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Soft and Hard Symplectic Geometry

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1. Basic definitions, examples, and problems.

1.1. *Symplectic forms and manifolds.* An exterior differential 2-form ω on a smooth manifold V is called *nonsingular* if the associated homomorphism between the tangent and cotangent bundles of V, denoted $I_\omega\colon T(V) \to T^*(V)$ and defined by $I_\omega(\tau) = \omega(\tau, \cdot)$, is an isomorphism. In this case, the dimension of V is necessarily even (we assume that V is finite-dimensional and all connected components of V have the same dimension), say $\dim V = m = 2n$, and the exterior power ω^n (which is a top-dimensional form on V) does not vanish on V. Conversely, if ω^n does not vanish, then ω is nonsingular. For example, every (oriented) area form on a *surface* is nonsingular.

A form ω on V is called *symplectic* if it is nonsingular and closed, that is, $d\omega = 0$. Then (V, ω) is called a *symplectic manifold.*

EXAMPLES. Every surface with an area form is a symplectic manifold. If (V_i, ω_i) are such surfaces for $i = 1, \dots, n$, then the Cartesian product $(V, \omega) = (V_1 \times V_2 \times \cdots \times V_n, \omega_1 \oplus \omega_2 \oplus \cdots \oplus \omega_n)$ is a $2n$-dimensional symplectic manifold. The *symplectic area* $\omega(S) = \int_S \omega$ of every surface S in this V equals the sum of the (oriented!) areas of the projections $S \to V_i$.

An important special case is the *symplectic space* $(\mathbf{R}^{2n}, \omega = \sum_{i=1}^n dx_i \wedge dy_i)$; that is, the sum of n copies of the (x, y)-plane $\mathbf{R}^2$ with the usual area form $dx \wedge dy$.

A less obvious example is the complex projective space $\mathbf{C}P^n$ which admits a unique (up to a scalar multiple) 2-form ω which is invariant under the action of the unitary group $\mathrm{U}(n+1)$ on $\mathbf{C}P^n$. This form is (easily seen to be) symplectic, and the symplectic area of every surface $S \subset \mathbf{C}P^n$ equals the average number of the intersection points (counted with algebraic multiplicity) of S with hyperplanes $p \subset \mathbf{C}P^n$,

$$\omega(S) = \int_P \#(S \cap p)\, dp,$$

where P is the (dual) projective space ($\approx \mathbf{C}P^n$) of hyperplanes p in $\mathbf{C}P^n$ and dp is a $\mathrm{U}(n+1)$-invariant measure on P.

1.2. *Symplectic (diffeo)morphisms.* A C^1-map $f\colon (V_1,\omega_1) \to (V_2,\omega_2)$ is called *symplectic* if the pull-back of ω_2 equals ω_1; that is, $f^*(\omega_2) = \omega_1$. Every such f necessarily is an *immersion*; that is, the differential $D_f\colon T(V_1) \to T(V_2)$ is injective on the tangent space $T_v(V_1)$ for all $v \in V_1$.

The group $\operatorname{Symp} V$ of symplectic diffeomorphisms of every symplectic manifold $V = (V,\omega)$ is infinite-dimensional for $\dim V \geq 2$. Indeed, the space Ω of closed 2-forms on V essentially is $\{1\text{-forms}/d(\text{functions})\}$ which makes "the functional dimension" of Ω equal $m-1$ (for $m = \dim V$), while "the functional dimension" of $\operatorname{Diff} V$ is m since every diffeomorphism $V \to V$ is determined by m functions on V. Hence, the expected "functional dimension" of the group $\operatorname{Symp}(V,\omega)$, which is the isotropy subgroup of $\omega \in \Omega$ for the natural action of $\operatorname{Diff} V$ on Ω, is one. This heuristic argument (which, in fact, can be made precise) suggests some correspondence between functions $V \to \mathbf{R}$ and symplectic diffeomorphisms of V. (Notice that for "sufficiently nondegenerate" forms ω of degree between 3 and $m-2$, the automorphism group of (V,ω) is finite-dimensional, and the same is true for *symmetric* differential forms of degree ≥ 2.)

Symplectic vector fields. A vector field X on (V,ω) is called *symplectic* if the Lie derivative $X\omega$ vanishes. Since

$$X\omega = dI_\omega(X) \tag{1}$$

for all fields X and closed 2-forms ω, the condition $X\omega = 0$ is equivalent to $dI_\omega(X) = 0$. Therefore, for every *closed* 1-form l on V, the field $I_\omega^{-1}(l)$ is symplectic. In particular, for every smooth function (Hamiltonian) $h\colon V \to \mathbf{R}$, the field $X = I_\omega^{-1}(dh)$, called a *Hamiltonian field*, is symplectic. *Integrable* Hamiltonian fields X (e.g., those where h has compact support) give us one-parametric subgroups $X_t \subset \operatorname{Symp}(V,\omega)$. This confirms the largeness of $\operatorname{Symp} V$ predicted by the above dimension argument.

Another property of $\operatorname{Symp}(V,\omega)$ which can be seen by looking at Hamiltonians is the *transitivity* of $\operatorname{Symp} V$ on k-tuples of disjoint points in V for every $k = 1,2,\ldots$, where we assume that V is connected. In particular, $\operatorname{Symp} V$ is transitive on V.

COROLLARY (DARBOUX). *Every two symplectic manifolds V_1 and V_2 of the same dimension are locally isomorphic.*

PROOF. If $U_i \subset V_i$, $i = 1,2$, are sufficiently small neighborhoods, then there obviously exists a *connected* symplectic manifold V', such that U_i are symplectically diffeomorphic to some neighborhoods $U_i' \subset V'$.

Examples of symplectic diffeomorphisms of $(\mathbf{R}^{2n}, \omega = \sum dx_i \wedge dy_i)$. (1) Every parallel translation of $\mathbf{R}^{2n}$ is symplectic.

(2) Let $z_i = x_i + \sqrt{-1}y_i$ and identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$. Then the form ω can be expressed with the Euclidean scalar product by

$$\omega(\tau_1,\tau_2) = \langle \tau_1, \sqrt{-1}\tau_2 \rangle$$

for all (tangent) vectors τ_1 and τ_2 in $\mathbf{R}^{2n}$. Therefore ω is invariant under unitary transformations of $\mathbf{C}^n$. In fact, the group $\mathrm{Sp}(2n)$ of *linear* symplectic transformations of $\mathbf{R}^{2n}$ *strictly* contains $\mathrm{U}(n)$, as

$$\dim \mathrm{U}(n) = n^2 < n(2n+1) = \dim \mathrm{Sp}(2n).$$

(3) Split $\mathbf{R}^{2n}$ into the sum of n copies of $(\mathbf{R}^2, dx \wedge dy)$ and let f_i for $i = 1, \dots, n$ be area-preserving transformations of $\mathbf{R}^2$. Then the Cartesian sum of f_i is symplectic on $\mathbf{R}^{2n}$.

(4) Identify $\mathbf{R}^{2n}$ with (the total space of) the cotangent bundle of $\mathbf{R}^n$ with coordinates $x_1, \dots, x_n$. Then the natural action of diffeomorphisms of $\mathbf{R}^n$ on $T^*(\mathbf{R}^n) = \mathbf{R}^{2n}$ is symplectic. Thus $\mathrm{Diff}\,\mathbf{R}^n$ embeds into $\mathrm{Symp}\,\mathbf{R}^{2n}$.

One can compose diffeomorphisms in (1)–(4) and also compose them with the above Hamiltonian diffeomorphisms X_t. Thus one obtains symplectic diffeomorphisms of $\mathbf{R}^{2n}$ which may look arbitrarily complicated. Yet, there is no $f \in \mathrm{Symp}\,\mathbf{R}^{2n}$ sending U_1 into U_2, where U_1 and U_2 are open subsets in $\mathbf{R}^{2n}$, such that $\mathrm{Vol}\, U_1 > \mathrm{Vol}\, U_2$. In fact, $\mathrm{Vol}\, U = \int_U \omega^n$ is invariant under $\mathrm{Symp}\,\mathbf{R}^{2n}$.

PROBLEM. Are there further obstructions besides $\mathrm{Vol}\, U_1 \le \mathrm{Vol}\, U_2$ for the existence of symplectic diffeomorphisms of $\mathbf{R}^{2n}$ mapping U_1 into U_2?

It is easy to see that no essentially new obstruction exists for *volume-preserving* diffeomorphisms of $\mathbf{R}^{2n}$. Namely, if U_1 is relatively compact, the sets $\mathbf{R}^{2n} \backslash U_1$ and U_2 are connected, and

$$\mathrm{Vol}\, U_1 < \mathrm{Vol}\, U_2 \qquad (*)$$

(the strict inequality removes an irrelevant problem of the boundary behavior of the maps), then there exists a smooth (even real analytic) diffeomorphism f of $\mathbf{R}^{2n}$ sending U_1 into U_2 and satisfying $f^*(\omega^n) = \omega^n$. However, such obstructions do exist for symplectic maps as seen in the following

EXAMPLE. Let U_1 be the round ball in $\mathbf{R}^{2n}$ of radius r, and let U_2 be the ε-neighborhood of a linear subspace $L \subset \mathbf{R}^{2n}$ with $\dim L < n$. Then *there is no symplectic* diffeomorphism (for $\omega = \sum_{i=1}^n dx_i \wedge dy_i$) sending $U_1 \to U_2$ for $\varepsilon < r$.

The proof (indicated in §4.1) relies on the geometry of *holomorphic curves* for some (nonintegrable) *almost complex* structure in $\mathbf{R}^{2n}$. No "soft" proof is known at present.

REMARK. If L is the n-dimensional linear subspace in $\mathbf{R}^{2n}$ given by the equations $x_i = 0$, for $i = 1, \dots, n$, then the unit ball U_1 goes to the ε-neighborhood U_2 of L by the symplectic maps $(x_i, y_i) \to (\varepsilon x_i, \varepsilon^{-1} y_i)$. However, if L is given by the equations $x_1 = 0$ and $y_1 = 0$ (here $\dim L = 2n - 2$), then no symplectic diffeomorphism $U_1 \to U_2$ exists for $r > \varepsilon$ (see §4.1).

1.3. *Isotropic immersions and Lagrange submanifolds.* A C^1-map $f\colon W \to (V, \omega)$ is called *isotropic* if $f^*(\omega) = 0$. We are especially interested in the case where $\dim W = n$ for $2n = \dim V$ and f is an immersion. These f are called *Lagrange immersions.* Similary, a submanifold $W \subset V$ is called *Lagrange* if $\omega \mid V = 0$.

EXAMPLES. (1) Let $f\colon (V_1, \omega_1) \to (V_2, \omega_2)$ be a symplectic map and $(V, \omega) = (V_1 \times V_2, \omega_1 \oplus -\omega_2)$. Then the graph $\Gamma_f\colon V_1 \to V$ for $\Gamma_f(v_1) = (v_1, f(v_2))$ is an isotropic immersion. In fact, the graph of every map is an embedding and $\Gamma_f(V_1)$ is a Lagrange submanifold in V if $\dim V_1 = \dim V_2$.

(2) If $\dim V = 2$ and $W \subset V$ has $\dim W = 1$, then obviously W is Lagrange for every area form ω on V. The Cartesian product of n copies of these,

$$W_1 \times \cdots \times W_n \subset (V_1 \times \cdots \times V_n, \omega_1 \oplus \cdots \oplus \omega_n),$$

also is Lagrange. In particular, the torus $T^n = \{x_i, y_i \mid x_i^2 + y_i^2 = 1\}$ is Lagrange in $(\mathbf{R}^{2n}, \sum_{i=1}^n dx_i \wedge dy_i)$.

(3) Let V be the (total space of the) cotangent bundle, $V = T^*(X)$ for some n-dimensional manifold X. Denote by σ the *canonical* 1-form on V defined by the identity

$$\alpha^*(\sigma) = \alpha, \qquad (**)$$

where $\alpha \colon X \to T^*(X) = V$ is an arbitrary C^1-section and where the same α on the right-hand side of $(**)$ is viewed as a 1-form on X. It is easy to see that the form $\omega = d\sigma$ is symplectic and that for $X = \mathbf{R}^n$ this V is symplectically isomorphic to $(\mathbf{R}^{2n} \sum_{i=1}^n dx_i \wedge dy_i)$. Now, a section $X \to T^*(X) = V$ is Lagrange if and only if the corresponding 1-form on X is closed. In particular, $dh \colon X \to T^*(X)$ is Lagrange for every smooth function h on X. (Notice that every $W \subset T^*(X)$, whose projection on V is a *diffeomorphism* of W *onto* V, is of this kind; $W = \alpha(X)$ for a unique 1-form $\alpha \colon X \to T^*(X) = V$.)

(4) The real projective space $\mathbf{R}P^n \subset \mathbf{C}P^n$ is Lagrange for the above $\mathrm{U}(n+1)$-invariant form ω on $\mathbf{C}P^n$. If $V \subset \mathbf{C}P^n$ is a nonsingular complex algebraic subvariety, then the induced form $\omega' = \omega \mid V$ clearly is nonsingular, and hence, symplectic on V. Furthermore, if V is defined over $\mathbf{R}$, then the *real locus* $W = V \cap \mathbf{R}P^n$ is isotropic in (V, ω'). This W is Lagrange if W is nonsingular and $\dim W = \dim_{\mathbf{C}} V$.

We shall see in §2.2 that the existence problem for Lagrange immersions $\varphi \colon W \to V$, for given W and $V = (V, \omega)$, belongs with the soft geometry. But the apparently similar problem of a possible topology of the image $\varphi(W) \subset V$ seems (at the present moment) of hard nature.

EXAMPLE (SEE §3.5). For every Lagrange immersion of a *closed* manifold into $\mathbf{R}^{2n} = \mathbf{C}^n$,

$$\varphi \colon W \to \left(\mathbf{R}^{2n}, \omega = \sum_{i=1}^n dx_i \wedge dy_i\right),$$

there exists a nonconstant holomorphic disk in $\mathbf{C}^n$ with boundary in $\varphi(W)$.

It easily follows that the relative cohomology class

$$[\omega] \in H^2(\mathbf{R}^{2n}, W, \mathbf{R})$$

does not vanish. In particular, if $H^1(W; \mathbf{R}) = 0$, then W admits no Lagrange *embedding* into $(\mathbf{R}^{2n}, \omega)$.

2. Symplectic immersions and embeddings.

2.1. *Topological obstructions for symplectic immersions.* We consider two symplectic manifolds (V, ω) and (W, ω') and ask ourselves when a given continuous map $\varphi: W \to V$ is homotopic to a symplectic map $f: W \to V$. There are two obvious obstructions for the existence of f. First, φ must respect the cohomology classes represented by ω and ω'. That is, the homomorphism $\varphi^*: H^2(V;\mathbf{R}) \to H^2(W;\mathbf{R})$ should send $[\omega]$ to $[\omega']$.

To see the second obstruction we observe that the differential of f, which is a fiber-wise linear map between tangent bundles $D_f: T(W) \to T(V)$, is *symplectic* in so far as f is symplectic. Here, we call a continuous fiber-wise linear map $\Delta: T(W) \to T(V)$ symplectic if $\Delta^*(\omega) = \omega'$. (Note that $f^*(\omega) = D_f^*(\omega)$ by the very definition of $f^*(\omega)$.)

Next we consider the space $\{\Delta\}$ of all symplectic fiber-wise linear maps $T(W) \to T(V)$ and the space $\{\varphi\}$ of continuous maps $W \to V$. It is easy to see that the projection $\{\Delta\} \to \{\varphi\}$, which sends each Δ to the underlying φ, is a *Serre fibration*. Hence, the existence of a homotopy between φ and f (where f lifts to $D_f \in \{\Delta\}$) implies the existence of a lift of φ to some $\Delta \in \{\Delta\}$. Note that such a lift is given by a *symplectic homomorphism* of bundles over W, say by

$$\delta: (T(W), \omega') \to \varphi^*(T(V), \omega)$$

(here $\varphi^*(\,)$ denotes the induced bundle and the symplecticity of δ is understood in the obvious sense), and these δ are sections of the fibration over W whose fiber at $w \in W$ consists of linear symplectic maps $T_w(W) \to T_{\varphi(w)}(V)$. Then we observe, for example, that every φ lifts to some Δ if the manifold W is contractible and $\dim W \le \dim V$. (This discussion depends only on the nonsingularity of ω and ω'. On the contrary, the earlier condition $\varphi^*[\omega] = [\omega']$ only needs the forms to be closed.)

2.2. Immersion Theorem (see **[Gr2]**, **[Gr4]**). *Let $\varphi: V = (V, \omega) \to W = (W, \omega')$ be a continuous map which admits a lift to a symplectic map $T(W) \to T(V)$ and which satisfies $\varphi^*[\omega] = [\omega']$. Then in the following two cases there exists a symplectic map $f: W \to V$ homotopic to φ.*

(i) Open case. *The manifold W is open (that is no connected component of W is a closed manifold).*

(ii) Extra dimension. $\dim W < \dim V$.

Remarks. This theorem, obviously, is false if W is closed, V is open, and $\dim W = \dim V$. In fact, not even a topological immersion $W \to V$ exists in this case.

The map f can be chosen essentially as smooth as the forms ω and ω'. For example, if ω and ω' are C^∞-smooth (real analytic), then there is some f which is C^∞ (real analytic).

COROLLARY A. *A $2n$-dimensional manifold (W, ω') admits a symplectic map into $(\mathbf{R}^{2n}, \omega = \sum dx_i \wedge dy_i)$ if and only if the following three conditions are satisfied:*

(a) *W is open;*

(b) *the form ω' is exact;*

(c) *the tangent bundle $(T(W), \omega')$ is a trivial $\mathrm{Sp}(2n)$-bundle. (This is equivalent to the existence of linearly independent vector fields, say X_i and Y_i on W for $i = 1, \dots, n$, such that $\omega'(X_i, Y_i) = 1$ and $\omega'(X_i, X_j) = \omega'(Y_i, Y_j) = \omega'(X_i, Y_j) = 0$ for $i \neq j$.)*

Note that (a), (b), and (c) are satisfied for every contractible (e.g., homeomorphic to $\mathbf{R}^{2n}$) manifold W.

COROLLARY B. *A smooth n-dimensional manifold X admits a Lagrange immersion into $\mathbf{R}^{2n}$ if and only if the complexification $T(X) \oplus \sqrt{-1}T(X)$ is a trivial $\mathrm{GL}_n(\mathbf{C})$-bundle over X.*

Here (as in Corollary A) the "only if" claim is trivial, and "if" follows from Corollary A applied to $W = T^*(X)$.

About the proof of 2.2. First, using a symplectic $\Delta: T(W) \to T(V)$, one easily constructs a family of local symplectic immersions $f_w: U_w \to V$ where U_w is a small ball in W around w and f_w is continuous in $w \in W$. Then the required f is assembled out of f_w by appropriately bending (or flexing) these locally defined maps f_w in order to make them agree on the intersections $U_{w_1} \cap U_{w_2}$. The assembling procedure uses (very soft) techniques of topological sheaves (see [**Gr4**]). It seems unlikely that such an f could be constructed by means of hard analysis.

2.3. *Imbeddings.* As we mentioned earlier, every symplectic map $W \to V$ is an *immersion* but not necessarily an *embedding*, where a map is called an embedding if it is a homeomorphism onto its image. (For example, every immersion without double points is an embedding, provided W is compact.)

We start with a *smooth embedding* $\varphi: W \to V$ (that is, φ is a smooth immersion as well as an embedding) and try to C^∞-*isotope* φ to a *symplectic embedding* $f: W \to V$. Note that the differential of such an isotopy is a homotopy of fiber-wise linear and fiber-wise injective maps $\Delta_t: T(W) \to T(V)$, where $\Delta_0 = D_\varphi$ and $\Delta_1 = D_f$ is symplectic.

Now, for a given embedding φ, we *assume* that there exists a homotopy of fiber-wise injective maps $\Delta_t: T(W) \to T(V)$ (which, for $t > 0$, do not have to be differentials of maps $V \to W$) such that $\Delta_0 = D$ and Δ_1 is symplectic.

THEOREM A. *If $\varphi^*[\omega] = [\omega']$, then in the following two cases the embedding φ is isotopic to a symplectic embedding $f: W \to V$.*

(i) *The manifold W is open and* $\dim W < \dim V$.

(ii) $\dim W \leq \dim V - 4$.

Note that (i) and (ii) exclude the following two cases allowed by the immersion theorem:

(a) W is open and $\dim W = \dim V$.

(b) W is closed and $\dim W = \dim V + 2$.

COROLLARY B. *If W is contractible and* $\dim W < \dim V$, *then every embedding $W \to V$ is isotopic to a symplectic one. In particular, if W is diffeomorphic to $\mathbf{R}^{2n-2}$, then there exists a symplectic embedding $(W, \omega') \to (\mathbf{R}^{2n}, \omega = \sum dx_i\, dy_i)$.*

REMARK. The above embedding theorem holds true for *proper* symplectic embeddings, provided $\dim W \leq \dim V - 4$. This allows a *proper* symplectic embedding $(W, \omega') \to (\mathbf{R}^{2n}, \omega = \sum dx_i \wedge dy_i)$ of every W homeomorphic to $\mathbf{R}^{2n-4}$.

About the proof of Theorem A. Symplectic embeddings $W \to V$ are obtained by incorporating a (soft) geometric construction of Nash [**N1**] (used for isometric C^1-immersions of Riemannian manifolds) into the framework of topological sheaves (see [**Gr4**]).

About (i) *and* (ii). The restrictions (i) and (ii) in Theorem A cannot be dropped. This is seen with (hard) analysis of holomorphic curves in V and W (see §4.2).

3. Holomorphic curves in almost complex manifolds.

3.1. Recall that each complex linear structure on $\mathbf{R}^{2n}$ is determined by an automorphism J on $\mathbf{R}^{2n}$, such that $J^2 = -\mathrm{Id}$, which corresponds to the multiplications by $\sqrt{-1}$. Next, *an almost complex structure* on a manifold V is given by an automorphism of $T(V)$, denoted by J or by $\sqrt{-1}$, such that $J^2 = -\mathrm{Id}$. Almost complex manifolds (V, J) with $\dim V = 2$ are called *Riemann surfaces*.

A C^1-map between two almost complex manifolds, say $f\colon (V_1, J_1) \to (V_2, J_2)$, is called *holomorphic* if the differential D_f is a *complex* linear map between the fibers $T_v(V_1) \to T_{f(v)}(V_2)$ for all $v \in V$. This is equivalent to the identity $D_f \circ J_1 = J_2 \circ D_f$.

If $\dim V_1 > 2$, then there is no nonconstant holomorphic map $V_1 \to V_2$ for generic J_1 and J_2. But if V_1 is a Riemann surface, then, at least locally, there are, roughly, as many holomorphic maps $V_1 \to V_2$ as in the case $V_1 = \mathbf{C}$ and $V_2 = \mathbf{C}^n$ for $2n = \dim V$. In fact, the equation $D_f \circ J_1 = J_2 \circ D_f$ is *elliptic* in this case. That is, the linearization of this equation is elliptic with the (principal) symbol at every point isomorphic to that of the Cauchy-Riemann equation $\bar\partial f = 0$ for maps $\mathbf{C} \to \mathbf{C}^n$.

DEFINITION. A (parametrized) *holomorphic curve* in an almost complex manifold V is a holomorphic map of a Riemann surface into V, say $f\colon S \to V$. Sometimes, we forget the parametrization and deal with *nonparametrized holomorphic curves* $f(S) \subset V$.

3.2. *Tame manifolds.* We say that a *closed* 2-form ω on V *tames* an almost complex structure J on V if ω is *J-positive*. That is,

$$\omega(\tau, J\tau) > 0 \tag{$*$}$$

for all nonzero tangent vectors $\tau \in T(V)$. Since every J-positive form (obviously) is nonsingular, this ω is symplectic.

EXAMPLES. (A) *Calibrating forms and Kähler manifolds.* Let g be a Riemannian metric on V invariant under J. Then the 2-form

$$\omega = \omega(\tau_1, \tau_2) \underset{\text{def}}{=} -g(\tau_1, J\tau_2)$$

satisfies $\omega(\tau, J\tau) = g(\tau, \tau)$. Hence ω is J-positive. Such an ω is called *calibrating* (compare [**H-L**]) if it is closed

If the structure J is *complex* (i.e., integrable) then calibrating forms also are called *Kähler*. For example, the induced form ω' (compare Example 4 in §1.3) on every complex submanifold in $\mathbf{C}P^m$ (obviously) is Kähler.

(B) *Convex functions.* A smooth function $h\colon V \to \mathbf{R}$ is called *strictly J-convex* (or plurisubharmonic) if the restriction of h to every holomorphic curve in (V, J) is subharmonic. It is obvious that J-convexity of h is equivalent to positivity of the exact 2-form $\omega = dJ\,dh$ on V and that a sufficiently small neighborhood U of each point $v \in V$ admits a strictly J-convex function $U \to \mathbf{R}$. Thus, every (V, J) can be locally tamed by some ω. (But the existence of a local *calibrating* form imposes a nontrivial partial differential condition on J.)

REMARK. Differential forms (of any degree) taming partial differential equations provide a major (if not the only) source of integro-differential inequalities needed for a priori estimates and vanishing theorems. These forms are defined on spaces of jets (of solutions of equations) and they are often (e.g., in Bochner-Weitzenbock formulas) exact and invariant under pertinent (infinitesimal) symmetry groups. Similarly, convex (in an appropriate sense) functions on spaces of jets are responsible for the maximum principles. A great part of hard analysis of P.D.E. will become redundant when the algebraic and geometric structure of taming forms and corresponding convex functions is clarified. (From the P.D.E. point of view, symplectic geometry appears as a taming device on the space of 0-jets of solutions of the Cauchy-Riemann equation.)

3.3. *Closed holomorphic curves.* If S is a closed Riemann surface, then the space Σ of holomorphic maps $f\colon S \to V = (V, J)$ is locally finite-dimensional, since the (elliptic!) Cauchy-Riemann operator is Fredholm. In general, the space Σ is far from compact (even if V is compact), but it admits a nice compactification provided V is a closed manifold (which can be tamed by some closed 2-form ω). This follows (see [**Gr3**]) from the obvious inequality

$$\operatorname{Area} f(S) \leq \text{const} \cdot \int_S f^*(\omega), \tag{$*$}$$

where the area is measured with a fixed Riemannian metric on V. Notice that the right-hand side of $(*)$ only depends on the homology class $[f(S)] \in H_2(V)$

and that an inequality similar to (*) holds true for the graph

$$\Gamma_f \colon S \to V \times S.$$

EXAMPLES. (A) Let V and S equal the Riemann sphere S^2. Then the space Σ_1 of holomorphic maps $f \colon S^2 \to S^2$ of degree one is noncompact. (This is the group $\mathrm{PGL}_2\mathbf{C}$.) Now we look at (the images of) the graphs of these maps which are smooth holomorphic curves $S_f \subset S^2 \times S^2$. If a sequence of maps $f_i \in \Sigma_1$ diverges, then there is a subsequence, say f_j, such that the curves $S_{f_j} \subset S^2 \times S^2$ converge (in an obvious sense) to a reducible curve in $S^2 \times S^2$ of the form $(S^2 \times s_2) \cup (s_1 \times S^2)$ for some points s_1 and s_2 in S^2.

(A′) Let S be any Riemann surface and V an arbitrary projective algebraic variety. Then the space Σ of holomorphic maps $f \colon S \to V$ in a fixed homology class is a quasiprojective variety which can be completed to a projective variety by adding to Σ (graphs of) reducible curves which are obtained by pinching some circles (vanishing cycles) in S.

(B) Let $V = (\mathrm{SL}_2\,\mathbf{C})/\Lambda$ for some cocompact lattice Λ in the group $\mathrm{SL}_2\,\mathbf{C}$. Take some nontorsion element $\lambda \in \Lambda$, and let $S = C_\lambda/Z_\lambda$ where $C_\lambda \subset \mathrm{SL}_2\,\mathbf{C}$ is the centralizer of λ (notice that $C_\lambda \approx \mathbf{C}^*$) and Z_λ is the infinite cyclic group generated by λ. Clearly, this S is a torus which is holomorphically mapped into V and the area of this torus can be made arbitrarily large by applying the (holomorphic) action of $\mathrm{SL}_2\,\mathbf{C}$ on V. This happens because the complex structure on V is not tame.

SCHWARZ LEMMA. *The bound on area of f provided by (*) allows one, in principle, to control the pointwise norm of the differential Df. For example, every holomorphic map f of the unit disk $B \to \mathbf{C}$ satisfies*

$$\left|\frac{df}{dz}(0)\right|^2 \le \pi^{-1}\int_B f^*(\omega) \tag{**}$$

for the area form $\omega = dx \wedge dy$ on $\mathbf{C}$. (In fact,

$$\left|\frac{df}{dz}(0)\right|^2 \le \pi^{-1}A,$$

for the area A of the minimal simply connected subset in $\mathbf{C}$ containing the image $f(B) \subset \mathbf{C}$.)

The inequality (**) generalizes to all manifolds (V, J) tamed by *exact* forms and implies (see [**Gr3**]) that the space of closed holomorphic curves S in every closed tame manifold (V, J) can *be compactified by adding (graphs of) singular curves obtained by pinching circles in S.*

REMARKS. (a) The pinching of circles in *minimal surfaces* was discovered by Sacks and Uhlenbeck [**B-U**] (compare [**Sch-Y**]).

(b) It seems that minimal surfaces become truly useful (for "soft purposes") in the presence of higher order taming (curvature) forms. Here are two examples.

(b_1) (Frankel Conjecture). *Every closed Kähler manifold V with positive bisectional curvature is biholomorphic to $\mathbf{C}P^n$.*

This is proven by Siu and Yau who start with an appropriate minimal surface S in V. Then they show S is holomorphic and admits as many holomorphic deformations as $\mathbf{C}P^1 \subset \mathbf{C}P^n$. (This, probably, generalizes to calibrated almost complex manifolds.)

Notice that the algebraic version (due to Hartshorne) of the Frankel conjecture (where "positive curvature" is replaced by "ample tangent bundle" and which is a soft proposition by algebra-geometric standards) was proven by Mori who used, as a hard tool, the action of Frobenius on curves in V (after reducing the problem to finite characteristic).

(b_2) Let V be a Riemannian manifold with positive curvature operator. It is shown in [**Mo**] and [**Mi**] that every minimal sphere in V has (Morse) index $\geq m/2 - 3/2$ for $m = \dim V$ and that this inequality implies the homotopy equivalence of the universal covering of V to the sphere S^m, provided that V is closed.

(c) Compactification theorems are known (Uhlenbeck) for many higher-dimensional conformally invariant elliptic systems.

(c′) The most fascinating (soft) application of such a compactification (for the Yang-Mills equation, where the hard part had been furnished by Uhlenbeck and Taubes) was discovered by Donaldson [**D1**]. Also, see [**D2**, **D3**, **D4**, **F-S**, **T1**, **T2**]. (Notice that the Yang-Mills equation on a given principal bundle P over a 4-manifold is tamed by the universal Pontryagin 4-form on the space of jets of connections on P.)

3.4. *Compactness and existence theorems for closed holomorphic curves.* If one rules out the pinching of circles described in the previous section, §3.3, then one concludes to *compactness* of a pertinent space Σ of holomorphic curves.

3.4. A. EXAMPLE. Let $S = S^2$ and consider holomorphic maps $f: S \to (V, J)$, such that the homology class $f_*[S] \in H_2(V)$ generates the image of the Hurewitz homomorphism $\pi_2(V) \to H_2(V)$. Then, assuming V is tamed by some ω, no pinching of curves in S is possible (as it would decompose $S \subset V$ into smaller holomorphic spheres). Hence, the space Σ is compact modulo conformal transformations of $S = S^2$. In other words, the space Σ' of corresponding nonparametrized holomorphic curves in V is compact for compact manifolds V. It easily follows (by Fredholm theory for nonlinear elliptic operators) that Σ' represents certain homology class, denoted $[\Sigma']$ in the space of all surfaces in V (here, V is a closed manifold), and that $[\Sigma']$ is invariant under homotopies J_t of J in-so-far as the homotoped structure J_t is tame (i.e., tamed by some ω_t) for all t. Since the space of the almost complex structures tamed by a fixed ω is contractible (this is trivial, see [**Gr3**]), *the class* $[\Sigma']$ *is an invariant of the symplectic form.*

Further invariants of (V, ω) are obtained by taking (the homology classes represented by) some subvarieties in Σ' such as the variety Σ'_0 of holomorphic curves $S \subset V$ passing through a fixed point $v_0 \in V$.

REMARK. Our $[\Sigma']$ and $[\Sigma'_0]$ are similar to Donaldson's invariants in gauge theory, but no direct link between the two types of invariants has been found so far. (Compare [**Hi**].)

3.4. B. Let us compute $[\Sigma'_0]$ in the simplest case, where

$$(V, J) = (V_0 \times S^2, J_0 \oplus J')$$

for a closed aspherical manifold (V_0, J_0) tamed by some form ω_0, and where (S^2, J') is the standard (Riemann) sphere. For any point $(v_0, s_0) \in V$ there exists a *unique* holomorphic sphere S_0 in V passing through this point and homologous to the sphere $v_0 \times S^2 \in V$. (In fact, $S_0 = v_0 \times S^2$.) It easily follows that the corresponding zero-dimensional class $[\Sigma'_0]$ is nontrivial. Hence, this class also is nontrivial for *every* almost complex structure J_1 on V tamed by the form $\omega = \omega_0 \oplus \omega'$ on V where ω_0 tames J_0 on V_0 and ω' is the standard area form on S^2.

3.4. B_1. COROLLARY. *For every almost complex structure J_1 on V tamed by ω there exists a holomorphic map $f\colon S^2 \to V$ which passes through a given point in V and which is homologous to the sphere $v_0 \times S^2 \subset V$.*

3.4. B_2. REMARK. This corollary is similar to *the Riemann mapping theorem for spheres* which states (in our language) that for every two almost complex structures J_0 and J' on S^2, there exists a holomorphic sphere

$$S \subset (S^2 \times S^2, J_0 \oplus J')$$

which is homologous to the diagonal in $S^2 \times S^2$ and which passes through three given points in $S^2 \times S^2$, provided that these points are not contained in $(s_1 \times S^2) \cup (S^2 \times s_2) \subset S^2 \times S^2$ for any pair $(s_1, s_2) \in S^2 \times S^2$. In fact, our proof of 3.4.B_1 delivers such a holomorphic S for *every* (nonsplit) structure J_1 on $S^2 \times S^2$ tamed by the form $\omega_0 \oplus \omega'$ on $S^2 \times S^2$, provided $\int_{S^2} \omega_0 = \int_{S^2} \omega'$ and assuming the three points in question are not contained in the union of any two holomorphic spheres S_1 and S_2 in $S^2 \times S^2$ homologous to $s_1 \times S^2$ and $S^2 \times s_2$. (The proof of 3.4.B_1 also yields holomorphic S_1 and S_2 through every point in $(S^2 \times S^2, J_1)$.) This generalization of the Riemann mapping theorem is similar to that due to Schapiro [**Sch**] and Laurentiev [**L**]. However, the results in [**Sch**] and [**L**] (which are quite general by "hard" standards) are stated and proven in a noninvariant form. (Geometrically speaking, it is assumed in [**Sch**] and [**L**] that the spheres $s_1 \times S^2$ and $S^2 \times s_2$ are J_1-holomorphic for all $(s_1, s_2) \in S^2 \times S^2$.) Besides enormously complicating statements and proofs, this makes these results unsuitable for "soft" applications, though (due to the existence of holomorphic spheres S_1 and S_2 in $(S^2 \times S^2, J_1)$ through every point (s_1, s_2) in $S^2 \times S^2$) the Riemann mapping theorem in [**Sch**] and [**L**] is essentially as general as our invariant version. (It seems that hard analytic theorems reach maturity and become truly interesting only when they are applied to a sufficiently large natural class of objects defined in a soft coordinate free language.)

3.4. C. *Solution of nonhomogeneous Cauchy-Riemann equation.* Consider the bundles $H = \mathrm{Hom}(T(S), T(V))$ and $H' = \mathrm{Hom}_{\mathbb{C}}(T(S), T(V))$ over $S \times V$ for given (almost) complex structures J' in $T(S)$ and J in $T(V)$ and let $h \to \bar{h}$ denote the quotient homomorphism $H \to \overline{H} = H/H'$. Then for every smooth section $\varphi\colon S \times V \to \overline{H}$ we consider the equation

$$\overline{D}_f = \varphi = \varphi(s, f(s)) \tag{$*$}$$

for smooth maps $f\colon S \to V$.

The graphs $S \to S \times V$ of solutions f of $(*)$ are holomorphic for some (naturally constructed) almost complex structure $J = J(J', J, \ )$ on $S \times V$. Hence, the earlier compactness and existence discussion extends to solutions of $(*)$. Furthermore, one shows (see [**Gr3**]) that the pinching of circles in $S \to S \times V$ necessarily creates holomorphic spheres in (V, J). This leads to the following

ALTERNATIVE. *If (V, J) is a closed tame manifold and $S = S^2$ then either there is a nonconstant holomorphic map $S \to V$ or the equation $(*)$ admits a homotopic to zero solution $f\colon S \to V$ for all φ. For example, $(*)$ is always solvable if V is aspherical.*

3.5. *Holomorphic curves with boundaries.* A submanifold $W \subset (V, J)$ is called *totally real* if $\dim W = \frac{1}{2} \dim V$ and the intersection $T_w(W) \cap JT_w(W) \subset T_w(V)$ equals zero for all $w \in W$. For example, every curve in a Riemann surface is totally real. We say that ω *tames* (V, W) if it tames (V, J) and $\omega \mid W = 0$. That is, W is Lagrange (see §1.3) in (V, ω).

Let S be a compact Riemann surface with boundary and look at holomorphic maps $S \to V$ sending ∂S to W. The structure of these maps $f\colon (S, \partial S) \to (V, W)$ is essentially the same as for the case of a closed surface S. Namely, this space is compactified by singular holomorphic curves obtained by pinching some circles and some arcs in S with boundary points in ∂S. To see the picture, consider a complex manifold V with an $\mathbf{R}$-structure given by an antiholomorphic involution of V whose fixed point set (the real locus) $W \subset V$ has $\dim W = \frac{1}{2} V$ and, hence, is totally real. Then we take the upper hemisphere $S \subset S^2$ and observe that every holomorphic map $(S, \partial S) \to (V, W)$ extends, by symmetry, to a (unique) holomorphic map $S^2 \to V$ commuting with the $\mathbf{R}$-structures in V and S^2 (where the $\mathbf{R}$-involution on S^2 is the symmetry in the equator). Thus, the space of holomorphic maps $(S, \partial S) \to (V, W)$ is identified with the real locus of the space of holomorphic maps $S^2 \to V$ and the compactification of the former equals the real locus of the latter.

The compactness and existence theorems in §3.4 easily generalize to curves with boundaries (see [**Gr3**]). In particular, the alternative in §3.4.C holds true for a class of Lagrange submanifolds $W \subset \mathbf{C}^n$ which includes all *closed* submanifolds. For $V = \mathbf{C}^n$ the equation $\overline{D}_f = \varphi$ amounts to $\overline{\partial} f = \varphi$, and if φ is a constant map $S \to \mathbf{C}^n$ the the solutions $f\colon S \to \mathbf{C}^n$ are harmonic. Therefore, for compact W, no such f exists if the norm $\|\varphi\|$ is sufficiently large. Hence, by our alternative, there exists a nonconstant holomorphic C^∞-map of the disk into $\mathbf{C}^n$ with the boundary sent to W for every closed Lagrange C^∞-submanifold $W \subset C^n$. A similar reasoning applies to *immersed* Lagrange submanifold W in

$\mathbf{C}^n$ and provides holomorphic disks mentioned in §1.3. Notice that holomorphic disks may be nonsmooth at the boundary points reaching double points of the immersed $W \subset \mathbf{C}^n$. But these disks are Hölder if W has normal crossings.

4. Applications of holomorphic curves in symplectic geometry.

4.1. *Embeddings of open manifolds.* Define *symplectic width* of (V, ω) as the lower bound of the numbers $a > 0$, such that for every almost complex structure J on V tamed by ω and for every point $v \in V$ there exists a nonconstant connected properly mapped J-holomorphic curve $f\colon S \to V$ passing through v, such that the symplectic area of f satisfies $\int_S f^*(\omega) \le a$.

Obviously (see [**Gr3**]), this width is monotone under *equidimensional* symplectic embeddings. Namely, if V_1 embeds into V_2, then

$$\operatorname{width} V_1 \le \operatorname{width} V_2.$$

EXAMPLES. (a) Let U_r be the ball of radius r in $\mathbf{C}^n$. Every holomorphic curve in U_r through the center has area $\ge r^2$ (this is well known and easy to prove). Hence,

$$\operatorname{width}(U_r, \omega = \sum dx_i \wedge dy_i) \ge \pi r^2.$$

(In fact, $\operatorname{width} U_r = \pi r^2$; see [**Gr3**].)

(b) Let $(V, \omega) = (V_0 \times S^2, \omega_0 \oplus \omega')$, where (V_0, ω_0) is a $(2n-2)$-dimensional closed aspherical manifold and ω' is an area form on S^2. They by 3.4.B$_1$,

$$\operatorname{width} V \le \operatorname{area}(S^2, \omega') \underset{\text{def}}{=} \int_{S^2} \omega'.$$

(Clearly, $\operatorname{width} V \ge \operatorname{area} S^2$.)

COROLLARY. *If the above ball U_r embeds into V, then* $\operatorname{area} S^2 \ge \pi r^2$.

Notice that the sphere S^2 minus a point, is symplectically isomorphic to the ε-disk in $\mathbf{C}$, such that $\pi\varepsilon^2 = \operatorname{area} S^2$ and every relatively compact subset in $\mathbf{C}^{n-1} \times S^2$ embeds into the closed manifold $(\mathbf{C}^{n-1}/\Lambda) \times S^2$ for some lattice Λ in $\mathbf{C}^{n-1}$. Thus, the corollary shows that U_r symplectically embeds into the ε-neighborhood of the subspace $\{z_1 = 0\} \subset \mathbf{C}^n$ if and only if $r \le \varepsilon$. This implies the nonembedding results started in §1.2 (compare [**Gr3**]).

4.2. *Codimension 2 embeddings.* Let V be a symplectic manifold and $V_0 \subset V$ a closed codimension two symplectic submanifold. By using an almost complex structure J on V for which the submanifold V_0 is J-complex, one can show that the invariants Σ' of V (see §3.4) restrict in some natural sense to those of V_0. A typical corollary one can obtain is as follows:

Let $V = \mathbf{C}P^2 \times \mathbf{R}^2$, where $\mathbf{C}P^2$ is endowed with the standard (U(3)*-invariant*) *symplectic structure and $\mathbf{R}^2 = (\mathbf{R}^2, dx \wedge dy)$. If a closed 4-dimensional symplectic manifold V_0 symplectically embeds into V, then V_0 is symplectically diffeomorphic to $\mathbf{C}P^2$, provided $\pi_2(V_0)$ is cyclic.*

4.3. *Existence, extension, and equivalence problems for symplectic structures.* Let ω_0 be a nonsingular 2-form on a connected manifold V.

PROBLEM. Does there exist a homotopy of nonsingular forms, say ω_t for $0 \le t \le 1$, such that the form ω_1 is closed (and hence, symplectic)?

If V is open, then the affirmative answer is provided by a sheaf theoretic version of the Smale-Hirsch immersion theory (see [**Gr1**]). If V is closed, there is an obvious obstruction for the existence of ω_t. Namely, there must exist a 2-dimensional cohomology class α on V, such that $\alpha^n \neq 0$ for $2n = \dim V$. One still does not know if there are further obstructions.

A similar problem is that of extension of a symplectic form from a subset in V to all of V. Here, holomorphic curves provide a nontrivial obstruction.

EXAMPLE (SEE [**Gr3**]). Let V be an open 4-dimensional manifold such that the Hurewitz homomorphism $\pi_2(V) \to H_2(V)$ is zero, and let ω_0 be a symplectic form on a neighborhood U of infinity in V. If (U, ω_0) is symplectically diffeomorphic to some neighborhood of infinity in $(\mathbf{R}^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$, and if ω_0 extends to a symplectic form on all of V, then V is diffeomorphic to $\mathbf{R}^4$.

Now, consider two symplectic forms ω_0 and ω_1 on a *closed* manifold V which represent the same cohomology class, $[\omega_0] = [\omega_1] \in H^2(V; \mathbf{R})$, and which can be joined by a homotopy of nonsingular forms ω_t. If ω_t is symplectic for all $t \in [0, 1]$, and the cohomology class $[\omega_t]$ is constant in t, then by the Darboux-Moser theorem the manifolds (V, ω_0) and (V, ω_1) are symplectically diffeomorphic. But if $[\omega_t]$ varies, then $[\Sigma']$-type invariants (see §3.4) may change (see [**McD**]) and then (V, ω_0) is not symplectically diffeomorphic to (V, ω_1).

4.4. *C^0-limits of symplectic diffeomorphisms.* It was probably assumed in the ("hard minded") classical mathematics (and mechanics, where symplectic diffeomorphisms are called *canonical transformations*) that symplectic diffeomorphisms can be distinguished from volume-preserving ones by certain global properties stable under uniform limits of diffeomorphisms. This belief (explicitly expressed by Arnold) was confirmed by Eliashberg (see [**E2**, **E3**]) who, in particular, proved (by a complicated combinatorial method stemming from Poincaré's "proof" of his last theorem) that every diffeomorphism of $\mathbf{R}^{2n}$ which is a C^0-limit of symplectic diffeomorphisms is symplectic. This C^0-stability also can be derived from the nonembedding results in §4.1 with use of Nash's implicit function theorem (see [**Gr4**]).

4.5. *Lagrange intersections and fixed points of symplectic diffeomorphisms.* As we have seen in §1.2, every symplectic diffeomorphism f lying in a one-parametric subgroup comes from some (generating) function (Hamiltonian) h on the underlying manifold V, provided $H^1(V; \mathbf{R}) = 0$, and the fixed points of f correspond to the critical points of h. Hence, the number of the fixed point of f can be bounded from below by Morse theory. A similar bound (conjectured by Arnold and generalizing the last Poincaré theorem) for exact (which generalizes, in a natural way, the notion of exactness for closed 1-forms corresponding to symplectic fields) area-preserving diffeomorphisms of surfaces was proved by Eliashberg [**E1**] by his combinatorial method. Then Conley and Zehnder [**C-Z**] proved another conjecture of Arnold: *every exact symplectic diffeomorphism f of the torus $T^{2n} = \mathbf{R}^{2n}/\mathbf{Z}^{2n}$ has at least $2n+1$ fixed points, and if f is generic* *then it has at least* 4^n *fixed points.* The proof involves an auxiliary (generating) function L (Lagrangian) on the space of contractible maps $S^1 \to T^{2n}$ where Conley and Zehnder apply variational techniques similar to those used for locating periodic orbits of Hamiltonian (i.e., symplectic) flows (see [**Rab**, **W2**, **Ber**]). Notice that this L has infinite Morse index (it looks like the quadratic function $\sum_{i=1}^{\infty} x_i y_i$ on $\mathbf{C}^{\infty}$) and the naive Morse theory does not apply to L. (In fact, there are only a few interesting variational problems where the Morse theory based on the Palais-Smale condition can be applied directly.) The method of Conley and Zehnder was cleaned of "hard" analysis by Chaperon who used, instead of L, a function on a finite-dimensional space similar to the space of broken geodesic in a Riemannian manifold. The hard regularity theorem reduces, in Chaperon's approach, to showing that every broken *extremal* curve is in fact unbroken.

The fixed points of a symplectic diffeomorphism f of (V,ω) correspond to the intersection points of two Lagrange submanifolds in $(V \times V, \omega \oplus -\omega)$, which are the graph of f and the diagonal in $V \times V$. The (Chaperon's rendition of) Conley-Zehnder method extends to some Lagrange manifolds besides symplectic graphs. For example, Laudenbach and Sikorav [**L-S**] established by this method a Morse theoretic lower bound on the number of intersection points of an *exact* (in a suitable sense) Lagrange submanifold in $T^*(X)$ with the zero section $X \subset T^*(X)$, where $T^*(X)$ is endowed with the canonical (see §1.3) symplectic structure. (Compare [**Ch**, **F-W**, **W1**, **Z**].)

An alternative approach to (self) intersections of immersed Lagrange submanifolds $W \subset V$ is provided by holomorphic curves $(S, \partial S) \to (V, W)$ (see §3.5 and [**Gr3**]). Similar curves (in a Morse theoretic framework) are used by Floer [**F1**] who proved a homological version of a general Arnold's conjecture. It seems that the method of holomorphic curves has an advantage of greater generality, while the symplectic Morse theory, whenever it applies, leads to finer (Morse) inequalities.

4.6. *Contact geometry.* A codimension one subbundle $\theta \subset T(x)$ is called *contact* if there exists an almost complex structure J on $X \times \mathbf{R}$, such that

$$\theta = T(X) \cap JT(X)$$

for $X = X \times 0 \subset X \times \mathbf{R}$, and such that X is *strictly J-convex* in $X \times \mathbf{R}$. That is, $X = X \times 0$ is a level of a strictly J-convex function (see §3.2) without critical points. Soft properties of contact manifolds (V, θ) are quite similar to those of symplectic manifolds (see [**Gr4**]). Fundamental rigidity theorems for contact 3-manifolds were proven by Bennequin [**B1**] using topological (knot theoretic) techniques. In particular, Bennequin proved the existence of exotic contact structures on $\mathbf{R}^3$ and on S^3. (The standard θ on S^3 is $T(S^3) \cap \sqrt{-1}T(S^3)$ for the usual embedding $S^3 \subset \mathbf{C}^2$. The standard $\mathbf{R}^3$ is obtained by removing a point from this S^3.)

Now let (V, J) be an almost complex manifold with J-convex boundary Y and let $\theta = T(Y) \cap JT(Y)$. If $W \subset Y$ is totally real in $V \supset Y$, then, under a suitable taming condition, holomorphic curves in V with boundaries in W behave like those in §3.5. This has a nontrivial effect on the contact geometry of (Y, θ) and, in particular, shows that not every (Y, θ) appears as a tame J-convex boundary of some (V, J). Using this, one can find, for example, an exotic contact $\mathbf{R}^{2n-1}$ for all $n \geq 1$ (see [**Gr3**, **B2**]) and can recapture finer results of Bennequin for 3-manifolds (a private communication by Eliashberg). Yet the holomorphic contact geometry is less understood at the present moment than the symplectic case.

Soft and hard historical remarks. It seems difficult (if possible at all) to assign a precise metamathematical meaning to the notions of relative softness and hardness of an argument or of a theory. Intuitively, "hard" refers to a strong and rigid structure of a given object, while "soft" suggests some weak general property of a vast class of objects. Thus, inequalities and estimates are softer than identities, (algebraic) number theory is harder than analysis (over locally compact fields and adeles), real analysis is softer than complex analysis. Semisimple Lie groups and symmetric spaces look, from a certain angle, almost as hard as the integers, while Riemannian geometry appears, on a whole, nearly as soft as differential topology. The proof of $x^3 + y^3 + z^3 \geq 3xyz$ for $x, y, z \geq 0$ by the convexity of $\log t$ seems softer than the proof by the identity $2(x^3 + y^3 + z^3 - 3xyz) = (x + y + z)((x - y)^2 + (x - z)^2 + (y - z)^2)$, although proofs in algebraic geometry which use elliptic P.D.E. are at a level of hardness with those using Frobenius.

"Soft" and "hard" in this talk are limited to the framework of the *global nonlinear analysis* concerning the geometry of spaces of maps between smooth manifolds. The modern approach to such spaces started with the soft homotopy touch by Serre [**S1**, **S2**], and soft techniques and ideas have dominated the theory ever since. It is still not known whether Serre's results (e.g., the finiteness of the stable homotopy groups of spheres) can be recaptured by hard means, although hard arguments have been occasionally used in similar problems (e.g., Morse theory in Bott periodicity, the action of Frobenius in Adams' conjecture, linear elliptic operators in the signature theorem and the higher signature conjecture). A similar softness also has prevailed in differential topology, since the flexibility of diffeomorphisms (Thom [**Th**]), immersions (Smale [**Sm**]), and surgeries allowed an essential reduction of Diff-problems to the homotopy theory. (This soft topological avalanche has been arrested by Thurston's geometrization of 3-manifolds and by Donaldson's gauge invariants of 4-manifolds.)

The softness discovered in topology could not, however, discourage the search for classical hard structures based on nonlinear partial differential equations. However, the naive dream of hard global analysis was destroyed overnight by Nash's discovery [**N1**, **N2**] of the amazing flexibility of solutions of certain nonlinear equations, namely of isometric immersions of Riemannian manifolds. (For example, *every* distance-decreasing embedding of the standard m-sphere into $\mathbf{R}^n$ admits, for $m < n$, a uniform approximation by isometric C^1-imbeddings; see [**N1**, **K**].) In fact, one could think (until the work of Donaldson) that Nash's phenomenon (which is ubiquitous for nonlinear P.D.E., see [**Gr4**]) totally rules out any hard P.D.E. structure in the soft vastness of nonlinear function spaces. (The hard structure of *linear* elliptic P.D.E. is firmly rooted today in soft topological soil as had been envisioned by Atiyah [**A**].) Now, holomorphic curves which are concerned with the most elementary equation (and which logically precede, though historically follow, gauge fields) appear as a first stepping stone to a comprehensive hard nonlinar theory.

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