Four Lectures on Scalar Curvature
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Unlike manifolds with controlled sectional and Ricci curvatures, those with their *scalar curvatures bounded from below* are not configured in specific rigid forms but display an uncertain variety of flexible shapes similar to what one sees in geometric topology.

Yet, there are definite limits to this flexibility, where determination of such limits crucially depends, at least in the known cases, on two seemingly unrelated analytic means: index theory of Dirac operators and the geometric measure theory in codimension one [1]

The nearest to what one sees in the emergent picture of the scalar curvature domain is reminiscent of symplectic geometry, but the former has yet to reach maturity enjoyed by the latter.

We start these lectures with a dozen pages, §§1 and 2, of elementary background material followed in §3 by a brief overview of main topics in spaces with their scalar curvatures bounded from below, that covers, I guess 70-80% of currently pursued directions. Then, in §§4 and 5 we give a more detailed exposition of several known and some new geometric constraints on spaces X implied by the lower bound $Sc(X) \geq \sigma$.

Contents

[1]Besides these, something is achieved with the *Hamilton's Ricci flow*, especially in dimension $n = 3$, and specifically 4-dimensional results are derived with a use of the *Seiberg-Witten equations*.

1 Geometrically Deceptive Definition.

The scalar curvature of a C^2-smooth Riemannian manifold $X = (X, g)$, denoted $Sc = Sc(X) = Sc(X, g) = Sc(g) = Sc_g$ is a continuous function on X, which is traditionally defined as

the sum of the values of the sectional curvatures at the $n(n-1)$ *ordered bivectors of an orthonormal frame in* X,

$$Sc(X)(x) = \sum_{i,j} \kappa_{ij}(x), \ i \neq j = 1, ..., n,$$

where this sum doesn't depend on the choice of this frame by the Pythagorean theorem.

But if you are geometrically minded, you want to have a geometric definition where the first attempt to find such a definition relies on the following properties of $Sc(X)$.

$\bullet_1$ *Additivity under Cartesian-Riemannian Products .*

$$Sc(X_1 \times X_2, g_1 \oplus g_2) = Sc(X_1, g_1) + Sc(X_2, g_2).$$

$\bullet_2$ *Quadratic Scaling.*

$$Sc(\lambda \cdot X) = \lambda^{-2} Sc(X), \text{ for all } \lambda > 0,$$

where

$$\lambda \cdot X = \lambda \cdot (X, dist_X) =_{def} (X, dist_{\lambda \cdot X}) \text{ for } dist_{\lambda \cdot X} = \lambda \cdot dist(X)$$

for all metric spaces $X = (X, dist_X)$ and where $dist \mapsto \lambda \cdot dist(X)$ corresponds to $g \mapsto \lambda^2 \cdot g$ for the Riemannian quadratic form g.

(This makes the Euclidean spaces scalar-flat: $Sc(\mathbb{R}^n) = 0$.)

$\bullet_3$ *Volume Comparison.* If the scalar curvatures of n-dimensional manifolds X and X' at some points $x \in X$ and $x' \in X'$ are related by the strict inequality

$$Sc(X)(x) < Sc(X')(x'),$$

then the Riemannian volumes of the ε-balls around these points satisfy

$$vol(B_x(X, \varepsilon)) > vol(B_{x'}(X', \varepsilon))$$

for all sufficiently small $\varepsilon > 0$.

Observe that this volume inequality is *additive under Riemannian products*: if

$$vol(B_{x_i}(X, \varepsilon)) > vol(B_{x'_i}(X'_i, \varepsilon)), \text{ for } \varepsilon \leq \varepsilon_0,$$

and *for all points* $x_i \in X_i$ and $x'_l \in X'_i$, $i = 1, 2$, then

$$vol_n(B_{(x_1,x_2)}(X_1 \times X_2, \varepsilon_0)) > vol_n(B_{(x'_1,x'_2)}(X'_1 \times X'_2, \varepsilon_0)$$

for all $(x_1, x_2) \in X_i \times X_2$ and $(x'_1, x'_2) \in X'_1 \times X'_2$.

This follows from the Pythagorean formula

$$dist_{X_1 \times X_2} = \sqrt{dist^2_{X_1} + dist^2_{X_2}}.$$

and the Fubini theorem applied to the "fibrations" of balls over balls:

$B_{(x_1,x_2)}(X_1 \times X_2, \varepsilon_0)) \to B_{x_1}(X_1, \varepsilon_0)$ and $B_{(x'_1,x'_2)}(X'_1 \times X'_2, \varepsilon_0)) \to B_{x_1}(X'_1, \varepsilon_0)$,

where the fibers are balls of radii $\varepsilon \in [0, \varepsilon_0]$ in X_2 and X'_2.

$\bullet_4$ *Normalisation/Convention for Surfaces with Constant Sectional Curvatures.* The unit spheres $S^2(1)$ have constant scalar curvature 2 and the hyperbolic plane $H^2(-1)$ with the sectional curvature -1 has scalar curvature -2

It is an elementary exercise to prove the following.

$\star_1$ *The function $Sc(X,g)(x)$ which satisfies $\bullet_1$-$\bullet_4$ exists and unique*;

$\star_2$ *The unit spheres and the hyperbolic spaces with sect.curv $= -1$ satisfy*

$$Sc(S^n(1)) = n(n-1) \text{ and } Sc(\mathbf{H}^n_{-1}) = -n(n-1).$$

Thus,

$$Sc(S^n(1) \times \mathbf{H}^n_{-1}) = 0 = Sc(\mathbb{R}^n),$$

which implies that the volumes of the small balls in $S^n(1) \times \mathbf{H}^n_{-1}$ are "very close" to the volumes of the Euclidean $2n$-balls.

Also it is elementary to show that the definition of the scalar curvature via volumes of balls agrees with the traditional $Sc = \sum \kappa_{ij}$, where the definition via volumes seem to have an advantage of being geometrically more usable.

But this is an illusion:

THERE IS NO SINGLE KNOWN (ARE THERE UNKNOWN?)
GEOMETRIC ARGUMENT WHICH WOULD MAKE USE OF THIS DEFINITION.

The immediate reason for this is *the infinitesimal* nature of the volume comparison property: it *doesn't integrate* to the corresponding property of balls of specified, let them be small, radii $r \leq \varepsilon > 0$. [2]

2 Useful Formulas.

The logic of most (all?) arguments concerning the global geometry of manifolds X with scalar curvatures bounded from below is, in general terms, as follows.

Firstly, one uses (or proves) the *existence theorems for solutions Φ of certain partial differential equations*, where the existence of these Φ and their properties depend on global, topological and/or geometric assumptions $\mathcal{A}$ on X, which are, a priori, unrelated to scalar curvature.

Secondly, one concocts some *algebraic-differential expressions $\mathcal{E}(\Phi, Sc(X))$*, where the crucial role is played by certain *algebraic formulae* and issuing inequalities satisfied by $\mathcal{E}(\Phi, Sc(X))$.

Then, assuming $Sc(X) \geq \sigma$ one arrives at *a contradiction*, thus showing that

the inequality $Sc(X) \geq \sigma$ implies properties of X opposite to $\mathcal{A}$.

[I] Historically the first Φ in this story were *harmonic spinors*, that are solutions s of $\mathcal{D}(s) = 0$, where $\mathcal{D}$ is the *Dirac operator*, the existence of which on certain manifolds X followed by the Atiyah-Singer index theorem of 1963,

[2] An attractive conjecture to the contrary appears in *Volumes of balls in large Riemannian manifolds* by Larry Guth in Annals of Mathematics173(2011), 51-76.

while the relevant formula involving $Sc(X)$ was an algebraic identity between the squared *Dirac operator* and the, *a priori positive*, (coarse) Laplace operator.

Confronting these, André Lichnerowicz [Lich 1963] found examples of closed $4k$-dimensional manifolds which admit no metrics with $Sc > 0$.

[II] Next Φ to come [SY(structure) 1979] were *smooth* stable minimal hypersurfaces in X for $n = dim(X) \leq 7$, the existence of which was proved by Federer [Fed 1970] relying on the *regularity* of volume minimising cones of dimensions ≤ 6 proved by Simons [Sim 1968], while the key algebraic identity employed by Schoen and Yau in [SY(structure) 1979] was a suitably rewritten Gauss formula, that lead, in particular, to

non-existence theorem of metrics with $Sc > 0$ on the 3-torus.

[III] The third kind of Φ are solutions to the 4-dimensional *Seiberg-Witten equation* of 1994, that is the Dirac equation coupled with a certain non-linear equation and where the relevant formula is essentially the same as in [II].

Using these, LeBrun [LeB 1999] established a non-trivial (as well as sharp) lower bound on $\int_X Sc(X,x)^2 dx$ for Riemannian manifolds X diffeomorphic to algebraic surfaces of general type.

In what follows in this section, we enlist classical formulae involved with [II] and indicate their (more or less) immediate applications.

2.1 Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) Riemannian Variation Formula. Let h_t, $t \in [0,\varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold Y and let us incorporate h_t to the metric $g = h_t + dt^2$ on $Y \times [0,\varepsilon]$.

Notice that an arbitrary Riemannin metric on an n-manifold X admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The t-derivative of h_t is equal to *twice the second fundamental form* of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0,\varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_\nu h = A^*,$$

where

ν is the unit normal field to Y defined as $\nu = \frac{d}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the *principal values* $\alpha_i^*(y)$, $i = 1,...,n-1$, of the quadratic form A_t^* on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal vectors $\tau_i^* \in T_i(Y)$, which *diagonalize* A^*, are called *the principal curvatures* of Y, and that the sum of these is called *the mean curvature* of Y,

$$mean.curv(Y,y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = trace(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames τ_i in $T_y(Y)$ by the Pythagorean theorem.

Also observe that A^* changes sign under reversion of the t-direction. Accordingly the sign of the quadratic form $A^*(Y)$ depends on the coorientation of Y in X, where our convention is such that

the boundaries of *convex* domains have *positive definite* second fundamental forms A^*, hence, positive mean curvatures.

(2.1.B) First Variation Formula. This concerns the t-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$[\circ_U] \qquad \partial_\nu vol_{n-1}(U) = \frac{dh_t}{dt} vol_{n-1}(U_t) = \int_{U_t} mean.curv(U_t) dy_t$$

where dy_t is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi\nu = \psi \cdot \nu$ for bounded Borel functions $\psi = \psi(y)$ as follows

$$[\circ_\psi] \qquad \partial_{\psi\nu} vol_{n-1}(Y_t) = \int_{Y_t} \psi(y) mean.curv(Y_t) dy_t$$

Now comes the first formula with the Riemannin curvature in it.

2.2 Gauss' Theorema Egregium

Let $Y \subset X$ be a smooth hypersurface in a Riemannin manifold X. Then the sectional curvatures of Y and X on a tangent 2-plane $\tau = \tau^2 \subset T_y(Y) \subset T)y(X)$ $y \in Y$, satisfy

$$\kappa(Y,\tau) = \kappa(X,\tau) + \wedge^2 A^*(\tau),$$

where $\wedge^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form form $A^* = A^*(Y) \subset X$ restricted to the plane τ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

This, with the definition the scalar curvature by the formula $Sc = \sum \kappa_{ij}$, implies that

$$Sc(Y,y) = Sc(X,y) + \sum_{i \neq j} \alpha_i^*(y)\alpha_j^*(y) - \sum_i \kappa_{\nu,i},$$

where:

• $\alpha_i^*(y)$, $i = 1, ..., n-1$ are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors τ_i in $T_y(Y)$;

• α^*-sum is taken over all ordered pairs (i,j) with $j \neq i$;

• $\kappa_{\nu,i}$ are the sectional curvatures of X on the bivectors (ν,τ_i) for ν being a unit (defined up to ±-sign) normal vector to Y;
• the sum of $\kappa_{\nu,i}$ is equal to the value of the Ricci curvature of X at ν,

$$\sum_i \kappa_{\nu,i} = Ricci_X(\nu,\nu).$$

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of Y and that in the case of $Y = S^{n-1} \subset \mathbb{R}^n = X$ this gives the correct value $Sc(S^{n-1}) = (n-1)(n-2)$.

Also observe that

$$\sum_{i\neq j} \alpha_i\alpha_j = \left(\sum_i \alpha_i\right)^2 - \sum_i \alpha_i^2,$$

which shows that

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu).$$

In particular, if $Sc(X) \geq 0$ and Y is *minimal*, that is $mean.curv(Y) = 0$, then

(Sc ≥ −2Ric) $$Sc(Y) \geq -2Ricci(\nu,\nu).$$

Example. The scalar curvature of a hypersurface $Y \subset \mathbb{R}^n$ is expressed in terms of the mean curvature of Y, the (point-wise) L_2-norm of the second fundamental form of Y as follows.

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2$$

for $||A^*(Y)||^2 = \sum_i (\alpha_i^*)^2$, while $Y \subset S^n$ satisfy

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n\max_i (c_i^*)^2.$$

It follows that *minimal* hypersurfaces Y in $\mathbb{R}^n$, i.e. these with $mean.curv(Y) = 0$, have *negative scalar curvatures*, while hypersurfaces in the n-spheres with all principal values $\leq \sqrt{n-2}$ have $Sc(Y) > 0$.

Let $A = A(Y)$ denote *the shape operator* that is the symmetric operator on $T(Y)$ associated with A^* via the Riemannin scalar product g restricted from $T(X)$ to $T(Y)$,

$$A^*(\tau,\tau) = \langle A(\tau),\tau\rangle_g \text{ for all } \tau \in T(Y).$$

2.3 Variation of the Curvature of Equidistant Hypersurfaces

(2.3.A) *The Second Main Formula of Riemannian Geometry.*[3] Let Y_t be a family of hypersurfaces t-equidistant to a given $Y = Y_0 \subset X$. Then the shape operators $A_t = A(Y_t)$ satisfy:

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

[3]The first main formula is *Gauss' Theorema Egregium.*

where B_t is the symmetric operator associated with the quadratic differential form B^* on Y_t, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the *sectional curvature* of g at (the 2-planes spanned by) the bivectors $\left(\tau, \nu = \frac{d}{dt}\right)$.

Remark. Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannin geometry. [4]

Tracing this formula yields

(2.3.B) *Hermann Weyl's Tube Formula.*

$$trace\left(\frac{dA_t}{dt}\right) = -||A^*||^2 - Ricci_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$trace(\partial_\nu A) = \partial_\nu trace(A) = -||A^*||^2 - Ricci(\nu, \nu),$$

where

$$||A^*||^2 = ||A||^2 = trace(A^2),$$

where, observe,

$$trace(A) = trace(A^*) = mean.curv = \sum_i \alpha_i^*$$

and where *Ricci* is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_x(X)$ is equal to the trace of the above B^*-form (or of the operator B) on the normal hyperplane $\nu^\perp \subset T_x(X)$ (where $\nu^\perp = T_x(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$Sc(X) = trace(Ricci)$$

and that the above formula $Sc(Y, y) = Sc(X, y) + \sum_{i \neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$Ricci(\nu, \nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i \neq j} \alpha_i^* \cdot \alpha_j^*\right) =$$

$$= \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + ||A^*||^2\right)$$

where, recall, $\alpha_i^* = \alpha_i^*(y)$, $y \in Y$, $i = 1, ..., n-1$, are the principal curvatures of $Y \subset X$, where $mean.curv(Y) = \sum_i \alpha_i^*$ and where $||A^*||^2 = \sum_i (\alpha_i^*)^2$.

[4] Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl's Tube Formula", appears in [Darm 1927] with the reference to Darboux' textbook of 1897.

2.4 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface $Y \subset X$ is called umbilic if all principal curvatures of Y are mutually equal at all points in Y.

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres $S^n_{\kappa>0}$, Euclidean spaces $\mathbb{R}^n$ and hyperbolic spaces $\mathbf{H}^n_{\kappa<0}$) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let $\varphi = \varphi(y) > 0$ be a smooth positive function on a Riemannian (n-1)-manifold $Y = (Y, h)$, and let $g = h_t + dt^2 = \varphi^2 h + dt^2$ be the corresponding metric on $X = Y \times [0, \varepsilon]$.

Then the hypersurfaces $Y_t = Y \times \{t\} \subset X$ are umbilic with the principal curvatures of Y_t equal to $\alpha_i^*(t) = \frac{\varphi'}{\varphi}$, $i = 1, ..., n-1$ for

$$A_t^* = \frac{\varphi'}{\varphi} h_t \text{ for } \varphi' = \frac{d\varphi}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi} .$$

The Weyl formula reads in this case as follows.

$$(n-1)\left(\frac{\varphi'}{\varphi}\right)' = -(n-1)^2\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{1}{2}\left(Sc(g) - Sc(h_t) - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2\right).$$

Therefore,

$$Sc(g) = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2 =$$

$$(\star) \qquad = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

where, recall, $n = dim(X) = dim(Y) + 1$ and the mean curvature of Y_t is

$$mean.curv(Y_t \subset X) = (n-1)\frac{\varphi'(t)}{\varphi(t)}.$$

Examples. (a) If $Y = (Y, h) = S^{n-1}$ is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

which for $\varphi = t^2$ makes the expected $Sc(g) = 0$, since $g = dt^2 + t^2 h$, $t \geq 0$, is the Euclidean metric in the polar coordinates.

If $g = dt^2 + \sin t^2 h$, $-\pi/2 \leq t \leq \pi/2$, then $Sc(g) = n(n-1)$ where this g is the spherical metric on S^n.

(b) If h is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}^n_{-1}).$$

What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2 h + dt^2$, where h is flat, is *constant positive*, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation[5]

Higher Warped Products. Let Y and S be Riemannian manifolds with the metrics denoted dy^2 (which now play the role of the above dt^2) and ds^2 (instead of h), let $\varphi > 0$ be a smooth function on Y, and let

$$g = \varphi^2(y)ds^2 + dy^2$$

be the corresponding warped metric on $Y \times S$,

Then

$$(\star\star)\quad Sc(g)(y,s) = Sc(Y)(y) + \frac{1}{\varphi^2} Sc(S)(s) - \frac{m(m-1)}{\varphi^2(y)} ||\nabla\varphi(y)||^2 - \frac{2m}{\varphi}\Delta\varphi(y),$$

where $m = dim(S)$ and $\Delta = \sum \nabla_{i,i}$ is the Laplace operator on Y.

To prove this, apply the above $(\star)$ to $l \times S$ for naturally parametrised geodesics $l \subset Y$ passing trough y and then average over the space of these l, that is the unit tangent sphere of Y at y.

The most relevant example of $(\star\star)$ is where S is the real line $\mathbb{R}$ or the circle S^1 also denoted $\mathbb{T}^1$ and where it reduces to

$$(\star\star)_1 \qquad Sc(g)(y,s) = Sc(Y)(y) - \frac{2}{\varphi}\Delta\varphi(y).$$

(The roles of Y and $S = \mathbb{R}$ and notationally reversed here with respect to those in $(\star)$.)

The basic feature of the metrics $\varphi^2(y)ds^2 + dy^2$ on $Y \times \mathbb{R}$ is that they are $\mathbb{R}$-*invariant*, where the quotients $(Y \times \mathbb{R})/\mathbb{Z} = Y \times \mathbb{T}^1$ carry the corresponding $\mathbb{T}^1$-*invariant* metrics, while *the* $\mathbb{R}$-*quotients are isometric to* Y.

Besides $\mathbb{R}$-invariance, a characteristic feature of warped product metrics is *integrability* of the tangent hyperplane field normal to the $\mathbb{R}$-orbits, where $Y \times \{0\} \subset Y \times \mathbb{R}$, being normal to these orbits, serves as an integral variety for this field.

Also notice that $Y = Y \times \{0\} \subset Y \times \mathbb{R}$ is totally geodesic with respect to the metric $\varphi^2(y)ds^2 + dy^2$, while the ($\mathbb{R}$-invariant) *curvature* (vector field) *of the* $\mathbb{R}$-*orbits* is equal to the *gradient field* $\nabla\varphi$ extended from Y to $Y \times \mathbb{R}$.

In what follows, we emphasise $\mathbb{R}$-invariance and interchangeably speak of $\mathbb{R}$-invariant metrics on $Y \times \mathbb{R}$ and metrics warped with factors φ^2 over Y.

2.5 Second Variation Formula

The Weyl formula also yields the following formula for the *second derivative* of the $(n-1)$-volume of a cooriented hypersurface $Y \subset X$ under a normal deformation of Y in X, where the scalar curvature of X plays an essential role.

The deformations we have in mind are by vector fields directed by geodesic normal to Y, where in the simplest case the norm of his field equals one.

In this case we have an equidistant motion $Y \mapsto Y_t$ as earlier and the second derivative of $vol_{n-1}(Y_t)$, denoted here $Vol = Vol_t$, is expressed in terms of of

[5] See §12 in [GL 1983].

the shape operator $A_t = A(Y_t)$ of Y_t and the Ricci curvature of X, where, recall $trace(A_t) = mean.curv(Y_t)$ and

$$\partial_\nu Vol = \int_Y mean.curv(Y)dy$$

by the first variation formula.

Then, by Leibniz' rule,

$$\partial_\nu^2 Vol = \partial_\nu \int_Y trace(A(y))dy = \int_Y trace^2(A(y))dy + \int_Y trace(\partial_\nu A(y))dy,$$

and where, by Weyl's formula,

$$trace(\partial_\nu A) = -trace(A^2) - Ricci(\nu,\nu)$$

for the normal unit field ν.

Thus,

$$\partial_\nu^2 Vol = \int_Y (mean.curv)^2 - trace(A^2) - Ricci(\nu,\nu),$$

which, combining this with the above expression

$$Ricci(\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + ||A^*||^2\right),$$

shows that

$$\partial_\nu^2 Vol = \int \frac{1}{2}\left(Sc(Y) - Sc(X) + mean.curv^2 - ||A^*||^2\right).$$

In particular, if $Sc(X) \geq 0$ and Y is minimal, then,

$(\int \mathrm{Sc} \geq 2\partial^2 \mathrm{V})$ $$\int_Y Sc(Y,y)dy \geq 2\partial_\nu^2$$

(compare with the $(Sc \geq -2Ric)$ in 2.2).

Warning. Unless Y is minimal and despite the notation ∂_ν^2, this derivative depends on how the normal filed on $Y \subset X$ is extended to a vector filed on (a neighbourhood of Y in) X.

Illuminative Exercise. Check up this formula for concentric spheres of radii t in the spaces with constant sectional curvatures that are S^n, $\mathbb{R}^n$ and $\mathbf{H}^n$.

Now, let us allow a non-constant geodesic field normal to Y, call it $\psi\nu$, where $\psi(y)$ is a smooth function on Y and write down the full second variation formula as follows:

$$\partial_{\psi\nu}^2 vol_{n-1}(Y) = \int_Y ||d\psi(y)||^2 dy + R(y)\psi^2(y)dy$$

for

$[\circ\circ]$ $$R(y) = \frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - ||A^*(Y)||^2\right),$$

where $M(y)$ stands for the mean curvature of Y at $y \in Y$ and $||A^*(Y)||^2 = \sum_i (\alpha^*)^2$, $i = 1,...,n-1$.

Notice, that the "new" term $\int_Y ||d\psi(y)||^2 dy$ depends only on the normal field itself, while the R-term depends on the extension of $\psi\nu$ to X, unless

Y is minimal, where [∘∘] *reduces to*

$$[\ast\ast]\qquad \partial^2_{\psi\nu} vol_{n-1}(Y) = \int_Y \|d\psi\|^2 + \frac{1}{2}\left(Sc(Y) - Sc(X) - \|A^*\|^2\right)\psi^2.$$

Furthermore, if Y is volume minimizing in its neighbourhood, then $\partial^2_{\psi\nu} vol_{n-1}(Y) \geq 0$; therefore,

$$[\star\star]\qquad \int_Y (\|d\psi\|^2 + \frac{1}{2}(Sc(Y))\psi^2 \geq \frac{1}{2}\int_Y (Sc(X,y) + \|A^*(Y)\|^2)\psi^2 dy$$

for all non-zero functions $\psi = \psi(y)$.

Then, if we recall that

$$\int_Y \|d\psi\|^2 dy = \int_Y \langle -\Delta\psi, \psi\rangle dy,$$

we will see that [⋆⋆] says that

the operator $\psi \mapsto -\Delta\psi + \frac{1}{2}Sc(Y)\psi$ *is greater than*[6] $\psi \mapsto \frac{1}{2}(Sc(X,y) + \|A^*(Y)\|^2)\psi$.

Consequently,

if $Sc(X) > 0$, *then the operator* $-\Delta + \frac{1}{2}Sc(Y)$ *on* Y *is positive.*

Justification of the $\|d\psi\|^2$ *Term.* Let $X = Y \times \mathbb{R}$ with the product metric and let $Y = Y_0 = Y \times \{0\}$ and $Y_{\varepsilon\psi} \subset X$ be the graph of the function $\varepsilon\psi$ on Y. Then

$$vol_{n-1}(Y_{\varepsilon\psi}) = \int_Y \sqrt{1 + \varepsilon^2\|d\psi\|^2} dy = vol_{n-1}(Y) + \frac{1}{2}\int_Y \varepsilon^2\|d\psi\|^2 + o(\varepsilon^2)$$

by the Pythagorean theorem

and

$$\frac{d^2 vol_{n-1}(Y_{\varepsilon\psi})}{d^2\varepsilon} = \|d\psi\|^2 + o(1).$$

by the binomial formula.

This proves [∘∘] for product manifolds and the general case follows by *linearity/naturality/functoriality* of the formula [∘∘].

Naturality Problem. All "true formulas" in the Riemannin geometry should be derived with minimal, if any, amount of calculation – only on the basis of their "naturality" and/or of their validity in simple examples, where these formulas are obvious.

Unfortunately, this "naturality principle" is absent from the textbooks on differential geometry, but, I guess, it may be found in some algebraic articles (books?).

Exercise. Derive the second main formula (above (IV) by pure thought from its manifestations in the examples in (VI).[7]

[6] $A \geq B$ for selfadjoint operators signifies that $A - B$ is positive semidefinite.
[7] I haven't myself solved this exercise.

2.6 Conformal Modification of Scalar Curvature.

Let (X_0, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and let $\varphi = \varphi(x)$ be a smooth positive function on X.

Then, by a straightforward calculation,[8]

$$Sc(\varphi^2 g_0) = \gamma_n^{-1} \varphi^{-\frac{n+2}{2}} L(\varphi^{\frac{n-2}{2}}),$$

where L is the *conformal Laplace operator* on (X_0, g_0)

$$L(f(x)) = -\Delta f(x) + \gamma_n Sc(g_0, x) f(x)$$

for the ordinary Laplace (Beltrami) $\Delta f = \Delta_{g_0} f = \sum_i \partial_{ii} f$ and $\gamma_n = \frac{n-2}{4(n-1)}$.

Thus, we conclude to the following.

Kazdan-Warner Conformal Change Theorem [KW 1975]. Let $X = (X, g_0)$ be a closed Riemannin manifold, such the the conformal Laplace operator L is positive.

Then X admits a Riemannin metric g (conformal to g_0) for which $Sc(g) > 0$.

Proof. Since L is positive, its first eigenfunction, say $f(x)$ is positive and since $L(f) = \lambda f$, $\lambda > 0$,

$$Sc\left(f^{\frac{4}{n-2}} g_0\right) = \gamma_n^{-1} L(f) f^{-\frac{n+2}{n-2}} = \gamma_n^{-1} f^{\frac{2n}{n-2}} > 0.$$

Example: the Schwarzschild metric. If (X_0, g_0) is the Euclidean 3-space, and $f = f(x)$ is positive function, then

the sign of $Sc(f^4 g_0)$ is equal to that of $-\Delta f$.

In particular, since the function $\frac{1}{r} = (x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}$, is harmonic, *the Schwarzschild metric $g_{Sw} = \left(1 + \frac{1}{r}\right)^4 g_0$ has zero scalar curvature.*

Question. What is the geometric/topological significance of positivity of the operator $-\Delta_X + \gamma Sc(X)$ for particular numbers γ, e.g, for those smaller than the above $\gamma_n = \frac{n-2}{4(n-1)}$?

For instance, do, for a given $\gamma < \gamma_n$, all n-manifolds X admit Riemannin metrics g with positive operators $-\Delta_g + \gamma Sc_g$?

(It is easy to see that all closed n-manifolds, $n \geq 2$, admit Riemannin metrics g with positive operators $-\Delta_g + \gamma Sc_g$ for all $\gamma < \frac{1}{n^{10n}}$.)

2.7 Applications to Minimal Surfaces and Hypersurfaces.

Let X be a three dimensional Riemannin manifold with $Sc(X) > 0$ and $Y \subset X$ be a cooriented surface with minimal area in its homology class.

Then the inequality $(\int Sc \geq 2\partial^2 V)$ from section 2.5, which says in the present case that

$$\int_Y Sc(Y, y) dy > 2\partial_\nu^2 area(Y),$$

implies that

Y must be a topological sphere.

In fact, minimality of Y makes $\partial_\nu^2 area(Y) \geq 0$, hence $\int_Y Sc(Y, y) dy > 0$, and the sphericity of Y follows by the Gauss-Bonnet theorem.

[8] There must be a better argument.

And since all integer homology classes in closed orientable Riemannin 3-manifolds admit area minimizing representatives by the geometric measure theory developed by Federer, Fleming and Almgren, we arrive at the following conclusion.

★₃ *Schoen-Yau Theorem. All integer 2D homology classes in closed Riemannian 3-manifolds with $Sc > 0$ are spherical.*

For instance, the 3-torus admits no metric with $Sc > 0$.

The above argument appears in Schoen-Yau's 15-page paper [SY(incompressible) 1979], most of which is occupied by an independent proof of the existence and regularity of minimal Y.

In fact, the existence of minimal surfaces and their regularity needed for the above argument has been known since late (early?) 60s[9] but, what was, probably, missing prior to the Schoen-Yau paper was the innocuously looking corollary of Gauss' formula in 2.2,

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu, \nu)$$

and the issuing inequality

$$Sc(Y) > -2Ricci(\nu, \nu)$$

for minimal Y in manifolds X with $Sc(X) > 0$.

For example, Burago and Toponogov, come close to the above argument in [BT 1973], where, they bound from below the injectivity radius of Riemannian 3-manifolds X with $sect.curv(X) \leq 1$ and $Ricci(X) \geq \rho > 0$ by

$$inj.rad(X) \geq 6e^{-\frac{6}{\rho}},$$

where this is done by carefully analysing minimal surfaces $Y \subset X$ bounded by, a priori very short, closed geodesics in X, and where an essential step in the proof is the lower bound on the first eigenvalue of Y by $\sqrt{Ricci(X)}$.

Exercises. Let X be homeomorphic to $Y \times S^1$, where Y is a closed orientable surface with the Euler number χ.

(a) Let $\chi > 0$, $Sc(X) \geq 2$ and show that there exists a surface $Y_o \subset X$ homologous to $Y \times \{s_0\}$, such that $area(Y_o) \leq 4\pi$.

(b) Let $\chi < 0$, $Sc(X) \geq -2$ and show that all surfaces $Y_* \in X$ homologous to $Y \times \{s_0\}$ have $area(Y_*) \geq -2\pi\chi$.

(c) Show that (a) remains valid for complete manifolds X homeomorphic to $Y \times \mathbb{R}$.[10]

Schoen-Yau Codimension 1 Descent Theorem. [SY(structure) 1979] . Let X be a compact orientable n-manifold with $Sc > 0$.

If $n \leq 7$, then all integer homology classes $h \in H_{n-1}(X)$ are representable by compact oriented $(n-1)$-submanifolds Y in X, which admit a metrics with $Sc > 0$.

Proof. Let Y be a volume minimizing hypersurface representing h, the existence and regularity of which is guaranteed by [Fed 1970] and recall that by

[9] Regularity of volume minimizing hypersurfaces in manifolds X of dimension $n \leq 7$, as we mentioned earlier, was proved by Herbert Federer in [Fed 1970], by reducing the general case of the problem to that of minimal cones resolved by Jim Simons in [Sim 1968].

[10] I haven't solved this exercise.

$\left[\star\star\right]$ in 2.5 the operator $-\Delta + \frac{1}{2}Sc(Y)$ is positive. Hence, the conformal Laplace operator $-\Delta + \gamma_n Sc(Y)$ is also positive for $\gamma_n = \frac{n-2}{4n-1} \leq \frac{1}{2}$ and the proof follows by Kazdan-Warner conformal change theorem.

$\bigstar_n$ *Corollary.* If a closed orientable n-manifold X admits a map to the torus $\mathbb{T}^n$ with *non-zero degree,* then X admits *no metric with* $Sc > 0$.

Indeed, if a closed submanifold Y^{n-1} is *non-homologous to zero* in this X then it (obviously) admits a map to $\mathbb{T}^{n-1}$ with non-zero degree. Thus, the above allows an inductive reduction of the problem to the case of $n = 2$, where the Gauss-Bonnet theorem applies.

Remarks. (a) The original argument by Schoen and Yau yields the following stronger topological constraints on X.

Call a closed orientable n-manifold *Schoen-Yau-Schick* if it admits a smooth map $f : X \to \mathbb{T}^{n-2}$, such that the homology class of the pullback of a generic point,

$$h = [f^{-1}(t)] \in H_2(X)$$

is *non-spherical,* i.e. it is not in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$.

What the above argument actually shows, is that

$\bigstar\bigstar_n$ *SYS-manifolds of dimensions $n \leq 7$ admit no metrics with $Sc > 0$.*

(b) *Exercise.* Construct examples of SYS manifolds of dimension $n \geq 4$, where all maps $X \to \mathbb{T}^n$ have zero degrees.

Hint: apply surgery to $\mathbb{T}^n$.

(c) The limitation $n \leq 7$ of the above argument is due a presence of singularities of minimal subvarieties in X for $dim(X) \geq 8$.

If $n = 8$, these singularities were proven to be unstable, (see [Smale 1993] and section 5.2), which improve $n \leq 7$ to $n \leq 8$ in $\bigstar\bigstar_n$

More recently, the dimension restriction was fully removed in [SY(singularities) 2017] and in [Loh(smoothing) 2018]; the arguments in both papers are difficult and I have not mastered them.

On the other hand, there are several short and technically simple (modulo standard index theorems) proofs of $\bigstar_n$ (but not of $\bigstar\bigstar_n$) for *spin* [11] manifolds X, e.g. for X homeomorphic to $\mathbb{T}^n$. (see section 3.2).

Also notice, that besides being short, the Dirac operator arguments deliver in some cases obstructions to $Sc > 0$ that lie fully beyond the range of the minimal surface techniques. For instance (see [G (positive) 1996] and [G(inequalities), 2018])

$\otimes$ *if a closed orientable manifold of dimension $dim(X) = 2k$ carries a closed 2-form ω (e.g. a symplectic one), such that $\int_X \omega^k \neq 0$, and if the universal cover $\tilde{X}$ is contractible,* [12] *then X admits no metric with $Sc > 0$.*

(This applies, for instance, to even dimensional tori and to aspherical 4-manifolds with $H^2(X, \mathbb{R}) \neq 0$.)

[11] A smooth connected n-manifolds X is spin if the frame bundle over X admits a double cover extending the natural double cover of a fiber, where such a fiber is equal to the linear group, (each of the two connected components of) which admits a a unique non-trivial double cover $\tilde{GL}(n) \to GL(n)$. For instance, all manifolds X with $H^2(X; \mathbb{Z}_2) = 0$ are spin.

[12] It's enough to have $\tilde{X}$ spin and the lift $\tilde{\omega}$ to $\tilde{X}$ exact.

3 Topics, Results, Problems

We present in this section a (very) brief overview of what is known and what is unknown about scalar curvature, where we illustrate general results by their simplest instances. The general formulations and the proofs will appear in the sections to come.

3.1 Closure and Density Theorems

Let X be a smooth Riemannian manifold, let $G = G^2(X)$ the space of C^2-smooth Riemannin metrics g on X and let $G_{Sc\geq\sigma} \subset G$ and $G_{Sc\leq\sigma} \subset G$, $-\infty < \sigma < \infty$, be the subsets of metrics g with $Sc(g) \geq \sigma$ and with $Sc(g) \leq \sigma$ respectively.

Then:

A: $\lim_{Sc\geq\sigma}$. The subset $G_{Sc\geq\sigma} \subset G$ is closed in G with respect to C^0-topology: *uniform limits $g = \lim g_i$ of metric g_i with $Sc(g_i) \geq \sigma$ have $Sc \geq \sigma$, provided these g are C^2-smooth in order to have their scalar curvature defined.*

B: $\lim_{Sc\leq\sigma}$.The subset $G_{Sc\leq\sigma} \subset G$ is dense in G with respect to C^0-topology. Moreover, *all $g \in G$ admit fine (which is stronger than uniform for non-compact X) approximations by metrics with scalar curvatures $\leq \sigma$.*

There are *two proofs* of A. The first one in [G(billiards) 2014] depends on *non-existence of metrics with $Sc > 0$ on tori* and the second one in [Bamler 2016] uses *Ricci flow.*

The proof of B is achieved by a (more or less) direct and elaborate geometric construction in [Lohkamp 1994], where it is, in fact, shown that the metics with *Ricci* < 0 are C^0-dense as well.

Observe that if contrary to A the space of metrics with $Sc \geq 0$ were dense, there would be no hope for a non-trivial geometry of such metrics, while A leads us to the following.

Problem. Study *continuous* Riemannian metrics which are C^0-limits of smooth g_i, such that $\liminf_{i\to\infty} Sc(g_i) \geq 0$.

Notice that the experience with a similar problem concerning *C^0-limits of symplectic diffeomorphisms* offers little expectations on geometry of such limits, but *stability* (see below) of basic geometric inequalities with $Sc \geq 0$ (e.g. as indicated in the next section below) points toward a more optimistic solution.)

3.2 $\mathbb{T}^n_{Sc\not>0}$: No Metrics with $Sc > 0$ on Tori

We have already explained (see section 2.7) Schoen-Yau's proof from [SY(structure) 1979] by an *inductive descent argument with minimal hypersurfaces* of the fact that

The tori $\mathbb{T}^n$, $n \leq 7$, admit no metrics with $Sc > 0$,

Schoen and Yau also show that

Riemannin metrics on these tori with $Sc \geq 0$ are Riemannin flat: the universal coverings of these tori are isometric to $\mathbb{R}^n$. (We shall explain this in section 5.8)

And as we mentioned earlier, the condition $n \geq 7$ was removed in the difficult papers [SY(singularities) 2017] and [Loh(smoothing) 2018].

An alternative proof of $\mathbb{T}^n_{Sc \ngtr 0}$, albeit very short and simple but lacking the geometric transparency of the Schoen-Yau argument, was given in [GL(fundamental group) 1980] for all n with a use of *twisted Dirac operators*[13] $\mathcal{D}$ *on* $\mathbb{T}^n$.

At the present moment there are (at least) five such proofs which rely on different versions of the *Atiyah-Singer index theorem* which guarantees the existence of *non-zero harmonic* representatives in various spaces of sections of *twisted spinors* on $\mathbb{T}^n$ (or on $\mathbb{R}^n$ which cover $\mathbb{T}^n$) with *arbitrary* metrics .

Then non-existence of a metric on $\mathbb{T}^n$ with $Sc > 0$ (eventually) follows from *Schroedinger-Lichnerowicz-Weitzenboeck* algebraic identity

$$\mathcal{D}^2 = \nabla^2 + \frac{1}{4} Sc$$

for a positive (*coarse Bochner Laplace*) operator ∇^2, [14]

that implies that *no non-zero harmonic spinor exists if* $Sc > 0$. (see §4 for more about it).

Stability Problem for $[Sc \geq -\varepsilon]$. Let a metric g on the torus have $Sc(g) > -\varepsilon$. Find additional conditions on g that would make it close to a flat metric.

The simplest expected result of this kind would be as follows:

if a sequence of smooth metrics g_i with $Sc(g_i) \geq -\varepsilon_i \underset{i\to\infty}{\to} 0$ uniformly converges to a continuous metric g, then this g is Riemannian flat.

(See section 5.8 for a possible approach to the proof of this)

3.3 Asymptotically Flat Spaces with $Sc \geq 0$

It was conjectured by Geroch for $n = 3$ [Ger 1975] that

The Euclidean metric on $\mathbb{R}^n$ admits no compactly supported perturbations with increase of the scalar curvature. Moreover,

If a metric g *on* $\mathbb{R}^n$ *with* $Sc(g \geq 0$ *is equal to* g_{Eucl} *outside a compact subset in* $\mathbb{R}^n$*, then* $(\mathbb{R}^n, g)$ *is isometric to* $(\mathbb{R}^n, g_{Eucl})$.

This, of course, trivially follows from the above $\mathbb{T}^n_{Sc \ngtr 0}$, since compactly supported perturbations of the flat metric on $\mathbb{R}^n$ yields similar perturbations of flat metrics on tori.

In fact, a more general version of this was originally proven by Schoen and Yau in [SY(positive mass) 1979] for a class of metrics g on 3-manifolds asymptotic to g_{Eu} under the name of *positive mass/energy theorem* (see sections 3.13) with a use of minimal surfaces.

Then Witten in [Witten 1981] (also see [Bartnik 1986]) suggested a proof with a use of a perturbation argument in the space of invariant (non-twisted) harmonic spinors on $\mathbb{R}^n$.

Later, Lohkamp [Loh(hammocks) 1999] found a (relatively) simple reduction of the general, and technically more challenging, case of the positive mass theorem to that of compactly supported perturbations, that in turn, (trivially) reduces to :$\mathbb{T}^n_{Sc \ngtr 0}$.

[13] The "untwisted" Dirac operator acts on the spin bundle $\mathbb{S}(X)$ and a "twisted" one operates on the tensor product of $\mathbb{S}(X)$ with some vector bundle L over X.

[14] Here and everywhere in our lectures, ∇^2 is an abbreviation for $\nabla\nabla^* = -\sum_i \nabla_i\nabla_i$, where ∇ is covariant differentiation operator in a Euclidean vector bundle with an orthogonal connection and where positivity of ∇^2 is seen via the relation $\int\langle\nabla^2\psi,\psi\rangle = \int ||\nabla\psi||^2$ for the sections ψ of our bundle.

Also notice that the *doubling property* for *mean convex manifolds* with boundaries (see [GL(fundamental group) 1980]) allows a reduction of the Geroch Conjecture and of similar more general results to the *Goette-Semmelmann theorem* [GS 2002] concerning *extremality/rigidity of the metrics g with positive curvature operators* (see $[X^{\to\bigcirc}]$ in section 3.5 below).

Problems. What are other (homogeneous?) Riemannian spaces that admit no (somehow) localised deformations with increase of the scalar curvatures?

What are most general asymptotic (or boundary) conditions on such deformations that would allow their localization?

Here is a definite result along these lines due to Michael Eichmair, Pengzi Miao and Xiadong Wang, [EMW 2009] generalizing an earlier result by Yuguang Shi and Luen-Fai Tam[ST 2002]

STEMW Rigidity Theorem. Let $\underline{X} \subset \mathbb{R}^n$ be a star convex domain, e.g. a convex one, such as the unit ball, for example, and let X be a compact Riemannin manifold the boundary $Y = \partial X$ of which is isometric to the boundary $\underline{Y} = \partial \underline{X}$.

If $Sc(X) \geq 0$ and if the total scalar curvature of Y is bounded from below by that of $\underline{Y}$,

$$\int_Y mean.curv(Y, y)dy \geq \int_{\underline{Y}} mean.curv(\underline{Y}, \underline{y})d\underline{y},$$

then X is isometric to $\underline{X}$.

Remark. Originally, this was proven for $n \leq 7$ but this restriction can be now removed in view of [SY(singularities) 2017] and/or of [Loh(smoothing) 2018].

Conjecture. Let X be a compact Riemannin manifold with $Sc \geq \sigma$. Then the integral mean curvature of the boundary $Y = \partial X$ is bounded by

$$\int_Y mean.curv(Y, y)dy \leq const,$$

where this $const$ depends on σ and on the (intrinsic) Riemannian metric on Y induced from that of $X \supset Y$.

(See section 3.6 for description of some results in this direction.)

3.4 Simply Connected Manifolds with and without $Sc > 0$

As we already stated earlier, according to Lichnerowicz [Lich 1963], the *Atiyah-Singer index theorem* for the Dirac operator $\mathcal{D}$ and the identity $\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$, imply that

there are smooth *closed simply connected* manifolds X of all dimensions $n = 4k$, $k > 0$, that admit no metrics with $Sc > 0$.

The simplest example of these for $n = 4$ is *the Kummer surface* given by the equation

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$$

in the complex projective space $\mathbb{C}P^3$.

Also by Lichnerowicz' theorem, other complex surfaces of *even degrees* $d \geq 4$ as well as their Cartesian products, e.g $X_{Ku} \times ... \times X_{Ku}$ admit no metrics with $Sc > 0$.

A decade later, using a more general index theorem by Atiyah and Singer, Hitchin [Hit 1974] pointed out that *there exist manifolds Σ homeomorphic (but no diffeomorphic!) to the spheres S^n, for all $n = 8k+1, 8k+2$, $k = 1, 2, 3...$, which admit no metrics with $Sc > 0$.*

Notice that, by *Yau's solution of the Calabi conjecture,* the Kummer surface admits a metric with $Sc = 0$, even with $Ricci = 0$, but, probably, (I guess this must be known) there is no metrics with $Sc = 0$ on these Σ.

The actual Lichnerowicz-Hitchin theorem says that if a certain topological invariant $\hat{\alpha}(X)$ *doesn't vanish*, then X *admits no metric with* $Sc > 0$, since, by the Atiyah and Singer index formulae,[15]

$$\hat{\alpha}(X) \neq 0 \Rightarrow Ind(\mathcal{D}_{|X}) \neq 0 \Rightarrow \exists \text{ harmonic spinor} \neq 0 \text{ on } X.$$

Conversely,

if X is a *simply connected* manifold of *dimension* $n \neq 4$, and if $\hat{\alpha}(X) = 0$ then it admits a metric with positive scalar curvature [GL(classification) 1980], [Stolz 1992].

Thus, for instance

all simply connected manifolds of dimension $n \neq 0, 1, 2, 4 \mod 8$ *admit metrics with* $Sc > 0$, since $\hat{\alpha}(X) = 0$ is known to vanish for these n.

A Few Words on $n = 4$ and on $\pi_1 \neq 0$. (See sections 3.11, 3.14) more about it.) If $n = 4$ then, besides vanishing of the $\hat{\alpha}$-invariant (which is equal to a non-zero multiple of first Pontryagin number for $n = 4$), positivity of the scalar curvature also implies the vanishing of the *Seiberg-Witten invariants* (See lecture notes by Dietmar Salamon, [Sal 1999]; also we say more about it in section 3.14).

If X is a closed spin manifold of dimension $n \geq 5$ with the fundamental group $\pi_1(X) = \Pi$, then

the existence/non-existence of a metric g on X with $Sc(g) > 0$ is an invariant of the spin bordism class $[X]_{sp} \in bord_{sp}(\mathsf{B}\Pi)$ in the classifying space $\mathsf{B}\Pi$,

where, recall, that (by definition of "classifying") the universal covering of $\mathsf{B}\Pi$ is contractible and $\pi_1(\mathsf{B}\Pi) = \Pi$. (See lecture notes [Stolz(survey) 2001].)

There is an avalanche of papers, most of them coming under the heading of "Novikov Conjecture", with various criteria for the class $[X]_{sp}$, and/or for the corresponding homology class $[X] \in H_n$ $(\mathsf{B}\Pi)$ (not) to admit g with $Sc(g) > 0$ on manifolds in this class, where these criteria usually (always?) linked to generalized index theorems for twisted Dirac operators on X with several levels of sophistication in arranging this "twisting". Yet, despite a significant progress in this direction, the following remains unsettled for $n \geq 4$.

Conjecture. No closed *aspherical*[16] manifold X admits a metric with $Sc > 0$.

Moreover,

if a closed oriented n-manifold X admits a continuous map to an aspherical space, that is $\mathsf{B}\Pi$ for some group Π, such that the image of the rational fundamental homology class of $[X]_{\mathbb{Q}}$ in the rational homology[17] homology $(\mathsf{B}\Pi; \mathbb{Q})$ doesn't vanish, then X admits no metic g with $Sc(g) > 0$.

(We shall briefly describe the status of this conjecture in section 3.11.)

[15]The Dirac operator is defined only on *spin* manifolds and to avoid entering into this at the present moment we postulate $\hat{\alpha}(X) = 0$ for non-spin manifolds X.

[16]*Aspherical* means that the universal covering is contractible.

[17]Bernhard Hanke pointed out to me that the role of homology with finite coefficients in prohibiting $Sc > 0$, especially for finite groups Π, remains obscure even on the level of conjectures.

3.5 Bounds on Size, Extremality, Rigidity

The inequality $Sc(X) \geq \sigma > 0$, as it becomes a *positive curvature* condition, imposes an *upper bound* on the size of X, where an instance of this can be expressed in terms of the *hyperspherical radius* $Rad_{S^n}(X)$, defined for *closed* Riemannian n-manifolds X as

the supremum of the radii $R > 0$ of n-spheres, such that X admits a non-contractible 1-Lipschitz, i.e. distance non-increasing, map $f : X \to S^n(R)$.

The existence of a non-trivial such bound,

$$Rad_{S^n}(X) \leq \frac{const_n}{\sqrt{\sigma}}, \ \ \sigma = \inf_{x \in X} Sc(X, x),$$

for orientable spin[18] manifolds X of even dimensions n [19] follows by confronting the index theorem with a "twisted version" of the formula $\mathcal{D}^2 = \nabla^2 + \frac{1}{4} Sc$ for the Dirac operator on X twisted with the f-pullback of a suitable vector bundle L over S^n [GL(fundamental) 1980], where

the optimal constant $const_n = \sqrt{n(n-1)} = \sqrt{Sc(S^n)}$ is achieved with L being the (complexified) positive spin bundle over S^n, (see [Llarull 1998] and section 4.2)

This sharp inequality, says, in particular, that one can't enlarge the spherical metric g_{sphr} on S^n without making the scalar curvature smaller at some point. That is if a metric g on S^n satisfies

$$g \geq g_{sphr} \text{ and } Sc(g) \geq n(n-1) = Sc(g_{sphr})$$

then, necessarily, $Sc(g) = n(n-1)$, which we express by saying that *spheres are extremal.*

In fact, Llarull's argument (we say a few words about it in section 4.2) shows that spheres are *rigid*:

$[g \geq g_{sph}] \& [Sc(g) \geq Sc(g_{sph})]$ implies that $g = g_{sph}$.

This extremality/rigidity property of spheres was generalised by Goette and Semmelmannto manifolds $\underline{X}$ with *positive curvature operators*, where the examples of such manifolds we are concerned with now are smooth *locally convex hypersurfaces* in Riemannin flat $(n+1)$-manifolds, e.g. products of convex hypersurfaces in $\mathbb{R}^{m+1}$ by the flat tori $\mathbb{T}^{n-m}$.

The (proof of the) main result in [GS 2002] implies in this case the following theorem.

$[X^{\to\bigcirc}]$ Let X be a connected orientable Riemannian n-manifold, let $\underline{X} \subset \mathbb{R}^{n+1}$ be a smooth closed locally convex hypersurface in a Riemannin flat $(n+1)$-manifold and let $f : X \to \underline{X}$ be a smooth map.

[18]All surfaces are spin and an orientable manifold X of dimension $n \geq 3$ is spin if and only if the restriction of the tangent bundle $T(X)$ to all surfaces $Y^2 \subset X$ are trivial, e.g. if $H^2(X; \mathbb{Z}_2) = 0$. The simplest examples or spin n-manifolds are smooth hypersurfaces in $\mathbb{R}^{n+1}$, such as product of spheres.

More interesting in this respect are complex projective spaces $\mathbb{C}P^m$ and smooth complex hypersurfaces $X \subset \mathbb{C}P^m$ of degree d: these X are spin if and only if $m + d$ is odd, as it the case for the Kummer surface, for instance.

[19]A trivial (and ungraceful) reduction to the even dimensional one follows taking X times the circle, but there is a better way of doing it.

Let the norm of the differential of f and the scalar curvatures of X and $\underline{X}$ be related by the inequality

$$Sc(X,x) \geq Sc(\underline{X}, f(x)) \cdot ||df(x)||^2, \; x \in X.$$

If X is orientable and the degree of f is non-zero, then, provided X is spin, this inequality becomes an equality:

$$Sc(X,x) = Sc(\underline{X}, f(x)) \cdot ||df(x)||^2,$$

at all points $x \in X$.

Notice that the above Llarull's theorem as well as non-existence of metrics with $Sc > 0$ on tori are special cases of $[X^{\to\bigcirc}]$.

Problem. What are further examples of extremal/rigid manifolds $\underline{X}$ with $Sc(\underline{X}) > 0$? (We shall meet a few later on.)

Do all closed manifolds which admit metrics with $Ricci > 0$ admit extremal/rigid metrics with $Sc > 0$?

3.6 Hypersurfaces with Large Mean Curvatures

Let $Y \subset \mathbb{R}^n$ be a smooth closed hypersurface with the mean curvature bounded from below by $\mu > 0$.

Then the hyperspherical radius of Y is bounded by

$$Rad_{S^{n-1}}(Y) \leq \frac{1}{n-1}.$$

Moreover, if $Rad_{S^{n-1}}(Y) = \frac{1}{n-1}$, then $mean.curv(Y) = n-1$,
which, by a theorem of A.D. Alexandrov, implies that

Y equals to the unit sphere $S^{n-1}_x \subset \mathbb{R}^n$ around some point $x \in \mathbb{R}^n$.

This is shown (see section 4.3) by applying $[X^{\to\bigcirc}]$, to a *smoothed double* $\mathbb{D}_\varepsilon(X)$ defined as follows.

Let

$$X_{1/2} \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$$

be the (closed) domain in $\subset \mathbb{R}^n$ bounded by Y and let $X_\varepsilon = \mathbb{D}_\varepsilon(X) \subset \mathbb{R}^{n+1}$ be a (more or less) naturally/canonically C^2-smoothed boundary of the ε-neighbourhood (which is only C^1-smooth) of $X_{1/2} \subset \mathbb{R}^{n+1}$.

Then let $\underline{X}_{1/2} \subset \mathbb{R}^{n+1}$ be the unit n-ball $B^n \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$ and let, accordingly, $\mathbb{D}_\varepsilon(\underline{X}) = \underline{X}_\varepsilon \subset \mathbb{R}^{n+1}$ be a (more or less) naturally/canonically C^2-smoothed boundary of its ε-neighbourhood.

Then maps $f : Y \to \underline{S}^{n-1}$ define maps

$$F_\varepsilon : X_\varepsilon \to \underline{X}_\varepsilon,$$

to which $[X^{\to\bigcirc}]$ applies and, when $\varepsilon \to 0$, it yields the inequality
$Rad_{S^{n-1}}(Y) \leq \frac{1}{n-1}$. (See [G(boundary) 2019] and section 4.3.)

Questions. Is there a direct proof of this inequality?

What exactly happens in the limit when $\varepsilon \to 0$ to the Dirac operator used in the proof of $[X^{\to\bigcirc}]$?

Exercise + Problem. Let $Y_0 \subset \mathbb{R}^n$ be a smooth compact cooriented submanifold with boundary $Z = \partial Y_0$.

If the mean curvature of Y_0 with respect to its coorientation satisfies

$$mean.curv(Y) \geq n-1 = mean.curv(S^{n-1}),$$

then every distance decreasing map

$$f : Z \to S^{n-2} \subset \mathbb{R}^{n-1}$$

is contractible, where "distance decreasing"refers to the distance functions on $Z \subset \mathbb{R}^n$ and on $S^{n-2} \subset \mathbb{R}^{n-1}$ coming from the ambient Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$.

Hint. Observe that the maximum of the principal curvatures of Y_0 is ≥ 1 and show that the filling radius of $Z \subset \mathbb{R}^n$ is ≤ 1.

Question. Does contractibility of f remains valid if the distance decreasing property of f is defined with the (intrinsic) spherical distance in S^{n-2} and with the distance in $Z \subset Y_0$ associated with the *intrinsic metric* in $Y_0 \supset Z$, where $dist_{Y_0}(y_1, y_2)$ is defined as the infimum of length of curves in Y_0 between y_1 and y_2?

Bringing Scalar Curvature into the Open. Our proof of the inequality

$$\inf_{y \in Y} mean.curv(Y, y) \leq \frac{1}{Rad_{S^{n-1}}(Y)}$$

applies not only to hypersurfaces in $\mathbb{R}^n$ but to

the boundaries $Y = \partial X$ of all compact Riemannin spin manifolds X with $Sc(X) \geq 0$.

This, suggests the following version of the conjecture following STEMW Rigidity Theorem in section 3.3.

Let the above $Y = \partial X$ be λ-bi-Lipschitz homeomorphic to the unit sphere S^n. Then, conjecturally,

$$\int_Y mean.curv(Y, y) dy \leq C(\lambda)(n-1) vol(S^n),$$

where – this might follows from the STEMW proof – $C(\lambda) \to 1$ for $\lambda \to 1$.

3.7 Widths of Riemannian Bands X with $Sc(X) \geq Sc(S^n)$

Bands, sometime we call them *capacitors*, are manifolds X with two distinguished disjoint non-empty subsets in the boundary $\partial(X)$, denoted

$$\partial_- = \partial_- X \subset \partial X \text{ and } \partial_+ = \partial_+ X \subset \partial X.$$

A band is called *proper* if $\partial_\pm$ are unions of connected components of ∂X and

$$\partial_- \cup \partial_+ = \partial X.$$

The basic instance of such a band is the segment $[-1, 1]$, where $\pm\partial = \{\pm 1\}$.

Furthermore, *cylinders* $X = X_0 \times [-1, 1]$ are also bands with $\pm\partial = X_0 \times \{\pm 1\}$, where such a band is proper if X_0 has no boundary.

Riemannian bands are those endowed with Riemannin metrics and

the width of a Riemannin band $X = (X, \partial_\pm)$ is defined as

$$width(X) = dist(\partial_-, \partial_+),$$

where this distance is understood as the infimum of length of curves in V between ∂_- and ∂_+.

We are concerned at this point with *proper compact Riemannin bands X of dimension n*, such that

no closed hypersurface $Y \subset X$, which separates ∂_- from ∂_+, admits a metric with strictly positive scalar curvature.

Simplest Examples of such bands are (we prove this in section 5.3)

$\bullet_{\mathbb{T}^{n-1}}$ *toric bands* which are homeomorphic to $X = \mathbb{T}^{n-1} \times [-1,1]$;

$\bullet_{\hat\alpha}$ these, called $\hat\alpha$ bands, are diffeomorphic to $Y_{-1} \times \times[-1,1]$, where the Y_{-1} is a closed spin $(n-1)$-manifold with non-vanishing $\hat\alpha$-invariant (see the IV above);

$\bullet_{\mathbb{T}^{n-1}\times\hat\alpha}$ these are bands *diffeomorphic to products* $X_{n-k}\times\mathbb{T}^k$, where $\hat\alpha(X_{n-k}) \neq 0$.

$\frac{2\pi}{n}$*-Inequality.* Let X be a *proper compact Riemannin bands X of dimension n with $Sc(X) \geq n(n-1) = Sc(S^n)$.*

If no closed hypersurface in X which separates ∂_- from ∂_+ admits a metric with positive scalar curvature, then

$$\left[\circledcirc_\pm \leq \tfrac{2\pi}{n}\right] \qquad width(V) \leq \frac{2\pi}{n}.$$

Moreover, the equality holds only for warped products $X = Y \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$[20] *with metrics $\varphi^2 h + dt^2$, where the metric h on Y has $Sc(h) = 0$ and where*

$$\varphi(t) = \exp\int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

as in VI of section 2.

Corollary. Let Y be a closed manifold of dimension $\neq 4$ (see 3.15 below about $n = 4$). Then the following three conditions are equivalent.

1: *the open cylinder $Y \times \mathbb{R}$ admits a complete metric g_1 with uniformly positive scalar curvature, i.e. with $\inf_{x\in X} Sc(g,x) > 0$;*

2: the open cylinder $Y \times \mathbb{R}$ admits a complete metric g_2 with positive scalar curvature which decays subquadratically:

$$\liminf_{x\to\infty} Sc(g_2, x) \cdot dist(x, x_0)^2 = \infty.$$

3: *the closed cylinder $Y \times [-1,1]$ admits a metric g_3 with $Sc(g_2) \geq n(n-1)$ and such that*

$$dist_{g_3}(Y \times \{-1\}, Y \times \{1\}) \geq \frac{2\pi}{n}.$$

[20]Here, since X is non-compact, the width is understood as the distance between the two ends of X.

Two words about the proof(s). There are two somewhat different proofs of $[\circledcirc_{\pm} \leq \frac{2\pi}{n}]$ which use the calculus of variation but advance along slightly different routes.

The first route follows an inductive descent with minimal hypersurfaces á la Schoen-Yau adapted to manifolds with boundaries similarly to that in [GL 1983]. This applies only to the toric and to similar bands, but not to $\hat{\alpha}$-bands. (See [G(inequalities) 2018].)

The second route proceeds with a use of *stable μ-bubbles* which are closed hypersurfaces in X with (prescribed) mean curvature μ, where $\mu = \mu(x)$ is a signed measure on X as in §$5\frac{5}{6}$ of [G(positive) 1996]. (See section 5)

This applies to all bands and it also improves certain results from [G(inequalities) 2018] obtained with the first proof.

Both proof, when it comes to $dim(X) = n \geq 9$ have to face the problem of possibly) stable singularities of minimal (and minimal-like) hypersurface in X.

I feel more comfortable in this respect with the first proof, where a direct application of theorem 4.6 from the recent Schoen-Yau paper [SY(singularities) 2017], (also see [Sch 2017]) is possible.

And as far the second proof for $n \geq 9$ is concerned, the argument from [Loh(smoothing) 2018] seems to be applicable to our case, but this seems harder than the analysis in [SY(singularities) 2017] (which, honestly, I haven't carefully studied, either).

3.8 Bound on Widths of Riemannian Cubes

Let g be a Riemannin metric on the cube $X = [-1,1]^n$ and let d_i, $i = 1,2,...,n$, denote the g-distances between the pairs of the opposite faces denoted $\partial_{i\pm} = \partial_{i\pm}(X)$ in this cube X, that are the length of the shortest curves between ∂_{i-} and ∂_{i+} in X.

$\square^n$*-Inequality. If $Sc(g) \geq n(n-1) = Sc(S^n)$, then*

$$\square_{\Sigma} \qquad \sum_{i=1}^{n} \frac{1}{d_i^2} \geq \frac{n^2}{4\pi^2}$$

In particular,

$$\square_{\min} \qquad \min_i dist(\partial_{i-}, \partial_{i+}) \leq \frac{2\pi}{\sqrt{n}}.$$

About the Proof. On the surface of things, this inequality is purely geometric with no topological strings attached. But in truth, the combinatorics of the cube fully reflects toric topology in it.

The proof of $\square_{\Sigma}$ indicated in section 5.4 proceeds along the above *second route* which, in fact, applies to more general "cube-like" manifolds X, such as $Y_{-m} \times [-1,1]^{n-m}$ and yields inequalities mediating between the above $\left[\circledcirc_{\pm} \leq \frac{2\pi}{n}\right]$ and $\square_{\Sigma}$.

But the proof of $\square_{\Sigma}$ as it stands for $m = n$ is also possible closely following t *the first route*, where the argument from [SY(singularities)] seems easily adaptable.

This makes the proof of $\square_{\Sigma}$ for $n \geq 9$ more tractable.

Corollary. Let X be a Riemannin manifold with $Sc(X) \geq n(n-1) = Sc(S^n)$, which admits a λ_n-Lipschitz[21] homeomorphism onto the hemisphere S^n_+,

$$f : X \to S^n_+.$$

Then

$$\lambda_n \geq \frac{\arcsin \beta_n}{\pi \beta_n} > \frac{1}{\pi} \text{ for } \beta_n = \frac{1}{\sqrt{n}}.$$

Proof. The hemisphere S^n_+ admits an obvious cubic decomposition with the (geodesic) edge length $2 \arcsin \frac{1}{\sqrt{n}}$ and $\square_{\min}$ applies to the pairs of the f-pullbacks of the faces of this decomposition.

Remarks. (a) This lower bound on λ_n improves those in §12 of [GL 1983] and in §3 of [G(inequalities) 2018].

Moreover the *sharp* inequality for Lipschitz maps to the punctured sphere stated in the next section implies that $\lambda_n \geq \frac{1}{2}$ for all n.

But it remains *problematic* if, in fact, $\lambda \geq 1$.

Exercise. Show that $\lambda_2 \geq 1$.

(b) The proof of the inequality $\square_\Sigma$ in section 5.4 applies to *proper* ((boundary→ boundary) *λ-Lipschitz* maps with *non-zero degrees* from all compact connected orientable manifolds X to S^n_+, while the proof via punctured spheres needs X to be spin.

Additional Exercises. (i) Show that the Riemannin metrics with sectional curvatures ≥ 1 on the square $[-1,1]^2$ satisfy

$\square^2_{\min}$.
$$\min_{i=1,2} dist(\partial_{i-}, \partial_{i+}) \leq \pi.$$

(ii) Construct iterated warped product metrics g_n on the n-cubes $[-1,1]^n$ with $Sc(g_n) = n(n-1)$, where, for $n = 2$, both d_i, $i = 1,2$, are equal to π and such that

$$d_i > 2 \arcsin \frac{1}{\sqrt{n}}, \;\; i = 1, ..., n, \;\; \text{for all } n = 3, 4, ..., \;.$$

(iii) Show, that $\square_{\min}$ is equivalent to the *over-torical* case of $\frac{2\pi}{n}$*-Inequality.* modulo constants. Namely,

A. If a Riemannin n-cube X has $\min_i dist(\partial_{i-}, \partial_{i+}) \geq d$, then it contains an n-dimensional Riemannin band $X_\circ \subset X$, where $dist(\partial_- X_\circ, \partial_+ X_\circ) \geq \varepsilon_n \cdot d$, $\varepsilon_n > 0$, and where $X_\circ$ admits a continuous map to the $(n-1)$ torus, $f_\circ : X_\circ \to \mathbb{T}^{n-1}$, such that all closed hypersurfaces $Y_\circ \subset X_\circ$ which separate $\partial_- X_\circ$ from $\partial_+ X_\circ$ are sent by $f_\circ$ to $\mathbb{T}^{n-1}$ with *non-zero degrees.*

B. Conversely, let X_o be a band, where $dist(\partial_- X_o, \partial_+ X_o) \geq d)$ and which admits a continuous map to the $(n-1)$ torus, such that the hypersurfaces $Y_o \subset X_o$, which separate $\partial_- X_o$ from $\partial_- X_o$, are sent to this torus with *non-zero degrees.*

Then there is a (finite if you wish) covering $\tilde{X}_o$ of X_o, which contains a domain $X_\square \subset \tilde{X}_o$, where this domain admits a continuous proper map of degree one onto the d-cube $f_\square : X_\square \to (0,d)^n$, such that the n coordinate projections of this map, $(f_\square)_i : X_\square \to (0,d)$, are distance decreasing.

[21] A map f between metric spaces is λ-Lipschitz if $dist(f(x)f(y)) \leq dist(x,y)$.

3.9 Extremality of Punctured Spheres

Let $(\underline{X}, \underline{g})$ be the unit sphere S^n minus two opposite points with the spherical Riemannin metric $\underline{g} = g_{sphe}$.

If a smooth metric g on X satisfies

$$g \geq \underline{g} \text{ and } Sc(g) \geq n(n-1) = Sc(\underline{g}),$$

then $g = \underline{g}$.

About the Proof. By following the above second route, one can reduce this to (a version of) Llarull's theorem (see section 5.5, where again I can fully vouch only for $n \leq 8$.

Remark. It follows by Llarull's argument *for all n* that

no *complete* metric g on the n-sphere minus a finite subset Σ can satisfy the inequalities $g \geq \underline{g}$ and $Sc(g) \geq n(n-1)$.

But this is *unknown* if one makes *no completeness assumption*, except for the empty Σ, a single point or a pair of opposite points.

Exercise. Prove with the above that no metric g on the hemisphere $(S^n_+, \underline{g})$ can satisfy the inequalities $g \geq 4\underline{g}$ and $Sc(g) > n(n-1)$. Then directly show that if $n = 2$ then the inequality $g \geq \underline{g}$ and $Sc(g) \geq 2$ imply that $g = \underline{g}$.

Question. Does the implication

$$[g \geq \underline{g}] \& [Sc(g) \geq n(n-1)] \Rightarrow g = \underline{g}$$

ever hold for $S^n \setminus \Sigma$ apart from the above cases?

3.10 Manifolds with Negative Scalar Curvature Bounded from Below

If a "topologically complicated" closed Riemannin manifolds X, e.g. an aspherical one with a *hyperbolic fundamental group*, has $Sc(X) \geq \sigma$ for $\sigma < 0$, then a certain "growth" of the universal covering $\tilde{X}$ of X is expected to be *bounded from above* by $const\sqrt{-\sigma}$ and accordingly, the "geometric size" – ideally $\sqrt[n]{vol(X)}$– must be *bounded from below* by $const'/\sqrt{-\sigma}$.

If $n = 3$ this kind of lower bound are easily available for areas of stable minimal surfaces of large genera via Gauss Bonnet theorem by the Schoen-Yau argument from [SY(incompressible) 1979].

Also Perelman's proof of the geometrization conjecture delivers a sharp bound of this kind for manifolds X with hyperbolic $\pi_1(X)$ and similar results for $n = 4$ are possible with the Seiberg-Witten theory for $n = 4$ (see section 3.15).

No such estimate has been established yet for $n \geq 5$ but the following results are available.

Min-Oo Hyperbolic Rigidity Theorem [Min(hyperbolic) 1989]. Let X be a complete Riemannin manifold, which is isometric at infinity (i.e. outside a compact subset in X) to the hyperbolic space $\mathbf{H}^n_{-1}$.

If $Sc(X) \geq -n(n-1) = Sc(\mathbf{H}^n_{-1})$, then X is isometric to $\mathbf{H}^n_{-1}$.

About the Proof. The original argument by Min-Oo, which generalizes Dirac-theoretic Witten's proof of the positive mass/energy theorem for asymptotically Euclidean (rather than hyperbolic) spaces, needs X to be *spin*.

But granted spin, Min-Oo's proof allows more general asymptotic (in some sense) agreement between X and $\mathbf{H}^n_{-1}$ at infinity.

In order to get rid of spin, one may use here either minimal hypersurfaces with boundaries (as in [G(inequalities) 2018]) or stable μ-bubbles (as in [G(positive) 1996]).

To accomplish this it is convenient, here as in the flat case, to pass to a quotient space $\mathbf{H}^n_{-1}/\Gamma$, where, instead of letting $\Gamma = \mathbb{Z}^n$ that allows a reduction of the rigidity of $\mathbb{R}^n$ to that of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, one takes a *parabolic* isometry group isomorphic to $\mathbb{Z}^{n-1}$ for Γ, for which the quotient $\mathbf{H}^n_{-1}/\Gamma$ is the *hyperbolic cusp-space*, that is $\mathbb{T}^{n-1} \times \mathbb{R}$ with the metric $e^{2r}dt^2 + dr^2$. (Here as earlier, when it comes to $n \geq 9$, I feel more comfortable with minimal hypersurfaces to which Schoen-Yau's theorem 4.6 from [SY(singularities) 2017] directly applies.)

Finally, a derivation of the *hyperbolic positive mass theorem* from the rigidity theorem follows by an extension of the Euclidean Lohkamp's argument from [Loh(hammocks) 1999] to the hyperbolic spaces (see [AndMinGal 2007]).

Ono-Davaux *Spectral Inequality* [Ono 1988], [Dav 2002]. Let X be a closed Riemannian manifold and let all smooth functions $f(\tilde{x})$ with compact supports on $\tilde{X}$ satisfy

$$\int_{\tilde{X}} f(\tilde{x})^2 d\tilde{x} \leq \frac{1}{\tilde{\lambda}_0^2} \int_{\tilde{X}} ||df(\tilde{x})||^2 d\tilde{x}.$$

(The maximal such $\tilde{\lambda}_0 \geq 0$ serves as the lower bound on the spectrum of the Laplace operator on the universal covering $\tilde{X}$ of X).

If $\tilde{X}$ is spin and if one of the following two conditions (A) or (B) is satisfied, then

$$[Sc/\tilde{\lambda}_0] \qquad \inf_{x \in X} Sc(X, x) \leq \frac{-4n\tilde{\lambda}_0}{n-1}.$$

Condition (A). The dimension of X is $n = 4k$ and the $\hat{\alpha}$-invariant from section 3.4 (that is a certain linear combinations Pontryagin number called $\hat{A}$-genus) doesn't vanish.

Condition (B). The manifold X is *enlargeable*: there exists a covering $\tilde{X}'$ of X, which admits a *proper distance decreasing* map $\tilde{X}' \to \mathbb{R}^n$ of *non-zero* degree.

Remarks. (a) The inequality $[Sc/\tilde{\lambda}_0]$ is sharp: if X has constant negative curvature -1, then

$$-n(n-1) = Sc(X) = \tfrac{-4n\tilde{\lambda}_0}{n-1}$$

for $\tilde{\lambda}_0 = \frac{(n-1)^2}{4}$, that is the bottom of the spectrum of $\mathbf{H}^n_{-1} = \tilde{X}$.

(b) The rigidity sharpening of $[Sc/\tilde{\lambda}_0]$ is proved in [Dav 2002] in the case A and it seems that a minor readjustment of the argument from [Dav 2002] would work in the case B as well. If so it would yield yet another proof of Min-Oo rigidity theorem in the spin case.

3.11 Positive Scalar Curvature, Index Theorems and the Novikov Conjecture

Given a proper (infinity goes to to infinity) smooth map between smooth oriented manifolds, $f : X \mapsto \underline{X}$ of dimensions $n = dim(X) = 4k + \underline{n}$ for $\underline{n} = dim(\underline{X})$, let $sign(f)$ denote the signature of the pullback $Y^{4k}_{\underline{x}} = f^{-1}(\underline{x})$ of a generic point $\underline{x} \in \underline{X}$, that is the signature of the (quadratic) intersection form on the homology $H_2(Y^{4k}_{\underline{x}}; \mathbb{R})$, where observe orientations of X and $\underline{X}$ define an orientation of $Y^{4k}_{\underline{x}}$ which is needed for the definition of the intersection index.

Since the f-pullbacks of generic (curved) segments $[\underline{x}_1, \underline{x}_2] \subset \underline{X}$ are manifolds with boundaries $Y^{4k}_{\underline{x}_1} - Y^{4k}_{\underline{x}_2}$, (the minus sign means the reversed orientation),

$$sign(Y^{4k}_{\underline{x}_1}) = sign(Y^{4k}_{\underline{x}_2}),$$

as it follows from the Poincaré duality for manifolds with boundary by a two-line argument. Similarly, one sees that $sign(f)$ depends only on the proper homotopy class $[f]_{hom}$ of f.

Thus, granted $\underline{X}$ and a proper homotopy class of maps f, the signature $sign[f]_{hom}$ serves as a *smooth invariant* denoted $sign_{[f]}(X)$, (which is actually equal to the value of some polynomial in Pontryagin classes of X at the homology class of $Y^{4k}_{\underline{x}_2}$ in the group $H_{4k}(X)$).

If X and $\underline{X}$ are closed manifolds, where $dim(X) > dim(\underline{X}) > 0$, and if $\underline{X}$, is *simply connected*, then, by the Browder-Novikov theory, as one varies the smooth structure of X in a given homotopy class $[X]_{hom}$ of X, the values of $sign_{[f]}(X)$ run through *all integers* $i = sign_{[f]}(X) \mod 100n!$ (we exaggerate for safety's sake), provided $dim(\underline{X}) > 0$ and $Y^{4k}_{\underline{x}} \subset X$ is non-homologous to zero.

However, according to the (illuminating special case of the) *Novikov conjecture*,

if $\underline{X}$ is a *closed aspherical* manifold[22] then this $sign_{[f]}(X)$ *depends only on the homotopy class of* X. [23]

Originally, in 1966, Novikov proved this, by an an elaborated surgery argument, for the torus $\underline{X} = \mathbb{T}^{\underline{n}}$, where $X = Y \times \mathbb{T}^{\underline{n}}$ and f is the projection $Y \times \mathbb{T}^{\underline{n}} \to \mathbb{T}^{\underline{n}}$.

Then in 1971, Gheorghe Lusztig found a proof for general X and maps $f : X \to T^{\underline{n}}$ based on *the Atiyah-Singer index theorem for families of differential operators* D_p *parametrised by topological spaces* P, where the index takes values not in $\mathbb{Z}$ anymore but in the K-theory of P, namely, this index is defined as the K-class of the (virtual) vector bundle over P with the fibers $ker(D_p) - coker(D_p)$, $p \in P$, (Since the operators D_p are Fredholm, this makes sense despite possible non-constancy of the ranks of $ker(D_p)$ and $coker(D_p)$.)

The family P in Lusztig's proof is composed of the *signature operators* on X twisted with complex line bundles L_p, $p = P$, over X, where these L are induced by a map $f : X \to T^{\underline{n}}$ from *flat* complex unitary line bundles $\underline{L}_p$ over $T^{\underline{n}}$ parametrised by P (which is the $\underline{n}$-torus of homomorphism $\pi_1(T^{\underline{n}}) = \mathbb{Z}^{\underline{n}} \to \mathbb{T}$).

Using the the Atiyah-Singer index formula, Lusztig computes the index of this operator, shows that it is equal to $sign(f)$ and deduce from this the homotopy invariance of $sign_{[f]}(X)$.

What is relevant for our purpose is that Lusztig's computation equally applies to the Dirac operator twisted with L_p and shows the following.

[22] *Aspherical* means that the universal cover of $\underline{X}$ is contractible

[23] Our topological formulation, which is motivated by the history of the Novikov conjecture, is is deceptive: in truth, Novikov conjecture is 90% about infinite groups, 9% about geometry and only 1% about manifolds.

Let X be a closed orientable spin manifolds of even dimension $\underline{n}$ and $f : X \to \mathbb{T}^{\underline{n}}$ be continuous map of non-zero degree. Then

$$ind(\mathcal{D}_{\otimes\{L_p\}}) \neq 0.$$

Therefore, there exits a point $p \in P$, such that X carries a harmonic L_p-twisted spinor

But if $Sc(X) > 0$, this is incompatible with the the Schroedinger-Lichnerowicz-Weitzenboeck formula from section 3.4 which says for *flat* L_p that

$$\mathcal{D}_{\otimes L_p} = \nabla^2_{\otimes L_p} + \frac{1}{4} Sc(X).$$

Thus, the existence of a map $f : X \to T^{\underline{n}}$ with $deg(f) \neq 0$ implies that X carries no metric with $Sc > 0$.

Moreover, Lusztig's computation applies to manifolds X of all dimensions $n = \underline{n} + 4k$, shows that if the generic pullback manifold $Y^4_p h - f^{-1})p) \subset X$ (here f is smooth) has *non-vanishing $\hat{\alpha}$-invariant* defined in section 3.4 (that is the $\hat{A}$-genus for $4k$-dimensional manifolds), then the index $ind(\mathcal{D}_{\otimes\{L_p\}})$ doesn't vanish either and, assuming X is spin, it *can't carry metrics with $Sc > 0$.*

Remark on $X = (X, g_0) = \mathbb{T}^{\underline{n}}$. If (X, g_0) is isometric to the torus, then the only g_0-harmonic L_p-twisted spinors on X are parallel ones, which allows a direct computation of the index of $\mathcal{D}_{\otimes\{L_p\}}$. Then the result of this computation extends to all Riemannin metrics g on $T^{\underline{n}}$ by the invariance of the index of $\mathcal{D}_{\otimes\{L_p\}}$ under deformations of $\mathcal{D}$, where the essential point is that, albeit the harmonic spinors of the (untwisted) $\mathcal{D}$ may (and typically do) disappear under a deformation $\mathcal{D}_{g_0} \rightsquigarrow \mathcal{D}_g$, they re-emerge as harmonic spinors of $\mathcal{D}_g$ twisted with a non-trivial flat bundle L_p.

The index theorem for families can be reformulated with P being replaced by the algebra $cont(P)$ of all continuous functions on P, where in Lusztig's case the algebra $cont(T^{\underline{n}})$ is Fourier isomorphic to the algebra $C^*(\mathbb{Z}^{\underline{n}})$ of bounded linear operators on the Hilbert space space $l_2(\mathbb{Z}^{\underline{n}})$ of square-summarable functions on the group $\mathbb{Z}^{\underline{n}}$, which commute with the action of $\mathbb{Z}^{\underline{n}}$ on this space.

A remarkable fact is that a significant portion of Lusztig's argument generalizes to all discrete groups Π instead of $\mathbb{Z}^{\underline{n}}$, where the algebra $C^*(\Pi)$ of bounded operators on $l_2(\Pi)$ regarded as algebra of functions on a (fictious) non-commutative space dual to Π (that is the actual space, namely that of of homomorphisms $\Pi \to \mathbb{T}$ for commutative Π.)

This allows a formulation of what is called in [Ros 1984] the *strong Novikov Conjecture*, the relevant for us special case of which reads as follows.

$\mathcal{D}_{\otimes C^*}$-Conjecture. If a smooth closed orientable Riemannin spin n-manifold X for n even admits a continuous map F to the classifying space $B\Pi$ of a group Π, such that the homology homomorphism F_* sends the fundamental homology class $[X] \in H_n(X; \mathbb{R})$ to *nonzero* element $h \in H_n(B\Pi; \mathbb{R})$, then

the Dirac operator on X twisted with some flat unitary Hilbert bundle over X has non-zero kernel.

(Here "unitary" means that the monodromy action of $\pi_1(X)$ on the Hilbert fiber $\mathcal{H}$ of this bundle is unitary and where an essential structure in this $\mathcal{H}$ is the action of the algebra $C^*(\Pi)$, which commute with the action of $\pi_1(X)$.)

This, if true, would imply, according to the Schroedinger-Lichnerowicz-Weitzenboeck formula, the spin case of the conjecture stated in section 3.4. saying that

X admits no metric with $Sc > 0$.

Also "Strong Novikov" would imply, as it was proved by Rosenberg, the validity of the

Zero in the Dirac Spectrum Conjecture. Let $\tilde{X}$ be a complete *contractible* Riemannin manifold the quotient of which under the action of the isometry group $iso(\tilde{X})$ is *compact.*

Then the spectrum of the Dirac operator $\tilde{\mathcal{D}}$ on $\tilde{X}$ contains zero, that is, for all $\varepsilon > 0$, there exist L_2-spinors $\tilde{s}$ on $\tilde{X}$, such that

$$||\tilde{\mathcal{D}}(\tilde{s}|| \leq \varepsilon||\tilde{s}||.$$

This, confronted with the Schroedinger-Lichnerowicz-Weitzenboeck formula, would show that $\tilde{X}$ can't have $Sc > 0$.

Are we to Believe in these Conjectures. A version of the *Strong Novikov conjecture* for a rather general class of groups, namely those which *admit discrete isometric actions on spaces with non-positive sectional curvatures*, was proven by Alexander Mishchenko in 1974.

Albeit this has been generalized since 1974 to many other cases groups Π and/or representatives $h \in H_n(\mathsf{B}\Pi; \mathbb{R})$, the sad truth is that one has a poor understanding of what these classes actually are, how much they overlap and what part of the world of groups they fairly represent.

At the moment, there is no basis for believing in this conjecture and there is no idea where to look for a counterexample either.

On the positive side, the C^*-algebras bring forth the following interesting perspective on *coarse geometry* of non-compact spaces proposed by John Roe.

Given a metric space Ξ, e.g. a discrete group with a word metric, let $\mathcal{T} = Tra(\Xi)$ be the semigroup of translations of M that are maps $\tau : \Xi \to \Xi$, such that

$$\sup_{\xi \in \Xi} dist(\xi, \tau(\xi)) < \infty.$$

The (reduced) Roe C^*-algebra $R^*(\Xi)$ is a certain completion of the semigroup algebra $\mathbb{C}[\mathcal{T}]$. For instance if Ξ is a group with a word metric for which, say the left action of Ξ on itself is isometric, then the right actions lie in $\mathcal{T}$ and $R^*(\Xi)$ is equal to the (reduced) algebra $C^*(\Xi)$.[24]

Using this algebra, Roe proves a *partitioned index theorem*, which implies, for example, that.

✏ *the toric half cylinder manifold $X = \mathbb{T}^{n-1} \times \mathbb{R}_+$ admits no complete Riemannin metric with $Sc \geq \sigma > 0$.*

The subtlety here is twofold:

(i) the presence of non-empty boundary which is poorly tolerated by Dirac operators,

(ii) the metric on this X may (can it?), similarly to the hyperbolic metric $dr^2 + e^{-2t}dt^2$, exponentially contract at infinity.

[24]"Reduced" refers to a minor technicality not relevant at the moment. A more serious problem – this is not joke – is impossibility of definition of "right" and "left" without an appeal to violation of mirror symmetry by weak interactions.

Notice in this regard that if X is sufficiently "thick at infinity", then this follows by a simple argument with twisted Dirac operators and the standard bound on the number of small eigenvalues in the spectrum of the Laplace (or directly of the Dirac) operator in vicinity of ∂X, which applies to all manifolds with boundaries and which yields, in particular, (see section 4.5) the following.

Let X be a complete oriented Riemannin spin n-manifold *with compact boundary*, such that there exists a sequence of smooth maps $f_i : X \to S^n$, which are constant in a (fixed) neighbourhood $V \subset X$ of the boundary ∂X as well as away from compact subsets $W_i \subset V$, which *decrease the areas* of all surfaces in X and which have

$$deg(f_i) \underset{i\to\infty}{\to} \infty.$$

Then the scalar curvature of X satisfies

$$\inf_{x\in X} Sc(Xx) \leq n(n-1).$$

Also notice, that according to corollary to the $\frac{2\pi}{n}$ -inequality in section 3.7, the scalar curvature on $\mathbb{T}^n$ not only approaches zero but it must decay quadratically fast.

Yet, different proofs of ✏display different geometric aspects of the scalar curvature which are interesting in their own rights, where "non-existence formulations" serve only an illustrative purpose in our picture.

Conclude by formulating the following.

Coarse $\mathcal{D}$-Spectrum Conjecture. Let $\hat{X}$ be a complete *uniformly contractible* Riemannian manifold, i.e. there exists a function $R(r) \geq r$, such that the ball $B_{\hat{x}}(r) \subset \hat{X}$, $x \in X$, of radius r is contractible in the concentric ball $B_{\hat{x}}(R(r))$ for all $\hat{x} \in \hat{X}$ and all radii $r > 0$.

Then the spectrum of the Dirac operator on $\hat{X}$ contains zero.

This conjecture, as it stands, must be, in view of [DRW 2003], false, but finding a counterexample becomes harder if we require the bounds $vol(B_{\hat{x}}(r)) \leq \exp r$ for all $\hat{x} \in \hat{X}$ and $r > 0$.

3.12 Foliations With Positive Scalar Curvature.

According to the philosophy (supported by results) of Alain Connes much of the geometry and topology of manifolds with discrete group actions, notably, those concerned with index theorems for Galois actions of fundamental groups on universal coverings of compact manifolds, can be extended to foliations.

In particular, Connes shows in [Con 1986] that compact manifolds X which carry foliations with leaf-wise Riemannin metrics with positive scalar curvatures behave in many respects as manifold which themselves admit such metrics.

As a specific result in this regard we mentioned here the following bound on the size of X that can be derived from Connes' theorem.

[$\star$] Let X be a complete Riemannin n-manifold with a smooth foliation such that scalar curvature of the induced metric on the leaves satisfies $Sc \geq \sigma > 0$.

If $H^2(X;\mathbb{Z}_2) = 0$,[25] *then X admits no distance non-increasing map $X \to \mathbb{R}^n$ with non-zero degree.*

[25] This condition, which safeguards *spin*, is, probably, redundant.

The Proof of this indicated in §§$9\frac{2}{3}, 1\frac{7}{8}$ in [G(positive) 1996], which follows [Con 1986], relies on *Connes-Scandalis longitudinal index theorem for foliations*[26], while similarresults all [Zhang [2018]) and [BH 2017] use somewhat different index theoretic arguments.

Although Connes theorem goes well beyond [$\star$], the geometry of foliated Riemannian manifolds X with leaf-wise positive scalar curvatures remains obscure.

For instance, the following remains *remains unclear.*

[$\star_?$] When/if do these X themselves, or closely geometrically related to X manifolds, admit metrics with $Sc > 0$.

What are geometric/topological effect of lower bounds on the scalar curvatures of the leaves by $\sigma < 0$?

For instance,

Do compact Riemannian n-manifolds with constant curvature -1 admit k-dimensional foliations, $2 \leq k \leq n-1$, such that the scalar curvatures of the induced Riemannian metrics in the leaves are bounded from below by $-\varepsilon$ for a given $\varepsilon > 0$?

In general, one expects most (all) aspects of constraints on geometry implied by $Sc \geq \sigma$ to have their counterparts for foliations, where certain formulations and sometimes proofs are transportable from individual manifolds to foliations, such, for instance as Ono-Davaux spectral inequality (see section 3.10), but in some cases, especially where minimal subvarieties are concerned, this look harder(compare [G(foliated) 1991]).

Also, surgery constructions of manifolds with $Sc > 0$ may go along with surgery of singularities in construction of foliations by Thurston and Eliashberg-Mishachev, see [EM 1998]/

3.13 Scalar Curvature in Dimension 3

If $n \geq 4$, then then all known bounds on the size of n-manifolds X with $Sc(X) \geq \sigma > 0$ are expressed by *non-existence* of "topologically complicated but geometrically simple" maps from these X to "standard manifolds" $\underline{X}$.

But if $n = 3$ the following two more satisfactory results are available.

Let X be a complete Riemannin 3-manifold with scalar curvature $\geq 6 = Sc(S^3)$.

Then

A. *There exists a continuous map $f : X \to P^1$, where P is a 1-dimensional polyhedral space (topological graph) such that the diameters of the pullback of all points are bounded by*

$$[width_{3-2}] \qquad diam(f^{-1}(p)) \leq 2\pi\sqrt{6}.$$

B. *If X is homeomorphic to S^3, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or $S^2 \times S^1$ then there exists a map $\Phi : S^2 \times T \to X$, where, either $T = \mathbb{R}$ or $T = S^1$ of degree 1*[27] *and such that the*

[26] I must admit that I didn't try now to reconstruct in memory all steps necessary for the proof of [$\star$].

[27] If $T = \mathbb{R}$ then "degree 1" here presupposes here that there are at most two points in X, such that if a compact subset $C \subset X$ doesn't contain either of these points, then the pullback $\Phi^{-1}(C) \subset S^2 \times T$ is compact.

areas (counted with multiplicities if you wish) of the images $\Phi(S^2 \times \{t\}, t \in S^1$ *satisfy*

$$[waist_{3-2}] \qquad area(\Phi(S^2 \times \{t\}) \leq 4\pi.$$

Remarks and Conjectures. (a) The proof of A (corollary 10.11 in [GL 1983]) relies on stable minimal surfaces in X, while B follows from the Marques-Neves estimate in [MN 2011] on the areas of surfaces with *Morse index* 1.

(b) The inequality $[waist_{3-2}]$, unlike $[width_{3-2}]$, is sharp, with the equality for the unit sphere S^3.

(c) The factor $\sqrt{6}$ in $[width_{3-2}]$ is, probably redundant, but even without this factor it wouldn't look as pretty as $[waist_{3-2}]$.[28]

(d) Proposition A, as it stands, (obviously) fails to be true for compact manifolds X with non-empty boundaries but, by the argument in §10 from GL 1983], it remains valid for the part of X within distance $d > 2\sqrt{6}\pi$ from the boundary .

(e) *Conjecturally,* all complete n-manifolds X with $Sc(X) \geq n(n-1)$ admit continuous maps to polyhedral spaces of dimension $n-2$, say, $F : X \to P^{n-2}$, such that

$$diam(F^{-1}(p)) \leq const_n \text{ and } vol_{n-2}(F^{-1}(p)) \leq const'_n \text{ for all } p \in P^{n-2}.$$

Probably, this can be shown for $n = 3$ by combining the arguments from [GL 1983] and [MN 2011].

(d) Let X be a complete Riemannin n-manifolds with a 3-dimensional foliation such that the scalar curvature of the induced leaf-wise metric is bounded from below by 6.

Does X admit a continuous map $F : X \to P^{n-2}$ with $diam(F^{-1}(p)) \leq const_n$, $p \in P^{n-2}$?

(If so, this would provide a geometric proof of [$\star$] from the previous section for 3-dimensional foliations.)

Penrose Inequality. Start with recalling that

the (space sliced) Schwarzschild metric with mass m

is defined on $\mathbb{R}^3$ minus the origin in polar coordinates as

$$g_{Sw_m} = g_{Sw} = \left(1 + \frac{\rho}{r}\right)^4 g_{Eucl}, \text{ for } \rho = \rho_m = \frac{m}{2},$$

and that the

scalar curvature of this metric is zero

by the conformal change formula IX in section 2.

Since the function $s(r) = r^2\left(1 + \frac{\rho}{r}\right)^4$ is invariant under the transformation

$$r \mapsto \frac{\rho^2}{r},$$

[28] The inequality $[width_{3-2}]$ says that X can be "sliced" by surfaces of small diameters, but it doesn't tell anything about topologies and/or areas and intrinsic diameters of thees surfaces.

this g_{Sw} is invariant under the (conformal) reflection of $\mathbb{R}^3$ around the sphere $S^2(R_m) \subset \mathbb{R}^3$ of radius $\rho = \frac{m}{2}$, that is

$$(s, r) \mapsto \left(s, \frac{\rho^2}{r}\right).$$

Thus the sphere $S(\rho)$ is totally geodesic in geometry of g_{Sw} with area

$$area_{g_{Sw}}(S(\rho) = \pi\rho^2 \left(1 + \frac{\rho}{\rho}\right)^4 = 16\pi m^2.$$

In 1973 Penrose formulated in [Pen 1973] a conjecture concerning black holes in general relativity with an evidence in its favour, that would, in particular imply the following

Special case of the Riemannian Penrose Inequality. Let X be complete Riemannin 3-manifolds with compact boundary $Y\partial X$, such that

• X is isometric at infinity to the Schwarzschild space of mass m at one of its two ends at infinity;

• the scalar curvature of X is everywhere non-negative: $Sc(X) \geq 0$;

• the boundary Y of X has zero mean curvature;[29]

• no minimal surface in X separates a connected component of Y from infinity.

Then the area of $Y = \partial X$ is bounded by the mass of the Schwarzschild space as follows.

$$area(Y) \leq 16\pi m^2.$$

This, in a greater generality was proven by Hubert Bray in [Bray 2009].

On Geometric Meaning of Mass. The Schwarzschild metric at infinity fast approaches the Euclidean metric, where the greater the mass the slower is the growth rate of this metric.

To get a rough idea, let is compare g_{Sw} with the conical metrics

$$g_a = a^2 \cdot r^2 ds^2 + dr^2.$$

If $a < 1$ these metrics have positive scalar curvatures (zero for $a = 1$) and if you compare them with g_{Sw} these have infinite masses, and would violate any kind of Penrose-like inequality.

But if $a > 1$, then these g_a have masses $-\infty$ and one can show, e.g. using the bound on Rad_{S^2} from section 3.6 for suitable surfaces at infinity, that such a fast growth rate of general Riemannian manifolds is incompatible with $Sc > 0$.

Moreover, the positive mass theorem says that even finite but negative mass of an asymptotically Euclidean metric needs a bit of negativity in its scalar curvature.

But I must admit I haven't thought through further the geometric meaning of what physicists call "mass" in general relativity.

[29]It suffices to assume that the the boundary is *mean convex*, i.e. its mean curvature relative to the normal field pointing outward is positive.

3.14 Scalar Curvature in Dimension 4

The simplest examples of 4-manifolds where non-existence of metrics with $Sc >$ follows from non-vanishing of Seiberg-Witten invariants are complex algebraic surfaces X in $\mathbb{C}P^3$ of degrees $d \geq 3$. (If d is even and these X are spin, this also follows from Lichnerowicz' theorem from section 3.4.)

In fact, it was shown by LeBrun (see [Sal 199] and references therein) that

no minimal (no lines with self-intersections one) *Kähler surface X admits a Riemannin metric with $Sc > 0$, unless X is diffeomorphic to $\mathbb{C}P^2$ or to a ruled surface .*

Furthermore, LeBrun shows in [LeB 1997] that

if such an X has *Kodaira dimension* 2, which is the case, for instance, for the algebraic surfaces $X \subset \mathbb{C}P^3$ of degree $d \geq 5$, then

the total squared scalar curvature is bounded by the first Chern number of X,

$$\int_X Sc(X,x)^2 dx \geq 32\pi^2 c_2(X),$$

where, moreover this inequality is sharp.

One may only dream of this kind of a bound on $\int_X Sc(X,x)^{\frac{n}{2}} dx$ for a manifolds X of dimension $n > 4$.

In fact the ideal bound, would be on $\int_X |Sc_-(X,x)|^{\frac{n}{2}} dx$ for $Sc_-(X,x) \min(Sc(X,x)$.

Conceivably(?), if a closed orientable Riemannin n-manifold X admits a map of non-zero degree to a closed locally symmetric manifold $\underline{X}$ with negative Ricci curvature, e.g. with constant negative curvature, then

$$\int_X |Sc_-(g,x)|^{\frac{n}{2}} dx \geq \int_{\underline{X}} |Sc(\underline{X},\underline{x})|^{\frac{n}{2}} d\underline{x}.$$

3.15 Topology and Geometry of Spaces of Metics with $Sc \geq \sigma$.

Non-triviality of the homotopy types of metrics with positive scalar curvatures, which was first proven by Nigel Hitchin in [Hit 1974], starts with the following observation.[30]

Let a closed n-manifold X be decomposed as $X_- \cup X_+$ where X_- and X_+ are smooth domains (n-submanifolds) in X with a common boundary $Y = \partial X_- = \partial X_+$ and where $X_\mp$ are equal to regular neighbourhoods of disjoint polyhedral subsets $P_\mp \subset X$ of dimensions $n_\mp$ such that $n_- + n_+ = n - 1$.

If $n_\mp \leq n-2$, then, by an easy elementary argument, both manifolds X_- and X_+ admit Riemannin metrics, say $g_\mp$, such that

the restrictions of these $g_\mp$ to Y, call them $h_\mp$, both have *positive* scalar curvatures.

And if X admits no metric with positive scalar curvature, e.g. if X is homeomorphic to the n-torus or to product of two Kummer surfaces, then h_- and h_+ can't be joined by a homotopy of metrics with positive scalar curvatures.

[30] Hitchin himself argued differently.

Indeed, such a homotopy, h_t, $t \in [-1,+1]$ could be easily transformed to a metric on the cylinder $Y \times [-1,+1]$ with positive scalar curvature and with relatively flat boundaries isometric to (Y, h_-) and (Y, h_+), which would then lead in obvious way to a metric on $X = X_- \cup Y \times [-1,+1] \cup X_+$ with $Sc > 0$ as well.

This kind of argument combined with surgery with $Sc > 0$ and empowered by index theorem(s) for Dirac operators leads, to the following results.

[HaSchSt 2014]. *If m is much greater than k then the kth homotopy group of the space of metrics with $Sc > 0$ on the sphere S^{4m-k-1} is infinite.*

[EbR-W 2017]. *There exists a compact Spin 6-manifold such that its space of positive scalar curvature metrics has each rational homotopy group infinite dimensional.* [31]

However, there is *no closed manifold* of dimension $n \geq 4$, which admits a metric with $Sc > 0$ and where the (rational) homotopy type, or even the set of connected components, of *the space of such metrics is fully determined.* [32]

Question. Given a Riemannian manifold $\underline{X}$ and a pair of numbers $(\lambda, \sigma) \in \mathbb{R}^2_+$, let $G(X; \underline{X}, \lambda, \sigma)$ be the space of pairs (g, f) where g is a Riemannian metrics on a X with $Sc(g) \geq \sigma$ and $f : X \to \underline{X}$ is a λ-Lipschitz map.

What is the topology and geometry of this space and of the natural embeddings

$$G(X; \underline{X}, \lambda_1, \sigma_1) \hookleftarrow G(X; \underline{X}, \lambda_2, \sigma_2)$$

for $\lambda_2 \geq \lambda_1$ and $\sigma_2 \geq \sigma_1$.

3.16 Manifolds with Corners.

Most (all?) theorems concerning closed manifolds X with $Sc \geq \sigma$ and, more visibly, manifolds with smooth boundaries $Y = \partial X$, have (some proven, some conjectural) counterparts for Riemannin manifolds X with corners at the boundary,

where the mean curvature $mean.curv(\partial X)$ for the smooth part of ∂X plays the role of *singular/distributional* scalar curvature supported on ∂X and where the *dihedral angles* $\angle$ along the corners, or rather $\pi - \angle$, can be regarded as *singular/distributional* mean curvature supported on the corners.

Below are two examples illustrating this idea.

Let X be a compact n-dimensional manifold with *simple*, also called *cosimplicial*, corners. This means that X is locally diffeomorphic at all points $x \in X$ to the positive cone $\mathbb{R}^n_+$ at some points $x' \in \mathbb{R}^n_+$, where the simples example of such an X is the n-cube $[0,1]^n$.

Call such an X *semihyperbolic* if whenever three $(n-1)$-faces of X pairwise meet then all three meet at some point in X.

Weak $\urcorner$-Reflection Rigidity. Let X be a semihyperbolic manifold X of dimension n with corners, assume for safety sake that all faces of X are contractible and let g be a Riemannin metric on X, such that

[31] It seems, judging by the references in [EbR-W 2017], that all published results in this direction depend on the Dirac operator techniques which do not cover the above example, if we take a *Schoen-Yau-Schick manifold* (see [G(inequalities) 2018]) for X.

[32] If $n = 3$ contractibility of this space, (if it is true) must follow from the known results on the Ricci flow á la Perelman.

$\bullet_n$ the scalar curvature of g is non-negative: $Sc(g) \geq 0$;

$\bullet_{n-1}$ the mean curvatures of all $(n-1)$-faces F_i of X are also *non-negative* : $mean.curv_g(F_i) \geq 0$;

$\bullet_{n-2}$ The dihedral angles $\angle_{ij}$ of X at all points of all $(n-2)$-faces, that are intersection of certain $(n-1)$ faces F_i and F_j, satisfy $\angle_{ij} \leq \frac{\pi}{2}$.

Then

$Sc(X) = 0$, $mean.curv(Y_{reg}) = 0$, all $\alpha = \frac{\pi}{2}$ and X itself admits a homeomorphism onto the n-cube $[0,1]^n$, which sends the faces of X onto faces of the cube.

About the Proof. This is shown by reflecting X around its $(n-1)$-faces, smoothing around the edges and applying the corresponding result for closed manifolds as it was done in [G(billiard] 2014] for cubical X, and where the general case needs an intervention of arguments from [G(inequalities) 2018], where the (non-spin) case $n \geq 9$ relies on [SY(singularities) 2017]. (Also see section 5.6).

Remarks. (a) There is little doubt that $\neg$-strong rigidity also holds for our X:

the conditions $\bullet_n$, $\bullet_{n-1}$, $\bullet_{n-2}$ should imply that X is isometric to a rectangular solid.

But there are few technical details still to settle in the proof.

(b) The weak $\neg$-rigidity for cubical (i.e. topologically isomorphic to cubes) yields, by an elementary argument, the C^0-closeness of spaces of metrics with $Sc \geq \sigma$ stated in section 3.1.

There are two major limitation to our $\urcorner$:

☹$_1$ the semihypebolicity condition rules out many promising spaces X, e.g. those isomorphic to n-simplices;

☹$_2$ condition $\bullet_{n-2}$ is unrealistically strong, e.g. for such X as planar k-gons with $k \geq 5$.

Below is an instance of where ☹$_1$ is partly appeased.

$\times\blacktriangle^i$*-Inequality.* Let $X_0 \subset \mathbb{R}^n$ be a polytope, i.e. convex compact polyhedron with non-empty interior, and let $X \subset \mathbb{R}^n$ be diffeomorphic to X_0.

Let all $(n-1)$-faces F_i of X have *positive mean curvatures*, e.g. the subset $X \subset \mathbb{R}^n$ is convex.

Let the dihedral angles between (the tangent spaces of) the faces F_i and F_j of X at all points in the $(n-2)$-faces where/if these faces meet, are bounded by the corresponding dihedral angles of X_0,

$$\angle_{ij}(X) \leq \angle_{ij}(X_0).$$

If all dihedral angles of X_0 are $\leq \frac{\pi}{2}$, then

$$\angle_{ij}(X) = \angle_{ij}(X_0).$$

This is shown, by doubling and smoothing X_0 and X and then applying $[X^{\to\bigcirc}]$ (see section 4.3 and 4.4).

Remarks/Exercises. (a) The only polytopes with $\angle_{ij} \leq \frac{\pi}{2}$ are products of simplices, such as the n-cubes $[0,1]^n$, for example.

(b) If both X_0 and X are n-simplices then the implication

$$\angle_{ij}(X) \leq \angle_{ij}(X_0) \Rightarrow \angle_{ij}(X) = \angle_{ij}(X_0)$$

follows from the Kirszbraun theorem with no need for the condition $\angle_{ji} \leq \pi/2$.

(c) There are cases where $\times\blacktriangle^i$-*inequality* is known to hold for certain polytopes e.g. for k-gonal prisms, where (some) dihedral angles may be $> \frac{\pi}{2}$ (an approach via minimal (hyper)surfaces is indicated in [G(billiards) 2014] and in [Li 2017]) but this remains problematic in general even for *simple* n-polytopes, where at most n faces of dimension $n-1$ may meet at the vertices.

Motivations for Corners. Besides opening avenues for generalisations of what is known for smooth manifolds, Riemannin manifolds with corners and $Sc \geq \sigma$ may do good to the following.

1. Suggesting new techniques, (calculus of variations, Dirac operator) for the study of Euclidean polyhedra.

2. Organising the totality of manifolds with $Sc \geq 0$ (or, more generally with $Sc \geq \sigma$) into a nice category (A_∞-category?) $\mathcal{P}^\square$, that would include, as objects manifolds Y with Riemannian metrics h and functions M on them and where *morphisms* are *(co)bordisms* (*h-cobordisms*?) (X, g), $\partial X = Y_0 \cup Y_1$, where g is a Riemannian metric on X with $Sc \geq 0$, which restricts to h_0 and to h_1 on Y_0 and Y_1 and where the the mean curvature of Y_0 with inward coorientation is equal to $-M_0$ while the mean curvature of Y_1 with the outward coorientation is equal to M_1.

Conceivably, the [SY]-variational techniques for "flags" of hypersurfaces or its generalisation(s), may have a meaningful interpretation in $\mathcal{P}^\square$, while a suitably adapted Dirac operator method may serve as a quantisation of $\mathcal{P}^\square$.

3.17 Who are you, Scalar Curvature?

There are two issues here.

1. What are most general geometric object that display features similar to these of manifolds with positive (bounded from below) scalar curvatures?

2. Is there a direct link between Dirac operators and minimal varieties or their joint appearance in the ambience of scalar curvature is purely accidental?

Notice in this regards that there are two divergent branches of the growing tree of scalar curvature.

A. The first one is concerned with the effects of $Sc > 0$ on the *differential structure* of spin (or spin$^{\mathbb{C}}$) manifolds X, such as their $\hat{\alpha}$ and Seiberg-Witten invariants.

B. The second aspect is about coarse geometry and topology of X with $Sc(X) \geq \sigma$, the (known) properties of which are derived by means of minimal varieties and twisted Dirac operators; here the spin condition, even when it is present, must be redundant.

To better visualise separation between A and to B, think of possible *singular spaces X with* $Sc(X) \geq 0$ corresponding to A and to B – these must be grossly different.

For instance, if X is an *Alexandrov space* with (generalised)sectional curvature $\geq \kappa > -\infty$ then the inequality $Sc \geq 0$ makes perfect sense and, probably most (all?) of B can be transplanted to these spaces. [33]

But nothing, from spin on, of what we know of A makes sense in singular Alexandrov spaces.

And if you start from the position of 2 you better go away from conventional spaces and start dreaming of geometric magic glass ball with ghosts of harmonic spinors and of minimal varieties dancing within.

In concrete terms one formulates two problems.

A. What is the largest class of spaces (singular, infinite dimensional ...) which display the basic features of manifolds with $Sc \geq 0$ and/or with $Sc \geq \sigma > -\infty$ and, more generally, of spaces X, where the properly understood operator $-\Delta + \frac{1}{2}Sc(X)$ is positive or, at least not too negative?

For instance, which (isolated) conical singularities and which singular volume minimising hypersurfaces belong to this class?

B. Is there a partial differential equation, or something more general, the solutions of which would mediate between twisted harmonic spinors and minimal hypersurfaces (flags of hypersurfaces?) and which would be non-trivially linked to scalar curvature?

Could, for instance, one non-trivially couple the twisted Dirac $\mathcal{D}_{\otimes L}$ with some equation on the connection in the bundle L the Dirac operator is twisted with in the spirit of the Seiberg-Witten equation?

4 Dirac Operator Bounds on the Size and Shape of Manifolds X with $Sc(X) \geq \sigma$

4.1 Spinors, Twisted Dirac Operators, and Distance Decreasing maps.

The Dirac operator $\mathcal{D}$ on a Riemannin manifold X tells you by itself preciously little about the geometry of X, but the same $\mathcal{D}$ twisted with vector bundles L over X carries the following message:

manifolds with scalar curvature $Sc \geq \sigma > 0$
can't be too large area-wise.

Albeit the best possible result of this kind (due to Marques and Neves, see B in section 3.13), which is known for X homeomorphic to S^3, which says that

if $Sc(X) \geq 6 = Sc(S^3)$, then X can be "swept over" by 2-spheres of areas $\leq 4\pi$,

was proven by means of minimal surfaces, all known bounds on "areas" of Riemannin manifolds of dimensions ≥ 4 depend on Dirac operators $\mathcal{D}$ twisted (or "non-linearly coupled" for n=4) with complex vector bundles L over X with unitary connections in L, where, don't forget it, the very definition of $\mathcal{D}$ needs X to be spin.

Recall that the twisted Dirac operator, denoted

$$\mathcal{D}_{\otimes L} : C^\infty(\mathbb{S} \otimes L) \to C^\infty(\mathbb{S} \otimes L),$$

[33]It seems, much of the geometric measure theory extends to Alexandrov spaces but it is unclear what would correspond to twisted Dirac operators on these spaces.

acts on the tensor product of the spinor bundle $\mathbb{S} \to X$ [34] with $L \to X$, where it is related to the (a priori, positive Bochner Laplace) operator in the bundle $\mathbb{S} \otimes L$ by the Schroedinger-Lichnerowicz-Weitzenboeck formula

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \tfrac{1}{4} Sc(X) + \mathcal{R}_{\otimes L},$$

where $\nabla_{\otimes L}$ denotes the covariant derivative operator in $\mathbb{S} \otimes L$ and $\mathcal{R}_{\otimes L}$ is a certain (zero order) operator which acts in the fibers of the twisted spin bundle $\mathbb{S} \otimes L$ and which is derived from the curvature of the connection in L.

If we are not concerned with the sharpness of constants , all we have to know is that $\mathcal{R}_{\otimes L}$ is controlled by

$$||\mathcal{R}_{\otimes L}|| \leq const \cdot ||curv(L)||$$

for $const = const(n, rank(L))$, where a little thought (no computation is needed) shows that, in fact, this constant depends only on $n = dim(X)$. (See [MarMin 2012] for details and references.)

We regard an *even dimensional* Riemannin manifold X as *area wise large*, if it carries a *homologically essential* bundle L over it with *small curvature*, where "homologically essential" signifies that the Chern character in the index formula guaranties non-vanishing of the cup product $\hat{A}(X) \smile Ch(L)$ evaluated at $[X]$,

$$(\hat{A}(X) \smile Ch(L))[X],$$

and, thus, by Atiyah-Singer theorem, the presence of non-zero harmonic twisted spinors, that are sections s of the bundle $\mathbb{S} \otimes L$ for which $\mathcal{D}_{\otimes L}(s) = 0$.

If the dimension n of X is odd, the above applies to $X \times S^1$ for a sufficiently long circle S^1.

For instance, n-manifolds, which admit area decreasing non-contractible maps to spheres $S^n(R)$ of large radii R are area wise large, where the relevant bundles L are induced from non trivial bundles over the spheres.

But if the scalar curvature of X is $\geq \sigma$ for a *large* $\sigma > 0$, where this *"large"* properly matches the above *"small"*, then by the Schroedinger-Lichnerowicz-Weitzenboeck formula the operator $\mathcal{D}_{\otimes L}$ is positive and no such harmonic twisted exists; therefore, a suitably defined "area"(X) must be bounded by $\frac{const}{\sigma}$.

(Recall that

$$\hat{A}(X) = 1 - \frac{1}{24} p_1 + \frac{1}{5760}(-4p_2 + 7p_1^2) + ... \in H^*(X)$$

is a certain polynomial in Pontryagin classes $p_i \in H^{4i}(X)$ of X and

$$Ch(L) = rank_{\mathbb{C}}(L) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + ... \in H^*(X)$$

is a polynomial in Chern classes $c_i \in H^{2i}(X)$ of L, while $[X] \in H_n(X)$ denotes the fundamental class of X.

[34] All you have to know about $\mathbb{S}(X)$ is that it is a vector bundle associated with the tangent bundle $T(X)$, which can be defined for spin manifolds X, where "spin" is needed, since the structure group of $\mathbb{S}(X)$ is the double cover of the orthogonal group $O(n)$ rather than $O(n)$ itself.

If $n = dim(X)$ is even, the spin bundle $\mathbb{S}$ naturally splits, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, the operator $\mathcal{D}_{\otimes L}$ also splits: $\mathcal{D}_{\otimes L} = \mathcal{D}^+_{\otimes L} \oplus \mathcal{D}^-_{\otimes L}$, for

$$\mathcal{D}^{\pm}_{\otimes L} : C^\infty(\mathbb{S}^{\pm} \otimes L) \to C^\infty(\mathbb{S}^{\mp} \otimes L)$$

and the index formula reads:

$$ind(\mathcal{D}^{\pm}_{\otimes L}) = \pm(\hat{A}(X) \smile Ch(L))[X].)$$

==================================

A characteristic topological corollary of the above reads:

If a closed orientable spin n-manifold X admits a map to a complete Riemannin manifold $\underline{X}$ with $sect.curv(X) \leq 0$,

$$f : X \to \underline{X},$$

such that the homology image $f_*[X] \in H_n(\underline{X};\mathbb{Q})$ doesn't vanish, then X admits no metric with $Sc(X) > 0$.

Two Words about the Proof. All we need of $sect.curv \leq 0$ is the existence of distance decreasing maps from the universal covering of $\underline{X}$ to (large) spheres,

$$F_{\underline{x}} : \underline{X} \to S^{\underline{n}}(R),\ \underline{n} = dim(\underline{X}),\ \underline{x} \in \underline{X},$$

which can be (trivially) obtained with a use of inverse exponential maps

$$\exp^{-1}_{\underline{x}} : \tilde{\underline{X}} \to T_{\underline{x}}(\underline{X}),\ \underline{x} \in \underline{X}.$$

To make the idea clear, let $\underline{X}$ be compact, the fundamental group of $\underline{X}$ be residually finite, (e.g. $\underline{X}$ having constant sectional curvature or, more generally being a locally symmetric space) and X be embedded to $\underline{X}$.

Let $X^\perp \subset \underline{X}$ be a closed oriented submanifold of dimension $m = \underline{n} - n$ for $\underline{n} = dim(\underline{X})$, which has *non-zero intersection index* with $X \subset \underline{X}$.

Also assume that the restriction of the tangent bundle of $\underline{X}$ to $\underline{X}^\perp \subset \underline{X}$ is trivial.

Then – this is rather obvious – there exist finite covers $\tilde{\underline{X}}_i \to \underline{X}$, such that the products of the lifts (i.e. pull-backs) of X and of $X^\perp$ to $\tilde{\underline{X}}_i$, denoted $\tilde{X}_i \times \tilde{X}^\perp_i$, admit smooth maps to the spheres of radii R_i,

$$F_i : \tilde{X}_i \times X^\perp_i \to S^{\underline{n}}(R_i),$$

where

$\bullet_1$ $R_i \to \infty$,

$\bullet_2$ $deg(F_i) \neq 0$,

• the maps F_i are *distance decreasing on the fibers* $\tilde{X}_i \times x^\perp$ for all $x^\perp \in X^\perp_i$ for the Riemannian metric in these fibers induced by the embedding $\tilde{X}_i \times x^\perp = \tilde{X}_i \subset \tilde{\underline{X}}_i$.

It follows that for *arbitrary* Riemannin metrics g and $g^\perp$ on X and on $X^\perp$ there exists (large) constants λ and C independent of i, such that

the maps F_i are *C-Lipschitz* with respect to the sum of the lift of the metric g to $\tilde{X}_i$ and the lift of $\lambda \cdot g^\perp$ to $\tilde{X}^\perp_i$ that is the metric

$$\tilde{g}_i \oplus \lambda \cdot \tilde{g}^\perp_i \text{ on } \tilde{X}_i \times \tilde{X}^\perp_i.$$

If $Sc(g) \geq \sigma > 0$, then also $Sc(\tilde{g}_i \oplus \lambda \cdot \tilde{g}_i^\perp) \geq \sigma' > 0$ for all sufficiently large λ, which, for large R_i, rules out non-zero harmonic spinors on $\tilde{X}_i \times \tilde{X}_i^\perp$ twisted with the bundle $L^* = F_i^*(L)$ induced from any given bundle L on $S^{\underline{n}}$.

But if $\underline{n} = 2k$ and the Chern class $c_k(L)$ is non-zero, then non-vanishing of $deg(F_i)$ implies non-vanishing of of $ind(\mathcal{D}_{\otimes L})$ via the index formula and the resulting contradiction delivers the proof for even $\underline{n}$ and the odd case follows with $\underline{X} \times S^1$.

Remarks. This argument, which is rooted in Mishchenko's proof of Novikov conjecture for the fundamental group of the above $\underline{X}$ and which was generalized/formalised in [CGM 1993], doesn't really need compactness of $\underline{X}$, residual finiteness of $\pi_1(\underline{X})$ and triviality of $T(\underline{X})|X^\perp$. Beside, the spin condition for X can be relaxed to that for the universal cover of X.

Moreover, since the bound on the size of $\tilde{X}_i \times \mathbb{T}^{\underline{n}-n}$ by $\frac{const}{\sqrt{\sigma}}$ can be obtained with the use of minimal hypersurfaces (see §12 in [GL 1983]), [G(inequalities) 2018] and section 5.4) the spin condition can be dropped altogether.

Question. Are there other topological non-spin obstructions to $Sc > 0$?

For instance, is the following true?

Conjecture. Let X be a closed orientable Riemannin n-manifold, such that no closed orientable n-manifold X' which admits a map $X' \to X$ with non-zero degree admits a metric with $Sc > 0$. Then there exists an integer m and a sequence of maps

$$F_i : \tilde{X} \times \mathbb{R}^m \to S^{n+m}(R_i),$$

where $\tilde{X}$ is some (possibly infinite) covering of X, such that

- the maps F_i are constant at infinity and they have non-zero degrees,
- $R_i \to \infty$,
- the maps F_i are distance decreasing on the fibers $\tilde{X} \times x^\perp$ for all $x^\perp \in \mathbb{R}^m$.

Apparently, there is no instance of a *specific* homotopy class $\mathcal{X}$ of closed manifolds X of dimension $n \geq 5$, where a Dirac theoretic proof of non existence of metrics with $Sc > 0$ on all $X \in \mathcal{X}$ couldn't be replaced by a proof via minimal hypersurfaces.

(This seems to disagree with what was said concerning $\otimes$ at the end of section 2.7.

In fact the *general* condition for $Sc \not> 0$ in $\otimes$, can't be treated, not as it stands, with minimal hypersurfaces, but this may be possible in all *specific examples*, where this condition was *proven to be* fulfilled.)

And it is conceivable when it comes to the Novikov conjecture, that its validity in all proven specific examples, can be derived by an elementary argument from the invariance of rational Pontryagin classes under ε-homeomorphisms.[35])

But even though the relevance of twisted Dirac theoretic methods is questionable as far as *topological* non-existence theorems are concerned, these methods seem irreplaceable when it comes to *geometry* of $Sc \geq \sigma$ as we shall see presently.

[35]The original proof of topological invariance of Pontryagin classes by Novikov, as well as simplified versions and modifications of his proof in [G(positive) 1996) automatically apply to ε-homeomorphisms and, sometimes, of homotopy equivalences

4.2 Sharp Estimates for Maps to Spheres and to Convex Surfaces.

Let us look closely at the last term in the Schroedinger-Lichnerowicz-Weitzenboeck formula
$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \frac{1}{4}Sc(X) + \mathcal{R}_{\otimes L},$$
for the twisted Dirac operator,
$$\mathcal{D}_{\otimes L} : C^\infty(\mathbb{S}\otimes L) \to C^\infty(\mathbb{S}\otimes L),$$
and write (see formula 1.3 in [GL(fundamental group) 1980]) thus term as follows.
$$\mathcal{R}_{\otimes L}(s\otimes l) = \frac{1}{2}\sum_{i,j}(e_i\cdot e_j\cdot s)\otimes R_{ij}(l),$$
where s and l are sections of the bundles $\mathbb{S}$ and L, where "$\cdot$" denotes the Clifford product, $e_i \in T_x(X)$, i=1,...n=dim(X), are orthonormal vectors at a point $x\in X$ and $R_{ij} : L_x \to L_x$ is the curvature of the (connection in) L at x.

Example of $L = \mathbb{S}$ *on* S^n. Since the norm of the curvature operator of (the Levi-Civita connection on) the tangent bundle is one, the norm of the curvature operators $R_{ij} : \mathbb{S}\to\mathbb{S}$ are at most (in fact, are to) $\frac{1}{2}$,
$$||R_{ij}(s)|| \le \frac{1}{2},$$
since the spin bundle $\mathbb{S}(X)$ serves as the "square root" of the tangent bundle $T(X)$, where this is literally true for $n = dim(X) = 2$, that formally implies the inequality $||R_{ij}(s)|| \le \frac{1}{2}$ for all $n\ge 2$.

And since the Clifford multiplication operators $s\mapsto e_i\cdot e_j\cdot s$ are unitary,
$$||\mathcal{R}_{\otimes L}(s\otimes l)|| \le \frac{1}{4}n(n-1) = \frac{1}{4}Sc(S^n)$$

This doesn't, a priori, imply this inequality for all (non-pure) vectors v on the tensor product $\mathbb{S}\otimes L$ for $L=\mathbb{S}$, but, by diagonalising the Clifford multiplication operators in a suitable basis and by employing the *essential constancy*[36] of the curvature R_{ij} of S^n, Llarull [Ll 1998] shows that
$$||\langle\mathcal{R}_{\otimes L}(\underline{\theta}),\underline{\theta}\rangle|| \ge -\frac{1}{4}n(n-1)$$
for all unit vectors $\underline{\theta}\in\mathbb{S}(S^n)\otimes\mathbb{S}(S^n)$.

This inequality for twisted spinors on S^n trivially yields the corresponding inequality on all manifolds X mapped to S^n, where the bundle $L\to X$ is the induced from the spin bundle $\mathbb{S}(S^n)$.

Namely, let $X=(X,g)$ be an n-dimensional Riemannin manifold, $f: X\to S^n$ be a smooth map, $L = f^*(\mathbb{S}(S^n))$, let $df : T(X)\to T(S^n)$ be the differential of f and
$$\wedge^2 df : \wedge^2 T(X) \to \wedge^2 T(S^n)$$

[36]Some eigenvalues of this operator are ±1 and some zero.

be the exterior square of df.[37]

Then the operator

$$\mathcal{R}_{\otimes L} : \mathbb{S}(X) \otimes L \to \mathbb{S}(X) \otimes L$$

satisfies

$$||\langle \mathcal{R}_{\otimes L}(\theta), \theta \rangle|| \geq -|| \wedge^2 df|| \frac{n(n-1)}{4}, \ L = f^*(\mathbb{S}(S^n)),$$

for all unit vectors $\theta \in \mathbb{S}(X) \otimes f^*(\mathbb{S}(S^n))$.

Moreover, – this is formula (4.6) in [Ll 1998]] –

$$||\langle \mathcal{R}_{\otimes L}(\theta), \theta \rangle|| \geq -\frac{1}{4}|trace \wedge^2 df|,$$

where $trace \wedge^2 df$ at a point $x \in X$ stands for

$$\sum_{i \neq j} \lambda_i \lambda_j,$$

for the differential $df : T_x(X) \to T_{f(x)}(S^n)$ diagonalised to the orthogonal sum of multiplications by λ_i.

This inequality, restricted to $L^+ = f^*(\mathbb{S}^+(S^n))$ together with the index formula, which says for this L_+ that

$$ind(\mathcal{D}_{\otimes L^+}) = \frac{|deg(f)|}{2} \chi(S^n),$$

provided X is a closed oriented spin manifold.

Thus we arrive at Llarull's theorem in the form suggested by Mario Listing [List 2010].

$\bigstar$ *$trace \wedge^2 df$-Inequality.* Let X be a closed orientable Riemannian spin n-manifold and $f : X \to S^n$ a smooth map of nonzero degree.

If

$$Sc(X, x) \geq \frac{1}{4}|trace \wedge^2 df(x)|$$

at all points $x \in X n$ then, in fact, $Sc(X) = \frac{1}{4}|trace \wedge^2 df|$ everywhere on X.

About the proof. If n is even and $\chi(S^n) = 2 \neq 0$, this follows from the above. And if n is odd, there are (at lest) three different reductions to the even dimensional one (see [Ll 1998], [List 2010], [G(inequalities 2018]). Also see see [Ll 1998] and [List 2010], for characterisation of maps f, where $Sc(X) = \frac{1}{4}|trace \wedge^2 df|$.

llarull's theorem, starting from his estimate for $\mathcal{R}_{\otimes f*(S(S^n))}$, was generalized by Goette and Semmelmann [GS 2002] to Riemannian manifolds $\underline{X}$ with *non-negative curvature operators* instead of S^n. We state below their result only the case of $\underline{X}$ homeomorphic to S^n, where our formulation follows that in [List 2010]

$\bigstar\bigstar$ *$\wedge^2 df$-Inequality.* Let $\underline{X} = (S^n, \underline{g})$ where $\underline{g}$ is a Riemannin metric with *non-negative curvature operator*, let X be a closed orientable Riemannin spin n-manifold and

$$f : X \to \underline{X}$$

[37] Recall that the norm $|| \wedge^2 df||$ measures by how f contracts/expands surfaces in X. For instance the inequality $|| \wedge^2 df||1$ signifies that f decreases the areas of the surfaces in X.

smooth map of non-zero degree.

If

$$Sc(X,x) \geq \| \wedge^2 df \| Sc(\underline{g}, f(x))$$

at all $x \in X$, *then* $Sc(X,x) = \| \wedge^2 df \| Sc(\underline{g}, f(x))$.

See [GS 2002] and [List 2010] for the proof, where the authors also identify the extremal cases, where f is an isometry or close to an isometry.

Examples. (a) The induced metrics on convex hypersurfaces $\underline{X} \subset \mathbb{R}^{n+1}$ have non-negative positive curvature operators.

Thus, $[X^{\to\bigcirc}]$ from section 3.5 is a special case of $\star\star$.

(b) By a theorem of Alan Weinstein [Wein 1970], the above (a) remains true for submanifolds $\underline{X}^n \subset \mathbb{R}^{n+2}$ with non-negative sectional curvatures of the induced metrics.

In particular,

the induced Riemannin metrics on convex hypersurfaces in S^{n+1} *and, more generally, on convex hypersurfaces* $\underline{X}^n \subset \Sigma^{n+1}$, *where* Σ^{n+1} *themselves are convex in* $\mathbb{R}^{n+2}$, *have non-negative curvature operators.*

Accordingly, $[X^{\to\bigcirc}]$ generalizes to this case.

$\star\star\star$ *Products and Stabilisation.* We shall need (see section 5.5, 5.6) a generalization of theorem $\star\star$ to maps

$$f : X \to \underline{X} = (S^m, \underline{g}) \times \mathbb{T}^{n-m},$$

where $\underline{g}$ is a metric with non-negative curvature operator and $\mathbb{T}^{n-m}$ is the torus with a Riemannin flat metric.

This is achieved (compare §$5\frac{4}{9}$ in [G(positive) 1996]) by replacing the bundle $\underline{L} = \mathbb{S}^{+}(S^n)$ in $\star\star$ by $\mathbb{S}^{+}(S^m) \otimes L_p$, where L_p are flat line bundles L_p over $\mathbb{T}^{n-m}$ as in section 3.11.[38]

Furthermore, whenever this kind of argument applies to $\underline{X}_1$ and $\underline{X}_2$, it goes over to maps $X \to \underline{X}_1 \times \underline{X}_2$.

Almost Example. Let $\underline{X}_1 = (S^m, \underline{g})$, where $\underline{g}$ is a metric with non-negative curvature operator and let $\underline{X}_2$ be a manifold which admits a complete metric with $sect.curv(X_2) \leq 0$.

Let X be a closed orientable Riemannian spin n-manifold and let

$$f : X \to X \to \underline{X}_1 \times \underline{X}_2$$

be a smooth map, such that the image of the fundamental class of X,

$$f_*[X] \in H^n(\underline{X}_1 \times \underline{X}_2; \mathbb{Q}),\ n = dim(X),$$

doesn't vanish.

If the composition of f *with the projection* $\underline{X}_1 \times \underline{X}_2 \to \underline{X}_1$, *that is* $f_1 : X \to \underline{X}_1$, *satisfies*

$$Sc(X,x) \geq \| \wedge^2 df_1 \| Sc(\underline{g}, f_1(x))$$

at all $x \in X$, *then* $Sc(X,x) = \| \wedge^2 df_1 \| Sc(\underline{g}, f_1(x))$.

[38] Instead of flat family L_p one can use individual almost flat bundles over the universal cover $\mathbb{R}^{n-m}$ of $\mathbb{T}^{n-m}$ or any other, possibly infinite dimensional, flat or almost flat bundle used in some proof of non-existence of metrics with $Sc > 0$ on tori.

Non-Spin Remark. If X is not assumed spin, then, by the arguments from section 5, one can prove the following rough bound on the Lipschitz constant of f_1.

Let $\underline{X}_1$ is the standard m-sphere with the metric of constant sectional curvature 1 and let $Sc(X) \geq m(m-1)$.

Then the X_1-components $f_1 : X \to \underline{X}_1 = S^m$ of the maps $f : X \to \underline{X}_1 \times \underline{X}_2$, such that $f_*[X] \neq 0$, satisfy

$$\|\|df_1\| \geq \frac{1}{\pi}.$$

And if $n = 4$ then - this follows from 5.5 – *the maps $f : X \to S^4$ of non-zero degrees satisfy the sharp inequality*

$$\|df_1\| \geq 1$$

(which is weaker than $\| \wedge^2 df_1\| \geq 1$) which holds in the spin case.

Categorical Remark. The above suggests that the geometry of Riemannian manifolds $X = (X, g)$, where $Sc(g) > 0$ is well depicted by the Sc-*normalised metric* $Sc(X) \cdot g$ and that maps, which are *1-Lipschitz with respect to the* Sc-*normalised metrics* can be taken for morphisms in the category of manifolds with $Sc > 0$.

4.3 Bounds on Mean Convex Hypersurfaces

Recall that the *hyperspherical radius* $Rad_{S^{n-1}}(Y)$ of a connected orientable Riemannin manifold of dimension $(n-1)$ is the supremum of the radii R of the spheres $S^{n-1}(R)$, such that X admits a distance decreasing map $f : Y \to S^{n-1}(R)$ of non-zero degree, where this f for non-compact Y this map is supposed to be constant at infinity.[39]

We already indicated in section 3.6 also see [G(boundary) 2019] that Goette-Semmlenann's theorem (above $\star\star$), applied to smoothed doubles $\mathbb{D}X$ and $\mathbb{D}\underline{X}$ yields the following corollary.

$\bigcirc^{n-1}$ *Let X be a compact orientable Riemannin manifold with boundary $Y = \partial X$.*

If $Sc(X) \geq 0$ and the mean curvature of Y is bounded from below by $mean.curv(Y) \geq \mu > 0$, then the hyperspherical radius of Y for the induced Riemannin metric is bounded by

$$Rad_{S^{n-1}}(Y) \leq \frac{n-1}{\mu}.$$

In fact, the proof of this indicated in section 3.6 (also see [G(boundary) 2019]) together with the above $\star\ \star\ \star$ yields the following more general theorem.

$\bullet_{n,n-1}$ Let X and $\underline{X}$ be compact connected orientable Riemannian n-manifolds with boundaries $Y = \partial X$ and $\underline{Y} = \partial \underline{X}$, and let $f : X \to \underline{X}$ be a smooth proper[40] map of *non-zero degree.*

---

[39] Alternatively, one might require f to be *locally* constant at infinity, or more generally, to have the limit set of codimension≥ 2 in $S^{n-1}(R)$.

[40] Here, "proper" means boundary$\to$ boundary.

Let $\underline{X}$ admit a locally convex isometric immersion to $\mathbb{T}^{n+1}$ and let the boundary $\underline{Y}$ of $\underline{X}$ be (geodesically) convex in $\underline{X}$.

If X is spin, if

SCAL $$Sc(X,x) \geq \| \wedge^2 df \| Sc(\underline{X}, f(x)) \text{ for all } x \in X$$

and if

MEAN $$mean.curv(Y,y) \geq \|df\| mean.curv(\underline{Y}, f(y)) \text{ for all } y \in Y,$$

then, in fact,

$$Sc(X,x) = \| \wedge^2 df \| Sc(\underline{X}, f(x))$$

and

$$mean.curv(Y,y) = \|df\| mean.curv(\underline{Y}, f(y)).$$

Remarks. (a) If $Sc(\underline{X}) = 0$, e.g. if $\underline{X}$ is a convex subset in $\mathbb{R}^{n+1}$, then the condition **SCAL** reduces to $Sc(X) \geq 0$.

(b) The above also yields some information on manifolds X with *negative scalar curvatures bounded from below.*

For instance, if $Sc(X) \geq -m(m-1)$, then $\bullet_{n,n-1}$ applies to maps from $f: X \times S^m$ to the $(m+n)$-balls $B^{m+n} \subset \mathbb{R}^{m+n}$ (see [G(boundary) 2019]).

However, the sharp inequalities for $Sc(X) < 0$, such, for instance, as *optimality* of the hyperspherical radii of the boundary spheres of balls $B^n(R)$ in the hyperbolic spaces $\mathbb{H}^n_{-1}$, remain *conjectural*.[41]

(c) It is unknown if the spin condition on X is necessary, but it can be relaxed by requiring the universal cover of X, rather than X itself, to be spin. In fact, $\bullet_{n,n-1}$ generalizes to non-compact complete manifolds with an extra attention to uniformity of the curvature inequalities involved.

And if one is content with a non-sharp bound

$$Rad_{S^{n-1}}(Y) \leq \frac{const_n}{\inf mean.curv(Y)},$$

then one and can prove this without the spin assumption by the "cubical type argument" from section 5.4.

4.4 Lower Bounds on the Dihedral angles of Curved Polyhedral Domains.

We want to generalise the above $\bullet_{n,n-1}$ to manifolds X with non-smooth boundaries with suitably defined mean curvatures bounded from below, where we limit ourself to manifolds with rather simple singularities at their boundaries.

Namely, let X and $\underline{X}$ be Riemannian n-manifolds *with corners*, which means that their boundaries $Y = \partial X$ and $\underline{Y} = \partial \underline{X}$ are decomposed into $(n-1)$-faces F_i and $\underline{F}_i$ correspondingly, where, locally, at all points $y \in Y$, and $\underline{y} \in \underline{Y}$ these decompositions are is diffeomorphic to such decomposition of the boundary of a convex n-dimensional polyhedron (polytope) in $\mathbb{R}^n$.

[41] This "optimality" means that if $Sc(X) \geq -n(n-1)$ and $mean.curv(\partial X) \geq mean.curv(\partial B^n(R))$ than $Rad_{S^{n-1}}(\partial X) \leq Rad_{S^{n-1}}(\partial B^n(R))$.

Let $f : X :\to \underline{X}$ be a smooth map, which is compatible with the corner structures in X and $\underline{X}$:

f sends the $(n-1)$-faces F_i of X to faces $\underline{F}_i$ of $\underline{X}$.

Assume as earlier that

$$[\geq]^{SCAL} \qquad Sc(X,x) \geq \| \wedge^2 df\| \cdot Sc(\underline{X}, f(x)) \text{ for all } x \in X$$

and replace **MEAN** by the corresponding condition applied to *for all faces* $F_i \subset Y$ individually,

$$[\geq]^{MEAN}_{\{i\}}, \qquad mean.curv(F_i, y) \geq \|df\| \cdot mean.curv(\underline{F}_i, f(y)) \text{ for all } y \in F_i\ .$$

Let $\angle_{i,j}(y)$ be the dihedral angle between the faces F_i and F_j at $y \in F_i \cap F_i$ and let us impose our main inequality between these $\angle_{i,j}(y)$ for all F_i and $F)j$ and the dihedral angles between the corresponding faces faces $\underline{F}_i$ and $\underline{F}_j$ at the points $f(y) \in \underline{F}_i \cap \underline{F}_j$:

$$\left[\leq\right]^{\angle_{ij}} \qquad \angle_{i,j}(y) \leq \angle_{i,j}(f(y)) \text{ for all } F_i, F_j \text{ and } y \in F_i \cap F_j.$$

Besides the above, we need to add the following condition the relevance of which remains unclear.

Call a point $y \in Y = \partial X$ *suspicious* if one of the following two conditions is satisfied

(i) the corner structure of X at y is *non-simple* (not cosimplicial), where simple means that a neighbourhood of y is diffeomorphic to a neighbourhood of a point in the n-cube, which is equivalent to transversality of the intersection of the $(n-1)$-faces which meet at $\underline{y}$;

(ii) there are two $(n-1)$-faces in X which contain y, say $F_i \ni y$ and $F_j \ni y$, such that the dihedral angle $\angle_{ij} = \angle(F_i, F_j$ is $> \frac{\pi}{2}$;

Then out final condition says that

$$\left[=\right]^{\angle_{ij}} \qquad \angle_{i,j}(y) = \angle_{i,j}(f(y)).$$

for all suspicious points y.

◆$_{\angle_{ij}}$ *Theorem.* Let $f : X \to \underline{X}$ be a smooth map between connected orientable n-dimensional Riemannian manifolds with corners, where this map respects the corner structure and satisfies the above conditions $[\geq]^{SCAL}$, $[\geq]^{MEAN}_{\{i\}}$, $\left[\leq\right]^{\angle_{ij}}$ and $\left[=\right]^{\angle_{ij}}$.

If X is spin, $\underline{X}$ admits a locally convex isometric immersion to $\mathbb{T}^{n+1}$, the boundary of $\underline{X}$ is convex and the map f has non-zero degree, then f satisfies the equalities corresponding to the inequalities $[\geq]^{SCAL}$, $[\geq]^{MEAN}_{\{i\}}$ and $\left[\leq\right]^{\angle_{ij}}$:

$$Sc(X,x) = \| \wedge^2 df\| \cdot Sc(\underline{X}, f(x)) \text{ for all } x \in X,$$

$$mean.curv(F_i, y) = \|df\| \cdot mean.curv(\underline{F}_i, f(y)) \text{ for all } y \in F_i,$$

$$\angle_{i,j}(y) = \angle_{i,j}(f(y)) \text{ for all } F_i, F_j \text{ and } y \in F_i \cap F_j.$$

About the Proof. This is shown by smoothing the boundaries of X and applying $\bullet^{n,n-1}$ from the previous section, where an essential feature of non-suspicious points follows from the following

Elementary Lemma. Let $\Delta \subset S^n$ be a spherical simplex with all edges of length $\geq l \geq \frac{\pi}{2}$. Then there exists a continuous family of simplices $\Delta_t \subset S^n$, $t \in [0,1]$ with the following properties.

- $\Delta_0 = \Delta$ *and* Δ_1 *is a regular simplex with the edge length* l*;*
- *all* Δ_t *have the edges of length* $\geq l$*;*
- $\Delta_{t_2} \subset \Delta_{t_1}$ *for* $t_2 \geq t_1$*;*
- *for each* $t < 1$ *there exists an* $\varepsilon > 0$*, such that* n *(out of* $n+1$*) vertices of* $\Delta_{t+\varepsilon}$ *coincide with those of* Δ_t*.*

The proof of the lemma is a high school exercise while *construction of adequate smoothing of* X with the help of this lemma, which is straightforward and boring, will be given elsewhere. $\blacklozenge_{\angle ij}$

Notice that the $\times\blacktriangle^i$-*Inequality* from section 3.10, which says that

convex polyhedra $\underline{X} \subset \mathbb{R}^n$ with the dihedral angles $\leq \frac{\pi}{2}$ admit no deformations which would *decrease their dihedral angles* and simultaneously *increase the mean curvatures of their faces,*

is an immediate corollary of $\blacklozenge_{\angle ij}$.

But it remains unclear what is the *full class* of polyhedra which enjoy this property.

Fundamental Domains of Reflection Groups. What underlies the double $\mathbb{D}$-construction, $X \rightsquigarrow \mathbb{D}X$ in the proof of the $\blacklozenge_{\angle ij}$ theorem is the doubling $S^n = \mathbb{D}S^n_+$, which is associated with the reflection of S^n with respect to the equatorial subsphere.

With this in mind, one can generalise everything from this section to general reflection groups, including spherical, Euclidean, "abstract" (semi)hyperbolic ones, (such as what we met in weak $\neg$-reflection rigidity theorem in section 3.16.) and also products of these.

Example. Let X be a manifold with corners, where the (combinatorial) corner structure is isomorphic to that of the product of an $(n-m)$-simplex $\blacktriangle$ with the rectangular fundamental domain $\blacksquare$ (orbifold) of a reflection group in an aspherical m-manifold which is *non-diffeomorphic to* $\mathbb{R}^m$. (These exist for all $m \geq 4$ by Michael Davis 1983 theorem, see his lectures [Dav 2008] and references therein.)

X admits no Riemannian metric with $Sc \geq 0$, with all faces having mean.curv ≥ 0 and with the dihedral angles smaller than those in the product of a regular Euclidean simplex $\blacktriangle$ by $\blacksquare$ with $\frac{\pi}{2}$ dihedral angles.

On Necessity of Condition $[=]^{\angle ij}$

4.5 Stability of Geometric Inequalities with $Sc \geq \sigma$ and Spectra of Twisted Dirac Operators.

Sharp geometric inequalities beg for being accompanied by their nearest neighbours.

For instance, the Euclidean isoperimetric inequality for bounded domains $X \subset \mathbb{R}^n$, which says that

$$vol_n(X) \le \gamma_n vol_{n-1}(\partial X)^{\frac{n}{n-1}} \text{ for } \gamma_n = \frac{vol(B_n)}{vol_{n-1}(S^{n-1})^{\frac{n}{n-1}}},$$

goes along with the following.

A. *Rigidity. If $vol_n(X) = \gamma_n vol_{n-1}(\partial X)^{\frac{n}{n-1}}$, then X is a ball.*

B. *Isoperimetric Stability.* Let $X \subset \mathbb{R}^n$ be a bounded domain with $vol_n(X) = vol_n(B^n)$ and $vol(\partial X) \le vol_{n-1}(S^n) + \varepsilon$.

Then there exists a ball $B = B^n_x(1+\delta) \subset \mathbb{R}^n$ of radius δ with center $x \in X$, where $\delta \underset{\varepsilon\to 0}{\to} 0$, such that the volume of the difference satisfies

$$vol_n(X \smallsetminus B) \le \delta_1,$$

and, moreover,

$$vol_{n-1}(\partial B \cap X) \le \delta_2, \text{ and } vol_{n-2}(\partial B \cap \partial X) \le \delta_3,$$

where

$$\delta_1, \delta_2, \delta_3 \underset{\varepsilon\to 0}{\to} 0.$$

(Unless $n = 2$ and X is connected, there is no bound on the diameter of X, but the constants $\delta, \delta_1, \delta_2, \delta_3$ can be explicitly evaluated even for moderately large ε.)

Turning to scalar curvature, observe, following Llarull, Min-Oo and Goette-Semmelmann, that their proofs (see [Ll 1998], [Min(Hermitian) 1998], [GS 2002]) (more or less) automatically deliver rigidity. For instance,

★ *if a manifold $\underline{X}$ homeomorphic to S^n, besides having $curv.oper(\underline{X}) \ge 0$ has $Ricci(\underline{X}) > 0$ and if X is a closed orientable spin Riemannin manifold with $Sc(X) \ge n(n-1)$ then, all* smooth 1-Lipschitz *maps $X \to \underline{X}$ of non-zero degrees are* isometries. [42]

What we want to understand next is what happens if the inequality $Sc(X) \ge n(n-1)$ is relaxed to $Sc(X) \ge n(n-1) - \varepsilon$ for a small $\varepsilon > 0$, where one has to keep in mind the following.

Example. (Compare [GL(classification) 1980], [BDS 2018] and section 2, and 23 in [G(questions) 2017].) Let $\Sigma \subset S^n$ be a compact smooth submanifold of dimension $\le n-3$. Then there exists an arbitrary small ε-neighbourhood $U_\varepsilon = U_\varepsilon(\Sigma) \subset S^n$ with a smooth boundary $\partial_\varepsilon = \partial U_\varepsilon$ and a family of smooth metrics $g_{\varepsilon,\epsilon}$ on the double

$$\mathbb{D}(S^n \smallsetminus U_\varepsilon) = (S^n \smallsetminus U_\varepsilon) \cup_{\partial_\varepsilon} (S^n \smallsetminus U_\varepsilon),$$

where $Sc(g_{\varepsilon,\epsilon}) \ge n(n-1) - \varepsilon - \epsilon$ and which, for $\epsilon \to 0$, *uniformly converge* to the natural continuous Riemannian metric on $\mathbb{D}(S^n \smallsetminus U_\varepsilon(\Sigma))$.

[42] Even if Ricci vanishes somewhere, one still may have a satisfactory description of the extremal cases. For instance, if $\underline{X} = (S^{n-m} \times \mathbb{R}^m)/\mathbb{Z}^m$, e.g. $\underline{X} = S^{n-m} \times \mathbb{T}^m$, then all (orientable spin) X with $Sc(X) \ge Sc(\underline{X}) = (n-m)(n-m-1)$, which admit maps $f : X \to \underline{X}$ with $deg(f) \ne 0$, are *locally isometric* to $\underline{X}$ (albeit the map f itself doesn't have to be a local isometry.

Moreover, if $\Sigma \subset S^n$ is contained in a hemisphere, then – this follows from the *spherical Kirszbraun theorem* – the (double) manifolds $\mathbb{D}(S^n \setminus U_\varepsilon, g_{\varepsilon,\epsilon})$ admit 1-Lipschitz maps to the sphere S^n with degrees one, for all sufficiently small $\varepsilon > 0$ and , $\epsilon = \epsilon(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$.

For instance, if $n \geq 3$ and Σ consists of a single point, then $\mathbb{D}(S^n \setminus U_\varepsilon)$, that is the connected sum $S^n \# S^n = S^n \#_{S^{n-1}(\varepsilon)} S^n$ of the sphere S^n with itself (where the ε-sphere $S^{n-1}(\varepsilon)$ serves as ∂_ε and $S^n \# S^n$ is homeomorphic to S^n), admits, for small ε, a 1-Lipschitz map to S^n with degree 2.

Furthermore, iteration of the connected sum construction, delivers manifolds (topologically spheres)

$$(S^n)^{k\#_\varepsilon} = \underbrace{S^n \#_{S^{n-1}(\varepsilon)} S^n \# ... \#_{S^{n-1}(\varepsilon)} S^n m}_{k}$$

which carry metrics with $Sc(S^n)^{k\#_\varepsilon} \geq n(n-1) - \varepsilon - \epsilon$ and, at the same time, admit maps to S^n of degree k, where these maps are 1-Lipschitz everywhere and which are locally isometric away from $\sqrt{\varepsilon}$-neighbourhoods of $k-1$ ε-spherical "necks" in $(S^n)^{k\#_\varepsilon}$.

(For general Σ and even k one has such maps f with $deg(f) = k/2$.

Conjecturally, this example faithfully represents possible geometries of closed Riemannian n-manifolds X with $Sc(X) \geq n(n-1) - \varepsilon$, which admit 1-Lipschitz maps to the unit sphere S^n, but only the following two, rather superficial, results of this kind are available.

1. Let $X = (X, g)$ be a closed oriented Riemannin spin n-manifold with $Sc(X) \geq n(n-1) - \varepsilon$ and let $f : X \to \underline{X} = S^n$ be a smooth 1-Lipshitz map of degree $d \neq 0$.

Denote by $\tilde{g}$ the (possibly singular) Riemannin metric on X induced by f from the spherical metric $\underline{g}$ on $\underline{X} = S^n$ and let $\underline{l}(f, x)$ be the minimum

$$\underline{l}_f(x) = \min_{\|\tau\|_g = 1} \|df(\tau)\|_{\underline{g}}, \ \tau \in T_x(X).$$

(Since f is 1-Lipschitz, $\underline{l}(f,x) \leq 1$ and $(\underline{l}(f,x))^{-1}$ measures the the distance from the differential $df(x) : T_x(X) \to T_{f(x)}(\underline{X})$ to an isometry.)

Let

$$\tilde{V} = \widetilde{vol}(X) = vol_{\tilde{g}}(X) = \int_{\underline{X}} card(f^{-1}(\underline{x}))d\underline{x}$$

be the $\tilde{g}$-volume of X.

Then the $\widetilde{g}$-volume of the subset $X_{\leq\lambda} \subset X$, $\lambda < 1$, where $\underline{l}_f(x) \leq \lambda$ satisfies

$$\widetilde{vol}(X_{\leq\lambda}) \leq c_{\lambda,n,\tilde{V}}(\varepsilon) \underset{\varepsilon\to 0}{\to} 0. \quad [|X_{\leq\lambda}| \leq]$$

Sketch of the Proof. Since the twisted Dirac operator $D_\otimes$ in Llarull's rigidity argument from [Ll 1998] has non-zero kernel, its square $D^2_\otimes$ is non-positive (we assume here that $n = dim(X) = dim(\underline{X})$ is even), and, by the Bochner-Schrödinger-Lichnerowicz-Weitzenböck formula (that is above $[D^2_\otimes]_f$), this implies non-positivity of

$$\nabla^2 + \frac{1}{4}Sc(X) + \mathcal{R}_\otimes.$$

Consequently, $-\Delta_g - \frac{1}{4}(\varepsilon + (1 - \underline{l}(x)))$, where Δ_g is an ordinary Laplace operator on $X = (X, g)$, also non-positive, since the coarse (Bochner) Laplacian

∇^2 is "more positive" than the (positive) Laplace(-Beltrami) operator $-\Delta$ as it follows from the *Kac-Feynman* formula and/or from the *Kato inequality.*

(In general, this applies in the context of the above rigidity theorem ★ and yields non-positivity of $-\Delta_g - \frac{1}{4}(\varepsilon + \underline{C}(1 - \underline{l}_f(x))$ with $\underline{C}$ depending on the smallest eigenvalue of $Ricci(\underline{X})$.)

In order to extract required geometric information concerning the metric $\tilde{g}$ from this property of the metric g, we observe that the essential part of X, that is the one, where we need to bound from below the L_2-norms of the g-gradients of functions $\phi(x)$ (to which the above Δ_g applies) is where

$$\lambda \geq \underline{l}_f(x) \geq \lambda_{\tilde{V}} > 0$$

for some $\lambda_{\tilde{V}} > 0$, and where the geometries of g and of $\tilde{g}$ are mutually $(\lambda_{\tilde{V}})^{-1}$-close.

Thus, the relevant lower g-gradient estimate for $\phi(x)$ comes from the isoperimetric inequality for $\tilde{g}$ which, in turn, follow from such an inequality in $\underline{X}$, that is the sphere in the present case. (Filling in the details is left to the reader.)

Remark. (a) The above example shows that the g-volume of $X_{\leq\lambda} \subset X$ can be large and that the bound on $\tilde{V}$ concerns not only the subset $X_{\leq\lambda}$ but its complement $X \setminus X_{\leq\lambda}$ as well.

Corollary + Question. (a) Let X be a closed orientable Riemannin spin n-manifold with $Sc(X) \geq n(n-1)$ and let $f : X \to S^n$ a (possibly non-smooth!) 1-Lipshitz map of degree $\neq 0$.

If the map Y is a homeomorphism, then it is an isometry.

(b) Is this remain true for *all* 1-Lipshitz maps?

The inequality $[|X_{\leq\lambda}| \leq]$ doesn't take advantage of $deg(f)$ when this is large, but the following proposition does just that.

2. Let X be a compact oriented Riemannian spin n-manifold with a boundary $Y = \partial X$, such that $Sc(X) \geq n(n-1) + \varepsilon$, $\varepsilon > 0$.

Let $f : X \to S^n$ be a smooth map, which is constant on Y, which is *area contracting away from the a neighbourhood $U \subset X$ of $Y = \partial X \subset X$,*

$$\| \wedge^2 df(x)\| \leq 1 \text{ for all } x \in X \setminus U,$$

and where

$$\| \wedge^2 df(x)\| \leq C_o \text{ for all } x \in X \setminus U \text{ and some constant } C_o > 0.$$

Then the degree of f is bounded by a constant d depending only on U and on C_o,

$$|deg(f)| \leq d = const_{U,C_o}.$$

Sketch of the Proof. (Compare with §§$5\frac{1}{2}$ and 6 in [G(positive) 1996].) Let $s(x)$ be the (Borel) function on X which equals to ε away from U and is equal to $E = -C_n \times C_o$ on U for some universal $C_n \approx n^n$.

Then arguing (essentially) as in the first part of the above proof, we conclude that the spectrum of the operator $-\Delta + s(x)$ on the (smoothed) double $\mathbb{D}(X)$ contains at least $d = deg(f)$ negative eigenvalues.

This an easy argument would deliver d eigenvalues λ_i of the operator $-\Delta$ on $\mathbb{D}(U)$, where the corresponding eigenfunctions vanish on the two copies of the boundary of U in X (but not, necessarily on Y), and such that $\lambda_i \lessapprox E$.

This would yield the required bound on d. (Here again, the details are left to the reader.)

Remark + Example + Two Problems. (a) If the boundary of $Y = \partial X$ admits an orientation reversing involution, then the constancy of f on Y can be relaxed to $dim(f(Y)) \leq n-2$, where the constant d will have to depend on the geometry of this involution and of the map $Y \to S^n$.

(It is unclear if the existence of such an involution is truly necessary.)

(b) This (a) apply, for instance, to coverings $X = \Sigma^2_{d,\delta}$ of the 2-sphere minus two δ-discs as well as to the products of these $\Sigma^2_{d,\delta}$ with the Euclidean ball $B^{n-1}(R)$ of radius $R > \pi$.

(c) What are the sharp and/or comprehensive versions of these **1** and **2**?

(d) Let Y be a homotopy sphere of dimension $4k-1$, which bounds a Riemannin manifold X with $Sc \geq \varepsilon > 0$. Give an *effective* bound on the $\hat{A}$-genus of X in terms of the geometry of Y and its second fundamental form $h = \mathrm{II}(Y \subset X)$ and study the resulting invariant

$$Inv_\varepsilon(Y,h) = \sup_X |\hat{A}(X)|, \text{ where } \partial X = Y,\ Sc(X) \geq \varepsilon,\ \mathrm{II}(Y \subset X) = h.$$

5 Stable μ-Bubbles in Manifolds with $Sc \geq \sigma$

5.1 Variation of Minimal Bubbles and Modification of their Metrics

Given a a Borel measure μ on an n-dimensional Riemannian manifold X, μ-*bubbles* are critical points of the following functional on a topologically defined class of domains $U \subset X$ with boundaries called $Y = \partial U$:

$$(U, Y) \mapsto vol_{n-1}(Y) - \mu(U).$$

Observe that in our examples, $\mu(U) = \int_U \mu(x)dx$ for (not necessarily positive) continuous functions μ on X and that $\mu(U)$ can be regarded as a *closed 1-form* on the space of cooriented hypersurfaces $Y \subset X$. Then $vol_{n-1}(Y) - \mu(U)$ also comes as such an 1-form which we denote $vol^{[-\mu]}_{n-1}(Y)(+const)$.

The first and the second variations of $vol^{[-\mu]}_{n-1}(Y)(+const)$ are the sums of these for $Vol_{-1}(Y)$ and of $vol(U)$ where the former were already computed in section 2.5.

And turning to the latter, it is obvious that the first derivative/variation of $\mu(U)$ under $\psi\nu$, where ν is the outward looking unit normal normal field to Y and $\psi(y)$ is a function on Y, is

$$\partial_{\psi\nu} \int_U \mu(x)dx = \int_Y \mu(y)\psi(y)dy$$

and the second derivative/variation is

$$\partial^2_{\psi\nu} \int_U \mu(x)dx = \partial_{\psi\nu} \int_Y \mu(y)\psi(y)dy = \int_Y (\partial_\nu \mu(y) + M(y)\mu(y))\psi^2(y)dy,$$

where the field ν is extended along normal geodesics to Y, (compare section 2.5) and where $M(y)$ denotes the mean curvature of Y in the direction of ν.

It follows that μ-bubbles Y, (critical points of $vol^{[-\mu]}_{n-1}(Y) = vol_{n-1}(Y) - \mu(U)$) have

$$mean.curv(Y) = \mu(y)$$

and that

second variation of *locally minimal bubbles* $Y \subset X$,

$$\partial_{\psi\nu}(vol^{[-\mu]}_{n-1}(Y)) = \partial_{\psi\nu}\left(vol_{n-1}(Y) - \int_U \mu(x)dx)\right),$$

is non-positive.

Then we recall, the formula [○○] from section 2.5

$$\partial^2_{\psi\nu} vol_{n-1}(Y) = \int_Y \|d\psi(y)\|^2 dy + R_-(y)\psi^2(y)dy$$

for

$$R_-(y) = -\frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - \sum_{i=1}^{n-1}\alpha_i(y)^2\right),$$

where $\alpha_i(y)$ are the principal curvatures of Y at y, and where $\sum \alpha_i^2$ is related to the mean curvature $M = \alpha_1 + ... + \alpha_{n-1}$, by the inequality

$$\sum \alpha_i^2 \geq \frac{M^2}{n-1}.$$

Thus, summing up all of the above, observing that

$$\partial_\nu \mu(x) \geq -\|d\mu(x)\|$$

and letting

[$R_+ =$] $$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + Sc(X,x),$$

we conclude that

if Y *locally minimises* $vol^{[-\mu]}_{n-1}(Y)(= vol_{n-1}(Y) - \mu(U))$, *then*

$$\int \|d\psi\|^2 dy + \left(\frac{1}{2}Sc(Y) - \frac{1}{2}R_+(y)\right)\psi^2(Y)dy \geq \partial_{\psi\nu} vol^{[-\mu]}_{n-1}(Y) \geq 0$$

for all functions ψ *on* Y.

Hence,

✤$_{\geq 0}$ *the operator* $-\Delta + \frac{1}{2}Sc(Y,y) - \frac{1}{2}R_+(y)$, *for* $\Delta = \sum_i \partial^2_{ii}$ *is positive on* Y.

Examples. (a) Let $X = \mathbb{R}^n$ and $\mu(x) = \frac{n-1}{r}$, that is the mean curvature of the sphere of radius r. Then

$$R_+(x) = \frac{n(n-1)}{R} - 2\frac{n-1}{r^2} + 0 = \frac{(n-1)(n-2)}{r^2} = Sc(S^{n-1}(r)).$$

(b) Let $X = \mathbb{R}^{n-1} \times \mathbb{R}$ be the hyperbolic space with the metric $g_{hyp} = e^{2r}g_{Eucl} + dr^2$ and let $\mu(x) = n-1$. Then

$$R_+(x) = n(n-1) - 0 + (-n(n-1)) = 0 = Sc(\mathbb{R}^n).$$

(c) Let $X = Y \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ with the metric $\varphi^2 h + dt^2$, where the metric h is a metric on Y and where

$$\varphi(t) = \exp \int_{-\pi/n}^{t} - \tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}.$$

Then a simple computation shows that

$$R_+(x) = \frac{n(n-1)}{R} - 2\frac{n-1}{r^2} + 0 = \frac{(n-1)(n-2)}{r^2} = Sc(S^{n-1}(r)).$$

$$\frac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + n(n-1) = 0.$$

Furthermore, if $Sc(h) = 0$, than $Sc(X(= n(n-1)$ and $R_+ = 0$.

Two relevant corollaries to ✤$_{\geq 0}$ are as follows.

Let X be a Riemannian manifold of dimension n, let $\mu(x)$ be a continuous function and Y be a smooth minimal μ-bubble in X.

✤$_{conf}$ *If*

$$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + Sc(X, x) > 0,$$

then by Kazdan-Warner conformal change theorem (see section 2.6) *Y admits a metric with $Sc > 0$.*

✤$_{warp}$ *There exists a metric $\hat{g}$ on the product $Y \times \mathbb{R}$ of the form $g_Y + \phi^2 dr^2$ for the metric g_Y on Y induced from X, such that*

$$Sc_{\hat{g}}(y, r) \geq R_+(y).$$

Proof. Let $\phi(y)$ be the first, necessarily positive eigenfunction of the operator $-\Delta + \frac{1}{2}Sc(g_Y, y) - R_+(y)$ and recall (see section 2.4) that $Sc(\hat{g}) = Sc(g_Y) - 2\frac{\Delta\phi}{\phi}$. Then

$$-\Delta\phi + \frac{1}{2}Sc(g_Y, y)\phi - \frac{1}{2}R_+(y)\phi = \lambda\phi, \ \lambda > 0,$$

$$\frac{\Delta\phi}{\phi} = -\lambda + \frac{1}{2}Sc(g_Y, y) - \frac{1}{2}R_+(y)$$

and

$$Sc(\hat{g}) = R_+ + 2\lambda,$$

which implies that $Sc_{\hat{g}}(y, r) \geq R_+(y)$, since $\lambda \geq 0$. QED.

5.2 On Existence and Regularity of Minimal Bubbles.

Let X be a compact connected Riemannian manifold of dimension n with boundary ∂X and let $\partial_- \subset \partial X$ and $\partial_+ \subset \partial X$ be disjoint compact domains in ∂X.

Example. Cylinders $Y \times [-1, 1]$ naturally come with such a $\partial_\mp$-pair for $\partial_- = Y \times \{-1\}$ and $\partial_+ = Y \times \{1\}$, where, observe, $\partial_- \cup \partial_+ = \partial(Y \times [-1, 1])$ if and only if Y is a manifold without boundary.

Let us agree that the mean curvature of ∂_- is evaluated with the incoming normal field and $mean.curv(\partial_+)$ is evaluated with the outbound field.

For instance, if the boundary of X is *concave*, as for instance for X equal to the sphere minus two small disjoint balls, t then $mean.curv(\partial_-) \geq 0$ and $mean.curv(\partial_+) \leq 0$.

Barrier $[\gtrless \mp mean]$*-Condition.* A continuous function $\mu(x)$ on X is said to satisfy $[\gtrless \mp mean]$-condition if

$$[\gtrless \mp mean] \qquad \mu(x) \geq mean.curv(\partial_-, v) \text{ and } \mu(x) \leq mean.curv(\partial_+, x)$$

for all $x \in \partial_- \cup \partial_+$.

It follows by the maximum principle in the geometric measure theory that

$\bigstar$ *the* $[\gtrless \mp mean]$*-condition ensures the existence of a minimal* μ*-bubble* $Y_{min} \subset X$*. which separates* ∂_- *from* ∂_-+*.*

If this condition is *strict*, i.e. if $\mu(x) > mean.curv(\partial_-)$ and $\mu(x) < mean.curv(\partial_+)$ and if X has no boundary apart from $\partial_\mp$, then $Y_{min} \subset X$ doesn't intersect $\partial\mp$; in general, the intersections $Y_{min} \cap \partial_\mp$ are contained in the *side boundary* of X that is the closure of the complement $\partial X \setminus (\partial_- \cup \partial_-)$. (This, slightly reformulated, remains true for non-strict $[\gtrless \mp mean]$.)

If $dim(X) = n \leq 7$, then, (this well known and easy to see) Federer's regularity theorem(see section 2.7) applies to minimal bubbles as well as to minimal subvarieties and the same can be said about Nathan Smale's theorem on non-stability of singularities for $n = 8$. Thus, in what follows we may assume our minimal bubbles smooth for $n \leq 8$.

Then, by the stability of Y_{min} (see section 5.1 above),

$\bullet_{\varphi_\circ}$: *there exits a function* $\phi_\circ = \phi_\circ(y) > 0$ *defined in the interior* $^\circ Y$ *of* Y, *i.e. on* $Y \setminus \partial X$, *such that the metric*

$$g_{\varphi_\circ} = \varphi_\circ^2 g_Y + dt^2 \text{ on the cylinder } {}^\circ Y \times \mathbb{R},$$

where g_Y *is the Riemannin metric on* Y *induced from* X, *satisfies*

$$\bigcirc \qquad Sc_{g_{\varphi_\circ}}(y,t) \geq Sc(X,y) + \frac{n\mu(y)^2}{n-1} - 2||d\mu(y)||$$

for all $y \in {}^\circ Y$.[43]

What if $n \geq 9$?.

The overall logic of the proof indicated in [Loh(smoothing) 2018] leads one to believe that, assuming strict $[\gtrless \mp mean]$, there always exists a smooth $Y_o \subset X$, which separates $\partial_\mp$ and and which admits a function $\phi_\circ$ with the property $\bigcirc$.

The proof of this, probably, is automatic, granted a full understanding Lohkamp's arguments. But since I have not seriously studied these arguments, everything which follows in sections 5.3-5.8 should be regarded as *conjectural* for $n \geq 9$.

Barrier $[\gtrless mean = \mp\infty]$*-Condition.* Let X be a non-compact, possibly non-complete, Riemannin manifold X and let the set of the ends of X is subdivided to $(\partial_\infty)_- = (\partial_\infty)_-(X)$ and $(\partial_\infty)_+ = (\partial_\infty)_+(X)$, where this can be accomplished, for instance, with a proper map from X to an open (finite or infinite) interval (a_-, a_+) where "convergence" $x_i \to (\partial_\infty)_\mp$, $x_i \in X$, is defined as $e(x_i) \to a_\mp$.

[43] Since the metric $g_{\varphi_\circ}$ is $\mathbb{R}$-*invariant* its scalar curvature is *constant* in $t \in \mathbb{R}$.

For example, if X is the open cylinder, $X = Y \times (a,b)$, where Y is a compact manifold, possibly with a boundary, this is done with the projection $Y \times (a_-,a_+) \to (a_-,a_+)$.

Obvious Useful Observation. If a function $\mu(x)$ satisfies

$$\mu(x_i) \to \pm\infty \text{ for } x_i \to (\partial_\infty)_\mp$$

then X can be exhausted by compact manifolds X_i with distinguished domains $(\partial_\mp)_i \subset \partial X_i$, such that

• these $(\partial_\mp)_i$ separate $(\partial_\infty)_-$ from $(\partial_\infty)_-$ for all i and

$$(\partial_\mp)_i \to (\partial_\infty)_\mp;$$

• restrictions of μ to $(X_i,(\partial_\mp)_i)$ satisfy the barrier $[\gtrless \mp mean]$-condition.

This ensures the existence of locally minimising μ-bubbles in X which separate $(\partial_\infty)_-$ from $(\partial_\infty)_+$.

5.3 Bounds on Widths of Riemannin Bands.

Let us prove the following version of the $\frac{2\pi}{n}$-inequality from section 2.6.

$\frac{2\pi}{n}$-Inequality.* Let X be an open, possibly non-complete Riemannian manifold of dimension n and let

$$f : X \to (-l,l)$$

be a proper (i.e. infinity → infinity) smooth distance non-increasing map, such that the pullback $f^{-1}(t_o) \subset X$ of a generic point t_o the interval $(-l,l)$ is non-homologous to zero in X.

If $Sc(X) \geq n(n-1) = Sc(S^n)$ and if the following condition $\nparallel_{Sc\ngtr 0}$ is satisfied, then

$$l \leq \frac{\pi}{n}.$$

$\nparallel_{Sc\ngtr 0}$ No smooth closed cooriented hypersurface in X homologous to $f^{-1}(t_o)$ admits a metric with $Sc > 0$.

Proof. Assume $l > \frac{\pi}{n}$. and let $\underline{\mu}(t)$ denote the mean curvature of the hypersurface $\underline{Y} \times \{t\}$ in the warped product metric $\varphi^2 h + dt^2$. on $\underline{Y} \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ for

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}$$

as in example (c) from the previous section.

Since $\underline{\mu}(t) \to \pm\infty$ for $t \to \mp\frac{\pi}{n}$, the barrier $[\gtrless mean = \mp\infty]$-condition from the section 5.2 guaranties the existence of a locally minimizing μ-bubble in X for μ being a slightly modified f-pullback of $\underline{\mu}$ to X.

Let us spell it out in detail.

Assume without loss of generality that the pullbacks $Y_\mp = f^{-1}\left(\mp\frac{\pi}{n}\right) \subset X$ are smooth, and let $\mu(x)$ be a smooth function on X with the following properties.

$\bullet_1$ $\mu(x)$ is constant on X on the complement of $f^{-1}\left(-\frac{\pi}{n},\frac{\pi}{n}\right)$ for $\left(-\frac{\pi}{n},\frac{\pi}{n}\right) \subset (-i,i)$;

$\bullet_2$ $\mu(x)$ is equal to $\underline{\mu} \circ f$ in the interval $\left(-\frac{\pi}{n}+\varepsilon, \frac{\pi}{n}-\varepsilon\right)$ for a given (small) $\varepsilon > 0$;

$\bullet_3$ the absolute values of the mean curvatures of the hypersurfaces $Y_\mp$ are everywhere smaller than the absolute values of μ;

$\bullet_4$ $\frac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + n(n-1) \geq 0$ at all points $x \in X$.

In fact, achieving $\bullet_3$ is possible, since $\underline{\mu}(t)$ is infinite at $\mp\frac{\pi}{n}$, while the mean curvatures of the hypersurfaces $Y_\mp$ and what is needed for $\bullet_4$ are the inequality $\|df\| \leq 1$ and the equality

$$\frac{n\underline{\mu}(t)^2}{n-1} - |\frac{d\underline{\mu}(t)}{dt}| + n(n-1) = 0$$

indicated in example(c) from section 5.1).

Because of $\bullet_3$, the submanifolds $Y_\mp$ serve as barriers for μ-bubbles (see the previous section) between them; this implies the existence of a minimal μ-bubble Y_{min} in the subset $f^{-1}\left(-\frac{\pi}{n}, \frac{\pi}{n}\right) \subset X$ homologous to Y_o. by $\bigstar$ in section 5.2.

Due to $\bullet_4$, the operator $\Delta + \frac{1}{2}Sc(Y)$ is positive by $\maltese_{\geq 0}$ from the section 5.1 .

Hence, by $\maltese_{conf}$ the manifold Y_{min} admits a metric with $Sc > 0$ and the inequality $l \leq \frac{\pi}{n}$ follows.

On Rigidity. A a close look at minimal μ-bubbles (see section 5.8) shows that

if $l = \frac{\pi}{n}$, then X is isometric to a warped product , $X = Y \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ with the metric $\varphi^2 h + dt^2$, where the metric h on Y has $Sc(h) = 0$ and where

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}.$$

5.4 Bounds on Distances Between Opposite Faces of Cubical Manifolds with $Sc > 0$

Let us see what kind of geometry Y_{min} may have if we drop the condition $\nparallel_{Sc\ngtr 0}$ and allow $l > \frac{\pi}{n}$.

$\square$-*Lemma.* Let X be a compact connected Riemannian manifold of dimension n with boundary ∂X and let $\partial_- \subset \partial X$ and $\partial_+ \subset \partial X$ be disjoint compact domains in ∂X as in section 5.2.

Let

$$Sc(X) \geq \sigma + \sigma_1, \ ,$$

where $\sigma_1 > 0$ is related to the distance $d = dist_X(\partial_-, \partial_+)$ by the inequality

$$\sigma_1 d^2 > \frac{4(n-1)\pi^2}{n}.$$

(If scaled to $\sigma_1 = n(n-1)$, this becomes $d > \frac{2\pi}{n}$.)

Then there exists a smooth hypersurface $Y_{-1} \subset X$,which separates ∂_- from ∂_+, and a smooth positive function ϕ_{-1} on the interior of Y_{-1}, such that the scalar curvature of the metric $g_{-1} = g_{Y_{-1}} + \phi_{-1}^2 dt^2$ on $Y_{-1} \times \mathbb{R}$ is bounded from below by

$$Sc(g_{-1}) \geq \sigma.$$

Proof. The general case of this reduces to that of $\sigma = n(n-1)$ by on obvious scaling/rescaling argument and when $\sigma = n(n-1)$ we use the same μ as above associated with $\varphi(t) = \exp\int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt$, $-\frac{\pi}{n} < t < \frac{\pi}{n}$. Then, as earlier, since

$$Sc_{g_{\varphi_o}}(y,t) \geq Sc(X,y) + \frac{n\mu(y)^2}{n-1} - 2||d\mu(y)||$$

by $\bigcirc$ from the previous section, the above equality $\frac{n\mu(t)^2}{n-1} - |\frac{d\mu(t)}{dt}| + n(n-1) = 0$ implies the requited bound $Sc(g_o) \geq \sigma_1$. QED.

Example. Let X be an *orientable spin* manifold, let $\partial_- \cup \partial_+ = \partial X$ and let $f : X \to S^{n-1} \times [-l, l]$ be a smooth map, such that $\partial_\mp \to S^{n-1} \times \{\mp l\}$.

Let the following conditions be satisfied.

• $deg(f) \neq 0$,

• the map $X \to S^{n-1}$, that is the composition of f with the projection $S^{n-1} \times [-l, l] \to S^{n-1}$, is *area decreasing;*

• $Sc(X) \geq (n-1)(n-2) + \sigma_1$ for some $\sigma_1 \geq 0$.

Then the above lemma in conjunction with the (stabilised) Llarull theorem shows that

$$dist(\partial_-, \partial_+) \leq \frac{2\pi}{n}\frac{n(n-1)}{\sqrt{\sigma}_1} = \frac{2\pi(n-1)}{\sqrt{\sigma}_1}.$$

Remark. This inequality if it looks sharp, then only for $\sigma_1 \to 0$, while sharp(er) inequality of this kind need different functions μ.

Equivariant $\square$-Lemma. Let X in the $\square$-Lemma be free isometrically acted upon by a unimodular Lie group G that preserves $\partial_\mp$.

Then there exists a submanifold $Y_{-1} \subset X$ and a function ϕ_{-1} on Y, which, besides enjoying all properties in the $\square$-Lemma, are also invariant under the action of G and the resulting metric on g_{-1} on $Y_{-1} \times \mathbb{R}$ is $G \times \mathbb{R}$-invariant.

In fact, the proof of the $\square$-Lemma applies to X/G.

Remark. This lemma may hold for all G, but what we need below is only the case of $G = \mathbb{R}^i$.

$\square^{n-m}$*-Theorem.* Let X be a compact connected orientable Riemannian manifold with boundary and let $\underline{X}_\bullet$ is a closed orientable manifold of dimension $n-m$, e.g. a single point $\bullet$ if $n = m$.

Let

$$f : X \to [-1,1]^m \times \underline{X}_\bullet$$

be a continuous map, which sends the boundary of X to the boundary of $[-1,1]^m \times \underline{X}_\bullet$ and which has *non-zero degree.*

Let $\partial_{i\pm} \subset X$, $i = 1, ..., m$, be the pullbacks of the pairs of the opposite faces of the cube $[-1,1]^m$ under the composition of f with the projection $[-1,1]^m \times \underline{X}_\bullet \to [-1,1]^m$.

Let X satisfy the following condition:

$\pitchfork^m_{Sc\not>0}$ No transversal intersection $Y_{-m\pitchfork} \subset X$ of m-hypersurfaces $Y_i \in X$ which separates ∂_{i-} from ∂_{i+}, admits a metric with $Sc > 0$; moreover, the products $Y_{-m\pitchfork} \times \mathbb{T}^m$ admit no metrics with $Sc > 0$ either.[44]

[44]This "moreover" is unnecessary, since the relevant for us case of stability of the $Sc \not> 0$ condition under multiplication by tori is more or less automatic. (The general case needs some effort.)

If $Sc(X) \geq n(n-1)$ that the distances $d_i = dist(\partial_{i-}, \partial_{i+})$ satisfy the following inequality (which generalise that from section 3.8).

$$\square_{\Sigma} \qquad \sum_{i=1}^{m} \frac{1}{d_i^2} \geq \frac{n^2}{4\pi^2}$$

Consequently

$$\square_{\min} \qquad min_i dist(\partial_{i-}, \partial_{i+}) \leq \sqrt{m}\frac{2\pi}{n}.$$

Proof. Let

$$\sigma_i' = \left(\frac{2\pi}{n}\right)^2 \frac{n(n-1)}{d^2} = \frac{4\pi^2(n-1)}{nd^2}$$

and rewrite $\square_{\Sigma}$ as

$$\sum_i \sigma_i' \geq n(n-1).$$

Assume $\sum_i \sigma_i' < n(n-1)$ and let $\sigma_i > \sigma_i'$ be such that $\sum_i \sigma_i < n(n-1)$.

Then, by induction on $i = 1, 2, ..., m$ and using $\mathbb{R}^{i-1}$-invariant $\square$-Lemma on the ith step, construct manifolds $X_{-i} = Y_{-i} \times \mathbb{R}^i$ with $\mathbb{R}^i$-invariant metrics g_{-i}, such that

$$Sc(X_{-i}) > n(n-1) - \sigma_1 - .. - \sigma_i.$$

The proof s concluded by observing that this for $i = m$ would contradict to $\Vert^m_{Sc \ngtr 0}$.

Remarks. (a) As we mentioned earlier, this inequality is non-sharp starting from $m = 2$, where where the sharp inequality

$$\square^2_{\min} \qquad min_{i=1,2} dist(\partial_{i-}, \partial_{i+}) \leq \pi.$$

for squares with Riemannin metrics on them with $Sc \geq 2$ follows by an elementary argument.

(b) One can show for all n that

$$min_i dist(\partial_{i-}, \partial_{i+}) \leq \sqrt{m}\frac{2\pi}{n} - \varepsilon_{m,n},$$

where $\varepsilon_{m,n} > 0$ for $m \geq 2$.

(c) A possible way for sharpening $\square_{\Sigma}$, say for the case $m = n$, is by using $n-2$ inductive steps instead of n and then generalizing the elementary proof of $\square^2_{\min}$ to $\mathbb{T}^{n-2}$-invariant metrics on $[-1,1]^2 \times \mathbb{T}^{n-2}$.

In fact, all theorems for surfaces X with positive (in general, bounded from below) sectional curvatures beg for their generalisations to $\mathbb{T}^{m-2}$-invariant metrics on $X \times \mathbb{T}^{m-2}$ with positive (and/or bounded from below) scalar curvatures.

5.5 Extremality and Rigidity of log-Concave Warped products.

The inequalities proven in section 5.3 say, in effect, that the metric

$$g_\phi = \phi^2 g_{flat} + dt^2 \text{ on } \mathbb{T}^{n-1} \times \mathbb{R} \text{ for } \phi(t) = \exp \int_{-\pi/n}^{t} -\tan\tfrac{nt}{2} dt$$

is *extremal*:

one can't increase g_ϕ without decreasing its scalar curvature,[45]

where the essential feature of ϕ (implicitly) used for this purpose was *log-concavity of* ϕ:

$$\frac{d^2 \log \phi(t)}{dt^2} < 0.$$

We show in this section that the same kind of extremality (accompanied by rigidity) holds for other log-concave functions, notably for $\varphi(t) = t^2$, $\varphi(t) = \sin t$ and $\varphi(t) = \sinh t$ which results in

rigidity of punctured Euclidean, spherical and hyperbolic spaces. vspace1mm

More generally, let $X = Y \times \mathbb{R}$ comes with the warped product metric $g_\phi = \phi^2 dg_y + dt^2$. Then the mean curvatures of the hypersurfaces $Y_t = Y \times \{t\}$, $t \in \mathbb{R}$, satisfy (see 2.4)

$$mean.curv(Y_t) = \mu(t) = (n-1)\frac{d \log \phi(t)}{dt} = \frac{\phi'(t)}{\phi(t)},$$

and, obviously, are these $Y_t \subset X$ are locally (non-strictly) minimizing μ-bubbles. [46]

Now, clearly, ϕ *is log-concave, if and only if*

$$\frac{d\mu}{dt} = -|\frac{d\mu}{dt}|.$$

Thus, R_+ defined in section 5.1 as

$$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + Sc(X, x)$$

is equal in the present case to

$$\frac{n\mu(t)^2}{n-1} + 2\mu'(t) + Sc(g_\phi(t)) = \frac{2(n-1)\phi''(t)}{\phi(t)} + (n-1)(n-2)\left(\frac{\phi'}{\phi}\right)^2 + Sc(g_\phi(t))$$

which implies, (see 5.1) that

$$(R_+)_{Y_t} = \frac{1}{\phi^2} Sc(g_{Y_t}) = Sc(g_{Y_t}) \text{ for } g_{Y_t} = \phi^2 g_Y.$$

Thus our operators $-\Delta_{Y_t} + \frac{1}{2}Sc(g_{Y_t}) - (R_+)_{Y_t}$ equal $-\Delta_{Y_t}$, the lowest eigenvalue of which are *zero* with *constant* corresponding eigenfunctions and the corresponding (S^1-invariant warped product) metrics on $Y_t \times S^1$ are (non-warped) $g_{Y_t} + ds^2$ for $Y_t = Y \times \{t\} \subset X = Y \times \mathbb{R}$ and all $t \in \mathbb{R}$.

(We "warp" with the circle S^1 rather than with $\mathbb{R}$ to avoid a confusion between two different $\mathbb{R}$.)

---

[45] To be precise, one should say that

one can't modify the metric, such that the *scalar curvature increases* but the metric itself *doesn't decrease.*

The relevance of this formulation is seen in the example of $X = S^n \times S^1$, where one can stretch the obvious product metric g in the S^1-direction without changing the scalar curvature, but one *can't increase* the scalar curvature by deformations that increase g.

[46] If Y is non-compact, the minimization is understood here for variations with compact supports.

This computation together with ✤$_{warp}$ in section 5.1 yield the following.

Comparison Lemma. Let $\underline{X} = \underline{Y} \times [a,b]$ be an $\underline{n}$-dimensional warped product manifold with the metric

$$g_{\underline{X}} = g_{\underline{\phi}} = \underline{\phi}^2 g_{\underline{Y}} + dt^2,\ t \in [a,b],$$

where $\underline{\phi}(t)$ is a smooth positive log-*concave* function on the segment $[a,b]$.

Let X be an n-dimensional Riemannian manifold, with a smooth function $\mu(x)$ on it and let $Y \subset X$ be a stable, e.g. locally minimising μ-bubble in X and let $g_\phi = \phi^2 g_Y + ds^2$ be the metric on $Y \times S^1$ where g_Y is the metric on Y induced from X, and where ϕ is the first eigenfunction of the operator $-\Delta + \frac{1}{2}Sc(g_Y, y) - R_+(y)$ for $R_+(x) = \frac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + Sc(X,x)$ (where ϕ is not assumed positive at this point).

Let $f : X \to \underline{X}$ be a smooth map let $f_{\underline{Y}} : X \to \underline{Y}$ denote the $\underline{Y}$-component of f, that is the composition of f with the projection $\underline{Y} \times [a,b] \to \underline{Y}$.

Let

$$f_{[a,b]} : X \to [a,b]$$

be the $[a,b]$-component of f, let

$$\underline{\mu}^*(x) = \underline{\mu} \circ f_{[a,b]}(x) \text{ for } \underline{\mu}(t) = (\underline{n}-1)\frac{d\log\underline{\phi}(t)}{dt} = mean.curv(\underline{Y}_t),\ t = f_{[a,b]}(x)$$

and let

$$\underline{\mu}'^* = \underline{\mu}' \circ f_{[a,b]}(x) \text{ where } \underline{\mu}' = \underline{\mu}'(t) = \frac{d\underline{\mu}(t)}{dt}.$$

Let

$$\underline{R}^*_+(x) = \frac{n\underline{\mu}^*(x)^2}{\underline{n}-1} - 2||d\underline{\mu}^*(x)|| + Sc(\underline{X}, f(x))$$

If

$$R_+(x) \geq \underline{R}^*_+(x),$$

then the function ϕ is positive and the scalar curvature of the metric g_ϕ on $Y \times S^1$ satisfies

$$Sc_{g_\phi}(y,s) \geq \frac{1}{(f_{[a,b]}(y))^2} Sc(\underline{Y}, f_{\underline{Y}}(y)) = Sc(\underline{Y}_t, f(y)) \textit{ for } Y_t \ni f(y).$$

The main case of this lemma, which we use below, is where

•$_{df_{[a,b]}}$ *the function $f_{[a,b]} : X \to [a,b]$ is 1-Lipschitz, i.e. $||df_{[a,b]}|| \leq 1$,*

and

•$_\mu$ *$\mu(x) = \underline{\mu} \circ f_{[a,b]}$, that is $\mu(x) = mean.curv(\underline{Y}_t, f(x))$ for $\underline{Y}_t \ni f(x)$*

and where the conclusion reads:

$$[Sc \geq]. \qquad Sc_{g_\phi}(y,s) \geq \frac{1}{(f_{[a,b]}(y))^2} Sc(\underline{Y}, f_{\underline{Y}}(y)) + Sc(X,y) - Sc(\underline{X}, f(y)).$$

Corollary. Let $\hat{X}$ denote the Riemannian (warped product) manifold $(Y \times S^1, g_\phi)$ and let $\hat{f} : X \to \underline{Y}$ is defined by $(y,s) \mapsto f_{\underline{Y}}(y)$.

If besides $\bullet_{df_{[a,b]}}$ *and* $\bullet_\mu$,

$$\|\wedge^2 df\| \le 1, \;\; e.g. \;\; \|df\| \le 1$$

and if

$$Sc(X,y) \ge Sc(\underline{X}, f(y)),$$

then the map $\hat{f}$ *satisfies*

$$Sc(\hat{X},\hat{x}) \ge \|d\hat{f}\|^2 Sc(\underline{Y},\hat{f}(\hat{x})) \ge \|\wedge^2 d\hat{f}\| Sc(\underline{Y},\hat{f}(\hat{x})).$$

Now, the existence of minimal bubbles under barrier [$\gtrless$ *mean* $= \mp\infty$]-condition (see section 5.2) and a combination of the above with the Llarull *trace* $\wedge^2$ *df*-inequality in section 4.2 yields the following.

$\odot_{S^n}$. *Extremality of Doubly Punctured Spheres.* Let X be an oriented Riemannian spin n-manifold, let $\underline{X}$ be the n-sphere with two opposite points removed and let $f: X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \ge n(n-1) = Sc(\underline{X}) = Sc(S^n)$, *then*

(A) the scalar curvature of X *is constant* $= n(n-1)$;

(B) the map f *is an isometry.*

Proof. The spherical metric on $\underline{X} = S^n \smallsetminus \{s,-s\}$ is the warped product $S^{n-1} \times \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ where the warping factor $\underline{\phi}(t) = \sin t$ which is logarithmically concave, where $\underline{\mu}(t) = \frac{d\log\underline{\phi}(t)}{dt} \to \pm\infty$ for $t \to \mp\frac{\pi}{2}$. [47]

This implies (A) while (B) needs a little extra argument indicated in section 5.8.

1-*Lipschitz Remark.* As it is clear from the proof, the 1-Lipshitz condition can be relaxed to the following one.

The radial component $f_{[-\frac{\pi}{2},\frac{\pi}{2}]}: X \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ of f, which corresponds to the signed distance function from the equator in $S^n \smallsetminus \{s,-s\}$ is 1-Lipschitz and the differential of the S^{n-1} component $f_{S^{n-1}}: X \to S^{n-1}$ satisfies

$$df_{S^{n-1}} \wedge^2 df(x) \le \left(\sin f_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)\right)^2.$$

Non-Spin Remark. If $n = 4$, one can drop the spin condition, since μ-bubbles $Y \in X$, being 3-manifolds, are spin.

Similarly to $\odot_{S^n}$ one shows the following.

$\odot_{\mathbb{R}^n}$. Let Let X be as above, let $\underline{X}$ be $\mathbb{R}^n$ with a point removed and let $f: X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \ge n(n-1) \ge 0$ *and if* X *is an isometry at infinity, then*

(A) $Sc(X) = 0$;

(B) the map f *is an isometry.*

[47] If a log-concave function ϕ on the segment $[-l,l]$ is positive for $-l < t < l$ and it vanishes at $-l$, then the logarithmic derivative of ϕ goes to ∞ for $t \to -l$; similarly,

$$\frac{\phi'}{\phi} \underset{t\to l}{\to} -\infty,$$

if ϕ vanishes at $t = l$.

$\odot_{\mathbf{H}^n}$. Let Let X be as above, let $\underline{X}$ be the hyperbolic space with a point removed and let $f : X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \geq -n(n-1)$ and if X is an isometry at infinity, then

(A) $Sc(X) = -n(n-1)$;

(B) the map f is an isometry.

Question. Let $d_0(\underline{x}) = dist(\underline{x}, \underline{x}_0)$ be the distance function in $\underline{X}$ from $\odot_{\mathbb{R}^n}$ or from $\odot_{\mathbf{H}^n}$ to the point $\underline{x}_0$, which was removed from $\mathbb{R}^n$ or from $\mathbf{H}^n$, and let $d_f(x) = d_0(f(x))$.

Can one relax the 1-Lipschitz condition in $\odot_{\mathbb{R}^n}$ and in $\odot_{\mathbf{H}^n}$ by requiring that not f but only the function $d_f(x)$ is 1-Lipschitz?

5.6 On Extremality of Warped Products of Manifolds with Boundaries and with Corners.

We explained in section 4.4 how reflection+ smoothing allows an extension of the Llarull and Goette-Semmelmann theorems from section 4.2 to manifolds with smooth boundaries and to a class of manifolds with corners. This, combined with the above, enlarges the class of manifolds with corners to which the conclusion of the extremality $\blacklozenge_{\angle_{ij}}$ *theorem* applies.

Here is an example.

Let $\triangle^{n-1} \subset S^{n-1}$ be the regular spherical simplex with flat faces and the dihedral angles $\frac{\pi}{2}$ and let $\mathsf{S}^*_*\triangle^{n-1} \subset S^n \subset S^{n-1}$ be the spherical suspension of $\triangle^{n-1}$ and let $\underline{X} = \mathsf{S}^b_a(\triangle^{n-1}) \subset \mathsf{S}^*_*\triangle^{n-1}$ be the region of $\mathsf{S}^*_*\triangle^{\mathsf{n}-1}$ between a pair of $(n-1)$-spheres concentric to our equatorial $S^{n-1} \subset S^n$.

Let X be an n-dimensional orientable Riemannin spin manifold with corners and let $f : X \to \underline{X}$ be a smooth 1-Lipschitz map which respects the corner structure and which has non-zero degree.

Spherical $\mathsf{S}^b_a(\triangle)$-Inequality. If $Sc(X) \geq Sc(\underline{X}) = n(n-1)$, if all $(n-1)$-faces $F_i \subset \partial X$ have their mean curvatures bounded from below by those of the corresponding faces in $\underline{X}$, [48]

$$mean.curv(F_i) \geq mean.curv(\underline{F}_i),$$

and if all dihedral angle of X are bounded by the corresponding ones of $\underline{X}$,

$$\angle_{ij} \leq \underline{\angle}_{ij} = \frac{\pi}{2},$$

then

$$Sc(X) = n(n-1),$$

$$mean.curv(F_i) = mean.curv(\underline{F}_i)$$

and

$$\angle_{ij} = \frac{\pi}{2}.$$

Exercise. Formulate and prove the Euclidean and the hyperbolic versions of the $\mathsf{S}^b_a(\triangle)$-inequality.

[48] All these but two have zero mean curvatures.

5.7 Disconcerting Problem with Boundaries of non-Spin Manifolds

Typically, μ-bubbles serve as well if not better than Dirac operators for manifolds with boundaries, but something goes wrong with a natural (naive?) approach to geometric bounds on $Y = \partial X$, where $Sc(X) \geq 0$ and $mean.curv(Y) \geq M > 0$, via μ-bubbles for non-spin manifolds X.

Albeit the existence and regularity theorems from section 5.1 extend to manifolds with boundaries, the second variation formula turns out a disappointment.

To see what happens, let X be a compact Riemannin manifolds with a boundary Y, let $\mu(y)$ be a continuous function $\mu : Y \to (-1,1)$ and let $\mathcal{Z}$ be the set of cooriented hypersurfaces $Z \subset X$ with boundaries $\Omega = \partial Z \subset Y = \partial X$, where the coorientation (unit normal) field ν is called *upward.*

Then such a Z is called a *μ-bubble* (compare 5.1), if it is extremal for

$$Z \mapsto vol_{n-1}^{[-\mu]}(Z) =_{def} vol_{n-1}(Z) - \int_{Y_-} \mu(y)dy,$$

in the class $\mathcal{Z}$, where $Y_- \subset Y$ the region in Y "below" $\Omega = \partial Z \subset Y$ and where our direction/coorientation/sign/angle convention is dictated by the following.

Encouraging Example. Let $X = B^n \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ be the unit ball, $Y = \partial B^n = S^{n-1}$ and let Z_θ, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, where θ is the *latitude parameter* on the sphere $Y = S^{n-1} \supset \partial Z_\theta$, be the horizontal discs, that are the ($S^{n-2}(R)$-spherical for $R = \cos\theta$) intersections

$$\Omega_\theta = \partial Z_\theta = B^n \cap \mathbb{R}^{n-1} \times \{t\},\ t = \sin\theta \in (-1,1) \subset \mathbb{R},$$

where – this is a matter of convention– the latitude parameter $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is related to the dihedral angle between the hypersurfaces Z_θ and Y along their intersection

$$\Omega_\theta = \partial Z_\theta = Z_\theta \cap Y,\ Y = \partial X = S^{n-1},\ \text{for } X = B^n,$$

by

$$\angle_\theta = \angle_{\Omega_\theta}(Z_\theta, Y) = \theta + \frac{\pi}{2}.$$

Next, let $\mu(y)$ for $y = (x,t) \in Y \subset \mathbb{R}^{n-1} \times \mathbb{R}$ be equal to the minus height t, i.e. $\mu(x,\theta) = -t = -\sin\theta$.

Then the *normal derivative* $\partial_\nu = \frac{d}{dt}$ of the volume of $Z_t = Z_{\sin\theta}$ is expressed in terms of

$$|\Omega_t| = vol_{n-2}(\Omega_t),\ t = \sin\theta, \text{ and the angle } \angle_\theta \in (0,\pi)$$

as follows

$$\partial_\nu vol_{n-1}(Z_t) = -|\Omega_t| \tan\theta = |\Omega_t| \cot\angle_\theta \text{ for } \theta = \arcsin t,$$

while the derivative of the μ-measure of the region $(Y_-)_t \subset Y$ below Z_t for the above $\mu(\theta) = -\sin\theta = -t = cos\angle_\theta$ is

$$\partial_\nu \mu((Y_-)_t) = \frac{|\Omega_t| \mu(t)}{\sin\angle_\theta} = |\Omega_t| \cot\angle_\theta.$$

Thus, Z_θ serves as μ-bubbles for this μ, and since they come in a "parallel" family they are *locally minimizing* ones.

Let us return to the general Riemannin manifold X with boundary $Y = \partial X$, a hypersurface $Z \subset X$, such that $\Omega = \partial Z \subset Y = \partial X$ and a function $\mu(y)$ on Y and observe the following.

First Variation Formula for $vol^{[-\mu]}_{n-1}(Z)$. . Let $\angle_\omega \in (0,\pi)$ denote the angle between Z and Y at $\omega \in \Omega = \partial Z = Z \cap Y$ and let us use the following abbreviations

$$\csc_\omega = \frac{1}{\sin \angle_\omega} \text{ and } \cos_\omega = \cos \angle_\omega.$$

Then

$$\partial_\nu vol_{n-1}(Z) = \int_Z mean.curv(Z,z)dz + \int_\Omega scs_\omega cos_\omega d\omega,$$

and

$$\partial_\nu \mu(Y_-) = \int_\Omega \csc_\omega \mu(\omega) d\omega.$$

and, since $vol^{[-\mu]}_{n-1}(Z) = vol_{n-1}(Z) - \mu(Y_-)$,

Z is a (stationary) μ-bubble, i.e. $\partial_{\psi\nu} vol^{[-\mu]}_{n-1}(Z) = 0$ for all smooth functions $\psi(z)$, if and only if

$$mean.curv(Z) = 0 \text{ and } \mu(\omega) = \cos_\omega.$$

Second Variation Formula for $vol^{[-\mu]}_{n-1}(Z)$. If Z is stationary then the ω contribution to the second variation/derivative $\partial^2_{\psi\nu} vol^{[-\mu]}_{n-1}(Z)$ is as follows

$$\partial_{\psi\nu} \int_\Omega \psi(\omega)(scs_\omega cos_\omega d\omega - \csc_\omega \mu(\omega))d\omega = \int_\Omega \psi^2(\omega)(-\csc_\omega(\varrho(\omega) - \partial_\nu \mu(\omega)))d\omega,$$

where $\varrho(\omega)$ is the curvature of $Y \subset Z$, i.e. the value of the second fundamental form of $Y \subset X$, on the unit tangent vector $\tau \in T_\omega(Y)$ normal to $T_\omega(\Omega) \subset T_\omega(Y)$.

(Our sign convention is such that this ϱ is positive for convex $Y = \partial X$ and negative for concave ones.)

Let $M_Y(\omega) = M(\Omega \subset Y, \omega)$ denote the mean curvature of $\Omega \subset Y$ and observe that ϱ equals the difference between the mean curvature of $Y \subset X$ and the values of the mean curvature (second fundamental form) of Ω on the unit normal bundle of $Y \subset X$, denotes $M(\Omega, T^\perp(Y \subset X))$,

$$\varrho = M(Y \subset X) - M(\Omega, T^\perp(Y \subset X)),$$

and that

$$M(\Omega, T^\perp(Y \subset X)) = \csc \cdot M(\Omega \subset Z) + \cos \cdot \csc \cdot M(\Omega \subset Y).$$

The essential problem, as I see it here, is that the mean curvature $M(\Omega \subset Y = \partial X)$ may (may not?) be uncontrollably $\pm$large and, unless $\mu = 0$, the positivity of the second variation operator doesn't yield a significant information on the intrinsic geometry of $Z \subset X$ at the boundary $\Omega = \partial Z$. (Am I missing something obvious?)

5.8 On Rigidity of Extremal Warped Products.

Let us explain, as a matter of example, that

doubly punctured sphere $\underline{X} = S^n \setminus \{\pm s\}$ *is rigid.*

This means (see (B) in $\odot_{S^n}$ of section 5.5) that

if an oriented Riemannin spin n-manifold X *with* $Sc(X) \geq n(n-1) = Sc(\underline{X} = Sc(S^n)$ *admits a smooth proper 1-Lipschitz map* $f : X \to \underline{X}$ *such that* $deg(f) \neq 0$, *then, in fact, such an* f *is an isometry.*

Proof. We know (see the the proof of $\odot_{S^n}$) that X contains a minimal μ-bubble Y, which separates the two (union of) ends of X, where $\mu(x)$ is the f-pullback of the mean curvature function of the concentric $(n-1)$-spheres in $\underline{X} = S^n \setminus \{\pm s\}$ between the two punctures and that this m-bubble must be umbilic, where we assume at this point that Y is non-singular, e.g. $n \leq 7$.

What we want to prove now is that these bubbles *foliate all of* X, namely they come in a continuous family of mutually disjoint minimal μ-bubbles Y_t, $t \in \left(-\frac{\pi}{2}\pi 2\right)$, which together cover X.

Indeed, if the *maximal* such family Y_t wouldn't cover X, then the would exists a small perturbation $\mu'(x)$ of $\mu(x)$ in the gap between two Y_t in the maximal family, such that $|\mu'| > |\mu|$ in this gap, while $||d\mu'|| = ||d\mu||$ in there and such that there would exist a minimal μ'-bubble Y' in this gap.

But then, by calculation in 5.5, the resulting warped product metric on $Y' \times S^1$ would be $> n(n-1)$, thus proving "no gap property" by contradiction.

Therefore, X itself is the warped product, $X = Y \times \left(-\frac{\pi}{2}\pi 2\right)$ with the metric $dt^2 = (sint)^2 g_Y$, where $Sc(g_Y) = n(n-1)$ and which by Llarull's rigidity theorem, has constant sectional curvature. QED.

Remarks (a) On the positive side, this argument is quite robust, which makes it compatible with approximation of bubble and metrics. For instance it nicely works for $n = 8$ in conjunction with Smale's generic regularity theorem and, probably, for all n with Lohkamp's smoothing theorem.

But it is not quite clear how to make this work for non-smooth limits of smooth metrics.

For instance (this was already formulated in section 3.2),

let g_i be a sequence of Riemannian metrics on the torus $\mathbb{T}^n$, such that

$$Sc(g_i) \geq -\varepsilon_i \underset{i\to\infty}{\to} 0$$

and such that g_i uniformly converge to a continuous metric g.

Is this g, *say for* $n \leq 7$, *Riemannian flat?*

(The above argument shows that, given an indivisible $(n-1)$-homology class in $\mathbb{T}^n$, there exists a foliation of $\mathbb{T}^n$ by g-minimal submanifolds from this class. But it is not immediately clear how to show that these submanifolds are totally geodesic.)

6 Problems, Generalisations, Speculations.

The most tantalising aspect of scalar curvature is it serving as a meeting point between two different branches of analysis: the index theory and the geometric measure theory, which suggests, on the one hand, the existence of a unified theory and, on the other hand a radical generalization of the concept of a space with the scalar curvature bounded from below.

This is a dream. In what follows we indicate what seems realistic, something lying within reach of the currently used techniques and ideas.

6.1 Moduli Spaces Everywhere

All topological and geometric constraints on metrics with $Sc \geq \sigma$ are accompanied by non-trivial homotopy theoretic properties of spaces of such metrics.

A manifestation of this principle is seen in how topological obstructions for the existence of metrics with $Sc > 0$ on closed manifolds X of dimension $n \geq 5$ give rise to

pairs (h_0, h_1) of metrics with $Sc \geq \sigma > 0$ on closed hypersurfaces $Y \subset X$ which *can't be joined by homotopies h_t with $Sc(h_t) > 0$.*

The elementary argument used for the proof of this (see section 3.15) also shows that (known) constraints on *geometry*, not only on topology, of manifolds with $Sc \geq \sigma$ play a similar role.

For instance, assuming for notational simplicity, $\sigma = n(n-1)$, and recalling the $\frac{2\pi}{n}$-inequality from sections 3.7, 5.3, we see that

(a) *if $l \geq \frac{2\pi}{n}$, then the pairs of metrics $h_0 \oplus dt^2$ and $h_1 \oplus dt^2$ on the cylinder $Y \times [-l, l]$, for the above Y and $l \geq \frac{2\pi}{n}$, can't be joined by homotopies of metrics h_t with $Sc(h_t) \geq n(n-1)$ and with $dist_{h_t}(Y \times \{-l\}, Y \times \{l\}) \geq \frac{2\pi}{n}$.*

This phenomenon is also observed for manifolds with *controlled mean curvatures of their boundaries*, e.g. for Riemannian bands X with $mean.curv(\partial_\mp X) \geq \mu_\mp$ and with $Sc(X) \geq \sigma$, whenever these inequalities imply that $dist(\partial_- X, \partial_+ X) \leq d = d(n, \sigma, \mu_\mp)$. (One may have $\sigma < 0$ here in some cases.)

Namely,

(b) *certain sub-bands $Y \subset X$ of codimension one with $\partial_\mp(Y) \subset \partial_\mp(X)$ admit pairs of metrics (h_0, h_1), such that $mean.curv_{h_0,h_1}(\partial_\mp Y) \geq \mu_\mp$ and $Sc_{h_0,h_1}(Y) \geq \sigma$ while $dist_{h_0,h_1}(\partial_-, \partial_+) \geq D$ for a given $D \geq d$. But these metrics can't be joined by homotopies h_t , which would keep these inequalities on the scalar and on the mean curvatures and have $dist_{h_t}(\partial_-, \partial_+) \geq d$ for all $t \in [0, 1]$.*

(c) This seems to persist (I haven't carefully checked it) for manifolds with corners, e.g. for cube-shaped manifolds X: these, apparently contain hypersurfaces $Y \subset X$, the boundaries of which $\partial Y \subset \partial X$ inherit the corner structure from that in X, and which admit pairs of "large" metrics h_0, h_1, which also have "large" scalar curvatures, "large" mean curvatures of the codimension one faces F_i in Y and "large" complementary $(\pi - \angle_{ij})$ dihedral angles along the codimension two faces F_{ij}, but where these h_0, h_1 can't be joint by homotopies of metrics h_t with comparable "largeness" properties.

It is unclear, in general, how to extend the π_0-non-triviality (disconnectedness) of our spaces of metrics to the higher homotopy groups, since the techniques currently used for this purpose rely entirely on the Dirac theoretic techniques (see [EbR-W 2017] and references therein), which are poorly adapted to manifolds with boundaries. But some of this is possible for closed manifolds.

For instance, let Y be a smooth closed spin manifold, and h_p, $p \in P$, be a homotopically non-trivial family of metrics with $Sc(h_p) \geq \sigma > 0$, where, for instance, P can be a k-dimensional sphere and non-triviality means non-contractibility.

Let $\mathcal{S}^m_\sigma(S^m \times Y)$ denote the space of pairs (g,f), where g is a Riemannian metric on $S^m \times Y$ with $Sc(g) \geq \sigma$ and $f:(S^m \times Y, g) \to S^m$ is a distance decreasing map homotopic to the projection $f_o : S^m \times Y \to S^m$.

If non-conractibility of the family h_p follows from non-vanishing of the index of some Dirac operator, then (the proof of) Llarull's theorem suggests that the corresponding family $(h_p + ds^2, f_o) \in \mathcal{S}^m_{\sigma_+}(S^m \times Y)$ for $\sigma_+ = \sigma + m(m-1)$ is non-contractible in the space

$$\mathcal{S}^m_{m(m-1)}(S^m \times Y) \supset \mathcal{S}^m_{\sigma_+}(S^m \times Y).$$

This is quite transparent in many cases, e.g. if $h_p = \{h_0, h_1\}$ is an above kind of pair of metrics with $Sc > 0$, say an embedded codimension one sphere in a Hitchin's homotopy sphere.

Remarks. (i) If *"distance decreasing"* of f is strengthened to *"ε_n-Lipschitz"* for a sufficiently small $\varepsilon_n > 0$, then the above disconnectedness of the space of pairs (g,f) follows for all X with a use of minimal hypersurfaces instead of Dirac operators.

(ii) The above definition of the space $\mathcal{S}^m_\sigma$ makes sense for all manifolds X instead of $S^m \times Y$, where one may allow $dim(X) < m$ as well as $> m$.

However, the following remains problematic in most cases.

For which closed manifolds X and numbers m, σ_1 and $\sigma_2 > \sigma_1 > 0$ is the inclusion $\mathcal{S}^m_{\sigma_2}(X) \leq \mathcal{S}^m_{\sigma_1}(X)$ homotopy equivalence?

Suggestion to the Reader. Browse through all theorems/inequalities in the previous as well as in the following sections, formulate their possible homotopy parametric versions and try to prove some of them.

6.2 Corners, Categories and Classifying Spaces.

It seems (I may be mistaken) that all known results concerning the homotopies of spaces with $Sc > 0$ are about the *iterated cobordisms* of manifolds with $Sc > 0$ rather than about spaces of metrics per se.

Namely, given a smooth closed manifold X, consider "all" Riemannin manifolds of the form $(X \times [0,1]^i, g)$, $i = 0,1,2,...$, such that $Sc(g) > 0$, and such that all metrics g in a small neighbourhoods of all faces $X \times F_j$, where F_j is are ($(i-1)$-cubical) codimension one faces in the cube $[0,1]^i$), split as Riemannin products: $g = g_{X \times F_j} \otimes dt^2$.

.... to be continued.

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