Scalar Curvature of Manifolds with Boundaries: Natural Questions and Artificial Constructions

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During the workshop *Emerging Topics: Scalar Curvature and Convergence* in October 15-19, 2018 at IAS in Princeton, Chao Li, Pengzi Miao, André Neves, Christina Sormani and myself discussed a possible formulation of and approach to the solution of the following problems.

Problem A. Let $Y = (Y, h)$ be a closed $(n-1)$-dimensional Riemannin manifold, which, besides a Riemannian metric h, carries a continuous function $M = M(y)$ on it.

Find condition(s) on (Y, h, M) that would allow/wouldn't allow

a filling of Y by a compact Riemannin $(n+1)$-dimensional manifold $X = (X, g)$, where "filling" means that

$$\partial X = Y,$$

where

- *the restriction of the Riemannin metric g to Y is equal to h;*
- *the mean curvature of Y in X is equal to M,*

(by our sign convention *convex* boundaries have $M \geq 0$) *and where the essential property required of X is*

- *non-negativity of the scalar curvature,*

$$Sc(X) = Sc(g) \geq 0,$$

or, more generally, a lower bound $Sc(X) \geq \sigma$ for a given $\sigma \in (-\infty, \infty)$.

Problem B. Granted that a filling X with $Sc(X) \geq 0$ (or with $Sc(X) \geq \sigma$) exists, what are the constraints on the geometry of such an X imposed by (Y, h, M)?

For example:

$\mathbf{A}_1$. Does *sufficient*, depending on (Y, g), *mean convexity*, i.e. "large positivity" of the mean curvature of Y, rule out such fillings?

$\mathbf{B}_1$. Is there a *lower bound on the volume* of filling manifolds X, in terms of (Y, h, M) and the lower bound on the scalar curvature of X (with a particular attention to the the case $Sc(X) \geq 0$)?[1]

C. Is there a relation(s) between $\mathbf{A}_1$ and *the lower bounds on the dihedral angles* of Riemannin manifolds X *with corners*, where

[1] This question was raised by Pengzi Miao, as I recall.

$$Sc(X) \geq 0 \text{ and } mean.curv(\partial X) \geq 0?$$[2]

What follows is (a rectified version of the first draft) of what came out of our conversation.

Contents

1 Bound on the Size of ∂X by the Scalar Curvature and the Mean Curvature

The following inequalities $[mean]_{Sc<0}$ and $[mean]_{Sc\geq 0}$, deliver a positive answer to $\mathbf{A}_1$ for *spin* manifolds X.

Let $Rad_{S^{n-1}}(Y)$, denote the the *(hyper)spherical radius* of a closed orientable Riemannin $(n-1)$-dimensional manifold Y, that is the maximal radius R of the sphere, such that Y admits a distance decreasing map $Y \to S^{n-1}(R)$ of non-zero degree.

Observe that

- $Rad_{S^{n-1+m}}(Y \times S^m(R)) = \min(R, Rad_{S^{n-1}}(Y))$;
- if we require our *distance decreasing* map to be *smooth*, we come up with the same (which could be, a priori, smaller) value of $Rad_{S^{n-1}}(Y)$.

Theorem. Let X be a compact orientable spin manifold of dimension n with boundary $Y = \partial X$.

If $Sc(X) \geq -\sigma$, $\sigma \geq 0$, then the infimum of the mean curvature of $Y \subset X$ is bounded by

$$[mean]_{Sc<0} \qquad \inf_{y\in Y} mean.curv(Y,y) \leq \max\left(\frac{n+m-1}{Rad_{S^{n-1}}(Y)}, \sqrt{\frac{\sigma}{m(m-1)}}\right).$$

for all $m \geq 2$.

Furthermore if $Sc(X) \geq 0$, then

[2]This question by Christina Sormani started our conversation.

$[mean]_{Sc\geq 0}$ $$\inf_{y\in Y} mean.curv(Y,y) \leq \frac{n-1}{Rad_{S^{n-1}}(Y)}.$$

About the Proof. Inequality $[mean]_{Sc\geq 0}$ is proven in the next section by mapping the double $X\cup_Y X$ to the sphere $S^n = B^n \cup_{S^{n-1}} B^n$ endowed with a radial metric, which has *positive curvature operator* and to which *Goette-Semmelmann's extremality theorem* [GS 2000] applies.

Then inequality $[mean]_{Sc<0}$ follows by applying $[mean]_{Sc\geq 0}$ to $X\times S^m(R)$ for $R=\sqrt{\frac{m(m-1)}{\sigma}}$.

Remarks, Examples, Questions. (a) The inequalities $[mean]_{Sc\geq 0}$ and $[mean]_{Sc<0}$ are already significant for closed *imbedded* hypersurfaces Y in the Euclidean space $\mathbb{R}^n$ (where they bound domains X with Sc(X)=0) and in the hyperbolic space $\mathbb{H}^n_{-1}$ (where $Sc(X)=-n(n-1)$).

There is no apparent *direct* proof of this with *no Dirac operators behind the scenes*, which are essential for the proof of Goette-Semmelmann's theorem.

Notice in this regard that a proof of such an inequality can't localise near Y. In fact,

immersed hypersurfaces with self-intersection in $\mathbb{R}^n$ e.g. finite coverings of a 2-torus in $\mathbb{R}^3$, with $mean.curv>0$, may have *arbitrarily large (hyper)spherical radii* regardless of their mean curvatures.

(b) The inequality $[mean]_{Sc\geq 0}$ is sharp, where balls $X=B^n(R)\subset\mathbb{R}^n$ with boundaries $Y=S^{n-1}(R)=\partial B^n(R)$ are *extremal* for this inequality, since $[mean]_{Sc\geq 0}$ turns to equality for balls:

$$mean.curv(S^{n-1}(R)) = \frac{n-1}{R=Rad_{S^{n-1}}(S^{n-1}(R))}.$$

Moreover, one can show as in 11.2 from [G 2018]) that

if $Sc\geq 0$ *and*
$$\inf_{y\in Y} mean.curv(Y,y) = \frac{n-1}{Rad_{S^{n-1}}(Y)},$$
then $Sc(X)=0$ *and the mean curvature of* Y *is constant* $=\frac{1}{Rad_{S^{n-1}}(Y)}$.

Furthermore,

if $Sc(X)=0$, then the equality $mean.curv(Y)=\frac{n-1}{Rad_{S^{n-1}}(Y)}$ implies that X is *isometric to the ball* $B^n(R)\subset\mathbb{R}^n$.

But this needs additional reasoning (with a use, at some point of the proof, of A.D. Alexandrov's theorem on *sphericity of hypersurfaces with constant mean curvature in* $\mathbb{R}^n$), where the complication is partly due to an approximation error in our argument, when we smooth the metric in the double $X\cup_Y X$.

Yet, there is a case, where the proof of the rigidity causes no problem, namely, when the manifold X near its boundary ∂X is isometric to a neighbourhood of a sphere in the Euclidean space $\mathbb{R}^n$ and where our theorem implies that

if a connected Riemannin spin manifold X^+ *with* $Sc(X^+)\geq 0$ *is isometric at infinity, i.e. outside a compact subset in* X^+, *to the complement of an* R*-ball in the Euclidean space* $\mathbb{R}^n$, *then* X^+ *is isometric to* $\mathbb{R}^n$.

(Probably, Goette-Semmelmann's Dirac theoretic argument behind our proof converges, when $R \to \infty$, to that by Witten in his proof of *the positive mass theorem.*[3])

(c) Unlike $[mean]_{Sc\geq 0}$, the inequality $[mean]_{Sc<0}$ is non-sharp.

Conjecturally, $\inf_{y\in Y} mean.curv(Y, y)$ must be bounded by the mean curvature of the ball of radius $R = Rad_{S^{n-1}}(Y)$ in the hyperbolic n-space with constant sectional curvature $-\sigma/n(n-1)$.

(d) The spin condition for X can be relaxed to *spin of the universal covering* $\tilde{X}$ (see remark (d) following formulation of the Goette-Semmelmann theorem in the next section and also section 10 in [G 2018]), but it remains *problematic* if this condition is necessary at all.

Yet, we indicate in section 6 an approach, although a very artificial one, to such (non-sharp) inequalities (with extra assumptions on the geometry of Y) by the techniques of *minimal hypersurfaces* where spin is irrelevant.

(e) One can't, in general, substitute $\inf_y mean.curv(Y, y)$ in the above inequalities by any integral characteristic of the function $mean.curv(Y, y)$ but this may be possible with a more subtle (spectral?) geometric invariant of Y than $Rad_{S^{n-1}}$. (Compare with ★ ★ in section 3.)

(f) What are simple (non-simple?) examples of *extremal/rigid* Riemannian n-manifolds X with boundaries, where you can't simultaneously increase $Sc(X)$, $mean.curv(\partial X)$ and $Rad_{S^{n-1}}(\partial X)$?

For instance what is the *sharp* bound on the mean curvature of $Y = \partial X$ by $Rad_{S^{n-1}}(Y)$ for manifolds X with $Sc(X) \geq n(n-1)$?

(The subtlety of this extremality/rigidity problem for balls in S^n is revealed by the results by Brendle, Marques and Neves [BMN 2010] and by Brendle and Marques [BM 2011]:

the existence/nonexistence of a *deformation of the spherical metric* in the interior of such a ball with *increase of the scalar curvature* depends of the *radius* of the ball.)

2 Extremal Metrics on Spheres, Doubles and the Proof of Inequality $[mean]_{Sc>0}$

Let X be a compact Riemannin n-manifold with boundary $Y = \partial X$ and let $X_\varepsilon X$ be the $[{}^{\varepsilon}\leftrightarrows_{\varepsilon}]^{\mathbb{D}}$*-rounding* of the double $X \cup_Y X$ of of X (compare p. 227 in [GL 1980] and section 11.4 in [G 2018]) that is the boundary of the ε-neighbourhood of $X_{-\varepsilon} = X_{-\varepsilon} \times \{0\} \subset X \times \mathbb{R}$,

$$X_\varepsilon X = \partial U_\varepsilon(X_{-\varepsilon}) \subset X \times \mathbb{R},$$

[3] The positive mass theorem, which claims the rigidity of $\mathbb{R}^n$ under a class of perturbations which may have *non-compact supports*, was proven in 1979 by Schoen and Yau for manifolds of dimensions ≤ 7 and in 1981 by Witten for spin manifolds of all dimensions, while the reduction to the, a priori more restrictive, Euclidean rigidity was found by J. Lohkamp in 1999.

The Euclidean rigidity also is an obvious corollary of

non-existence of non-flat metrics with $Sc \geq 0$ *on the torus and on overtorical manifolds,*

that is now established (see [SL 2017] and [L 2018] for, all possibly non-spin, manifolds of all dimensions n.

where $X_{-\varepsilon} \subset X$ is the complement of the ε-neighbourhood of the boundary $\partial X \subset X$.

The hypersurface $X_\varepsilon X \subset X \times \mathbb{R}$ consists of two 2ε-equidistantly parallel copies of $X_{-\varepsilon}$ and a semicircular part $\mathbb{D}_\varepsilon = \mathbb{D}_\varepsilon(X) = X_{-\varepsilon} \times S^1_+(\varepsilon)$, that is a half of the boundary of the ε-neighbourhood of $X_{-\varepsilon} \times \mathbb{R}$, as depicted in figure 8 on p. 227 in [GL 1980]).

The principal curvatures $\lambda_1, ..., \lambda_n$ of $\mathbb{D}_\varepsilon$ are evaluated, for small $\varepsilon \to 0$, in terms of the principal curvatures $\mu_1, ..., \mu_{n-1}$ of the boundary ∂X as follows.

$$\lambda_i = (\mu_i + O(\varepsilon)) \cdot \cos\theta + o(\varepsilon) \text{ for } i = 1, ..., n-1 \text{ and } \lambda_n = \varepsilon^{-1} + O(1),$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ denotes the angular parameter of the (right) semicircle $S^1_+(\varepsilon)$ and the scalar curvature of $\mathbb{D}_\varepsilon$, which is expressed by the Gauss theorema egregium, satisfies

$$Sc(\mathbb{D}_\varepsilon)(\underline{x}, \theta) = Sc(X)(\underline{x}) + (2\varepsilon^{-1} m(\underline{x}) + O(1)) \cdot \cos\theta + o(1),$$

where $m(v) = mean.curv(\partial X)(\underline{x})$, $x \in \partial X$, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Smoothing Remark. The submanifold $X_\varepsilon X \subset X \times \mathbb{R}$ is only C^1-smooth and the induced metric in it is only continuous. But it can be smoothed as in section 11.1 of [G 2018] with an arbitrary small perturbation of the metric of this submanifold and of its scalar curvature.

Let f be a smooth map from the boundary $Y = \partial X$ to the R-sphere $S^{n-1}(R) \subset \mathbb{R}^n$ and let $F : X \to \underline{X} = B^n(R) \subset \mathbb{R}^n$ be a smooth extension of f to a map from X to the ball bounded by $S^{n-1}(R)$.

Observe that, for small ε, the map F naturally "suspends" to a map between the "round doubles" of these manifolds, denoted

$$F_{\mathbb{D}_\varepsilon} : X_\varepsilon X \to \underline{X}_\varepsilon \underline{X}.$$

Clearly, the differential of $F_{\mathbb{D}_\varepsilon}$ away from $\mathbb{D}_\varepsilon(X)$ is the same as dF, while the norm of $F_{\mathbb{D}_\varepsilon}$ at $(x, \theta) \in \mathbb{D}_\varepsilon(X)$ satisfies

$$||dF_{\mathbb{D}_\varepsilon}(x, \theta)|| \leq ||df(x)||(1 + O(\varepsilon)).$$

Now, these estimates for $X_\varepsilon X$ along with similar ones for $\underline{X}_\varepsilon \underline{X}$, where $\underline{X} = B^n(R)$, and with the above smoothing remark allows reduction of the desired inequality $[mean]_{Sc \geq 0}$, written as

$$Rad_{S^{n-1}}(\partial(X)) \leq \frac{n-1}{\inf mean.curv(\partial X)}.$$

to the following,

Goette-Semmelmann Extremality Theorem [GS 2000]. Let $\underline{Z} = (S^n, \underline{g})$ be the n-sphere with a Riemannin metric $\underline{g}$ which has a *non-negative curvature operator*, e.g. $\underline{Z} \subset \mathbb{R}^{n+1}$ is a smooth convex hypersurface obtained by smoothing $\underline{X}_\varepsilon \underline{X}_\varepsilon \subset \mathbb{R}^{n+1}$ for $\underline{X}_\varepsilon = B^n \subset \mathbb{R}^n$, with the induced Riemannin metric. [4]

Let X be a closed connected orientable n-manifold and $f : X \to \underline{X}$ be a smooth map.

[4]Manifolds with positive sectional curvatures isometrically embeddable to $\mathbb{R}^{n+2}$, have positive curvature operators by Weinstein's theorem [W 1970], but for $X \subset \mathbb{R}^{n+1}$ this, probably, was known prior to Weinstein's paper.

If

- *X is spin and the map f has non-zero degree;*
- *$||df(x)|| \leq 1$ at all points $x \in X$, where $Sc(\underline{X}, f(x)) > 0$;*
- *$Sc(X, x) \geq Sc(\underline{X}, f(x))$. for all $x \in X$,*

then

$$Sc(X, x) = Sc(\underline{X}, f(x))$$

for all $x \in X$.

Thus, the inequality $[mean]_{Sc\geq 0}$, hence, $[mean]_{Sc<0}$, is established.

Remarks. (a) There are further (less straightforward) applications of the Goette-Semmelmann theorem to manifolds with boundaries but we shall discuss these somewhere else.

(b) A non-sharp version of the inequality $[mean]_{Sc\geq 0}$ can be derived from a *rough hypersphericity inequality* [GL 1980], the proof of which, unlike the Goette-Semmelmann's theorem. doesn't depend on a non-trivial curvature computation.

All we actually need is the lower bound $Sc(X_\varepsilon X) \geq \frac{1}{3\varepsilon}$ in the $\frac{\varepsilon}{2}$-neighbourhood of $\partial X \subset X_\varepsilon X$ for a small $\varepsilon > 0$, where we observe that

smooth maps $f : \partial X \to S^{n-1}$, $n = dim(X)$, extend to smooth maps

$$F_\varepsilon : X_\varepsilon X \to S^n \supset S^{n-1},$$

such that the two components of the complement to the normal $\frac{\varepsilon}{2}$-neighbourhood $\partial X \times \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] \subset X_\varepsilon X$ of $\partial X \subset X_\varepsilon X$ collapse to the two poles of S^n and such that the second exterior power of the differential of F_ε is bounded by the norm of the differential of f on ∂X as follows.

$$|| \wedge^2 (dF_\varepsilon(x, \delta)|| \leq \frac{10}{\varepsilon} ||df_{|\partial X}(x)||$$

for all points $(x, \delta) \in \partial X \times \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$, $x \in \partial X$, $\delta \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right]$.

Since $deg(F_\varepsilon) = deg(f)$, the rough bound

$$Sc(X_\varepsilon X, (x, \delta) \leq const_n || \wedge^2 (dF_\varepsilon(x, \delta)|| \cdot Sc(S^n)$$

from [GL 1980] yields the corresponding bound

$$mean.curv(\partial X, x) \leq const'_n ||df_{\partial X}|| \cdot mean.curv(S^{n-1}).$$

Notice, that one could apply here *Llarull's inequality* [Ll 1998] to F_ε with *sharp* $const_n = 1$, but this would not make the resulting bound for f sharp anyway.

(c) If X is homeomorphic to the 3-ball, one could also use a (finer than Llarull's) version of the hypersphericity inequality due to the Marques and Neves, that is

a bound on areas of spherical minimal surfaces in $X_{\delta\delta}$ with Morse indices one.[5]

(d) A suitable form of the L_2-index theorem (see [G 1996]) allows a generalisations of Goette-Semmelmann' theorem to manifolds Z the universal covering

[5] This is theorem 4.9 in [MN 2011], which implies a sharp(!) bound on *the 2-waist* of $X \cup_Y X$, which is significantly more precise than Llarull's inequality, where, conceivably, there is a version of this bound applicable to non-spherical 3-manifolds.

of which, rather than Z themselves, are spin. Accordingly, the inequalities $[mean]_{Sc\geq 0}$ and $[mean]_{Sc<0}$ generalise to manifolds X the universal covering of which are spin.

And, in a similar fashion, these inequalities generalise to complete non-compact manifolds X.

Warning Exercise. Find a counterexample to the following "proposition" and then find the mistake in its "proof".

Let X be a closed orientable Riemannin spin n-manifold with $Sc(X) \geq -\varepsilon$ and let $U \subset X$ be an open connected subset, such that the scalar curvature (function) of X is large on U, say

$$Sc(X,u) \geq 1000n(n-1) \text{ for all } u \in U.$$

.

"Proposition". *If $\varepsilon > 0$, is sufficiently small, say $\varepsilon \leq 0.001$, then X admits no area decreasing map $f : X \to S^n = S^n(1)$ of non-zero degree that sends the complement of U to a point $s_0 \in S^n$.*

"Proof". Assume otherwise, let $X' = X \times S^2$ and compose the map $f' = (f, id) : X' \to S^n \times S^2$ with a smooth *distance decreasing* map $S^n \times S^2 \to S^{n+2}\left(\frac{1}{10}\right)$ of degree one, which collapses $\{s_0\} \times S^2$ to a point. Since the resulting map $F' : X' \to S^{n+2}$ is *area decreasing* and since the scalar curvature of X' is *everywhere positive*, say for $\varepsilon < 1$, and since

$$Sc(X') > 100(n+1)(n+2) = Sc\left(S^{n+3}\left(\frac{1}{10}\right)\right)$$

at *all points where the differential of F' doesn't vanish*, namely in $U \times S^2$, the "proof" follows by the contradiction with rough hypersphericity inequality.

Hint. The above argument becomes valid if "no area decreasing map f" is replaced by "no distance decreasing map f".[6]

3 Rigidity and Extendability of Metrics with Lower Bounds on the Scalar Curvature.

To put our inequalities $[mean]_{Sc\geq 0}$ and $[mean]_{Sc<0}$ to a perspective, let us remind several earlier results by Miao, Shi-Tam, and Mantoulidis-Miao.

Let Y_0 be a smooth closed hypersurface in the Euclidean space $\mathbb{R}^n$, let X be a compact n-manifold with the boundary Y isometric to Y_0 and let $mc_0 = mc_0(y)$, $y \in Y$, be the mean curvature of Y_0 transported to Y by our isometry $Y \leftrightarrow Y_0$.

$\star$Mean Curvature Rigidity Theorem. *If $Sc(X) \geq 0$, then the mean curvature of $Y \subset X$ can't be greater than that of $Y_0 \subset \mathbb{R}^n$,*

$$mean.curv(Y,y) \ngeq mc_0(y),$$

unless X is isometric to the domain $X_0 \subset \mathbb{R}^n$ bounded by $Y_0 \subset \mathbb{R}^n$.

[6] The method of minimal hypersurfaces (which needs no spin) applied to U allows an alternative proof of this in most (but not all!) cases, see [GL 1983] and [G 2018].

This theorem, which in the case $Y_0 = S^{n-1}$ can be regraded as a (partial) upgrade of $[mean]_{Sc\geq 0}$ from extremality to rigidity, is proved in [M 2003] and in [ST 2003]) in three steps, roughly, as follows.

$\star$ Attach X to the complement $Z_0 = \mathbb{R}^n \smallsetminus X_0$ by the "$\leftrightarrow$"-isometry

$$Y = \partial X \leftrightarrow \partial Z = Y_0.$$

$\star\star$ Smooth the Y-corner in the resulting manifold

$$W = X \cup_Y Z$$

while keeping $Sc(W)$ everywhere ≥ 0.

$\star\star\star$ Use Euclidean Rigidity theorem, which we have already met in section 1 and which says in the present notation the following.

$\ast$ *If a complete C^3-smooth Riemannin manifold W with $Sc(W) \geq 0$ is isometric to $\mathbb{R}^n$ at infinity, then W is isometric to $\mathbb{R}^n$.*

The following more satisfactory version of $\bigstar$ is proven in [ST 2003]), for convex hypersurfaces.

$\bigstar\bigstar$ Integral Mean Curvature Rigidity Theorem. *If Y_0 is convex, then*

$$\int_Y M(y)dy \leq \int_Y M_0(y)dy,$$

where the equality $\int M(y) = \int M_0(y)$ implies that X is isometric to X_0.

Hyperbolic Remarks. ($\mathbf{H}_1$) It is shown in [MM 2016], Proposition 3.2, that,

Min-Oo's rigidity and positive mass theorems for hyperbolic spaces
allow

$\bigstar_{-\kappa}$ *extensions of theorems $\bigstar$ and $\bigstar\bigstar$ to hypersurfaces Y_0 in the hyperbolic spaces $\mathbb{H}^n_{-\kappa}$ with the sectional curvature $-\kappa$ and manifolds X with $Sc(X) \geq -n(n-1)\kappa$.*

This is can be reformulated in an especially pleasant manner for $n = 2$ if $Y = \partial X$ is homeomorphic to S^2 and has $sect.curv(Y) \geq -\kappa$ due to

Pogorelov's existence (and essential uniqueness) theorem for isometric embeddings of spherical surfaces with metrics having sectional curvatures $> -\kappa$ to $\mathbb{H}^3_{-\kappa}$.

($\mathbf{H}_2$) The above mentioned results from [M 2003], [ST 2003] and [MM 2016] allow small perturbations of the metrics on Y which can be deformed back to their original values with a minimal loss of positivity of the scalar and the mean curvatures.

Thus, one easily shows, for instance, that

given a smooth closed hypersurface Y_0 in $\mathbb{H}^n_{-\kappa}$, there is an effectively computable $\Delta = \Delta_{N,\kappa}(Y) > 0$, such that if a metrics h on Y_0 is C^2-close to the original metric h_0 on $Y_0 \subset \mathbb{H}^n_{-\kappa}$ induced from $\mathbb{H}^n_{-\kappa}$

$$||h - h_0||_{C^2} \leq \delta < \Delta,$$

then Y admits no filling by manifolds X with $Sc(X) \geq -n(n-1)\kappa$ and with the mean curvature of $Y = \partial X$ in X bounded from below by M_δ for function M_δ on Y_0 which converges to the mean curvature of $Y_0 \subset \mathbb{H}^n_{-\kappa}$ for $\delta \to 0$.

Whenever applies, this inequality is sharper than our $[mean]_{Sc<0}$, but, unfortunately this Δ is pretty small, something like $1/n^2$, even for the unit sphere $S^{n-1} \subset \mathbb{H}^n_{-1}$.

($\mathbf{H}_3$) Min-Oo's proof of his rigidity and the positive mass theorems uses a version of Witten's Dirac operator method, thus, it needs the spin condition. Accordingly, the original proofs in [M 2003], [ST 2003] and [MM 2016] needed X to be *spin*, but, as we know now, the spin condition is redundant according to [SY 2017].

Moreover, Min-Oo's rigidity theorem remains valid for manifolds $Z = \mathbb{H}^n_{-1}/\Gamma$ for *all discrete parabolic* isometry groups,[7] where it reads as follows.

(*) Generalised Min-Oo Rigidity Theorem. *Parabolic quotient manifolds* $Z = H^n_{-1}/\Gamma$ *admit no non-trivial "deformations" with compact supports and with* $Sc \geq -n(n-1)$,

where these "deformations" may arbitrarily change the topology of (a compact region in) Z with *no condition on spin.*

Notice, that the original Min-Oo rigidity theorem corresponds to the case where $\Gamma = \{id\}$ and that the rigidity for all Γ, including Min-Oo rigidity itself, trivially follows from the rigidity of "hyperbolic cusps", i.e. where Γ is isomorphic to $\mathbb{Z}^n$.

Thus, all of (*) reduces to the case of cusps, where the proof follows by a Schoen-Yau style argument with a use of minimal μ-bubbles or of minimal surfaces with boundary as it is briefly explained in section 4 below.

4 Manifolds with Negative Scalar Curvatures Bounded From Below.

We discuss in in this section an approach to the extremality/rigidity problem based on calculus of variation, that allows sharp inequalities for $Sc(X) < 0$ in some cases.

Let X be a compact orientable Riemannin n-manifold with, possibly disconnected, boundary, such that

• the scalar curvature of X is bounded from below by that of the hyperbolic n-space H^n_{-1},

$$Sc(X) \geq -n(n-1),$$

• the mean curvature of the boundary is bounded from below by that of the complement of a horoball in H^n_{-1},

$$mean.curv(\partial X) \geq -(n-1);$$

• there is a connected component of the boundary ∂X, call it Y_+, the mean curvature of which is bounded from below by $(n-1)$,

$$mean.curv(Y_+) \geq n-1.$$

[7] An isometry group Γ of H^n_{-1} is *parabolic* if there is a horosphere in H^n_{-1} invariant under the action of Γ, or, equivalently, if all isometries $\gamma \in \Gamma$ except *id* keep fixed a *unique common fixed pint* in the ideal boundary of H^n_{-1}.

Besides, in the present context, we also require that the actions of Γ on H^n_{-1} are *free*. But this, in fact, becomes unnecessary, if instead of deformations of H^n_{-1}/Γ, we speak of Γ-equivariant deformations of H^n_{-1}, such that the Γ-quotients of the supports of these deformations are compact.

●$_-$ *If the inclusion homomorphism between the fundamental groups*

$$\pi_1(Y_+) \to \pi_1(X)$$

is injective, then Y_+ *admits no map with non-zero degree to the* $(n-1)$*-torus unless the universal covering of* X *is isometric to the (infinite) "band" between too equidistant horospheres in* $\mathbb{H}^n_{-1}$.

This is proved in §$5\frac{5}{6}$ of [G 1996] for $n \leq 7$ by a "soap bubble"argument which extends to all n in view of [SY 2017] and where the argument from [L 2018] may be also applicable.

Example and Generalisation. Let X_0 be a compact *hyperbolic* (i.e. $sect.curv(X_0) = -1$) manifold of dimension n with *totally geodesic* boundary Y_0.

Observe that the equidistant deformations Y_d $d \geq 0$ of Y_0 in the ambient complete hyperbolic manifold $X_+ \supset X$ have $mean.curv(Y_d) \to n-1$ for $d \to \infty$. But, the above implies that

no Riemannin manifold X homeomorphic to X_0, or just admitting a map with non-zero degree to X_0, can have $Sc(X) \geq -n(n-1)$ and $mean.curv(\partial X) \geq n-1$, provided,

$Y_0 = \partial X_0$*, or a finite covering of it admits a non-zero degree map to the* $(n-1)$*-torus,*

This argument doesn't directly apply if no map $Y_0 \to \mathbb{T}^n$ with $deg \neq 0$ exists. However, since X_0 is *isoenlargeable* in the sense of [G 2018] a combination of the arguments from [G 1996] and [G 2018] (with a use of [SY 2017] for $n+1 \geq 9$) shows that

the above conclusion still holds for *all compact hyperbolic* manifolds X_0 with *totally geodesic* boundaries. Namely,

$$Sc(X) \geq -n(n-1) \Rightarrow mean.curv(\partial X) \not\geq n-1.$$

for all X homeomorphic to X_0.

Application of Symmetrization. In many (all?) cases ●$_-$ can be proved by the torical symmetrization argument from [G 2018] with no use of "bubbles" at all. A characteristic example is as follows.

Let X be an n-dimensional Riemannin manifold homeomorphic to $Y \times [0,1]$ for $Y = \mathbb{T}^{n-1}$.

If $Sc(X) \geq -n(n-1)$ *and if the mean curvature of* $Y_1 = Y \times \{1\} \subset \partial X$ *is bounded from below by* $M_1 > n-1$*, then*

$$dist(Y_0 = Y \times \{0\}, Y_1) \leq l = \frac{2}{n}\coth^{-1}\frac{M_1}{n-1},$$

where $\coth^{-1}$ denotes the inverse function of $\coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}}$.

In fact (see [G 2018]), the extremal Riemannin metric g_{ext} with the maximal l is defined on the manifold $\mathbb{T}^{n-1} \times (0,l]$, where it is invariant under the obvious action of the torus $\mathbb{T}^{n-1}$. Hence, $g_{ext} = dt^2 + \varphi^2 h_{flat}$, where one may assume that $\phi(t) \to 0$ for $t \to 0$.

Since,

$$mean.curv(g_{ext}) = \frac{(n-1)\varphi}{\phi'}$$

by the *first variation formula* and $Sc(g_{ext}) = -n(n-1)$, *Weyl's Riemannian tube formula* shows that

$$-n(n-1) = Sc(g_{ext}) = -2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\frac{\varphi'^2}{\varphi^2}.$$

It follows that the function $f = \frac{\phi}{\phi'}$ satisfies the equation

$$\frac{f'}{1-f^2} = (\coth^{-1} f)' = \frac{n}{2},$$

which implies that

$$f(t) = \coth\frac{tn}{2},$$

where the domain of definition of this f is the segment $(0,l]$ for $l = \frac{2}{n}\coth^{-1}\frac{M_1}{n-1}$.

Cuspidal Boundary Rigidity Theorem. Let X be a complete orientable Riemannian n-manifold with compact boundary Y, such that $Sc(X) \geq -n(n-1)$ and $mean.curv(Y) \geq n-1$.

$\bigstar_-$ If some connected component Y_0 of Y admits a map to the $(n-1)$-torus $\mathbb{T}^{n-1}$ with non-zero degree, which continuously extends to a map $X \to \mathbb{T}^{n-1}$, then the above argument shows that

the universal covering of X is isometric to a horoball in the hyperbolic space $\mathbb{H}^n_{-1}$.

(If X is compact and $Y = \partial X$ is connected, then maps $Y \to \mathbb{T}^{n-1}$ with non-zero degrees do not extend to X, but this can be sometimes remedied by passing to an infinite coverings $p : \tilde{X} \to X$, such that the pullbacks $p^{-1}(Y) \subset \tilde{X}$ are disjoint union of compact manifolds.)

Remark. The advantage of $\bigstar_-$ over the above $\bullet_-$ is admission of *complete non-compact* manifolds X. In fact, one may also allow here additional boundary components with $mean.curv(\geq -(n-1)$, and then $\bigstar_-$ will imply $\bullet_-$.

Question. Is there a Dirac operator proof of $\bigstar_-$ à la Witten and Min-Oo?

5 Problems with $B^2 \times \mathbb{T}^{n-2}$-Manifolds.

Let us discuss a possibility of extension of the above to a class of compact orientable n-dimensional Riemannian manifolds X with boundaries $Y = \partial X$, where the inclusion homomorphism of the fundamental groups is non-injective.

Namely,

$(*_n)$ *Let an X admit a map with non-zero degree to $B^2 \times \mathbb{T}^{n-2}$, where B^2 denotes the disc.*

If $Sc(X) \geq n(n-1)$ and $mean.curv(Y) \geq M > n-1$, than, conjecturally, *the lengths L of the closed curves in Y, which are non-homologous to zero in Y but homologous to zero in $X \supset Y$, are bounded by the length of the circle Y_0 in the hyperbolic plane $\mathbb{H}^2_{-1}$ with $mean.curv(Y_0) = M_0 = \frac{M}{n-1}$.*

If $n = 2$, this is a baby version of the Miao-Shi-Tam rigidity theorem which reads as follows.

($*_2$) *Let a compact connected surface X with boundary Y has $Sc(X) \geq -2$ and $mean.curv \geq M > 1$. Then X is homeomorphic to the disc and the length of Y is bounded by that of the circle Y_0 in the hyperbolic plane $\mathbb{H}^2_{-1}$ with $Mncurv(Y_0) = M$.*

In fact, this is seen by attaching X to the complement of the ball in $\mathbb{H}^2_{-1}$ with the boundary length equal the length of Y and using the (obvious) rigidity of of the hyperbolic metric under curvature increasing deformations, which have compact supports in $\mathbb{H}^2_{-1}$.

Exercise. State and prove the "dual" inequality for metrics with sectional curvatures bounded from above.

If $n \geq 3$, then some (very limited) results can be derived by Miao-Shi-Tam gluing argument combined with the rigidity of the manifold $Z = \mathbb{H}^n_{-1}/\mathbb{Z}^{n-2}$ that was stated in ($\mathbf{H}_3$) in section 3

Example. Think of $\mathbb{H}^n_{-1}/\mathbb{Z}^{n-2}$ as $\mathbb{H}^2_{-1} \times \mathbb{T}^{n-2}$ with a warped (in particular, $\mathbb{T}^{n-2}$-invariant) metric and let $X_0 = B^2_0 \times \mathbb{T}^{n-2}$ for some disc $B^2_0 \subset \mathbb{H}^2_{-1}$.

Notice that the boundary $Y_0 = \partial X_0$ is the n-torus $(\partial B^2_0 = S^1) \times \mathbb{T}^{n-2}$ with a warped product metric where the certain warping function on the closed curve $S^1 = \partial B^2_0$ depends on the position and shape of this curve in the hyperbolic plane $\mathbb{H}^2_{-1}$.

Then, by the rigidity of $Z = \mathbb{H}^n/\mathbb{Z}^{n-2}$,

no Riemannian manifold with the boundary isometric to Y_0 can have the scalar curvature $\geq -n(n-1)$ and the mean curvature of the boundary greater than that of the original $mean.curv(Y_0 \subset Z)$.

Also notice that here, similarly to what was indicated in ($\mathbf{H}_3$) in section 3, one may allow a C^2-small perturbation h_{new} of the original metric h_0 in Y_0 induced from Z. Namely,

no Riemannian manifold with the boundary isometric to (Y_0, h_{new}) can have the scalar curvature $\geq -n(n-1)$ and the mean curvature of the boundary greater than that of the original $mean.curv(Y_0 \subset Z)$ plus one.

Our formulation of ($*_2$) as well as distinguishing the case of $\mathbb{T}^{n-1}$-invariant torical domains is motivated by the possibility of symmetrisation of an arbitrary X with no decrease of its scalar curvature and of the mean curvature of the boundary ∂X.

Recall (see [G 2018]), that "the ultimately" symmetric manifold Z_0 carries the (warped product) metric $dt^2 + (\sinh^2 t) \cdot h_{flat}$, $0 < t < \infty$, which has *constant sectional* curvature *only* for $n = 2$, where, if $n = 2$, this metric remains non-singular at $t = 0$ if h_{flat} is the metric of the circle $S^1_{2\pi}$ of length (exactly) 2π.

In general, we assign h_{flat} to the torus $\mathbb{T}^{n-1} \times S^1_{2\pi}$ and think of Z_0 as a torical warped product over the hyperbolic plane minus a point.

But we don't know

whether such metrics are rigid with respect to deformations with compact (bounded?) supports and $Sc \geq -n(n-1)$.

Another, even more annoying, problem is that the symmetrization in the present case terminates at a warped product metric g_0 on $X_0 = \Sigma^2 \times \mathbb{T}^{n-1}$, where Σ^2 is a disc with a Riemannin metric $\underline{g}_0$, where we don't control either $\underline{g}_0$ or the warping factor that is a function on Σ^2.

Ideally, we would like the warping factor $w(\sigma)$, $\sigma \in \Sigma^2$, of this g_0 on the (circular) boundary of Σ^2 to match the warping function on $\partial B_0^2 \subset \mathbb{H}^2_{-1}$ in the above example.

But, apparently, $w(\sigma)$ for $\sigma \in \partial\Sigma^2$ depends on the geometry of all of X, not only on ∂X. This prevents us from a proof of an even non-sharp version of $(*_2)$ by the gluing argument used for $n = 2$.

On the other hand one may think of an alternative geometric approach, which, in particular, would deliver a proof for the case of $n = 2$ by analysing the intrinsic geometry of the surface X from $(*_2)$ rather than by attaching something to it.[8]

On Surfaces in 3-Manifolds. If $n = 3$, then additional possibilities are opened by the existence of isometric imbeddings of "many" non rotationally symmetric Riemannin metric on 2-tori to $\mathbb{H}^3/\mathbb{Z}$. (Probably, the space of the embeddable metrics has codimension one in the space of all metrics.)

The situation is more satisfactory for imbedding into $\mathbb{H}^3/\mathbb{Z}^2$, where instead of an ambiguous "many" one can definitely say "all".

In fact, let $Y = (\mathbb{T}^2, h)$ be a 2-torus with a Riemannin metric where $sectcurv(h) > -1$. Then, by the *torical version of Pogorelov's isometric embedding theorem,*

there exists a hyperbolic cusp with the sectional curvature -1 *and an (essentially unique) isometric embedding* $Y \to Z$.[9]

(This can be exploited similarly to how that was done in [MM 2016] and mentioned in $\bigstar_{-\kappa}$ in section 4.)

6 Manifolds with Corners

Let us briefly discuss here what can be done about Sormani's question **C** from the introduction.

Basic examples of Riemannian n-manifolds with corners are (convex) polyhedra in $\mathbb{R}^n$ with smooth Riemannian metrics on them.

In general, a corner structure P on a smooth manifold X with boundary $Y = \partial X$ is given by *the shadow* $\underline{P}$ pf P, that is a partition of Y into locally closed submanifolds corresponding to the actual faces of P.

For example, the shadow of the corner structure of the cube $[-1,1]^{n+1}$ on the unit n-sphere $S^{n-1} \subset \mathbb{R}^n$ can be seen by radially projecting the boundary of the cube to this sphere.

More generally, let X be a smooth n-manifold with boundary Y and let $f : Y \to S^{n-1}$ be a smooth map which is transversal to all faces of such a shadow $\underline{P}$ in S^{n-1}. Then the pullbacks of these faces define a shadow of a corner structure on X, where the corresponding corner structure on X is called *cubical* if X is orientable and the map f has *non-zero degree.*

$(\square_\circ)$ The simplest instance of this, that we shall use below, is where the map f is a diffeomorphism from the interior of a ball $B \subset Y$ onto S^{n-1} minus the

[8] A non-sharp inequality can be derived from "strong generalised concavity" of the distance function $dist_X(s_1, s_2)$ for $s_1, s_2 \in \partial X$, as in the "proof" of unproven corollary in section 9.

[9] The set of the isometry classes of these "cusps" that are the quotient manifolds $\mathbb{H}^3_{-1}/\Gamma$, $\Gamma = \mathbb{Z}^2$, is naturally parametrised by the modular curve of conformal structures on $\mathbb{T}^2$, where the modular parameter of Z, which receives an isometric embedding of (T^2, h), depends on h.

south pole $s_\star \in S^{n-1}$ and where all of the complement of B goes to $s_\star$.

Besides the scalar curvature of X and the mean curvatures of its $(n-1)$-faces an essential geometry of a Riemannin manifold with corners is carried by the *dihedral angles* α at the *"edges"* that are the $(n-2)$-faces of X, where the difference $\pi - \alpha$ plays the role of the mean curvature.

An essential motivation of what we try to do in this section is the following.

Hyperbolic Subrectangular Theorem. Let X be a compact cubical Riemannian n-manifold, where

(i) the dihedral angles are $\leq \frac{\pi}{2}$;
(ii) all $(n-1)$-faces but one have positive mean curvatures;
(iii) the exceptional face have $mean.curv \geq -(n-1)$;
(iv) the opposite face, which is also called "exceptional", has $mean.curv > n-1$.

Then the scalar curvature of X satisfies:

$$\inf_{x\in X} Sc(X,x) < -n(n-1).$$

Idea of the proof. Reflect X around $2(n-1)$ non-exceptional faces, smooth the resulting C^0-metric corners and apply $\bullet_-$ from section 6 to the resulting "sub-hyperbolic band".

In truth, however, such a smoothing is a mess; a technically less demanding approach is explained in [G 2014] and in section 11.10 of [G 2018].

(This proof of the theorem is immediate for $n=2$, where it is seen by looking at minimal geodesic segments between two exceptional faces).

Rigidity Question. If (iv) is relaxed to $Mncurv \geq n$, and if $Sc(X,x) \geq -n(n-1)$, then, most likely, X is isometric to a *parabolic rectangular solid* in $\mathbb{H}^n_{-1}$, that is defined in the *horospherical coordinates* $x_1, ...x_n$ by the inequalities

$$0 \leq x_i \leq a_i,\ i = 1, ..., n,$$

where, recall, the hyperbolic metric g in these coordinates is

$$g = dx_1^2 + e^{2x_1} \sum_{i=2}^{n} dx_i^2.$$

We fail short of directly proving this because of, a priori possible, presence of singularities of the boundaries of extremal X, which is due to a use of the Schoen-Yau style variational argument in an essential step of our proof.

On the other hand, a suitable adapted *Kazdan-Warner perturbation argument* would, probably, reduce the rigidity problem for $Sc(X) \geq -n(n-1)$ to that, where all faces but one are convex and $Ricci(X) \geq -(n-1)g$; then the proof would follow by Weyl's tube formula.

Unproven Corollary. Let X be a compact Riemannian n-manifold with boundary $Y = \partial X$, such that

$$Sc(X) \geq -1 \text{ and } mean.curv(Y) \geq -1.$$

Let $B = B_y(r) \subset Y$, $y \in Y$, be a ball of radius r, such that the sectional curvatures of Y in this ball are bounded in absolute values by a constant $\kappa > 0$;

the exponential map $T_y(Y) \to Y$ is one-to-one on the ball $B_0(r) \in T_y(Y)$ to B.

Then

the infimum of the mean curvature of Y in the ball B satisfies

$$[\square_{-1}] \qquad \inf_{y \in B} mean.curv(Y, y) \le \mathscr{M}_n(\kappa + r^{-1})$$

for some (possibly very large) universal continuous function $\mathscr{M}_n$.

How we Want to Prove it. The *actual proof* for $n = 2$ is easy. In fact, let X be a surface with circular boundary $Y = \partial X$, and let $Y_+ \subset Y$ be a segment, such that the following conditions are satisfied.

(1) sect.curv$(X) \ge -1$;

(2) the curvature of the segment Y_+ is $\ge M_+ = 1 + \varepsilon_+$, $\varepsilon_+ > 0$;

(3) the curvature of Y in the complement of Y_+ is ≥ -1.

Then the length of Y_+ is bounded by 1 000 *times the length of the circle $Y_{M_+} \subset \mathbb{H}^2_{-1}$ with the curvature $curv(Y_{M_+}) = M_+$.*

Proof. Assume, this is easy to justify, that the curvature of Y is constant on Y_+ and let us endow the cylinder $Y \times \mathbb{R}_+$ to X with a (canonical) metric with sectional curvature -1, and such that the boundary $Y \times \{0\}$ of this cylinder admits an isometry, i.e. a length preserving map

$$Y \times \{0\} \leftrightarrow Y,$$

where this isometry matches the curvature of these curves, which makes the resulting metric on the extended manifold

$$X_+ = X \cup_Y Y \times \mathbb{R}_+$$

C^1-smooth. (This is essential only on Y_+.)

If the length of the segment Y_+, now positioned in $Y \times \mathbb{R}_+$, were sufficiently long then, by an elementary argument, there would exist a deformation of Y in $Y \times \mathbb{R}_+$ to a curve

$$Y' \subset Y \times \mathbb{R}_+ \subset X_+,$$

such that

• there are four 90° corners on this curve Y'.

• the curvatures of one pair of disjoint segments bounded by the corner points are ≥ 0, while the curvatures of the second pair of segments, call them Y'_+ and Y'_- are bounded by

$$curv(Y'_+) \ge 1 + 0,01\varepsilon_+ \text{ and } curv(Y'_+) \ge -1.$$

This contradicts the (obvious in this case) the above (sub)rectangular theorem, applied to the domain X' bounded by the curve Y' in X_+, and thus, the proof is concluded.

Questions. (i) Which functions can be realised as curvatures of boundaries $Y = \partial X$ of surfaces X, where $sect.curv(X) \ge \kappa$?

To get an idea of the expected patterns of segments of Y with large and small curvatures , let $Y \subset \mathbb{R}^2$ be a simple closed curve, which contains a segment Y_+ of length L_+, where the curvature of Y is ≥ 1.

It is not hard to show that Y must contain an open subset Y_- (possibly, with arbitrarily many connected components) of length L_-, such that $L_- \geq (1-\varepsilon)L_+$ and

$$\inf curv(Y_-) = \inf_{y \in Y_-} curv(Y, y) \leq -1 + \delta,$$

where $\varepsilon, \delta \to 0$ for $L_+ \to \infty$.

Probably, all surfaces X with $sectcurv \geq \kappa$ display similar patterns of distributions of curvatures of their boundaries Y for large $length(Y)$.

(ii) What would be analogues of the above for (distribution of) the mean curvatures of boundaries Y of n-manifolds X with lower bounds on the sectional, Ricci or scalar curvatures of X?

(The case of bounded domains $X \subset \mathbb{R}^n$, $n \geq 3$, already seems interesting.)

Idea of a Possible Proof of $[\square_{-1}]$. We want to extend the above "cornering" argument to $n \geq 3$ and create a (sub)rectangular corner structure on X, the shadow of which will be induced by a map $B \to S^{n-1}$ as in the above $(\square_\circ)$.

It seems manageable to bend Y along individual edges, that are the $(n-2)$-faces of X (compare with the constructions in section 11.3, 11.4 in [G 2018]), but it less clear how to do it (I think it is unpleasant but possible) at the corners, where the edges meet.

However, similarly how this was bypassed in [G 2014[and in [G 2018] one can, probably, avoid this problem by performing the constructions of corners, reflections and smoothings *interchangeably.*

For instance, let X, topologically, be the 3-ball and let us create two circular corners on its (spherical) boundary corresponding to the (circular) boundaries of two non-exceptional faces.

Then double the resulting X' over this pair, smooth it and arrive at X_1, which is now homeomorphic to $B^2 \times S^1$, where the shadow of the corner structure is induced from that on the disc B^2.

Proceed as before, now with the remaining pair of non-exceptional faces and, thus, arrive at X_2, which is homeomorphic to $S^1 \times S^1 \times [0,1]$ and to which $\bullet_-$ from section 6 applies.

Conceivably, the proof of $[\square_{-1}]$ can be achieved along these lines. But artificiality of the argument and the issuing non-sharpness of the result leave the problem of the correct (re)formulation and of the proof of $[\square_{-1}]$ open.

On the other hand, the geometry behind this may be interesting in its own right as suggested by following.

Question. Let $X = (X, g)$ be a Riemannin manifold diffeomorphic to the n-ball and let the scalar curvature of X in a neighbourhood $U \subset X$ of $Y = \partial X$ be positive. Let V be a closed smooth n-manifold.

A packing with $Sc > 0$ of V by (copies of) X is a Riemannian metric g_+ on V, along with isometric embeddings of several copies of (X, g) to (V, g_+), such the metric g_+ has

positive scalar curvature in the complement of the images of these embeddings.

Under what conditions on the intrinsic geometry of $Y = \partial X$, on the mean curvature of Y and on the topology of V, can V be packed with $Sc > 0$ by X?

More specifically, let $locgeo(Y)$ denote the maximum of two invariants of Y

$$locgeo(Y) = \max\left(+\sqrt{|sect.curv(V)|}, \frac{1}{injrad(V)}\right),$$

let $top(V)$ means "topology of V" and let $\mathscr{M}_{top(V)}(*)$ be a (large) continuous function.

For which V, is the inequality $mean.curv(Y) \geq \mathscr{M}_{top(V)}(locgeo(Y))$ sufficient for the existence of such packing?

Notice in this regard, that our cornering argument, if it works, implies the existence of such packings of the n-torus, provided $\mathscr{M} = \mathscr{M}_{\mathbb{T}^n}$ is taken sufficiently large.[10]

Also, such packings are likely to exist in other "reflection manifolds".

Contrariwise, the largeness of the curvature of the boundary curves serves as an obstruction for packing with $Sc > 0$ of 2-spheres by discs.

More generally, *aspherical*[11] manifolds V of all dimensions have a chance to be packable, (more realistically, *approximately* packable in some sense), under similar conditions but it is rather improbable for the spheres, even if "$locgeo$" is replaced by a more comprehensive global invariant of X. But finding relevant examples of non-extendability of metrics with $Sc > 0$ seems difficult.

(All what matters for the existence of an extension with $Sc > 0$ is the geometry of X near $Y = \partial X$. But what happens in X far from Y may harbour obstructions to such extensions.)

Bound on Hight of Fat Cylinders. Let us indicate another application of the cornering construction, where there is an instance where no higher order corners difficulty appears.

Let X be a compact connected orientable n-manifold with boundary Y and let $Y_{bot} \subset$ and Y_{top} be two disjoint smooth connected domains in $Y = \partial X$ called the *bottom* and the *top*.

The basic example is where X is the cylinder, $X_0 = B^{n-1} \times [0,1]$; in general, $X = (X, Y_{bot}, X_{top})$ is called "cylinder" if it admits a continuous map $f : X \to B^{n-1} \times [0,1]$, such that ∂X is sent to ∂X_0,

$$f^{-1}(B^{n-1} \times \{0\}) = Y_{bot}, \quad f^{-1}(B^{n-1} \times \{1\}) = Y_{top}$$

and $deg(f) \neq 0$.

Denote by $Y_{\circledcirc} \subset Y$ the "side" of the "cylinder" X, that is the pullback $f^{-1}(S^{n-2} \times [0,1])$ for the boundary sphere $S^{n-2} = \partial B^{n-1}$ and let the following conditions be satisfied.

- $Sc(X) \geq n(n-1)$,
- $locgeo(Y_{\circledcirc}) \leq 1$,
- $mean.curv(Y_{\circledcirc}) \geq \underline{M}$.

Second Unproven Corollary. *The "hight" of X is bounded by a universal function of $\underline{M}$:*

$$dist_X(Y_{bot}, Y_{top}) \leq \mathscr{H}_n(\underline{M}),$$

[10]The packings associated with such "cornering", may have, if you wish, the complements to the balls contained in the ε-neighbourhood of these balls for arbitrarily small $\varepsilon > 0$.

[11]A topological space is called aspherical if its universal covering is contractible.

such that

$$\mathscr{H}_n(\underline{M}) \to \frac{2\pi}{n} \textit{ for } \underline{M} \to \infty.$$

Idea of a Possible Proof. If $n = 3$, bend $Y_\circ$ along four segments between the top and the bottom of X and, thus, produce four edges with dihedral angles $\le \frac{\pi}{2}$. This seems non-difficult.

More problematic is to create similar n-cubical structure for $n \ge 4$ along the lines indicated in the above "possible proof" of $[\square_{-1}]$.

But if this is accepted, then the *sub-rectangular* $\frac{2\pi}{n}$*-inequality* from section 11.10 in [G 2018] could be applied and the proof would follow.

Remarks/Questions. (a) The apparent examples suggest that

$$dist_X(Y_{bot}, Y_{top}) \to 0 \text{ for } \underline{M} \to \infty,$$

where it is, indeed, so for $n = 2$ by an elementary argument.

But it is unclear whether this is true or false for $n \ge 3$.

(b) Can one replace the bound on $locgeo(Y_{\circledcirc})$ by a lower bound on some kind of size of $Y_{\circledcirc}$?

Such a "size" can be conveniently defined with the above map f in the definition of the " cylinder" structure on X as follows.

Restrict this f to $Y_{\circledcirc}$, observe that it sends $Y_{\circledcirc} \to S^{n-2} \times [0,1]$, compose this with the projection $S^{n-2} \times [0,1] \to S^{n-2}$ and denote the resulting map by $\underline{f} : Y_{\circledcirc} \to S^{n-2}$.

If $n \ge 3$, denote by $\underline{R}ad_{S_{n-2}}(Y_{\circledcirc})$ the supremum of the numbers R such that $\underline{f}$ is homotopic to a smooth map $\underline{f}' : Y_{\circledcirc} \to S^{n-2}$, where the norm of the differential of $\underline{f}'$ is bounded by

$$\sup_{y \in Y_{\circledcirc}} \|d\underline{f}'(y)\| \le \frac{1}{R}.$$

And If $n = 2$, let $\underline{R}ad_{S_{n-1}}(Y_{\circledcirc}) = 1$ for all X.

Conjecture. Let $Sc(X) \ge 0$ and $mean.curv(Y) \ge \underline{M}$. If the product

$$\Theta(X) = \underline{M} \cdot \underline{R}ad_{S_{n-2}}(Y_{\circledcirc})$$

(this Θ speaks for "fatness" of X) is sufficiently large, say

$$\Theta(X) > \Theta_{crit},$$

then

$$dist_X(Y_{bot}, Y_{top}) \le \lambda_n(\Theta(X))$$

for some continuous monotone decreasing function $\lambda_n(\Theta)$ defined for $\Theta > \Theta_{crit}$.

The best one may expect here is that

$$\Theta_{crit} = n - 2$$

and that

$$\lambda_n(\Theta) \to 0 \text{ for } \Theta \to \infty.$$

But any bound $\Theta_{crit} \le const_n$ will be welcome and the inequality

$$\lim_{\Theta\to\infty} \lambda_n(\Theta) \le \frac{2\pi}{n-1}$$

will be quite satisfactory.

And it is conceivable that all of the above holds for $Sc(X) \ge \sigma$, for all $-\infty < \sigma < \infty$, with $\Theta_{crit} = \Theta_{crit}(\sigma)$ and $\lambda_n = \lambda_n(\Theta, \sigma)$.

7 Further Conjectures and Problems.

Probably, all we know (and don't know) about manifolds with boundaries extends to manifolds with corners, albeit by no means automatically. In fact, it will be more productive to study manifolds with corners for their own sake rather than as intermediates in the arguments concerning smooth manifolds.

For instance, problem **A** from the introduction becomes even more interesting for manifolds with corners where it reads as follows.

Let X be a compact n-dimensional Riemannin manifold with corners and Y_0 be an $(n-1)$-dimensional face of X.

Find an upper bound on the mean curvature of this face, in terms of lower bounds on the scalar curvature of X and the mean curvatures of the remaining $(n-1)$-faces of X and the dihedral angles along $(n-2)$-faces, as well the some geometric invariants of the face Y_0, e.g. its "size" and its own scalar curvature.

The topologically simplest instances of this are as follows.

• The half-ball B^n_+, that is the intersection of the ball with the halfspace $x_1 \ge 0$ where the n-ball $B^n_+ \cap \mathbb{R}^{n-1}$ is taken for Y_0;

• the *cylinder* $X = B^{n-1} \times [0,1]$, where $B^{n-1} \times \{0\}$ is taken for Y_0;

• products of smooth manifolds with boundaries, e.g. cubes $[0,1]^n$ and close relatives of cubes – n-diamonds and n-simplices.

But it is more challenging to understand the geometry of X when the combinatorics of the corner structure becomes more complicated.

For instance, define $CombRad_{n-1}(Y)$, for a closed orientable $(n-1)$-manifold Y partitioned into faces, as the supremum of the numbers R, such that Y admits a continuous map of non-zero degree to the sphere $S^{n-1}(R)$, such that the images of all faces have diameters at most one.

⎔ CONJECTURE. Let X be a compact orientable Riemannin n-manifold with corners, such that $Sc(X) \ge 0$, and all $n-1$-faces have positive mean curvatures. Let $\overline{\mathscr{A}}(X)$ denote the supremum of the dihedral angles of X taken over all points of all edges ($(n-2)$-faces) of X.

Then

$$\pi - \overline{\mathscr{A}}(X) \le \frac{const_n}{CombRad_{n-1}(\partial X)}$$

for some universal (possibly large) constant $const_n$.

Admission. I haven't check this even in the case of convex polyhedra $X \subset \mathbb{R}^n$.

Below is another kind of conjecture, where the difficulty resides in the gap between a use of Dirac theoretic methods and those of minimal hypersurfaces.

○= Conjecture. There exists a (possibly very large) constant $\sigma_n > 0$ with the following property.

If a compact orientable Riemannian n-manifold with boundary Y has $Sc(X) \geq \sigma_n$, then every smooth area decreasing map f from X to the unit n-sphere, which sends the 1-neighbourhood (unit collar) of $Y \subset X$ to a single point, has degree $deg(f) = 0$.

We conclude by formulating an instance of a general *stability problem for $Sc \geq \sigma$*, in the spirit of [S 2016], which may have a satisfactory solution, at least in dimension 3.

Spherical Stability Problem. Describe the geometry of closed Riemannian orientable n-dimensional manifolds X_i, such that

$$\inf Sc(X_i) \to n(n-1) \text{ and } Rad_{S^n}(X_i) \to 1.$$

One expects here, in view of Llarull's theorem, that manifold X_i with $Sc(X_i) \geq n(n-1)-\varepsilon_i$ and $Rad_{S^n}(X_i) \geq 1-\varepsilon_i$ must look, approximately, as the unit sphere S^n with an extra staff attached by "ε_i-narrow bridges" to it.

In other words, certain equidimensional submanifolds $U_i \subset X_i$ with (small?) boundaries must somehow converge (sub-converge?), e.g. in *the intrinsic flat topology* (see [S 2016]), to S^n.

Besides S^n, the stability problem arises for all, not only spherical, *length extremal* and *area extremal* closed manifolds with positive scalar curvatures as well as for extremal manifolds with boundaries (as in (d) of section 1), where the situation is even less clear.

It seems helpful for developing an idea of what happens to scalar curvature in this regard, up to a point of formulating conjectures, to look at the corresponding problem(s) for hypersurfaces with positive mean curvatures.

For instance,

what is the geometry of (limits of sequences of) smooth bounded domains $U_i \subset \mathbb{R}^{n+1}$, which contain the unit ball $B_0^{n+1}(1) \subset \mathbb{R}^{n+1}$ and such that the mean curvatures of the boundaries of these domains are bounded from below by $M_i \to n$ for $i \to \infty$.

We conclude by reiterating the problem suggested by Pengzi Miao:

Is there a *lower bound on the volumes* of manifolds X with $Sc(X) \geq \sigma$, say for $\sigma = -1, 0, +1$, in terms of

intrinsic geometries of their boundaries $Y = \partial X$ and mean curvatures of Y?

For instance,

Let a compact Riemannian n-manifold X satisfies:

- $Rad_{S^{n-1}}(\partial X) \geq 1$;
- $mean.curv(\partial(X)) \geq n-1$;
- $Sc(X) \geq -\varepsilon$.

Does the volume of X relate to that of the unit ball as follows.

$$vol(X) \geq vol(B^n(1)) - \delta \text{ for } \delta = \delta(\varepsilon) \underset{\varepsilon \to 0}{\to} 0?$$

This seems unclear (do I miss something obvious?) even if X has sectional curvature $\geq -\varepsilon$.

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