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SMALL CANCELLATION, UNFOLDED HYPERBOLICITY, AND TRANSVERSAL MEASURES

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To my friend André Haefliger

The (word) hyperbolic groups Γ are characterized by hyperbolicity of *all globally* minimal surfaces in the (hyperbolic) spaces *the groups* act upon. There is a distinguished class of these Γ, including the classical *small cancellation groups* acting on 2-polyhedra, where all *locally* minimal surfaces are hyperbolic. (These groups and the corresponding spaces appear in 6.8.U. of [$\mathrm{Gro}_{\mathrm{HG}}$] under the hastily concocted and unappropriate heading of "local hyperbolicity".) We shall give in this article a description of this subclass in terms of the transversal measures in the spaces of *unfolded* surfaces in 2-polyhedra where "unfolded" encodes "locally minimal" in our category.

1. VARIATION AND FOLDS

Given a continuous map $f: S \to X$, we measure its *variation* by the cardinalities of the sets of the connected components of the pull-backs $f^{-1}(x)$, $x \in X$:

$$\mathrm{var}_f(x) = \mathrm{card}\ \mathrm{Conn} f^{-1}(x).$$

If $\mathrm{var}_f(x) < \infty$ for all $x \in X$ (i.e. each $f^{-1}(x)$ has finitely many components) this gives a reasonable order relation on maps. In particular, f is called *minimal* if every homotopy f_t of $f = f_0$ with $\mathrm{var}_{f_t} \leq \mathrm{var}_f$ necessarily has $\mathrm{var}_{f_t} = \mathrm{var}_f$, where "var" is thought of as a function $X \to \mathbf{Z}_+$ and $\mathrm{var} \leq \mathrm{var}'$ refers to the pointwise order.

In general, one may allow infinite variation, though it is irrelevant for the picture at hand, and use the decomposition $f = f_\circ f_\bullet$,

$$S \xrightarrow[f_\bullet]{} S_\circ \xrightarrow[f_\circ]{} X,$$

where the map $f_\bullet$ is surjective and *connected*, i.e. $f_\bullet^{-1}(s)$ is connected for all $s \in S_\circ$, while $f_\circ$ is 0-dimensional, i.e. $f^{-1}(x)$ is totally disconnected for all $x \in X$. Then one compares maps f and f', possibly with different domains S and S', according to the "sizes" of $S_\circ$ and $S'_\circ$, saying that $f' < f$ if there is a continuous map $S'_\circ \to S_\circ$ commuting with $f_\circ$ and $f'_\circ$,

$$\begin{array}{ccc} S'_\circ & \longrightarrow & S_\circ \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Then minimality of f means that every homotopy $f_t < f$ of f satisfies $f_t > f$ (where one may insist that the implied maps $S_\circ \to S_\circ(t)$ are inverse to the maps $S_\circ(t) \to S_\circ$).

Next, f is called *locally minimal* or *unfolded* if each point $s \in S$ admits a neighbourhood $U \subset S$ of s, such that the above property is satisfied for homotopies f_t *constant in* t *outside* U.

If S and U are polyhedra (of arbitrary dimension) and S is compact, then every piece-wise linear map f can be (obviously) homotoped to a minimal $f = f_1$ by a homotopy $f_t < f$. Furthermore, if S is a surface (possibly with a boundary), then the resulting (unfolded!) map f_1 has a very simple structure: there exists a 2-polyhedron $S_1^* = S_1^1 \cup S_1^2$ with $\dim(S_1^1 \cap S_1^2) = 0$, where $\dim S_1^1 = 1$ and where S_1^2 is a (possibly disconnected) surface, and a decomposition of f_1 into two maps $f_1^\bullet$ and f_1^*,

$$\begin{tikzcd} S \arrow[r, "f_1^\bullet"] \arrow[rr, bend right, "f_1"'] & S_1^* \arrow[r, "f_1^*"] & X \end{tikzcd}$$

satisfying the following conditions.

(a) There exists an open subset $S^\vee \subset S$ with piece-wise linear boundary, such that $f_1^\bullet$ is $1-1$ on $S^\vee$ and sends the closure of $S^\vee$ in S *onto* S_1^2, where each connected component of the topological boundary of $S^\vee \subset S$ goes to a single point.

(b) The map f_1^* is 0-dimensional (i.e. with finite pull-backs of points) and unfolded on S_1^2.

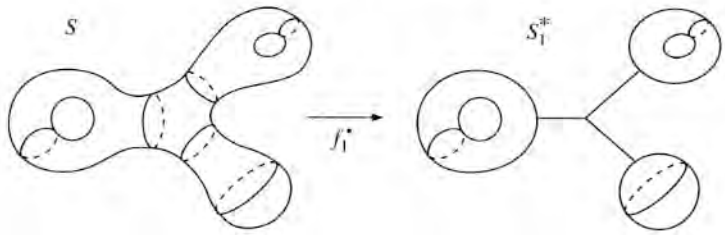

FIGURE 1

Furthermore, if S has non-empty boundary, one can take f_t constant on the boundary coming up with f_1^* minimal *relative* to the boundary of S_1^2, i.e. being unfolded at the *interior* points in S_1^2.

2. SIMPLICIAL HYPERBOLICITY

If X is a simplicial polyhedron, we consider

$$\mathrm{Var}_\Delta f = \sup_{x\in\Delta} \operatorname{card} \operatorname{Conn} f^{-1}(x)$$

for $f\colon S \to X$ and all open simplices $\Delta \subset X$. Then we define

$$|\mathrm{Var}_f|_* \underset{def}{=} \sum_\Delta \mathrm{Var}_\Delta f$$

summing up over all $\Delta \subset X$; we also introduce $|\mathrm{Var}_f|_k$, $k = 0, 1, 2, \ldots$, by limiting the sum to the set of the open *k-dimensional* simplices Δ in X. Thus $|\mathrm{Var}_f|_* = \sum_k |\mathrm{Var}_f|_k$ over $k = 0, 1, \ldots$.

Next, for every compact surface mapped to X, say $f\colon S \to X$, we define its *average Euler characteristics*

$$\chi_{\mathrm{av}}(S) = \chi(S)/\,|\mathrm{Var}_f|_2\,,$$

and introduce *the (supremal) unfolded characteristic* of X as

$$\chi^+_{\mathrm{unf}}(X) = \sup_S \chi_{\mathrm{av}}(S)\,,$$

over all *closed cofinite unfolded* surfaces $f\colon S \to X$, where "cofinite" means f is finite-to-one.

2.1 REMARKS

(a) We shall see below that the condition

$$\chi_{\mathrm{unf}}(X) < 0$$

is equivalent to the "local hyperbolicity" in [$\mathrm{Gro}_{\mathrm{HG}}$] and to *unfolded hyperbolicity* in Section 7.

(b) If one wants to pass to k-dimensional unfolded manifolds and pseudomanifolds S over k-dimensional polyhedra, one may use the *simplicial norm* $|S|$ of S instead of the Euler characteristics (see [$\mathrm{Gro}_{\mathrm{VBH}}$]). It is unclear if it makes much sense for $k > 2$, but the following definition seems rather promising. Given a continuous map f of a closed oriented pseudomanifold S of dimension k into X (here X may have any dimension), consider all oriented pseudomanifolds $\widetilde{S}_d$ along with maps $\widetilde{f}_d\colon \widetilde{S}_d \to S$ of degree d and for every such $\widetilde{S}_d$ define

$$\left|\widetilde{S}_d/X\right|_k = \inf_{f_d} d^{-1}\,|\mathrm{Var}_{f_d}|_k$$

where f_d ranges over all maps $\widetilde{S}_d \to X$ homotopic to $f \circ \widetilde{f}_d\colon \widetilde{S}_d \to X$. Then take

$$|\mathrm{Var}_f|_k^{\sim} \underset{def}{=} \inf_{\widetilde{S}_d} \left|\widetilde{S}_d/X\right|_k$$

over all $\widetilde{S}_d \to S$ and all $d = 1, 2, \ldots$. Now we define

$$(*)_k^{\sim} \qquad |X|_k^{\sim} = \inf_S |S|/\,|\mathrm{Var}_f|_k$$

over all oriented k-pseudomanifolds S and all continuous maps $f\colon S \to X$.

This definition is justified by the fact that the compact aspherical polyhedra X with word-hyperbolic fundamental groups have $|X|_k^{\sim} > 0$ in all dimensions $k \geq 2$ (by the discussion in [$\mathrm{Gro}_{\mathrm{HG}}$]) but the ultimate notion(s) may be somewhat different from the above.

(c) Simplicial unfolded finite-to-one maps between 2-polyhedra, $f\colon S \to X$, can be thought of as generalized ramified coverings of X: the "unfolded" condition at each vertex $s \in S$ says, in effect, that the corresponding map of the link $\mathrm{Li}_s \subset S$ (consisting of all edges opposite to s in the triangles containing s), say $f_s\colon \mathrm{Li}_s \to \mathrm{Li}_x \subset X$ for $x = f(s)$, is *locally* one-to-one.

Unfolded 2-polyhedra S over X can be seen in the *universal ramified cover* $\widetilde{X} \to X$ that is the universal cover of X minus the vertex set (where we tacitly assume that every simplex of dimension < 2 is a face of a 2-simplex): every

connected simplicial unfolded $S \to X$ lifts to an *embedding*, i.e. a 1-1 map, from some ramified covering $\widetilde{S}$ of S to $\widetilde{X}$, where this $\widetilde{S} \hookrightarrow \widetilde{X}$ is unique up-to the action of the (free !) Galois group G of the covering $\widetilde{X} \to X$. Conversely, every $\widetilde{S} \subset \widetilde{X}$ gives rise to an unfolded polyhedron over S, where this S is compact provided X is compact and $\widetilde{S}$ is invariant under a "sufficiently large" subgroup in G.

(d) *On normalization of area.* The denominator $|\mathrm{var}_f|_2$ in the definition of $\chi_{\mathrm{av}}(S)$ is nothing else but the *area* of $S \to X$ counted with multiplicity, where the area structure in X chosen in such a way that every 2-simplex has unit area. Take another Borel measure on X which does not vanish on the 2-simplices, say area$'$ different from area $= |\mathrm{var}|_2$. Then the corresponding averaged Euler characteristic $\chi'_{\mathrm{av}} = \chi(S)/\,\mathrm{area}'(S)$ differs from $\chi_{\mathrm{av}} = \chi(S)/\,\mathrm{area}(S)$ by a positive multiplicative constant

$$C^{-1}\chi_{\mathrm{av}}(S) \leq \chi'_{\mathrm{av}}(S) \leq C\chi_{\mathrm{av}}(S)$$

where C is independent of S, and so the sign of the corresponding $\chi_{\mathrm{unf}}(X)$ and the notion of unfolded hyperbolicity do not depend on a particular choice of the area (structure in X). Yet it would be nice to have a canonical choice of the area and something can be done in this regard. Denote by $X^2_{\mathrm{reg}} \subset X$ the subset where X is locally homeomorphic to $\mathbf{R}^2$ and let $X^2_{\mathrm{reg}}(i)$, $i = 1, \ldots, n$, be the connected components of X^2_{reg}. What matters for the areas of unfolded surfaces are the total areas $\mathrm{area}(X^2_{\mathrm{reg}}(i))$, $i = 1, \ldots, n$, but not particular distributions of the areas (measures) within each X^2_{reg} (actually if two such measures are sufficiently regular they can be brought one into another by a self-homeomorphism of X, provided they give equal areas to $X^2_{\mathrm{reg}}(i)$). Thus the ambiguity in χ_{av} is reduced to n parameters, i.e. we have a non-ambiguous function $\chi_{\mathrm{av}}(S, a_1, \ldots, a_n)$ depending on $f\colon S \to X$ and $a_i = \mathrm{area}\, X^2_{\mathrm{reg}}(i)$. Consequently $\chi_{\mathrm{unf}}(X)$ becomes a function $\mathbf{R}^n_+ \to \mathbf{R}$ *canonically* defined by the topology of X.

This is all very good as far as we deal with a fixed X, but if we turn to families of 2-polyhedra where we may ask for *uniform* unfolded hyperbolicity, the matter of normalization becomes more delicate. Let, for example, X be obtained by attaching 2-cells to a fixed graph X_1 (i.e. 1-polyhedron) where the number of cells and the attaching maps may vary (as happens for families of groups presented by fixed generators and various relations). Here a natural area in X comes from "coning" a given *length structure* in X_1 : the area of a cell C in X is declared equal 1/2 (the length of the attaching map $\partial C \to X_1$).

3. Unfolded foliations

Every unfolded surface $f: S \to X$ defines a (combinatorial) *sheaf* Σ over X given by the following data:

(1) A set $\Sigma(\Delta)$ assigned to each 2-simplex $\Delta \subset X$. The elements of $\Sigma(\Delta)$ are the 2-simplices $\widetilde{\Delta} \subset S$ with $f(\widetilde{\Delta}) = \Delta$.

(2) A set $\Sigma(\Delta, \Delta') = \Sigma(\Delta', \Delta)$ assigned to every pair of 2-simplices Δ and Δ' in X with a common edge $E = \Delta \cap \Delta'$. The elements of $\Sigma(\Delta, \Delta')$ are the edges $\widetilde{E}$ of S, such that $f(\widetilde{E}) = E$ and the two 2-simplices adjacent to $\widetilde{E}$ in S go to Δ and Δ' under f.

(3) Bijective maps $\Sigma(\Delta, E): \Sigma(\Delta) \to \bigsqcup_{\{\Delta_E\}} \Sigma(\Delta, \Delta_E)$, for all $\Delta \subset X$ and edges $E \subset \Delta$, where $\{\Delta_E\}$ denotes the set of all $\Delta' \subset X$ with $\Delta \cap \Delta' = E$. This corresponds to taking the 2-simplex $\widetilde{\Delta}_{\widetilde{E}}$ adjacent to $\widetilde{\Delta}$ in S across the edge $\widetilde{E} \subset \widetilde{\Delta}$.

(3′) There is an alternative description of Σ not using $\Sigma(\Delta, \Delta_E)$ at all, where instead of $\Sigma(\Delta, E)$ one takes the disjoint unions $\Sigma^{\sqcup}(E) = \bigsqcup_{\Delta \supset E} \Sigma(\Delta)$ and then define Σ with involutive maps $\Sigma_E: \Sigma^{\sqcup}(E) \to \Sigma^{\sqcup}(E)$ for all edges E in X, where the "unfolded" condition is expressed by $\Sigma_E(\Sigma(\Delta)) \cap \Sigma(\Delta) = \varnothing$ for all $\Delta \supset E$.

In what follows we shall be alternatively describing Σ by $\{\Sigma(\Delta),\ \Sigma^{\sqcup}(E)\}$ or by $\{\Sigma(\Delta),\ \Sigma(\Delta, \Delta_E), \Sigma(\Delta, E)\}$. Given Σ, there is a unique (possibly disconnected) unfolded surface $f: S \to X$ realizing this sheaf. (This remains valid and obvious in all dimensions $k = 1, 2, 3, \ldots$, with k-dimensional unfolded pseudomanifolds instead of surfaces.)

If S is compact then the sets $\Sigma(\Delta)$ and $\Sigma(\Delta, \Delta')$ are finite and we set

$$N(\Delta) = |\Sigma(\Delta)| \underset{def}{=} \operatorname{card} \Sigma_2(\Delta)$$

and

$$M(E) = \left\{ |\Sigma(\Delta, \Delta')| \;\middle|\; \Delta \cap \Delta' = E \right\}.$$

One regards $M(E)$ as a symmetric matrix associated to the edge E where the rows and columns are indexed by the 2-simplices $\Delta,\ \Delta',\ \Delta'', \ldots$, adjacent to E. The "unfolded" condition says that the diagonal entries of M are zero (since M is positive this is equivalent to trace $M = 0$) while the surface (or cycle) property is reflected in the *Kirchhoff-Markov* kind of relation saying that the *sum of* (the entries in) *each row equals* $N(\Delta)$ *for* Δ *representing this row.*

$$|\Sigma(\Delta)| = |\Sigma(\Delta, \Delta'_E)| + |\Sigma(\Delta, \Delta''_E)| + \dots$$

for all 2-simplices $\Delta \subset X$, all edges $E \subset X$, and all $\Delta'_E, \Delta''_E, \dots$ meeting Δ at E.

Conversely, given positive integers $N(\Delta)$ and positive traceless integer symmetric matrices $M(E)$ satisfying Kirchhoff-Markov relations, there obviously exists a sheaf Σ of finite sets over X with $|\Sigma(\Delta)| = N(\Delta)$ and $\{|\Sigma^{\sqcup}_1(E)|\} = M(E)$ representing an unfolded surface S over X, but this Σ (and S) is by no means unique: there may be many bijective selfmappings $\Sigma^{\sqcup}(E) \to \Sigma^{\sqcup}(E)$.

3.1 From surfaces to foliations

Let us allow arbitrary positive *real* numbers $N(\Delta)$ and positive real traceless symmetric matrices $M(E)$ satisfying the Kirchhoff-Markov relations, called *Kirchhoff-Markov* data and consider *measurable unfolded foliations* Σ *by surfaces* over X defined by the following "extended Markov data".

(1) Measure spaces $\Sigma(\Delta)$ for all 2-simplices $\Delta \subset X$.

(2) Measure preserving involution $\Sigma_E \colon \Sigma^{\sqcup}(E) \to \Sigma^{\sqcup}(E)$ for $\Sigma^{\sqcup}(E) = \bigsqcup_{\Delta \supset E} \Sigma(\Delta)$, where

$$\Sigma_E(\Sigma(\Delta)) \cap \Sigma(\Delta) = \varnothing$$

for all $\Delta \supset E$ and $E \subset X$.

Such a foliation defines numbers

$$N(\Delta) = |\Sigma(\Delta)| \underset{def}{=} \text{the total measure of } \Sigma(\Delta)$$

and matrices

$$M(E) = \{|\Sigma(\Delta, \Delta')|\} \text{ for } \Delta \cap \Delta' = E$$

satisfying the above Kirchhoff-Markov properties. And arbitrarily Markov data $\{N(\Delta), M(E)\}$ can be (obviously) implemented (non-uniquely) by a measurable unfolded foliation over X. (There is a canonical *Markovian* representation of Kirchhoff data but it is not relevant for us at the moment.)

3.2 Remark about measure spaces

The category of measure spaces and (injective) measurable maps "implements" the category $\mathbf{R}_+$ of positive real numbers, where the morphisms are inequalities $x \leq y$, in the same way as the category of finite sets implements $\mathbf{Z}_+$. Intuitively, a single point of mass $x \in \mathbf{R}_+$ is regarded as a measure space of total mass x (one may take the "string" $[0, x]$) with "internal symmetries":

the automorphisms of the measure space. This elevates arithmetic operations in $\mathbf{R}_+$ to the categorical constructions of the measure theory. For example, sums $x = x' + x'' + \dots$ become measurable partitions, positive matrices turn into bijections between partitioned (by rows and columns) spaces, etc.

3.3 On stratified measurable foliations

The above Σ makes a *measurable foliation* by surfaces away from the vertices of X. The *transversal measurable structure* of Σ over each 2-simplex Δ is given by $\Sigma(\Delta)$, while over an edge E this is the quotient space $\Sigma^{\sqcup}(E)/\Sigma_E$.

Now let us see what happens over the vertices $x \in X$. The restriction of Σ to the link $\mathrm{Li}_x \subset X$ makes an unfolded measurable foliation by curves (or paths) over the graph Li_x. The global measure space where these curves reside is the disjoint union

$$\Sigma^{\sqcup}(x) \underset{def}{=} \bigsqcup_{\Delta \supset x} \Sigma(\Delta),$$

where the leaves are curves of two kinds: some of them are closed, homeomorphic to S^1 which correspond to certain cycles in Li_x, and the remaining curves are copies of $\mathbf{R}$ projecting to double infinite paths in Li_x. The space of all these curves can be thought as a "*non-commutative, or singular measure space*" (see [Con]), that is the quotient space

$$\Sigma(x) \underset{def}{=} \Sigma^{\sqcup} x)/\sim,$$

where two points in $\Sigma^{\sqcup}(x)$ are declared equivalent by "$\sim$" if and only if they are contained in the same curve (leaf). The theory of measurable foliations (which are singular measure spaces anyway, see [Con]) can be augmented by allowing singular transversal measurables structures compatible with a certain stratification of the global space. (Such structures are seen, for example, in higher dimensional unfolded Σ and in measure foliations associated to *ramified* coverings of complex manifolds.) The theory of measurable foliations (due to Alain Connes), e.g. the index and Riemann-Roch theorems, measurable simplicial volume(s) etc., perfectly blends with stratified structures as we shall see somewhere else. Here, however, we only concerned with surfaces and the *regular* part of $\Sigma(x)$, corresponding to closed leaves (curves) in there.

4. Euler characteristic of Σ

Let $|\Sigma(\Delta)|$, $|\Sigma(E)|$ and $|\Sigma(x)|$ denote the total masses of the corresponding measure spaces, where the latter takes into account only the regular part of

$\Sigma|\,\mathrm{Li}_x$; namely

$$|\Sigma(x)| \underset{def}{=} |\Sigma^{\sqcup}_{\mathrm{reg}}(x)/\sim|$$

where $\Sigma^{\sqcup}_{\mathrm{reg}}$ is the union of the closed leaves in $\Sigma^{\sqcup}(x)$. (One could define $|\mathcal{M}/\sim|$ in general as the infimum of the measures of the subsets in $\mathcal{M}$ meeting all equivalence classes.) Now we define the Euler characteristic of Σ by taking the alternating sum of the above masses (weights) of the simplices x, E and Δ in X,

$$\chi(\Sigma) \underset{def}{=} \sum_x |\Sigma(x)| - \sum_E |\Sigma(E)| + \sum_\Delta |\Sigma(\Delta)|\,,$$

and sometimes we normalize to

$$\chi_{\mathrm{av}}(\Sigma) = \chi(\Sigma)/\sum_\Delta |\Sigma(\Delta)|\,.$$

This $\chi(\Sigma)$ enjoys the usual features of the Euler characteristic: the Gauss-Bonnet formula (for piece-wise smooth metrics in the leaves), the Euler-Poincaré formula (for the L_2-cohomolgy), the proportionality to the simplicial volume, etc. What is most relevant to us at the moment is the agreement of this with χ_{av} for closed unfolded surfaces over X.

5. Rational foliations

If the measure spaces $\Sigma(\Delta)$ and $\Sigma(E)$ defining Σ are *finite*, i.e. are collections of finitely many points, called *atoms*, of a certain mass > 0, then the total space $[\Sigma]$ of Σ decomposes into finitely many *weighted surfaces*, that are the leaves of Σ, say

$$[\Sigma] = \bigcup_k p_k S_k\,,$$

where p_k is the (transversal) measure of the leaf $S_k \subset [\Sigma]$. In particular, if all atoms have *rational* weights, the numbers p_k are also rational > 0 and $[\Sigma]$ is uniquely represented as $w^{-1}S$ for an unfolded (typically disconnected) surface S over X with a weight w^{-1} for an integer w. Clearly,

$$\chi_{\mathrm{av}}(\Sigma) = \sum_k p_k \chi_{\mathrm{av}}(S_k)/\sum_k p_k\,.$$

6. Surfaces with boundary and sub-Markov data

If $S \to X$ is an unfolded surface with a non-empty boundary, one defines as earlier positive numbers $N(\Delta)$ and symmetric positive traceless matrices $M(E)$, where the Kirchhoff-Markov equalities do not hold anymore but are replaced by *sub-Markovian inequalities*:

The sum of the entries in each column (row) of $M(E)$ is $\leq N(\Delta)$ for Δ corresponding the column (row) in question. Then we define (unfolded measurable) foliations Σ by *surfaces with boundaries*, where $\Sigma(\Delta)$ and $\Sigma^{\sqcup}(E)$ are as earlier but where the involutions Σ_E are only *partially* defined, say on subsets $\Sigma^{\mathrm{Int}}(E) \subset \Sigma^{\sqcup}(E)$

$$\Sigma_E : E^{\mathrm{Int}}(E) \to \Sigma^{\mathrm{Int}}(E) .$$

Clearly, the corresponding numbers $N(\Sigma)$ and matrices $M(E)$ make sub-Markov data for these foliations.

Next we introduce the *boundary* $\partial\Sigma$ which is the assignment of the measurable set

$$\partial\Sigma(\Delta, E) \underset{def}{=} \Sigma(\Delta) \backslash \Sigma^{\mathrm{Int}} \subset \Sigma^{\sqcup}(E)$$

to each pair $(\Delta, E \subset \Delta)$.

6.1 Remark

There is more than a mere measure to $\partial\Sigma$: it carries a natural foliated structure over the graph $\widetilde{X}_1$ where the vertex set equals that of X and where the edges between vertices $x_1, x_2 \in X_1$ correspond to pairs $(\Delta, E \subset \Delta)$ with the edge E joining x_1 with x_2 in X. (Thus the number of edges between x_1 and x_2 in $\widetilde{X}_1$ equals the number of 2-simplices containing an edge joining x_1 with x_2 in X.) Let

$$\widetilde{\Sigma}_x = \bigsqcup_{\widetilde{E}_x} \Sigma(\Delta(\widetilde{E}_x)) ,$$

where $\widetilde{E}_x$ runs over the edges in $\widetilde{X}$ adjacent to x and where $\Delta(\widetilde{E})$ refers to the 2-simplex of the pair $(\Delta, E) = \widetilde{E}$. The foliated structure in Σ induces a measure preserving automorphism of $\widetilde{\Sigma}_x$, say $\tilde{\alpha}_x$, defined as follows. Denote by $\Sigma_E(\Delta) \subset \widetilde{\Sigma}_x$ the copy of $\Sigma(\Delta)$ corresponding to some $\widetilde{E}_x = (\Delta, E)$ and observe that for each point $\sigma \in \Sigma_E(\Delta)$ there are two possibilities:

(1) $\sigma \in \Sigma^{\mathrm{Int}}$.

In this case the map $\Sigma(E)$ sends σ to some $\sigma' \underset{def}{=} \tilde{\alpha}_x(\sigma) \in \Sigma_E(\Delta') \subset \widetilde{\Sigma}_x$ where $\Delta' \neq \Delta$ is a certain 2-simplex adjacent to E (and thus to x).

(2) $\sigma \in \partial\Sigma(\Delta, E)$.

Here we take the second edge $E' \subset \Delta$ adjacent to x and send σ to $\tilde{\alpha}_x(\sigma) \in \Sigma_{E'}(\Delta)$ via the identity map $\Sigma_E(\Delta) \to \Sigma_{E'}(\Delta)$.

We construct the foliated structure $\widetilde{\Sigma} = \partial\Sigma$ over $\widetilde{X}_1$ by taking $\widetilde{\Sigma}(\widetilde{E}) = \partial\Sigma(\Delta, E)$ for $(\Delta, E) = \widetilde{E}$, and we observe the natural embedding $\widetilde{\Sigma}^{\sqcup}(x) \subset \widetilde{\Sigma}_x$. Then we define the (foliation) structure maps $\widetilde{\Sigma}_{\widetilde{E}}\colon \widetilde{\Sigma}^{\sqcup}(x) \to \widetilde{\Sigma}^{\sqcup}(x)$ as the *first return maps* of $\tilde{\alpha}_x$. Namely $\widetilde{\Sigma}_{\widetilde{E}}(\sigma)$ equals $\underbrace{\tilde{\alpha}_x \circ \tilde{\alpha}_x \circ \ldots \circ \tilde{\alpha}_x}_{k}(\sigma)$ for the first integer $k \geq 1$ where this composed map sends σ to $\widetilde{\Sigma}_{\widetilde{E}}(\sigma)$. Since we (always tacitly) assume our measure spaces to be of finite measure, this $\widetilde{\Sigma}_{\widetilde{E}}$ is a.e. defined on $\widetilde{\Sigma}^{\sqcup}(x)$ and does the job.

Clearly $\partial\partial = 0$ (for all dimensions ≥ 2). This can be given a more algebraic flavour by passing to the Grothendieck group of measurable k-dimensional sub-Markovian (i.e. with boundaries) foliations over X, where the sum corresponds to the disjoint union, so that these groups, say $\mathcal{F}^k$, make a chain complex under ∂. The homology of this complex surjects to the ordinary homology (with real coefficients) and in many cases this surjection is also injective. (It is typically a non-injection in the top dimension but this can be probably regained somehow stabilizing X.)

6.2 Filling-in the boundary

An edge E in X is called *free* if there is at most one 2-simplex containing it. Observe that every X contains a unique maximal subpolyhedron $X_\circ \subset X$ with no free edges. And if X is connected with all edges contained in 2-simplices, then X homotopy retracts on $X_\circ$.

6.2.A. Canonical Markov data. If X has no free edges, then we define $N_c(\Delta) = c$ for a given $c \geq 0$ and all 2-simplices Δ in X and $M_c(E) = \{c(d(E)-1)^{-1}\}$, where $d = d(E)$ denotes the number of 2-simplices adjacent to E and $\{(d-1)^{-1}\}$ is the matrix with all off-diagonal entries $(d-1)^{-1}$ and the zero diagonal. Clearly (N_c, M_c) make Markov data and if X is finite, then for arbitrary sub-Markov (N, M) there exists $c \geq 0$, such that $N_c \geq N$ and $M_c \geq M$ where the latter inequality refers to the corresponding inequalities between the entries of the matrices. It follows that for every Σ

with boundary, there exists $\Sigma_c^+ \supset \Sigma$ without boundary with $|\Sigma_c^+(\Delta)| = N_c(\Delta)$ and $|\Sigma_c^+(E)| = M_c(E)$ for all Δ and E in X. In fact, passing from numbers to foliations, amounts here to extending partial involution to involutions defined everywhere.

Let us take $(N_c, M_c) \geq (N, M)$ with the minimal possible c, denote this by (N_+, M_+) and let $N_- = N_+ - N$. Clearly, N_- is bounded by the "sub-Markovian errors" of (N, M) and if (N, M) is rational then so is N_-. It follows that the complementary foliation(with boundary) $\Sigma_- = \Sigma_+ \backslash \Sigma$ has $|\Sigma_-| \leq b|\partial\Sigma|$ for some $b \geq 0$ depending on the total number of 2-simplices in X. In other words, *the boundary $\partial\Sigma$ of Σ can be filled in by Σ_- satisfying the linear isoperimetric inequality.*

6.3 Reduced foliations

Remove the vertices of our 2-polyhedron X and let $\widetilde{X}^\bullet$ be the universal covering of X minus the vertices. Every leaf of an unfolded foliation Σ over X lifts to a connected (embedded!) surface in $\widetilde{X}^\bullet$ that is unique up to the action of the Galois group Γ of $\widetilde{X}^\bullet \to X$. Thus the measurable structure of Σ induces a measure μ on the space of connected surfaces in $\widetilde{X}^\bullet$. To set it up straight, denote by $\tilde{2}^X$ the space of all sub-polyhedra in $\widetilde{X}^\bullet$ with the obvious (totally disconnected) topology and let $\mathcal{C} \subset \tilde{2}^X$ consist of the (strongly) *connected* non-empty subpolyhedra. (Our surfaces are *strongly* connected with the relevant neighbourhood relation requiring a common edge between 2-simplices, and when we return to surfaces we shall read "connected" as "strongly connected".) Observe that $\mathcal{C}$ is *not* a closed subset in $\tilde{2}^X$ and to remedy this we introduce another (partial) topology in $\mathcal{C}$, different from the one induced from $\tilde{2}^X$. Pick a simplex Δ and let $\mathcal{C}_\Delta \subset \mathcal{C}$ consist of subsets $S \in \mathcal{C}$ with $\Delta \in \mathcal{C}$. Similarly, for each sub-polyhedron $Y \subset \widetilde{X}^\bullet$ containing Δ, we introduce the set $\mathcal{C}_\Delta(Y)$ consisting of all sub-polyhedra in Y containing Δ and for every $Y' \subset Y$ we map $\mathcal{C}_\Delta(Y) \to \mathcal{C}_\Delta(Y')$ by sending each $S \in \mathcal{C}_\Delta(Y)$ to the *connected component* of Δ in the intersection $S \cap Y'$, i.e. $S \mapsto S' = \mathrm{Conn}_\Delta(S \cap Y')$. Thus $\mathcal{C}_\Delta$ emerges as the projective limit of the finite sets $\mathcal{C}_\Delta(Y)$ for all finite sub-polyhedra $Y \subset \widetilde{X}^\bullet$ and we give the projective limit topology to $\mathcal{C}_\Delta$, called *$\mathcal{C}$-topology*. (Recall that the topology coming from $\tilde{2}^X$ is the projective limit for the same sets $\mathcal{C}_\Delta(Y)$ but with different maps, namely $S \mapsto S \cap Y'$.)

6.3.A. Definition. A *(reduced) $\mathcal{S}$-foliation over X* is a Γ-invariant measure μ on $\mathcal{C} \subset \tilde{2}^X$, such that the restriction of this μ to each subset $\mathcal{C}_\Delta \subset \mathcal{C}$, $\Delta \subset X$ is a Borel measure on the (compact!) space $\mathcal{C}_\Delta$ supported

on the (closed!) subset $\mathcal{S}_\Delta \subset \mathcal{C}_\Delta$ of connected surfaces in X containing Δ. Clearly, every $\mathcal{S}$-foliation canonically defines our old measurable foliation over X.

An advantage of the new definition is a possibility to take *limits* of $\mathcal{S}$-foliations understood as weak limits of the corresponding measures. Clearly, the space of $\mathcal{S}$-foliations with uniformly bounded masses of the corresponding $\mathcal{S}_\Delta$ for all $\Delta \subset X$ is *weakly compact* for the corresponding weak topology.

7. Unfolded hyperbolicity

A finite 2-polyhedron X is called *unfolded hyperbolic* or *un-hyperbolic* if it satisfies one of the four conditions in the following

THEOREM 7.1. *The conditions (1)–(4) below are mutually equivalent.*

(1) $\chi^+_{\mathrm{unf}}(X) < 0$, *i.e. the Euler characteristic of every closed unfolded surface $S \to X$ satisfies $\chi(S) \leq \kappa_-$ area (S) for some $\kappa_- = \kappa_-(X) < 0$, where the area of S refers to the number of 2-simplices in S.*

(2) *Every measurable unfolded foliation Σ over X by surfaces without boundaries has $\chi(\Sigma) < 0$. (Here we do not a priori exclude $\sup_\Sigma \chi(\Sigma) = 0$ but this cannot happen as becomes clear below.)*

(3) *Every unfolded disk D over X has*

$$\mathrm{area}(D) \leq \lambda\, \mathrm{length}(\partial D)$$

for some $\lambda = \lambda(X) > 0$ with the length of the boundary measured by the number of edges in ∂D.

(4) *Every unfolded foliation Σ with boundary over X satisfies*

$$|\Sigma| \leq \lambda |\partial\Sigma| - \theta\chi(\Sigma)$$

for some positive λ, θ (depending on X), where $|\Sigma|$ denotes the sum of the masses of $\Sigma(\Delta)$ over all 2-simplices $\Delta \subset X$ and $|\partial\Sigma|$ the sum of masses of $\partial\Sigma(\Delta, E)$ over all pairs $(\Delta, E \subset \Delta)$ in X.

(This theorem brings some psychological satisfaction without being truly useful as the standard criteria for un-hyperbolicity deliver all four properties at the same time as we shall see below.)

Proof. Every unfolded surface S defines a (rational) foliation Σ over X which can be reduced to a measure μ on $\mathcal{C}$ which we normalize by $\sup_{\Delta} \mu(\mathcal{C}_\Delta) = 1$, $\Delta \in \widetilde{X}^\bullet$, and denote Σ_μ over X. Then, for every sequence of surfaces Σ_i, there is a subsequence for which the corresponding sequence of measures μ_i weakly converges to a measure μ_∞ on $\mathcal{C}$ where the corresponding foliation Σ_∞ clearly satisfies:

if $\chi(S_i)/\operatorname{area}(S_i) \to 0$, then $\chi(\Sigma_\infty) = 0$;

if length $\partial S_i/\operatorname{area} S_i \to 0$, then $|\partial\Sigma_\infty| = 0$, i.e. Σ_∞ has no boundary. Thus,

$$(2) \Rightarrow (1) \text{ and } (2) \Rightarrow (3).$$

Let us show that (1) $\Rightarrow$ (3). To achieve this we complement the foliation Σ_D corresponding to an unfolded disk by a rational foliation $\Sigma_+ \supset \Sigma_D$ without boundary with $|\Sigma_- = \Sigma_+ \backslash \Sigma_D| \leq b|\partial\Sigma_D|$. (This works if X has no free edges and the general case reduces to this by looking at $X_0 \subset X$.) Next we turn to (3) $\Rightarrow$ (2). Suppose we have Σ with $\chi(\Sigma) = 0$ (the case $\chi(\Sigma) > 0$ is trivial as such Σ contains spherical leaves). Then the Kobayashi metric in the leaves necessarily degenerates thus making (almost) all leaves parabolic and therefore containing disks violating (3) by Ahlfors' lemma (combined with an obvious piece-wise linear approximation of the boundaries, see [$\mathrm{Gro}_{\mathrm{MS}}$]). Finally, we observe that (2) yields inequality $\chi(\Sigma) \leq \kappa_- < 0$ by reducing Σ's to measures on $\mathcal{C}$ and then passing to weak limits. Then we see that (2) $\Rightarrow$ (4) by the complementation argument (used for (1) $\Rightarrow$ (3)) and since (4) obviously implies (1), (2) and (3) the proof is concluded. $\square$

7.2. REMARKS

(a) A closed *topological* unramified surface S over X can be turned into a *simplicial* one by moving the ramification points (where the implied map $S \to X$ is not locally one-to-one) to the vertices of X. It follows that unfolded hyperbolicty is a *topological* invariant of compact 2-polyhedra albeit not a homotopical one.

(b) If $\dim X > 2$, one can consider a class $\mathcal{S}$ of surfaces distinguished by some (local) data and define the *$\mathcal{S}$-hyperbolicity* with the corresponding $\chi^+_{\mathcal{S}}(X)$. For example, compact spaces X with negative curvature have $\chi^+_{\mathcal{S}}(X) < 0$ for the class $\mathcal{S}$ of all *saddle* surfaces in X; thus our unfolded (semi-local) hyperbolicity mediates between the (local) condition $K < 0$ and the ordinary (global) hyperbolicity.

Another interesting class $\mathcal{S}$ is constituted by *holomorphic* curves in an (almost) *complex manifold* X, where the $\mathcal{S}$-hyperbolicity is similar to the

Kobayashi hyperbolicity and where the above equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) remain problematic.

QUESTION. What happens to (1)–(4) for k-dimensional unfolded pseudo-manifolds S over k-dimensional polyhedra with something like the simplicial volume replacing $\chi(S)$ for $k \geq 3$?

8. NUMERICAL CRITERIA FOR UN-HYPERBOLICITY

It is unclear if there is an effective criterion for deciding whether $\chi^{+}_{\mathrm{unf}}(X) < 0$ (where the difficulty appears already for the spaces with $K(X) \leq 0$ and prevents one from developping a comprehensive "semi-hyperbolic theory"). On the other hand, there are several sufficient *small cancellation conditions* for un-hyperbolicity. These impose certain restrictions on small (pieces of) surfaces over X implying $\chi \leq -\kappa < 0$ for *all* surfaces.

NMR-data. The information needed for reconstruction of the Euler characteristic of Σ involves three sets of numbers:

(i) $\{N(\Delta) = |\Sigma(\Delta)|\}$ for all 2-simplices Δ;

(ii) $\{|M(E)| = |\Sigma(E)|\}$ for all edges E, where the norm $|M|$ of a (positive symmetric) matrix M is defined as $\frac{1}{2}$ (the sum of the entries of M);

(iii) $\{|\Sigma(x)| = |\Sigma^{\sqcup}(x)/\sim|\}$ for all vertices $x \in X$, where the latter can be decomposed, at each x, into the following sum over all *cycles* C in Li_x. Here the cycles are represented by *unfolded* maps of the circle into (the graph) Li_x, where two such maps define the same C if and only if they differ by a (reparametrization) homeomorphism of the circle.

Suppose C consists of k edges corresponding to some 2-simplices cyclically arranged around x, say

$$\Delta_1 \ni x,\ \Delta_2 \ni x,\ \ldots,\ \Delta_k \ni x,\ \Delta_{k+1} = \Delta_1\,,$$

and let $\Sigma^{\sqcup}(C) \subset \Sigma(\Delta_1) \sqcup \Sigma(\Delta_2) \sqcup \ldots \sqcup \Sigma(\Delta_k)$ consist of the (cyclic) sequences of points $\sigma_i \in \Sigma(\Delta_i)$, $i = 1, \ldots, k$, corresponding to the closed curves (leaves) in the foliation $\Sigma|\,\mathrm{Li}_x$ which projects to Li_x according to (the circle map parametrizing) C. The measure $|\Sigma^{\sqcup}(C) \cap \Sigma(\Delta_i)|$, $i = 1, \ldots, k$ is independent of i and is denoted

$$R(C) = k^{-1}|\Sigma^{\sqcup}(C)|\,.$$

Clearly,

$$(*) \qquad |\Sigma(x)| = \sum_C R(C)$$

summed over all cycles C in Li_x; furthermore

$$(**) \qquad |\Sigma(\Delta_x)| = \sum_C (C \times \Delta_x) R(C) + R_\infty(\Delta_x)$$

where $C \times \Delta_x$ denotes the multiplicity of the entrance of (the edge opposite to x in) Δ_x in C, i.e. the multiplicity of the corresponding map $S^1 \to \mathrm{Li}_x$ over the edge corresponding to Δ_x and where $R_\infty(\Delta_x)$ denotes the measure of non-compact leaves (curves) in $\Sigma^{\sqcup}(x)$ intersecting $\Sigma(\Delta_x) \subset \Sigma^{\sqcup}(x)$. In other words $R_\infty(\Delta_x) = |(\Sigma^{\sqcup}(x) \backslash \Sigma^{\sqcup}_{\mathrm{reg}}(x)) \cap \Sigma(\Delta_x)|$.

Similarly, the M-data at each edge E issuing from x is expressible in terms of $R(C)$ for the cycles C in Li_x passing through the (chains of) pairs of edges opposite to x in 2-simplices Δ_x and Δ'_x with $\Delta_x \cap \Delta'_x = E$. Namely

$$\binom{**}{*} \qquad |\Sigma(\Delta_x \cap \Delta'_x)| = \sum_C \left(C \times (\overline{\Delta_x \Delta'_x})\right) R(C) + R_\infty(\overline{\Delta_x \Delta'_x})$$

where $\overline{\Delta_x \Delta'_x}$ denotes the path of the edges in Li_x opposite to x in Δ_x and Δ'_x, where $C \times (\overline{\Delta_x \Delta'_x})$ denotes the number of passages of C through this chain and $R_\infty(\overline{\Delta_x \Delta'_x})$ stands for the measure of the infinite leaves (curves) passing over $\overline{\Delta_x \Delta'_x}$.

Notice that the terms $R_\infty(\Delta_x)$ in $(**)$ and $R_\infty(\Delta_x \Delta'_x)$ in $\binom{**}{*}$ are *positive* numbers. If these are not specified, then $(**)$ and $\binom{**}{*}$ become inequalities, rather than identities. These R_∞-terms can be bounded in terms of $R(C)$ since

$$\sum_{\overline{\Delta_x \Delta'_x}} R_\infty(\overline{\Delta_x \Delta'_x}) = \sum_{\Delta_x} R_\infty(\Delta_x) = \sum_{D_x} |\Sigma(\Delta_x)| - \sum_C k(C) R(C)$$

for $k(C)$ denoting the number of edges in C.

8.1 Definition

A collection of numbers $\{N(\Delta)\}$, $\{M(E)\}$ and $\{R(C)\}$ for all $C \in \mathrm{Li}_x$ and vertices x in X constitute NMR-data if the matrices $M(E)$ satisfy the Kirchhoff-Markov relations and

$$(\star) \qquad M(\Delta_x, \Delta'_x) \leq \sum_C (C \times (\overline{\Delta_x \Delta'_x}) R(C))$$

for all vertices $x \in X$ all edges E issuing from x and all pairs of 2-simplices with $\Delta_x \cap \Delta'_x = E$. NMR-data are called *saturated* if the inequalities in $(\star)$

are replaced by equalities. (These correspond to Σ with finite ramifications at the vertices $x \in X$.)

The space of all NMR-data constitute a convex cone, denoted $\overline{NMR}(X)$ in the linear space with the basis marked by the countable set

$$\{\Delta,\ E,\ C \subset \mathrm{Li}_x,\ y \in X\}\,.$$

We say that NMR-data are *exact* if they come from some Σ over X. These exact data $NMR = NMR(\Sigma)$, for all possible Σ, make a *convex* subcone in $\overline{NMR}(X)$, denoted $\overline{ENMR}(X) \subset \overline{NMR}(X)$.

Recall that the Euler characteristic of Σ is expressible by

$$(+) \qquad \chi(\Sigma) = |N| - |M| + |R|$$

for $|N| = \sum_{\Delta} N(\Delta)$, $M = \sum_{E} |M(E)|$ and $|R| = \sum_{x} |R(x)|$, where $R(x) = \sum_{C} R(C)$ for $C \subset \mathrm{Li}_x$, and we extend this to all of $\overline{NMR}(X) \supset \overline{ENMR}(X)$ using $(+)$ as the *definition*:

$$(+^{\bullet}) \qquad \chi(NMR) = |N| - |M| + |R|\,.$$

Then we introduce

$$\chi_{\mathrm{unf}}(X) = \sup_{NMR} \chi(NMR)$$

over all $NMR \in \overline{NMR}(X)$ normalized by $|N| = 1$ (where "1" refers to 1-layer Van-Kampen diagrams). Clearly, $\chi'_{\mathrm{unf}} \geq \chi^{+}_{\mathrm{unf}}$ and so negativity of $\chi_{\mathrm{unf}}(X)$ is sufficient for the unfolded hyperbolicity of X.

The computation of $\chi'_{\mathrm{unf}}(X)$ may first seem a transcendental task as there are infinitely many cycles C. However, one only needs *simple* cycles in Li_x, i.e. those which pass through every edge of the graph Li_x at most once. Namely, let $\overline{NMS}(X) \subset \overline{NMR}(X)$ by the (finite dimensional!) subpolyhedron where $R(C)$ vanishes on all non-simple cycles and let $\chi^{s}_{\mathrm{unf}}(X)$ be defined as the supremum of $\chi(NMR)$ over $\overline{NMS}(X) \subset \overline{NMR}(X)$ normalized as earlier by $|N| = 1$. Obviously $\chi^{s}_{\mathrm{unf}} \leq \chi'_{\mathrm{unf}}$ and a little thought reveals that, in fact, $\chi^{s}_{\mathrm{unf}} = \chi'_{\mathrm{unf}}$. Indeed every non-simple cycle decomposes into shorter simple ones and such decomposition increases R and thus χ. Therefore our "numerical Euler characteristic" χ'_{unf} is (and rather easily) computable for finite 2-polyhedra X and the condition $\chi'_{\mathrm{unf}} < 0$ provides an effective sufficient condition for un-hyperbolicity of X.

8.2 REMARKS

(a) The condition $\chi'_{\text{unf}} < 0$ mediates between unfolded hyperbolicity and the existence of a metric of negative curvature on X. In fact, it is equivalent to the existence of a *conformal structure* of negative curvature (see [Ger], [Gro_{HG}]) as seen by dualizing the (linear/convex) maximization problem for χ over $\overline{NNS}(X)$.

(b) The condition $\chi' < 0$ is not general enough to imply the standard small cancellation conditions. These will be taken care of below.

(c) The characteristic $\chi'_{\text{unf}}(X)$ is obviously invariant under subdivisions of X and so it is a topological invariant of X. In fact, the whole polyhedron $\overline{NMR}(X)$ behaves in a transparent way under subdivisions (and moving ramification points from the interiors of the simplices Δ and E to vertices) and can be used for extracting further topological invariants of X. Eventually, these invariants can be computed with the canonical topological stratification of X.

(d) The structure of $\overline{ENMR}(X)$ is by far more intricate than that of $\overline{NMR}(X)$ as it encodes the *global* topology of unfolded surfaces over X, not only the local data. One can elaborate further and try to look at the *conformal* structure of these surfaces in the framework of topological foliations by surfaces. Similarly, one is interested in (various classes of) such foliations made of (compact as well as infinite) holomorphic curves in algebraic manifolds (over $\mathbf{C}$).

9. TWO-LAYER DIAGRAMS AND THE $^1/_{6+\varepsilon}$-CONDITION

The NMR-data encodes the measures of germs of surfaces at singular points in X: N counts 2-simplices, M is responsible for the edges, and R takes care of the vertices. One can go one step further and numerically encode the information on the measures of germs of neighbourhood of certain pieces of surfaces, e.g. of simplices, cells etc. Thus one obtains various intermediate convex polyhedral cones $\mathcal{C}$ pinched between $\overline{ENMR}$ and $\overline{NNR} \supset \overline{ENMR}$, where the corresponding (maximal) Euler characteristic $\chi^{\mathcal{C}}_{\text{unf}}(X)$ satisfies

$$\chi^{+}_{\text{unf}} \leq \chi^{\mathcal{C}}_{\text{unf}} \leq \chi'_{\text{unf}}$$

and $\chi^{\mathcal{C}}_{\text{unf}}$ converges to χ^{+}_{unf} as we take into account larger and larger pieces of surfaces over X as we shall see somewhere else. We shall need below $\mathcal{C}$ corresponding to pieces consisting of small neighbourhood of certain

(unramified) cells over X and we shall evaluate the arising characteristic $\chi^{\text{cell}}_{\text{unf}}(X)$ in a particular useful case.

Start with a closed surface S decomposed into cells C_i, $i = 1, \ldots, n$. Actually, $C_i \subset S$ can be arbitrary connected disjoint open subsets with piecewise smooth boundaries such that the closure of their union equals S and their complement makes a finite graph $S_1 \subset S$. The graph S_1 may contain isolated vertices with no edges adjacent to them as well as *free vertices* with a single edge adjacent to such vertex. In this case we take the maximal subgraph $S_1^\bullet \subset S_1$ where each vertex contains at least two edges attached to it and enlarge C_i accordingly by taking $C_i^\bullet$ equal the closure of C_i in the complement $S \backslash S_1^\bullet$. The *vertices* of $S_1^\bullet$ are those points $s \in S$ where at least *three* cells $C_i^\bullet$ meet and where the self-intersection of the closure of a single $C_i^\bullet$ counts. In other words, $s \in S_1^\bullet$ is a vertex if and only if for each neighbourhood $U_s \subset S$ of s the intersection $U_s \cap (\bigcup_{i=1}^{n} C_i^\bullet)$ contains at least three connected components. The remaining part of $S_1^\bullet$ consists of the union of topological intervals, that are *edges*, and, possibly, several circles. If we look at the boundary of a $C_i^\bullet$ from inside, then every connected component of it looks like a circle, denoted $\partial_{\text{int}} C_i^\bullet$, with a distinguished finite subset of the vertex points corresponding to the vertices in $S_1^\bullet$. Several $C_i^\bullet$-vertices may correspond to a single vertex in $S_1^\bullet$: this happens when the closure of $S_i^\bullet$ meets itself.

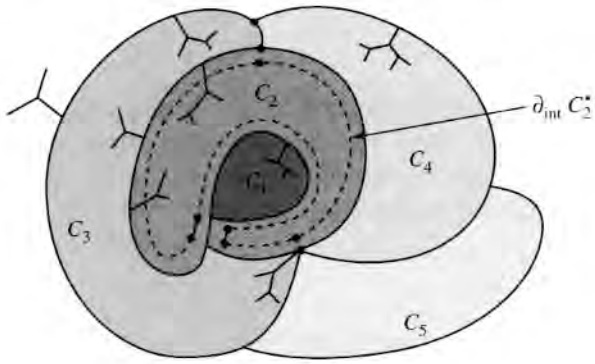

FIGURE 2

We denote by $|E|(C_i^\bullet)$ the number of the edges into which the $C_i^\bullet$-vertices divide $\partial_{\text{int}} C_i^\bullet$ (where undivided circular components do not count) and observe the corresponding number of vertices seen from inside of each $C_i^\bullet$ equals

$|E|(C_i)$. We assign to each vertex $s \in S_1^\bullet$ its "meeting multiplicity", that is the minimal possible number of components in $U_s \cap (\bigcup_i C_i^\bullet)$ or, equivalently the degree (or valency) of the graph $S_1^\bullet$ at s. Then we define the *meeting (Euler) characteristic* of C_i by

$$\chi_{\mathrm{mee}}(C_i) \underset{def}{=} \chi_{\mathrm{mee}}(C_i^\bullet) = \chi(C_i^\bullet) - \frac{1}{2}|E|C_i^\bullet + \sum_j \deg^{-1}\left(s_j(C_i^\bullet)\right)$$

where the latter sum is taken over all $C_i^\bullet$-vertices s_j of $C_i^\bullet$. This $\chi_{\mathrm{mee}}(C_i)$ depends only on the patterns of the cells meeting C_i and it tends to be small for large $|E|$, since the number of summands in $\sum_j$ equals $|E|$ while all degrees there are at least three. For example C_1 and C_2 in Fig. 2 have $\chi_{\mathrm{mee}}(C_1) = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $\chi_{\mathrm{mee}}(C_2) = 1 - \frac{6}{2} + (\frac{5}{3} + \frac{1}{4}) = -\frac{1}{12}$. Obviously, these meeting characteristics add up to the ordinary Euler characteristic of S,

$$\sum_{i=1}^{n} \chi_{\mathrm{mee}}(C_i) = \chi(S).$$

In particular, if $|E|(C_i) \geq 6$ for all i, then $\chi(S) \leq 0$ since $\chi_{\mathrm{mee}}(C) \leq 1 - {}^{|E|}/_2 + {}^{|E|}/_3$, and if $|E|(C_i) \geq 7$, then $\chi(S) \leq -{}^{n}/_6$ where n is the number of the cells C_i.

9.1 Remark

In the traditional small cancellation theory one emphasizes disks D rather than closed surfaces S, where the above formulae appear with a correction term corresponding to the boundary of D; and so one has there $C_i = C_i^\bullet$ as one deals with cyclically irreducible words.

9.2 Remark

Let X_1 and $X_1^\sim$ be finite graphs and $a\colon X_1^\sim \to X_1$ be a simplicial map. Let $\mathrm{Cyl}(a)$ denote the cylinder of a, i.e. the cylinder $X_1^\sim \times [0, 1]$ with the "bottom" $X_1^\sim \times 1$ attached to X_1 via a, and let $\mathrm{Cone}(a)$ be obtained from $\mathrm{Cyl}(a)$ by shrinking the "top" $= X_1^\sim \times 0$ to a point. A map is called *un-hyperbolic* if the polyhedron $\mathrm{Conc}(a)$ is un-hyperbolic. Notice that this property depends only on the topology of X_1, $X_1^\sim$ and a, but not on the particular simplicial structure.

SIMPLE OBSERVATION. *Let X be an arbitrary 2-polyhedron containing $X_1^\sim$ and look at the space $X \bigcup_a X_1$ obtained by attaching X to X_1 via the map a,*

$$X \supset X_1^\sim \xrightarrow{a} X_1 .$$

If X and a are un-hyperbolic then also $X \bigcup_a X_1$ is un-hyperbolic.

In fact, every unfolded surface in $X \bigcup_a X_1$ crossing X_1 is obtained from a similar surface in $\mathrm{Cone}(a)$ by replacing honest cells by more general open surfaces C_i with $\chi(C_i) \leq 1$.

9.3 REMARKS

(a) The above suggests more general cutting and pasting (in)equalities for χ^+_{unf}.

(b) The un-hyperbolic maps respect the ordinary hyperbolicity as well. For example, if $\pi_1(X)$ is hyperbolic then so is $\pi_1(X \bigcup_a X_1)$ for all un-hyperbolic a. This leads to the notion of a *hyperbolic homomorphism between groups*, $\alpha\colon \Gamma_1^\sim \to \Gamma_1$ respecting hyperbolicity of ambient groups $\Gamma \supset \Gamma_1^\sim$ (where actually the inclusion homomorphism $\Gamma_1^\sim \to \Gamma$ does not need to be injective).

Let us define $\chi^+_{\mathrm{mee}}(a)$ as the supremum of $\mathrm{area}^{-1}(C)\chi_{\mathrm{mee}}(C)$ for all possible "unfolded cells" C over $\mathrm{Cone}(a)$. Here the area in $\mathrm{Cone}(a)$ is obtained by coning some length structure in X_1 where the area of the cone over an arc of length ℓ in X_1 is agreed to be $^1/_2\,\ell$ and where the area of a C mapped to $\mathrm{Cone}(a)$ is counted with the multiplicity (as we always do). Clearly

$$\chi^+_{\mathrm{unf}}(\mathrm{Cone}(a)) \leq \chi^+_{\mathrm{mee}}(a),$$

where χ^+_{unf} is defined here via $\mathrm{area}^{-1}(S)\chi(S)$ with the above area rather than the number of the simplices.

To make the definition clear, look at an *unfolded finite-to-one* map f of a closed surface S into $\mathrm{Cone}(a)$; remove $X_1 \subset \mathrm{Cone}(a)$ from $\mathrm{Cone}(a)$ and take the connected components of the pull-back $f^{-1}(\mathrm{Cone}(a)\backslash X_1)$ for our cells C_i. In other words, we use $f^{-1}(X_1)$ for the graph $S_1 \subset S$. If the map f is simplicial for some triangulations of $\mathrm{Cone}(a)$ and X_1, this is indeed a finite graph and we can speak of the meeting characteristic $\chi_{\mathrm{mee}}(C)$ of every cell C among C_i. Then $\chi^+_{\mathrm{mee}}(a)$ is defined as the supremum of $\mathrm{area}^{-1}(C)\chi_{\mathrm{mee}}(C)$ over all f and C.

This may look not very helpful as it involves *all* unfolded surfaces S over $\mathrm{Cone}(a)$, but in fact, the evaluation of $\chi^+_{\mathrm{mee}}(a)$ does not need f on all of S but

only on a narrow band around the boundary of $C^\bullet \supset C$ (compare above). To simplify the matter we assume that the map $a: X_1^\sim \to X_1$ is locally one-to-one (which corresponds to the cyclic irreducibility of words in the traditional small cancellation theory) and consider a graph G in an annulus A as in Fig. 3.

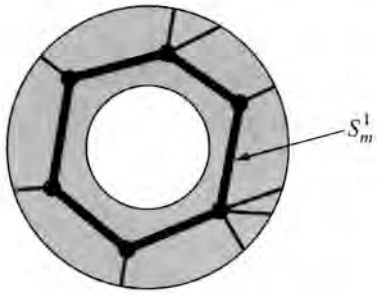

FIGURE 3

In other words, G consists of the middle circle S_m^1 with several edges going from it to the outer circle of the annulus. We define the meeting characteristic $\chi_{\text{mee}}(G)$ of this graph to be equal the meeting characteristic of the cell bounded by the middle circle, i.e. $1 - \frac{1}{2}$ (the number of the edges of S_m^1) + (the sum of the inverse degrees of the vertices of G on S_m^1).

We say that such a G *belongs* to a, if there exists a map $f: A \to \text{Cyl}(a)$ with the following properties:

(a) f is finite-to-one and unfolded;

(b) $f^{-1}(X_1) = G$ for $X_1 \subset \text{Cyl}(a)$;

(c) the map f is locally one-to-one away from the vertices of G on S_m^1;

(d) the inner circle is mapped by f to the top of the cylinder, i.e. to $X_1^\sim \subset \text{Cyl}(a)$.

Now we define $\chi_{\text{mee}}^0(a)$ as the supremum of $\frac{1}{2}$ (length $S_m^1)^{-1}\chi_{\text{mee}}(G)$ over all G belonging to a (where the "$^1/_2$" factor is to make this length equal the cone area). It is clear that $\chi_{\text{mee}}^+(a) \leq \chi_{\text{mee}}^0(a)$ since the local injectivity of a enforces the equality $C^\bullet = C$ and then a small neighbourhood of ∂C in S parametrizes by the annulus $A = \partial_{\text{int}} C \times [0, 1]$.

9.4 REMARK

It may happen that $\chi_{\text{mee}}^+(a)$ is *strictly* smaller than $\chi_{\text{mee}}^0(a)$ since we allow maps $A \to \text{Cyl}(a)$ which do not necessarily factor through a map of

A into any surface. One can take this into account by replacing individual maps $A \to \mathrm{Cyl}(a)$ by (transversal) measures in the space of these, where appropriate Kirchhoff-Markov relations encode the pattern of matching of edges in $S^1_m \to \mathrm{Cyl}(a)$ and make the supremum of χ_{mee} over such measures smaller than $\chi^{\circ}_{\mathrm{mee}}$ defined with individual maps.

9.5 Main example

The $^1/_{6+\varepsilon}$-condition. Given a map $a\colon X_1^{\sim} \to X$, a *connected* subgraph $P \subset X_1^{\sim}$ is called a *piece* if there exists a *fixed point free* continuous injective map $\mu\colon P \to Q \subset \widetilde{X}^1$, such that $a \circ \mu = a|P$. Then a is called $^1/_{6+\varepsilon}$ if, for every subgraph $Y \subset X_1^{\sim}$ that is contained in the union of at most 6 pieces, the restricted map $a|Y\colon Y \to X_1$ is contractible.

9.5.A. Remark. If a is locally one-to-one (which is usually the case in most examples), then we do not have to pass to X_1 in order to speak of the contractibility, since the homomorphism $a_*\colon \pi_1(X_1^{\sim}) \to \pi_1(X_1)$ is injective. Yet such a passage is needed for more general situations admitting X_1 and $X_1^{\sim}$ of dimensions > 1.

9.6 Continuation of the $^1/_{6+\varepsilon}$-discussion

If a is *locally one-to-one* and $^1/_{6+\varepsilon}$, then every *unfolded* $f\colon A \to \mathrm{Cyl}(a)$ clearly contains at least 7 vertices on S^1_m; thus $\chi_{\mathrm{mee}}(G) \le -\frac{1}{6}$. This can be improved for "long" $S^1_m \to X_1$ by observing that the minimal number of pieces needed to cover S^1_m grows roughly proportionally to the length of $S^1_m \to X_1$ for a given length structure in X_1. In fact, *no* cycle $C^{\sim}$ in $X_1^{\sim}$ of X_1-length L can be decomposed into less than $N \cong N^{\sim}(C^{\sim})$ pieces in $X_1^{\sim}$ for

$$(*) \qquad N^{\sim} \le 6 + \varepsilon L,$$

for some $\varepsilon > 0$ (equal, if you wish, the ε in $^1/_{6+\varepsilon}$). Hence, the corresponding χ_{mee} satisfies,

$$\chi_{\mathrm{mee}} \le 1 - \frac{6+\varepsilon L}{2} + \frac{6+\varepsilon L}{3} \le -\frac{L\varepsilon}{6},$$

and so

$$(**) \qquad \chi^{+}_{\mathrm{unf}}(X) \le -\frac{\varepsilon}{3},$$

where this χ^{+}_{unf} is defined with $\chi_{\mathrm{av}}(S) = \chi(S)/\operatorname{area} S$ for the area "coned" from the length structure in X_1.

9.7 Remarks

(a) If we deal with an individual map $a\colon X_1^\sim \to X_1$ then all we care about ε is its being positive. But if we have a family, e.g. a sequence $a(i)\colon X_1^\sim(i) \to X_1$, where X_1 is kept fixed along with its length structure and such that the X_1-length of $X_1^\sim(i) \to X_1$ goes to infinity for $i \to \infty$, then one may easily have all $\varepsilon(i)$ positive, yet converging to zero for $i \to \infty$. This would indicate a (possible) non-uniformity of hyperbolicity of the spaces $\mathrm{Cone}(a(i))$.

(b) All of the above easily generalizes to non locally injective maps a. On the other hand, every map a can be homotoped to a locally injective a' and thus the hyperbolicity (but not un-hyperbolicity) of the universal covering of the cone $\mathrm{Cone}(a)$ reduces to that for $\mathrm{Cone}(a')$.

10. Hyperbolic quotients of trees

Let us concentrate on the hyperbolicity of the universal covering $\mathrm{Cone}(a)$ of ($\mathrm{Cone}(a)$ for ${}^1/_{6+\varepsilon}$-maps $a\colon X_1^\sim \to X_1$. We represent this covering as a quotient space of a "coned tree" as follows. Let T be the universal covering of X_1. This is a tree acted upon by the (free) fundamental group $\Gamma = \pi_1(X_1)$. If a is locally injective and $X_1^\sim$ is connected, then each lift of a to the universal covering U of X_1, *embedds* U to T. Pick up such a lift and the corresponding embedding $U \subset T$. This is a subtree, where the fundamental group $\Gamma_U = \pi_1(X_1^\sim)$ acts on T keeping U invariant. Every $\gamma \in \Gamma$ acting on T moves U to another subtree, say $U' = \gamma(U) \subset T$ acted upon by the subgroup $\Gamma_{U'} = \gamma\Gamma_U\gamma^{-1}$, and so we have a collection of subtrees, $\mathcal{U} = \{U\}$ in T and groups Γ_U, $U \in \mathcal{U}$, where each Γ_U acts on T keeping U invariant. And if $X_1^\sim$ is disconnected, we consider all lifts of a to the universal covers of the connected components of $X_1^\sim$ and arrive at the same picture.

10.1 Translation $\mathcal{U}$-families of groups

Such a family is given by a collection $\mathcal{U}$ of subsets $U \subset T$ (where T may be an arbitrary metric space for this matter) and groups Γ_U, $U \in \mathcal{U}$, isometrically acting on T, where each Γ_U keeps U invariant and also preserves the family $\mathcal{U}$. That is every $\gamma \in \Gamma_U$ maps each $U' \in \mathcal{U}$ onto some $U'' = U''(U', \gamma) \in \Gamma_U$ and conjugates $\Gamma_{U''}$ to $\Gamma_{U'}$.

Then we take the cones $\operatorname{Cone} U = U \times [0,1]/U \times 0$, attach them to T via the embeddings $U \times 1 = U \hookrightarrow T$, and denote the resulting space by $T^\bullet \supset T$,

$$T^\bullet = \Big(\bigcup_{U \in \mathcal{U}} \operatorname{Cone} U\Big) \cup T.$$

Now each group Γ_U acts on $T^\bullet$ keeping the cone $\operatorname{Cone} U \subset \operatorname{Cone} \mathcal{U}$ invariant and its apex u (that is $U \times 0$ shrunk to u fixed).

Coming back to the tree T covering X_1, we see that

$$\widetilde{\operatorname{Cone}}(a) = T^\bullet/\Gamma_{\mathcal{U}},$$

where $\Gamma_{\mathcal{U}}$ denotes the group generated by all Γ_U, $U \in \mathcal{U}$. We are concerned with the hyperbolicity of the group π_1 (Cone(a)) reflected in the hyperbolicity of the space $\widetilde{\operatorname{Cone}}(a)$. In the present case the action of each Γ_U on U is *co-compact*; thus the space $T^\bullet/\Gamma_U$ is quasi-isometric to T/Γ_U and the hyperbolicity of $T^\bullet/\Gamma_U$ is equivalent to that of T/Γ_U. Furthermore, since T/Γ_U equals the Cayley graph of $\pi_1(\operatorname{Cone}(a)) = \Gamma/\Gamma_{\mathcal{U}}$, the latter hyperbolicity amounts to *the hyperbolicity of the fundamental group* $\pi_1(\operatorname{Cone}(a))$, where the latter group *is* hyperbolic as it comes from a $^1/_{6+\varepsilon}$-map.

10.2 Final remarks and questions

(a) The above treatment of the small cancellation theory is a deviation from my paper on random groups [$\mathrm{Gro}_{\mathrm{RG}}$], where we need more general translation families acting on trees and hyperbolic spaces and resulting in *hyperbolic* quotients $T^\bullet/\Gamma_{\mathcal{U}}$ and $T/\Gamma_{\mathcal{U}}$. In fact, every "*local*" small cancellation criterion for groups acting on trees can be "δ-perturbed" and thus extended to all δ-hyperbolic spaces (compare [Del]). Then this can be applied to quotients of these spaces, quotients of quotients etc. eventually leading to *fractally hyperbolic spaces* where many interesting groups act (such as infinite Burnside groups and random groups, see [$\mathrm{Gro}_{\mathrm{SQ}}$], [$\mathrm{Gro}_{\mathrm{RG}}$]). On the other hand, the geometry of transversal measures on locally defined spaces of (smooth and/or combinatorial) subvarieties seems to be pointing into a different direction which we tried to elucidate in this paper.

(b) One can go beyond $^1/_{6+\varepsilon}$ and, following the lead of the classical theory, introduce $^1/_{6-\varepsilon}$-maps (as well as the corresponding polyhedra and groups) where every union of at most 5 (rather than 6) pieces is contractible. Moreover, one can "δ-perturbe" this (a priori unstable) class of groups thus arriving at a meaningful (semi-hyperbolic) theory of $^1/_{6-\varepsilon}$-$\mathcal{U}$-families acting on hyperbolic spaces.

(c) Besides (semi)-hyperbolicity, $^1/_{6\pm\varepsilon}$-$\mathcal{U}$-families of groups acting on trees (and general hyperbolic spaces for this matter) enjoy the full package of the ordinary small cancellation properties. For example, $\mathcal{U}$ admits a subset $\mathcal{V} \subset \mathcal{U}$, such that the groups Γ_V, $V \in \mathcal{V}$ *freely* generate $\Gamma_{\mathcal{U}}$; furthermore, the quotient space (and consequently the group $\Gamma/\Gamma_{\mathcal{U}} = \pi_1(\mathrm{Cone}(a))$) is infinite of exponential growth apart from a few obvious exceptions. However, all this remains unclear for general unfolded hyperbolic polyhedra and the corresponding $\mathcal{U}$-families of groups.

(d) The notion of un-hyperbolicity can be adjusted to torsion by switching from polyhedra to orbihedra.

(e) Let X be a finite connected polyhedron (of arbitrary dimension) with $\pi_2(X) = 0$. If the fundamental group $\Gamma = \pi_1(X)$ is word hyperbolic, then every closed surface mapped to X, say $f_0 \colon S \to X$, can be homotoped to $f_1 \colon S \to X$, where

$$\mathrm{area}(S, f_1) \leq \lambda \ \mathrm{genus}\ (S)$$

for a constant $\lambda = \lambda(X)$ independent of S and f_0. Does this property characterize word hyperbolic groups? In other words, does the implication (1) $\Rightarrow$ (3) from 7.1 hold true in the hyperbolic (instead of un-hyperbolic) framework, i.e. with minimal surfaces rather than unfolded ones. (A similar problem appears in the algebraic geometry where one wishes to derive the Kobayashi hyperbolicity from a linear lower bound on the genera of the curves by their degree.)

(f) The above can be given more (algebra) topological flavour by excluding area (and allowing infinite polyhedra X) by looking at *incompressible* surfaces $f \colon S \to X$, where the kernel of the homomorphism $f_* \colon \pi_1(S) \to \pi_1(X)$ contains no *simple* (i.e. without selfintersection) non-contractible curve in S. Two such surfaces are called *equivalent* if the corresponding homomorphisms differ by a conjugation in $\Gamma = \pi_1(X)$ followed by an automorphism of $\pi_1(S)$. If Γ is word hyperbolic, one knows that the number of equivalence classes of surfaces of genus g, say $\sharp(X; g)$ is bounded by a constant $C = C(X, g)$. Then, given a function $C(g)$ (for example, $C(g) \equiv 1$), one wants to characterize the groups $\Gamma = \pi_1(X)$, where $\sharp(X, g) \leq C(g)$ for all $g = 1, 2, \ldots$.

(g) Let us look at surfaces $S \to X$ with boundaries consisting of several closed curves representing powers of some elements in Γ, say $\gamma_1^{d_1}, \ldots, \gamma_k^{d_k}$. In the hyperbolic case there are various lower bounds on genus(S) in terms of the exponents d_i and one wishes to characterize hyperbolicity by such a property. (The simplest instance of the above is the lower linear bound on

genus(S), where the boundary consists of a single component γ^d and γ is non-torsion,

$$\text{genus}(S) \geq \varepsilon d$$

for some $\varepsilon = \varepsilon(\Gamma) > 0$.)

(h) Keeping track of surfaces and of bounds on their genera in X refines the *bounded* cohomology structure in dimension two. One can treat similarly higher dimensional pseudomanifolds, e.g. those filling-in closed surfaces made out of two surfaces with common boundary etc., where the genus is replaced by the simplicial (or similar, as in $[\text{Gro}]_{\text{VBC}}$) norm. The patterns observed here are reminiscent of those seen in the topological field theory but I do not know how to incorporate the picture into a clean logical framework.

(i) The above (f) and (g) indicate a possible purely algebraic notion of hyperbolicity defined entirely within the category of groups. A similar problem arises if one cares to pinpoint hyperbolicity for a *commutative diagram of groups*, or a *small category* of *groups*. The relevant example of this is the diagram $\underline{\Gamma} \xleftarrow{\underline{f}} \Gamma_0 \xrightarrow{\bar{f}} \overline{\Gamma}$, where the group $\prod$ *presented* by the diagram is the quotient

$$\prod \underset{def}{=} \left(\underline{\Gamma} * \Gamma_0 * \overline{\Gamma}\right) / \{\underline{\gamma} = \underline{f}(\gamma), \overline{\gamma} = \bar{f}(\gamma)\}$$

for all $\gamma \in \Gamma_0$, where the traditional presentations appear for Γ_0 and $\overline{\Gamma}$ being free, $\underline{\Gamma} = \{id\}$ and $\bar{f}$ injective, while the diagrams, where both $\underline{f}$ and $\bar{f}$ are injective, presents the amalgamated product $\underline{\Gamma} \underset{\Gamma_0}{*} \overline{\Gamma}$. (In general, given a diagram or category of groups one passes to the corresponding diagram of classifying spaces having the same commutativity as the original diagram. One represents each arrow by the cylinder of the corresponding map, then fills-in every commuting triangle by an actual triangle etc., thus arriving at a topological space whose fundamental group $\prod$ is *presented* by the diagram, see [Bri-Hae].)

To stay within our category we replace "area" by the following notion of (absolute) degree. Given a space X and a subspace Y we define the *absolute degree* of a homotopy class $[f]$ of maps f of a surface S to X as follows:

$$|\deg_Y[f]| \underset{def}{=} \inf_{f \in [f]} \text{card Conn} f^{-1}(Y),$$

and call (X, Y) *algebraically hyperbolic* if

$$|\deg_Y[f]| \leq \text{const} \cdot \text{genus}(S)$$

for some const $=$ const(X, Y). (Sometimes one can refine the absolute degree by looking at f over a small "band" B around Y, i.e. over $B = U \backslash Y$ for a small neighbourhood $U \subset X$ of Y. One imagines that $f^{-1}(B)$ consists of several annuli around the components of $f^{-1}(Y) \subset S$ and keeps track of the conjugacy classes of the images of the π_1(annuli) in $\pi_1(B)$.)

Turning to the above diagram, we take the classifying spaces $\underline{X} = K(\underline{\Gamma}; 1)$, $X_0 = K(\Gamma_0; 1)$ and $\bar{X} = K(\bar{\Gamma}; 1)$, represent the arrows by the corresponding continuous maps,

$$\underline{X} \xleftarrow{\underline{F}} X_0 \xrightarrow{\bar{F}} \bar{X}$$

and make X out of the cylinders of these maps,

$$X = \mathrm{Cyl}(\underline{F}) \cup \mathrm{Cyl}(\bar{F}),$$

where, observe, $\mathrm{Cyl}(\underline{F}) \cap \mathrm{Cyl}(\bar{F}) = X_0$.

Finally, we call the diagram (algebraically) *hyperbolic* (relative to $\underline{f}$) if the pair $(X, \underline{X})$ is algebraically hyperbolic. By browsing through various presentations (diagrams) one convinces himself that this definition agrees (albeit not fully) with the intuitive idea of hyperbolicity.

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