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Rigid transformations groups

§0 INTRODUCTION

Let a manifold V be endowed with a geometric structure g (see 0.1.) and consider the isometry group Is(V,g) which is the subgroup of diffeomorphisms of V preserving g. We are especially interested in the case where the isometry group is non-compact while V is compact or carries a finite invariant measure.

0.1. Infinitesimal geometric structures. Consider germs of diffeomorphisms of $\mathbb{R}^n$ at the origin fixing the origin $0 \in \mathbb{R}^n$ and let $\mathcal{D}^r = \mathcal{D}^r(\mathbb{R}^n, 0)$ denote the set of r-th order jets of these germs. This $\mathcal{D}^r$ comes with a natural structure of a Lie group of dimension

$$\dim \mathcal{D}^r = n\left(n + \frac{n(n+1)}{2} + \dots + \frac{(n+r-1)!}{(n-1)!\,r!}\right).$$

Moreover, $\mathcal{D}^r$ naturally is a linear real algebraic group. For example, $\mathcal{D}^1$ equals the full linear group $GL_n\mathbb{R}$.

Next, take a smooth manifold G with a smooth action of $\mathcal{D}^r$ on G and define structures of type G on n-dimensional manifolds V as follows. Denote by $\mathcal{D}^r(V) \to (V)$ the natural principal $\mathcal{D}^r$-bundle whose fiber $\mathcal{D}^r_v$ over $v \in V$ consists of the r-jets of germs of local coordinate systems in V around v. These are C^∞-diffeomorphisms of small neighborhoods $U \subset V$ of v into $\mathbb{R}^n$, such that $v \mapsto 0$ and the jets are taken at v. Let $G(V) \to V$ be the associated bundle with fiber G. Now, smooth sections $G : V \to G(V)$ are called geometric structures of order r and type G on V. Notice that sections $V \to G(V)$ naturally correspond to $\mathcal{D}^r$-equivariant maps $\mathcal{D}^r(V) \to G$.

(1) Let g be a <u>full frame</u> on V that is a system of vector fields which are independent and span the tangent space $T_v(V)$ for all $v \in V$. In other words, g is a section $V \to \mathcal{D}^1(V)$. Obviously, the group $Is^1(v)$ is trivial for all v and so g is 0-rigid.

(2) Every affine connection g on V canonically defines a full frame g' on $\mathcal{D}^1(V)$ and the 0-rigidity of g' implies 1-rigidity of g .

(3) Let g be a pseudo-Riemannian metric. Then the 1-rigidity of the Levi-Civitta connection yields 1-rigidity of g .

(4) One knows that conformal pseudo-Riemannian structures on V are 2-rigid for $\dim V \geq 3$ but conformal structures on surfaces are not rigid.

(5) The complex analytic, symplectic and contact structures are not rigid.

<u>Remark</u>. Rigid structures generalize structures g of <u>finite type</u> of Cartan. These are <u>G-structures</u>, which means $G = \mathcal{D}^r/G$ for a given subgroup $G \subset \mathcal{D}^r$, and the finite type condition requires a full $\mathcal{D}^r$-invariant frame on $\mathcal{D}^r(V)$ canonically associated to g . In fact, our r-rigidity is equivalent to the existence of a canonical full frame on $\mathcal{D}^r(V)$ which is not required to be $\mathcal{D}^r$-invariant.

0.3.A. If g is (r+i)-rigid, then by a trivial argument the homomorphism $Is^{r+j+1}(v) \to Is^{r+j}(v)$ is injective with a closed image for all $j = i, i+1, \dots \infty$. The same is true for the homomorphisms of the local isotropy group $Is^{loc}(v)$ into the infinitesimal isometry groups. Thus we have $Is^{loc}(v) \subset Is^{\infty}(v) \subset Is^{r+i}(v) \subset \mathcal{D}^{r+i}(v) \approx \mathcal{D}^{r+i}$, for all points $v \in V$. A more interesting (but still rather trivial, see 1.6.F.) is the following

<u>Stabilization of</u> Is^{r+j}. <u>There exists an integer</u> j_o, <u>which depends only on</u> $n = \dim V$ and $r+i$, <u>such that</u> $Is(v) = Is^{\infty}(v) = Is^{r+j}(v)$, <u>for all</u> $j \geq j_o$ <u>and for all</u> $v \in U \subset V$, <u>where</u> U <u>is an open dense subset in</u> V .
<u>Moreover</u>, <u>if</u> g <u>is real analytic</u>, <u>then</u> $Is^{loc}(v) = Is^{\infty}(v)$ for all $v \in V$.

0.4. Isometries of rigid structures. It is classically known (since S. Lie) and easy to prove (see 1.6.H.) that the full isometry group $Is = Is(V,g)$, for every rigid g, carries a unique structure of a (possible disconnected) Lie group, such that the action of Is on V is smooth and Is is closed in Diff V. Furthermore, the dimension of Is is bounded by that of $\mathcal{D}^{r+i}(V)$ (here and below we assume $r+i \geq 0$) via the following (trivial, see 1.5.B)

Proposition. If g is $(r+i)$-rigid then the homomorphism of the isotropy subgroup $Is_v \subset Is$ into the infinitesimal group $Is^{r+i}(v)$ is injective for all $v \in V$.

In fact, this proposition says that the action of Is on $\mathcal{D}^{r+i}(V)$ (induced by the action of Is on V) is free.

The freedom of the action does not tell much about Is if, for example, Is is discrete, but the following (trivial, see 1.5.B.) proposition is quite useful.

Properness of the action. If g is $(r+i)$-rigid, then the action of Is on $\mathcal{D}^{r+i}(V)$ is proper as well as free.

Recall that an action of a topological group H on X is called proper if for every compact subset $Y \subset X$ the intersection $h(Y) \cap Y$ is empty for all $h \in H$ outside some compact subset $K = K(Y) \subset H$. In the smooth category, properness is equivalent to the existence of an equivariant Riemannian metric μ on X, such that $H = Is(X,\mu)$.

It is worth noticing that the freedom and properness of an action on $\mathcal{D}^{r+i-1}(V)$ is characteristic for rigidity. Namely, let H be a Lie group smoothly acting on V, such that the associated action of H on $\mathcal{D}^k(V)$ is free and proper for some k. Then $H = Is(V,g)$ for some rigid structure g on V.

Proof. Since the actions of H and $\mathcal{D}^k$ on $\mathcal{D}^k(V)$ commute the quotient map $g : \mathcal{D}^k(V) \to \mathcal{D}^k(V)/H$ is $\mathcal{D}^k$-equivariant. If H acts freely and properly on $\mathcal{D}^k(V)$, then $\mathcal{D}^k(V)/H$ is a smooth manifold and so g is a structure on V in our sense. This g on V is H-equivariant but not always rigid (take, for example the action on $\mathcal{D}^1(\mathbb{R}^n)$ of the group H of affine transformations of $\mathbb{R}^n$). To achieve rigidity we take the action of H on $\mathcal{D}^{k+1}(V)$ and then observe that the kernel $\Delta^{k+1} \subset \mathcal{D}^{k+1}$ of the natural homomorphism $\mathcal{D}^{k+1} \to \mathcal{D}^k$ acts freely on $\mathcal{D}^{k+1}(V)/H$. This is obviously equivalent to $(k+1)$-rigidity of the corresponding structure $g' : \mathcal{D}^{k+1}(V) \to \mathcal{D}^{k+1}(V)/H$.

Definition. Actions on V which are proper on $\mathcal{D}^k(V)$ are called k-*rigid*; "*rigid*" means k-rigid for some k.

Rigidity of an action imposes strong restrictions on the underlying group H, especially if H preserves an additional structure, such as a finite measure or a particular A-structure. We postpone a detailed discussion until §4 and only present here two examples.

0.4.A. (See 4.4.) *Let Γ be the fundamental group of a closed aspherical (i.e. with contractible universal covering) manifold of dimension M. If Γ admits a C^k-smooth k-rigid action on a compact n-dimensional manifold, then*

$$M \leq N = \dim \mathcal{D}^k - \frac{n(n-1)}{2} .$$

Recall that

$$\dim \mathcal{D}^k = n\left(n + \frac{n(n+1)}{2} + \ldots + \frac{(n+k-1)!}{(n-1)!k!}\right)$$

and observe that $\frac{n(n-1)}{2}$ Is the dimension of the maximal compact subgroup $O(n) \subset \mathcal{D}^k$.

Note also that the inequality $M \leq N$ remains valid for every proper action of Γ on $\mathcal{D}^k(V)$ which commutes with the action of $\mathcal{D}^r$ and does not necessarily come from any action on V (see §4).

0.4.B. *Let Γ be the group of integral $(m \times m)$-matrices with $\det = 1$, that is $\Gamma = SL_m \mathbb{Z}$. If Γ admits a rigid action on V preserving a finite measure μ on V then $m \leq \dim V$. If, moreover, Γ preserves some A-structure g on V then the Lie algebra of the infinitesimal isotropy group, say $LIs^{\infty}(v)$, contains $s\ell_m \mathbb{R}$ as a subalgebra for almost all (for the measure μ) points $v \in V$.*

The first claim (apart from "easy dimensions" $m = 1,2$) is a special case of a deep theorem by Zimmer $[Z]_1$ generalizing Margulis' superrigidity. The A-refinement is a simple corollary of Zimmer's theorem (see §4).

Note that $SL_m \mathbb{Z}$ acts by automorphisms on the torus $T^m = \mathbb{R}^m / \mathbb{Z}^m$. This action is 1-rigid ; it preserves Lebesgue measure and the standard (flat) affine structure on T^n . In fact, the equality cases for inequalities like $M \leq N$ and $m \leq n$ can be often identified with some standard examples (see 0.8.).

Also, note that rigidity appears typical for "small groups" such as $\mathbb{Z}$ and $\mathbb{R}$ acting on V . For example, every axiom A action is 1-rigid.

0.4.C. Rigidity for connected groups. Let G be a connected Lie group such that

(a) The adjoint group $G' \subset \mathrm{Aut}\, L(G)$ is closed in the group $\mathrm{Aut}\, L(G)$ of automorphisms of the Lie algebra $L(G)$. Note that every algebraic group G has this property.

(b) The center of G is compact. Then our first remark is

$0.4.C_1$. Every locally free C^1-action of G on a smooth manifold V is 1-rigid.

Proof. Let $L \subset T(V)$ be the subbundle generated by the vector fields corresponding to a basis in $L(G)$. Then the action of G of L is given by the adjoint representation and hence, proper.

$0.4.C_2$. Every faithful C^{an}-action (i.e. real analytic) of G on a compact manifold V is rigid.

Proof. The lift of the action to $\mathcal{D}^k(V)$ is locally free for a sufficiently large k and then the 1-rigidity of this action yields $(k+1)$-rigidity of the original action on V .

$0.4.C_3$. If G is semisimple with finite center, then every faithful C^3-action on V is 3-rigid.

Proof. It is easy to see that the isotropy subgroups of the associated action of G on $\mathcal{D}^1(V)$ are unipotent which implies, in turn, that the action is locally free on $\mathcal{D}^2(V)$. Hence, the action on V is 3-rigid by $0.4.C_1$.

0.4.D. <u>An example of a non-rigid action</u>. Take a small ball $B^n \subset V$, $n = \dim V$, and let f_1 and f_2 be generic C^∞-diffeomorphisms of V sending B^n into itself and fixing the complement $V \setminus B^n$. The group generated by f_1 and f_2 obviously is free. Next take a diffeomorphism $f : V \to V$ such that the images $f^j(B^n)$ are disjoint for the iterates f^j of f. Then the group Γ generated by f, f_1 and f_2 contains an infinite product of free groups as a subgroup. Hence, Γ admits no rigid action at all on any compact manifold by 0.4.A. We shall see in §4 that this Γ admits no rigid action on a non-compact V preserving a finite Borel measure. But it is uncelear if there is some real analytic action of Γ on some compact manifold.

0.5. <u>Standard A-rigid actions</u>. An action of a Lie group on V is called A-<u>rigid</u> if it preserves some rigid A-structure. Here is a list of examples.

0.5.A. Let G be a real algebraic group algebraically acting on a real algebraic manifold V. It is well known (and easy to show) that this action is rigid. In fact, this action preserves the tautological <u>rigid</u> A-structure $g : \mathcal{D}^r(V) \to \mathcal{D}^r(V)/G$, where r is sufficiently large (compare 0.4.).

0.5.B. Let $V \to W$ be a principal torus bundle and let $f_j : W \to \mathbb{R}^m$, $j = 1,\dots,k$, be C^∞-maps where the space $\mathbb{R}^m$ is identified with the Lie algebra of the fiber (torus) T^m. Then each f_j defines a vector field, say X_j on V commuting with the action of T^m on V. Let Let $G \subset \operatorname{Diff} V$ be the (Abelian Lie) group generated by T^m and by these fields. It is easy to show that the following conditions are equivalent

1. The action of G is A-rigid.
2. The action is rigid.
3. The induced action of G on $\mathcal{D}^r(V)$ is free for some r.
4. There exists an r, such that :

if a linear combination $f = \sum_{j=1}^{k} \lambda_j f_j$ has $\partial^I f(v_o) = 0$ for $|I| = 1,\dots,r$, where v_o is some point in V (and the partial derivatives are taken in some local coordiantes around v_o), then f is constant on the connected component of v_o in V.

0.5.B_1. Let $V = \mathbb{R}^{n-m} \times T^m$ with the standard (product) affine structure and take for G the group of affine transformations of V mapping each torus $x \times T^m$ into itself by some rotation of T^m. Clearly, G is isomorphic to $T^m \times \mathbb{R}^{m(n-m)}$.

0.5.B_2. Remark. The actions described in 0.5.B. are not algebraic unless $G = T^m$. In fact, they violate the following

Property of algebraic actions. If an action of G on V is algebraic, then there is an (Zariski) open dense subset U in the Cartesian product $\underbrace{V \times \ldots \times V}_{k}$ for some (sufficiently large) k, such that the action of G on U is proper. (See 2.3.).

0.5.C. Let a connected Lie group G faithfully and transitively act on V. Clearly, this action is rigid and an easy argument shows it is A-rigid as well. In fact, one can "split" such an action into three pieces", where the first and the second ones are as 0.5.A. and 0.5.B. respectively, and the most interesting third piece is as follows.

0.5.D. Let g be a bi-invariant structure on G and let $\Gamma \subset G$ be a discrete subgroup. Then the action of G on G/Γ preserves g. Here are specific examples :

0.5.D_1. Let G be semisimple and g be the Killing form on G. Then take a cocompact lattice $\Gamma \subset G$ and thus obtain a compact pseudo-Riemannian manifold $V = G/\Gamma$ with the isometry group G. Note that the isotropy subgroup $G_v, v \in V$ is conjugate to Γ for all $v \in V$, but the local isotropy group $Is^{loc}(v)$ is isomorphic to G. In fact, if G has no compact factor groups, then for any G-invariant A-structure one has $L(G) \subset LIs^{loc}(v)$ (see §5).

The first interesting case is $G = SL_2\mathbb{R}$, where the Killing form is of the type (++-).

0.5.D_2. Let $g = GL_m$ with the flat affine structure induced from $\mathbb{R}^{m^2} \supset GL_m$. Here we get compact affine flat manifolds of dimension m^2 acted upon by GL_m. If we pass to PSL_m we obtain compact projectively flat manifolds with PSL_m-action. Notice that this projective structure for $m \geq 3$ does not come from the Killing pseudo-Riemannian structure.

0.5.C. One can "twist" the above examples with some proper actions. Namely, let G_1 freely and properly act on V_1 (and thus admit an invariant Riemannian metric). Take a lattice $\Gamma \subset G \times G_1$ and observe that G acts on $G \times V_1/\Gamma$ preserving (local) product structures $g \times g_1$ for all biinvariant g on G and G_1-invariant g_1 on V.

0.5.D. Let V be a <u>locally homogeneous space</u> modelled on a homogeneous space $W = E/I$ where E is a connected Lie Group and $I \subset E$ is a closed connected subgroup. Recall that V admits a unique minimal Galois' covering $\tilde{V} \to V$ called the <u>holonomy covering</u>, a locally diffeomorphic map $\delta : \tilde{V} \to W$, called the <u>developing</u> map, and a monomorphism of the Galois group π of $\tilde{V} \to V$ into E, say $h : \pi \to E$, such that $\delta(\alpha\tilde{v}) = h(\alpha)\delta(\tilde{v})$ for all $\alpha \in \pi$ and $\tilde{v} \to \tilde{V}$. A diffeomorphism f of V is called <u>isometric</u> if it locally lifts (usually non-uniquely) to some diffeomorphism $\tilde{f}$ of $\tilde{V}$, $\delta \circ \tilde{f} = e \circ \delta$ for some $e \in E$. It is not hard to see that the group Is(V) is A-rigid.

Standard examples of locally homogeneous spaces are

(i) Affine flat manifolds. Here $W = \mathbb{R}^n$ and E is the group of affine transformations of $\mathbb{R}^n$.

(ii) Projectively flat manifolds, where $W = P^n$ and $E = PL_{n+1}$,

(iii) Conformally flat manifolds of type (n_+, n_-). Here E is the orthogonal group $O(n_+ + 1, n_- + 1)$ acting on the quadric W of isotropic lines in $\mathbb{R}^{n_+ + 1, n_- + 1}$.

The isometry group Is(V) can be arbitrarily complicated if no compactness (or finiteness of volume) assumption is imposed on V. For example, for any countable group Γ one can take a domain $U \subset \mathbb{R}^4$ with $\pi_1(U) = \Gamma$ and then Γ isometrically acts on the universal covering V of U. A more convincing example is as follows. Start with some Riemannian metric g_o on a surface V, such that $Is(V, g_o) = \Gamma$ and then conformally change g_o to get a complete metric g on V with <u>constant</u> negative curvature. Then $Is(V,g) \supset \Gamma$ and it is easy to make $Is(V,g) = \Gamma$ as well.

If V is compact and the action of Is(V) on V preserves the volume, then the connected part of Is(V) is more or less manageable. The difficulty for discrete Is comes from the action of Is on the fundamental group of V and on the <u>holonomy group</u> $h(\pi) \subset E$.

All known examples of interesting discrete actions are essentially of arithmetic origin (e.g. automorphisms of nil-manifolds) and can be eventually reduced to the standard action of $SL_n \mathbb{Z}$ on the (flat affine) torus $\mathbb{R}^n/\mathbb{Z}^n$. Here is a specific

Question. Let E be an algebraic group without center and assume the action of E on W is faithful. Can a compact manifold V with Zariski dense holonomy $h(\pi) \subset E$ have (discrete) infinite isometry group ?

0.5.E. Let V be the unit tangent bundle of a complete (e.g. compact) locally symmetric space X . Consider $G = \mathbb{R}$ acting by the geodesic flow. If rank X = 1 (e.g. X has constant sectional curvature), then V admits a natural locally homogeneous structure compatible with the flow. But for rank $X \geq 2$ the manifold V is not (at least in a natural way) locally homogeneous. Yet it is partitioned into locally homogeneous "fibers" of generic codimension k = rank X . In fact, the quotient space of this partition can be easily identified with S^{k-1}/W, where S^{k-1} is the unit tangent sphere to some flat in X and W is the Weil group.

Note that this action of $\mathbb{R}$ on X admits an invariant rigid A-structure (this is easy to show) and the partition into locally homogeneous pieces is typical for such actions. (Compare 0.6. below).

0.6. Isometries of simply connected analytic A-manifolds. We shall see in 6.3. that (isometry group of) any rigid A-structure is built, in a certain way, of (the subgroups from) the above examples. The most important part of isometry comes from locally homogeneous manifolds which, unfortunately, are poorly understood. Besides, many basic facts are proven only for C^{an}-manifolds (C^{an} means real analytic). Here are some examples,

Let g be a C^{an}-smooth rigid A-structure on a compact manifold V .

0.6.A. (see 3.5.C.). If V is simply connected then the group Is = Is(V,g) has at most finitely many connected components. Furthermore, each orbit of Is is embedded in V . In fact the orbits are semianalytic subsets in V and there exists at least one orbit which is a closed C^{an}-submanifold in V .

This makes group Is very similar to those in O.5.A. and O.5.B.

O.6.B. (see 3.7.). Assume as earlier $\pi_1(V) = 0$ and let Is preserve a smooth volume element on V. Then all orbits of Is are compact and there is a connected normal Abelian subgroup $A \subset Is$, such that Is/A is compact. In particular, the group Is has polynomial growth.

This result is closely related to the following theorem by D'Ambra (see [DA]) on isometries of Lorentz manifolds that are pseudo-Riemannian manifolds of type (+--...-).

O.6.C. If V is a compact simply connected C^{an}-Lorentz manifold, then Is(V) is compact.

It is unclear if this remains true for C^∞-manifolds, but the conditions $\pi_1 = 0$ and Lorentz are essential. In fact, there are many C^{an}-metrics of type (++-------) on $S^3 \times S^3 \times S^3$ with the isometry group $T^3 \times \mathbb{R}$ (which is as in O.5.B.), but no such manifold of dimension ≤ 8 exists.

Note that O.6.C is similar to the following theorem by Obata and Lelong-Ferrand which solves a conjecture by Lichneriowicz,

If a compact Riemannian manifold V is not isoconformal to the standard sphere, then the group of conformal transformations of V is compact.

O.7. Isometries of non-simply connected manifolds. Let V be A-rigid real analytic and let us give three examples of an upper bound on Is V in terms of the fundamental group $\pi_1(V)$.

O.7.A. (See 6.2. and compare [Ti]). Let the fundamental group $\pi_1(V)$ contain no free subgroup F_2 and consider a closed subgroup $G \subset Is\ V$ preserving a smooth finite measure on V. Then

(a) the Lie algebra L(G) contains no $s\ell_2\mathbb{R}$;

(b) If G is non amenable, then the image of the natural (continuous) homomorphism of G to the group of exterior automorphisms of $\pi_1(V)$ contains F_2.

Remark. A connected Lie group G contains a closed subgroup isomorphic to F_2 if and only if L(G) contains $s\ell_2\mathbb{R}$.

0.7.B. If V is compact and the isometry group contains a connected subgroup G which has exponential growth and such that the isotropy subgroup $G_v \subset G$ is discrete for all $v \in V$, then $\pi_1(V)$ also has exponential growth. Furthermore $\pi_1(V)/F$ is infinite for every free subgroup $F \subset \pi_1(V)$. (See 6.1.).

0.7.C. Let $\mathrm{Is}(V,g)$ contain $SL_n \mathbb{R}$ for $n \geq 3$, whose action preserves a smooth finite measure on V and let $\pi_1(V)$ be (isomorphic to) a subgroup in $SL_n \mathbb{Z}$. Then $\pi_1(V)$ has finite index in $SL_n \mathbb{Z}$.

0.8. Extremal cases. There is a huge gap between available examples of A-rigid actions and our descriptive results concerning general actions. A "first approximation" conjecture is that every A-rigid action on a compact (or finite volume) manifold is "essentially" contained in the list of examples in 0.5. One can prove this conjecture in some equality cases for the inequalities $\mathrm{Is}(V) \leq \pi_1(V)$ from the previous section and for the inequalities $\mathrm{Is}(V) \leq \mathrm{Is}^{loc}(v_0)$. Here are two examples.

0.8.A. Let V be as in 0.7.C. Then V is not compact. In fact, V is "essentially" $SL_n \mathbb{R}/SL_n \mathbb{Z}$ with the natural $SL_n \mathbb{R}$-action (See 6.3. for precise statements and proofs).

0.8.B. Let (V,g) be a Lorentz manifold of finite volume with an isometric $SL_2 \mathbb{R}$-action. Then this action is everywhere locally free (i.e. the isotropy is discrete at all $v \in V$). The metric g is non-singular on the (3-dimensional) orbits and the normal subbundle to the orbits is integrable with totally geodesic leaves. Furthermore, some infinite covering $\tilde{V}$ of V is split by the lifts to V of the two (into orbits and into normal leaves) foliations. (see 5.4.).

0.9. The structure of the paper. The following §1 contains a background on local and infinitesimal geometry of rigid structures. In §2 we review some elementary properties of actions of algebraic groups on real algebraic verieties. We present, in particular, Zimmer's generalization of the Borel's density theorem. In §3, we study a kind of Gauss map h of an A-manifold V assigning to each $v \in V$ the infinitesimal isometry class of V at v. With this h we obtain a fair picture of isometries of V provided V is C^{an} and $\pi_1(V) = 0$. In §4 we discuss certain correspondences between (actions of) topological groups, called placements and mes-placements $G \to H$, which generalize monomorphisms $G \to H$ and specialize topological and measurable cocycles $G \to H$. If G is an isometry group of an A-rigid manifold, we obtain such a placement of G into the local isotropy group $Is^{loc}(v)$ for some $v \in V$. In some cases, we also have a placement $G \to \pi_1(V)$. The placement relation $G \to Is^{loc}$ is sharpened in §5 for connected isometry groups G. This allows a satisfactory local (and then global) characterization of manifolds V with "sufficiently large" isometry groups.

In the final §6 we study placements of G into $\pi_1(V)$ and into the (Zariski closure of the) holonomy group of V for A-rigid C^{an}-manifolds V.

It should be noted that two subjects discussed in my talk are not included in here due to the lack of time and space. The first is a study of geometric compactifications of (rigid) diffeomorphisms groups and of more general spaces of maps between smooth manifolds satisfying (elliptic) systems of P.D.E. The second subject deals with recurrency of the action of Diff V on the space of rigid structures on V and with geometric invariants of these actions. In fact, isometry groups, which are isotropy subgroups in Diff, provide the simplest examples of recurrency. The presence of the recurrency, say in the space of Pseudo-Riemannian metrics, makes the invariants of these quite different from Riemannian invariants (such as diameter, injectivity radius etc.).

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§1 INTEGRATION OF INFINITESIMAL ISOMETRIES.

If the isometry condition for maps $f : V \to V$ is expressed by a system of partial differential equation, then rigidity becomes equivalent to completeness of this system.

1.0. Definition (18^{th} century). A system S of P.D.E. of order $s + 1$ on V is called complete if it "expresses" each partial derivative of order $s + 1$ of unknown f in terms of derivatives of order $\leq s$.

It follows that for every smooth curve $C \subset V$ the s-jet (i.e. the string of the partial derivatives of order $\leq s$) of f satisfies along C some system of O.D.E.,

$$\frac{\partial}{\partial c} J_f^s(c) = S_C(J_f^s(c)) \qquad (*)$$

Therefore $f|C$ is uniquely determined by a single value of J_f^s on C, say by $J_f^s(v_o)$ for some $v_o \in C$. Furthermore, if V is connected, we join each v with v_o by some curve $C = C(v_o,v)$ in V and then see that the jet $J_f^s(v_o)$ uniquely determines $f(v) = f_C(v)$ for all $v \in V$. If so determined value $f(v)$ does not depend on a choice of $C(v_o,v)$, for all $v \in V$, then S is called consistent. More precisely, the consistency claims that $f_C(v)$ only depends on the homotopy class of curves C between v_o and v.

That is

$$\frac{d}{dt} f_{C_t}(v) = 0 \qquad (**)$$

for all smooth one-parameter families of curves C between v_o and v.

By expanding (**) at every point $v \in V$ for all potential solutions of S one expressed (**) by some first order system C of equations on the "coefficients" of S. This C is, in fact, equivalent to the infinitesimal of order $s + 2$ solvability of S at all $v \in V$. The implication

$$[S \text{ satisfies } C] \Rightarrow [S \text{ is locally solvable}] \qquad (***)$$

is sometimes called Frobenius'theorem. (The terminological jump to the 19^{th} century is probably due to the timing of the first general existence proof for O.D.E.).

1.1. Language of jets. Let $X \to V$ be a smooth fibration, denote by X^s the space of s-jets of (germs of) sections $V \to X$ and let $p_r^s : X^s \to X^r$ be the natural projections for all $r < s$ (including $r = 0$ corresponding to $X = X^\circ$).

Holonomy. A section $\varphi: V \to X^s$ is called holonomic if it is the jet of some section $f : V \to X$. That is $\varphi = J_f^s$ where f (necessarily) equals $p^s \circ \varphi$. An n-plane, where $n = \dim V$, in the tangent bundle $T(X^s)$ is called holonomic if it is tangent to some holonomic C^1-section $V \to X^s$. The space of all holonomic n-planes in $T(X^s)$ is canonically (and obviously) diffeomorphic to X^{s+1} .

Now, instead of systems of P.D.E. of order s we speak of P.D. relations which by definitions are subsets $R \subset X^s$. Solutions of R are sections $f : V \to X$ whose jets $J_f^s : V \to X^s$ send V into R. Equivalently, solutions are holonomic sections $\varphi: V \to R$.

1.2. Uniqueness of solutions of complete relations. Say that $R' \subset X^{s+1}$ is complete if it is homeomorphically projected by p_s^{s+1} onto some $R \subset X^s$. Call R' Lipschitz complete if the inverse map $R \to R'$ is locally Lipschitz with respect to some smooth Riemannian metrics in X^s and X^{s+1} .

1.2.A. If R' is Lipschitz complete, then for each point $x' \in R'$ there is at most one germ of continuous holonomic section $\varphi : V \to R$ passing through x' .

Here "passing through x' " means $\varphi(v) = x'$ for $v \in V$ under x' and "germs" indicates a small connected neighborhood $U \subset V$ of v where φ is actually defined.

Proof. The inversion $R \to R'$ extends near $x \in R$ under x' to a Lipschitz map $X^s \to X^{s+1}$ which defines a Lipschitz field of n-planes in X^s near x . Since the s-jet of every C^{s+1}-solution of R' is tangent to this field our claim follows from the uniqueness theorem for O.D.E. with Lipschitz coefficients.

Say that R' is C^i-complete if the map $R \to R'$ extends near each point $x \in R$ to a C^i-map $X^s \to X^{s+1}$. It is clear that

1.2.B. If R' is C^i-complete, then every continuous holonomic section $V \to R$ is at least C^{i-1}-smooth.

Denote by F^i the space of global C^j-solutions $f : V \to X$ of R' with the topology of uniform C^i-convergence on compact subsets in V . Then we recall that solutions of smooth O.D.E. continuously depend on initial data and conclude

1.2.C. If R' is C^i-complete and R is a closed subset in X^s , then the jet evaluation map $J^s : V \times F^j \to R$ for $(v,f) \to J_f^s(v)$ is a homeomorphism of $V \times F$ onto a closed subset in R for all

$$s \leq j \leq i - s - 1 .$$

1.3. Consistency and local solvability. Say that an n-plane $\tau' \in T_{x'}(X^{s+1})$ for $x' \in R'$ is tangent to R' if there exists a germ of a section $V \to R'$ passing through X' and tangent to τ'. (If R' is a C^1-submanifold, this is the usual tangency, $\tau' \subset T_{x'}(R')$). Call R' consistent if for each $x' \in R'$ there exists a holonomic n-plane tangent to R' at x'.

1.3.A. Frobenius' theorem. Let $R' \subset X^{s+1}$ be locally closed (i.e. open $\cap$ closed) C^2-complete and consistent. Then there is a germ of a holonomic C^1-solution $V \to R'$ passing through a given point $x' \in R'$.

Proof. Consider a C^2-smooth field τ of n-planes in X^s near the point $x \in R$ under x'. Take a small split neighborhood $U = U_o \times \mathbb{R} \subset V$ around $v \in V$ under x and assume by induction on n that there exists a C^2-section $\varphi_o : U_o \times 0 \to X^s$ passing through x and tangent to τ (i.e. $T(U_o \times 0) \subset \tau$). Then, by solving O.D.E., extend φ_o to a C^2-section $\varphi: U \to X^s$, such that every line $\varphi(u_o \times \mathbb{R}) \subset X^s$, $u_o \in U_o$, is tangent to τ. If R is a C^1-submanifold, then obviously $\varphi(U) \subset R$, since U is small. For general R (this case is not needed for our applications) the inclusion $\varphi(U) \subset R$ follows from the following (trivial but mildy assuming)

Lemma. Consider a vectorfield ∂ on a manifold Y an let $R \subset Y$ be a closed subset, such that every C^∞-function $f : Y \to \mathbb{R}$ which is non-negative on R satisfies $\partial f(y) \geq 0$ for all those $y \in R$ where $f(y) = 0$. Then every flow line of ∂ starting in R at $t = 0$ remains in R for $t > 0$.

Finally, a trivial check up by induction on dim V shows that $p_o^s \circ \varphi: U \to X$ is the required solution of R'.

1.4. C^{an}-relations. Let the fibration $X \to V$ be real analytic and R be the zero set of a system of C^{an}-functions $\psi_1, \psi_2, \ldots$, defined in some neighborhood $Y \subset X$ of R. Denote by Ψ the ideal spanned by $\psi_1, \psi_2, \ldots$ and call R the support of Ψ. A holonomic n-plane $\tau \subset T_x(X^s)$ for $x \in R$ is called C^{an}-tangent to R if (the differential) $d\psi|\tau = 0$ for all $\psi \in \Psi$. Now we prolong R to X^{s+1} by defining $R' \subset X^{s+1}$ as the set of (the $(s+1)$-jets corresponding to) all holonomic n-planes tangent to R. Next, we prolong Ψ to an ideal of functions on $Y' = (P_s^{s+1})^{-1}(Y) \subset X^{s+1}$ as follows. Consider the n-dimensional vector bundle over Y' whose fiber at $y \in Y'$ equals the holonomic n-plane in Y corresponding to y. Sections of this bundle are C^{an}-vectorfields in Y depending on $y \in Y'$. Now we define Ψ' as the span of the functions $P_s^{s+1} \circ \psi$ and $\partial\Psi$ for all $\psi \in \Psi$ and all sections ∂ of the above bundle. Note that R' equals the support of Ψ'.

Let R' be C^{an}-complete and say that it is also C^{an}-*consistent* if the implied lift $Y \to Y'$ pulls the ideal Ψ' back to Ψ.

1.4.A. C^{an}-*Frobenius*. *If* R' *is* C^{an}-*complete and consistent, then there is a unique holonomic* C^{an}-*germ* $V \to R'$ *through every point in* R'.

The proof is the same as in the smooth case.

1.4.B. *Linear* C^{an}-*relations*. Let X and Z be C^{an}-vector bundles over V and $\Delta : X^s \to Z$ a C^{an}-homomorphism. Denote by Ψ the pull-back to X^s under Δ of the ideal on Z defining the zero section $V \hookrightarrow Z$. Note that the solutions f of $R = \operatorname{supp} \Psi \subset X^s$ are exactly the solutions to the P.D.E. system $\Delta \circ J_f^s = 0$.

1.4.C. *Linear* C^{an}-*Frobenius*. *If* R' *is* C^{an}-complete, *then the sheaf of* C^{an}-*solutions of* R' *is locally constant on* V. *In particular, if* $\pi_1(V) = 0$, *then every local solution of* R' *uniquely extends to all of* V.

Proof. Take $R_1 = P_s^{s+1}(R') \subset R$, then $R_2 = P_s^{s+1}(R_1')$ and so on. These R_i stabilize to a certain C^{an}-relation $R_\infty \subset X^s$ for which R'_∞ is complete and consistent. Then, by a trivial argument, this R_∞ is a *subbundle* of X^s foliated by solutions of R'_∞. See $[\mathrm{Gr}]_1$ for (trivial) details.

1.5. *Completeness of the isometry relation*. Let $X = V \times V \to V$ for $(v_1, v_2) \mapsto v_1$ be the trivial fibration and let $Y^s \subset X^s$ be the (open) subset consisting of jets of *locally diffeomorphic* maps $V \to V$. Then, for a given structure g on V of order $r \leq s$ we denote by $I^s \subset Y^s$ the subset of jets of diffeomorphisms which preserve jets J_g^{s-r}. Observe that X^s and hence I^s is naturally mapped to $V \times V$ and denote by $\mathrm{Is}^s(v_1, v_2) \subset I^s$ the pull-back of a point. Note that $\mathrm{Is}^s(v,v)$ equals the infinitesimal isotropy group $\mathrm{Is}^s(v)$

(see 0.3.) and, in general, we view $Is^s(v_1,v_2)$ as the set of <u>infinitesimal isometries</u> sending $v_1 \mapsto v_2$. Observe that solutions of I^s are exactly isometries of (V,g).

Warning. In general, $I^{s+1} \neq (I^s)'$ even if g is analytic and rigid.

1.5.A. <u>If g is $(r+i)$-rigid and $C^\infty(C^{an})$-smooth, then I^{r+i+1} is $C^\infty(C^{an})$-complete. In fact, the projection $I^{r+i+1} \to Y^{r+i}$ is a $C^\infty(C^{an})$-invertible homeomorphism of $I' = I^{r+i+1}$ onto a closed subset I in I^{r+i}</u> (Compare 1.2.).

<u>Proof</u>. The group $Is^s(v_1)$ freely and transitively acts on $Is^s(v_1,v_2)$ for all v_1 and v_2 in V and all $s \geq r$. Hence, the injectivity of the map $Is^{r+i+1}(v) \to Is^{r+i}(v)$ for all $v \in V$ (see 0.3.) implies that for the map $I' \to Y^{r+i}$.

To invert $I \to I'$, we need the spaces G and $G(V) \to V$, and also the space $G^{i+1}(V) \to V$ of $(i+1)$-th jets of germs of sections $V \to G(V)$. Note that $G^{i+1}(V)$ is fibered over V, where the fiber is the $\mathcal{D}^{r+i+1}$-space G^{i+1} mentioned in 0.3. Set

$$Z' = \mathcal{D}^{r+i+1} \times G^{i+1} \times G^{i+1}$$

and let $I'_o \subset Z'$ be the <u>graph</u> of the action (map)

$$\mathcal{D}^{r+i+1} \times G^{i+1} \to G^{i+1}.$$

That is

$$I'_o = \{d,g_1,g_2 \mid dg_1 = g_2\} \subset Z' .$$

Recall the homomorphism $\mathcal{D}^{r+i+1} \to \mathcal{D}^{r+i}$, denoted $d \to \bar{d}$, with the kernel $\Delta^{r+i+1} \subset \mathcal{D}^{r+i+1}$ and consider the (trivial) fibration $Z' \to Z = Z'/\Delta^{r+i+1}$ for $(d,g_1,g_2) \to (\bar{d},g_1,g_2)$.

Since g is rigid, we may assume (see 0.3.A.) that the action of Δ^{r+i+1} is free and proper on G^{i+1}. The freedom implies (in fact, is equivalent to what follows) that the projection $Z' \to Z$ is an injective $C^\infty(C^{an})$-immersion of I'_o onto some $I_o \subset Z$. Similarly, the properness of the Δ^{r+i+1}-action shows that the projection $I'_o \to Z$ is a proper embedding. Hence, I_o is a properly embedded $C^\infty(C^{an})$-submanifold in Z and <u>the map $I'_o \to I_o$ is $C^\infty(C^{an})$-invertible</u>, by a $C^\infty(C^{an})$ extension to some neighborhood in Z).

Now, to (locally) extend the original map $I \to I'$ to $Y^{r+i} \supset I$ we locally induce the fibration $Y^{r+i+1} \to Y^{r+i}$ from $Z' \to Z$ as follows. Fix local coordinates in some neighborhoods U_1 and U_2 in V around given points v_1 and v_2 and thus obtain sections $U_j \to \mathcal{D}^{r+i+1}(V)$, $j = 1,2$, . Compose these sections with the equivariant map $\mathcal{D}^{r+i+1}(V) \to G^{i+1}$ corresponding to the $(i+1)$-th jet of the underlying structure g and thus obtain a map $U_1 \times U_2 \to G^{i+1} \times G^{i+1}$. This map induces (via the local coordinates) the fibration $Y^{r+i+1} \to V \times V$ restricted to $U_1 \times U_2$ from $Z' \to G^{i+1} \times G^{i+1}$, such that I_o' pulls back to I'. Now the projection $Y^{r+i+1} \to Y^{r+i}$ over $U_1 \times U_2$ is induced from $Z' \to Z$ and a local extension $Y^{r+i} \to Y^{r+i+1}$ of the map $I \to I'$ (over $U_1 \times U_2$) is obtained by pulling back some (local) map $Z \to Z'$ extending $I_o \to I_o'$. Q.E.D.

1.5.B. Corollary. (Compare 0.4.). The isometry group $Is = Is(V,g)$ acts freely and properly on $\mathcal{D}^{r+i}(V)$. In particular, the orbit map $Is \to Is(x_o) \subset \mathcal{D}^{r+i}(V)$ is a topological embedding for each $x_o \in \mathcal{D}^{r+i}(V)$.

1.6. Consistency and local solvability of $I = Is$. Denote by $g^j : \mathcal{D}^{r+j}(V) \to G^j$ the $\mathcal{D}^{r+j}$-equivariant map corresponding to the jet $J_g^j : V \to G^j(V)$ and denote by $\mathrm{cor}_v\, g^j$ for $v \in V$, the corank of g^j at some point $x \in \mathcal{D}^{r+j}(V)$ over v, where $\mathrm{corank}_x\, g^j$ is the dimension of the kernel of the differential $D_x\, g^j$ and where the equivariance of g^j makes this corank a function on V.

1.6.A. Lemma. If $\mathrm{cor}_v\, g^j$ is constant in $v \in V$ then the relation $I^{r+j} \subset Y^{r+j}$ is smooth submanifold and the projection $I^{r+j} \to V$ is a submersion with fibers I_v^{r+j} of dimension $= \mathrm{cor}\, g^j$. (Here g is not assumed rigid).

Proof. Since the problem is local, we assume there is a coordinate system on V. This defines a section $s : V \to \mathcal{D}^{r+j}(V)$ and also a splitting $Y^{r+j} = V \times \mathcal{D}^{r+j}(V)$. Now the relation $I^{r+j} \subset V \times \mathcal{D}^{r+j}$ is given by the equation $g^j \circ s(v) = g^j(d)$. That is

$$I^{r+j} = \{v,d \mid g^j(s(v)) = g^j(d)\} \subset V \times \mathcal{D}^{r+j}(V). \tag{*}$$

If corank g^j is constant, then g^j (locally) decomposes into a submersion followed by an immersion. This implies the Lemma where I_v^{r+j} are identified with the fibers of this submersion.

The following lemma is obvious by now.

1.6.B. Take a the holonomic n-plane $\tau \in T_y(Y^{r+j})$ corresponding to some point $x' \in I^{r+j+1} \subset Y^{r+j+1}$. If cor g^j is constant, then τ is tangent to $I^{r+j} \subset Y^{r+j}$.

Next, we invoke Frobenius' theorem (see 1.3.A.) and obtain the following

1.6.C. Corollary. If g is $(r+i)$-rigid and cor g^j = cor g^{j+1} = const for some $j \geq i+1$, then the relation I^{r+j+1} is consistent. Hence every infinitesimal isometry of order $j+1$ extends to a local isometry.

Proof. The $(r+i)$-rigidity of g (obviously) implies the $(r+k)$-rigidity for all $k \geq i$. Then C^∞-completeness of Is^{r+j+1} and 1.6.A. show that the map $Is^{r+j+1} \to Is^{r+j}$ is a diffeomorphism onto an open subset $Is' \subset Is^{r+j}$. This Is' is consistent by 1.6.B. which implies local solvability (as well as the consistency of Is^{r+j+1}).

1.6.D. Drop the assumption corank ≡const and denote by $\mathrm{Reg}_k \subset V$ the subset where cor g^k is locally constant. Note that Reg_k is an open dense subset in V since $\mathrm{cor}_v\, g^k$ is semicontinuous in $v \in V$. Let $\mathrm{Reg}_k^* \subset \mathrm{Reg}_k \cap \mathrm{Reg}_{k+1}$ consist of the points v where $\mathrm{cor}_v\, g^k = \mathrm{cor}_v\, g^{k+1}$.

1.6.E. Lemma. If g is $(r+i)$-rigid and $k > \dim \mathcal{D}^{r+i}(V)$, then the subset Reg_k^* is dense (as well as open) in V.

Proof. Let $U = \bigcap_{j=i+1}^{k+1} \mathrm{Reg}_j$ and show that $U \subset \mathrm{Reg}_k^*$. First, $(r+i)$-rigidity obviously implies $(r+j)$-rigidity for all $j \geq i$ and therefore the functions cor g^j on U satisfy,

$$\dim \mathcal{D}^{r+i}(V) \geq \mathrm{cor}\, g^{i+1} \geq \mathrm{cor}\, g^{i+2} \geq \ldots \geq \mathrm{cor}\, g^{k+1} \geq 0 .$$

Hence, for each $v \in U$ there are two consecutive numbers, say j and $j+1$ in the interval $i+1 \leq j \leq k$, such that $\mathrm{cor}_v\, g^j = \mathrm{cor}_v\, g^{j+1}$. Now, by 1.6.C. every infinitesimal isometry of order $j+1$ extends to a local one and thus to an infinitesimal isometry of order k+1 .(This conclusion is purely algebraic and rather obvious without 1.6.C.). Hence,

$$\mathrm{cor}_v\, g^j = \mathrm{cor}_v\, g^{j+1} = \mathrm{cor}_v\, g^{j+2} = \dots = \mathrm{cor}_v\, g^{k+1} .$$

Therefore $v \in \mathrm{Reg}_k^*$ and the Lemma follows.

1.6.F. Theorem (Compare 0.3.A.). If g is (r+i)-rigid, then there exists an open dense subset $V_o \subset V$, such that every infinitesimal isometry in $\mathrm{Is}^{r+k+2}(v_o,v)$ extends to a unique local isometry (sending $v_o \mapsto v$) for all $v_o \in V_o$ and $v \in V$, where k is an arbitrary integer $> \dim \mathcal{D}^{r+i}(V)$.

Proof. The above subset $\mathrm{Reg}_k^* \subset V$ is invariant under infinitesimal isometries of order k+2 since the differential Dg^j is expressible by J_g^{j+1} for all j. Therefore, the existence of some infinitesimal isometry in $\mathrm{Is}^{r+k+2}(v_o,v)$ for $v_o \in \mathrm{Reg}_k^*$ implies $v \in \mathrm{Reg}_k^*$. Now the theorem follows from 1.6.C. applied to $V_o = \mathrm{Reg}_k^*$.

1.6.G. Corollary (Compare [S]). If Is^{r+k+2} is transitive on V (i.e. the set $\mathrm{Is}^{r+k+2}(v_1,v_2)$ is non-empty for all v_1 and v_2 in V) then V is locally homogeneous, that is the pseudogroup of local isometries is transitive on V.

1.6.H. Lie structure in Is(V,g). Take a point $v_o \in V_o$ and consider the local group Is^{loc} of isometries sending $v_o \mapsto V_o$ close to v_o. This is a local Lie group of dimension $= \mathrm{cor}_{v_o}\, g^{r+k+2}$. The isometry group Is(V,g) is locally embedded into Is^{loc} with a closed (local) image. This induces a Lie group structure on Is(V,g) mentioned in §0.4.

1.7. Killing fields. Embed V to $G^i(V)$ by the jet $J_g^i : V \to G^i(V)$ of a given structure g on V and denote by $Z_i \to V$ the normal bundle of V in $G^i(V)$ that is the quotient of the tangent bundles, $Z_i = \tilde{T}_i | T(V)$, where $\tilde{T}_i = T(G^i(V))|V = J_g^i(V)$. Next, we recall the natural lift of tangent fields from V to $G^i(V)$ (corresponding to the natural action of Diff V on $G^i(V)$) and compose this lift with the quotient homomorphism $\tilde{T}_i \to Z_i$. Thus we obtain a linear differential operator of order $r+i$ on vectorfields on V, say $\Delta^i : T(V) \to Z_i$. Clearly, a vectorfield ∂ on V satisfies

$\Delta^i \circ J_g^{r+i} = 0$ if and only if ∂ generates a one-parameter pseudo-group of <u>isometries</u> of (V,g). These ∂ are called <u>Killing fields</u> on (V,g).

Denote by $T^{r+i} \to V$ the bundle of $(r+i)$-jets of sections of the tangent bundle and let $K^i \subset T^{r+i}$ be the kernel of Δ^i, where K^i is taken along with the defining ideal (see 1.4.) in the analytic case. It is clear that $K^{i+1} = (K^i)'$ (that is the prolongation of K^i) and that K^{j+1} is complete for i-rigid G and $j \geq i$ as the lift we constructed for I^j automatically lift $K^j \to K^{j+1}$. Thus by 1.4.B., <u>if</u> g <u>is</u> C^{an}<u>-rigid then the sheaf of Killing fields is locally constant and the stalk at every point</u> $v \in V$ <u>equals the fiber of</u> K_v^i <u>for all</u> $v \in V$ <u>and all sufficiently large</u> $j \geq j_o(v)$. <u>In fact</u>, <u>if</u> V <u>is compact</u>, <u>then there is a universal</u> $j_o = j_o(V,g)$.

1.7.A. <u>Corollary</u>. <u>Let</u> (V,g) <u>be compact rigid real analytic</u>. <u>Then the map</u> $g^j : \mathcal{D}^{r+j}(V) \to G^j$ has constant rank for <u>large</u> $j \geq j_o(V,g)$ and $\mathrm{cor}_v\, g^j = \dim K_v^j$.

<u>Proof</u>. It suffices to observe that (the $(r+j)$-jet of) a tangent field ∂ on V satisfies $\Delta^o \circ J_g^{r+j}(v) = 0$, if and only if the natural lift of ∂ to $\mathcal{D}^{r+j}(V)$ is annihilated by the differential of the map g^j at all $y \in \mathcal{D}^{r+j}(V)$ over v.

Now we combine this corollary with 1.6.C. and conclude

1.7.B. <u>Every infinitesimal isometry of</u> (V,g) <u>of order</u> $\geq j_o$ <u>extends to a local isometry</u>.

1.7.C. <u>Killing orbits</u>. If the sheaf of Killing fields is locally constant on an open subset $U \to V$, then it is constant on the universal covering $\tilde{U} \to U$ and the fundamental group of U acts on the Lie $\tilde{L}$ of Killing fields on $\tilde{U}$. If this action is trivial (e.g. if $\pi_1(U)$ admits no non-trivial linear representation at all), then the Killing fields are globally defined on U itself. Orbits of Killing fields on $\tilde{U}$ projected to U are called <u>Killing orbits</u> in U and every Killing vector $\tau \in T_u(U)$ defines a unique maximal (connected) Killing orbit tangent to τ. If $U = V$ and V is compact without boundary, then every Killing orbit has infinite length with respect to any fixed Riemannian metric g_o in V. Yet such an orbit is not always complete : If we follow an orbit with a tangent Killing field, denoted τ_t, then the image I of this orbit in $\mathbb{R}$ under a map sending τ_t to $\frac{\partial}{\partial t}$ may be a proper subinterval in $\mathbb{R}$. This, however, does not happen if $\sup_t \|\tau_t\|_{g_o} < \infty$, e.g. if the original structure g was Riemannian.

If $U \neq V$, then the g_o-length of a Killing orbit in U may be finite as the orbit can run into the boundary of U. Yet this does not happen if U equals the subset where the function cor g^k achieves global minimum for some $k \geq 2 + \dim \mathcal{D}^{r+k}(V)$. This completeness-like property (whose proof is easy) seems useful but no application is known at the moment.

1.7.D. A lower bound on the exceptional set. Denote by $V_1 \subset V$ the (closed) subset where the sheaf of Killing fields is not locally constant. This V_1 may be strictly smaller than the set where cor g^j is not locally constant. For example, if a structure g has no local Killing fields at all on an open dense $U \subset V$, then there is no Killing fields at the points $v \in V \setminus U$ as well, though $\mathrm{cor}_v\, g^j$ may be positive for all $v \in V \setminus U$.

Let us show, that if V_1 is non-empty, then it must be sufficiently large as explained below.

Definition. A point v in a closed subset $V_1 \subset V$ is called non-essential if there is a C^∞-map $f : B \times [0,1] \to V$, for the ball B in some Euclidean space, such that

(a) the image $f(B \times 0) \subset V$ does not meet V_1;

(b) the image $f(B \times [0,1]) \subset V$ contains a neighborhood $U_v \subset V$ of v;

(c) there exists a dense subset $B' \subset B$ such that the image $f(b \times [0,1]) \subset V$ does not intersect V_1 for all $b \in B'$.

If such an f does not exist, then v is called essential.

Example. Let $V_o \subset V$ be a compact smooth connected submanifold of positive codimension. If codim $V_o \geq 2$ or if V_o has a non-empty boundary, then all points in V_o are non-essential (here we assume that V itself has no boundary). Otherwise all points in V_o are essential.

1.7.E. Proposition. Let $V_1 \subset V$ be the (exceptional) set where the sheaf of Killing fields is not locally constant. Then all points in V_1 are essential. For example if V_1 is contained in some compact connected submanifold $V_o \subset V$ of positive codimension and with non-empty boundary, then V_1 is empty.

Proof. Consider a smooth path $p : [0,1] \to V$ where $p(0) \in V \setminus V_1$. We assume the image of p is contained in a local coordinate chart and then we have a differential equation along the path which provides a canonical prolongation of Killing fields from $p(0)$ to $p([0,1])$. The prolonged fields are Killing if $p([0,1])$ does not meet V_1, but in general they are not Killing. For example, if p is a geodesic in a pseudo-Riemannian manifold, then we can prolong Killing fields at $p(0)$ by Jacoby fields along $p([0,1])$.

Now we apply such a prolongation to the paths $f(b \times [0,1])$ for the above f, for all $b \in B$ and some Killing field defined in a neighborhood of the image $f(B \times 0) \subset V$. To simplify the matter we assume that f is an embedding (the general case can be trivially reduced to embeddings) and thus we get a smooth field on $f(B \times [0,1]) \subset V$ which is Killing on an open dense subset in $f(B \times [0,1])$ according to the property (c). Hence this field is Killing on all of $f(B \times [0,1])$ (as the field is smooth) and so it is Killing in some neighborhood of the point b. Thus we constructed a homomorphism of the stalk of our sheaf at some $v' \in V \setminus V_1$ to the stalk at v. This is clearly an isomorphism and so the sheaf is constant near v. Q.E.D.

§2. ALGEBRAIC ACTIONS.

We briefly discuss here standard facts on algebraic groups which are used in the following sections. Missing details and references can be found in $[Z]_1$.

2.1. Rosenlicht theorem. Let G be a real algebraic group regularly acting on a smooth real algebraic variety V. Then there exists an invariant Zariski open subset $V_o \subset V$, such that the quotient space V_o/G is Hausdorff. Moreover, V_o/G admits a natural structure of smooth algebraic variety such that the quotient map $V_o \to V_o/G$ is a smooth fibration.

Proof. Let m be the maximal dimension of an orbit and let $T \subset V$ be a smooth algebraic subvariety of dimension $\dim V - m$ transversal to some orbit of dimension m. By Bezout theorem, there is an integer k and an invariant open $U \subset V$, such that every orbit in U has dimension m and is transversal to T and the number of intersection points counted over $\mathbb{C}$ equals k. This defines a map of U to the k-th symmetric power of T and this map is a smooth fibration over some Zariski open subset in this symmetric power. Q.E.D.

2.2. By applying 2.1. to non-singular strata of the complement, we obtain the following

Stratification of V. There exists a finite G-invariant partition of V into smooth subvarieties,

$$V = V_0 \cup V_1 \cup \ldots \cup V_s ,$$

such that the union $V_r \cup \ldots \cup V_s$ is Zariski closed in V for all $r \leq s$ and such that the action of G on every stratum V_j is as nice as on V_0. That is the quotient V_j/G is a smooth algebraic variety and the quotient map $V_j \to V_j/G$ is a smooth fibration for all $j = 0,\ldots,s$.

2.2.A. **Corollary**. Every orbit is embedded into V.

2.2.B. **Remark**. Every orbit is a semialgebraic subset in V.

In fact, every orbit is the image of G under the (regular!) orbit map.

2.2.C. **Corollary**. The closure of each orbit contains a compact orbit.

2.3. If the action of G on V is faithful, then by 2.1. the action is free and proper on a Zariski open subset U in some Cartesian power V^N of V (Compare 0.5.B.).This immediately yields the following

2.3.A. **Furstenberg-Tits Lemma**. If the action preserves a finite Borel measure in V with a Zariski dense support, then G is compact.

2.3.B. **Corollary**. Let G preserve a finite measure μ and let $V_\mu \subset V$ denote the Zariski closure of the support of μ. (Now we do not assume $V_\mu = V$). Then the (normal algebraic) subgroup $G_\mu \subset G$ which fixes V_μ is cocompact in G (i.e. G/G_μ is compact).

A well known corollary of this is

2.3.C. BDT (Borel density theorem). Let H be a closed subgroup in an algebraic group G, such that G/H admits a finite invariant measure and let $\bar{H} \subset G$ denote the Zariski closure of H. Then $\bar{H}$ contains a Zariski closed subgroup G' which is cocompact and normal in G.

Proof. The obvious map $G/H \to G/\bar{H}$ pushes forward the measure from G/H to a G-invariant measure μ on the (algebraic!) space $G/\bar{H}$ and $G' = G_\mu$ is cocompact by 2.3.B.

2.3.C_1. Remark. Usually, BDT is stated for connected (semisimple) groups G without compact factor groups. Then it reads.

BDT_1. The subgroup H is Zariski dense in G and the connected component of the identity $H_o \subset H$ is normal in G. In particular, if G is simple, then H is discrete.

2.4. Zimmer's version of B.D.T. Let a (possibly disconnected) Lie group G act on a smooth manifold V preserving a finite measure μ on V. Denote by $L = L(G)$ the Lie algebra of G and consider the adjoint homomorphism $ad = ad_G : G \to \mathrm{Aut}\, L \subset GL_m \mathbb{R}$ for $m = \dim G$. Take the Zariski closure $G^a \subset \mathrm{Aut}\, L$ of $ad\, G$ and let $G^b \subset G^a$ be the (unique) minimal normal algebraic subgroup for which G^a/G^b is compact. Set $N^c = ad^{-1}(G^b) \subset G$ and observe that N^c is a closed normal subgroup in G containing the center of G, such that the Lie algebra $L(G/N^c)$ is compact.

Next, for each $v \in V$ we consider the isotropy subgroup $G_v \subset G$ and denote by $N_v \subset G$ the maximal subgroup such that

(a) N_v normalizes the identity component $G_v^\circ \subset G_v$;

(b) the group $ad_G N_v \subset \mathrm{Aut}\, L$ normalizes the Zariski closure $G_v^a \subset \mathrm{Aut}\, L$ of $ad_G G_v \subset \mathrm{Aut}\, L$;

(c) the group $ad_G N_v$ normalizes the identity component of the topological closure G_v^t of $ad_G G_v \subset \mathrm{Aut}\, L$.

The following theorem is a straightforward generalization of Zimmer's theorem concerning actions of semisimple groups G. (see $[Z]_3$).

2.4.A. <u>For almost all</u> $v \in V$ <u>the subgroup</u> $N_v \subset G$ <u>contains the (normal) subgroup</u> $N^c \subset G$.

<u>Proof</u>. Assign to each $v \in V$ the triple $(L(G_v), G_v^a, L(G_v^t))$. The set A of these triples is, in a natural way, a disjoint union of countably many algebraic varieties and our map, say $h : V \to A$ is Borel. Furthermore, the group ad G naturally acts on A , such that h is equivariant for this action. Now, 2.4.A. follows from 2.3.B. applied to the push-forward measure $h_*(\mu)$ on A .

2.4.A₁. <u>Remarks</u>. (a) The level of the map h at each $v \in V$, that is $h^{-1}(h(v)) \subset V$ is the fixed point set of the subgroups G_v^o , $ad^{-1}(G_v^a)$ and $ad^{-1}(G_v^t)^o$. These levels are especially nice if G preserves some geometric structure f on V . For example, if f is an affine (e.g. pseudo-Riemannian) connection, then these levels are totally geodesics submanifolds in V .

(b) If the map h is continuous at some point v in the support $V_\mu \subset V$ of μ , then $N_v \supset N^c$. The same applies to each (out of three) component of h . For example if $\dim G_v$ is constant near v, then the normalizer NG_v contains N^c .

(c) If G is a connected non-compact simple, then 2.4.A. implies that there are only three possibilities for almost all $v \in V$ (compare $[Z]_3$).

1) G_v is contained in the center of G .

2) G_v is discrete and ad $G_v \subset$ ad G is Zariski dense.

3) $G_v = G$.

(d) As another example, let G be the subgroup of affine transformations of $\mathbb{R}^m$ generated by the parallel translations and some countable subgroup $\Gamma \subset GL_m \mathbb{R}$.

<u>If</u> Γ <u>is Zariski dense in</u> $GL_m \mathbb{R}$ <u>then either</u> $G_v \supset \mathbb{R}^m$ <u>or</u> G_v <u>is discrete for almost all</u> $v \in V$.

(e) One can suppress any measure in stating (and proving) 2.4.A. by appealing directly to 2.3. rather than to the Fürstenberg-Tits lemma. For example, if G is non-compact simple acting on V then the resulting topological version of the above (c) gives us the following

Alternative. Either there exists an open dense G-invariant subset $U \subset V$, such that $G_v \subset G$ satisfies 1), 2) or 3) of (c) for all $v \in U$. Or there exists a non-empty open subset W in some Cartesian power V^N, such that W is invariant for the diagonal action of G and the action is free and proper on W. Furthermore, the same holds true for every invariant subset $V_o \subset V$.

Note that (an obvious modification of) this alternative remains true for an action of the Lie algebra $L = L(G)$ which does not necessarily "integrate" to an action of G.

2.4.B. If G is connected, the action of G on V is transitive and G_v has finitely many components, then 2.4.A. reduces to the following well known fact.

The manifold V is compact. In fact, by a theorem of Tits [Ti] V remains compact if $G_v | G_v^o$ is infinite but contains no free subgroup on two generators.

2.4.C. Assume the measure μ is smooth (i.e. everywhere Lebesgue in some local coordinates) and the action of $L = L(G)$ is infinitesimally faithful. That is the vectorfield $\ell(v)$ has no zero of infinite multiplicity for all non-zero $\ell \in L$ (compare 5.1.). For example, if the action is a.e. faithful and real analytic, then it is infinitesimally faithful (compare 0.4.C.).

2.4.C_1. Lemma. Let the dimension of the orbit $G(v) \subset G$ be constant near $v \in V$ and the isotropy Lie algebra $L_v = L(G_v)$ be normal (i.e. ad G-stable. Then L_v is nilpotent. Moreover the operator $\text{ad}\,\ell : L \to L$ is nilpotent for all $\ell \in L_v$.

Proof. Denote by Δ the Lie algebra of smooth vectorfields in V tangent to the G-orbits and vanishing on $G(v)$. Then the algebra of k-jets in Δ at v, say Δ_v^k obviously is nilpotent and L_v embeds into Δ_v^k for large k as the action is infinitesimally faithful. Similarly, one sees the nilpotency of $\text{ad}\,\ell$ (Compare 0.4.).

2.4.C_2. Remark. The condition dim G(v) is locally constant and can be (obviously) replaced by the following weaker condition: the subset

$$D_v = \{v' \in V \mid \dim G(v') = \dim G(v)\} \subset V$$

has non-zero μ-density at v.

Note that this condition is satisfied for almost all $v \in V$.

2.4.C_3. Corollary. Let the Zariski closure $G^a \subset \text{Aut } L$ of ad $G \subset \text{Aut } L$ have no cocompact normal algebraic subgroup. Then LG_v is nilpotent for almost all $v \in V$.

Proof. LG_v is a.e. normal by 2.4.A. and 2.4.C_2 applies

2.4.C_4. Proposition. Let the nil-radical R of the above $L = L(G)$ have one-dimensional center. Then G_v is discrete for almost all $v \in V$.

Proof. If $\dim LG_v > 0$, then LG_v contains the center of R for almost all $v \in V$ which would make the action non-faithful.

Example. The condition $\dim CR = 1$ is satisfied by the Heisenberg group and by the group of affine transformations of $\mathbb{R}^1$.

2.4.C_5. Remarks. (a) If $\dim CR > 1$, one can impose additional conditions which would insure the validity of 2.4.C_2. For example, if $\dim CR = 2$, the conclusion of 2.4.C_4 remains true unless $[\ell, c] = \lambda_o c$ for all $\lambda \in L$, $c \in CR$ and some constant $\lambda_o \in \mathbb{R}$.

(b) If one makes no assumption at all on G^a , one still sees with the above discussion that the identity component $G_v^o \in G_v$ is "small" for almost all $v \in V$. For example, G_v^o has polynomial growth and $G_v^o | R_v$ is compact $\times$ Abelian for some normal nilpotent subgroup $R_v \subset G_v^o$.

2.5. The structure of minimal sets. Let a connected Lie group G smoothly act on V and assume that G has a compact invariant subset V_o (instead of an invariant measure). Then we take a compact minimal invariant subset $V' \in V_o$ and apply the map h (see the proof of 2.4.A.) to V' . This map is continuous on V' and the image is a compact subset in A which is invariant and minimal for the action of G/N' on $h(V')$, where $N' = \bigcap_{v \in V'} N_v$ (compare 2.4.).

2.5.A. Suppose the adjoint group ad $G \subset \text{Aut } L(G)$ is algebraic. Then $h(V') \subset A$ is homogeneous under G/N' , since G/N' also is algebraic. Thus, the action of G on V' is induced from the action of $N_{v'}$ on the fiber $F_{v'} \subset V'$ of the map h . That is

$F_{v'} = h^{-1}(h(v'))$ and $N_{v'}$ minimally acts on $F_{v'}$.

2.5.A_1. Example. Let G be algebraic and contain no cocompact algebraic subgroup of positive codimension. Then $h(V')$ consists of a single point and $N' = G$. Hence, the isotropy Lie algebra $L(G_v) \subset L(G)$ is normal (i.e. an ideal in $L(G)$) for all $v \in V'$ .The same conclusion remains true for all G from 2.5.A. if a maximal compact subgroup $K \subset G$ fixes a point in V' .

2.5.B. Remark. If the action of G is C^{an}-smooth and $V_o \subset V$ is a compact analytic subset, then the previous discussion applies to C^{an}-minimal subsets $V' \subset V_o$ which are minimal invariant real analytic subsets.

§3. PARTITIONS OF A-MANIFOLDS INTO Is-ORBITS AND EQUIMETRIC STRATIFICATIONS.

Let V be a manifold with a C^∞-(C^{an})-smooth (possibly non-rigid) A-structure g of order r . Then for each j we have the partition of V into the orbits under infinitesimal isometries of order j , called Is^{r+j}-orbits : two points v_1 and v_2 are in the same orbit iff the set $Is^{r+j}(v_1,v_2)$ (see 1.5.) is non-empty. The quotient space for this Is^{r+j}-partition is denoted by V/Is^{r+j} . Similarly, we define Is^{loc}-orbits referring to the local isometry pseudogroup of V .

3.1. Equimetric stratification theorem. For every $j = 1,2,\ldots,$ there exists a finite partition of V into locally closed subsets,

$$V = V_o^j \cup V_1^j \cup \ldots \cup V_{s_j}^j ,$$

with the following six properties

(1) V_o^j is an open dense subset in V and the union $V_\ell^j \cup V_{\ell+1}^j \cup \ldots \cup V_{s_j}^j$ is a closed subset in V for all $\ell = 1,\ldots,s_j$.

(2) The partition is Is^{r+j}-invariant, that is each V_ℓ^j is saturated for the Is^{r+j} partition.

(3) Every quotient space V_ℓ^j/Is^{r+j} is Hausdorff. In particular all Is^{r+j}-orbits in $V_{s_j}^j$ are closed in V . These are compact

if V is compact.

(4) The V^{j+1}-partition is a refinement of V^j-partition for all $j = 1,\ldots$.

(5) The functions $\operatorname{cor} g^i$ and $\dim \mathrm{Is}^{r+i}(v)$ are locally constant on each stratum V^j_ℓ for all $i < j$.

(6) Is^{r+j-1}-orbits in V^j_o are smooth submanifolds in V^j_o . In fact, there exists a $C^\infty(C^{an})$-map h^j_o of V^j_o , $h^j_o : V^j_o \to X^j_o$ for some smooth algebraic variety X^j_o , such that corank $h^j_o = \operatorname{cor} g^{j-1}$ and the partition of V into the pull-backs of points $x \in X^j_o$ equals the Is^{j-1}-partition. Similarly, there exist smooth algebraic X^j_ℓ and $C^\infty(C^{an})$-smooth maps $h^j_\ell : V^j_\ell \to X^j_\ell$, such that the pull-back $(h^j_\ell)^{-1}(x)$-partition equals the Is^{r+j-1}-partition for all $\ell = 0,\ldots,s_j$.

Proof. The map $g^j : \mathcal{D}^{r+j}(V) \to G^j$ induces a map $h^j \to G^j/\mathcal{D}^{r+j}$. The Rosenlicht stratification of G^j induces a partition of $G^j/\mathcal{D}^{r+j}$ which pulls back to a partition of V satisfying (1),(2) and (3). The properties (4),(5) and (6) are obtained with an obvious refinement of this pulled back partition.Q.E.D.

3.1.A. Corollary. Let a group G isometrically act on V . Then

(i) If there is a dense G-orbit, then there exists an open dense Is^k-orbit for all $k \geq r$.

(ii) If the G-action is minimal (i.e. if there is no non-trivial G-invariant closed subsets) then the action of Is^k is transitive on V for all $k \geq r$.

(iii) Every compact G-invariant subset $V' \subset V$ contains another compact G-invariant subset $V'' \subset V'$, such that Is^k is transitive on V'' for all $k \geq r$.

Proof. The above partition as well as each Is^k-orbit are G-invariant. This immediately yields (i) and (ii), while (iii) is satisfied with every closed G-minimal subset $V'' \subset V'$.

3.2. C^{an}-case. If g is C^{an}, then by the proof of 3.1., the union $V^j_\ell \cup V^j_{\ell+1} \cup\ldots\cup V^j_{s_j}$ is a closed analytic subset in V and the map $h^j_\ell : V^j_\ell \to X^j_\ell$ extends to a meromorphic map of this union into some complete algebraic variety $\bar{X}^j \supset X^j$. It follows that every Is^{r+j}-orbit, say $Is^{r+j}[v] \subset V$ for all $v \in V$ is a locally closed semi-analytic subset in V .

One knows that every semianalytic subset in a compact manifold has finite Betti numbers. In particular the number of components $b_o(Is^{r+j}[v])$ is finite. In fact, the meromorphic extension of h^j_ℓ provides a universal bound. Namely, $b_o(Is^{r+j}[v]) \leq C = C_j(V,g)$ for all $v \in V$, provided V is compact. Similarly, one bounds the topology of the infinitesimal isotropy groups,

$$b_o(Is^{r+j}(v)) \leq C = C(G^j)$$

since $Is^{r+j}(v)$ equals the isotropy subgroup $\mathcal{D}^{r+j}_x \subset \mathcal{D}^{r+j}$ for some $x = x(v) \subset G^j$. In fact, all topological invariants of the spaces $Is^{r+j}[v]$, $Is^{r+j}(v)$ and $(g^j)^{-1}(x) \subset \mathcal{D}^{r+j}(V)$ for all $v \in V$ and $x \in G^j$ are uniformely bounded for compact V by Thom's equisingularity theory for (semi) analytic sets.

3.3. Rigid C^∞-case. If g is rigid, then the Is^j-partitions stabilize for $j \geq k = 2 + \mathrm{Dim}\, \mathcal{D}^{r+j}$ (see 1.6.) and equal the Is^{loc}-partition on an open dense subset $V_o \subset U^k_o$.

3.3.A. Corollary. (Compare 1.6.G.). If Is^k-partition is minimal (e.g. V admits a minimal isometric action of some group G) then V is locally homogeneous. Furthermore, if some isometry group G has a dense orbit in V , then V contains an open and dense locally homogeneous subset $V_o \subset V$.

Remark. The existence of such a $V_o \subset V$ seems a very strong condition on (V,g) which, however, is not at all understood. The first question is whether (or when) the existence of an open dense locally homogeneous subset $V_o \in V$ implies that the (A-rigid, apriori C^∞-smooth) structure g is C^{an} .

3.4. Rigid C^{an}-case. Here, Is^j-partition stabilizes to Is^{loc}-partition on every compact subset in V as $j \to \infty$. In particular, if V is compact, we obtain for some j a semianalytic stratification $V = V^j_o \cup \ldots \cup V^j_{s_j}$, such that the partition of each stratum into Is^{loc}-orbits comes from a C^{an}-map of constant rank, $V^j_\ell \to X^j_\ell$, for all $\ell = 0,1,\ldots,s_j$, extendible to a meromorphic map of V into some completion of X^j_ℓ .

This allows us to extend the infinitesimal results in 3.2. to local orbits $Is^{loc}[v]$ and isotropy groups $Is^{loc}(v)$. In particular the number of components of these is bounded.

3.4.A. <u>If</u> V <u>is compact, then</u> $b_o(Is^{loc}[v])+b_o(Is^{loc}(v)) \leq C(V,g)$, <u>for all</u> $v \in V$.

Here is another useful property of compact rigid C^{an}-manifolds,

3.4.B. <u>The closure of every</u> Is^{loc}<u>-orbit contains a compact</u> Is^{loc}<u>-orbit</u>.

3.4.C. <u>Manifolds without Killing fields</u>. If a rigid manifold (V,g) carries no local Killing field, then the local orbits and local isotropy groups are discrete which implies discreteness of the isometry group $Is = Is(V,g)$. Furthermore, if V is compact and g is a C^{an}-structure of algebraic type, then 3.4.A. implies

3.4.C_1. <u>The group</u> Is <u>is finite</u>.

<u>Remarks</u>. (i) It is unclear whether this holds true for C^∞-smooth A-manifolds.

(ii) No finiteness conclusion can be obtained from local information for non-A-structures.

(iii) If V is non-compact and g is A-rigid, then one sees with 3.4.A. that the orbit map $f \mapsto f(x_o)$ for $f \in Is$ is a <u>proper</u> map $Is \to V$ for all $v_o \in V$. Yet it is unclear whether the action of Is on V is proper. Here is the simplest potential counterexample which does not work.

(iv) <u>Example</u>. Let $V = \mathbb{R}^2 \setminus \{o\}$ and let $\mathbb{Z}$ act on V by $(x,y) \mapsto (2x, \frac{1}{2}y)$. This action has proper orbits but is not proper. Let us show that every C^{an}-smooth $\mathbb{Z}$-invariant function $g(x,y)$ necessarily is $\mathbb{R}^\times$-invariant for the action $(x,y) \mapsto (tx, t^{-1}(y))$. In fact, if g is continuous, it is constant on the lines $\{xy = 0\}$ as for any two points $(x_o, 0)$ and $(0, y_o)$ there are points arbitrarily close to these two and lying on the same $\mathbb{Z}$-orbit. Hence $g = c_o + xy\, g_1$. Then by induction

$$g = c_o + xy(c_1 + xy(c_2 + \dots)) ,$$

which implies the $\mathbb{R}$-invariance of g in C^{an}-case.

3.4.C_2. Extension problem. Let V and V' be connected A-rigid C^{an}-manifolds of the same dimension with structures of the same type, where V is compact without boundary and V' is simply connected. Take a connected open subset $U' \subset V'$ and let $f_o : U' \to V$ be an isometric immersion.

Question. Under what condition does this f_o extend to an isometric immersion $f : V' \to V$?

There is no extension, for example, if V is affine flat non-complete and $V' = \mathbb{R}^n$. On the other hand, one can show that f does exist for many types of structures (including affine and (n_+,n_-)-conformal), provided V has no local Killing field. Probably the extension is possible if V has trivial holonomy (compare 3.5. below).

3.5. Manifolds with trivial holonomy. Call a rigid manifold (V,g) regular if the sheaf K of Killing fields on V is locally constant. Recall (see 1.7.) that C^{an}-manifolds are regular and every C^∞-manifold contains an open dense regular subset. If V is regular, there is a unique minimal Galois covering, called the holonomy covering, $\tilde{V} \to V$, such that K lifts to a trivial sheaf over V , say $\tilde{K} \times \tilde{V}$, where $\tilde{K}$ is the Lie algebra of Killing fields on $\tilde{V}$.

Example. Let V be the flat Riemannian torus T^n . Then $\tilde{V} = \mathbb{R}^n$ and $\tilde{K}$ is the algebra of all (including rotations) infinitesimal rigid motions of $\mathbb{R}^n$.

The Galois group π of the covering $\tilde{V} \to V$ faithfully acts on $\tilde{K}$ by automorphisms of $\tilde{K}$ and the image $\pi \subset \operatorname{Aut} \tilde{K}$ is called the holonomy group of V . The Lie algebra of (global) Killing fields on V is canonically isomorphic to the subalgebra in $\tilde{K}$ fixed by π .

Now we concentrate on the case where π is trivial. This is so, for example, if the fundamental group $\pi_1(V)$ is trivial, or generally, if π_1 admits no linear representation of dimension $\leq \dim \tilde{K}$.

3.5.A. Theorem. Let (V,g) be a connected C^∞-smooth rigid A-manifold. Suppose V is regular (e.g. V is C^{an}), the holonomy is trivial (e.g. $\pi_1(V) = 0$) and all Killing fields on V are integrable (e.g. V is a closed manifold). Then there exists an open dense subset $U \subset V$ invariant under the group $Is = Is(V,g)$ and a C^∞-map $h : U \to W$, where W is some smooth manifold, such that

(i) h is Is-invariant on U.

(ii) h has locally constant rank.

(iii) the connected component Is_o of the identity in Is acts transitively on each connected component of the pull-back $h^{-1}(w) \subset U$ for all $w \in W$.

Furthermore, the isotropy subgroup $Is_u \subset Is$ has finite index in $Is^j(u)$ for all $u \in U$ and all sufficiently large j. (Namely, $j \geq 2 + \dim \mathcal{D}^{r+i}(V)$, compare 1.6.). In particular, Is_u has at most finitely many connected components.

Proof. Under our assumptions $Is_o = Is_o^{loc}$, and so the set $U = V_o^j$ (see 3.1.) is O.K. for large j.

3.5.B. Corollary. Every Is-orbit of u is properly embedded into U for all $u \in U$ and the isotropy subgroup $Is_u \subset Is$ is naturally embedded into $\mathcal{D}^{r+i} = \mathcal{D}_u^{r+i}(U)$ (here g is an i-rigid structure of order r) as an algebraic subgroup in $\mathcal{D}^{r+i}$ for all $u \in U$. Moreover, the number of connected components in Is_u is uniformely bounded on U.

Let us assume in addition to 3.5.A. that V is a compact C^{an}-manifolds. Then we have the following stronger

3.5.C. C^{an}-Theorem. The group $Is(V,g)$ has at most finitely many connected components, and each $Is_v \subset Is^{r+j}(v) \subset \mathcal{D}^{r+j}$ has finite index in $Is^{r+j}(v)$ for $j \geq j_o(V,g)$. Furthermore, every orbit of Is is embedded into V and is a locally closed semianalytic subset in V. Moreover, the closure of each non-compact Is-orbit, say $Is(v_o) \subset V$, contains a compact orbit of dimension $< \dim Is(v_o)$.

The proof is immediate with 3.4.

3.5.D. Corollary. Let $Is(V,g)$ contain no compact subgroup of positive dimension. Then the connected component of the identity $Is_o \subset Is$ is a simply connected solvable group and the action of Is on V has a finite orbit.

Proof. The maximal compact subgroup $K_o \subset Is_o$ acts transitively (see the explanation below) on each compact orbit of Is_o. Hence, there exists a point $v_o \in V$ fixed by Is_o whose Is-orbit is finite. The existence of a fixed v_o also rules out the universal covering $\widetilde{SL}_2 \mathbb{R}$ among subgroups of Is_o. Therefore Is_o is simply connected solvable.

3.5.D'. Explanation. The maximal compact subgroup K in a connected Lie group G acts transitively on every compact homogeneous space G/H, provoded H is a connected subgroup in G.

Indeed, K and G have some homological dimension and since G/H is compact and H is connected, this dimension satisfies

$$\dim \hom G = \dim \hom H + \dim G/H .$$

On the other hand, every K-orbit X in G/H satisfies

$$\dim X \geq \dim \hom K - \dim \hom H$$

as $\dim \hom H \geq \dim \hom H \cap K$. Hence, $\dim X = \dim G/H$. Q.E.D.

3.6. Disjoint sums of geometric structures and A-subgroups in $Iq(V,g)$. Given two structures g_1 and g_2 on V, we denote by $g_1 \dotplus g_2$ the structure which is given by the pair (g_1, g_2). The type of $g_1 \dotplus g_2$ is the Cartesian product of the corresponding g_1 and g_2.

If G is a connected group of isometries of (V,g) then, besides g, the action of G preserves some natural A-structures associated to G. (Compare 5.1.).

3.6.A. Examples. (i) Let the action of G be free. Then G preserves the subbundle of vectors in $T(V)$ tangent to the orbits of G.

(ii) Let L_o be a linear subspace in the Lie algebra $L(G)$ invariant under ad G. Then G preserves the subbundle in $T(V)$ corresponding to L_o (assuming the vectors in T_o correspond to linearly independent vectors in V).

(iii) Let the above L_o be in the center of $L(G)$. Then G preserves (i.e. commute with) the vectorfields in V coming from L_o.

3.6.B. Definition. Call some $H \subset Is(V,g)$ an A-subgroup if there exists an A-structure g_1 on V, such that $H = Is(g \dotplus g_1)$.

3.6.B_1. Examples. (i) The centralizer of Is_o in Is (Is_o is the connected component of the identity) is an A-subgroup in Is .

(ii) Let $A_o \subset Is_o$ be a maximal connected Abelian subgroup. Then the centralizer of A_o in Is is A . In particular, the normalizer of the identity component of the isotropy subgroup $Is_v \subset Is$ is an A-subgroup for all $v \in V$.

The results of the previous section obviously generalize to A-subgroups in $Is(V,g)$. Here is a corollary.

3.6.C. Let (V,g) be a compact A-rigid C^{an}-manifold with trivial holonomy (e.g. $\pi_1(V) = 0$), let $A_1 \subset Is(V,g)$ be a one parameter subgroup and consider the A_1-orbit $X_1 \subset V$ of some point $v \in V$. Then, there are only the following four possibilities.

(i) X_1 consists of a single point.

(ii) X_1 is an embedded circle $S' \subset V$.

(iii) The closure of X_1 in V is homeomorphic to an m-torus for some $m > 1$. In this case there exists an m-torus $T^m \subset Is(V,g)$ whose v-orbit equals the closure of X_1 and A_1 acts on this T^m-orbit by an irrational rotation of T^m .

(iv) The orbit X_1 is homeomorphic to $\mathbb{R}$ and is embedded into V . Then X_1 is locally closed in V and the closure of X_1 contains an orbit of one of the types (i), (ii) or (iii).

Proof. Take a maximal connected Abelian subgroup $A_o \subset Is_o(V,g)$ containing A_1 and apply 3.5.C. to the centralizer $A \subset Is_o(V,g)$ of A_o (which is just a finite extension of A_o). Then X_1 lies in the A_o-orbit of v that is a cylinder $T^m \times \mathbb{R}^k \subset V$, and X_1 is, essentially, a one parameter subgroup in this cylinder. The proof is concluded by recalling that the isotropy subgroup of A_o must have finitely many component. This yields,

$$T^m \times \mathbb{R}^k = A_o \,/\, (\text{connected subgroup in } A_o) .$$

3.6.C_1. Remark. (a) It is clear that the type(iv) orbits constitute an open subset in V , which is either dense in V or empty, provided V is connected. The complement of this set, say $V' \subset V$ is A_o-invariant. In fact, V' is invariant under the centralizer of $A_1 \subset Is(V,g)$. Furthermore, V' is an analytic subset in V .

It consists of those $v \in V$, where the tangent field corresponding to A_1 is tangent to an orbit of the maximal torus $T^\ell \subset A_o$. Also note that every finite A_1-invariant measure in V has support in V', as it is clear from the definition of $V \setminus V'$.

(b) One can generalize 3.6.C. to an arbitrary connected group A_1 isometrically acting on an A-rigid (possibly non-compact) C^{an}-manifold V with trivial holonomy. The picture is more complicated in the general case but one gets a fair idea of what happens by looking at the following examples;

1) V_1 is a Lie group and A_1 is a (possibly non-closed) subgroup in V_1.

2) V_2 is an algebraic manifold and A_1 is an algebraic transformation group of V_2.

3) $V_3 = V_1 \times V_2$ with the diagonal action of A_1.

3.7. Measure preserving isometries. Let V be an A-rigid C^∞-manifold and μ be a measure on V with support V. Suppose the isometry group Is of V preserves μ.

3.7.A. Theorem. If V is regular (e.g. C^{an}), the holonomy is trivial (e.g. $\pi_1(V) = 0$) and the Killing fields on V are integrable (e.g. V is compact) then,

(i) The orbit Is(v) is compact for all $v \in V$.

(ii) The group Is has at most finitely many connected components.

(iii) The identity component $Is_o \subset Is$ contains a connected Abelian normal subgroup $A \subset Is_o$, such that Is_o/A is compact.

Proof. Recall the open dense Is-invariant subset $U \subset V$ from 3.5.A. and the Is-invariant map $h : U \to W$. Then for almost all $u \in U$, the measure on the fiber $F_u = h^{-1}(h(u)) \subset U$ and hence, such a measure μ_u on the orbit Is(u) which is (see 3.5.A.) is the union of some connected components of F_u. (see Explanation I below). Since Is_u has finitely many components, we derive from $\mu_u\, Is(u) < \infty$ the compactness of Is(u) for almost all u (see 2.4.B.) as well as the finiteness of the number of components in Is. Hence, for almost all $v \in V$ the maximal compact subgroup in Is_o is transitive on Is(v) (see 3.5.D'.) which implies, by continuity, this property for all $v \in V$. This completes the proof of (i) and (ii).

To prove (iii) we recall the map $h : V \to A$ from 2.4.A. for $G = Is_o$. The group G acts on $h(V) \subset A$ via a factor-group, say $G_* = G/N = K \times \mathbb{R}^m$ where K is compact, and each G_*-orbit in $h(V)$ is compact. Take some $v_i \in V$, $i = 1,\dots,k$, let $N_i \subset G$ denote the normalizer of the identity component G_i^o of the isotropy subgroup $G_{v_i} \subset G$, and observe that the subgroup $H_* \subset G_*$ fixing $h(v_1),\dots,h(v_k)$ equals H/N for $H = \bigcap_{i=1}^{k} N_i \subset H$. Since the subgroups N_i are A (see 3.6.B_1.) their intersection H also is A and by applying (i) to H we conclude that the orbits of H (which come form orbits of H) are compact. Then it follows (see Explanation II below) that G/N is compact.

Finally, since $G = Is_o$ acts faithfully on V, the group N embeds into a finite Cartesian product of <u>compact</u> groups $N/G_i^o \cap N$ (see 2.4.B_1) and so $N = K_o \times \mathbb{R}^{m_o}$ <u>for a compact group</u> K_o. <u>Since</u> G/N <u>is compact</u>, <u>the maximal connected normal Abelian subgroup in</u> N <u>is normal and co-compact in</u> G. Q.E.D.

3.7.A_1. <u>Explanation I</u>. Let $h : U \to W$ be a continuous map between manifolds and μ be a probability measure on U. Denote by μ_* the push-forward of μ to W. Then, for almost all $w \in W$ there exists a unique (a.e.) probability measure μ_w on $h^{-1}(w) \subset U$, such that

$$\mu(A) = \int_W \mu_w\,(A \cap h^{-1}(w))\,d\mu_*$$

for all Borel subsets $A \subset V$. This is the first (and nearly the last) theorem of general measure theory.

3.7.A_2. <u>Explanation II</u>. Let a connected Lie group H_o act on some W with compact orbits, such that the subgroup $H_* \subset H_o$ fixing some points $w_1,\dots,w_k$ in W also has compact orbits for all finite systems of points in W. Then, obviously, H_o is compact.

3.7.A_3. <u>Remark</u>. Theorem 3.7.A. shows that the action of Is_o on V is essentially the one described in Example 0.5.B. Namely, the maximal torus $T^m \subset A$ is central in Is_o and the orbits of A_o equal those of T^m. Therefore, the action of Is_o is determined by the action of the maximal compact subgroup $K_o \subset Is_o$ on V and by an

ℓ-dimensional, for $\ell = \dim A/T^m$, linear subspace Z in the space of $C^\infty(C^{an})$-maps of V into the Lie algebra $L(T^m) = \mathbb{R}^m$, such that Z is K_o-invariant and fixed by T^m. Each $z \in Z$ naturally defines a vectorfield on V (tangent to T^m-orbits) and Is_o is generated by K_o and these fields.

3.7.B. Let us indicate a generalization of 3.7.A. where we replace the assumption that the support of μ, called $V_\mu \subset V$, equals V, by the following. There exists a dense subset $U_\mu \subset V_\mu$ such that for all $u \in U_\mu$, the function cor g^k (see 1.6.) is locally constant on V at u for infinitely many values of k. Then

(i)' The orbit $Is(v)$ is compact for all $v \in V_\mu$.

(ii)' The group Is has finitely many components.

(iii)' Let $Is_{V_\mu} \subset Is$ be the (normal) subgroup fixing V_μ. Then the factor group Is/Is_{V_μ} contains a normal connected and co-compact Abelian subgroup.

The proof is the same as of 3.7.A. Note that 3.7.B. applies to the (trivial) case of a δ-measure μ, where 3.7.A. does not apply. Another (only slightly less trivial) application of 3.7.B. is

3.7.B_1. <u>Let</u> V <u>be a compact</u> A-<u>rigid</u> C^{an}-<u>manifold with trivial holonomy</u>. <u>Then every diffeomorphism</u> $f \in Is(V)$ <u>has zero topological entropy</u>.

Note that 3.7.B_1 can be also derived from 3.6.C. In fact, the orbit study in 3.6.C. leads to another proof 3.7.B.

3.7.C. There are other generalizations (besides 3.7.B.) of 3.7.A. For example, a large part of the conclusion of 3.7.A remains true if one allows non-integrable Killing fields. One may also relax the holonomy condition by only requiring that the holonomy group trivially acts on the <u>centralizer</u> (in the algebra of local fields) of $L(G)$ (compare 5.2.). Another class of manifolds where a version of 3.7.A. holds true are those where the holonomy group is precompact as it acts on local Killing fields. On the other hand, it is unclear how to remove the regularity condition in 3.7.A.

§4. PLACING GROUPS INTO GROUPS.

A homomorphism $h : H \to G$ defines a left action of H on G which commutes with the right action of G on itself. This suggests the notion of a virtual homomorphism of H to G which is given by a space X (in an appropriate category) and by commuting actions of H and G on X. Such homomorphisms have been extensively studied in topology (see [B-C]) and in measure theory (see $[Z]_1$, [C-S] where they are called measurable cocycles). Here we indicate basic (and for the most part well known) properties of a special class of virtual homomorphisms called placements (or placings) $H \to G$.

Pertinent examples are

(1) The lift of an action of H from V to $\mathcal{D}^k(V)$, where it commutes with the action of $\mathcal{D}^k$.

(2) Certain lifts of H to some normal coverings $\tilde{V} \to V$, such that the lifted action commutes with deck transformations

4.1. Uniformely bounded maps and placements. A (possibly discontinuous) map between metric spaces, $f : X \to Y$ is called UB if

$$\left[\mathrm{dist}_Y(f(x_i), f(x_i')) \underset{i\to\infty}{\to} \infty\right] \Rightarrow \Rightarrow \left[\mathrm{dist}_X(x_i, x_i') \to \infty\right], \qquad (*)$$

for all sequences x_i and x_i' in X.

4.1.A. Example. Let H and G be locally compact groups with proper left invariant metrics, where "proper" means compactness of all closed balls of finite radius. Clearly, every continuous homomorphism $H \to G$ is U.B. It is equally clear that the UB property of a map $H \to G$ does not depend on a choice of proper left invariant metrics in G and H.

4.1.A$_1$. Warning. If dist_ℓ is left invariant on G and dist_r is right invariant, then the identity map $(G, \mathrm{dist}_\ell) \to (G, \mathrm{dist}_r)$ is not, in general, UB. But the map $g \to g^{-1}$ is U.B.

4.1.B. Call a UB map $f : X \to Y$ a UB-equivalence if there exists a UB-map $f_* : Y \to X$ such that the selfmaps ff_* and f_*f are translations which means

$$\mathrm{dist}(x, f_* \circ f(x)) \le \mathrm{const} < \infty$$

for all $x \in X$ and the same for $f \circ f_*$.

4.1.B'. Example. Let $H \quad G$ be a closed cocompact (i.e. G/H is compact) subgroup. Then G and H are UB-equivalent.

4.1.C. A UB map $f: X \to Y$ is called a placing (or placement) if

$$\left[\mathrm{dist}_X(x_i, x_i') \underset{i \to \infty}{\to \infty}\right] \Rightarrow$$
$$\Rightarrow \left[\mathrm{dist}_Y(f(x_i), f(x_i')) \underset{i \to \infty}{\to \infty}\right] \qquad (**)$$

for all x_i and x_i' in X (compare (*) above).

Clearly, every U.B. equivalence is a placing.

4.1.D. Example. A continuous homomorphism $H \to G$ is a placing iff the kernel is compact and the image is closed.

4.1.C. Main example. Consider proper commuting actions of G and $\bar{G}$ on a space X. If $X/\bar{G}$ is compact, then there exists a placing $G \to \bar{G}$.

Proof. Fix a compact subset $K \subset X$, such that $\bar{G}K = X$ and take some $x_o \in K$. Assign to each $g \in G$ some $\bar{g} \in \bar{G}$ for which $(\bar{g}g)^{-1} x_o \in K$. Then we have the following implications for any two sequences $g = g_i$ and $h = h_i$ in G,

$$\bar{g}^{-1}\,\bar{h} \to \infty \Leftrightarrow \bar{h}^{-1}\,\bar{g} \to \infty \Leftrightarrow$$
$$\bar{h}^{-1}\,\bar{g}(\bar{g}g)^{-1}\,x_o \to \infty \Leftrightarrow g^{-1}\,\bar{h}^{-1}\,x_o \to \infty$$
$$\Leftrightarrow g^{-1}\,h(\bar{h}h^{-1})\,x_o \to \infty \Leftrightarrow g^{-1}\,h \to \infty .$$

Q.E.D.

4.1.C_1. Remarks (a) In the more general case where the action of G is not necessarily proper, the above argument shows that $g \mapsto \bar{g}$ is a UB-map $G \to \bar{G}$.

(b) Let $x_o : G \to \bar{G}$ be a placing and consider the set X_o of the $(\bar{G} \times G)$-translates of x_o. This X_o admits a metric, such that the actions of G and $\bar{G}$ are proper on the completion X of X_o and X/G is compact. Thus every placing comes from commuting actions on some X .

4.1.D. Corollary. Let G be a k-rigid action on V . Then G can be placed into $\mathcal{D}^k$. Similarly, if the action of G lifts to a proper action of some Galois' covering $\tilde{V} \to V$ and the lifted action commutes with deck transformation, then G can be placed into the Galois group of $\tilde{V}$.

4.2. Uniform dimension. A subset $X_o \subset X$ is called a net if $\mathrm{dist}(x, X_o) \leq \mathrm{const} < \infty$ for all $x \in X$. Clearly, X_o is a net iff the inclusion $X_o \hookrightarrow X$ is a U-equivalence. A metric space $Y \supset X$ is called a thickening of X if $\mathrm{dist}_X = \mathrm{dist}_Y | X$ and X is a net in Y .

Definition. The dimension Udim X is the minimal number $d = 0, 1, \ldots, \infty$, such that for each net $X_o \subset X$ every thickening $\bar{X}$ of X_o which is a finite dimensional polyhedron can be compressed to a d-dimensional polyhedron Y . That is, there exists a continuous map $p : \bar{X} \to Y$, such that $\mathrm{Diam}\, p^{-1}(S) \leq \mathrm{const} < \infty$ for the stars $S \subset Y$ of all simplices in Y .

Clearly, Udim is a U-invariant. Moreover, if X_1 admits a placing in X_2 , then $\mathrm{Udim}\, X_1 \leq \mathrm{Udim}\, X_2$.

A metric space X is called U-contractible (or geometrically contractible, see $[\mathrm{Gr}]_2$) if there exists a function $\rho(\delta) = \rho_X(\delta)$, such that every bounded subset $X_o \subset X$ is contractible within its ρ-neighborhood $X_\rho \subset X$ for $\rho = \rho(\mathrm{Diam}\, X_o)$.

4.2.A. Example. If X is contractible and if the isometry group is cocompact on X (i.e. $X/\mathrm{Is}(X)$ is compact), then X is U-contractible. Thus the universal covering of a compact manifold is U-contractible iff it is contractible. More generally, if all leaves of a foliation of a compact manifold are contractible, then the leaves are U-contractible for the Riemannian metric induced from the ambient manifold.

The following proposition allows one to evaluate Udim in the above examples.

4.2.B. If X is a d-dimensional U-contractible polyhedron, then $\mathrm{Udim}\, X \leq d$. Furthermore, if X is a manifold (or pseudo-manifold without boundary), then $\mathrm{Udim}\, X \geq d$.

The (obvious) proof can be found in $[Gr]_2$.

4.2.C. One has with 4.2.A. and 4.2.B. many examples of spaces and groups admitting no U-placing one into another . For example, if G is a connected Lie group, then it isometrically acts on G/K for the maximal compact subgroup K in G . Since G/K is a contractible manifold, it is U-contractible and so Udim G= Udim G/K = dim G/K. Thus G_1 admits no U-placing into G_2 for $\dim G_1/K_1 > \dim G_2/K_2$.

In general, if X_1 and X_2 are U-contractible pseudo-manifolds, where X_1 is placed into X_2 , then $\dim X_1 \leq \dim X_2$. Furthermore, (by a trivial argument, compare $[Gr]_2$) if $\dim X_1 = \dim X_2$ then every placing $X_1 \to X_2$ is a U-equivalence.

4.2.D. The above statement has a simple (and well known) version in the category of proper G-spaces fibered upon a given G-space V . Namely, let $X \to V$ be a fibration and G properly act on X , such that the fibers go to fibers under the action. Let V be compact, the fibers of $X \to V$ be contractible locally polyhedral spaces of dimension N and let G be U-equivalent to an M-dimensional U-contractible manifold. Then $N \geq M$ and the equality $M = N$ implies X/G is compact, provided the fibers of $X \to V$ are pseudo-manifolds.

The proof is an exercise in the elementary obstruction theory and is left to the reader.

4.3. Growth and rank. There are other invariants besides Udim which are monotone under placings The simplest is the growth of a group or of a manifold. Namely, if V is a complete Riemannian manifold, one defines Vol(V,R) as the supremum of the volumes of the balls in V of radius R . If G is a connected Lie group, one defines this volume growth function for a left invariant Riemannian metric on G . If G is discrete with finitely many generators, then Vol(G,R) signifies the number of $g \in G$ representable by words of length $\leq$ R.

Recall that the growth of every connected Lie group G is either polynomial or exponential. Namely, there are only two possibilities,

(1) $\mathrm{Vol}(G,R) \leq R^d + \text{const}$, for some $d \leq (\mathrm{U}\dim G)^2$.

(2) $\mathrm{Vol}(G,R) \geq C^R - \text{const}$, for some $C > 1$.

Proof. Consider the adjoint action of the Lie algebra L of G .

If the spectrum of some $\ell \in L$ contains a point $a + \sqrt{-1}\, b \in \mathbb{C}$ with $a \neq 0$, then G contains a solvable subgroup $G_o \subset G$ of exponential growth (where $\dim G_o = 2$ if $b = 0$ and $\dim G_o = 3$ for $b \neq 0$). Otherwise, G has polynomial growth of degree $\leq$ $(U\dim G)^2$.

If G is discrete finitely generated, then it is polynomial if and only if there is a nilpotent subgroup $N \subset G$ of finite index (see $[Gr]_1$). In this case, the degree d is bounded by $(U \dim G)^2$.

If G contains a non-Abelian free group, then, obviously, the growth is exponential. There are examples (see [G]) of finitely generated groups with the growth function like $\exp \sqrt{R}$, but all known finitely presented groups are either polynomial or exponential.

4.3.A. If G_1 can be placed into G_2 then the growth of G_2 dominates that of G_1. In particular, if G_2 is polynomial or subexponential, then G_1 also has such growth.

The proof is obvious.

4.3.A_1. Remark. The same monotonicity of growth is satisfied by placings between complete Riemannian manifolds, whose Ricci curvature is away from $-\infty$. That is Ricci $\geq$ const $> -\infty$.

4.3.B. One can define, in a certain way, the rank of a metric space X and show this rank to be monotone under placings. We shall only give here the following

Example. Let V be a complete simply connected Riemannian manifold with pinched negative curvature, $-\infty < -C_1 \leq K(V) \leq C_2 < 0$ (this signifies rank $V = 1$). Let G_1 and G_2 be infinite finitely generated groups, such that $G_1 \times G_2$ can be placed into V. Then G_1 and G_2 have polynomial growth.

The idea of the proof. Study placings $f : \mathbb{R}^2 \to V$ and show that $\mathrm{dist}(f(x), f(y) \leq C_1 \log(1+\mathrm{dist}(x,y)) + C_2$ for some constants C_1 and C_2 and all points x and y in $\mathbb{R}^2$ which are sufficiently far apart.

4.4. Rigid actions. Consider a k-rigid (see 0.4.) action of G on V preserving some compact subset $V_o \subset V$. Then G can be placed into the group $\mathcal{D}^k$. Therefore (compare 0.4.A.)

$$U\dim G \leq U\dim \mathcal{D}^k = \dim \mathcal{D}^k - \frac{n(n-1)}{2},$$

for $n = \dim V$. In particular, G contains no closed subgroups isomorphic to the fundamental group of a compact aspherical manifold of dimension > $U\dim \mathcal{D}^k$. For example, a compact V cannot be k-rigidly acted upon by $\mathbb{Z}^N$ for $N > U\dim \mathcal{D}^k$.

Another restriction on G comes from the notion of rank which we do not define in this paper. One can show, for example, that G contains no closed subgroup isomorphic to the Cartesian product of $m > n$ copies of the free group F_2 . Note that the torus T^n admits a 2-rigid action by the product of n free groups F_2 , as T^n is acted upon by n commuting $SL_2\mathbb{R}$.

4.4.A. A-actions. Suppose the action of G preserves an A-structure. Then the infinitesimal isometry "group" Is^j is transitive on each minimal set $V' \subset V_0$ for all $j = 1,2,\ldots$. (see 3.1.). Every Is^j-orbit in $\mathcal{D}^j(V)|V'$ is a principle bundle over V' whose fiber is the infinitesimal isotropy group $Is^j(v)$. The action of G on $\mathcal{D}^j(V)$ preserves these orbits and commute with $Is^j(v)$. Therefore, the group G is placed into $Is^j(v)$ for all $j \geq k$ and all $v \in V'$. Furthermore, if the structure in question is C^{an}-rigid, then G is placed into $Is^{loc}(v)$.

4.4.A_1. Corollaries. (a) If G is non-compact, then $Is^j(v)$ also is non-compact for all $v \in V'$ and $j \geq k$.

(b) If G has exponential growth, then so has $Is^j(v)$ for $j \geq k$.

(c) $U\dim Is^j(v) \geq U\dim G$ for $j \geq k$.

(d) If G is isomorphic to the Cartesian product of m free groups F_2 then the $\mathbb{R}$-rank of $Is^j(v)$ is at least m .

Recall that the $\mathbb{R}$-rank of an algebraic group is the dimension of the maximal $\mathbb{R}$-split torus. Note that $\operatorname{rank}_{\mathbb{R}} Is^j(v) \leq \operatorname{rank}_{\mathbb{R}} Is^1(v)$ for all $j \geq 1$.

4.4.A_2. Examples. (1) Let the structure be a conformal one of type (n_+,n_-) for $n_+ + n_- = n = \dim V \geq 3$. Then every closed subgroup in $Is(V)$ is 2-rigid and (by an easy argument)

$$U\dim Is^j(v) \leq U\dim O(n_+ +1, n_- +1) = n_+ n_- + n_- + n_- + 1 ,$$

and

$$\operatorname{rank}_{\mathbb{R}} Is^j(v) \leq \operatorname{rank}_{\mathbb{R}} O(n_+ +1, n_- +1) = \min(n_+ +1, n_- +1).$$

for all j and $v \in V$.

This makes the above (c) and (d) more specific for conformal structures.

(2) Let $\dim V = 2m + 1$ and the structure in question be the <u>contact</u> one. The contact structure is not rigid and the rigidity of G must be either assumed or derived from another source. The groups $Is^j(v)$ satisfy,

$$U\dim Is^j(v) = \dim Is^j(v) - \dim U(m) =$$

$$= \frac{n(n+1)}{2} + \dots + \frac{(n+j)!}{(n-1)!j!}, \text{ for } n = 2m-1,$$

and

$$\operatorname{rank}_{\mathbb{R}} Is^j(v) = \operatorname{rank}_{\mathbb{R}} GL_m\mathbb{C} = m .$$

(3) Take the standard n-frame field on $\mathbb{R}^n$ and conformally compactify $\mathbb{R}^n$ to the sphere S^n. The frame becomes a C^{an}-rigid A-structure on S^n with the isometry group $= \mathbb{R}^n$. The group $Is^j(v)$ is trivial for all j and all points $v \in S^n$ but the north pole $v_o \in S^n$, where $Is^1(v_o)$ is trivial and

$$Is^2(v_o) = Is^3(v_o) = \dots = Is^{loc}(v_o) = \mathbb{R}^n .$$

4.4.B. Let (V,g) be Lorentz of type $(n-1,1)$ and $G = \mathbb{R}^m$ be a closed subgroup in $Is(V,g)$. Then the pertinent structure is $g \dotplus L$ where L is the system of tangent fields corresponding to a basis in the Lie algebra of G. Denote by k the minimal dimension of the orbits in V' and obtain with 4.4.A. the following conclusion.

<u>There exists a closed connected subgroup</u> $I \subset O(n-1,1)$ <u>with</u> $U\dim I \geq m$ <u>which fixes a</u> k-<u>dimensional subspace in</u> $\mathbb{R}^{n-1,1}$. <u>Therefore</u>, $m \leq n-k-1$.

<u>Remark</u>. This estimate is not sharp for $k = 0$ as every Abelian subgroup in $O(n-1,1)$ has dimension $\leq n-2$ (rather than $n-1$). In fact, if the metric g is non-singular on the G-orbits in V', then $m \leq n-k-2$. (Otherwise, we would have a placing $\mathbb{R}^m \to (O(m,1))$).

4.5. <u>Measurable placings</u>. Let G_1 and G_2 be locally compact countably compact groups and V be a probability space with the measure denoted μ. Let μ_r be the right Haar measure on G_2.

Definition. A mes-map $G_1 \to G_2$ is a measure preserving action of G_1 on $X = (V,\mu) \times (G_2,\mu_r)$ commuting with the left action of G_2. This $G_1 \to G_2$ is called a mes-placing (or a mes-placement) if the action of of G_1 on V is proper. That is the isotropy subgroup of each point is compact and the partition of X into G_1-orbits is measurable. This means the action on each ergodic component is isomorphic to the natural action on $G_1/H \times Y$ for some compact subgroup $H \subset G_1$ and some measure space Y. A mes-placing is called a mes-equivalence if the action of G_1 is co-bounded. That is there exists a discrete subset $\Gamma_o \subset G_1$ and a subset $X_o \subset X$ of finite measure, such that $\Gamma_o X_o = X$.

Example. Every continuous homomorphism $f : G_1 \to G_2$ is a mes-map for V = single atom. This is a mes-placing if and only if the kernel of f is compact and the image $f(G_1) \subset G_2$ is closed. Such an f is a mes-equivalence, if and only if $G_2/f(G_1)$ has a finite G_2-invariant measure. In particular, every lattice in G_2 is mes-equivalent to G_2.

4.5.A. In what follows, we state basic properties of mes-placing, most of which are due to Zimmer (see $[Z]_1,[Z]_4$), where mes-placings are called measurable cocycles, and where the reader can find a complete discussion with proofs).

We start with the following trivial

4.5.B. Proposition. Let G_1 be mes-placed into G_2. Then,

(1) If G_2 is compact, then also G_1 is compact.

(2) If G_2 is amenable, then so is G_1.

(3) If G_1 and G_2 are mes-equivalent and one of them is unimodular, then so is the other.

4.5.B$_1$. Remark. It is not hard to show that any two countable amenable groups are mes-equivalent. This can also be derived from a (deep) theorem of Connes-Feldman-Ornstein-Weiss, claiming that all ergodic measure preserving actions of such groups on probability spaces are orbit equivalent. In fact, our mes-equivalence amounts to the orbit equivalence of some actions.

4.5.C. Induced representations and a-T-menable groups. If G_2 acts on a space L and $G_1 \to G_2$ is a mes-map given by an action of G_1

on $X = V \times G_2$, then G_1 acts on $V \times L$ as follows. Consider the space $Y = X \times L$ with the diagonal action of G_2 and the (commuting) action of G_1 coming from the first factor. Then Y/G_2 can be identified with $V \times L$ such that the resulting action of G_1 on $V \times L$ commutes with the action of G_1 on $V = X/G_2$ via the natural projection $X \to V$. Denote by L' the space of maps $V \to L$ whose graphs $V \to V \times L$ are G_1-equivariant and observe that G_1 naturally acts on L'. Furthermore, if L is a linear G_2-space, then L' is a linear G_1-space. Moreover, if G_2 preserves some norm $\| \ \|$ on L, then G_1 preserves the norm $\| \ \|' = \left(\int_V \|\varphi(v)\|^2\right)^{\frac{1}{2}}$ on L'.

<u>In particular, every unitary representation of</u> G_2 <u>induces that of</u> G_1.

Observe that the inducing operation preserves the following

<u>Almost fixed points</u>. Let a group G act on a metric space S. <u>An almost fixed point in</u> S is a sequence of points $s_i \in S$, such that the sequence of functions φ_i on G defined by $\varphi_i(g) = \text{dist}(s_i, g(s_i))$ uniformely converges to zero for $i \to \infty$ on each compact subset in G.

Now we have the following obvious

Lemma (Compare $[Z]_5$). <u>If</u> G_2 <u>has almost fixed points on the</u> $\| \ \|$<u>-unit sphere in</u> L <u>then</u> G_1 <u>has such a point on the</u> $\| \ \|'$<u>-sphere in</u> L'.

Next, assume G_1 is mes-placed into G_1 and the representation in question is unitary. Then we have another obvious

Lemma. <u>If the matrix coefficients of the</u> G_2<u>-representation decay, then so do the coefficients of the representation of</u> G_1 <u>on</u> L'.

Recall that matrix coefficients of a G acting on a Hilbert space L are functions on G,

$$\langle \ell_1 , g(\ell_2) \rangle ,$$

for fixed ℓ_1 and ℓ_2 on G and "decay" means $\to 0$ for $g \to \infty$.

The following difinition and proposition are implicit in $[Z]_5$.

A group G is called a-T-<u>amenable</u> if it admits a non-trivial unitary representation with decaying coefficients and with an almost fixed unit vector.

4.5.C_1. If a group G_1 can be mes-placed into an a-T-menable group, then G_1 is a-T-menable.

This is obvious with the above lemmas.

4.5.C_2. Remarks and example. The a-T-menability is a strengthened negation of Kazdan's property T which says :

If some unitary representation of G has an almost fixed unit vector, then it has a fixed unit vector.

Note that decaying coefficients rule out fixed vectors for non compact groups G . One knows (see $[Z]_5$ for details and references) that all semisimple Lie groups without O(n,1) and U(n,1) factors are T . On the other hand, O(n,1) and U(n,1) are a-T-menable, and Cartesian products of these and compact groups are the only a-T-menable semisimple Lie groups.

4.5.C_3. Corollary (see $[Z]_5$). If a non-compact simple Lie group G can be mes-placed into a product of groups O(n,1) and U(n,1), then G is (locally) such a product.

Also note that all our discussion is mes-invariant and thus extends to lattices in semisimple groups. For example, no lattice in Sp(n,1) for $n \geq 2$ can be mes-placed into O(N,1) or U(N,1).

Question. Does there exist any mes-placing $U(n,1) \to O(N,1)$ for $n \geq 2$? Or $O(n+1,1) \to O(n,1)$?

Remark. The notion of a-T-menability admits a non-Hilbert version. For example, one can use L_p-classes of spaces for $p \neq 2$ closed under subquotients and continuous Hilbert sums. Unfortunately, ther are no meaningful examples of groups G where one can prove (or disprove) a-T_p-menability (or T_p-property) in order to find further obstructions for mes-placings $G_1 \to G_2$.

4.5.D. Groups with $\mathbb{R}$-rank ≥ 2 . Let G_1 be a semisimple group all simple components of which have $\mathbb{R}$-rank ≥ 2 . If G_1 admits a mes- placing into an algebraic group G_2 , then the Lie algebra $L(G_1)$ isomorphically embeds into $L(G_2)$. Moreover, if $L(G_2)$ is isomorphic to $L(G_1)$, then every mes-placing $G_1 \to G_2$ is a mes-equivalence.

This is Zimmer's version of Margulis' super-rigidity (see $[Z]_1$).

Using Zimmer's (Fürstenberg-Margulis) techniques, one can extend 4.4.D. to a class of groups including lattices in semisimple groups and their products with fundamental groups of manifolds with negative curvature and finite volume.

4.5.D_1. Example. Let V_1 be the Cartesian product of k_1 copies of manifolds with Vol $< \infty$ and the curvatures between two negative constants. If V_2 is the product k_2 of such manifolds and if there is a mes-placement between the fundamental groups, $\pi_1(V_1) \to \pi_1(V_2)$, then $k_1 \le k_2$.

4.6. Rigid actions preserving a measure. If G admits a k-rigid action on V preserving a finite measure, the V can be mes-placed into $\mathcal{D}^k$. If the measure in question is smooth at some point $v \in V$, then there is a mes-placement $G \to S\mathcal{D}^k$, where $S\mathcal{D}^k \subset \mathcal{D}^k$ denotes the subgroup of k-th jets of volume preserving diffeomorphisms. The groups $\mathcal{D}^k$ are a-T-menable for $\dim V = 2$, which rules out, for example, lattices in T-groups for the role of G, if $n = 2$. (see $[Z]_4$).

4.6.A. Let the above G preserve an A-structure as well as a smooth finite measure. Then G is mes-placed into $Is^j(v)$ for almost all $v \in V$ and all $j \ge k$ (compare 4.4.). Therefore, non-compactness, non-amenability and non-a-T-menability of G implies the corresponding properties of $Is^j(v)$ for $j \ge k$. Note that the subset of those $v \in V$ where $Is^j(v)$ is non-compact, is closed in V because the number of connected components of $Is^j(v)$ is uniformely founded on V. It follows that $Is^j(v)$ is non-compact for all $v \in V$, provided G is non-compact (compare (3) in 4.4.).

A more interesting relation comes from (a slight generalization of a corollary of) Zimmer's theorem,

If G is isomorphic to a lattice in a semisimple Lie group of $\mathbb{R}$-rank $= m$ (e.g. G is the Cartesian product of m free groups F_2) then $\operatorname{rank}_{\mathbb{R}} Is^j(v) \ge m$ for all $j \ge k$ and almost all $v \in V$. (If all simple components of the semisimple group $H \supset G$ are of $\mathbb{R}$-rank ≥ 2, then, by Zimmer, H is locally contained in $Is^j(v)$).

4.6.A_1. Examples. (1) Let V be a pseudo-Riemannian manifold of finite volume and type (n_+, n_-). Then $\operatorname{rank}_{\mathbb{R}} Is^j(v) \le \min(n_+, n_-)$ for all $v \in V$ and $j = 1,2,\ldots$. Also note that $Is^j(v)$ is a-T-menable if $\min(n_+,n_-) = 1$.

(2) Let V be symplectic of dimension $n = 2m$. Here $\mathrm{rank}_{\mathbb{R}} Is^j(v) = m$ for all $j = 1,2,\ldots$. This bounds the rank of the semisimple group H containing $G \subset Is(V)$ in-so-far as the action is rigid and $\mathrm{Vol}\, V < \infty$.

(3) Let V be a manifold with an affine connection and G be a lattice in a simple Lie group H of $\mathbb{R}$-rank ≥ 2 . Let G isometrically act on V preserving a smooth finite measure. Then the group H (or rather a finite covering of H) admits a non-trivial representation into the infinitesimal isometry group, $\rho_v: H \to Is^\infty(v) \subset SL_n\mathbb{R}$ for $n = \dim V$ and almost all $v \in V$. If the connection has non-zero torsion at v , then the representations ρ_v and the exterior power $\Lambda^2\rho_v: H \to SL_N\mathbb{R}$ for $N = \frac{n(n-1)}{2}$ have a common irreducible component. Similarly, if the curvature tensor does not vanish at $v \in V$, then $\Lambda^2\rho_v$ and the adjoint representation $\mathrm{ad}\,\rho_v : H \to \mathrm{Aut}\, s\ell_n\mathbb{R} \subset SL_M\mathbb{R}$ for $M = n^2 - 1$ have a common component (compare 5.3.D.). In particular, if $H = SL_m\mathbb{R}$ then $m \leq n$ and if $m = n$, then V is an affine flat manifold. Then one can show that sufficiently many $\gamma \in G$ act on V with non-zero Liapunov exponents and such that the stable manifolds have codimension one in V. Using these, one can prove that V is a flat torus (compare $[Z]_6$).

4.6.B. The above results can be sharpened by replacing $Is^j(v)$ by a smaller group $Is^{loc}_G(v) \subset Is^{loc}(v)$ defined as follows. Take a connected neighborhood $U \subset V$, let $L(U)$ be the Lie algebra of Killing fields on U and $G(U)$ the simply connected Lie group with the Lie algebra $L(U)$. We assume that the A-structure on V is regular at a point v and then observe that the groups $G(U)$ are canonically isomorphic for all small connected neighborhoods $U \subset V$ of v . Denote by $G_v(U)$ the connected subgroup corresponding to the isotropy subalgebra $L_v(U)$ of v and observe that this is a closed subgroup in $G(U)$ since the structure is A and the point v is regular. Furthermore, there is a canonical homomorphism of $G_v(U)$ onto the identity component of the local isotropy group $Is^{loc}(v)$, whose kernel $C_v \subset G_v(U)$ is discrete and contained in the center of $G_v(U)$. Since (by definition) local isometries are faithful on U , the subgroup C_v is central in $G(U)$ and we may pass to $\bar{G} = G(U)/C_v$.

The subgroup $\overline{G}_v = G_v(U)/C_v$ is closed in $\overline{G}$ and is canonically isomorphic to the identity component in $Is^{loc}(v)$. Let us assume that $Is^{loc}(v)$ is connected. (Non-connected cases we shall treat later on). Take a small neighborhood $\overline{U} \subset \overline{G}$ of the identity element and let $\overline{W}$ be the product $\overline{U} \cdot \overline{G}_v \subset \overline{G}$. There is a natural local action of $\overline{U}$ on V defined with the action of Killing fields and $\overline{G}_v$ acts by local isometries in $Is^{loc}(v)$. Then we have the product action of $\overline{W}$ where for every local isometry g of V sending v close to v there exists a unique $\overline{w}_g \in \overline{W}$, such that $\overline{w}_g(v) = g(v)$.

Now, let a group G isometrically act on V. For each regular $v \in V$ we take the subset $\overline{W}_U = \{\overline{w}_g\} \subset \overline{G}$ for those $g \subset G$ which map v to a small neighborhood $U \subset V$ of v. Then we take the subgroup $G_U \subset \overline{G}$ generated by $\overline{W}_U$, take the Zariski closure of the $ad_{\overline{G}}$-image of $\overline{G}_U$ in $\mathrm{Aut}\, L(\overline{G})$ and let $\overline{G}_U \subset \overline{G}$ be the $ad_{\overline{G}}$-pullback of this closure. Next, we intersect $\overline{G}_U$ with $\overline{G}_v$ and take the Zariski closure, say $G_v(U) \subset \overline{G}_v = Is^{loc}(v) \subset \mathcal{D}^{\infty}(v)$ of this intersection $\overline{G}_U \cap \overline{G}_v$. Finally, we consider the <u>reduced local isotropy</u> $Is_G^{loc}(v) \subset Is^{loc}(v)$ which is the intersection of the groups $G_v(U)$ over a fundamental system of neighborhoods U of v. Note that our $Is_G^{loc}(v)$ is a geometric version of <u>Mackey's range of a measurable cocycle</u>.

4.6.B$_1$. <u>Let</u> G <u>preserve a smooth finite measure on</u> V. <u>Then</u> G <u>admits a mes-placement into</u> $Is_G^{loc}(v)$ <u>for almost all regular points</u> <u>$v \in V$. (Recall that regular points form a dense open subset in V).</u>

<u>Proof</u>. If $Is^{loc}(v)$ is a.e. connected, then the proof consists in following through the definition of $Is_v^{loc}(G)$. In the disconnected case, we pass to a finite covering of the regular locus of V corresponding to the connected components of $Is^{loc}(v)$ and lift G to an action of a commensurable group on this covering. We leave to the reader to fill in the details.

4.6.B$_2$. <u>Corollary</u> (Compare [Ti]). <u>If</u> G <u>is a non-amenable closed subgroup in</u> $Is(V)$ <u>then the subgroup</u> $G_U = G_U(v) \subset \overline{G}$ <u>generated by</u> $\overline{W}_U$ <u>contains a discrete subgroup isomorphic to the free group</u> F_2 <u>for almost all regular</u> $v \in V$ and all neighborhoods $U \subset V$ of v.

§5. CONNECTED ISOMETRY GROUPS.

If a connected group G acts on an A-manifold $V = (V,f)$, then the placing $G \to Is^{\infty}(v)$ can be realized (see 5.2.) by a homomorphism of a subgroup $\subset Is^{\infty}(v)$ into ad G. This sharpens the results of the previous section and also allows a placing of G into the holonomy group of V (see §6).

5.1. Lie structures. Consider an s-tuple $L = (\ell_1, \dots, \ell_i, \dots, \ell_s)$ of C^{∞}-vectorfields on V. This is an A-structure on V and $Is(V,L)$ equals the centralizer of these fields. If the fields commute, then $Is(V,L)$ contains the subgroup generated by the fields but, in general, the group gerated by ℓ_i is not in $Is(V,L)$.

We are interested in the case where ℓ_i are given by an action of an s-dimensional Lie algebra on V which is also called (with some abuse of notations) L. The action is given by a homomorphism of bundles $V \times L \to T(V)$ over V and ℓ_i correspond to a fixed basis in the Lie algebra. Denote by $L^k : V \times L \to T^k(V)$ the k-th jet of this homomorphism, that is the s-tuple of the jets $J^k_{\ell_i}$, where $T^k(V)$ denotes the space of k-jets of tangent fields in V. This is again an A-structure whose type is denoted by $\mathcal{L}^k$. Note that the linear group $GL_s \mathbb{R}$ naturally acts on $\mathcal{L}^k$ and this action commutes with that of $\mathcal{D}^{k+1}$. In particular the group Aut $L \subset GL_s \mathbb{R}$ of automorphisms of L acts on $\mathcal{L}^k$.

Call the action of L on V k-faithful if the rank of the homomorphism L^k equals $s = \dim L$. In this case the image $|L^k| \subset T^k(V)$ of L^k is an s-dimensional subbundle and the field of fibers, $v \to |L^k|_v$, is an A-structure on V, also called $|L^k|$, whose type is $\mathcal{L}^k/GL_s \mathbb{R}$. The j-th infinitesimal isometry group of $|L^k|$ does not depend on k for $j > k$ and is denoted $Is^j_{|L|}(v)$.

Call our action infinitesimally faithful if it is k-faithful near each $v \in V$ for some $k = k(v)$. (If V is compact, then "infinitesimally faithful" amounts to "k-faithful" for large k). For such actions we can non-ambiguously define the group $Is^{\infty}_{|L|}(v)$ for all $v \in V$. Observe that $I^{\infty}_{|L|}(v)$ naturally acts on L and denote by $\delta_v : Is^{\infty}_{|L|}(v) \to GL_s \mathbb{R}$ the corresponding homomorphism.

Example. Let L be the Lie algebra of a group G acting on V. If G is Abelian, then the action preserves L^k. In general, G preserves $|L^k|$ and acts on L^k via the adjoint representation. That is $(g\, L^k_v)_w = (\mathrm{ad}\, g)\, L^k_w$, for all v and w in V and transformations $g \subset G$ of V which send $v \to W$.

Remarks. (a) The structures L^k and $|L^k|$ usually are not rigid. However, if L acts transitively on V which means surjectivity of the homomorphism $L^{\circ} : V \times L \to T(V)$, then the structure $|L^k|$ is rigid for $k > \dim L$ as a simple argument shows. Thus (locally) homogeneous spaces fit into the framework of rigid A-structures (compare 0.5.D.).

(b) It is sometimes useful to refine $|L^k|$ by using $L^k/\mathrm{Aut}\, L$ instead of $L^k/GL_s\mathbb{R}$, or even better, L^k/G^a, where $G^a \subset \mathrm{Aut}\, L$ is the Zariski closure of $\mathrm{ad}\, G \subset \mathrm{Aut}\, L$.

5.2. Refined Z.D.T. (compare 2.4.). Consider a Lie algebra L isometrically acting on a C^∞-smooth A-manifold $V = (V,f)$ and let G be an isometry group of V which normalizes L. An important example is $L = L(G)$. But in general, we do not even assume that the action of L integrates to an action of any Lie group. In any case, we have a homomorphism $a : G \to \mathrm{Aut}\, L$. We denote by $G^a \subset \mathrm{Aut}\, L$ the Zariski closure of $a(G) \subset \mathrm{Aut}\, L$ and we consider the minimal normal cocompact algebraic subgroup $G^b \subset G^a$.

Consider the group of infinitesimal isometries of $|L^k|$ for large k which normalize L, say $\mathrm{Is}^\infty_L(v) \subset \mathcal{D}^\infty(v)$ for $v \in V$. If the action of L is infinitesimally faithful at v, we have a natural homomorphism $\delta_v : \mathrm{Is}^\infty_L(v) \to \mathrm{Aut}\, L$. Next, we intersect $\mathrm{Is}^\infty_L(v)$ with the infinitesimal isometry group of f and set $\bar{N}_v = \delta_v(\mathrm{Is}^\infty_L(v) \cap \mathrm{Is}^\infty(v)) \subset \mathrm{Aut}\, L$, for $\mathrm{Is}^\infty(v) = \mathrm{Is}^\infty_f(v)$, where f (instead of g in 0.3.) denotes the A-structure on V. Observe that $\bar{N}_v$ is a Zariski closed subgroup in $\mathrm{Aut}\, L$.

5.2.A. Theorem. Let the action of L be infinitesimally faithful and the action of G preserve a finite measure μ on V. Then $\bar{N}_v \supset G^b$ for almost all $v \in V$.

Proof. To simplify notations we assume L is r-faithful for r equal the order of f. (By passing to jets we can increase r and eventually make $r \to \infty$). The structure $f \dotplus L^r$ is given by a

$\mathcal{D}^r$-equivariant map $\mathcal{D}^r(V) \to F \times \mathcal{L}^r$ which gives us a map $h = h^r : V \to X_L = (F \times \mathcal{L}^r)/\mathcal{D}^r$. Note that the action of Aut L on $\mathcal{L}^r$ induces an action of Aut L on X_L and the map h is G-equivariant for the homomorphism $a : G \to \text{Aut } L$. The space X_L is not an algebraic variety. Yet it is a finite union of such varieties which allows us to conclude the proof using the argument in 2.4.A.

5.2.A_1. Remarks. (a) It is not hard to extend 5.2.A. to not necessarily infinitesimally faithful actions.

(b) Theorem 5.2.A. essentially reduces to 2.4.A. if the structure f is void.

(c) Let $G^b = G^a$. For example, $L = L(G)$ where G is semisimple without compact factors. Also assume that $G^a \subset \text{Aut } L$ acts irreducibly on L (which happens, for example if $L = L(G)$ for a simple G). Then we know (by 2.4.A.) that the action of L is almost everywhere locally free on V. Now, additionally assume that the structure f is C^{an}-rigid and observe that 5.2.A. becomes equivalent to the following property of the Lie algebra L' of local (defined near v) isometric (Killing) vectorfields on V commuting with L.

5.2.A_2. The local orbit $L(v) \subset V$ of v is contained in the orbit $L'(v)$ for almost all $v \in V$.

5.3. Examples. Let $L = L(G)$ for a semisimple Lie group without compact factors. Then, the action of G is a.e. locally free (by 2.4.A. as well as by 5.2.A. as we assume the action locally faithful) and L is isomorphic to a subquotient of $L(\text{Is}_f^\infty(v))$ for almost all $v \in V$. In fact, since L is semisimple, there exists a subalgebra $L_v \subset L(\text{Is}_f^\infty(v))$ isomorphic to $L(G)$ which stabilizes the orbit $G(v) \subset V$ for almost all $v \in V$ and such that the resulting action of L_v on $L = T_v\, G(v)$ equals the adjoint action of $L = L_v$. (This is a slight refinement of a theorem by Zimmer, see $[Z]_4$).

5.3.A. Let (V,f) be pseudo-Riemannian of type (n_+, n_-). Then the form f is ad G-invairant on almost all, and hence, on all orbits of G. If G is simple, then $f|G(v)$ must be either non-singular or identically zero and the same remains true for any linear combination of $f|G$ with the Killing form, say κ on G. This implies, 5.3.B (compare $[Z]_4$). If G <u>is simple and</u> $\dim G > n_o = \min(n_+, n_-)$ <u>(this n_o equals the dimension of a maximal isotropy subspace in</u> V) <u>then the Killing form of</u> G <u>is of the type</u> (n_+, n_-) <u>or</u> (n_-, n_+) .

(If $\mathbb{C}G$ is simple, then $f|G$ is proportional to κ , compare $[Z]_4$).

For example, if $n_o \leq 2$, then G must be locally isomorphic to $SL_2\mathbb{R}$ (compare $[Z]_4$).

5.3.C. Let f be a symplectic form and G simple. Since G admits no biinvariant symplectic structure (this is well known and easy to prove), <u>form</u> f <u>vanishes on all</u> G-<u>orbits</u>.

5.3.D. Suppose the action is locally free and preserves a subbundle $S \subset T(V)$ of dimension $m = \dim V - \dim G$ transversal to the orbits of G . Denote by $L_1, \ldots, L_i, \ldots, L_k$ the simple components of the Lie algebra $L = L(G)$, take some m-dimensional representation ρ_i of L_i and let ρ_i' be an irreducible component of the exterior power $\Lambda_2 \rho_i$ on $\Lambda_2 \mathbb{R}^m$.

<u>Let</u> $\mathrm{ad}\, L_i$ <u>be non-equivalent to any</u> ρ_i' <u>for all</u> $i = 1, \ldots, k$ <u>and</u> ρ_i . <u>Then the subbundle</u> S <u>is integrable</u>.

<u>Proof</u>. The obstruction to integrability is given, at each $v \in V$, by the curvature operator $\Omega_v : \Lambda_2 T_v(S) \to T_v(G(v))$, which commutes with (infinitesimal) diffeomorphisms in $\mathcal{D}^\infty(v)$ which preserve S and the orbit $G(v)$. In particular Ω_v intertwines the representation of L_v on $\Lambda_2 T_v(S)$ with the adjoint representation of L_v . Our assumption on L allows no non-trivial intertwining operator, and hence $\Omega \equiv 0$. Q.E.D.

<u>Remarks</u>. (a) If the representation of L_v on $T_v(S)$ is trivial and infinitesimal isometries in L_v integrate to local isometries, then the integral leaf of S through v equals the fixed points set of L_v .

(b) The foliation S and the orbits of G split some covering $\tilde{V} \to V$. In fact, for every leaf S_o of S the map $S_o \times G \to V$ for $(s,g) \to g(s)$ is a covering map. (See [B-H]. Note that such a splitting takes place for all actions of connected Lie groups with transversal invariant foliations).

(c) A much deeper integrability theorem is proven in [C-G] without assuming G preserves any measure.

Examples. (1) If $m = \dim S = 2$, then S is integrable since every representation of G on $\Lambda_2 \mathbb{R}^2 = \mathbb{R}^1$ is trivial. This is a (trivial) special case of an integrability theorem in [C-G].

(2) If G is simple, then S is integrable for $\dim G \geq \frac{m(m-1)}{2}$.

(3) Let the action preserve an (n_+, n_-)-form f on V and $S \subset T(V)$ be the orthogonal bundle to the orbits (which is not apriori transversal to the orbits). Let (m_+^i, m_-^i) denote the types of the Killing forms of the simple components L_i of $L(G)$.

The bundle S is integrable unless $L(G)$ admits a non-trivial homomorphism into the orthogonal Lie algebra $O(p_+, p_-)$ for some (p_+, p_-) of the form

$$p_+ = n_+ - \sum_i \mu_i, \quad p_- = n_- - \sum_i \mu_i'$$

where each (μ_i, μ_i') equals either (m_+, m_-) or (m_-, m_+).

Note that the bundle S is totally geodesic (whether it is integrable or not) as every geodesic normal to one orbit is normal to all orbits it meets.

5.3.E. Let $V = (V, f)$ be a Lorentz $(n-1,1)$-manifold of finite volume and $G \subset \mathrm{Is}\, V$ be a closed connected subgroup. Denote by $G_v^\circ \subset G_v \subset G$ the identity component in the isotropy subgroup G_v of $v \in V$ and let $N_v \subset G$ be the normalizer of G_v°. Take a sequence of generic points $v_1, v_2, \ldots, v_i$ in V and set $N(i) = \bigcap_{j=1}^{i} N_{v_j}$.

5.3.E$_1$. The subgroup $G_v^\circ \subset G$ is compact for almost all $v \in V$. The (normal) subgroup $N(i)$ does not depend on i for large i; neither $N = N(i)$ for large i depends on a particular senquence v_i. The factor group G/N is compact $\times$ Abelian and N acts a.e. freely on V.

Proof. Let $h_1 : V \to \mathrm{Gr}\, L$ be the first component of the map h from 2.4.A. That is h_1 assign to each $v \in V$ the Lie subalgebra $L(G_v) \subset L = L(G)$. Then $N \subset G$ consists of the isometries $g \in G$ fixing h_1 and the group $G/N \subset \mathrm{Aut}\, \mathrm{Gr}\, L$ is precompact (compare 2.4.A.).

If an isometry $g \in G$ fixes some $v \in V$, then by definition of N, this g fixes the fiber of h_1 through v. The following lemma implies that the differential g' of such a g at v fixes $T_v(V)$ and so $g = Id$.

5.3.E_2. Lemma (D'Ambra, see [DA]). *If a linear isometry* $g' \in O(n-1,1)$ *of* $\mathbb{R}^n = \mathbb{R}^{n-1,1}$ *fixes a linear subspace* $S = \mathbb{R}^n$ *as well as the factor-space* $\mathbb{R}^n/S$ *then* $g' = Id$.

Now, we see that G_v° injects into the isotropy subgroup of G/N acting on $\overline{h(V)} \subset Gr\ L$, where $\overline{h(V)}$ denotes the Zariski closure of the h_1-image of V. Since the (sub)group $G/N \subset Aut\ \overline{h(V)}$ is precompact, the (sub)group $G_v^\circ \subset Aut\ T_v(V)$ also is precompact (by an easy argument) and since G is closed in $Is(V,f)$ this G_v° is, in fact, compact.

5.3.E_3. Corollary to the proof (compare $[Z]_4$) *Let* G *have no compact non-commutative factor group. Then the isotropy subgroup* $G_v \subset G$ *is discrete for almost all* $v \in V$.

5.3.E_4. Consider a (generic) point $v \in V$ where $\dim G_v = 0$ and identify the tangent space to the orbit, $T_v\, G(v) \subset T_v(V)$ with the Lie algebra $L(G)$. According to 5.2.A., there exists an algebraic subgroup $I \subset Is_f^\infty(v) \subset O(n-1,1)$, such that

(i) I keeps invariant the tangent space $T_v(G(v)) \subset T_v(V)$;

(ii) I fixes the center $C \subset L(G) = T_v(G(v))$;

(iii) The image N_1 of the natural (restriction to $T_v(G(v)) \subset T_v(V)$) homomorphism $I \to Aut\ L(G)$ is contained in the Zariski closure $G^a \subset Aut\ L(G)$ of $ad\ G \subset Aut\ L(G)$.

(iv) The above N_1 is normal and cocompact in G^a.

Since $I \subset O(n-1,1)$ the restricted form $f|L(G) = T_v(G(v))$ is N_1-invariant which imposes non-trivial restrictions on N_1. To see these, we observe that the form $f|L(G)$ may be of the following five types.

(1) Negative definite. In this case the orbits of G are 1-dimensional, and if the action is locally free, then G is Abelian of dimension one. Furthermore, the closure of G in $Is(V,G)$ is compact, since the infinitesimal isotropy of the structure $f \dot{+} L$ is compact.

(2) $f|L(G) = 0$. Here again G is Abelian 1-dimensional, but the closure of G in $Is(V,f)$ may be non-compact.

(3) $f|L(G)$ is positive definite. Then N_1 is compact and (by an easy argument) G is compact × Abelian.

(4) $f|L(G)$ is positive semidefinite with 1-dimensional kernel $L_o \subset L(G)$. Then, we may assume without loss of generality that the subgroup $I \subset Is_f(v)$ stabilizes $L_o \subset L(G) \subset T_v(V)$. Denote by $L_1 \subset L(G)$ the Lie algebra of the pull-back of N_1 under the adjoint homomorphism $ad : G \to Aut\, L(G)$ and observe that $f|L(G)$ is L_1-invariant,

$$f([\ell_1, \ell], \ell') + f(\ell, [\ell_1, \ell']) = 0$$

for all $\ell_1 \in L_1$ and $\ell, \ell' \in L(G)$. If L_1 does not contain L_o , then $f|L_1$ is positive and so L_1 is compact. Moreover, since L/L_1 is compact, one can perturb f to a positive form f' on L which also is L_1-invariant. Hence, the adjoint action of L_1 on L is semisimple with purely imaginary spectrum.

If $L_1 \supset L_o$, then L_1/L_o is compact. Moreover, if L_o is not normal in L , then L_1 is compact as $f + (ad g)f$ is positive on L_1 for some $g \in G$.

Finally, if L_o is normal in L , then L/L_o is compact.

(5) $f|L(G)$ is non-singular of type $(m,1)$ for $m > 0$. If $f|L_1(G)$ is semidefinite, then L_1 is compact unless there is some one-dimensional $L_o' \subset L_1$ which is f-isotropic and normal in L . In this case L_1/L_o' is compact.

If $f|L_1$ is of the type $(m_1,1)$, then there are two possibilities.

(a) L_1 contains $s\ell_2\mathbb{R}$ + compact . Then $L(G) = s\ell_2\mathbb{R}$ + compact (see $[Z]_5$ and 5.3. .).

(b) L_1 does not contain $s\ell_2\mathbb{R}$. Then obviously, there is a 1-dimensional normal subalgebra $L_o'' \subset L_1$ which is f-isotropic. This L_o'' lies in the center $C_1 \subset L_1$ by 5.3.E_2. In fact, every normal Abelian subalgebra in L_1 is central. The f-orthogonal complement $C_1^{\perp} \subset L_1$ of C_1 is a normal subalgebra in L_1 of codimension one and $C_1^{\perp}/C_1$ is compact. Also note that L_1/C_1 embeds into the Lie algebra of rigid motions of $\mathbb{R}^{m_1-1}$ for $m_1 = \dim L_1 - 1$.

Example. Consider the 4-dimensional Lie algebra L generated by x,y,z,t, where z is central and $[x,y] = z$, $[t,x] = y$ and $[t,x] = -x$. Then the form $x^2 + y^2 + zt$ is ad L-invariant as a straightforward check-up shows. This L is the Lie algebra of the semidirect product G of S^1 with the Heisenberg group H^3, where S^1 acts on H^3 according to the action of $t \in L(S')$ on $L(H^3)$ generated by x,y,z. Then, every lattice Γ in H^3 is cocompact in G and so we get a compact $(3,1)$-manifold G/Γ acted upon by G. Note that the center $C \subset \Gamma$ acts trivially on G/Γ and the group G/C acts faithfully and 1-rigidly on G/Γ.

5.3.E_5. Let us specialize the above discussion to a group G, such that the Zariski closure $G^a \subset \mathrm{Aut}\, L(G)$ of ad G contains no normal algebraic cocompact subgroup. (This condition is satisfied, for example, by all nilpotent groups and by solvable simply connected algebraic groups).

If G admits a locally faithful action on a $(n-1,1)$-manifold V of finite volume, then the isotropy subgroup $G_v \subset G$ is discrete for almost $v \in V$ and either $L(G) = s\ell_2 \mathbb{R} \oplus$ Abelian, or there is a 1-dimensional ideal $L_o \subset L(G)$, such that $L(G)/L_o$ is Abelian.

Example (compare $[Z]_4$). If G is nilpotent, then $L(G) = L_1 \oplus L_2$, where L_1 is Abelian and L_2 is Heisenberg.

5.4. Splitting theorems. If an action of a group G on V preserves a smooth finite measure, then the action is a.e. locally free (i.e. the isotropy $G_v \subset G$ is discrete for almost all $v \in V$) if the Lie algebra $L(G)$ has no compact factor algebra (see 2.4.). Examples seem to indicate that such an action of every non-compact simple G is everywhere locally free and we establish below this freedom for certain geometric actions.

5.4.A. Let V be a connected C^∞-Lorentz manifold (of type $(n-1,1)$) of finite volume and G a non-compact simple Lie group. Then every faithful isometric action of G on V is everywhere locally free on V.

Proof. We already know that $L(G) = s\ell_2 \mathbb{R}$ and that each orbit map $G \to V$ for $g \mapsto g(v_o)$, induces a bi-invariant form on G. Hence, every (degenerate) orbit $G(v)$ with $\dim G(v) < 3$ is isotropic in V

and therefore $\dim G(v) \leq 1$. Assume for the moment that V is regular (e.g. real analytic). Then, near each point $v \in V$ there exists an isometric action of $L' = s\ell_2 \mathbb{R}$ on V commuting with the original action, such that the local orbits of L' equal those of G at the points v' in V (close to v) where G acts locally and freely. It follows that the Lorentz form is bi-invariant on the L'-orbits and so this form is non-degenerate of type (2,1) on the non-degenerate L'-orbits, while the degenerate orbits are isotropic.

Let $L'(v) \subset V$ be an L'-orbit through some $v \in V$ and look at the identity component $G_v^\circ \subset G_v \subset G$. Since L' commute with G , the group G_v° fixes L'(v) and therefore the action of G_v° on $T_v(V) = \mathbb{R}^{n-1,1}$ fixes the subspace $T' = T(L'(v)) \subset \mathbb{R}^{n-1,1}$. Now consider two cases.

(1) $\dim T' > 0$. Then the Lorentz form $f_v|T'$ is either (2,1) (if $\dim L'(v) = 3$) or zero (if $\dim L'(v) = 1$). Hence, G_v° fixes an isotropic vector in $\mathbb{R}^{n-1,1}$. On the other hand, $\dim G_v^\circ \geq 2$, and so G_v° contains the subgroup H of affine transformation of $\mathbb{R}^1$. This leads to a contradiction as no subgroup in $O(n-1,1)$ isomorphic to H fixes an isotropic vector.

(2) $\dim T' = 0$. Then we would have a faithful representation of $L' \oplus L(H)$ in the Lie algebra $so(n-1,1)$ which is clearly impossible.

Now, let us drop the regularity assumption. Then the local action of L' is defined on an open dense subset in V and, by continuity, we get an infinitesimal action of L' at all $v \in V$. This obviously suffices for the above argument.Q.E.D.

5.4.A. Corollary. Some covering $\tilde{V}$ of V admits a splitting, $\tilde{V} = S \times G$, where the fibers $S \times g$ are totally geodesic submanifolds, normal to the G-orbits.

This follows from the discussion in 5.3.D. . Note that 0.8.B immediately follows from 5.4.A. Also observe that the group $G \subset Is(V)$ has a finite center by 5.3.E_4. It is also clear with 5.3.D_2, that Is(V) has finitely many components and that the identity component is $G \times$ compact (compare $[Z]_5$).

Let us generalize 5.4.A. to general pseudo-Riemannian manifolds (V,f) of type (n_+,n_-) . Start with the following.

5.4.B. Lemma. Let an isometric action of a simple non-compact Lie group G on V preserve a finite measure μ with compact support $V_\mu \in V$. Assume $L(G)$ admits no non-trivial homomorphism into $o(p_+,p_-)$ for $p_+ = n_+ - \dim G$, $p_- = n_- - \dim G$ and suppose there is a point $v_o \in V_\mu$ where the isotropy subgroup $G_v \subset G$ is non-discrete. Then G fixes some point $v_1 \in V_\mu$.

Proof. Fix a basis of bi-invariant forms f_i on G, $i = 1,\dots,k$, (note that $k = 1$ if $\mathbb{C}G$ is simple and $k = 2$ otherwise), denote by f^V the pull-back of f to G via the orbit map $g \to g(v)$ and write $f^V = \sum_i a_i(v) f_i$. Set $a(v) = \sum_i a_i^2(v)$. Note that $a(v_o) = 0$. Consider two cases.

(1) $a(v) = 0$ for all $v \in V_\mu$. Then G fixes V_μ. Otherwise, we would have a subset $U \subset V_\mu$ of positive measure where G_v is discrete, $f^V = 0$ and $L(G)$ admits a homomorphism into $o(n_+,n_-)$ stabilizing an isotropic subspace (corresponding to $T_v(G(v)) \subset T_v(V) = \mathbb{R}^{n_+,n_-}$) of dimension = $\dim G$. This would give us a homomorphism into $o(p_+,p_-)$.

(2) $a|V_\mu \neq 0$. In this case the subset $\{v \in V_\mu | a(v) = 0\}$ also has an invariant measure which is a weak limit of normalized measures $\mu | \{v \in V_\mu | a(v) \leq \varepsilon\}$ for $\varepsilon \to 0$. Thus the situation is reduced to that in (1) and the existence of a fixed point is proven.

5.4.C. Consider an isometric action of G on V near a fixed point $v_1 \in V$ and suppose there exists an open dense subset $U \subset V$ (which may not contain v_1) such that

(a) $\dim G_v = 0$ for all $v \in U$,

(b) denote by L' the Lie algebra of Killing fields on U commuting with G. Then the orbit $G(v)$ is locally contained in the orbit $L'(v)$ for all $v \in U$.

Note that (a) and (b) are satisfied if $\mathrm{Vol}(V) < \infty$ by 5.2.A.

Let $v_1 \in V$ be a fixed point and denote by ρ the isotropy action of G on $\mathbb{R}^{n_+,n_-} = T_{v_1}(V)$ and let $L_1' \subset o(n_+,n_-)$ be the centralizer of the ρ-action of G. We want to show that the action of G and L_1' on $T_{v'}(V)$ satisfy the above (a) and (b). Namely

(a)' Almost all orbits $G(\tau) \subset T_{v'}(V)$ have $\dim G(\tau) = \dim G$ and (b)' $L_1'(\tau) \supset G(\tau)$.

This is easy to see in the following two cases.

(1) The point v_1 is regular (e.g. the manifold (V,f) is C^{an}). Then the action of L' is defined in some neighborhood of v_1 and the isotropy subalgebra $L'_{v_1} \supset L_1'$ satisfies $L_1'(\tau) \in G(\tau)$ for almost all $\tau \in T_{v_1}(V)$. In fact, this remains true for an arbitrary 1-rigid structure on V , where v_1 is regular, as an obvious argument shows. (Probably, this remians valid without the assumption of regularity).

(2) The manifold (V,f) and the group G satisfy the assumptions of (3) in 5.3.D. Then L' contains a unique subalgebra $L'' \subset L'$ isomorphic to $L(G)$, such that the (local) orbits of L'' are equal to those of G . Then the action of L'' extends to some neighborhood of v_1 and the isotropy algebra $L''_{v_1} = L'' = L(G)$ satisfies $L''_{v_1}(\tau) = G(\tau)$ for almost all τ .

Next, we observe the following elementary

5.4.D. Lemma. Let ρ be a linear representation of a simple Lie group G on some space $\mathbb{R}^n$ and ρ' be a representation of the Lie algebra $L' = L(G)$ on the same $\mathbb{R}^n$ commuting with $\rho(G)$ such that generic orbits of G are (locally) equal to those of L' (compare the above (a)' and (b)'). If ρ and ρ' preserve some non-singular quadratic form f' on $\mathbb{R}^n$, then $\rho = \rho_o + \rho_i$, $i = 1,2,3,$ where ρ_o is the trivial representation and

(1) ρ_1 is the action of $SL_2\mathbb{R}$ on $\mathbb{R}^4$ by right (or left) multiplication on (2×2)-matrices.

(2) ρ_2 is the same action of $SL_2\mathbb{C}$ on $\mathbb{R}^8 = \mathbb{C}^4$.

(3) ρ_3 is the action of $SU(2)$ on $\mathbb{R}^4 = \mathbb{Q}^1$ by the left multiplication of quatornians.

The form f' is the determinant on (2×2)-matrices in (1), it is Re Det (or Im Det) in (2) and the Euclidean scalar product in (3). Note that $\mathbb{C}\rho_1 = \mathbb{C}\rho_3 = \rho_2$.

5.3.E. Theorem. Let V be a compact pseudo-Riemannian (n_+,n_-)-manifold and $G \subset Is(V)$ is a connected simple Lie group whose Killing form has type (m_+,m_-) , such that

$$\text{rank}_{\mathbb{R}} G > n_o - m_o \qquad (*)$$

for $n_o = \min(n_+, n_-)$ and $m_o = \min(m_+, m_-)$. Then the action of G on V is locally free, and some covering $\tilde{V}$ equivariantly splits, $\tilde{V} = S_o \times G$, where the fibers $S_o \times g$ project to totally geodesic leaves in V normal to the G-orbits.

Proof. The equality (*) is sufficient for (3) in 5.3.B_3 and then 5.3.D_3. and D_4. apply.

Remarks. (a) Theorem 5.3.D_5 yields 5.3.D_2 for compact manifold V. The proof of 5.3.D_2 for non-compact V with $\text{Vol}\, V < \infty$ extends to general types (n_+, n_-) but the condition on G (I was able to obtain this way) is somewhat more restrictive than (*).

(b) The proof of 5.3.D_5 covers, in fact, a wider range of possibilities than those covered by (*).

(c) A non-split example. Take $V = O(m+1,1)/\Gamma$ for some cocompact lattice Γ and $G = O(m,1)$. Here $n = \dim V = \frac{(m+1)(m+2)}{2}$, $\text{rank}_{\mathbb{R}} G = 1$, $n_o = m+1$ for $m \geq 2$, and $m_o = m$ for $m \geq 3$ which shows (*) is sharp for this G and the Killing form on V.

5.5. Minimal actions. Let an isometric action of connected Lie group G on an A-manifold V have a compact invariant set $V_o \subset V$. Then there exists a minimal invariant subset in V (compare 2.5.) and to save notations we assume the action of G on V_o is minimal to start with. We assume the action of $L(G)$ is infinitesimally faithful (see 5.1.), and invoke the groups $\text{Is}^{\infty}_{|L|}(v) \subset \mathcal{D}^{\infty}(v)$ for $L = L(G)$ and $\bar{N}_v \subset \text{Aut}\, L$. Recall that $\bar{N}_v$ is an algebraic subquotient of $\text{Is}^{\infty}_f(v)$ as well as of $\text{Is}^{\infty}_{|L|}(v)$. In particular, $\bar{N}_v$ is an algebraic subgroup in Aut L (see 5.1., 5.2.).

5.4.A. Theorem. Let $G^a \subset \text{Aut}\, L$ be the Zariski closure of ad G. Then the intersection $G^a \cap \bar{N}_v$ is a cocompact subgroup in G^a for all $v \in V_o$. (Compare 5.2.A_2.).

Proof. Combine the proof of 5.2.A. with the discussion in 2.5.

5.4.B. Corollary. Let G be an algebraic group with finite center. Then G is isomorphic to an algebraic subquotient of $\text{Is}^{\infty}_f(v)$ (i.e. to such a subquotient of $\text{Is}^{k}_f(v)$ for all sufficiently large k)

for all $v \in V_o$. In particular, $U\dim Is_f^\infty(v) \geq U\dim G$ (compare 4.2.) and $\operatorname{rank}_{\mathbb{R}} Is_f^\infty(v) \geq \operatorname{rank}_{\mathbb{R}} G$ (compare $[Z]_2$) .

Example. Let (V,f) be Lorentz of type $(n-1,1)$ and G be a simply connected solvable group with a trivial center, such that $\operatorname{ad} G \subset \operatorname{Aut} L(G)$ is algebraic. Then G is a subquotient of $O(n-1,1)$; hence, G embeds into $O(n-1,1)$.

Remark. The above discussion extends to C^{an}-minimal subset of C^{an}-action (compare 2.5.B.).

5.5. Remarks on discrete groups. The method of this section does not give us anything new compared with the abstract placing approach in §4, if the isometry group in question is disconnected. We shall postpone a geometric refinement of placings till another paper. Here, we indicate another (easier) kind of relation between local and global isometries.

5.5.A. Let the Lie algebra L of local Killing fields on V be Abelian (nilpotent, solvable) near every regular point $v \in V$. Then, the isometry group G of V preserving a smooth finite measure contains an Abelian (nilpotent, solvable) subgroup of finite index. Furthermore, the dimension, nilpotency degree etc. of L bound the corresponding invariants of G .

Proof. Let L be Abelian. Then the regular part of V is sliced into infinitesimal orbits S of L of dimension $k \leq \dim L$ and G preserves these slices. Furthermore, the restricted action $G \to \operatorname{Is} S$ is compact for all S as the corresponding isotropy subgroups are finite on S . Since the kernel of $G \to \operatorname{Is} S$ is (obviously) virtually Abelian, the proof is concluded. The argument is only slightly harder in the nilpotent and solvable cases.

5.6. Rationality questions. If a connected Lie group G isometrically acts on V with a "sufficient recurrency", say preserving a smooth finite measure, then the orbits of the (local) centralizing algebra L' must be, roughly, as large as those of G . (See 5.2.). There are further restrictions on L' saying, in effect, that the Lie group G' with the Lie algebra L' (and some related groups) have "sufficiently large" discrete subgroups. We shall not give here any precise statement (or proof) but rather look at the following

Example. Let $ad\, G \subset Aut\, L$ be algebraic without cocompact normal algebraic subgroups. We know that almost all orbits of L' contain those of G, and in many cases the action of G is a.e. locally free. Now, suppose $\dim L' = \dim G$. Then, one can see that (at least on an open invariant subset) there is a foliation transversal to the G-orbits, such that the two foliations split a finite covering of V. It follows tha the orbits have finite volume which requires a lattice in the group G' (which is here locally isomorphic to G).

A more general algebraic pattern which emmerges from our geometry is that of a subgroup $G \subset G'$ of positive codimension whose action on G'/Γ' is "sufficiently recurrent" for some discrete subgroup $\Gamma' \subset G'$.

§6. HOLONOMY COVERINGS OF REGULAR MANIFOLDS.

Let $V = (V,f)$ be an A-rigid regular (e.g. C^{an}) manifold and $p : W \to V$ a Galois covering of V. Denote by $L = L(W)$ the Lie algebra of Killing fields on W and consider the group $Is(W,\Gamma)$ of isometries of W which normalize the Galois group Γ of W. This group can be viewed as a lift of a certain subgroup $Is\, V$ to W. The identity component $\overset{\circ}{Is}(W,\Gamma) \subset Is(W,\Gamma)$ centrilizers Γ which makes lifts of connected isometry group especially easy.

Recall that rigidity of V means that the sheaf K of Killing fields on V is locally constant. The minimal covering of V which makes this sheaf constant is called the holonomy covering of V. Similarly, one has a holonomy covering for every subsheaf of Lie algebras of K.

6.1. Lifts of connected isometry groups. Let G be a connected isometry group faithfully acting on V and let $\tilde{G} \to G$ be the minimal covering of G whose action lifts to a given covering $W \to V$. Denote by $\tilde{L}' \subset L = L(W)$ the centralizer of G in L and suppose that $\tilde{L}'$ acts transitively on the orbits of G. Then one has the following obvious

6.1.A. Lemma. *If the adjoint homomorphism $ad : \tilde{G} \to L(\tilde{G}) = L(G)$ is proper (i.e. the center of G is finite and $ad\, \tilde{G}$ is a closed subgroup in $Aut\, L(G)$), then the action of $\tilde{G}$ is proper on W.*

This lemma combines with 5.2. and leads to the following corollary (compare §4).

6.1.B. Let the Zariski closure $G^a \subset \text{Aut } L(G)$ of ad $G \subset \text{Aut } L(G)$ have no cocompact normal algebraic subgroup. If G preserves a smooth finite measure on V and if the universal covering $\hat{G}$ of G is finite, then the lifted action of $\hat{G}$ on W is mes-proper (see §4) for every covering W lying over the holonomy covering of the sheaf (of Lie algebras on V) centralizing $L(G)$ on V. In particular, $\hat{G}$ acts mes-properly on the universal and the holonomy coverings of V.

This is worthwhile to compare with the following easy

6.1.B_1. Proposition. Let W be an A-rigid C^{an}-manifold with trivial holonomy (e.g. W is simply connected). Let G be a connected semisimple subgroup in Is W, such that the action of G is locally free everywhere on W. If G has finite center, then the action of G is proper on some open dense G-invariant subset $U \subset W$. Moreover, W admits an analytic stratification, such that the action is proper on every stratum. In particular, if W is compact, then also G is compact.

6.1.C. Remark and examples. (a) An important application of 6.1.B. is the existence of a mes-placement of G (as well as of $\hat{G}$) into the fundamental group $\pi_1(V)$ and into the holonomy group. This gives a non-trivial "lower bound" on these groups in terms of G (see §4) and yields, for example, theorem 0.7.C.

(b) The finiteness assumption on $\pi_1(G)$ can be removed by appropriately generalizing 6.1.B.

6.1.D. Example. Let G be a solvable Lie group for which the adjoint homomorphism $\text{ad} : G \to \text{Aut } L(G)$ is proper and such that the Zariski closure $G^a \subset \text{Aut } L(G)$ of $\text{ad}(G)$ contains no cocompact algebraic subgroup of positive codimension. Consider an action of G on V which has a compact invariant semi-analytic subset (compare 2.5.B.) $V_o \subset V$ and assume the structure f to be C^{an} near V_o. Then, there exists a C^{an}-minimal analytic subset $V' \subset V_o$. Assume the action is locally free on V_o. Then, the lifted action of G to the covering $\tilde{V}' \to V'$ induced by the holonomy (or the universal) covering of V satisfies the assumptions of 6.1.B.

It follows that G admits (now a topological) placement into $\pi_1(V)$ as well as into the holonomy group of V (compare §4). For example, if G is the group of affine transformations of $\mathbb{R}$, then

$$\mathrm{U\,dim}\,\pi_1(V) \geq \mathrm{Udim}\, G = 2$$

and $\pi_1(V)$ has exponential growth. The holonomy group Γ of V also has exponential growth and by Tits' theorem either Γ is virtually solvable or contains the free group F_2. In the latter case, the Killing (Lie) algebra of the universal covering of V contains $s\ell_2\mathbb{R}$. In both cases $\pi_1(V)$ contains no free subgroup F of finite index as Udim F = 1.

This conclusion remains valid for every connected group G of exponential growth as is seen by looking at solvable subgroups in G.

6.2. <u>Algebraic factorisaiton of</u> W. Let us apply the basic construction in the proof of 5.2.A. to the manifold W rather than to V. Then we obtain an open dense subset $U \subset W$ invariant under $G = \mathrm{Is}(W,\Gamma)$ and a C^∞-map $h : h_L : U \to X_L$ where X_L is an algebraic manifold acted upon by Aut L for the Killing algebra L=L(W), such that the map h is equivariant for the homomorphism $a : G \to \mathrm{Aut}\, L$.

Observe that the fibers $h^{-1}(x) \subset U$ are disjoint unions of orbits of local Killing fields on W which centralize L. In particular, if W is the universal (or the holonomy) covering of V, then these are the orbits of the center C_L of L. One can always assume (by making U smaller if necessary) that the map h has constant corank equal to the dimension of the fibers $h^{-1}(x)$.

6.2.A. <u>Remark</u>. If V is <u>compact</u> without boundary, then (by an easy argument) the Lie algebra C_L on $W = \hat{V}$ integrates to an action of a connected Abelian Lie group on W. However, the holonomy group may be infinite on C_L and so this group does not necessarily descend to any <u>finite</u> covering of V.

6.2.A_1. <u>Example</u>. Start with a Lie group $G = \mathrm{Is}(V,f)$ for some compact regular A-rigid manifold V. Take a maximal connected Abelian subgroup $A \subset G$ and let some $\ell \in L(G)$ normalize but not centralize A. Take $V' = V \times S^1$ with the structure given by the following data

(a) the original structure $f = f_s$ on every fiber $V \times s \in V \times S^1$

(b) the projection $V \times S^1 \to S^1$.

(c) a frame of vectorfields a_i on $V \times S^1$ corresponding to a basis in $L(A)$

(d) the field $\tau = \ell + \frac{\partial}{\partial s}$.

Every field a_i on a fixed fiber, say on $V \times s_o$ locally extends to a unique field, say a_i' on $V \times S^1$ commuting with τ. These a_i' are Killing on $V' = V \times S^1$ and they are contained (because of (c)) in the center of the local lagebra of the Killing fields. However, such an a_i' does not extend to all of V' unless it commutes with ℓ.

6.2.B. The vectorfields a_i' in the above example admit a global extension on the cyclic covering $V \times \mathbb{R}$ of $V' = V \times S^1$. Of course, this $V \times \mathbb{R}$ is non-compact but the isometry group of $V \times \mathbb{R}$ apart from the deck transformation group preserves each (compact!) subset $V \times [a,b] \subset V \times \mathbb{R}$.

6.2.C. What happens in the above example is due to the fact that the (cyclic part of the) holonomy is "transversal" to the orbits of the infinitesimal isometry pseudo-group of the manifold.

Now, take a generic point v_o in an A-rigid manifold V and let $V_o \subset V$ be the union of connected components of the infinitesimal orbits meeting a small connected neighborhood $U_o \subset V$ of v_o. Since the local isotropy group has finitely many components "almost all" local isometries of V_o come from local Killing fields. Namely

6.2.C_1. <u>Let $\hat{V}_o$ be the universal covering of V_o and $\hat{G} = Is(\hat{V}_o, \pi_1(V_o))$. Then the adjoint group</u> ad $L \subset$ Aut L <u>intersects</u> $a(\hat{G}) \subset$ Aut L <u>by a subgroup of finite index in</u> $a(\hat{G})$. <u>In particular, the image of $\hat{G}$ in</u> Aut C_L <u>is finite</u>.

Next, let G be some isometry group of V preserving a smooth finite measure. Then, obviously, there is a subgroup $G_o \subset G$ of finite index transforming $V_o \to V_o$. Thus the study of $Is(V)$ reduces to $Is(V_o)$ and to save notations we assume below that $V_o = V$.

6.2.D. Now we return to the situation in the beginning of 6.2. with the universal (or holonomy) covering $\hat{V}$ of V. We assume that $\hat{G} = G(\hat{V},\Gamma)$ preserves a smooth measure on $\hat{V}$ coming from a smooth measure on V. Then we consider the Zariski closure $\bar{\Gamma}$ of the image of Γ in (the image of Aut L in) Aut X_L. Then $X_L/\bar{\Gamma}$ is a finite union of algebraic varieties and the action of $\hat{G}$ on $X_L/\bar{\Gamma}$ preserves a finite measure coming from a natural map $V \to X_L/\Gamma \to X_L/\bar{\Gamma}$. Thus we have the same "domination" of orbits of $\hat{G}$ in X_L by those of $\bar{\Gamma}$ as in 5.2.A_2., where the orbits of G were dominated by the orbits of the centralizing Killing algebra L'.

Here are some corollaries.

6.2.D_1. _Let_ G _be a connected isometry group of_ V _preserving a finite measure and such that_ ad $G \subset$ Aut L(G) _is an algebraic group without cocompact normal algebraic subgroup. If_ L(G) _has no center and if the center of the nil-radical of_ L(G) _is at most one-dimensional, then the Lie algebra_ L(G) _is a quotient of_ $L(\bar{\Gamma})$. (Compare 2.4.C_4.).

Proof. The action of G lifts to $\hat{V}$ where it commutes with Γ. Since G has no center, the "projection" of this action to X_L is locally faithful as the kernel should contain the center of L (and hence of L(G)). Since the action of G on $X_L/\bar{\Gamma}$ preserves a measure, it is trivial and the proof follows as in 5.2.A.

6.2.D_2. Suppose Γ contains no free subgroup F_2. Then $\bar{\Gamma}$ is virtually solvable (see [Ti]). Now let $\hat{G}' \subset \hat{G}$ be a non-amenable subgroup which projects onto a discrete subgroup in Is(V). If the (conjugation) homomorphism of $\hat{G}'$ to Ext Aut Γ contains no F_2 in its image, then by [Ti] there is a non-amenable subgroup $\hat{G}'' \subset \hat{G}'$ commuting with Γ. Next, we conclude using 4.5. and [Ti] again that the image of $\hat{G}''$ in Aut X_L contains a discrete F_2. If we apply the proof of 6.2.D_1. to the Zariski closure of this $F_2 \subset$ Aut X_L, we arrive at a contradiction which gives us the proof of 0.7.A.

6.3. _The proof of 0.7.C and 0.8.A._ According to 6.1. the group $G = SL_n\mathbb{R}$ is mes-placed into $\Gamma = \pi_1(V) \subset SL_n\mathbb{Z}$ which by Zimmer's theory (see $[Z]_1$) implies that $SL_n\mathbb{Z}/\Gamma$ is finite. It also follows that $\hat{V}/\hat{G}$ has finite volume, where $\hat{V}$ is the universal covering of V (or rather of the part of V, where the action is locally free) and $\tilde{G}$ is a covering (of order one or two as $\pi_1(SL_n \mathbb{R}) = \mathbb{Z}_2$ for $n \geq 3$) of $SL_n \mathbb{R}$ acting on $\hat{V}$. Furthermore, the group $\bar{\Gamma} \subset \mathrm{Aut}\, X_L$ is locally isomorphic to $G \times K$ where K is compact semisimple. In fact, K is trivial for lattices in $SL_n \mathbb{Z}$ as they contain infinite non-Abelian nilpotent subgroups for $n \geq 3$ (This was pointed out to me by J. Tits). Yet it is worthwhile to keep K in mind as it may appear for more general lattices.

The action of the identity component $\bar{\Gamma}_o \subset \bar{\Gamma}$ lifts to a unique isometric action on $\hat{V}$ commuting with G, such that $\Gamma' = \bar{\Gamma}_o \cap \Gamma$ is a lattice in $\bar{\Gamma}_o$ and of finite index in Γ.

Finally, we observe that on the (G-invariant) subset $U \subset V$, where the action of G is locally free, there exists a (unique) foliation S transversal to the G-orbits and such that the lifted foliation $\hat{S}$ on $\hat{U}$ is $G \times \bar{\Gamma}_o$-invariant. Thus $\hat{U}$ is split by $\hat{S}$ and the orbits of $\tilde{G}$. It follows from the previous discussion that the leaves of $\hat{S}$ have finite volume. Since the orbits of G in $U \subset V$ also have finite volume (by Tits' remark) some finite covering of U, say U' is of the form $S'_o \times (SL_n \mathbb{R}/\Gamma)$ acts in an obvious way.

Finally, assume V is C^{an}. Then the action of $\bar{\Gamma}_o$ is defined on all of $\hat{V}$ (and not only on $\hat{U}$). Since the <u>lattice</u> $\Gamma \subset \bar{\Gamma}_o$ is discrete on V, it acts discretely on each orbit of $\bar{\Gamma}_o$. Hence, the isotropy subgroup of $\bar{\Gamma}_o$ is finite at all $v \in \tilde{V}$. It follows that $\tilde{G}$ also has finite isotropy and so the action of G is locally free on all of V. In other words, $U = V$ and our description of (V,G) is completed. (Probably a similar argument yields local freedom of G in the C^{∞}-case as well).

<u>Remark</u>. The above argument applies to every semisimple Lie group G of A-rigid isometries where $\pi_1(V)$ (or the holonomy group of V) is isomorphic to a discrete subgroup $\Gamma \subset G$.

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