Quantitative Homotopy Theory

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I once attended lectures about cosmology by two topologists, great topologists, concerning the possible shape of the Universe (it was about 25 years ago), and I asked the first one whether the Universe was simply connected or not. When asked this question he said "It is clear that the Universe cannot be but simply connected, for non-simple connectedness would imply some high-scale periodicity, which is ridiculous." The other's talk was entitled, "Is the Universe simply connected?" When I told him what the first said, he responded, "Who cares, it's still a meaningful question, like it or not." What I have to say is not exactly related to this, but is motivated by the naive question of whether or not it makes sense to ask of something that it be simply connected. By sense, I mean physical, in the spirit of Aristotle—everyday physics. When we ask this question we want a "physically" meaningful answer. So, we consider the question this way: take a loop in the Universe, a reasonably short loop compared to the size of the Universe, say of no more than 10^{10} to 10^{12} light years long and ask if it is contractible. And, to be realistic, we pick a certain time, for example 10^{30} years, and ask if it is contractible within this time. So you are allowed to move the loop around, say at the speed of light, and try to determine whether or not it can be contracted within this time. The point is, even imagining our space to be some topological 3-sphere S^3, we can organize an innocuous enough metric on S^3 so that it takes more than 10^{30} years to contract certain loops in this sphere and in the course of contraction we need to stretch the loop to something like 10^{30} light years in size. So, if 10^{30} years is all the time you have, you conclude that the loop is not contractible and whether or not $\pi(S^3) = 0$ becomes a matter of opinion.

My point is that when you have a space of maps, like the space of maps of a circle into a compact three manifold, there is, in the homotopy theory, some extra structure coming from geometry as the one just illustrated. Namely, when we speak of homotopy, we try to keep track of the sizes of maps and of homotopies. Here we encounter new questions, some of which I am going to discuss.

Consider compact finite dimensional spaces X and Y, say finite polyhedra. Now, a finite polyhedron has essentially a unique piece-wise Euclidean metric, unique meaning that if you have two metrics on X, then (perhaps after a little wiggling) we can construct a bi-Lipschitz homeomorphism between the two. In particular, for a compact manifold, two metrics differ by a multiplicative constant, and from our point of view, which will be concerned with orders of magnitude, these are essentially the same.

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So we are given metric spaces X and Y and we study the space of continuous maps $\mathrm{Map}(X \to Y)$. A basic characteristic of a map $f : X \to Y$ is its dilation or *Lipschitz constant* which measures by how much f stretches (the curves in) X. Namely,

$$\mathrm{Lip}(f) = \sup_{x_1 \neq x_2} \frac{\mathrm{dist}\big(f(x_1), f(x_2)\big)}{\mathrm{dist}(x_1, x_2)}.$$

Now we want to understand the structure of this function $f \mapsto \mathrm{Lip}(f)$ for $f \in \mathrm{Map}(X \to Y)$ pretending we perfectly understand the topology of the space of maps. For example, let X be the sphere S^n so that the connected components of $\mathrm{Map}(X \to Y)$ are represented by the homotopy group $\pi_n(Y)$ and suppose this group is isomorphic to $\mathbb{Z}$. We represent each $h \in \pi_n(Y) = \mathbb{Z}$ by a map f of minimal dilation and think of this dilation $h \mapsto \inf_{[f]=h} \mathrm{Lip}(f)$ as a kind of norm $\|h\|$ on $\pi_n(Y)$.

Notice that the individual values $\|h\|$, $h \in \mathbb{Z}$, of this "norm" depend on our metrics on S^n and Y but the asymptotics for $h \to \infty$ are essentially the same for all commonly used metrics.

Example. Consider maps $f : S^2 \to S^2$. Every such f is characterized, up to homotopy, by the degree $d \in \mathbb{Z} = \pi_2(S^2)$. One easily constructs maps f of degree d and $\mathrm{Lip}(f) \leq 100\sqrt{d}$ for the standard metric on S^2 and all $d \in \mathbb{Z}$, thus showing $\|d\| = O(\sqrt{d})$ for $d \to \infty$ and for *all* Riemannian metrics on S^2. On the other hand, $\|d\|$ can not be much smaller than $\sqrt{d}$ since

$$d = \deg(f) = \int_{S^2} \mathrm{Jac}(f)\, ds,$$

where the Jacobian of f is quadratic in the partial derivatives of f and thus is bounded by $\big(\mathrm{Lip}(f)\big)^2$. Therefore,

$$\|d\| \sim \sqrt{d}.$$

Generalization. Take an $h \in \pi_n(Y)$ such that the *Hurewicz homomorphism* does not vanish on h, not even after tensoring with $\mathbb{R}$. Then, our "Lipschitz norm" of the powers $h^d \in \pi_n(X)$ grows as $\mathrm{const}\, d^{\frac{1}{n}}$, where the upper bound on $\|h^d\|$ is obtained with easy maps $S^n \to S^n$ of degree d and dilation $\leq C_n d^n$, while the lower bound appeals to the volume growth of maps f_d representing h^d, or, equivalently to the growth of the integrals $\int_{S^n} f^*(\omega)$ for some closed n-form ω on Y for which $\int_{S^n} f_d^*(\omega) \neq 0$.

Hopf maps. This map, $f : S^3 \to S^2$ is homologous to zero (i.e. Hurewicz $[f] = 0$) as $H_3(S^2) = 0$, yet it represents a *nontrivial* class $h \in \pi_2(S^3) = \mathbb{Z}$. The self-mappings $\varphi : S^2 \to S^2$ of degree k transform this h to h^{k^2} which easily shows that the norm $\|h^d\|$ grows no faster than $\mathrm{const}\, d^{\frac{1}{4}}$ (for all d, not only for those of the form k^2). And the lower bound follows from the following definition of the Hopf invariant $h(f) \in \mathbb{Z} = \pi_3(S^2)$ due to Whitehead. Take the area form ω on S^2 and let ω'_f be a primitive (1-form) of the pull-back $\omega_f^* = f^*(\omega)$ on S^3, i.e. $d\omega'_f = \omega_f^*$. Then, according to Whitehead,

$$h(f) = \int_{S^3} \omega_f^* \wedge \omega'_f,$$

and the required bound

$$h(f) \leq \text{const}\big(\text{Lip}(f)\big)^4$$

follows, since $\|\omega_f^*\| \leq \big(\text{Lip}(f)\big)^2\|\omega\|$ and since one can always find a primitive ω' of ω^* satisfying

$$\sup_{x\in S^3} \|\omega'(x)\| \leq \text{const} \sup_{x\in S^3} \|\omega^*(x)\|\,,$$

as an elementary argument shows.

Sullivan's minimal models. These generalize the above construction and, assuming Y is simply connected, express *all* $\mathbb{R}$-valued invariants (i.e. homomorphisms) of the group $\pi_n(Y)$ as integrals of products of some pulled back differential forms on Y and their consecutive primitives. This leads to the bound

$$\|h^d\| \geq \text{const}\, d^\alpha$$

for every non-torsion element $h \in \pi_n(Y)$ where $\alpha = \alpha(Y,h) \neq 0$ is some *rational* number coming out of a computation with the minimal model.

Conjecture. Let Y be a compact simply connected Riemannian manifold (or a more general compact simply connected space with a "reasonable" metric). Then every non-torsion element $h \in \pi_n(Y)$ satisfies

$$\|h^d\| \sim d^\alpha$$

for the above mentioned rational number α provided by the minimal model.

What is unclear here is how to construct maps $f : S^n \to Y$ representing $[h^d]$ with $\text{Lip}(f) = O(d^\alpha)$. (Many such maps come via the Whitehead product and similar higher order products, but these seem to be unsufficient for our purpose.)

A special case of the above problem reads as follows. Let $h \in \pi_n(Y)$ be homologous to zero. Show that

$$\|h^d\| = O\big(d^{1/(n+1)}\big) \text{ for } d \to \infty\,.$$

In fact I do not even see how to get $\|h^d\| = o(d^{1/n})$ in this case.

Counting maps X $\to$ Y. Denote by $\#(\lambda)$ the number of mutually non-homotopic maps $f : X \to Y$ with $\text{Lip}(f) \leq \lambda$. The above discussion tells us something about the asymptotics of $\#(\lambda)$ for $\lambda \to \infty$ in the case $X = S^n$ and in general, the minimal model method applies to all X and simply connected Y. Thus one can show

$$\#(\lambda) = O(\lambda^A),\ \lambda \to \infty\,,$$

for some rational number A depending on the minimal models of X and Y, but one has a poor idea of how to generate sufficiently many homotopically distinct maps $f : X \to Y$ with small dilations in-so-far as the minimal model theory allows us to do it.

Controlled homotopy. Now, following the logic of our introductory remark on π_1 (Universe), we want to study the geometry of *homotopies* between maps $X \to Y$. For example, given a *contractible* map $f : X \to Y$ with $\mathrm{Lip}(f) \le \lambda$, we want to find a homotopy f_t of $f = f_0$ to a point where each map $f_t : X \to Y$, $t \in [0,1]$, has

$$\mathrm{Lip}(f_t) \le \Lambda(\lambda)$$

for some "reasonable" function $\Lambda(\lambda)$. For example, if $X = S^n$, $Y = S^m$, where $n \ne m$, $2m-1$, then the composed map

$$S^n \underset{\varphi}{\longrightarrow} S^n \underset{f_0}{\longrightarrow} S^m,$$

for some specific map $\varphi : S^n \to S^n$ of degree $k = k(m,n)$, is always contractible and we may ask our question for $f = \varphi \circ f_0$ where f_0 is an arbitrary map $S^n \to S^m$ with $\mathrm{Lip}(f_0) \le \lambda_0$.

If Y is a non-simply connected space, then the (best possible) function $\Lambda(\lambda)$ may be essentially as complicated as any other recursive function, as follows from a recent (yet unpublished) work by Rips and Sapir. But for simply connected compact Riemannian manifolds one conjectures that

$$\Lambda(\lambda) \le \mathrm{const}\, \lambda^p$$

for some p depending on the rational homotopy types of X and Y, where the minimal such $p = p(X,Y)$ is expected to be a rational number. (Actually, I have not worked out any example where $p > 1$.) What we know from the general principles of the homotopy theory is the bound

$$\Lambda(\lambda) \le \exp\big(\exp \ldots (\exp(\lambda))\big)$$

where exp is iterated about dim X times. These exponents appear any time we appeal to the Serre fibration property as (uncareful) lifting of homotopies exponentially enlarges the dilation. However certain examples (such as S^{2n+1} fibred over $\mathbb{CP}^n$) suggest that one can get away with the polynomial enlargement.

Filling Riemannian manifolds. The (conjectural) bound $\Lambda(\lambda) = O(\lambda^p)$ has a (also conjectural) counterpart in the cobordism theory where we want to fill-in a closed Riemannian manifold V by W with a suitably defined size of W controlled by that of V. Here one may use the volumes of V and W for the size, provided one restricts to manifolds with appropriate bounds on their local geometries. Then one expects that every n-dimensional V bounds W, i.e. $\partial W = V$, such that

$$\mathrm{Vol}_{n+1} W \le \mathrm{const}_n \, \mathrm{Vol}_n V\,,$$

where we assume that V is null-cobordant to start with. This conjecture is easy to prove for $n = 2$ while the only supporting evidence for $n \ge 3$ comes from our old result with Jeff Cheeger claiming that the η-invariant of a manifold V with bounded local geometry is bounded by const $\mathrm{Vol}(V)$ for const $= \mathrm{const}_n$ (bound on local geometry).

Morse landscape of the function Lip on the space Map(X $\to$ Y). This generalizes our previous perspective of counting homotopy classes of map $f : X \to Y$ with $\mathrm{Lip}(f) \le \lambda$ as well as the controlled homotopy discussion. The Morse theoretic shape of the function Lip (e.g. the positions of its deep minima) is essentially independent of the specific metrics in X and Y and is determined by the homotopy types of these spaces. The question is how to effectively determine this shape for given X and Y.

Other dilation functions. Instead of Lip which measures the stretch of curves in X one can study some function

$$f \mapsto \mathrm{Lip}_j(f), \ f \in \mathrm{Map}(X \to Y),$$

measuring the stretch of j-dimensional submanifolds in X under f. In fact, one can look at several (all?) such functions Lip_j simultaneously and study the topology of the resulting map of the space $\mathrm{Map}(X \to Y)$ to some $\mathbb{R}^k$.

Noncompact spaces. Many interesting non-compact (e.g. infinite dimensional) spaces, (especially those of operator theoretic origin) come along with natural (classes of) metrics to which the above discussion applies. In fact, some information concerning the Lipschitz (and more general uniform) homotopy theoretic information can be derived from the classical isoperimetric inequality (via the measure concentration phenomenon of Levy–Milman) as we observed with Milman about 20 years ago. But this (measure theoretic) approach seems to be rather far removed from the above discussion which is more topological in origin.

References. See my forthcoming book "Metric structures for Riemannian and non-Riemannian spaces", Birkhäuser, 1998.

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