

# Scalar Curvature, IHES 2019. Lecture 1

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## 1 Geometrically Deceptive Definition.

The scalar curvature of a  $C^2$ -smooth Riemannian manifold  $X = (X, g)$ , denoted  $Sc = Sc(X) = Sc(X, g) = Sc(g)$  is a continuous function on  $X$ , which is traditionally defined as

*the sum of the values of the sectional curvatures at the  $n(n-1)$  ordered bivectors of an orthonormal frame in  $X$ ,*

$$Sc(X)(x) = \sum_{i,j} \kappa_{ij}(x), \quad i \neq j = 1, \dots, n,$$

where this sum doesn't depend on the choice of this frame by the Pythagorean theorem.

But if you are geometrically minded, you want to have a geometric definition where the first attempt to find such a definition relies on the following properties of  $Sc(X)$ .

- <sub>1</sub> *Additivity under Cartesian-Riemannian Products .*

$$Sc(X_1 \times X_2, g_1 \oplus g_2) = Sc(X_1, g_1) + Sc(X_2, g_2).$$

- <sub>2</sub> *Quadratic Scaling.*

$$Sc(\lambda \cdot X) = \lambda^{-2} Sc(X), \quad \text{for all } \lambda > 0,$$

where

$$\lambda \cdot X = \lambda \cdot (X, dist_X) =_{def} (X, dist_{\lambda \cdot X}) \quad \text{for } dist_{\lambda \cdot X} = \lambda \cdot dist(X)$$

for all metric spaces  $X = (X, dist_X)$  and where  $dist \mapsto \lambda \cdot dist(X)$  corresponds to  $g \mapsto \lambda^2 \cdot g$  for the Riemannian quadratic form  $g$ .

(This makes the Euclidean spaces scalar-flat:  $Sc(\mathbb{R}^n) = 0$ .)

- <sub>3</sub> *Volume Comparison.* If the scalar curvatures of  $n$ -dimensional manifolds  $X$  and  $X'$  at some points  $x \in X$  and  $x' \in X'$  are related by the strict inequality

$$Sc(X)(x) < Sc(X')(x'),$$

then the Riemannian volumes of small balls around these points satisfy

$$vol(B_x(X, \varepsilon)) > vol(B_{x'}(X', \varepsilon))$$

for all sufficiently small  $\varepsilon > 0$ .

Observe that this volume inequality is *additive under Riemannian products*:  
if

$$\text{vol}(B_{x_i}(X, \varepsilon)) > \text{vol}(B_{x'_i}(X'_i, \varepsilon)), \text{ for } \varepsilon \leq \varepsilon_0,$$

and for all points  $x_i \in X_i$  and  $x'_i \in X'_i$ ,  $i = 1, 2$ , then

$$\text{vol}_n(B_{(x_1, x_2)}(X_1 \times X_2, \varepsilon_0)) > \text{vol}_n(B_{(x'_1, x'_2)}(X'_1 \times X'_2, \varepsilon_0))$$

for all  $(x_1, x_2) \in X_1 \times X_2$  and  $(x'_1, x'_2) \in X'_1 \times X'_2$ .

This follows from the Pythagorean formula

$$\text{dist}_{X_1 \times X_2} = \sqrt{\text{dist}_{X_1}^2 + \text{dist}_{X_2}^2}.$$

and the Fubini theorem applied to the "fibrations" of balls over balls:

$$B_{(x_1, x_2)}(X_1 \times X_2, \varepsilon_0) \rightarrow B_{x_1}(X_1, \varepsilon_0) \text{ and } B_{(x'_1, x'_2)}(X'_1 \times X'_2, \varepsilon_0) \rightarrow B_{x'_1}(X'_1, \varepsilon_0),$$

where the fibers are balls of radii  $\varepsilon \in [0, \varepsilon_0]$  in  $X_2$  and  $X'_2$ .

•<sub>4</sub> *Normalisation/Convention for Surfaces with Constant Sectional Curvatures.* The unit spheres  $S^2(1)$  have constant scalar curvature 2 and the hyperbolic plane  $H^2(-1)$  with the sectional curvature  $-1$  has scalar curvature  $-2$

It is an elementary exercise. to prove the following.

★<sub>1</sub> *The function  $Sc(X, g)(x)$  which satisfies •<sub>1</sub>-•<sub>4</sub> exists and unique;*

★<sub>2</sub> *The unit spheres and the hyperbolic spaces with  $\text{sect.curv} = -1$  satisfy*

$$Sc(S^n(1)) = n(n-1) \text{ and } Sc(\mathbf{H}_{-1}^n) = -n(n-1).$$

Thus,

$$Sc(S^n(1) \times \mathbf{H}_{-1}^n) = 0 = Sc(\mathbb{R}^n),$$

which implies that the volumes of the small balls in  $S^n(1) \times \mathbf{H}_{-1}^n$  are "very close" to the volumes of the Euclidean  $2n$ -balls.

Also it is elementary to show that the definition of the scalar curvature via volumes of balls agrees with the traditional  $Sc = \sum \kappa_{ij}$ , where the definition via volumes seem to have an advantage of being geometrically more usable.

But this is an illusion:

THERE IS NO SINGLE KNOWN (ARE THERE UNKNOWN?)

GEOMETRIC ARGUMENT WHICH WOULD MAKE USE OF THIS DEFINITION.

The immediate reason for this is *the infinitesimal* nature of the volume comparison property: it *doesn't integrate* to the corresponding property of balls of specified, let them be small, radii  $r \leq \varepsilon > 0$ .<sup>1</sup>

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<sup>1</sup>An attractive conjecture to the contrary appears in *Volumes of balls in large Riemannian manifolds* by Larry Guth in *Annals of Mathematics*173(2011), 51-76.

## 2 Eight Useful Formulas.

Let  $h_t$ ,  $t \in [0, \varepsilon]$ , be a family of Riemannian metric on an  $(n-1)$ -dimensional manifold  $Y$  and let us incorporate  $h_t$  to the metric  $g = h_t + dt^2$  on  $Y \times [0, \varepsilon]$ .

Notice that an arbitrary Riemannian metric on an  $n$ -manifold  $X$  admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface  $Y \subset X$ .

**(I) Riemannian Variation Formula.** The  $t$ -derivative of  $h_t$  is equal to *twice the second fundamental form* of the hypersurface  $Y_t = Y \times \{t\} \subset Y \times [0, \varepsilon]$ , denoted and regarded as a quadratic differential form on  $Y = Y_t$ , denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on  $Y = Y_t$ .

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_\nu h = A^*,$$

where

$$\nu \text{ is the unit normal field to } Y \text{ defined as } \nu = \frac{d}{dt}.$$

In fact, if you wish, you can take this formula for the definition of the second fundamental form of  $Y^{n-1} \subset X^n$ .

Recall, that the *principal values*  $\alpha_i^*(y)$ ,  $i = 1, \dots, n-1$ , of the quadratic form  $A_t^*$  on the tangent space  $T_y(Y)$ , that are the values of this form on the orthonormal vectors  $\tau_i^* \in T_i(Y)$ , which *diagonalize*  $A^*$ , are called *the principal curvatures* of  $Y$ , and that the sum of these is called *the mean curvature* of  $Y$ ,

$$\text{mean.curv}(Y, y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = \text{trace}(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames  $\tau_i$  in  $T_y(Y)$  by the Pythagorean theorem.

Also observe that  $A^*$  changes sign under reversion of the  $t$ -direction. Accordingly the sign of the quadratic form  $A^*(Y)$  depends on the coorientation of  $Y$  in  $X$ , where our convention is such that

the boundaries of *convex* domains have *positive definite* second fundamental forms  $A^*$ , hence, positive mean curvatures.

**(II) First Variation Formula.** This concerns the  $t$ -derivatives of the  $(n-1)$ -volumes of domains  $U_t = U \times \{t\} \subset Y_t$ , which are computed by tracing the above **(I)** and which are related to the mean curvatures as follows.

$$[\circ] \quad \partial_\nu \text{vol}_{n-1}(Y) = \frac{dh_t}{dt} \text{vol}_{n-1}(U_t) = \int_{U_t} \text{mean.curv}(U_t) dy_t$$

where  $dy_t$  is the volume element in  $Y_t \supset U_t$ .

Now comes the first formula with the Riemannian curvature in it.

(III) **Gauss' Theorema Egregium.** Let  $Y \subset X$  be a smooth hypersurface in a Riemannian manifold  $X$ . Then the sectional curvatures of  $Y$  and  $X$  on a tangent 2-plane  $\tau = \tau^2 \subset T_y(Y) \subset T_y(X)$   $y \in Y$ , satisfy

$$\kappa(Y, \tau) = \kappa(X, \tau) + \wedge^2 A^*(\tau),$$

where  $\wedge^2 A^*(\tau)$  stands for the product of the two principal values of the second fundamental form  $A^* = A^*(Y) \subset X$  restricted to the plane  $\tau$ ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

This, with the definition the scalar curvature by the formula  $Sc = \sum \kappa_{ij}$ , implies that

$$Sc(Y, y) = Sc(X, y) + \sum_{i \neq j} \alpha_i^*(y) \alpha_j^*(y) - \sum_i \kappa_{\nu, i}$$

where  $\alpha_i^*(y)$ ,  $i = 1, \dots, n-1$  are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors  $\tau_i$  in  $T_y(Y)$ , where the  $\alpha^*$ -sum is taken over all ordered pairs  $(i, j)$  with  $j \neq i$  and where  $\kappa_{\nu, i}$  are the sectional curvatures of  $X$  on the bivectors  $(\nu, \tau_i)$  for  $\nu$  being a unit (defined up to  $\pm$ -sign) normal vector to  $Y$ .

Observe that both sums are independent of coorientation of  $Y$  and that in the case of  $Y = S^{n-1} \subset \mathbb{R}^n = X$  this gives the correct value  $Sc(S^{n-1}) = (n-1)(n-2)$ .

Also observe that

$$\sum_{i \neq j} \alpha_i \alpha_j = \left( \sum_i \alpha_i \right)^2 - \sum_i \alpha_i^2.$$

Thus, for example, the scalar curvature of a hypersurface  $Y \subset \mathbb{R}^n$  is expressed in terms of the mean curvature and the (point-wise)  $L_2$ -norm of their second fundamental form of  $Y$  as follows.

$$Sc(Y) = (\text{mean.curv}(Y))^2 - \|A^*(Y)\|^2$$

for  $\|A^*(Y)\|^2 = \sum_i (\alpha_i^*)^2$ , while  $Y \subset S^n$  satisfy

$$Sc(Y) = (\text{mean.curv}(Y))^2 - \|A^*(Y)\|^2 + (n-1)(n-2) \geq (n-1)(n-2) - n \max_i (c_i^*)^2.$$

It follows that *minimal* hypersurfaces  $Y$  in  $\mathbb{R}^n$ , i.e. these with  $\text{mean.curv}(Y) = 0$ , have *negative scalar curvatures*, while hypersurfaces in the  $n$ -spheres with all principal values  $\leq \sqrt{n-2}$  have  $Sc(Y) > 0$ .

Let  $A = A(Y)$  denote the *shape operator* that is the symmetric operator on  $T(Y)$  associated with  $A^*$  via the Riemannian scalar product  $g$  restricted from  $T(X)$  to  $T(Y)$ ,

$$A^*(\tau, \tau) = \langle A(\tau), \tau \rangle_g \text{ for all } \tau \in T(Y).$$

(IV) *The Second Main Formula of Riemannian Geometry.*<sup>2</sup> Let  $Y_t$  be a family of hypersurfaces  $t$ -equidistant to a given  $Y = Y_0 \subset X$ . Then the shape operators  $A_t = A(Y_t)$  satisfy:

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

<sup>2</sup>The first main formula is *Gauss' Theorema Egregium*.

where  $B_t$  is the symmetric operator associated with the quadratic differential form  $B^*$  on  $Y_t$ , the values of which on the tangent unit vectors  $\tau \in T_{y,t}(Y_t)$  are equal to the values of the *sectional curvature* of  $g$  at (the 2-planes spanned by) the bivectors  $(\tau, \nu = \frac{d}{dt})$ .

*Remark.* Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannian geometry.

Tracing this formula yields

(V) *Hermann Weyl's Tube Formula.*

$$\text{trace}\left(\frac{dA_t}{dt}\right) = -\|A^*\|^2 - \text{Ricci}_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$\text{trace}(\partial_\nu A) = \partial_\nu \text{trace}(A) = -\|A^*\|^2 - \text{Ricci}(\nu, \nu),$$

where

$$\|A^*\|^2 = \|A\|^2 = \text{trace}(A^2),$$

where, observe,

$$\text{trace}(A) = \text{trace}(A^*) = \text{mean.curv} = \sum_i \alpha_i^*$$

and where *Ricci* is the quadratic form on  $T(X)$  the value of which on a unit vector  $\nu \in T_x(X)$  is equal to the trace of the above  $B^*$ -form (or of the operator  $B$ ) on the normal hyperplane  $\nu^\perp \subset T_x(X)$  (where  $\nu^\perp = T_x(Y)$  in the present case).

Also observe – this follows from the definition of the scalar curvature as  $\sum \kappa_{ij}$  – that

$$Sc(X) = \text{trace}(\text{Ricci})$$

and that the above formula  $Sc(Y, y) = Sc(X, y) + \sum_{i \neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu, i}$  can be rewritten as

$$\begin{aligned} \text{Ricci}(\nu, \nu) &= \frac{1}{2} \left( Sc(X) - Sc(Y) - \sum_{i \neq j} \alpha_i^* \cdot \alpha_j^* \right) = \\ &= \frac{1}{2} \left( Sc(X) - Sc(Y) - (\text{mean.curv}(Y))^2 + \|A^*\|^2 \right) \end{aligned}$$

where, recall,  $\alpha_i^* = \alpha_i^*(y)$ ,  $y \in Y$ ,  $i = 1, \dots, n-1$ , are the principal curvatures of  $Y \subset X$ , where  $\text{mean.curv}(Y) = \sum_i \alpha_i^*$  and where  $\|A^*\|^2 = \sum_i (\alpha_i^*)^2$ .

(VI) *Umbilic Hypersurfaces and Warped Product Metrics* A hypersurface  $Y \subset X$  is called umbilic if all principal curvatures of  $Y$  are mutually equal at all points in  $Y$ .

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres  $S_{\kappa > 0}^n$ , Euclidean spaces  $\mathbb{R}^n$  and hyperbolic spaces  $\mathbf{H}_{\kappa < 0}^n$ ) are umbilic.

In fact these are special case of the following class of spaces .

*Warped Products.* Let  $\varphi = \varphi(y) > 0$  be a smooth positive function on a Riemannian  $(n-1)$ -manifold  $Y = (Y, h)$ , and let  $g = h_t + dt^2 = \varphi^2 h + dt^2$  be the corresponding metric on  $X = Y \times [0, \varepsilon]$ .

Then the hypersurfaces  $Y_t = Y \times \{t\} \subset X$  are umbilic with the principal curvatures of  $Y_t$  equal to  $\alpha_i^*(t) = \frac{\varphi'}{\varphi}$ ,  $i = 1, \dots, n-1$  for

$$A_t^* = \frac{\varphi'}{\varphi} h_t \text{ for } \varphi' = \frac{d\varphi}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}.$$

The Weyl formula reads in this case as follows.

$$(n-1) \left( \frac{\varphi'}{\varphi} \right)' = -(n-1)^2 \left( \frac{\varphi'}{\varphi} \right)^2 - \frac{1}{2} \left( Sc(g) - Sc(h_t) - (n-1)(n-2) \left( \frac{\varphi'}{\varphi} \right)^2 \right).$$

Therefore,

$$\begin{aligned} Sc(g) &= \frac{1}{\varphi^2} Sc(h) - 2(n-1) \left( \frac{\varphi'}{\varphi} \right)' - n(n-1) \left( \frac{\varphi'}{\varphi} \right)^2 = \\ (\star) \quad &= \frac{1}{\varphi^2} Sc(h) - 2(n-1) \frac{\varphi''}{\varphi} - (n-1)(n-2) \left( \frac{\varphi'}{\varphi} \right)^2, \end{aligned}$$

where, recall,  $n = \dim(X) = \dim(Y) + 1$  and the mean curvature of  $Y_t$  is

$$\text{mean.curv}(Y_t \subset X) = (n-1) \frac{\varphi'(t)}{\varphi(t)}.$$

*Examples.* (a) If  $Y = (Y, h) = S^{n-1}$  is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1) \frac{\varphi''}{\varphi} - (n-1)(n-2) \left( \frac{\varphi'}{\varphi} \right)^2,$$

which for  $\varphi = t^2$  makes the expected  $Sc(g) = 0$ , since  $g = dt^2 + t^2 h$ ,  $t \geq 0$ , is the Euclidean metric in the polar coordinates.

If  $g = dt^2 + \sin^2 t h$ ,  $-\pi/2 \leq t \leq \pi/2$ , then  $Sc(g) = n(n-1)$  where this  $g$  is the spherical metric on  $S^n$ .

(b) If  $h$  is the (flat) Euclidean metric on  $\mathbb{R}^{n-1}$  and  $\varphi = \exp t$ , then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}_{-1}^n).$$

What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_{-\pi/n}^t -\tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric  $\varphi^2 h + dt^2$ , where  $h$  is flat, is *constant positive*, namely  $Sc(g) = n(n-1) = Sc(S^n)$ , by elementary calculation<sup>3</sup>

(VII) *Higher Warped Products.* Let  $Y$  and  $S$  be Riemannian manifolds with the metrics denoted  $dy^2$  (which now play the role of the above  $dt^2$ ) and  $ds^2$  (instead of  $h$ ), let  $\varphi > 0$  be a smooth function on  $Y$ , and let

$$g = \varphi^2(y) ds^2 + dy^2$$

<sup>3</sup>See §12 in [GL 1983].

be the corresponding warped metric on  $Y \times S$ ,

Then

$$(\star\star) \quad Sc(g)(y, s) = Sc(Y)(y) + \frac{1}{\varphi^2} Sc(S)(s) - \frac{m(m-1)}{\varphi^2(y)} \|\nabla\varphi(y)\|^2 - \frac{2m}{\varphi} \Delta\varphi(y),$$

where  $m = \dim(S)$  and  $\Delta = \sum \nabla_{i,i}$  is the Laplace operator on  $Y$ .

To prove this, apply the above  $(\star)$  to  $l \times S$  for naturally parametrised geodesics  $l \subset Y$  passing through  $y$  and then average over the space of these  $l$ , that is the unit tangent sphere of  $Y$  at  $y$ .

The most relevant example of  $(\star\star)$  is where  $S$  is the real line  $\mathbb{R}$  or the circle  $S^1$  and where it reduces to

$$(\star\star)_1 \quad Sc(g)(y, s) = Sc(Y)(y) - \frac{2}{\varphi} \Delta\varphi(y).$$

(The roles of  $Y$  and  $S = \mathbb{R}$  and notationally reversed here with respect to those in  $(\star)$ .)

**(VIII) Second Variation Formula.** The Weyl formula also yields the following formula for the *second derivative* of the  $(n-1)$ -volume of a cooriented hypersurface  $Y \subset X$  under a normal deformation of  $Y$  in  $X$ , where the scalar curvature of  $X$  plays an essential role.

The deformations we have in mind are by vector fields directed by geodesic normal to  $Y$ , where in the simplest case the norm of this field equals one.

In this case we have an equidistant motion  $Y \mapsto Y_t$  as earlier and the second derivative of  $vol_{n-1}(Y_t)$ , denoted here  $V = V_t$ , is expressed in terms of the shape operator  $A_t = A(Y_t)$  of  $Y_t$  and the Ricci curvature of  $X$ , where, recall  $trace(A_t) = mean.curv(Y_t)$  and

$$\partial_\nu V = \int_Y mean.curv(Y) dy$$

by the first variation formula.

Then, by Leibniz' rule,

$$\partial_\nu^2 V = \partial_\nu \left( \int_Y trace(A(y)) dy \right) = \int_Y trace(A^2(y)) dy + \int_Y trace(\partial_\nu A(y)) dy,$$

and where, by Weyl's formula,

$$trace(\partial_\nu A) = -trace(A^2) - Ricci(\nu, \nu)$$

for the normal unit field  $\nu$ .

Thus,

$$\partial_\nu^2 V = \int_Y (mean.curv)^2 - trace(A^2) - Ricci(\nu, \nu),$$

which, combining this with the above expression

$$Ricci(\nu) = \frac{1}{2} (Sc(X) - Sc(Y) - (mean.curv(Y))^2 + \|A^*\|^2),$$

shows that

$$\partial_\nu^2 V = \int \frac{1}{2} (Sc(Y) - Sc(X) + mean.curv^2 - \|A^*\|^2).$$

**Warning.** Unless  $Y$  is minimal and despite the notation  $\partial_\nu^2$ , this derivative depends on how the normal field on  $Y \subset X$  is extended to a vector field on (a neighbourhood of  $Y$  in)  $X$ .

Now, let us allow a non-constant geodesic field normal to  $Y$ , call it  $\psi\nu$ , where  $\psi(y)$  is a smooth function on  $Y$  and write down the full second variation formula as follows:

$$\partial_{\psi\nu}^2 vol_{n-1}(Y) = \int_Y \|d\psi(y)\|^2 dy + R(y)\psi^2(y)dy$$

for

$$[\circ\circ] \quad R(y) = \frac{1}{2} (Sc(Y, y) - Sc(X, y) + \mu(y) - \|A^*(Y)\|^2),$$

where  $\mu(y)$  stands for the mean curvature of  $Y$  and  $\|A^*(Y)\|^2 = \sum_i (\alpha^i)^2$ ,  $i = 1, \dots, n-1$ .

Notice, that the "new" term  $\int_Y \|d\psi(y)\|^2 dy$  depends only on the normal field itself, while the  $R$ -term depends on the extension of  $\psi\nu$  to  $X$ , unless

$Y$  is minimal, where  $[\circ\circ]$  reduces to

$$[*\ast] \quad \partial_{\psi\nu}^2 vol_{n-1}(Y) = \int_Y \|d\psi\|^2 + \frac{1}{2} (Sc(Y) - Sc(X) - \|A^*\|^2) \psi^2.$$

*Justification of the  $\|d\psi\|^2$  Term.* Let  $X = Y \times \mathbb{R}$  with the product metric and let  $Y = Y_0 = Y \times \{0\}$  and  $Y_{\varepsilon\psi} \subset X$  be the graph of the function  $\varepsilon\psi$  on  $Y$ . Then

$$vol_{n-1}(Y_{\varepsilon\psi}) = \int_Y \sqrt{1 + \varepsilon^2 \|d\psi\|^2} dy = vol_{n-1}(Y) + \frac{1}{2} \int_Y \varepsilon^2 \|d\psi\|^2 + o(\varepsilon^2)$$

by the Pythagorean theorem

and

$$\frac{d^2 vol_{n-1}(Y_{\varepsilon\psi})}{d^2 \varepsilon} = \|d\psi\|^2 + o(1).$$

by the binomial formula.

This proves  $[\circ\circ]$  for product manifolds and the general case follows by *linearity/naturality/functoriality* of the formula  $[\circ\circ]$ .

**Naturality Problem.** All "true formulas" in the Riemannian geometry should be derived with minimal, if any, amount of calculation – only on the basis of their "naturality" and/or of their validity in simple examples, where these formulas are obvious.

Unfortunately, this "naturality principle" is absent from the textbooks on differential geometry, but, I guess, it may be found in some algebraic articles (books?).

*Exercise.* Derive the second main formula (above (IV)) by pure thought from its manifestations in the examples in (VI).<sup>4</sup>

### 3 Twelve Samples of Theorems

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<sup>4</sup>I haven't myself solved this exercise.

**I: Closure and Density Theorems.** Let  $X$  be a smooth Riemannian manifold, let  $G = G^2(X)$  the space of  $C^2$ -smooth Riemannian metrics  $g$  on  $X$  and let  $G_{Sc \geq \sigma} \subset G$  and  $G_{Sc \leq \sigma} \subset G$ ,  $-\infty < \sigma < \infty$ , be the subsets of metrics  $g$  with  $Sc(g) \geq \sigma$  and with  $Sc(g) \leq \sigma$  respectively.

Then:

**I:  $\lim_{Sc \geq \sigma}$ .** The subset  $G_{Sc \geq \sigma} \subset G$  is closed in  $G$  with respect to  $C^0$ -topology: uniform limits  $g = \lim g_i$  of metric  $g_i$  with  $Sc(g_i) \geq \sigma$  have  $Sc \geq \sigma$ , provided these  $g$  are  $C^2$ -smooth in order to have their scalar curvature defined.

**I:  $\lim_{Sc \leq \sigma}$ .** The subset  $G_{Sc \leq \sigma} \subset G$  is dense in  $G$  with respect to  $C^0$ -topology. Moreover, all  $g \in G$  admit fine (which is stronger than uniform for non-compact  $X$ ) approximations by metrics with scalar curvatures  $\leq \sigma$ .

There are two proofs of **I:  $\lim_{Sc \geq \sigma}$** . The first one in [Billiard 2014] depends on non-existence of metrics with  $Sc > 0$  on tori (see ??? below) and the second one in [Bamler 2016] uses Ricci flow.

The proof of **I:  $\lim_{Sc \leq \sigma}$**  is achieved by (more or less) direct geometric construction in [Lohkamp 1994], where it is also shown that metrics with Ricci  $< 0$  are  $C^0$ -dense as well.

**II:  $\mathbb{T}_{Sc > 0}^n$ .** Tori  $T^n$  admit no metrics with  $Sc > 0$ . Moreover, Riemannian metrics on  $T^n$  with  $Sc \geq 0$  are Riemannian flat: the universal coverings of these tori are isometric to  $\mathbb{R}^n$ .

This was proven by Schoen and Yau in [SYstr 1979] for  $n \leq 7$  by an inductive descent argument with minimal hypersurfaces, see section ???. Then in [GLfund 1980] for all  $n$  with a use of twisted Dirac operators<sup>5</sup>  $\mathcal{D}$  on  $T^n$ .

At the present moment there are (at least) five such proofs which rely on different versions of the Atiyah-Singer index theorem which guarantees the existence of non-zero harmonic representatives in various spaces of sections of twisted spinors on  $T^n$  (or on  $\mathbb{R}^n$ , which cover  $T^n$ ) with arbitrary.

Then non-existence of a metric on  $T^n$  with  $Sc > 0$  follows from Schroedinger-Lichnerowicz-Weitzenboeck algebraic identity

$$\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$$

for a positive (coarse Bochner Laplacian) operator  $\nabla^2$ , that implies that no non-zero harmonic spinor exists.

(see sections???)

**III: (Formerly) Geroch Conjecture.** The Euclidean metric  $g_{Eu}$  on  $\mathbb{R}^n$  admits no compactly supported perturbations  $g$  with  $Sc(g) > 0$ . Moreover,

If a metric  $g$  on  $\mathbb{R}^n$  with  $Sc(g) \geq 0$  is equal to  $g_{Eu}$  outside a compact subset in  $\mathbb{R}^n$ , then  $(\mathbb{R}^n, g)$  is isometric to  $(\mathbb{R}^n, g_{Eu})$ .

This, of course, trivially follows from the above **II:  $\mathbb{T}_{Sc > 0}^n$** , since compactly supported perturbations of the flat metric on  $\mathbb{R}^n$  yields similar perturbations of flat metrics on tori.

But a more general version of this was originally proven by Schoen and Yau in [SYpm 1979] for a class of metrics  $g$  on 3-manifolds asymptotic to  $g_{Eu}$  under the name of positive mass/energy theorem (see section ???) with a use of minimal surfaces.

<sup>5</sup>The "untwisted" Dirac operator acts on the spin bundle  $S(X)$  and a "twisted" one operates on the tensor product of  $S(X)$  with some vector bundle  $L$  over  $X$ .

Then Witten in [Witten 1981] (also see [Bartnik 1986]) suggested a proof with a use of a perturbation argument in the space of invariant (non-twisted) harmonic spinors on  $\mathbb{R}^n$ .

Later, Lohkamp [Loh 1999] found a (relatively) simple reduction of the general, and technically more challenging, case of the positive mass theorem to that of compactly supported perturbations, that in turn, (trivially) reduces to  $\mathbb{T}_{Sc>0}^n$ .

Also notice that the *doubling property* for *mean convex manifolds* with boundaries (see [GL 1980] and section ???) allows a reduction of the Geroch Conjecture and of similar more general results to the *Goette-Semmelmann theorem* [GS 2002] concerning *extremality/rigidity of the metrics  $g$  with positive curvature operators* (see section  $[X \rightarrow \bigcirc]$  below).

**IV: Simply Connected Manifolds with  $Sc > 0$ .** In 1963, by confronting the *Atiyah-Singer index theorem* with the identity  $\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$ , Lichnerowicz [Li 1963] has shown that

there are smooth *closed simply connected* manifolds  $X$  of all dimensions  $n = 4k$ ,  $k > 0$ , that admit no metrics with  $Sc > 0$ .

The simplest example of these for  $n = 4$  is *the Kummer surface* given by the equation

$$z_1^4 + z_2^4 + z_3^4 + z_4^4$$

in the complex projective space  $\mathbb{C}P^3$ .

Also the Cartesian powers  $X_{Ku} \times \dots \times X_{Ku}$  admit no metrics with  $Sc > 0$ .

Later, using a more general index theorem by Atiyah and Singer, Hitchin [Hit 1974] pointed out that

*there exist manifolds  $\Sigma$  homeomorphic (but no diffeomorphic!) to the spheres  $S^n$ , for all  $n = 8k + 1, 8k + 2$ ,  $k = 1, 2, 3, \dots$ , which admits no metrics with  $Sc > 0$ .*

Notice that, by *Yau's solution of the Calabi conjecture*, the Kummer surface admits a metric with  $Sc = 0$ , even with  $Ricci = 0$ , but, probably, (I guess this must be known) there is no metrics with  $Sc = 0$  on these  $\Sigma$ .

The actual Lichnerowicz-Hitchin theorem says that if a certain topological invariant  $\hat{\alpha}(X)$  *doesn't vanish* then  $X$  *admits no metric with  $Sc > 0$* , since, by the Atiyah and Singer index formulae,<sup>6</sup>

$$\hat{\alpha}(X) \neq 0 \Rightarrow Ind(\mathcal{D}|_X) \neq 0 \Rightarrow \exists \text{ harmonic spinor } \neq 0 \text{ on } X.$$

Conversely,

if  $X$  is a *simply connected* manifold of *dimension  $n \neq 4$* , and if  $\hat{\alpha}(X) = 0$  then it admits a metric with positive scalar curvature. [GLcl 1980], [Stolz 1992]

Thus, for instance

*all simply connected manifolds of dimension  $n \neq 0, 1, 2, 4 \pmod{8}$  admit metrics with  $Sc > 0$* , since  $\hat{\alpha}(X) = 0$  is known to vanish for these  $n$ .

**V. Bounds on Size, Extremality, Rigidity.** The inequality  $Sc(X) \geq \sigma > 0$ , as it becomes a *positive curvature* condition, imposes an *upper bound* on the size of  $X$ , where an instance of this can be expressed in terms of the *hyperspherical radius*  $Rad_{S^n}(X)$ , defined for *closed* Riemannian  $n$ -manifolds  $X$  as

<sup>6</sup>The Dirac operator is defined only on *spin* manifolds and to avoid entering into this at the present moment we postulate  $\hat{\alpha}(X) = 0$  for non-spin manifolds  $X$ .

the supremum of the numbers  $R > 0$ , such that  $X$  admits a non-contractible 1-Lipschitz, i.e. distance non-increasing, map  $f : X \rightarrow S^n$ .

The existence of a non-trivial such bound,

$$Rad_{S^n}(X) \leq \frac{const_n}{\sqrt{\sigma}}, \quad \sigma = \inf_{x \in X} Sc(X, x),$$

for orientable spin<sup>7</sup> manifolds  $X$  of even dimensions  $n$ <sup>8</sup>, follows by confronting the index theorem with a "twisted version" of the formula  $\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$  for the Dirac operator on  $X$  twisted with the  $f$ -pullback of a suitable vector bundle  $L$  over  $S^n$  [GLfund 1980], where

the optimal constant  $const_n = \sqrt{n(n-1)} = \sqrt{Sc(S^n)}$  is achieved with  $L$  being the (complexified) positive spin bundle over  $S^n$ , (see [Llarull 1998a] and section ???)

This sharp inequality, says, in particular, that one can't enlarge the spherical metric  $g_{sph}$  on  $S^n$  without making the scalar curvature smaller at some point. That is if a metric  $g$  on  $S^n$  satisfies

$$g \geq g_{sph} \text{ and } Sc(g) \geq n(n-1) = Sc(g_{sph})$$

then, necessarily,  $Sc(g) = n(n-1)$ , which we express by saying that *spheres are extremal*.

In fact, Llarull's argument shows that spheres are *rigid*:

$[g \geq g_{sph}] \& [Sc(g) \geq Sc(g_{sph})]$  implies that  $g = g_{sph}$ .

This extremality/rigidity property of spheres was generalised by Goette and Semmelmann to manifolds  $\underline{X}$  with *positive curvature operators*, where the examples of such manifolds we are concerned with now are smooth *locally convex hypersurfaces* in Riemannian flat  $(n+1)$ -manifolds, e.g. products of convex hypersurfaces in  $\mathbb{R}^{m+1}$  by the flat tori  $\mathbb{T}^{n-m}$ .

The (proof of the) main result in [GS 2002] implies in this case the following theorem.

$[X \rightarrow \bigcirc]$  Let  $X$  be a connected orientable Riemannian  $n$ -manifold, let  $\underline{X} \subset \mathbb{R}^{n+1}$  be a smooth closed locally convex hypersurface in a Riemannian flat  $(n+1)$ -manifold and let  $f : X \rightarrow \underline{X}$  be a smooth map.

Let the norm of the differential of  $f$  and the scalar curvatures of  $X$  and  $\underline{X}$  be related by the inequality

$$Sc(X, x) \geq Sc(\underline{X}, f(x)) \cdot \|df(x)\|^2, \quad x \in X.$$

If  $X$  is orientable and the degree of  $f$  is non-zero, then, provided  $X$  is spin, this inequality becomes an equality:

$$Sc(X, x) = Sc(\underline{X}, f(x)) \cdot \|df(x)\|^2,$$

---

<sup>7</sup>All surfaces are spin and an orientable manifold  $X$  of dimension  $n \geq 3$  is spin if and only if the restriction of the tangent bundle  $T(X)$  to all surfaces  $Y^2 \subset X$  are trivial, e.g. if  $H^2(X; \mathbb{Z}_2) = 0$ . The simplest examples of spin  $n$ -manifolds are smooth hypersurfaces in  $\mathbb{R}^{n+1}$ , such as product of spheres.

More interesting in this respect are complex projective spaces  $\mathbb{C}P^m$  and smooth complex hypersurfaces  $X \subset \mathbb{C}P^m$  of degree  $d$ : these  $X$  are spin if and only if  $m+d$  is odd, as it the case for the Kummer surface, for instance.

<sup>8</sup>Odd dimensions need extra attention.

at all points  $x \in X$ .

Notice that the above Llarull's theorem as well as non-existence of metrics with  $Sc > 0$  on tori are special cases of  $[X \rightarrow \mathcal{O}]$ .

**VI. Two Euclidean applications of  $[X \rightarrow \mathcal{O}]$ .** Let  $Y \subset \mathbb{R}^n$  be a smooth closed hypersurface with the mean curvature bounded from below by  $\mu > 0$ .

Then the hyperspherical radius of  $Y$  is bounded by

$$Rad_{S^{n-1}}(Y) \leq \frac{1}{n-1}.$$

Moreover, if  $Rad_{S^{n-1}}(Y) = \frac{1}{n-1}$ , then  $Y$  then  $mean.curv(Y) = n-1$ , which, by a theorem of A.D. Alexandrov, implies that

$Y$  equals to the unit sphere  $S_x^{n-1} \subset \mathbb{R}^n$  around some point  $x \in \mathbb{R}^n$ .

To relate this to  $[X \rightarrow \mathcal{O}]$ , let

$$X_{1/2} \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$$

be the (closed) domain in  $\mathbb{R}^n$  bounded by  $Y$  and let  $X_\varepsilon \subset \mathbb{R}^{n+1}$  be a (more or less) naturally/canonically  $C^2$ -smoothed boundary of the  $\varepsilon$ -neighbourhood (which is only  $C^1$ -smooth) of  $X_{1/2} \subset \mathbb{R}^{n+1}$ .

Then let  $\underline{X}_{1/2} \subset \mathbb{R}^{n+1}$  be the unit  $n$ -ball  $B^n \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$  and let, accordingly,  $\underline{X}_\varepsilon \subset \mathbb{R}^{n+1}$  be a (more or less) naturally/canonically  $C^2$ -smoothed boundary of its  $\varepsilon$ -neighbourhood.

Then maps  $f : Y \rightarrow \underline{S}^{n-1}$  define maps

$$F_\varepsilon : X_\varepsilon \rightarrow \underline{X}_\varepsilon,$$

to which  $[X \rightarrow \mathcal{O}]$  applies and, when  $\varepsilon \rightarrow 0$ , it yields the inequality  $Rad_{S^{n-1}}(Y) \leq \frac{1}{n-1}$  as we explain in detail in lecture???

**Questions.** Is there a direct proof of this inequality?

What exactly happens in the limit when  $\varepsilon \rightarrow 0$  to the Dirac operator used in the proof of  $[X \rightarrow \mathcal{O}]$ ?

**▲<sup>m</sup>-Inequality.** Let  $X_0 \subset \mathbb{R}^n$  be a polytope, i.e. convex compact polyhedron with non-empty interior, and let  $X \subset \mathbb{R}^n$  be diffeomorphic to  $X_0$ .

Let all  $(n-1)$ -faces  $F_i$  of  $X$  have positive mean curvatures, e.g. the subset  $X \subset \mathbb{R}^n$  is convex.

Let the dihedral angles between (the tangent spaces of) the faces  $F_i$  and  $F_j$  of  $X$  at all points in the  $(n-2)$ -faces where/if these faces meet, are bounded by the corresponding dihedral angles of  $X_0$ ,

$$\angle_{ij}(X) \leq \angle_{ij}(X_0).$$

If all dihedral angles of  $X_0$  are  $\leq \frac{\pi}{2}$ , then

$$\angle_{ij}(X) = \angle_{ij}(X_0).$$

This is shown, by doubling and smoothing  $X_0$  and  $X$  and then applying  $[X \rightarrow \mathcal{O}]$  (see Lecture ???).

*Remarks/Exercises.* (a) The only polytopes with  $\angle_{ij} \leq \frac{\pi}{2}$  are products of simplices, such as the  $n$ -cubes  $[0, 1]^n$ , for example.

(b) If both  $X_0$  and  $X$  are  $n$ -simplices then the implication

$$\angle_{ij}(X) \leq \angle_{ij}(X_0) \Rightarrow \angle_{ij}(X) = \angle_{ij}(X_0)$$

follows from the Kirszbraun theorem with no condition  $\angle_{ji} \leq \pi/2$ .

**MORE TO COME....**

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**MORE TO COME....**