic Varieties.

Let $\tilde{X}$ be an n-dimensionalal Riemannian symmetric space of *non-compact type* (i.e. with negative Ricci curvature) and let Γ_i, $i = 1, 2, ...$, be free discrete, e.g. arithmetic, isometry groups acting on $\tilde{X}$ with compact quotients, denoted $X_i = \tilde{X}/\Gamma_i$, such that $\cap_1 \Gamma_i = id$ for the common identity element $id \in \Gamma_i$.

Let $waist_m(X_i)$ denote the the *infimum of the numbers* $w > 0$ such that X_i admits a continuous map $X_i \to \mathbb{R}^{n-m}$, such that the Minkowski measures of the pullbacks of all points $p \in \mathbb{R}^{n-m}$ satisfy

$$Mink_m(f^{-1}(p)) \le w.$$

Question. What are possible asymptotics of $waist_m(X_i)$ for $i \to \infty$?

One knows in this respect that many sequences X_i are *expanders* and they satisfy *linear wast inequalities* that are lower bounds on waists by $const_{\tilde{X}} \cdot vol(X_i)$.

For, example if $\tilde{X}$ is an irreducible symmetric space with $rank_{\mathbb{R}} \ge 2$ then its isometry group is Kazhdan T and the quotient spaces $X_i \tilde{X} \Gamma_i$ are expanders. Consequently,

$$\liminf_{i\to\infty} \frac{\log waist_{n-1}(X_i)}{\log vol_n(X_i)} = 1.$$

It is also not hard to show that if $\tilde{X}$ is an irreducible symmetric space with $2 \le rank_{\mathbb{R}}(\tilde{(X)} \le m - 1$, then

$$\liminf_{i\to\infty} \frac{\log waist_m(X_i)}{\log vol_n(X_i)} \ge c > 0.$$

Questions. Does the limit $\lim_{i\to\infty} \frac{\log waist_m(X_i)}{\log vol_n(X_i)}$ exist for all "natural sequences $X_i = \tilde{X}/\Gamma_i$?

Are there sequences X_i such that

$$\liminf_{i\to\infty} \frac{\log waist_m(X_i)}{\log vol_n(X_i)} = 1$$

for some $m < n - 1$?

For instance, let $\tilde{X}$ be irreducible symmetric space of rank r, and let $m \ge n - r/2$. Is then

$$\liminf_{i\to\infty} \frac{\log waist_m(X_i)}{\log vol_n(X_i)} = 1?$$

This example is motivated by possible behaviour of wasts of Cartesian products where one may(?) expect (sometimes?) $waist_{m'+m''}(X'_i \times X''_i)$ to be bounded *from below* by ("reasonable", e.g. linear, functions of)

$$waist_{m'}(X'_i) \cdot waist_{m''}(X''_i).$$

The above questions also make sense for quotients of Bruhat-Tits buildings $\tilde{X}$, where the lower linear bound on $(n-1)$-waists of $X_i = \tilde{X}/\Gamma_i$ is available for buildings $\tilde{X}$ with no 1-dimensional factors but the case $m < n - 1$ remains problematic

In general, it is unclear what should be a class of n-dimensiopnal spaces X_i where one may expect interesting, let them be conjectural, lower bounds on waists$_m$ for $m < n-1$,

Waists of Miscellaneous Spaces. Even if X is a Riemannian manifold with apparently simple and transparent geometry, evaluation of $waist_m(X)$ may be difficult, especially, for $2 \leq m \leq dim(X) - 2$.

Among specific examples we mention the following:

(a) compact homogeneous, e.g. symmetric, spaces X, such as the real, complex and quaternionic projective spaces;

(b) R-balls in compact and, especially, in non-compact symmetric spaces.

One can hardly *precisely* evaluate the waists of these spaces, but it would be interesting to establish non-trivial upper and lower bounds on their waists.

Also one wishes to understand the behaviour of waists under basic geometric constructions such as

- Cartesian and more general warped products;
- geometric suspensions and geometric joins;
- ramified coverings and similar maps;
- quotients under actions of compact isometry groups.

2.7 Fundamental Geometric Inequalities and Geometric Measure Theory in Alexandrov Spaces with Curvatures bounded from below.

A metric space X is called a geodesic and/or length space if every pair of points $x_1, x_2 \in X$ can be joined by a distance minimising geodesic that is an isometric map from the interval $[0, d = dist(x_1, x_2)]$ to X with its ends going to x_1 and x_2.

X is called *Alexandrov space with curvatures* $\kappa \geq \kappa_0$ for $\kappa_0 \geq 0$) if every 1-Lipschitz (i.e. distance non strictly decreasing map from any subset $X_0 \subset X$ to the hemisphere $S^N_+(R)$ of radius R for $R = \kappa^{-2}$ and for all $N = 1, 2, 3, ...$ extends to a 1-Lipshitz map of all of X to $S^N_+(R)$, where we agree that $S^N_+(R = \infty) = \mathbb{R}^N$).

The main example of such spaces X are the spheres $S^n(R)$, where the 1-Lipschitz extension property goes back to Kirszbraun.

In general, this concept is the dual to that of CAT spaces (see section 2.3) and if such an X happens to be a smooth Riemannian manifold this is equivalent to X having its sectional curvatures $\kappa \geq 0$ by the generalisation of Kirszbraun's theorem by Lang and Schroeder.

Buyalo-Hentze-Karcer-Weyl Tube's Volume Bound. The volumes of the ε-neighbourhoods $U_\varepsilon(Y) \subset X$ of closed m-dimensional Riemmanin submanifolds Y in Riemannian spaces X with sectional curvatures $\kappa \geq 1$ are bounded in terms of $vol_m(Y)$ and the supremum of the mean curvatures of Y, where the corresponding inequality becomes equality for round sub-spheres $S^m \subset S^n$. For instance, the ε-neighbourhoods of submanifolds $Y \subset X$ with zero mean curvatures are bounded by $C(m, n, \varepsilon) \cdot vol_m(X)$ where the factor C is such that this becomes equality for equatorial spheres $S^m \subset S^n$.

The tube's volume bound extend to *singular subvarieties* $Y \subset X$ in-so-far as their singularities $\Sigma \subset X$ are such that the measure of the set of points $x \in X$

for which $dist(x, \Sigma) = dist(x, Y)$ equals zero.

In particular, these bounds apply to *minimal subvarieties* where the above property follows from general regularity theorems of Almgren and Allard. It follows [112] that Almgen's lower bound on waists of spheres as his optimal isoperimetric/filling inequality extend to manifolds X with sectional curvatures $\kappa \geq 1$.

Recall that Almgren's bound on waists applies to m-volumes of members of "topologically significant" families $Y_p \subset X$ but the corresponding bound for waists defined with with n-volumes of the ε-neighbourhoods of Y_p in X remains open. Namely, we have the following

Conjecture. Let X be a closed Riemannin n-manifold with the sectional curvatures $\kappa \geq 1$, let $f : X \to \mathbb{R}^{n-m}$ be a continuos map, and let $\varepsilon > 0$ be given. Then there exits a point $p \in \mathbb{R}^{n-m}$ such that the ε-neighbourhood of the "fiber" $Y_p = f^{-1}(p) \subset X$ satisfies

$$\frac{vol_n(U_\varepsilon(Y_p))}{vol_n(X)} \geq \frac{vol_n(U_\varepsilon(S^m))}{vol_n(S^n)}$$

where S^n is the unit sphere and $S^m \subset S^n$ is an equatorial subsphere.

Remarks. (a) As we mentioned earlier, the limiting case for $\varepsilon \to 0$ of this conjecture for generic smooth maps f follows by Almgren's minmax argument that delivers a minimal subvariety Y_{min} such that $vol_m(Y_{min}) \leq vol_m(Y_p)$ for all $p \in \mathbb{R}^{n-m}$. This goes along with the*Buyalo-Hentze-Karcer-Weyl tube's volume bound* and shows that

$$\frac{vol_m(Y_p)}{vol_n(X)} \geq \frac{vol_m(S^m)}{vol_n(S^n)}.$$

Notice in this regard that the function $\varepsilon \mapsto vol_n(U_\varepsilon(Y))$ is not only bounded but it *grows slower* and, moreover, it is "*more concave*" than the corresponding function for $S^m \subset S^n$. This gives us a sharp lower bound on the volume $U_\varepsilon(Y_{min})$ but this does not(?) tell us anything about the volumes of $U_\varepsilon(Y_p)$.

(b) If $X \neq S^n$, it seems to much to ask for a p that would serves all $\varepsilon > 0$ simultaneously.

(c) Let us see what happens if $\varepsilon = \pi/2$, where the conjecture says that $U_\varepsilon(Y_p) = X$ for some p.

In this case the complements of $X \setminus U_{\pi/2}(Y_p) = X$ are convex and if all of them were non-empty one would get a continuous map $X \to X$ that sends each subset Y_p to a point in its complement.

If, for instance, X is homeomorphic to S^n such a map, necessarily must be contractible and it can be ruled out by the fixed point theorem, but I am not certain what happens in general.

Plateau Problem in Singular Alexandrov Spaces. Most likely, the geometric measure theory can be extended to *singular Alexandrov spaces with curvatures bounded from below* to a point where Almgren's argument will match the tube's volume bounds and would, in particular, deliver Almgren's waist inequality for spaces with $\kappa \geq 1$ and his optimal isoperimetric/filling inequality for complete non-compact manifolds X with $\kappa \geq 0$.

Let us indicate two specific corollaries of such an extension that we formulate below in least general terms in order to use the minimal amount of definitions and to make the ideas clearer.

[112] See *A Note on the Geometry of Positively-Curved Riemannian Manifolds* by Memarian and sections 3.3-3.5 in part 2 of my "*Singularities, Expanders....*

[1] Area Shrinking Conjecture. (Compare section 2.3.) Let $W = V \times [R_1, R_2]$ be a compact Riemannian n-manifold with two boundary components, $V_1 = V \times R_1$ and $V_2 = V \times R_2$, with the metric $dR^2 + \theta^2 R^2 dv^2$ for some Riemannian metric dv^2 on V and some positive constant $\theta \le 1$.

Let X be an N-dimensional Alexandrov space with curvatures ≤ 0, such that the volumes of the balls $B(R) \subset X$ of radii R around a fixed point $x_0 \in X$ are related to the volumes $vol_{\mathbb{R}^N} = R^n vol_{\mathbb{R}^N}(1)$ of the Euclidean balls by

$$\limsup_{R\to\infty} \frac{vol_N(B(R))}{vol_{\mathbb{R}^N}(R)} \ge \theta^N.$$

where vol_N is understood as the N-dimensional Hausdorff measure normalised as usual (i.e. such that the unit Euclidean cubes have measure 1.).

Then, conjecturally, every $(n-1)$-volume contracting map $V_2 \to X$ extends to a n-volume contracting map $f : W \to X$ such that the restriction of f to V_1 is $(n-1)$-volume contracting.[113]

Lower Bound on the Waist. Let X be an n-dimensional Alexandrov space with curvatures $\kappa \ge 1$, let Z be an n-dimensional pseudomanifold with piecewise smooth Riemannian metric and let $f : Z \to \mathbb{R}^{n-m}$ be a piecewise linear map with (at most) m-dimensional "fibers" $Z_p = f^{-1}(p)$. $p \in \mathbb{R}^{n-m}$.

Let $g : Z \to X$ be a Lipshitz map that is m-volume contracting on all Z_p.[114]

If g induces a non-trivial isomorphism on the n-dimensional (co)homology, say, the homomrphism

$$g^* : H^n(X; \mathbb{Z}_2) \to H^n(Z; \mathbb{Z}_2)$$

is non-zero, then, conjecturally, there exists a point $p \in \mathbb{R}^{n-m}$, such that the m-volume of Z_p is bounded from below in terms of that of the equatorial spheres $S^m \subset S^n$ by

$$\frac{vol_m(Z_p)}{vol_n(X)} \ge \frac{vol_m(S^m)}{vol_n(S^n)}.$$

Evidence. If X has only *orbi-singularities*, i.e. such as in smooth manifolds divided by compact isometry groups of $\tilde{X}$, than the Plateau problems can be handled by the classical theory; also *piecewise smooth* Alexandrov metrics seem within reach.

In general, the Plateau problem is "weakly solvable" in Alexandrov spaces X with curvatures bounded from below (by possibly negative continuous functions on X), since these spaces are *locally Lipschitz contractible.*[115]

In fact, Lipschitz contractibility of an X implies (non-sharp) *filling inequalities*[116] which suffices for the existence of "weak solutions" of Plateau type problems by the standard compactness argument, where these solutions Y appear

[113] If X is smooth Riemannian this follows by Almgren's argument as it is explained in Part 2 of my "*Singularities, Expanders...*".

[114] This means g can be reparametrized by volume preserving self maps of the open faces of Z_p to 1-Lipschitz maps as in section 2.3.

[115] See *Locally Lipschitz contractibility of Alexandrov spaces and its applications* where this is proven by Mitsuishi and Yamaguchi with a use of Perelman-Petrunin gradient flow.

[116] See *Fillimg Riemannan Manifolds.*

as certain limits of Lipschitz maps of polyhedra into X.[117]

Conjecturally, minimal subvarieties Y in Alexandrov spaces X are as regular as it is conceivably possible and also they share other properties with minimal varieties in smooth manifolds.

For instance one expects

the monotonicity inequality for the volumes of intersections of Y with R-balls $B_y(R) \subset X$), where this inequality must be as *sharp* as it is allowed by obvious examples.

Also

all notions of m-volume, $m = dim(Y)$, must be equivalent for these Y, including

(1) Hausdorff m-dimensional measure $Hau_m(Y)$.

(2) The "entropic volume" $entvol_m(Y)$ defined with covering Y by balls of *equal radii* ε, where eventually $\varepsilon \to 0$.

(3) The Hilbert volume $Hilb_m(Y)$ defined via families of Lipshitz functions on X.[118]

The equivalence of these (1), (2), (3) seems easy. For instance, it would follow from the existence of a bi-Lipshitz embedding of X into a Hilbert space. (I am not certain if this has been proved for Alexandrov spaces.)

But it is unclear how to properly formulate the corresponding property of the *Minkowski volume* that is defined via the n-volumes of the ε-neighbourhoods of Y in X, for $n = dim(X)$:

on the one hand, minimal subvarieties tend to be *transversal* to the singular locus $\Sigma \subset X$;

on the other hand, subvarieties minimising the Minkovski volumes may end up *inside* Σ , as it happens, for instance, for $X = X_0 \times Y$.

Some geometry of a minimal $Y \subset X$ at a point $y_0 \in Y$ can be seen in the tangent cone X'_{y_0} of X at y_0 with a tangent minimal (sub)cone $Y'_{y_0} \subset X'$. (The existence of these is obvious in the present case.)

For instance, let x_0 be an *isolated singular* point in X where the tangent cone to X at x_0 equals the cone over a non-singular manifold S with sectional curvatures ≥ 1, that is *not isometric* to the unit sphere S^{n-1} and let $Y \subset X$ be an m-volume minimizing subvariety passing through x_0.

Then Y satisfies the tube volume inequality near x_0, since the point x_0 is "overshadowed" by the rest of Y according to the following (easy to prove) strict inequality.

$$dist(x, Y) < dist(x, x_0) \text{ for all } x \in X,\ x \neq x_0.$$

About Scalar Curvature. The scalar curvature of an Alexandrov space X is defined at almost all points $x \in X$ (where X is regular) and it may be assumed equal $+\infty$ at the singular points of X. Thus, one may speak of Alexandrov spaces with their *scale curvatures bounded from below* by a given constant κ or, more generally, by a continuous function $\kappa(x)$ on X.

[117]See *Currents in metric spaces* by Ambrosio and Kirchheim and *The intrinsic flat distance between Riemannian manifolds and other integral current spaces* by Sormani and Wenger for a higher perspective.

[118]See my *Hilbert Volume in Metric Spaces* where the definition is similar (but, probably, not identical) to the mass introduced by Ambrosio and Kirchheim.

Do geometric inequalities extend from smooth manifolds with $scal(X) \geq \kappa$ to singular Alexandrov spaces?

For instance, let X_i, $i = 1, 2, 3, ...$, be a sequence of n-dimensional Alexandrov spaces that are all orientable topological pseudomanafolds and let $X_{i+1} \to X_i$ by 1/2-Lipschitz maps of degree one that induce isomorphisms on the 2-dimensional cohomology groups with $\mathbb{Z}_2$-coefficients.

Then, conjecturally, there exits points $x_i \in X_i$, such that

$$\limsup_{i\to\infty} scal_{x_i}(X_i) \leq 0.$$

Notce that the proof in the smooth case (with the 1/2-Lipschitz condition generalised to the area contraction property) from my paper *Positive curvature, macroscopic dimension, spectral gaps and higher signatures* applies to all complete manifolds. An extension of this argument to "sufficiently large" *manifolds with boundaries* can, conceivably, be applied to "large almost non-singular regions" in singular spaces X_i for large i and lead to the proof of the above assertion.

Finer results depending on the Dirac operator seems harder to adapt to singular spaces, in part, because the spin condition (which, conjecturally, is unnecessary in most cases anyway) does not quite make sense for singular X. But the Schoen-Yau proof of the Geroch conjecture with a use of minimal hypersurfaces, is likely to extend to the singular case:

Conjectuarlly,

if a topological n-psedomanifold X admits a maps of positive degree to the n-torus $\mathbb{T}^n$, then X can not carry an Alexandrov metric with $scal > 0$.

(This and related geometric results depending on minilamal hypersurfaces[119] were originally proven for $n \leq 7$. Then the singularity problem for $n = 8$ was removed by Natan Smale [120] and the techniques developed by Lohkamp for "going around singularities of minimal hypersurfaces" in smooth n-manifolds for $n \geq 9$ may be applicable to singular Alexandrov spaces as well.)

Conclude by noticing that the concept of positive scalar curvature goes beyond Alexandrov spaces as we shall explain in section (???).

2.8 Regularization and Geometrization of the Geometric Measure Theory.

Can one directly prove the tube volume bound for quasiminimaizing sub varieties?

On the other hand, it is less clear what should be "the calculus of variation" for the functional $vol_n(U_\varepsilon(Y_{n-m}))$, $\varepsilon > 0$, on $(n-m)$-cycles $Y_{n-m} \subset X$ instead of $vol_{n-m}(Y^{n-m})$.[121]

[119] See the last chapters in our paper with Lawson, *Positive scalar curvature and the Dirac operator on complete riemannian manifolds.*

[120] See his *Generic regularity of homologically area minimizing hypersurfaces in eight dimensional manifolds.*

[121] Such calculus for $m = 1$ would justify Paul Lévy's proof of his isoperimetric inequality.

2.9 Equidistribution Arguments, Transportation of Measures and Complexified Isoperimetry.

2.10 Concentration, Waists and Spaces of Cocycles.

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2.11 Selected References.

References

[1] Burago-Zalgaller

[2] Isaac Chavel. Isoperimetric Inequalities: Differential Geometric and Analytic Perspectives

[3] Philippe Castillon. Submanifolds, Isoperimetric Inequalities and Optimal Transportation (references pointed to me out by Simon Brendle)

[4] Sanatalo

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3 Homology Measures and Linearized Isoperimetry.

Let $\phi : X \to Y$ be a continuous map and let $U \mapsto \mathcal{H}^*_\phi(U) =_{def} H^*(\phi^{-1}(U); \mathbb{F})$ for some field $\mathbb{F}$ be the corresponding cohomology sheaf over Y. The "set function" $\mathcal{H}^*_\phi(U)$ behaves in some respect as a *Borel measure* on Y that suggests the following

Vague General Question. *Which geometric inequalities concerning measures of U, such as isoperimetric (and/or spectral* inequalities, for instance. have their counterparts for $\mathcal{H}^*_\phi(U)$?

The Ideal Valued Measures. The sheaf $\mathcal{H}^*_\phi$, probably, is a bona fide mathematical object for specific "nice" maps ϕ, e.g. for *perfect Morse functions*, but it is unstable under small perturbation of maps in the uniform topology. and is quite unruly for *general continuous* maps.

A more robust measure-like set function on Y is defined via the cohomology restriction homomorphisms from the cohomology of X to those of the pullbacks of the complements of open subsets $U \subset Y$, say

$$Re^*_{\setminus U} : H^*(X; \mathbb{F}) \to H^*(\phi^{-1}(Y \setminus U); \mathbb{F}),$$

with the *ambient (co)homology mass* of U measured by the graded ideal

$$I^*_\phi(U) = ker(Re^*_{\setminus U}) \subset H^*(X; \mathbb{F}).$$

The basic numerical characteristic of such an ideal is its *graded rank*, that is the set of numbers

$$rank^*(I^*) = rank^*(I^*_\phi(U)) = \{r_n\}_{n=0,1,2,3,...} \text{ for } r_n = rank_{\mathbb{F}}(I^n_\phi(U))$$

where $I^n_\phi(U)) \subset H^n(X;\mathbb{F})$ denotes the n-graded subspace in $I^*_\phi(U)$.

Sample Question. Let X equals the Cartesian I-power, $X = X_0^I$, for a set I of a finite cardinality $card(I) = N$, and let Y be a topological space covered by finitely many open subsets, $Y = \bigcup_k$, $k \in K$.

What are universal inequalities satisfied by the numbers

$$r_{n,k} = rank(I^n_\phi(U_k))?$$

(Here "universal" means dependence only on X_0, on N and on $(Y, \{U_k\})$, but not on the map $\phi : X \to Y$. Also notice that the most relevant feature of Y is encoded in the combinatorics of the nerve of the covering $\{U_k\}$ and that one may allow "universal" be dependent on the homotopy type of ϕ or of the corresponding map to the simplicial complex representing the nerve of $\{U_k\}$.[122])

A concise answer to such a question may be expected for $N \to \infty$ which suggests looking at *infinite* Cartesian powers $X = X_0^I$ where, for instance, the set I; hence, the space $X = X_0^I$ as well, is acted upon by an amenable (sofic?) group and the properly renormalised graded ranks can be defined.

3.1 Homological Waist Inequalities.

Given a topological space X let $N_{cell}[X]$ denote the *cell number* of X that is minimum of the numbers of cells in the cellular spaces that are homotopy equivalent to X. Observe that this number is bounded from below by the totality of the Betti numbers of X,

$$N_{cell}[X] \geq \sum_{i\geq 0} rank(H_i(X;\mathbb{F})) \text{ for all fields } \mathbb{F}.$$

Digression: Stable Cell Number. Let X be a closed oriented n-manifold and consider all closed oriented n-manifolds X' that admit continuous maps $f' : X' \to \underline{X}$ of degrees $deg(f') = d' > 0$. Define

$$N^{stbl}_{cell}[X] = \inf_{X'} \frac{1}{d'} N_{cell}[X'].$$

Questions. 1. What are manifolds X where $N^{stbl}_{cell}[X] \neq 0$?

2. When does $N^{stbl}_{cell}(X_1 \times X_2)$ equal $N^{stbl}_{cell}(X_1) \cdot N^{stbl}_{cell}(X_2)$?

3. Are there *pseudomanifolds* X, such that $N^{stbl}_{cell}[X] \neq 0$ and such that all *n-manifolds* $\underline{X}$ that receive maps $X \to \underline{X}$ of non-zero degrees have $N^{stbl}_{cell}(\underline{X}) = 0$?

The simplest (yet, non-trivial for $n \geq 4$) examples of manifolds X where $N^{stbl}_{cell}[X] \neq 0$ are product of surfaces with negative Euler characteristics and where this follows from a lower bound on the *rank norm* of the *Wall groups* of these X that generalises by a Lustig theorem to the other quotient manifolds $X = \tilde{X}/\Gamma$ for the symmetric space of non-compact type with non-zero Euler form, that where $\chi(X) \neq 0$.[123]

[122] Some inequalities of this kind for the tori $\mathbb{T}^N$ are proven in the second part of my ?Singularities&Expanders? 2009 paper in GAFA, where they are used to bound from below *max-cell numbers* of continuous maps defined in the next section.

[123] See section $8\frac{1}{2}$ in my *"Positive Curvature, Macroscopic Dimensin..."* and section ??? where we discuss norms on various kind of (co)-homology.

It remains unknown

whether there are odd-dimensional manifolds X with $N^{stbl}_{cell}[X] \neq 0$,

but recent results by Wise and Agol rule out 3-dimensional manifolds and compact quotients $X = \tilde{X}/\Gamma$ for other odd dimensional symmetric spaces $\tilde{X}$, even those and without 3-dimensional factors, are likely(?) to have $N^{stbl}_{cell}[X] = 0$ as well.

Given a continuous map $f : X \to Y$ define its max-cell number as the supremum of the cell numbers of the fibers $X_y = f^{-1}(y)$,

$$N_{cell}(f) = \sup_{y \in Y} N_{cell}[X_y].$$

Denote by $MIN_{cell}\left(X_{/Y}\right)$ *the minimal cell number of X over Y* that is the minimum of the numbers $N_{cell}(f)$ over all continuous maps $f : X \to Y$, and let $MIN_{cell}\left([X]_{/Y}\right)$ be the infimum of $MIN_{cell}\left(X'_{/Y}\right)$ over all X' that are homotopy equivalent to X.

Torus Problem. *Evaluate the minimal cell numbers $MIN_{cell}\left([\mathbb{T}^N]_{/\mathbb{R}^m}\right)$ of the (homotopy types of the) N-tori over the Euclidean m-spaces for large N and $m << N$.*

The currently known (easy) lower bound on this MIN_{cell}, that is obtained by an evaluation of the maximum $\max_{y\in Y}$ of the ranks of the restriction cohomology homomorphisms $H^*(X) \to H^*(X_y)$ (via the above $rank^*(I^*)$ of the kernels of the restriction cohomology homomorphisms $H^*(\mathbb{T}^N) \to H^*(f^{-1}(U))$, $U \subset \mathbb{R}^m$) reads:[124]

$$MIN_{cell}\left([T^N]_{/\mathbb{R}^m}\right) \geq 2^{\frac{N}{m+1}}$$

which means (somewhat more than) that every continuous map $f : \mathbb{T}^N \to \mathbb{R}^m$ admits a point $y \in \mathbb{R}^m$ such that the fiber $X_y = f^{-1}(y) \subset \mathbb{T}^N$ *can not be decomposed* into less than $2^{\frac{N}{m+1}}$ cells. Moreover if $m = 1$, then, for every $i < N/2$, there exists a point $y \in \mathbb{R}$ (that depends on i) such that all cellular decompositions the fiber X_y contain at least $\left(1 - \frac{2i}{N}\right)\binom{N}{i}$ cells of dimension i.

k-Connected Homotopy Expanders. *Can a "simple" space X have a "relatively large" $MIN_{cell}\left(X_{/\mathbb{R}^m}\right)$?*

Let us make this precise with the following definitions.

Local Boundedness. Call a family $\mathcal{P}$ of simplicial polyhedra P *locally bounded* if all $P \in \mathcal{P}$ have *uniformly bounded* local geometries: the numbers of simplices at all vertices in all $P \in \mathcal{P}$ are bounded by a constant $C = C(\mathcal{P})$.

k-Connectedness A polyhedron P called *k-connected* if its *homotopy groups $\pi_i(P)$ vanish* for $i \leq k$, or, equivalently, if all continuous maps $Q \to P$, for *all k-dimensional* polyhedra Q, are *contractible.*

Let k, m and N be positive integers and let $\mathcal{P}$ be a **locally bounded** family $\mathcal{P}$ of finite N-dimensional simplicial polyhedra, such that

- the number of different homotopy types $[P]$ of polyhedra $P \in \mathcal{P}$ is **infinite**;
- all $P \in \mathcal{P}$ are **k-connected**;

[124] See part 1 of my *"Singularities&Expanders"* 2009 paper in GAFA with an improvement for $m = 1$ in part 2.

Question. *Can the minimal cell numbers of homotopy types of the members of such a family over $\mathbb{R}^m$ satisfy the following inequality?*

Linear Lower Bound on MIN_{cell}**.**

$$MIN_{cell}\left([P]_{/\mathbb{R}^m}\right) \geq \lambda \cdot N_{simpl}(P) \text{ for all } P \in \mathcal{P} \text{ and some } \lambda = \lambda(\mathcal{P}) > 0.$$

In other words. every continuous map of a polyhedron P', that is homotopy equivalent to some $P \in \mathcal{P}$, to $\mathbb{R}^m$, say $f : P' \to \mathbb{R}^m$ must have a fiber $P'_y = f^{-1}(y) \subset P'$ that is almost as complicated as P:

the minimal number of cells needed for a cellular decomposition of P'_y satisfies

$$N_{cell}(P'_y) \geq \lambda \cdot N_{simpl}(P).$$

This kind of linear bound is (obviously) satisfied by $MIN_{cell}\left(P_{/\mathbb{R}}\right)$, that is by the spaces P themselves rather than by their homotopy types,[125] for (families of) *expander graphs* P (where $k = 0$ and $N = 1$). More interestingly, families with the linear lower bound on $MIN_{cell}\left([P]_{/\mathbb{R}}\right)$ *do exist*[126] for $k = 1$ (and $m = 1$), where one may have all $P \in \mathcal{P}$ being *closed* (simply connected, since $k = 1$) 6-*manifolds*.

But if $k \geq 2$, then it is hard to say whether the k-connectedness condition drastically restricts possibilities of P and/or by how much the "homotopy size" of the fibers of maps to $\mathbb{R}^m$ suffers for large m; yet, one is inclined toward the positive answer to the above question, if one allows P of relatively large dimensions, say $dim(P) = N > 2m(k + 1)$ or something like this.

In any case, if the answer is negative, one is faced with the problem of finding an optimal *non-linear* low bound on $MIN_{cell}\left([P]_{/\mathbb{R}^m}\right)$ by $N_{simpl}(P)$ where this may be, in the interesting cases, of the form

$$MIN_{cell}\left([P]_{/\mathbb{R}^m}\right) \geq \lambda \cdot N^{\alpha}_{simpl}(P) \text{ for some } \alpha = \alpha(k, m, N) > 0.$$

4 Morse Spectra, Spaces of Cycles and Parametric Packing Problems.

An "ensemble" $\mathcal{A} = \mathcal{A}(X)$ of (finitely or infinitely many) particles in a space X, e.g. in the Euclidean 3-space, is customary characterised by the set function

$$U \mapsto ent_U(\mathcal{A}) = ent(\mathcal{A}_{|U}),\ U \subset X,$$

that assigns the *entropies* of *the U-reductions $\mathcal{A}_{|U}$ of* $\mathcal{A}$, to all bounded open subsets $U \subset X$. In the physicists' parlance, this entropy is

"*the logarithm of the number of the states of* $\mathcal{E}$

[125]If $N = k + 1$, then N-dimensional k-connected polyhedra P (e.g. connected graphs) are homotopic to a *joins of spheres*; hence, $MIN_{cell}\left([P]_{/\mathbb{R}}\right) \leq 2$ for all these P. Probably, one can (completely?) describe the families $\mathcal{P} = \{P\}$ where $MIN_{cell}\left([P]_{/\mathbb{R}}\right) \leq const$ for all P.

[126]The construction of such $\mathcal{P}$ that is suggested in Part 2 of "*Singularities&Expanders*" relies on quite peculiar properties of *Margulis' expanders*.

that are effectively observable from U",

This "definition", in the context of mathematical statistical mechanics, is customary translated to the language of the measure/probability theory.[127]

General Question. What happens if "*effectively observable number of states*" is replaced by

"*the number of effective degrees of freedom*
of ensembles of moving balls".

In the classical packing problem, one is predominantly concerned with *maximally dense* packings of spaces X by *disjoint balls* that do not move much. We are, on the contrary, concerned with families of balls that are far from being dense and can move a lot.

For instance, let X is a *compact* Riemannian n-manifold and let $A = A(N,r)$ be the space of N-tuples of mutually disjoint r-balls $U_i = U(x_i,r) \subset X$, $i = 1,...,N$, with centres $x_i \in X$, where, A embeds into the Cartesian power $X^N = X^{1,...,N}$ for $U_i \mapsto x_i$.

In fact, since the balls are assumed disjoint, A lies in the complement $X^N \setminus diag$ that is the set of N-tuples $(x_i)_{i=1,...,N}$ such that $x_i \neq x_j$ for $i \neq j$.

Assume that r *is rather small relative to* N, say

$$r = \left(\frac{c \cdot vol(X)}{N}\right)^{\frac{1}{\delta}} \text{ for some constants } c > 0 \text{ and } \delta < n = dim(X),$$

where N is large (eventually, $N \to \infty$) and, accordingly, r is small; in fact, significantly smaller than $(1/N)^{\frac{1}{n}}$.

Then the total volume of the balls U_i is much smaller than the volume of X, and our N-tuples of (mutually disjoint!) balls are *far from being maximally dense in X.*

In this case the subset $A = A(N,r) \subset X^N$ constitutes a "topologicaly significant" part of $X^N \setminus diag$.

Asymptotic Parametric Packing Problem. Give a precese formulation and a quantitative estimate of this "significant" as a function of α and c for $N \to \infty$.[128]

Our discussion of this and other *parametric packing problems* borrows ideas from several different sources that include the following.

- *Classical (Non-parametic) Sphere Packings.*
- *Homotopy and Cohomotopy Energy Spectra.*
- *Homotopy Dimension, Cell Numbers and Cohomology Valued Measures.*
- *Infinite Packings and Equivariant Topology of Infinite Dimensional Spaces Acted upon by Non-compact Groups.*
- *Bi-Parametric Pairing between Spaces of Packings and Spaces*

[127]See: Lanford's *Entropy and equilibrium states in classical statistical mechanics, Lecture Notes in Physics, Volume 20, pp. 1-113, 1973* and Ruelle's *Thermodynamic formalism : the mathematical structures of classical equilibrium statistical mechanics*, 2nd Edition, Cambridge Mathematical Library 2004, where the emphasis is laid upon (discrete) *lattice* systems. Also a *categorical rendition* of Boltzmann-Shannon entropy is suggested in "*In a Search for a Structure, Part 1: On Entropy*", www.ihes.fr/~gromov/PDF/structre-serch-entropy-july5-2012.pdf

[128]An approach to this problems via the Morse singularity theory is suggested in *Min-type Morse theory for configuration spaces of hard spheres* by Baryshnikov-Bubenik-Kahle, arXiv:1108.3061 and http://www.math.ncsu.edu/TLC/TLC-kahle.pdf

of Cycles.[129]
- *Non-spherical Packings, Spaces of Partitions and Bounds on Waists.*
- *Symplecting Packings.*
- *Parametric coverings.*

4.1 Non-parametric Packings.

Recall, that *a sphere packing* or, more precisely, *a packing of a metric space X by balls of radii* r_i, $i \in I$, $r_i > 0$, for a given *indexing set I* of finite or countable cardinality $N = card(I)$ is, by definition, a collection of (closed or open) balls $U_{x_i}(r_i) \subset X$, $x_i \in X$, with mutually non-intersecting interiors.

Basic Problem. What is the *maximal radius* $r = r_{max}(X;N)$ such that X admits a packing by N balls of radius r?

In particular,

what is the asymptotics of $r_{max}(X;N)$ *for* $N \to \infty$*?*

If X is a compact n-dimensional Riemannian manifold (possibly with boundary), then the principal term of this asymptotics depends only on the volume of X, namely, one has the following (nearly obvious)

Asymptotic Packing Equality.

$$\lim_{N\to\infty} \frac{N \cdot r_{max}(X;N)^m}{vol_n(X)} = \kappa_m,$$

where $\kappa_m > 0$ is a universal (i.e. independent of X) *Euclidean packing constant* that corresponds in an obvious way to the *optimal density* of the sphere packings of the Euclidean space $\mathbb{R}^m$.

(Probably, the full asymptotic expansion of $r_{max}(X;N)_{N\to\infty}$ is expressible in terms of the derivatives of the curvature of X and derivatives the curvature similarly to Minakshisundaram-Pleijel formulae for spectral asymptotics.)

The explicit value of κ_m is known only for $n = 1,2,3$. In fact, the optimal, i.e. maximal, packing density of $\mathbb{R}^m$ for $m \le 3$ can be implemented by a $\mathbb{Z}^m$-*periodic* (i.e. invariant under some discrete action of $\mathbb{Z}^m$ on $\mathbb{R}^m$) packing, where the case $m = 1$ is obvious, the case $m = 2$ is due to Lagrange (who proved that the optimal packing is the hexagonal one) and the case of $m = 3$, conjectured by Kepler, was resolved by Thomas Hales.

(Notice that $\mathbb{R}^3$, unlike $\mathbb{R}^2$ where *the only* densest packing is the hexagonal one, admits *infinitely many* different packings; most of these are *not* $\mathbb{Z}^3$*-periodic*, albeit they are $\mathbb{Z}_2$-periodic.

Probably, none of densest packings of $\mathbb{R}^m$ is $\mathbb{Z}^m$-periodic for large m, possibly for $m \ge 4$. Moreover, *the topological entropy* of the action of $\mathbb{R}^m$ on the space of optimal packings may be non-zero.

Also, there may be infinitely many algebraically independent numbers among $\kappa_1, \kappa_2, ...$; moreover, the number of algebraically independent among $\kappa_1, \kappa_2, ..., \kappa_m$ may grow as $const \cdot m$, $const > 0$.)

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[129] An innovating use of such pairing for evaluation of Hermann Weyl's kind of asymptotics of *the Morse (co)homology spectra* of the *k-volume function* on the space of the *k-cycles* in the Euclidean m-ball, $m = k+1, k+2, ...$ is due to Larry Guth's, as it is exposed in his paper *Minimax problems related to cup powers and Steenrod squares*, Geometric and Functional Analysis, 18 (6), 1917-1987 (2009).

Many packing problems can be expressed in terms of invariants of (the boundaries of) the *I-packing shadows* denoted

$$pack_I(X) \subset \mathbb{R}_+^I = \bigtimes_{i \in I} (\mathbb{R}_+)_i = \underbrace{\mathbb{R}_+ \times \mathbb{R}_+ \times \dots \times \mathbb{R}_+}_{I}$$

that are defined as the subsets of those I-tuples of positive numbers r_i for which an X admits a packing by balls of radii r_i. 4. These invariants carry more information about the geometry of X than those associated with with mutually equal balls (corresponding to the intersection of $pack_I(X) \subset \mathbb{R}_+^I$ with the main diagonal ${}^{dia}\mathbb{R}_+ \subset \mathbb{R}_+^I$).

For instance, "simple" metric spaces, e.g. compact locally homogeneous Riemannian manifolds X, or at least those of constant curvatures, must be (almost?) uniquely determined by their I-packing shadows for sufficiently large (depending on X) finite sets I.

But the geometry of the shadows $pack_I(X) \subset \mathbb{R}_+^I$, e.g. the algebra-geometric complexity of the singularities of the boundaries of these shadows for $N \to \infty$ (and oven less so for $dim(X) = n \to \infty$) remains poorly understood, even for such manifolds X as n-tori and n-spheres.

Non-Spherical Packings. Besides round balls, one may look for packing of a (not necessary metric) space X by domains $U_i \subset X$ with certain constrains on their shapes.

Question. Let $U \subset \mathbb{R}^n$ be a bounded open subset, e.g. a generic semialgebraic one. Is there an effective sufficient conditions for the densest packing of $\mathbb{R}^n$ by isometric copies of U to be periodic, or, on the contrary, non-periodic?

Symplectic Packings. If X is a symplectic manifold, one is concerned with symplectically invariant rather than metric constrains on U, where definite results are available for U_i that are required to be symplectomorphic to round balls in the Euclidean space $\mathbb{R}^{2m}$ with a translationally invariant symplectic structure.

The central issue in the non-parametric case is to decide when X admits a density one packing by symplectic copies of U.[130] In general, one is concerned with the homotopy structure of spaces of symplectic embeddins of disjoint unions $\bigsqcup_i U_i \to X$, where (the only known) geometric constrains come via psedoholomorphic curves similarly to the non-parametric case.

4.2 Homotopy Perspective on Dirichlet's, Plateu's and Packing Spectra.

Let A be a topological space and $E : A \to \mathbb{R}$ a continuous real valued function, that is thought of as an energy $E(a)$ of states $a \in A$ or as a Morse-like function on A.

The subsets

$$A_r = A_{\leq r} = E^{-1}(\infty, r] \subset A, \ r \in \mathbb{R},$$

[130]See *Symplectic packings and algebraic geometry* by Dusa McDuff and Leonid Polterovich, *From Symplectic Packing to Algebraic Geometry and Back* by Paul Biran,
Symplectic embeddings and continued fractions: a survey by Dusa McDuff, *A Lagrangian quantum homology* by P.Biran and O. Cornea in *New Perspectives and Challenges in Symplectic Field Theory* edited by: Miguel Abreu and Franois Lalonde,
A maximal relative symplectic packing construction by L. Buhovsky.

are called the (closed) *r-sublevels of E*.

A number $r_\circ \in \mathbb{R}$ is said *to lie in the homotopy spectrum of E* if the homotopy type of A_r undergoes an *essential*, that is *irreversible, change* as r passes through the value $r = r_\circ$.

Prior giving precise definitions, representative examples and clarifyying remarks are in order.

Quadratic Example. Let A be an infinite dimensional projective space and E equal the ratio of two quadratic functionals. More specifically, let E_{Dir} be the Dirichlet function(al) on differentiable functions $a = a(x)$ normalised by the L_2-norm on a compact Riemannian manifold X,

$$E_{Dir}(a) = \frac{||da||^2_{L_2}}{||a||^2_{L_2}} = \frac{\int_X ||da(x)||^2 dx}{\int_X a^2(x) dx}.$$

The eigenvalues $r_0, r_1, r_2, ..., r_n, ...$ of E_{Dir} (i.e. of the corresponding Laplace operator) are *homotopy essential* since the rank of the inclusion homology homomorphism $H_*(A_r; \mathbb{Z}_2) \to H_*(A; \mathbb{Z}_2)$ *strictly* increases (for $* = n$) as r passes through r_n.

Volume as Energy. Besides Dirichlet's there are other natural "energies" on spaces A of continuous maps between Riemannian manifolds, $a : X \to R$. The most relevant for the moment is *the k-volume*[131] *of the pullback of a subset* $R_0 \subset R$,

$$a \mapsto vol_k(a^{-1}(R_0)), \ k = dim(R_0) + (dim(X) - dim(R)).$$

Notice that only the topology of R enters this definition, while some symmetry group of the pair (R, R_0) may be essential. For instance if $R = \mathbb{R}$ and $k = dim(X) - 1$ then one works with the (infinite projective) space of non-zero continuous functions $a : X \to \mathbb{R}$ divided by the involution $a \leftrightarrow -a$.

A more sophisticated version of the above is the k-volume function on the

space $C_k(X; \Pi)$ *of k-dimensional* Π*-cycles in a Riemannian manifold* X,

where Π is an Abelian group with a norm-like function on it, e.g. $\Pi = \mathbb{Z}$ or $\Gamma = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.

These spaces of (rectifiable) cycles with natural (flat) topologies are homotopy equivalent to products of Eilenberg McLain spaces that have quite rich homology structures that makes the homotopy spectra of the volume energies on these spaces,

$$E = vol_k : C_k(X; \Pi) \to \mathbb{R}_+,$$

quite non-trivial.

Packing Energy. Let X be a *metric space* and $A = A_N(X)$ be the *set of subsets* $a \subset X$ of finite cardinality N. Let

$$\rho(a) = \min_{x,y \in a, x \neq y} dist(x, y)$$

and define *packing energy* as

$$E_N(a) = \frac{1}{\rho(a)}$$

[131] This "volume" may be understood as the corresponding *Hausdorff measure* but if $k \geq 2$ than it is easier to work with *the Minkovski* measure.

for the energy of a.

Sublevel $A_{1/r}$ of this energy are exactly packings of X by r-balls.[132]

Permutation Symmetry and the Fundamental Group. The space $A_N(X)$ of unordered(!) N-tuples of points in X can be seen as the quotient space of the space $X^N \setminus diag$ that is, in other words, the space $X^{I_{inj}} \subset X^I$ of injective maps of a set I of cardinality N into X, by the permutation group $S_N = Sym(I)$,

$$A_N = X^{I_{inj}}/Sym(I),\ card(I) = N.$$

This suggest a G-equivariant setting for the homotopy spectrum for energy functions $E(x_1, x_2, ..., x_N)$ on X^I that are invariant under subgroups $G \subset S_N$, where even for fully symmetric E it may be profitable to use subgroups $G \subsetneq S_N$ containing only special permutations.[133]

Since the action of $S_N = Sym(I)$ on $X^{I_{inj}}$, (unlike the corresponding action of $Sym(I)$ on the Cartesian power X^I) is *free* the group S_N is seen in the fundamental group of $A_N(X)$, provided, for instance, X is a connected manifold of dimension $n \geq 2$. And if X equals the Euclidean n-space, the n-ball or the n-sphere, for $n \geq 3$, then

the fundamental group $\pi_1(A_N(X))$ is isomorphic to the permutation group S_N and the main contribution to the homotopy complexity of the space $A = A_N(X)$ comes from this fundamental group.

Finding a general setting embracing qualitative properties of the G-equivariant homotopy spectra of G-invariant energies E on X^I, $G \subset Sym(I)$, for "simple" spaces X, especially for $N \to \infty$, is an essential (but not the only) aspect of the parametric packing problem.

On Numbers and on Orders. The role of real numbers in the concept of "homotopy essential spectrum" reduces to indexing the subsets $A_r \subset A$ according to their *order by inclusion:* $A_{r_1} \subset A_{r_2}$ for $r_1 \leq r_2$.

In fact, our "spectra" make sense for functions $X \to R$ where R is in an arbitrary *partially ordered* set, where it is convenient to assume that R is a lattice i.e. it admits inf and sup.

Additivity, that is the most essential feature of physical energy, becomes visible only for spaces A that split as $A = A_1 \times A_2$ for$E(a_1, a_2) = E(a_1) + E(a_2)$.

On Stable and Unstable Critical points. If E is a Morse function on a smooth manifold A, then the homotopy type of A_r does change at all critical values r_{cri} of E. However, only exceptionally rarely, for the so called *perfect Morse functions*, such as for the above quadratic energies, these changes are irreversible. In fact, every value $r_0 \in \mathbb{R}$ can be made critical by an *arbitrary small C^0-perturbation*[134] E' of a smooth function $E(a)$, such that E' equals E outside the subset $E^{-1}[r_0 - \varepsilon, r_0 + \varepsilon] \subset A$; thus, the topology change of the sublevels of E' at r_0 is "non-essential".

[132]One could use, instead of $\frac{1}{\rho}$, an obituary positive monotone decreasing function in ρ.

[133]The word "permutation" usually applies to the set $I = \{1, 2, 3, ..., N\}$, with the points in $X^{I=\{1,2,...,N\}}$ written as $(x_1, x_2, ...x_N)$. But often one deals with (more) general categories of (finite or infinite) sets I where the group of invertible self morphisms of an I is denoted $aut(I)$.

[134]"C^0" refers to the uniform topology in the space of continuos functions.

4.3 Spectra of Induced Energies on Homotopies and Homologies.

Category $\mathcal{H}_\circ(A)$, Induced Energy $E_\circ$ and Homotopy Spectrum. Let $\mathcal{S}$ be a class of topologicl spaces S and let $\mathcal{H}_\circ(A) = \mathcal{H}_\circ(A;\mathcal{S})$ be the category where the objects are homotopy classes of continuous maps $\phi : S \to A$ and morphisms are homotopy classes of maps $\psi_{12} : S_1 \to S_2$, such that the corresponding triangular diagrams are (homotopy) commutative, i.e. the composed maps $\phi_2 \circ \psi_{12} : S_1 \to A$ are homotopic to ϕ_1.

Extend functions $E : A \to \mathbb{R}$ from A to $\mathcal{H}_\circ(A)$ as follows. Given a continuous map $\phi : S \to A$ let

$$E(\phi) = E_{max}(\phi) = \sup_{s\in S} E \circ \phi(s),$$

denote by $[\phi] = [\phi]_{hmt}$ the homotopy class of ϕ. and set

$$E_\circ[\phi] = E_{mnmx}[\phi] = \inf_{\phi\in[\phi]} E(\phi).$$

In other words,
$E_\circ[\phi] \le e \in \mathbb{R}$ *if and only if the map* $\phi = \phi_0$ *admits a homotopy of maps* $\phi_t : S \to A$, $0 \le t \le 1$, *such that* ϕ_1 *sends* S *to the sublevel* $A_e = E^{-1}(-\infty, e] \subset A$.

Definition. *The covariant (homotopy) $\mathcal{S}$-spectrum of E* is the set of values $E_\circ[\phi]$ for some class $\mathcal{S}$ of (homotopy types of) topological spaces S and (all) continuous maps $\phi : S \to A$.

For instance, one may take for $\mathcal{S}$ the set of homemorphism classes of countable (or just finite) cellular spaces. In fact, the set of sublevels A_r, $r \in \mathbb{R}$, themselves is sufficient for most purposes.

Category $\mathcal{H}^\circ(A)$, Induced Energy E° and Cohomotopy $\mathcal{S}$-Spectrum. Now, instead of $\mathcal{H}_\circ(A)$ we extend E to the category $\mathcal{H}^\circ(A)$ of homotopy classes of maps $\psi : A \to T$, $T \in \mathcal{S}$, by defining $E^\circ[\psi]$ as the supremum of those $r \in \mathbb{R}$ for which the restriction map of ψ to A_r,

$$\psi_{|A_r} : A_r \to T,$$

is *contractible.*[135] Then the set of the values $E^\circ[\psi]$, is called *the contravariant homotopy (or cohomotopy) $\mathcal{S}$-spectrum of E.*

For instance, if $\mathcal{S}$ is comprised of the *Eilenberg-MacLane $K(\Pi, n)$-spaces*, $n = 1, 2, 3, ...$, then this is called *the Π-cohomology spectrum of E.*

Relaxing Contractibility via Cohomotopy Operations. Let us express "contractible" in writing as $[\psi] = 0$, let $\sigma : T \to T'$ be a continuous map and let us regard the (homotopy classes of the) compositions of σ with $\psi : A \to T$ as an *operation* $[\psi] \overset{\sigma}{\mapsto} [\sigma \circ \psi]$.

Then define $E^\circ[\psi]_\sigma \ge E^\circ[\psi]$ by maximising over those r where $[\sigma \circ \psi_{|A_r}] = 0$ rather than $[\psi_{A_r}] = 0$.

Pairing between Homotopy and Cohomotopy. Given a pair of maps (ϕ, ψ), where $\phi : S \to A$ and $\psi : A \to T$, write

[135] In some cases, e.g. for maps ψ into discrete spaces T such as Eilenberg-MacLane's $K(\Pi; 0)$, *"contractible"*, must be replaced by *"contractible to a marked point serving as zero"* in T that is expressed in writing as $[\psi] = 0$.

$[\psi \circ \phi] = 0$ *if the composed map* $S \to T$ *is contractible,*
$[\psi \circ \phi] \neq 0$ *otherwise.*

Think of this as a function with value "0" and " $\neq 0$" on these pairs.[136]

Induced Energies E_**,* E^* *and (Co)homology Spectra.* If h is a homology class in the space A then $E_*(h)$ denotes *the infimum* of $E_\circ[\phi]$ over all (homotopy classes) of maps $\phi : S \to A$ such that h is *contaned in the image* of the homology homomorphism induced by ϕ.

Dually, the energy $E^*(h)$ on a cohomology class $h \in H^*(A;\Pi)$ for an Abelian group Π, is defined as $E^\circ[\psi_h]$ for the h-inducing map from A to the product of Eilenberg-MacLane spaces:

$$\psi_h : A \to \bigtimes_n K(\Pi, n),\ n = 0, 1, 2, \dots .$$

In simple words, $E^*(h)$ equals the supremum of those r for which h vanishes on $A_r = E^{-1}(\infty, r] \subset A$.[137]

Then one defines the (co)homology spectra as the sets of values of these energies E_* and E^* on homology and on cohomology.

On Multidimensional Cohomology and Cohomotopy Spectra $\{\Sigma_h\} \subset \mathbb{R}^l$. Given spaces A_k, $k = 1, ..., l$, functions E_k on A_k and a cohomology class h on the Cartesian product space $A_1 \times ... \times A_l$, one define the *spectral hypersurface* $\Sigma_h \subset \mathbb{R}^l$ in the Euclidean space $\mathbb{R}^l = \mathbb{R}^{\{1,...,l\}}$ as the boundary of the subset $\Omega_h \subset \mathbb{R}^l$ of the l-tuples of numbers $(e_1, ..., e_k, ..., e_l)$ such that the class h vanishes on the product of the subsets $A_{e_k} = E_k^{-1}(-\infty, e_k) \subset A_k$,

$$\Sigma_h = \partial\Omega_h,\ \Omega_h = \{e_1, ..., e_k, ..., e_l\}_{h|A_{e_1} \times ... \times A_{e_k} \times ... \times A_{e_l}} = 0.$$

This also make sense for general cohomotopy classes h in $A_1 \times ... \times A_l$, with $h = 0$ understood as contractibility of the map $\psi : A \to T$ that represent h to a marked (zero) point in T. (Marking is unnecessary for connected spaces T.)

On Positive and Negative Spectra. Our definitions of homotopy and homology spectra are best adapted to functions $E(a)$ bounded from below but they can be adjusted to more general functions E such as $E(x) = \sum_k a_k x_k^2$ where there may be infinitely many negative as well as positive numbers among a_k.

For instance, one may define the spectrum of a E unbounded from below as the limit of the homotopy spectra of $E_\sigma = E_\sigma(a) = \max(E(a), -\sigma)$ for $\sigma \to +\infty$.

But often, e.g. for the action-like functions in the symplectic geometry, one needs something more sophisticated than a simple minded cut-off of "undesirable infinities".[138]

On Continuous Homotopy Spectra. There also is a homotopy theoretic rendition/generalisation of *continuous spectra* with some Fredholm-like notion of homotopy,[139] such that, for instance, the natural inclusion of the projectivised Hilbert subspace $PL_2[0, t] \subset PL_2[0, 1]$, $0 < t < 1$, would not contract to any $PL_2[0, t - \varepsilon]$.

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[136]If the space T is *disconnected*, it should be better endowed with a marking $t_0 \in T$ with "contractible" understood as "contractible to t_0".

[137]The definitions of energy on homology and cohomology obviously extend to generalised homology and and cohomology theories.

[138]It seems, however, that neither a general theory nor a comprehensive list of examples exit for the moment.

[139]See *On the uniqueness of degree in infinite dimension* by P. Benevieri and M. Furi, http://sugarcane.icmc.usp.br/PDFs/icmc-giugno2013-short.pdf.

4.4 Homotopy Height, Cell Numbers and Homology.

The homotopy spectral values $r \in \mathbb{R}$ of $E(a)$ are "named" after (indexed by) the homotopy classes $[\phi]$ of maps $\phi : S \to A$, where $r = r_{[\phi]}$ is, by definition, the minimal r such that $[\phi]$ comes from a map $S \to A_r \subset A$ for $A_r = E^{-1}(-\infty, r]$. In fact, such a "name" depends only on the partially ordered set, cal it $\mathcal{H}_{\gtrless}(A)$, that is *the maximal partially ordered reduction* of $\mathcal{H}_{\circ}(A)$ defined as follows.

Write $[\phi_1] \prec [\phi_2]$ if there is a morphism $\psi_{12} : [\phi_1] \to [\phi_2]$ in $\mathcal{H}_{\circ}(A)$ and turn this into a partial order by identifying objects, say $[\phi]$ and $[\phi']$, whenever $[\phi] \prec [\phi']$ as well as $[\phi'] \prec [\phi]$.

Perfect Example. If X is (homotopy equivalent to) the real projective space P^∞ then the partially ordered set $\mathcal{H}_{\gtrless}(A)$ is isomorphic to the set of nonnegative integers $\mathbb{Z}_+ = \{0, 1, 2, 3, ...\}$. This is why spectral (eigen) values are indexed by integers in the classical case.

In general the set $\mathcal{H}_{\gtrless}(A)$ may have undesirable(?) "twists". For instance, if A is homotopy equivalent to the circle, then $\mathcal{H}_{\gtrless}(A)$ is isomorphic to set $\mathbb{Z}_+$ with the *division order*, where $m \succ n$ signifies that m divides n. (Thus, 1 is the maximal element here and 0 as the minimal one.)

Similarly, one can determine $\mathcal{H}_{\gtrless}(A)$ for general Eilenberg-MacLane spaces $A = K(\Pi, n)$. This seems transparent for Abelian groups Π. But if a space A, not necessarily a $K(\Pi, 1)$, has a non-Abelian fundamental group $\Pi = \pi_1(A)$, such as the above space $A_N(X)$ of subsets $a \subset X$ of cardinality N, then keeping track of the conjugacy classes of subgroups $\Pi' \subset \Pi$ and maps $\phi : S \to A$ that send $\pi_1(S)$ to these Π' becomes more difficult.

If one wishes (simple mindedly?) to remain with integer valued spectral values, one has to pass to some numerical invariant that that takes values in a quotient of $\mathcal{H}_{\gtrless}$ isomorphic to $\mathbb{Z}_+$, e.g. as follows.

Homotopy Height. Define *the homotopy (dimension) height* of a homotopy class $[\phi]$ of continuous map $\phi : S \to A$ as the minimal integer n such that the $[\phi]$ factors as $S \to K \to A$, where K is a cell complex of dimension (at most) n.

"Stratification" of Homotopy Cohomotopy Spectra by Hight. This "hight" or a similar hight-like function defines a partition of the homotopy spectrum into the subsets, call them $Hei_n \subset \mathbb{R}$, $n = 0, 1, 2, ...$, of the values of the energy $E[\phi] \in \mathbb{R}$ on the homotopy classes $[\phi]$ with homotopy heights n, where either the *supremum* or the *infimum* of the numbers $r \in Hei_n$ may serve as the *"n-th HH-eigenvalue of a"*.

One also may "stratify" cohomptopy spectra by replacing "*contractibility condition* of maps $\psi_{|A_r} : A_r \to T$ by $\psi_{|A_r} \leq n$.

In the classical case of $A = P^\infty$ any such "stratification" of homotopy "eigenvalues" lead the usual indexing of the spectrum. where, besides the homotopy hight, among other hight-like invariant invariants we indicate the following.

Example 1: Total Cell Number. Define $N_{cell}[\phi]$ as the minimal N such that $[\phi]$ factors as $S \to D \to A$, where D is a cell complex with (at most) N cells in it.

Example 2: Homology Rank. Define $rank_{H_*}[\phi]$ as the maximum over all fields $\mathbb{F}$ of the the $\mathbb{F}$-ranks of the induced homology homomorphisms
$[\phi]_* : H_*(S; \mathbb{F}) \to H_*(A; \mathbb{F})$.

On Essentiality of Homology. There are other prominent spaces, X, besides the infinite dimensional projective spaces $X = P^\infty$, and energy functions on them, such as

spaces A of loops $a : S^1 \to X$ in simply connected Riemannian manifolds X with length(a) taken for $E(a)$[140],

where the cell numbers and the homology ranks spectra for $E(a) = lenght(a)$ are "essentially determined" by the homotopy height. (This is why the homotopy height was singled out under the name of "essential dimension" in my paper *Dimension, Non-liners Spectra and Width.*)

However, the homology carries significantly more information than the homotopy hight for the k-volume function on the *spaces of k-cycles of codimensions* ≥ 2 as it was revealed by Larry Guth in his paper *Minimax problems related to cup powers and Steenrod squares.*

On Height and the Cell Numbers of Cartesian Products. If the homotopy heights and/or cell numbers of maps $\phi_i : S_i \to A_i$, $i = 1, ..., k$, can be expressed in terms of the corresponding homology homomorphisms over some filed $\mathbb{F}$ independent of i, then, according to *Künneth formula,* the homotopy hight of the Cartesian product of maps,

$$\phi_1 \times ... \times \phi_k : S_1 \times ... \times S_k \to A_1 \times ... \times A_k,$$

is additive

$$height[\phi_1 \times ... \times \phi_k] = height[\phi_1] + ... + height[\phi_k]$$

and the cell number is multiplicative

$$N_{cell}[\phi_1 \times ... \times \phi_k] = N_{cell}[\phi_1] \times ... \times N_{cell}[\phi_k].$$

What are other cases where these relation remain valid?

Specifically, we want to know what happens in this regard to the following classes of maps:

(a) *maps between spheres* $\phi_i : S^{m_i+n_i} \to S^{m_i}$,

(b) *maps between locally symmetric spaces*, e.g. compact manifolds of *constant negative curvatures*,

(c) *high Cartesian powers* $\phi^{\times N} : S^{\times N} \to A^{\times N}$ of a single map $\phi : S \to A$.

When do, for instance, the limits

$$\lim_{N\to\infty} \frac{height[\phi^{\times N}]}{N} \text{ and } \lim_{N\to\infty} \frac{\log N_{cell}[\phi^{\times N}]}{N}$$

not vanish? (These limits exist, since the the hight and the logarithm of the cell number are sub-additive under Cartesian product of maps.)

Probably, the general question for *"rational homotopy classes* $[...]_{\mathbb{Q}}$*"* (instead of "full" homotopy classes" $[...] = [...]_{\mathbb{Z}}$) of maps into *simply connected* spaces A_i is easily solvable with *Sullivan's minimal models.*

Also, the question may be more manageable for *homotopy classes* mod p.

[140]This instance of essentiality of the homotopy heights is explained in my article *Homotopical Effects of Dilatation*, while the full range of this property among "natural" spaces A of maps a between Riemannian manifolds and energies $E(a)$ remains unknown.

4.5 Graded Ranks, Poincare Polynomials, Ideal Valued Measures. and Spectral $\smile$-Inequality

The images as well as kernels of (co)homology homomorphisms that are induced by continuous maps are *graded* Abelian groups and their ranks are properly represented not by individual numbers but by *Poincaré polynomials*.

Thus, sublevel $A_r = E^{-1}(-\infty, r] \subset A$ of energy functions $E(a)$ are characterised by the *polynomials* Poincaré$_r(t;\mathbb{F})$ of the the inclusion homomorphisms $\phi_i(r) : H_i(A_r;\mathbb{F}) \to H_i(A;\mathbb{F})$, that are

$$\text{Poincaré}_r = \text{Poincaré}_r(t;\mathbb{F}) = \sum_{i=0,1,2,\dots} t^i rank_{\mathbb{F}} \phi_i(r).$$

Accordingly, the homology spectra, that are the sets of those $r \in \mathbb{R}$ where the ranks of $\phi_*(r)$ change, are indexed by such polynomials with positive integer coefficient. (The semiring structure on the set of such polynomials coarsely agrees with basic topological/geometric constructions, such as taking $E(a) = E(a_1) + E(a_2)$ on $A = A_1 \times A_2$.)

The set function $D \mapsto$ Poincaré$_D$ that assigns these Poincaré polynomials to subsets $D \subset A$, (e.g. $D = A_r$) has some measure-like properties that become more pronounced for the set function

$$A \supset D \mapsto \mu(D) = \mu^*(D;\Pi) = \mathbf{0}^{\smallsetminus *}(D;\Pi) \subset H^*(A;\Pi),$$

where Π is an Abelian (homology coefficient) group, e.g. a field $\mathbb{F}$, and $\mathbf{0}^{\smallsetminus *}(D;\Pi)$ is the *kernel* of the cohomology restriction homomorphism for the complement $A \smallsetminus D \subset A$,

$$H^*(A;\Pi) \to H^*(A \smallsetminus D;\Pi).$$

Since the cohomology classes $h \in \mathbf{0}^{\smallsetminus *}(D;\Pi) \subset H^*(A:\Pi)$ are representable by cochains with the support in D [141]

additive for the sum-of-subsets in $H^*(A;\Pi)$ and super-multiplicative for the the $\smile$-product of ideals in the case Π is a commutative ring:

$$[\cup +] \qquad \mu(D_1 \cup D_2) = \mu(D_i) + \mu(D_2)$$

for *disjoint* open subsets D_1 and D_2 in A, and

$$[\cap \smile] \qquad \mu(D_1 \cap D_2) \supset \mu(D_1) \smile \mu(D_2)$$

for all open $D_1, D_2 \subset A$.[142]

The relation $[\cap \smile]$, applied to $D_{r,i} = E_i^{-1}(r,\infty) \subset A$ can be equivalently expressed in terms of cohomomoly spectra as follows.

$[\min \smile]$-*Inequality.* Let $E_1, ..., E_i, .., E_N : A \to \mathbb{R}$ be continuous functions/energies and let $E_{min} : A \to \mathbb{R}$ be the minimum of these,

$$E_{min}(a) = \min_{i=1,\dots,N} E_i(a), \ a \in A.$$

[141] This property suggests an extension of μ to multi-sheated *domains D over A* where D go to A by non-injective, e.g. locally homeomorphic finite to one, maps $D \to A$.

[142] See section 4 of my article *Singularities, Expanders and Topology of Maps. Part 2.* for further properties and applications of these "measures" .

Let $h_i \in H^{k_i}(A;\Pi)$ be cohomology classes, where Π is a commutative ring, and let

$$h_\smile \in H^{\sum_i k_i}(A;\Pi)$$

be the $\smile$-product of these classes,

$$h_\smile = h_1 \smile ... \smile h_i \smile ... \smile h_N.$$

Then

[min $\smile$] $$E^*_{min}(h_\smile) \geq \min_{1=1,...,N} E^*_i(h_i).$$

Consequently, the value of the "*total energy*"

$$E_\Sigma = \sum_{i=1,...,N} E_i : A \to \mathbb{R}$$

on this cohomology class $h_\smile \in H^*(A;\Pi)$ is bounded from below by

$$E^*_\Sigma(h_\smile) \geq \sum_{i=1,...,N} E^*_i(h_i).$$

On $\wedge$-Product. The (obvious) proof of [$\cap\smile$] (and of [min $\smile$]) relies on locality of the $\smile$-product that, in homotopy theoretic terms, amounts to factorisation of $\smile$ via $\wedge$ that is the *smash product* of (marked) Eilenberg-MacLane spaces that represent cohomology, where, recall, the *smash product* of spaces with marked points, say $T_1 = (T_1, t_1)$ and $T_2 = (T_2, t_2)$ is

$$T_1 \wedge T_2 = T_1 \times T_2 / T_1 \vee T_2$$

where the factorisation "$/T_1 \vee T_2$" means "*with the subset* $(T_1 \times t_2) \cup (t_1 \times T_2) \subset T_1 \times T_2$ *shrunk to a point*" (that serves to mark $T_1 \wedge T_2$).

In fact, general cohomotopy "measures" (see 4.9) and spectra defined with maps $A \to T$ satisfy natural (obviously defined) counterparts/generalizations of [$\cap\smile$] and [min $\smile$], call them [$\cap\wedge$] and [min $\wedge$] that are

On Grading Cell Numbers. Denote by $N_{i\_cell}[\phi]$ the minimal number N_i such that homotopy class $[\phi]$ of maps $S \to A$ factors as $S \to K \to A$ where K is a cell complex with (at most) N_i cells *of dimension* i and observe that the total cell number is bounded by the sum of these,

$$N_{cell}[\phi] \leq \sum_{i=0,1,2,...} N_{i\_cell}[\phi].$$

Under what conditions on ϕ does the sum $\sum_i N_{i\_cell}[\phi]$ (approximately) equal $N_{cell}[\phi]$?

What are relations between the cell numbers of the covering maps ϕ between (arithmetic) locally symmetric spaces A besides $N_{cell} \leq \sum_i N_{i\_cell}$?[143]

[143]The identity maps $\phi = id : A \to A$ of locally symmetric spaces A seem quite nontrivial in this regard. On the other hand, general locally isometric maps $\phi : A_1 \to A_2$ between symmetric spaces as well as continuous maps $S \to A$ of positive degrees, where S and A are equidimensional manifolds with only A being locally symmetric, are also interesting.

4.6 Homotopy Spectra in Families.

Topological spaces with (energy) functions on them often come in families. The simplest class of such families A_q is defined via continuous maps F from a space $A = A_Q$ to Q where the "fibers" $A_q = F^{-1}(q) \subset A$, $q \in Q$, serve as the members of these kind of families and where the energies E_q on A_q are obtained by restricting functions from A to $A_q \subset A$.[144]

Homotopy spectra in this situation are defined with continuous families of spaces S_q that are "fiberes" of continuous maps $S \to Q$ and where the relevant maps $\phi : S \to A$ send $S_q \to A_q$ for all $q \in Q$ with these maps denoted $\phi_q = \phi_{|S_q}$.

Then the energy of the fibered homotopy class $[\phi]_Q$ of such a fiber preserving map ϕ is defined as earlier as

$$E[\phi]_Q = \inf_{\phi \in [\phi]_Q} \sup_{s \in S} E \circ \phi(s) \leq \sup_{q \in Q} E_q[\phi_q],$$

where the latter inequality is, in fact, an equality in many cases.

Example 1: k-Cycles in Moving Subsets. Let U_q be a Q-family of open subsets in a Riemannian manifold X. An instance of this is the family of the ρ-balls $U_x = U_x(\rho) \in X$ for a given $\rho \geq 0$ where X itself plays the role of Q.

Define $A = A_Q$ as the space $C_k\{U_q;\Pi\}_{q \in Q}$ of k-dimensional Π-cycles[145] $c = c_q$ in U_q for all $q \in Q$, that is $A = A_Q$ equals the space of pairs $(q \in Q, c_q \in C_k(U_q;\Pi))$, where, as earlier, Π is an Abelian (coefficient) group with a norm-like function; then we take $E(c) = E_q(c_q) = vol_k(c)$ for the energy.

Example 2: Cycles in Spaces Mapped to an X. Here, instead of subsets in X we take locally diffeomorphic maps y from a fixed Riemannian manifold U into X and take the Cartesian product $C_k(U;\Pi) \times Q$ for $A = A_Q$.

Example 1+2: Maps with variable domains. One may deal families of spaces U_q (e.g. "fibers" $U_q = \psi^{-1}(q)$ of a map between smooth manifolds $\psi : Z \to Q$) along with maps $y_q : U_q \to X$.

On Reduction 1 ⇒ 2. There are cases, where the spaces $A_Q = C_k\{U_q;\Pi\}_{q \in Q}$ of cycles in moving subsets $U_q \subset X$ topologically split:

$$A_Q = C_k(U;\Pi) \times Q, \text{ for a fixed manifold } U.$$

A simple, yet representative, example is where $Q = X$ for the m-torus, $X = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, where $B = U_0$ is an open subset in $\mathbb{T}^m$ and where $Y = X = \mathbb{T}^m$ equals the space of translates $U_0 \mapsto U_0 + x$, $x \in \mathbb{T}^m$.

For instance, if U_0 is a ball of radius $\varepsilon \leq 1/2$, then it can be identified with the Euclidean ε-ball $B = B(\varepsilon) \subset \mathbb{R}^m$.

Similar splitting is also possible for *parallelizable* manifolds X with injectivity radii $> \varepsilon$ where moving ε-balls $U_x \subset X$ are obtained via the exponential maps $exp_q : T_q = \mathbb{R}^m \to X$ from a fixed ball $B = B(\varepsilon) \subset \mathbb{R}^m$.

In general, if X is *non-parallelizable*, one may take the space of the tangent orthonormal frames in X for Q, where, the product space $C_k(B;\Pi) \times Q$, where

[144] In general, one may have functions with the range also depending on q, say $a_q : A_q \to R_q$ and one may generalise further by defining families as some (topological) sheaves over Grothendieck sites.

[145] "k-Cycle" in $U \subset X$ means a *relative* k-cycle in $(U, \partial U)$, that is a k-chain with boundary contained in the boundary of U. Alternatively, if U is an open non compact subset, "k-cycle" means an *infinite* k-cycle, i.e. with (a priori) *non-compact support*.

$B = B(\varepsilon) \subset \mathbb{R}^m$, makes a principle $O(m)$- fibration, $m = dim(X)$, over the space $C_k\{U_x(\varepsilon); \Pi\}_{x\in X}$ of cycles of moving ε-balls $U_x(\varepsilon) \subset X$.[146]

4.7 Symmetries, Equivariant Spectra and Symmetrization.

. If the energy function E on A is invariant under a continuous action of a group G on A – this happens frequently – then the relevant category is that of *G-spaces S*, i.e. of topological spaces S acted upon by G, where one works with *G-equivariant* continuous maps $\phi : S \to A$, *equivariant* homotopies, equaivariinat (co)homologies, decompositions, etc.

Relevant examples of this are provided by symmetric energies $E = E(x_1, ..., x_N)$ on Cartesian powers of spaces, $A = X^{\{1,...,N\}}$, such as our (ad hoc) packing energy for a metric space X,

$$E\{x_1, ..., x_i, ..., x_N\} = \sup_{i\neq j=1,...,N} dist^{-1}(x_i, x_j)$$

that is invariant under the symmetric group Sym_N. It is often profitable, as we shall see later on, to exploit the symmetry under certain *subgroups $G \subset Sym_N$*.

Besides the group Sym_N, energies E on $X^{\{1,...,N\}}$ are often invariant under some groups H acting on X, such as the isometry group $Is(X)$ in the case of packings.

If such a group H is compact, than its role is less significant than that of Sym_N, especially for large $N \to \infty$; yet, if H properly acts on a *non-compact* space X, such as $X = \mathbb{R}^m$ that is acted upon by its isometry group, then H and its action become essential.

MIN-Symmetrized Energy. An arbitrary function E on a G-space A can be rendered G-invariant by taking a symmetric function of the numbers $e_g = E(g(a)) \in \mathbb{R}$, $g \in G$. Since we are mostly concerned with the order structure in $\mathbb{R}$, our preferred symmetrisation is

$$E(a) \mapsto \inf_{g\in G} E(g(a)).$$

Minimization with Partitions. This inf-symmetrization does not fully depends on the action of G but rather on the partition of A into orbits of G. In fact, given an arbitrary partition α of A into subsets that we call *α-slices*, one defines the function

$$E_{inf_\alpha} = inf_\alpha E : A \to \mathbb{R}$$

where $E_{inf_\alpha}(a)$ equals the infimum of E on the α-slice that contains a for all $a \in A$. Similarly, one defines $E_{sup_\alpha} = sup_\alpha E$ with E_{min_α} and E_{max_α} understood accordingly.

Example: Energies on Cartesian Powers. The energy E on A induces N energies on the space $A^{\{1,...,N\}}$ of N-tuples $\{a_1, ..., a_i, ..., a_N\}$, that are

$$E_i : \{a_1, ..., a_i, ..., a_N\} \mapsto E(a_i).$$

It is natural, both from a geometric as well as from a physical prospective, to symmetrize by taking *the total energy* $E_{total} = \sum_i E_i$. But in what follows we

[146]Vanishing of Stiefel-Whitney classes seems to suffice for (homological) splitting of this vibration in the case $\Pi = Z_2$ as in section 6.3 of my article *"Isoperimetry of Waists...")* in GAFA

shall resort to $E_{min} = \min_i E_i = \min_i E(a_i)$ and use it for bounding the total energy from below by

$$E_{total} \geq N \cdot E_{min}.$$

For instance, we shall do it for families of N-tuples of balls U_i in a Riemannian manifold V, thus bounding the k-volumes of k-cycles c in the unions $\cup_i U_i$, where, observe,

$$vol_k(c) = \sum_i vol_k(c \cap U_i)$$

if the balls U_i *do not intersect*.

$$E(c) = \sum_i c \cap U_{x_i}.$$

This, albeit obvious, leads, as we shall see later on, to non-vacous relations between

homotopy/homology spectrum of the vol_k-energy on the space $C_k(X;\Pi)$
and
equivariant homotopy/homology of the spaces of packings of X by ε-balls.

4.8 Equivariant Homotopies of Infinite Dimensional Spaces.

If we want to understand homotopy spectra of spaces of "natural energies" on spaces of infinitely many particles in non-compact manifolds, e.g. in Euclidean spaces, we need to extend the concept of the homotopy and homology spectra to infinite dimensional spaces A, where infinite dimensionality is compensated by an additional structure, e.g. by an action of an infinite group Υ on A.

The simplest instance of this is where Υ is a countable group that we prefer to call Γ and $A = B^\Gamma$ be the space of maps $\Gamma \to B$ with the (obvious) *shift action* of Γ on this A, motivates the following definition.[147]

Let H^* be a graded algebra (over some field) acted upon by a countable amenable group Γ. Exhaust Γ by finite *Følner subsets* $\Delta_i \subset \Gamma$, $i = 1, 2, ...,$ and, given a finite dimensional graded subalgebra $K = K^* \subset H^*$, let $P_{i,K}(t)$ denote the Poincare polynomial of the graded subalgebra in H^* generated by the γ-transforms $\gamma^{-1}(H^*_K) \subset H^*$ for all $\gamma \in \Delta_i$.

Define *polynomial entropy* of the action of Γ on H^* as follows.

$$Poly.ent(H^* : \Gamma) = \sup_K \lim_{i \to \infty} \frac{1}{card(\Delta_i)} \log P_{i,K}(t).$$

Something of this kind could be applied to subalgebras $H^* \subset H^*(A;\mathbb{F})$, such as images and/or kernels of the restriction cohomology homomorphisms for (the energies sublevel) subsets $U \subset A$, IF the following issues are settled.

1. In our example of moving balls or particles in $\mathbb{R}^m$ the relevant groups Υ, such as the group of the orientation preserving Euclidean isometries are connected and act *trivially* on the cohomologies of our spaces A.

For instance, let $\Gamma \subset \Upsilon$ be a discrete subgroups and A equal the *dynamic Υ-suspension of* B^Γ, that is $B^\Gamma \times \Upsilon$ divided by the diagonal action of Γ.

$$A = \left(B^\Gamma \times \Upsilon\right)/\Gamma.$$

[147]See our paper with Melanie Bertelson *Dynamical Morse entropy*.

The (ordinary) cohomology of this space A are bounded by those of B tensored by the cohomology of Υ/Γ that would give *zero* polynomial entropy for finitely generated cohomology algebras $H^*(B)$.

In order to have something more interesting, e.g. the *mean Poincaré polynomial* equal that of B^Γ, which is the ordinary Poincaré$(H^*(B))$, one needs a definition of some *mean (logarithm) of the Poincaré polynomial* that might be *far from zero* even if the ordinary cohomology of A vanish.

There are several candidates for such *mean Poincaré polynomials*, e.g the one is suggested in section1.15 of my article *Topological Invariants of Dynamical Systems and Spaces of Holomorphic Maps.*

Another possibility that is applicable to the above $A = \left(B^\Gamma \times \Upsilon\right)/\Gamma$ with *residually finite* groups Γ is using finite i-sheeted covering $\tilde{A}_i$ corresponding to subgroups $\Gamma_i \subset \Gamma$ of order i and taking the limit of

$$\lim_{i\to\infty} \frac{1}{i} \log \mathrm{Poincaré}(H^*(B)).$$

,

(Algebraically, in terms of actions of groups Γ on abstract graded algebras H^*, this corresponds to taking the normalised limit of logarithms of Γ_i-*invariant* sub-algebras in H^*; this brings to one's mind a possibility of a generalisation of the above polynomial entropies to *sofic groups*.[148])

2. The above numerical definitions of the polynomial entropy and of the mean Poincaré polynomials beg to be rendered in categorical terms similarly to the ordinary entropy.[149]

3. The spaces $A_\infty(X)$ of (discrete) infinite countable subsets $a \subset X$ that are meant to represent infinite ensembles of particles in non-compact manifolds X, such as $X = \mathbb{R}^m$, are more complicated than $A = B^\Gamma$, $A = \left(B^\Gamma \times \Upsilon\right)/\Gamma$ and other "product like" spaces studied eralier.

These $A_\infty(X)$ may be seen as as limits of finite spaces $A_N(X_N)$ for $N \to \infty$ of N-tuples of points in compact manifolds X_N where one has to chose suitable approximating sequences X_N.

For instance, if $X = \mathbb{R}^m$ acted upon by some isometry group Υ of $\mathbb{R}^m$ one may use either the balls $B^m(R_N) \subset \mathbb{R}^m$ of radii $R_N = const \cdot R^{N/\beta}$ in $\mathbb{R}^m$, $\beta > 0$ for X_N or the tori $\mathbb{R}^m/\Gamma_N$ with the lattices $\Gamma_N = const \cdot M \cdot \mathbb{Z}^m$ with $M = M_N \approx N^{\frac{1}{\beta}}$ for some $\beta > 0$.[150]

Defining such limits and working out functional definitions of relevant structures the limit spaces, collectively callused $A_\infty(X)$ are the problems we need to solve where, in particular, we need to

• incorporate actions of the group Υ coherently with (some subgroup) of the infinite permutations group acting on subsets $a \subset X$ of particles in X that represent points in $A_\infty(X)$

and

• define (stochastic?) homotopies and (co)homologies in the spaces $A_\infty(X)$, where these may be associated to limits of families of n-tuples $a_{P_N} \subset X_N$

[148] Compare *Linear sofic groups and algebras* by Arzhantseva&Paunescu, arXiv:1212.6780.

[149] See my paper *In a Search for a Structure, Part 1: On Entropy.*

[150] The natural value is $\beta = m$ that make the volumes of X_N proportional to N but smaller values, that correspond to ensembles of points in $\mathbb{R}^m$ of *zero densities*, also make sense as we shall see later on.

parametrised by some P_N where $dim(P_N)$ may tend to infinity for $N \to \infty$.[151]

4. Most natural energies E on infinite particle spaces $A_\infty(X)$ are everywhere infinite[152] and defining "sublevels" of such E needs attention.

4.9 Symmetries, Families and Operations Acting on Cohomotopy Measures.

Cohomotopy "Measures". Let T be a space with a distinguished *marking point* $t_0 \in T$, let $H^\circ(A;T)$ denote the set of homotopy classes of maps $A \to T$ and define the "T-measure" of an open subset $U \subset A$,

$$\mu^T(U) \subset H^\circ(A;T),$$

as the set of homotopy classes of maps $A \to T$ that send the complement $A \setminus U$ to t_0.

For instance, if T is the Cartesian product of Eilenberg-MacLane spaces $K(\Pi;n)$, $n = 0,1,2,...,$, then $H^\circ(A;T) = H^*(A;\Pi)$ and μ^T identifies with the (graded cohomological) ideal valued "measure" $U \mapsto \mu^*(U;\Pi) \subset H^*(A;\Pi)$ from section 4.5.

Next, given a category $\mathcal{T}$ of marked spaces T and homotopy classes of maps between them, denote by $\mu^{\mathcal{T}}(U)$ the totality of the sets $\mu^T(U)$, $T \in \mathcal{T}$, where the category $\mathcal{T}$ acts on $\mu^{\mathcal{T}}(U)$ via composition $A \overset{m}{\to} T_1 \overset{\tau}{\to} T_2$ for all $m \in \mu^T(U)$ and $\tau \in \mathcal{T}$.

For instance, if $\mathcal{T}$ is a category of Eilenberg-MacLane spaces $K(\Pi;n)$, this amounts to the natural action of the (unary) cohomology operations (such as Steenrod squares Sq^i in the case $\Pi = \mathbb{Z}_2$) on ideal valued measures.

The above definition can be adjusted for spaces A endowed with additional structures.

For example, if A represents a *family of spaces* by being endowed with a partition β into closed subsets – call them *β-slices* or *fibers* – then one restricts to the space of maps $A \to T$ *constant on these slices* (if T is also partitioned, it would be logical to deal with maps sending slices to slices) defines $H^\circ_\beta(A;T)$ as the set of the homotopy classes of these slice-preserving maps and accordingly defines $\mu^{\mathcal{T}}_\beta(U) \subset H^\circ_\beta(A;T)$.

Another kind of a relevant structure is an action of a group G on A. Then one may (or may not) work with categories $\mathcal{T}$ of G-spaces T (i.e. acted upon by G) and perform homotopy, including (co)homology, constructions equivariantly. Thus, one defines equivariant T-measures $\mu^T_G(U)$ for G-invariant subsets $U \subset A$.

(A group action on a space , defines a partition o this space into orbits, but this is a weaker structure than that of the the action itself.)

Guth' Vanishing Lemma. The supermultiplicativity property of the cohomology measures with arbitrary coefficients Π (see 4.5) for spaces A acted upon by finite groups G implies that

$$\mu^*\left(\bigcap_{g\in G} g(U;\Pi)\right) \supset \smile_{g\in G} \mu^*(g(U;\Pi))$$

[151]We shall meet families of dimensions $dim(P_N) \sim N^{\frac{1}{\gamma}}$ where $\gamma + \beta = m$ for the above β.

[152]In the optical astronomy, this is called *Olbers' dark night sky paradox.*

for all open subset $U \subset A$.

This, in the case $\Pi = \mathbb{Z}_2$ was generalised by Larry Guth for families of spaces parametrised by spheres S^j as follows.

Given a space A endowed with a partition α, we say that a subset in A is *α-saturated* if it equals the union of some α-slices in A and define two operations on subsets $U \subset A$,

$$U \mapsto \cap_\alpha(U) \subset U \text{ and } U \mapsto \cup_\alpha(U) \supset U,$$

where

$\cap_\alpha(U)$ is the *maximal α-saturated subset that is contained in U*

and

$\cup_\alpha(U)$ is the *minimal α-saturated subset that contains U.*

Let, as in the case considered by Guth, $A = A_0 \times S^j$ where $S^j \subset \mathbb{R}^{j+1}$ is the j-dimensional sphere, let α be the partition into the orbits of $\mathbb{Z}_2$-action on A by $(a_0, s) \mapsto (a_0, -s)$ (thus, "α-saturated" means "$\mathbb{Z}_2$-invariant") and let β be the partition into the fibres of the projection $A \to A_0$ (and "β-saturated" means "equal the pullback of a subset in A_0").

Following Guth, define

$$Sq_j : H^{*\geq j/2}(A; \mathbb{Z}_2) \to H^*(A; \mathbb{Z}_2) \text{ by } Sq^j : H^p \to H^{2p-j}$$

and formulate his "Vanishing Lemma" in μ_β-terms as follows,[153]

$$[\cup\cap] \qquad \mu^*_\beta(\cup_\beta(\cap_\alpha(U)); \mathbb{Z}_2) \supset Sq_j(\mu^*_\beta(U; \mathbb{Z}_2)) \subset H^*_\beta(A; \mathbb{Z}_2),$$

where, according to our notation, $H^*_\beta(A; \mathbb{Z}_2) \subset H^*(A; \mathbb{Z}_2)$ equals the image of $H^*(A_0; \mathbb{Z}_2)$ under the cohomology homomorphism induced by the projection $A \to A_0$.

If $E : A \to \mathbb{R}$ is an energy function, this lemma yields the lower bound on the *maxmin*-energy[154]

$$E_{max_\beta min_\alpha} = max_\beta min_\alpha E$$

evaluated at the cohomology class $St_j(h)$, $h \in H^*_\beta(A; \mathbb{Z}_2)$:

$$[maxmin] \qquad E^*_{max_\beta min_\alpha}(St_j(h)) \geq E^*(h).$$

.

Question. What are generalisations of $[\cup\cap]$ and $[maxmin]$ to other cohomology and cohomotopy measures on spaces with partitions α, β, γ,...?

4.10 Pairing Inequality for Cohomotopy Spectra.

Let A_1, A_2 and B be topological spaces and let

$$A_1 \times A_2 \overset{\circledast}{\to} B$$

be a continuous map where we write

$$b = a_1 \circledast a_2 \text{ for } b = \circledast(a_1, a_2).$$

[153] Guth formulates his lemma in terms of the complementary set $V = A \setminus U$:
if a cohomology class $h \in H^*_\beta(A; \mathbb{Z}_2)$ vanished on V, then $St_j(h)$ vanishes on $\cap_\beta(\cup_\alpha(V))$.

[154] Recall that $min_\alpha E(a)$, $a \in A$, denotes the minimum of E on the α-slice containing a and max_β stands for similar maximisation with β (see 4.7).

For instance, composition $a_1 \circ a_2 : X \to Z$ of morphisms $X \overset{a_1}{\to} Y \overset{a_2}{\to} Z$ in a topological category defines such a map between sets of morphisms,

$$mor(X \to Y) \times mor(Y \to Z) \overset{\circledast}{\to} mor(X \to Z).$$

A more relevant example for us is the following

Cycles $\times$ packings.

Here, A_1 is a space of locally diffeomorphic maps $U \to X$ between manifolds U and X,

A_2 is the space of cycles in X with some coefficients Π,

B is the space of cycles U with the same coefficients,

$\circledast$ stands for *"pullback"*

$$b = a_1 \circledast a_2 =_{def} a_1^{-1}(a_2) \in B.$$

This U may equal the disjoint unions of N manifolds U_i that, in the spherical packing problems, would go to balls in X that do not intersect for injective maps a.

Explanatory Remarks. (a) Our *"cycles"* are defined as *subsets* in relevant manifolds X and/or U with Π-*valued functions* on these subsets.

(b) In the case of *open* manifolds, we speak of cycles with *infinite supports*, that, in the case of compact manifolds with boundaries or of open subsets $U \subset X$, are, essentially, *cycles modulo the boundaries* ∂X.

(c) *"Pullbacks of cycles"* that preserve their codimensions are defined, following Poincaré,[155] for a wide class of smooth *generic* (not necessarily equividimensional) maps $U \to X$.

(d) It is easier to work with *cocycles* (rather than with cycles) where contravariant functoriality needs no extra assumptions on spaces and maps in question.[156]

Let h^T be a (preferably non-zero) cohomotopy class in B, that is a homotopy class of non-contractible maps $B \to T$ for some space T, (where *"cohomotopy"* reads *"cohomology"* if T is an Eilenberg-MacLane space) and let

$$h^{\circledast} = \circledast \circ h^T : [A_1 \times A_2 \to T]$$

be the induced class on $A_1 \times A_2$, that is the homotopy class of the composition of the maps $A_1 \times A_2 \overset{\circledast}{\to} B \overset{h^T}{\to} T$.

(Here and below, we do not always notationally distinguish *maps* and *homotopy classes* of maps.)

Let h_1 and h_2 be homotopy classes of maps $S_1 \to A_1$ and $S_2 \to A_2$ for some spaces S_i, $i = 1, 2$.

(In the case where h^T is a *cohomology* class, these h_i may be replaced by *homology* – rather than homotopy – classes represented by these maps.)

Compose the three maps,

$$S_1 \times S_2 \overset{h_1 \times h_2}{\to} A_1 \times A_2 \overset{\circledast}{\to} B \overset{h^T}{\to} T,$$

[155] This is spelled out in my article *Manifolds: Where Do We Come From?...* .

[156] See my paper *Singularities, expanders and topology of maps. Part 2.*

and denote the homotopy class of the resulting map $S_1 \times S_2 \to T$ by

$$[h_1 \circledast h_2]_{h^T} = h^{\circledast} \circ (h_1 \times h_2) : [S_1 \times S_2 \to T]$$

Let $\chi = \chi(e_1, e_2)$ be a function in two real variables that is monotone unceasing in each variable. Let $E_i : A_i \to \mathbb{R}$, $i = 1, 2$, and $F : B \to \mathbb{R}$ be (energy) functions on the spaces A_1, A_2 and B, such that the $\circledast$-pullback of F to $A \times B$ denoted

$$F^{\circledast} = F \circ \circledast : A_1 \times A_2 \to \mathbb{R}$$

satisfies

$$F^{\circledast}(a_1, a_2) \leq \chi(E(a_1), E(a_2)).$$

In other words, the $\circledast$-image of the product of the sublevels

$$(A_1)_{e_1} = E_1^{-1}(-\infty, e_1) \subset A_1 \text{ and } (A_2)_{e_2} = E_2^{-1}(-\infty, e_2) \subset A_2$$

is contained in the f-sublevel $B_f = F^{-1}(-\infty, f) \subset B$ for $f = \chi(e_1, e_2)$,

$$\circledast((A_1)_{e_1} \times (A_2)_{e_2}) \subset B_{f=\chi(e_1,e_2)}.$$

$\circledast$-Pairing Inequality.

Let $[h_1 \circledast h_2]_{h^T} \neq 0$, that is *the composed map*

$$S_1 \times S_2 \to A_1 \times A_2 \to B \to T$$

is non-contractible. Then the values of E_1 and E_2 on the homotopy classes h_1 and h_2 are *bounded from below* in terms of a lower bound on $F^{\circ}[h^T]$ as follows.

$$[_{\circ\circ}\geq^{\circ}] \qquad \chi(E_{1\circ}[h_1], E_{2\circ}[h_2]) \geq F^{\circ}[h^T].$$

In other words

$$\big(E_{1_\circ}[h_1] \leq e_1\big) \& \big(E_{2\circ}[h_2] \leq e_2\big) \Rightarrow \big(F^{\circ}[h^T] \leq \chi(e_1, e_2)\big)$$

for all real numbers e_1 and e_2; thus,

upper *bound* $E_1^{\circ}[h_1] \leq e_1$ + ***lower*** *bound* $F^{\circ}[h^T] \geq \chi(e_1, e_2)$

yield

upper *bound* $E_2^{\circ}[h_2] \geq e_2$,

where, observe, E_1 and E_2 are interchangeable in this relation.

Proof. All one needs for $[_{\circ\circ}\geq^{\circ}]$ is unfolding the definitions.

Also $[_{\circ\circ}\geq^{\circ}]$ can be visualised without an explicit use of χ by looking at the $h^{\circledast}$*-spectral line* in the (e_1, e_2)-plane

$$\Sigma_{h^{\circledast}} = \partial\Omega_{h^{\circledast}} \subset \mathbb{R}^2$$

(we met this Σ section 4.3) where $\Omega_{h^{\circledast}} \subset \mathbb{R}^2$ consists of the pairs $(e_1, e_2) \in \mathbb{R}^2$ such that the restriction of $h^{\circledast}$ to the Cartesian product of the sublevels $A_{1e_1} = E_1^{-1}(-\infty, e_1) \subset A_1$ and $A_{2e_2} = E_2^{-1}(-\infty, e_2) \subset A_2$ vanishes,

$$h^{\circledast}_{|A_{1e_1} \times A_{2e_2}} = 0.$$

4.11 Pairing Inequalities between Families of Cycles and of Packings.

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4.12 Symplectic Parametric Packings.

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4.13 Reconstruction of Geometries of Spaces X by Homotopies of Parametric Packings of X.

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5 Isoperimetry and Vershik-Følner Profile of Algebras.

Let X be a set and G a set of maps $g : X \to X$, where our primarily case of interest here is where

X is a *linear space* over some field $\mathbb{F}$
and
G is a set of *linear operators* on X.

"G-Interior" $inv_G(V)$ and "G-Boundary" $\partial_G(U)$. Given a subset $Y \subset X$, e.g. a linear subspace in X when this makes sense, define the G-invariant part (kind of "interior") $inv_G(Y) \subset Y$ as the subset of those $y \in Y$ for which $g(y) \in y$ for all $g \in G$. Notice that $inv_G(Y) \subset Y$ is a *linear* subspace in the linear case.

Now, assuming everything is linear, define the *G-boundary* of Y as the quotient space

$$\partial_G(Y) =_{def} Y/inv_G(Y).$$

The (isoperimetric) rank profile of (X, G) over $\mathbb{F}$ is, by definition, a positive integer function denoted $r^\partial_{min}(R) = r^\partial_{min}(R; \mathbb{F})$, $R \in \mathbb{Z}_+$, that equals

the minimum of the ranks of the boundaries
of all linear subspaces $Y \subset X$ of rank R,

$$r^\partial_{min}(R) = r^\partial_{min}(R; \mathbb{F}) = \inf_{rank_{\mathbb{F}}(Y)=R} rank(\partial_G Y).$$

The Profile $n^\partial_{min}(R)$. The traditional *combinatorial* isoperimetric profile, denoted $n^\partial_{min}(R)$, concerns a *non-linear* action of G on a set X with the boundary of Y being defined as the *complement* $Y \smallsetminus inv(Y)$ and with the *cardinalities* of Y and its boundary rather than with $rank_{\mathbb{F}}$.

General Problem 3. *Evaluate $r^\partial_{min}(R)$ for specific (classes of) linear spaces X with sets of linear operators acting on them.*

In particular, let X be some "natural" space of $\mathbb{F}$-valued functions on a set S and G be a set of maps $g : S \to S$, such that X is invariant under the induced action of G on functions $S \to \mathbb{F}$.

What is the relation between the ***combinatorial*** *isoperimetric profile* $n^{\partial}_{min}(R)$ *of the action on* S *and the* ***rank profile*** $r^{\partial}_{min}(R)$ *of the corresponding* G*-action on* X*?*

When, for instance, are the two profiles equal,

$$r^{\partial}_{min}(R) = n^{\partial}_{min}(R)?$$

Most of what we know about relations between combinatorial and rank profiles[157] concern the case where S is a *group generated by a finite set* $G \subset S$ with the action $g(s) = g \cdot s$, with the point of depart being the concept of *(non)amenability*.

Recall that a finitely generated group S is called *non-amenable* if its combinatorial profile with respect to some (and hence, any) finite generating subset $G \subset S$ is (asymptotically) *linear*:

$$n^{\partial}_{min}(R) \geq \lambda \cdot R \text{ for some constant } \lambda = \lambda(S, G) > 0.$$

Then

• *the rank profile* $r^{\partial}_{min}(R) = r^{\partial}_{min}(R; \mathbb{F})_{fnt}$ *of* G *acting on the space of* $\mathbb{F}$*-valued functions on* S *with finite supports is linear for all fields* $\mathbb{F}$*,* [Bartholdi, 2007];

• *the* l_2*-rank profile* $r^{\partial}_{min}(R) = r^{\partial}_{min}(R; \mathbb{C})_{l_2}$ *of* G *acting on the space* X *of complex valued square summable functions* $x(s)$ *on* S *is linear.*[158] [Elek, 2006].

Now, if a group S *is amenable*, then its combinatorial profile expresses a quantitative measure of this amenability and the rank profile of the linear action of S on functions $x(s)$ with finite supports on S plays a similar role; but these two measures (profiles) may be quite different and the rank profile for different fields $\mathcal{F}$ may, a priori, differ as well.

Question [A]. *What are (classes of) finitely generated groups* S *where the rank profiles* $r^{\partial}_{min}(R) = r^{\partial}_{min}(R; \mathbb{F})_{fnt}$ *of the spaces* $X = X_{fnt}$ *of functions with finite supports on* S *is "reasonably bounded"*[159] *from below*[160] *by the combinatorial isoperimetric profile* $n^{\partial}_{min}(R)$ *?*

What are relations between rank profiles over different fields $\mathbb{F}$*?*

Discussion. A significant difference between the two profiles is that the combinatorial $n^{\partial}_{min}(R)$ does not depend on the maps $g : S \to S$ themselves but rather on the (directed) *graph* on the vertex set S with the edge set being

$$E = E_G =_{def} \{(x, g(x)\}_{x \in X, g \in G} \subset S \times S.$$

where many different sets G of transformations $g : S \to S$ have identical graphs.

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[157]More general algebras appear in Elek's paper *The amenability of affine algebras* and, I guess, in *Entropy and Følner function in algebras* by M D'Adderio, but I failed to find open access to the later article on the web.

[158]One may replace $\sum_s |s(x)|^2 < \infty$ in this case by the weaker condition $\sum_s |s(x)|^p < \infty$ for an arbitrary large $p < \infty$.

[159]A most satisfactory instance of such a "reasonable lower bound" would be $r(R) \geq c_1 \cdot n(c_2 \cdot R)$ for $c_1, c_2 > 0$.

[160]The upper bound $r^{\partial}_{min}(R) \leq n^{\partial}_{min}(R)$ is obvious since subsets $T \subset S$ correspond to *coordinate subspaces* in linear spaces of functions on S.

In fact, this graph $E = E_G$ can be represented by the transformation set $G^{\approx} = G^{\approx}(E) \supset G$ of *"E-bounded quasi-parallel translations"* of X that is, by definitions, the *maximal set* of transformations of X such that $E^{\approx}_G = E$.

Although this set $G^{\approx}$ of transformations $S \to S$ is much larger than G, (it is *uncountable* unless $g(s) = x$, $g \in G$, for all but finitely many $s \in s$), it has the same combinatorial profile as G.

But, non-surprisingly, the rank profiles of $G^{\approx}$ may be much larger than that of G, since the linear span of a G-orbit of a vector $x = x(s) \in X$ contains non-zero vectors $x_e \in X$ supported on the pairs $\{s, g(s)\} \subset S$ for all those edges $e = (s, g(s)) \in E_G$ where $x(s) \neq x(g(s))$. Such vectors x_e may span a lot since they are *linearly independent* for *disjoint* edges $e \in E_G$.

In fact,

there are finitely generated groups S where the rank profiles for their actions on functions $S \to \mathbb{F}$ with finite supports,

$$r^{\partial}_{min}(R) = r^{\partial}_{min}(R; \mathbb{F})_{fnt}$$

are ***bounded*** *while the combinatorial profile grows* ***almost as fast as*** *R.*

That is, given an arbitrarily ***slowly decaying*** *function $\lambda(R) \to 0$, $R \to \infty$, there exists a finitely generated group S, where*

$$n^{\partial}_{min}(R) \geq \lambda(R) \cdot R \text{ for large } R$$

while the corresponding rank profile functions $r^{\partial}_{min}(R) = r^{\partial}_{min}(R; \mathbb{F})_{fnt}$ are bounded for all fields $\mathbb{F}$.

Conjecture 1. *If a finitely generated group S has bounded rank profile $r^{\partial}_{min}(R) = r^{\partial}_{min}(R; \mathbb{F})_{fnt}$ for some field $\mathbb{F}$, then S equals a semidirect product of a locally finite group by a cyclic group.*

This, probably, is not very difficult to show but a similar description of groups S where $r^{\partial}_{min}(R) \sim R^{\varepsilon}$ may be hard (if possible at all) even for small ε, e.g. for $\varepsilon < 1/2$.

There are several classes of groups S, including (besides non-amenable groups) *Grigorchuk's pure torsion groups* and *wreath products* of these, where the lower bounds on the rank profiles $r^{\partial}_{min}(R) = r^{\partial}_{min}(R; \mathbb{F})_{fnt}$ for the spaces X_{fnt} of functions $x(s)$ with finite supports on S are close to what we know of the combinatorial profiles $n^{\partial}_{min}(R)$ of these groups. in particular, the rank profiles $r^{\partial}_{min}(R)$ of these (pure torsion) groups may grow as fast as $\frac{R}{\log \log R}$.

But the following remains problematic.

Conjecture 2. *There are finitely generated (finitely presented?) amenable pure torsion groups, possibly among iterated wreath product of Grigorchuk's groups, where the rank profiles grow as fast as $\frac{R}{\log \log \ldots \log R}$.*

In a somewhat opposite direction:

Conjecture 3. *There are (many) finitely generated groups* without torsion *where the rank profile is much smaller than the combinatorial one.*[161]

At the other extreme one has the following

[161] This is akin to asking for a counterexample to *Kaplansky's conjecture* on divisors of zero in group algebras.

Equality Problem. *Identify finitely generated groups G groups where the two profiles are equal for all generating subsets $G \subset X$.*

It seems the only known groups in this class are the *left orderable* ones as it is explained below and, possibly, there is no other such groups.

Transformations that Preserve Orders.[162] If a linear *G-space* X (meaning that X is acted upon by a set G of linear transformations $g : X \to X$) admits a G-equivariant map τ to a G-set S, such that

$$card(\tau(Y)) = rank_{\mathbb{F}}(Y)$$

for all linear subspaces $Y \in X$ of finite rank, then, clearly,

$$r^{\partial}_{min}(R) \geq n^{\partial}_{min}(R),$$

and if S carries *an order structure* on it, one gets such a map on functions $x = x(s)$ with *finite supports* by sending each $x = x(s)$ to the *minimal* s in the support of $x(s)$.

Thus,

if all $g : S \to S$, $g \in G$, preserve an order on S

then

the rank profile of the space X of functions on S with finite supports equals the combinatorial isoperimetric profile.

Observe that the order preserving property of G is hard (impossible?) to express in terms of the graph $E_G \subset X \times S$ of G.

Question [B]. *Does the rank profile of the space X of decaying (i.e. converging to zero at infinity) functions on groups (on more general S?) with values in an ultrametric field is reasonably bounded from below by the profile of the space $X_0 \subset X$ of functions with finite supports?*

*Are such lower bounds satisfied by the l_2-**profiles** $r^{\partial}_{min}(R) = r^{\partial}_{min}(R;\mathbb{C})_{l_2}$ that are the rank profiles of the Hilbert spaces of square summable complex functions on S?*

The equality between the rank profiles of the Hilbert spaces and those of functions with finite supports hold for *free Abelian* groups of finite rank k, where these profile are $\sim R^{\frac{k-1}{k}}$, while the equivalence[163] between these two profiles is established for the *non-virtually nilpotent polycyclic* groups where these profile are $\sim \frac{R}{\log R}$. But

there is no known instance where the l_2-profile grows faster than $\frac{R}{\log\log R}$.

These and other results/conjectures on linearized isoperimetry for *groups* acting on spaces of functions on them are collected in my 2008 article *Entropy and isoperimetry for linear and non-linear group actions*, where we do not address the following

General Question. *Which (proven and/or conjectural) inequalities for rank profiles $r^{\partial}_{min}(R)$ extend from spaces functions on **groups** S to **more general sets** S with transformations $g : S \to S$?*

[162] I picked up the idea of using order in this kind of problems from Dima Grigoriev.

[163] Functions $r_1(R)$ and $r_2(R)$ are regarded as *equivalent* if $r_2(R) \geq c_1 \cdot r_1(c_2 \cdot R)$ and $r_1(R) \geq c_3 \cdot r_2(c_4 \cdot R)$ for $c_1, c_2, c_3, c_4 > 0$.

Basic examples here are actions of groups Γ on their quotients $S = \Gamma/\Gamma_0$. Combinatorially, such a quotient S can be seen as a directed graph on the vertex set S with its arrow-edges colored by a set G, such that there is a *single* r*-colored edge* (possibly a loop) issuing from $s \in S$ for all $s \in S$. Then each $g \in G$ defines a map $S \to S$ by moving all $s \in S$ along the g-colored arrows.

5.0.1 Sofic and Linear Sofic Profiles.

There is class of group called *sofic* that naturally includes both *amenable* and *residually finite* groups.

Definitions. Let $\mathcal{G} = \{G_i\}, i \in I$, be a family of groups with metrics $dist_i$ on them.

A quasi-embedding of a group Γ to $\mathcal{G}$ is given by a family of subsets $\Delta_j \subset \Gamma$, $j = 1, 2, 3, ...$ that *exhaust* Γ (i.e. $\Delta_{j+1} \supset \Delta_j$ and $\cup_j \Delta_j = \Gamma$) and a family of maps $h_j : \Delta_i \to G_{i_j}$ for some sequence i_j in I that satisfy the following three conditions expressed with the distances $dist_{i_j}$ in G_{i_j}.

[1] $$dist_{i_j}(h_j(\delta) \cdot h_j(\delta'), h_j(\delta \cdot \delta')) < \varepsilon_j \to 0 \text{ for all } \delta, \delta' \in \Delta_j;$$

[2] $$dist_{i_j}(h_j(id_\Gamma), id_{G_{i_j}}) \to 0, \text{ for } j \to \infty;$$

[3] $$dist_{i_j}(h_j(\delta), h_j(\delta')) \geq \lambda > 0 \text{ for all } \delta \neq \delta' \in \Delta_j \text{ and all } j.$$

The two relevant examples of $\mathcal{G}$ are

(I) The family of *permutations groups* $G_i = \Pi_i$ acting on sets S_i of cardinalities i, with the *normalized Hamming distances*, that are defined via the subsets $eql(g, g') \subset S_i$ where $g(s) = g'(s)$ as

$$dist(g, g') = \frac{card\,(S_i \smallsetminus eql(g, g'))}{i}.$$

(II) The family of the *linear groups*, $G_i = GL_i(\mathbb{F})$ acting on $\mathbb{F}^i$ where

$$dist(g, g') = \frac{rank_{\mathbb{F}}(\mathbb{F}^i/eql(g, g'))}{i}.$$

A group S is called *sofic* if it is quasi-embeddable into the family $\{\Pi_i\}$ and *linear sofic* (over $\mathbb{F}$) if it is quasi-embeddable into $\{GL_i(\mathbb{F})\}$.[164]

General Question. *What are "sofic inequalities" that quantify the notions of soficity in the combinatorial and in the linear case and what are relations between these "quantifications"?*

Notice that it is not even known if "linear sofic" implies "sofic". In fact,

an example of a non-sofic group

is yet to be constructed and/or identified.

[164] See *An introduction to hyperlinear and sofic groups* by Pestov & Kwiatkowska, arXiv:0911.4266, and *Linear sofic groups and algebras* by Arzhantseva&Paunescu, arXiv:1212.6780.

6 Hyperbolic Hypersurfaces and the Gauss Map.

A smooth map $f : X \to Y$ is called *immersion* if its differential $D_f : T(X) \to T(Y)$ is *injective* on all tangent spaces $T_x(X)$. We we think of an immersed X in Y as a "smooth submanifold with (usually transversal) self intersection" and suppress f from notation.

An immersed hypersurface $X = X^{n-1}$ in a Riemannian n-manifold $Y = Y^n$ is called *strictly hyperbolic* if the *second fundamental form is nowhere singular*, without being positive or negative: there are $n_+ > 0$ strictly positive and $n_- = (n-1) - n_+ > 0$ strictly negative principal curvatures at all points in X.

The Euclidean Case. If the ambient space Y equals the Euclidean $\mathbb{R}^n$, non-degenaricy of the second fundamental form is equivalent to the *Gauss map* being an (equvidimensional) *immersion* $X \to S^{n-1}$, where, being interested in hyperbolicity, we exclude locally convex immersions from consideration.

Recall that the *Gauss map* $G : X \to S^{n-1}$ of an *immersed oriented hypersurface* X in the Euclidean space $\mathbb{R}^n$ to the unit sphere $S^{n-1} \subset \mathbb{R}^n$ sends $x \mapsto s = G(x) \in S^{n-1}$, such that the tangent space $T_s(S^{n-1})$ is parallel to the tangent space $T_x(X)$ where both tangent spaces are seen as (affine) hyperplanes in $\mathbb{R}^n$. If X is *non-oriented*, then the Gauss map lands in the *projective space* $P_{n-1} = S^{n-1}/\{\pm 1\}$.

The Spherical Case. If $Y = S^n$, then non-degenaricy of the second fundamental form is equivalent to the *Legendre map* being an *immersion*, where the Legendre map sends X to the dual sphere $S^{n\perp}$ of equatorial hyperspheres $S^{n-1} \subset S^n$ by assigning to each $x \in X$ the equatorial S^{n-1} that is tangent to X at x. In the metric terms, the Legendre map (normally) moves $X \subset S^n$ to the *equidistant hypersurface* with distance $\pi/2$ from X.

6.1 Simple Hyperbolic Hypersurfaces in $\mathbb{R}^{n+1}$ and in S^{n+1}.

General Problem 5. *Is there a natural class of "simple" strictly hyperbolic hypersufaces comparable in their geometric rigidity and in richness of their properties to convex hypersurfaces?*

What are strictly hyperbolic hypersurfaces that are related to unbounded Euclidean quadrics similarly to how compact convex surfaces are related to ellipsoids.?

This problem was intensely discussed by Leningrad's geometers – Burago, Sen'kin, Verner, Zalgaller – in the early sixties where a specific question I recall (and that may have been resolved) was as follows:

Let a strictly hyperbolic surface $X \subset \mathbb{R}^3$ homeomorphic to the cylinder $S^1 \times [0,1]$ be pinched between two parallel planes, with the boundary circles of X being closed convex curves in these planes.

> *Does there exit a straight line "going through inside of X" (i.e. missing X and linked with both boundary curves)?*

In this period, A. Verner[165] introduced the class of hypersurfaces X that are *one-sheeted over the sphere*, i.e. such that

the Gauss map is a diffeomorphism of X onto its spherical image $U \subset S^n$,

[165] A. L. Verner, "Topological structure of complete surfaces of non-negative curvature with one-to-one spherical map", Vesting LGU, 60, 1965, pp 16-29.

and he proved in the case $n = 2$ that

if an open subset $U \subset S^2$ serves as the diffeomorphic image of the Gauss map of a complete connected hyperbolic surface X in $\mathbb{R}^3$ (one sheeted over S^2) then U has ***at most two ends****: it is either a topological disc or an annulus.*[166]

Completeness and Properness. Here and below an immersed hypersurface X in a Riemannian n-manifold Y is called *complete* if the induces Riemannian metric in X is *complete.* This is the weakest (topological, despite its geometric garments) condition of this kind that one imposes on open hypersurfaces, where the strongest one is X being *properly embedded* into Y, i.e X being *closed as a subset* in Y.

Questions. *Do Verner's results generalize to $n \geq 3$?*

When does an open subset $U \subset S^n$ serve as the diffeomorphic image of a complete hyperbolic hypersurface in $\mathbb{R}^{n+1}$?

Is there a bound on the topology, e.g. on the number of ends of U?

Can one say anything of substance about hypersurfaces $X \subset \mathbb{R}^{n+1}$ that are k-sheeted[167] *over S^n for a given $k < \infty$?*

Another natural class of "simple" hyperbolic Euclidean hypersurfaces consists of *regularly hyperbolically compactifiable* $X \subset \mathbb{R}^{n+1}$, i.e. those that admit extensions to closed smooth strictly (non-strictly?) hyperbolic hypersurface $\overline{X}$ in the projective space $P^{n+1} \supset \mathbb{R}^{n+1}$, where this makes sense since hyperbolicity is a projectively invariant property.

Slightly more generally, one starts with a *closed connected hyperbolic hypersurface* $\overline{X} \subset S^{n+1}$, cuts it by an equatorial hemisphere from it and take the remaining part of $\overline{X}$, that is $\overline{X} \setminus S^n$, in one of the open hemispheres bounded by this equatorial S^n, where this hemisphere is projectively identified with $\mathbb{R}^{n+1}$. This leave us with the following

Questions. *What are possible closed strictly hyperbolic hypersurfaces in the sphere?*

Is there a universal, depending only on n, bound on the topology, e.g. on the Betti numbers of these $\overline{X}$?

Is there a universal, depending only on n, bound on the multiplicity of the Gauss map for $X = \overline{X} \setminus S^n$?

Does every connected closed strictly hyperbolic $\overline{X}$ admits a small perturbation $\overline{X}_\varepsilon$, $\varepsilon > 0$, such that all principal curvatures of $\overline{X}_\varepsilon$ are bounded by ε?

Constructions and Obstructions. The simplest spherical hyperbolic hypersurfaces are codimension two orbits of isometry groups acting on spheres. The first amuzing instance of this (if one disregards the obvious $S^k \times S^{n-k-1} \subset S^n$)

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[166]In the following paper On the extrinsic geometry of elementary complete surfaces with nonpositive curvature. I,II, Mat. Sb. (N.S.), 74(116):2 (1967), 218-240 and 75(117):1 (1968), 112-139. Verner studies the *geometry* (rather than just topology) of X and U, and shows, for instance, that the immersion $X \to \mathbb{R}^3$ is *proper* and that one of the components of the boundary $\partial(U) \subset S^2$ is an *equatorial circle* in many cases.

[167]This means that the Gauss map $X \to S^n$ is *at most k-to-one*, where the hypersurfaces $X \subset \mathbb{R}^{n+1}$ are assumed *complete* as earlier but where one is tempted to drop "strictly hyperbolic" condition.

is delivered by the action of the orthogonal group $O(3)$ on the unit sphere S^4 in the Euclidean 5-space of quadratic polynomials on $\mathbb{R}^3$, the $3d$-orbits of which equal $O(3)/\{\pm 1\}$.

(This disagrees with several conjectures made by Arnold[168] who suggested that hyperbolic hypersurfaces in projective spaces must be similar to quadrics $(S^k \times S^{n-k-1})/\{\pm 1\} \subset P^n$.)

More surprising occurrence of strict hyperbolicity in the spheres is seen in Cartan's *isoparametric* hypersurfaces $\overline{X} \subset S^n$; these, without being homogeneous, have nevertheless *constant* principal courvatures.[169]

I guess, all *known* strictly hyperbolic hypersurfaces X in S^n are deformations[170] of isoparametric ones but there is no reason to believe in the absence of other examples.

6.2 Jacobians of Gauss Maps.

A celebrated theorem by Efimov (1964) says that

the Jacobian of the Gauss map $G : X \to S^2$ of a connected complete non-compact surface X in $\mathbb{R}^3$ must, necessarily, approach zero,

$$\inf_{x \in X} |Jac(G)(x)| = 0,$$

where the interesting case is where X is strictly hyperbolic.

Efimov's proof has resisted multiple attempts to simplify it[171] and finding a transparent proof would be most welcome.

Also one would like to extend Efimov's theorem to higher dimensional spaces but one hardly can adjust his argument to $n \geq 4$.[172]; possibly(?), all $\mathbb{R}^n$, $n \geq 4$, contain complete strictly hyperbolic hypersurfaces X with

$$Jac(G)(x) \geq const > 0 \text{ for all } x \in X.$$

On the other hand one may achieve something by introducing additional constrains on X by requiring, for example, that its Gauss map $G : X \to S^n$ is "special", e.g. being the universal covering map of a "simple" subset U in the sphere, say $U = S^n \setminus \Sigma^{n-2}$ for a subvariety $\Sigma^{n-2} \subset S^n$ of codimension 2 in S^n.

Reconstruction of Hypersurfaces $X \subset \mathbb{R}^n$ from their Gauss Maps $G : X \to S^n$.

Immersions with quasi conformal Gauss maps.

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[168] V.I. Arnold, A branched covering of $\mathbb{C}P^2 \to S^4$, hyperbolicity and projectivity topology; Journal: Siberian Mathematical Journal Volume 29, Issue 5 , pp 717-726, http://link.springer.com/article/10.1007%2FBF00970265#page-1; also see the paper by A. G. Khovanskii, D. Novikov, On affine hypersurfaces with everywhere nondegenerate second quadratic form, Mosc. Math. J., 6:1 (2006), 135-152, where the authors justify Arnold's idea in some case.

[169] E. Cartan, Sur des familles remarquables d'hypersurfaces isoparamtriques dans les espaces sphériques, Math. Z. 45 (1939), 335-367. Also see U. Abresh, Isoparametric hypersurfaces with four or six distinct principal curvatures. Math. Ann. 264(1983), 283-302.

[170] "Deformation" is a C^2-continuous family X_t of *strictly hyperbolic* hypersurfaces with $X_0 = X$ and where X_1 is isoparametric.

[171] See the recent survey: V. Alexandrov, On a differential test of homeomorphism found by N.V. Efimov, arxiv.org/pdf/1010.3637

[172] Reformulation of Efimov's theorem in terms of the Ricci curvature does extend to all n, see B. Smith, F. Xavier, Efimov's theorem in dimension greater than two, Invent. Math. 90 (l987), 443-450.

7 Curvature, Geometry, Topology.

Conjecture. *A closed Riemannian n-manifold X with* ***non-negative*** *sectional curvatures has the totality of its rational Betti numbers bounded by those of the n-torus,*

$$\sum_{i=0,1,\ldots,n} rank(H_i(X;\mathbb{Q})) \leq 2^n.$$

....

8 Holomorphic, Quasiconformal and Similar Classes of Maps.

8.1 Rough Weierstrass factorization theorem for holomorphic and pseudoholomorphic curves

The rough form of the Weierstrass theorem provides (multiplicative) Fredholm correspondence between *judicially chosen* spaces of holomorphic functions $\mathbb{C} \to \mathbb{C}$ with certain discrete subsets in $\mathbb{C}$.

Are there similar correspondences for spaces of (pseudo)holomorphic maps $X \to Y$, at least for $dim_{\mathbb{X}} = 1$ and $Y = \mathbb{C}P^n$ for $n > 1$?

8.2 Topological Uniformization Problem.

Does every orientable topological n-manifold X is homeomorphic to the Euclidean space $\mathbb{R}^n$ divided by a group of homeomorphisms, where this group acts on $\mathbb{R}^n$ discretely but not necessarily freely?

Discussion.

A continuous map between Riemannian manifolds, $f : X \to Y$, is called *quasiconformal*[173] if the f-images of the small balls in X are pinched between approximately equal balls in Y. Namely,

there is a constant $\kappa \geq 1$ and a continuous positive function $\varepsilon_0(x) > 0$ on X, such that the images of the ε-balls $B_x(\varepsilon) \subset X$ lie between concentric balls $B_{f(x)}(r) \subset Y$ and $B_{f(x)}(\kappa r) \subset Y$ for some r,

$$B_{f(x)}(r) \subset f(B_x(\varepsilon) \subset B_{f(x)}(\kappa r) \text{ for all } x \in X \text{ and } \varepsilon \leq \varepsilon_0(x)$$

8.3 Topological Picard Problem.

Are there smooth closed ***simply connected*** *manifolds X that admit no non-constant quasiconformal map $f : \mathbb{R}^n \to X$?*

Discussion.

[173] If $dim(X) = dim(Y) \geq 2$, then such maps are usually called *quasiregular*.

8.4 Codimension minus one Liouville Problem.

Let $f:\mathbb{R}^{n+1}\to\mathbb{R}^n$ be a smooth submersion between Euclidean spaces that is *quasiconformal transversally to the fibers* of f, i.e. where the images of tie unit balls of the differential of f at all points are ellipsoids with the ratios of their principal axises bounded by some constant.

Can the image $f(\mathbb{R}^{n+1})\subset\mathbb{R}^n$ be bounded?

Discussion

*What are examples of $\mathbb{Q}$-**essential** e.g. aspherical, complex algebraic manifolds X that admit surjective holomorphic maps $\mathbb{C}^n\to X$?*

Recall, that X is called $\mathbb{Q}$-essential if its fundamental $\mathbb{Q}$-homology class *does not vanish* under the classifying map from X to the classifying space of the fundamental group of X.)

Discussion.

9 Range of the h-Principle.

Let us start with two concrete seemingly simple questions.

9.1 Smooth Maps with Small Spherical Images and Maps with Large Osculating Spaces.

Let $Gr_n^{ori}(\mathbb{R}^{n+k})$ denote the Grassmann manifold of *oriented n-planes* in $\mathbb{R}^{n+k}$ passing through the origin and let $Gr_n(\mathbb{R}^{n+k})=Gr_n^{ori}(\mathbb{R}^{n+k})/\pm1$ be the Grassmann manifold of *non-oriented* n-planes.

For instance, $Gr_n^{ori}(\mathbb{R}^{n+1})$ equals the n-sphere S^n and $Gr_n(\mathbb{R}^{n+1})$ is the projective space $P^n=S^n\{\pm1\}$ where the group $\mathbb{Z}_2=\{\pm1\}$ acts on the sphere by $s\leftrightarrow -s$.

Given a smooth *immersion* f of an oriented n-manifold X to $\mathbb{R}^{n+k}$ let $G=G_f:X\to Gr_n^{ori}(\mathbb{R}^{n+k})$ denote the *oriented tangential Gauss map* that sends each $x\in X$ to the n-subspace in $\mathbb{R}^{n+k}$ through the origin that is *parallel to tangent space* $T_x(X)\subset\mathbb{R}^{n+k}$.

The spherical image of X is just the image $G(X)\subset Gr_n^{ori}(\mathbb{R}^{n+k})$.

(Recall that "immersion" $f:X\to Y$ means that the differential $D_f:T(X)\to T(Y)$ is *injective* on all tangent spaces of X; thus $f(X)$ may be thought of as a "smooth non-singular hypersurface in Y with self intersections", where the above "$T_x(X)\subset\mathbb{R}^{n+k}$" is a shorthand for $D_f(T_x(X))\subset\mathbb{R}^{n+k}$ for $Y=\mathbb{R}^{n+k}$.)

Directed Immersions. A subset $U\subset Gr_n^{ori}(\mathbb{R}^{n+k})$ is said to *direct* an immersion $X=X^n\to\mathbb{R}^{n+k}$ if the spherical image $G(X)\subset Gr_n^{ori}(\mathbb{R}^{n+k})$ is *contained in* U.

$\pm s$-Condition. *Say that a subset $U\subset S^n$ satisfies the $\pm s$-condition if the quotient map $S^n\to P^n=S^n/\{\pm1\}$ sends U **onto** P^n.*

If U directs a closed, i.e. compact without boundary, immersed hypersurface X in the Euclidaen $(n+1)$-space, i.e. $U\supset G(X)$, then U does satisfy the $\pm s$-condition.

In fact, U must contain *either s or $-s$* in S^n, as one of these two points equals $G(x)$ for the point(s) $x = x_{max}(s) \in X$ where the *normal projection function* $p_s(x)$ from X to the line $l_s = \mathbb{R}$ in $\mathbb{R}^{n+1}$ that linearly extends the (unit) vector $s \in \mathbb{R}^{n+1}$ *achieves its maximum* (or minimum).

What is not so apparent is that the *oriented* Gauss map $G : X \to S^n$ of an *immersion with a self intersection* (unlike those of imbeddings that have no selfintesections) may be *non-onto*.

The simplest instance of this is the above squeezed figure ∞ in the plane the spherical image of which in S^1 lies in a small neighbourhood of a *half* circle. Then by consecutive axial rotations of $\mathbb{R}^2$ in $\mathbb{R}^3$, of $\mathbb{R}^3$ in $\mathbb{R}^4$, etc. one constructs immersions of the n-torus $X = \mathbb{T}^n$, $n = 2, 3, ...$, into the Euclidean space $\mathbb{R}^{n+1}$ that are directed by arbitrary small neighborhoods $U \subset S^n$ of half spheres. But there are many more subsets in S^n for $n \geq 2$ that satisfy the $\pm s$-condition besides neighborhoods of half spheres.

Question 1. *What kind of subsets $U \subset S^n$ can direct immersions of* ***closed oriented*** *n-manifolds into $\mathbb{R}^{n+1}$?*

Does a U that directs an immersed n-torus necessarily directs immersions of ***all parallelizable***[174]*n-manifolds?*

Is, the $\pm s$-condition on an open connected subset $U \subset S^n$ ***sufficient*** *for the existence of an immersion of the n-torus $X = \mathbb{T}^n$ (all parallelazible X?) into the Euclidean space, such that the oriented Gauss map $G : X \to S^n$ sends X to U?*

The positive answer for certain U, e.g. for the complements to finite subsets in the S^n, is delivered by *convex integration* that applies, besides tori, to all parallelizable manifolds.[175]

Osculating Spaces. Given a C^k-smooth map $f : X \to \mathbb{R}^N$ the *k-th osculating space* $T^k_x \subset \mathbb{R}^N$ at $f(x) \in \mathbb{R}^N$ is the *minimal affine subspace* $T \subset \mathbb{R}^N$, such that $f(X)$ is *tangent* to T at $f(x)$ with *order k*:

$$dist_{\mathbb{R}^m}(f(y), T) \leq const \cdot \varepsilon^{k+1}$$

for all $y \in X$ with $dist_X(y, x) \leq \varepsilon$ and all small $\varepsilon \geq 0$.

For instance, the first osculating space of an immersed $X \to \mathbb{R}^m$ at $x \in X$ is the tangent space $T_x(X) \subset \mathbb{R}^m$.

The osculation space T^k_x (or rather the liner subspace in $\mathbb{R}^m$ parallel to it) can be equivalently defined with local coordinates in X as being *linearly spanned* by the *values of the partial derivatives* of f of orders $\leq k$ at x in these coordinates (where these partial derivatives are maps $X \to \mathbb{R}^m$); thus the dimension of T^k_x is bounded by

$$N(k) = n + \frac{n(n+1)}{2} + ... + \frac{n(n+1)...(n+k-1)}{k!} \text{ for } n = dim(X).$$

[174]"Parallelizable" means with *trivial* tangent bundle. Obviously, all manifolds $X = X^n$ that admit immersions to $\mathbb{R}^{n+1}$ directed by a $U \subsetneq S^n$ are parallelizable.

[175]An explicit construction of a class of directed immersions is suggested by M Ghomi in *Directed immersions of closed manifolds*, Geometry & Topology 15 (2011) 699-705.

Question 2. *Does there exist a C^2-differentiable map f from the n-torus $X = \mathbb{T}^n$ to $\mathbb{R}^N$ for $N = n + \frac{n(n+1)}{2}$, such that the second osculating spaces of f have* ***maximal possible*** *dimensions that is $dim(T^2_x) = n + \frac{n(n+1)}{2}$ at all points $x \in X$?*

In other words,

Can the vectors

$$\frac{\partial f}{dt_i}(x) \in \mathbb{R}^N \text{ and } \frac{\partial^2 f}{dt_i dt_j}(x) \in \mathbb{R}^n,\ i,j = 1,2,...,n,\ i \geq j,$$

of the first and the second partial derivatives of f with respect to the cyclic coordinates in the torus, be ***linearly independent*** *at all $x \in X$?*

Discussion.

9.2 Best Regularity of Solutions Obtained by Convex Integration.

9.3 Soft and Rigid Isometric Immersions.

Isometric immersions of Riemannian manifolds X to Euclidean spaces $\mathbb{R}^N$ have been faithful customers of the h-principle for more than forty years but the extent of this relationship, however, has not been fully clarified.

Below is the list of "standard conjectures" with motivations and definitions to follow.

1. Cartan-Janet Local C^∞-Immersion Conjecture. A small neighbourhood $U = U_x \subset X$ of a point $x \in X$ in an arbitrary C^∞-smooth Riemannian manifold X admits an isometric C^∞-immersion into $\mathbb{R}^N$ for $N = \frac{n(n+1)}{2}$.

2. Cartan Local Rigidity Conjecture. *C^∞-Smooth* ***generic*** *submanifolds X^n in $\mathbb{R}^N$ are* ***locally metrically rigid for all*** *$N < \frac{n(n+1)}{2}$.*

3. Dual Flexibility Conjecture. *C^∞-Smooth* ***generic*** *submanifolds X^n in $\mathbb{R}^N$* ***are not*** *locally metrically rigid for $N \geq \frac{n(n+1)}{2}$.*

Moreover, if $N > \frac{n(n+1)}{2}$, then generic C^∞-smooth n-dimensional submanifolds in $\mathbb{R}^N$ are ***microflexible****.*

4. Let τ_1, τ_1, ..., τ_n be linearly independent real analytic tangent vector fields on a real analytic manifold X of certain dimension $M \geq n$.

Orthonormal Frame Conjecture. *There exists a real analytic map $F : X \to \mathbb{R}^N$ for $N = \frac{n(n+1)}{2} + 1$, such that the images of these fields under the differential $: T(X) \to T(\mathbb{R}^N)$ are* ***orthonormal*** *at all $x \in X$.*

5. Unbelievable C^2-Immersion Conjecture. *All C^∞-smooth Riemannian manifolds of sufficiently high dimension n, say $n \geq 10$, admit (local?) isometric C^2-immersions to $\mathbb{R}^N$ for $N = \frac{n(n+1)}{2} - 1$.*

6. Immersions of Flat Manifolds Conjecture. *All* ***flat*** *(parallelizable?) Riemannian n-manifolds X admit real analytic isometric immersions to the Euclidean space $\mathbb{R}^{2n}$.*

9.4 Algebric Inversion of Differential Operators.

9.5 Holomorphic h-Principle for Strongly Underdetermined PDE.

9.6 Hurwitz Averaging, Waring's problem, Undetermined Diophantine Equations and Soft Non-Linear PDE.

9.7 Donaldson's h-Principle Associated with the $\overline{\partial}$-Operator.

9.8 Lokhamps Theorem and the Curvature h-Principles.

9.9 Softness in Symplectic Geometry.

9.10 Concepts and Constructions.

The first recorded manifestation of the h-principle, attributed by Hassler Whitney in his 1937 paper to W.C. Graustein, reads:

if two closed immersed planar curves, $f_0, f_1 : X = S^1 \to \mathbb{R}^2$, have equal degrees of their tangential Gauss maps $G_{f_0}, G_{f_1} : X \to S^1$, then they can be joined by a smooth homotopy of immersions $f_t : X \to \mathbb{R}^2$, $t \in [0,1]$.

9.11 Perspective on Geometric PDE Inspired by Liposomes.

10 Dilation of Maps and Homotopy.

10.1 Weak Topologies Detecting Topological Invariants.

10.2 Asymptotic Sharpness of the Dilation Bound via Sullivan's Minimal Model with Differential Forms.

Let Y be a Riemannian manifold, let f be a continuous map of the k-sphere S^k to Y, let $h_d : S^k \to S^k$ be continuous maps of degrees $d = 1, 2, ...$ and let $f \circ h_d : S^k \to Y$ be the composed maps.

Conjecture. *If Y is **compact simply connected** and if the induced homology homomorphism $f_* : H_k(S^k) = \mathbb{Z} \to H_k(Y)$ **vanishes** then*

the maps $f \circ h_d$ are homotopic to continuous maps $f_d : S^k \to Y$ such that the dilations of these maps are bounded by:

$$Dil(f_d) =_{def} \sup_{s_1 \neq s_2} \frac{dist_Y(f_d(s_1), f_d(s_2))}{dist_{S^k}(s_1, s_2)} \leq C \cdot d^{\frac{1}{k} - \varepsilon}$$

for all $d = 1, 2, ...$ and some constants $\varepsilon > 0$ and $C > 0$, where ε, depends only on the homotopy type of Y and C depends on the metric in Y as well.

10.3 Distortion of Knots.

10.4 Metric Geometry of Families of Homogeneous Manifolds and Local Geometry of Carnot-Caratheodory Spaces.

11 Large Manifolds, Novikov Higher Signature Conjecture and Scalar Curvature.

Can one directly construct open subsets U in manifolds V with nomologically interesting covering $\tilde{U}$ (e.g. as in the proof of the topological invariance of Pontryagin classes) *without* a use of the Serre theorem on finiteness of the homotopy groups of spheres?

Can one do that with some geometric model of $K(\mathbb{Z}; n)$ (or $K(\mathbb{Q}; n)$?) spaces, e.g. represented by 0-cycles in S^n?

Can one replace such coverings by construction of some foliataions or of some linearised infinite dimensional counterparts thereof as it is implicit in the index theoretic argument in this context?

Do closed Riemannian manifolds X of positive scalar curvature have zero simplicial volume?

More generally, is the simplicial volumes of closed Riemannian n-manifolds X with $scal.curv(X) \geq -1$ bounded by $const_n \cdot volume(X)$?

11.1 Link between Plateau and Dirac Equations Suggested by Scalar Curvature.

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12 Infinite Groups.

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12.1 Small Cancelation and Hyperbolic Groups.

Non-Residual finiteness as obstruction for existence of "many" high dimensional hyperbolic groups.

Do Rips groups with universal relations in words in pairs of generators, e.g. where all subgroups generated by two elements satisfy something like

$$[[[...[w_1, w_2], [w_1', w_2'], ..., [w_1'', w_2''], [w_1''', w_2'''] ...]]] = id.$$

are limits of (graded?) relatively hyperbolic ones?

And/or

(Rips) Construct groups in 3 generators of exponential growth where all subgroups generated by two elements are nilpotent of nil potency degree $\leq 10^{100000}$?

And/or

Construct groups in 3 generators of exponential growth where all subgroups generated by two elements are finite of cardinalities $\leq 10^{100000}$?

Combination Problem. Let X_1 and X_2 be hyperbolic manifolds of constant curvature and $Y_i \subset X_i$, $i = 1, 2$ be two mutually isometric submanifolds.

When does there exist a hyperbolic manifold Z of given dimension $m \geq dim(X_1) + dim(X_2) - dim(Y_1)$ that contains $X_1 \cup X_2$ glued over $Y_1 \leftrightarrow Y_2$?

Generalization. Given more than two X_i, realise, whenever possible, their union in some Z with a given intersection pattern between these $X_i \subset Z$.

Specification. Answer the same questions where all manifolds are assumed/required to be compact.

Infinite Dimensional Riemannian-Hilbertian Locally Symmetric Spaces X.

Are there such X that are not built in an obvious manner from (increasing countable unions of) finite dimensional spaces and flat spaces?

For instance are there complete infinite dimensional manifolds of constant negative curvatures with $Injrad > 0$ and $Diam < \infty$?

Consider compact n-manifolds X^n of constant negative curvatures. What is, roughly, the maximum

$$D_n = \max_{X^n} \frac{Diam(X^n)}{Injrad(X^n)}?$$

Does (how fast) $D_n \to \infty$ for $n \to \infty$?

12.2 Homology of Finite Coverings of Arithmetic Varieties.

Can one define some kind of dimension of L_p-cohomology using the contributions of their products to the L_2-cohomology, e.g. for L_n-cohomology of the hyperbolic spaces H^n?

Do the homology away from the middle dimension of d-sheeted congruence covering $\tilde{V}_d \to V$, say with the $\mathbb{Z}_2$-coefficients, grow slower than $const \cdot d$, $d \to \infty$?

What happens at the middle dimension?

If $V = H^n/\Gamma$ how much of its homology may come from intersections of totally geodesic hypersurfaces?

12.3 Amenability, Commutativity and Growth.

What are a amenable groups without torsion where all Abelian subgroups are cyclic?

What if, moreover, all cyclic subgroups are undistorted?

Are there non-virtually solvable amenable groups that admit a discrete action on $\mathbb{R}^n$?

What if, moreover, such an action is free and/or cocompact?

Can such groups be limits of relatively hyperbolic groups?

12.4 Sofic Groups and Dynamical Systems.

12.5 Sparse Systems of Relations: Deterministic and Random.

12.6 Scarceness of High Dimensional Groups and Constrains on Homologies of their Quotient Groups.

12.7 Scale Limits of Markov Spaces, of Leaves of Foliations and of Dynamical Systems.

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Do the (Marcovian Hyperbolic!) actions of hyperbolic groups Γ on their ideal boundaries B admit "Markov partitions" that are Γ-equaivariant finite-to-one morphisms $A \to B$ for A being (Markovian) Γ-subshifts of finite type?

If yes, would this (or something of this kind) imply Rips theorem on narrow triangles with finitely many bubbles in them?

12.8 Norms on Homologies, on Bordisms and on K-Functors

Is there a counterpart of simplicial volume that is multiplicative under Cartesian products of spaces? [???Multidimensional fillings of subpolyhedra]

Can a closed aspherical manifold X with non-zero Euler characteristic have zero simplicial volume?

("Aspherical" means that the universal covering is *contractible.*)

Superrigidity with Harmonic Maps into Infinite Dimensional Symmetric spaces. ....

12.9 Reconstructing Spaces from their Fundamental Groups.

(*Recollection: What Is Quasi-isometry?*[176] Call a map between metric spaces, $f : \Gamma \to \Delta$, *large scale Lipschitz* if

$$dist_\Delta(f(\gamma_1), f(\gamma_2) \leq const \cdot dist_\Gamma(\gamma_1, \gamma_2),\ \gamma_1, \gamma_2 \in \Gamma,\ dist(\gamma_1, \gamma_2) \geq 1.$$

Say that $f_1, f_2 : \Gamma \to \Delta$ are *quasi-parallel*[177] if

$$dist_\Delta(f_1(\gamma), f_2(\gamma)) \leq const < \infty \text{ for all } \gamma \in \Gamma.$$

and define qiasi-isometry as

isomorphism in the category of metric spaces and qiasi-parallelism classes

of large scale Lipschitz maps.)

Apply this to the vertex sets of (undirected) graphs with the (discrete) metrics defined by the minimal lengths of shortest edge paths between verices.)

13 Large Dimensions

13.1 Is There Interesting Geometry in Infinite Dimensions?

13.2 Topology of Infinite Cartesian Products.

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[176]The concept, suitably adapted to general metric spaces, is at least 43 years old as it already appears in Margulis' 1970 paper *The isometry of closed manifolds of constant negative curvature with the same fundamental group.*

[177]*An essential Example.* Lifts of *homotopic* maps between *compact spaces* to their universal coverings are quasi parallel.

13.3 Surjunctivity of Symbolic Algebraic Endomorphisms and Middle Dimensional Homologies of Infinite Dimensional Spaces.

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13.4 Parametric Packing Problem Revisited.

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13.5 Protein Folding Problem and Percolation in High Dimension.

13.6 Percolation Style Problems for Finite and Infinite n-Cycles where $n > 1$.

13.7 What is "Geometric Space"?

Sturtevant-Koncevich

14 Miscellaneous.

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14.1 Metrics Categorical.

14.2 Combinatorics in Geometric Categories.

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14.3 Measurable Polyhedra and the Constrained Moment Problem.

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14.4 Crofton Geometry and Desarguesian Spaces.

14.5 Counting Spaces with Given Topology.

Given a set $\mathcal{X}$ of homotopy classes of n-dimensional simplicial polyhedra X, let

$$N(k) = N_{\mathcal{X}}(k, n)$$

denote the number of contractible polyhedra made of at most k simplices, such that the homotopy classes of X itself and of the stars of all simplices in X lie in $\mathcal{X}$.

What is the asymptotic behaviour of $N(k)$ *for* $k \to \infty$*?*
Is, for instance the growth of the number $N(k)$ *of triangulations of the* n*-ball into* k*-simplices is* ***super exponential***
(i.e. $k^{-1} \log N(k) \to \infty$*) starting from* n *equal* 3*?* 4*?* 5*?*

Discussion.

14.6 Unitarity vesus Bistochasticity.

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14.7 Genericity and Inaccessibility of "Generic".

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14.8 Scale Limits Functors of Particles Evolution Equations

14.9 Upper Bound on Laplacian Spectra of Riemann Surfaces.

Can one recover Hersch theorem by Waffa-Witten method?

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14.10 Miscellaneous Miscellaneous.