Four Lectures on Scalar Curvature IHES, Spring 2019 (Still unfinished and unedited)

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Unlike manifolds with controlled sectional and Ricci curvatures, those with their *scalar curvatures bounded from below* are not configured in specific rigid forms but display an uncertain variety of flexible shapes similar to what one sees in geometric topology.

Yet, there are definite limits to this flexibility, where determination of such limits crucially depends, at least in the known cases, on two seemingly unrelated analytic means: *index theory of Dirac operators* and the *geometric measure theory*, [1] where there are two distinct aspects of the role of the Dirac operator and two (less sharply) separate one from another aspects to applications of the geometric measure theory.

I. *Positivity of the scalar curvature* of a metric g on a (closed spin) manifold X implies, according to *Lichnerowicz' theorem, vanishing of the index* of the Dirac operator $\mathcal{D}_g$, where this index, which is *independent of* g, identifies, by the the *Atiyah-Singer theorem*, with a certain (smooth) topological invariant, denoted $\hat{\alpha}(X)$. Thus (see section 3.4).

$$Sc(g) > 0 \textit{ implies that } \hat{\alpha}(X) = 0.$$

II. Both, Lichnerowicz' and Atiyah-Singer's theorems apply to *twisted Dirac operators* $(\mathcal{D}_g)_{\otimes L} = \mathcal{D}_{\otimes L}$ that act on *spinors with values in (coefficients with) vector bundles L over X with unitary connections.*

But now,

vanishing/non-vanishing of $ind(\mathcal{D}_{\otimes L})$ depends on a balance between geometric invariants of g and of (connections on) L,

thus delivering information on *geometry* of (X, g) issuing from positivity of the scalar curvature. This in turn, yields non-trivial topological information for *non-simply connected manifolds X*.[2]

I. Somewhat similarly, information concerning *closed* manifolds X with $Sc(X) > 0$, which is obtained with the use of the geometric measure theory–

Schoen-Yau's inductive decent method with minimal hypersurfaces,

[1] Besides these, something is achieved with the *Hamilton's Ricci flow*, especially in dimension $n = 3$, and specifically 4-dimensional results are derived with a use of the *Seiberg-Witten equations*.

[2] The geometry of (X, g) may be hidden in algebraic constructions with the fundamental groups of X, which deliver (finite or infinite dimensional) vector bundle L with *flat* unitary connections in them (see section 3.11).

which is mainly concerned with the (abelianised) fundamental groups of X, albeit being quite different from what comes along with non-twisted Dirac operators, is also of predominantly (not entirely) *topological nature*.

II. A variation of the method of minimal hypersurfaces, applied to *manifolds with boundaries*, yields *geometric information* which is analogous (but not fully identical) to that delivered by *twisted* Dirac operators (see section 3.5) and additional information of the same kind follows with a use of *stable μ-bubbles*, that are *stable* (in a subtle sense) hypersurfaces in Riemannian manifolds X with *prescribed mean curvatures* $\mu = \mu(x)$ (see sections 3.7-3.9).

(The emergent picture of manifolds with positive scalar curvature, where topology and geometry are intimately intertwined, is reminiscent of what happens in symplectic geometry; but the former has yet to reach maturity enjoyed by the latter.[3]

We start these lectures with a dozen pages (§§1 and 2) of elementary background material followed in §3 by a brief overview of the main topics in spaces with their scalar curvatures bounded from below, that covers, I guess 70-80% of currently pursued directions. Then, in §§4 and 5 we give a more detailed exposition of several known and some new geometric constraints on spaces X implied by the lower bound $Sc(X) \geq \sigma$.

Finally, in §6, we sketch a few connective links between different faces of the scalar curvature shown in the earlier sections and formulate several general problems.

Contents

[3]Geometric invariants associated with the scalar curvature, such as the *K-area* (we call it *K-wast*$_2$ in the present paper) are linked with the symplectic invariants (see [G(positive) 1996], [Polt(rigidity) 1996], [Ent(Hofer) 2001], [Sav(jumping) 2012[), but this link is still poorly understood.

1 Geometrically Deceptive Definition.

The scalar curvature of a C^2-smooth Riemannian manifold $X = (X, g)$, denoted $Sc = Sc(X) = Sc(X, g) = Sc(g) = Sc_g$ is a continuous function on X, which is traditionally defined as

the sum of the values of the sectional curvatures at the $n(n-1)$ ordered bivectors of an orthonormal frame in X,

$$Sc(X,x) = Sc(X)(x) = \sum_{i,j} \kappa_{ij}(x),\ i \neq j = 1,...,n,$$

where this sum doesn't depend on the choice of this frame by the Pythagorean theorem.

But if you are geometrically minded, you want to have a geometric definition where the first attempt to find such a definition relies on the following properties of $Sc(X)$.

$\bullet_1$ *Additivity under Cartesian-Riemannian Products.*

$$Sc(X_1 \times X_2, g_1 \oplus g_2) = Sc(X_1, g_1) + Sc(X_2, g_2).$$

$\bullet_2$ *Quadratic Scaling.*

$$Sc(\lambda \cdot X) = \lambda^{-2} Sc(X), \text{ for all } \lambda > 0,$$

where

$$\lambda \cdot X = \lambda \cdot (X, dist_X) =_{def} (X, dist_{\lambda \cdot X}) \text{ for } dist_{\lambda \cdot X} = \lambda \cdot dist(X)$$

for all metric spaces $X = (X, dist_X)$ and where $dist \mapsto \lambda \cdot dist(X)$ corresponds to $g \mapsto \lambda^2 \cdot g$ for the Riemannian quadratic form g.

(This makes the Euclidean spaces scalar-flat: $Sc(\mathbb{R}^n) = 0$.)

$\bullet_3$ *Volume Comparison.* If the scalar curvatures of n-dimensional manifolds X and X' at some points $x \in X$ and $x' \in X'$ are related by the strict inequality

$$Sc(X)(x) < Sc(X')(x'),$$

then the Riemannian volumes of the ε-balls around these points satisfy

$$vol(B_x(X,\varepsilon)) > vol(B_{x'}(X',\varepsilon))$$

for all sufficiently small $\varepsilon > 0$.

Observe that this volume inequality is *additive under Riemannian products*: if

$$vol(B_{x_i}(X,\varepsilon)) > vol(B_{x'_i}(X'_i,\varepsilon)), \text{ for } \varepsilon \leq \varepsilon_0,$$

and *for all points* $x_i \in X_i$ and $x'_l \in X'_i$, $i = 1,2$, then

$$vol_n(B_{(x_1,x_2)}(X_1 \times X_2, \varepsilon_0)) > vol_n(B_{(x'_1,x'_2)}(X'_1 \times X'_2, \varepsilon_0)$$

for all $(x_1,x_2) \in X_i \times X_2$ and $(x'_1,x'_2) \in X'_1 \times X'_2$.

This follows from the Pythagorean formula

$$dist_{X_1 \times X_2} = \sqrt{dist^2_{X_1} + dist^2_{X_2}}.$$

and the Fubini theorem applied to the "fibrations" of balls over balls:

$$B_{(x_1,x_2)}(X_1 \times X_2, \varepsilon_0)) \to B_{x_1}(X_1,\varepsilon_0) \text{ and } B_{(x'_1,x'_2)}(X'_1 \times X'_2, \varepsilon_0)) \to B_{x_1}(X'_1,\varepsilon_0),$$

where the fibers are balls of radii $\varepsilon \in [0, \varepsilon_0]$ in X_2 and X_2'.

$\bullet_4$ *Normalisation/Convention for Surfaces with Constant Sectional Curvatures.* The unit spheres $S^2(1)$ have constant scalar curvature 2 and the hyperbolic plane $H^2(-1)$ with the sectional curvature -1 has scalar curvature -2

It is an elementary exercise to prove the following.

$\star_1$ *The function $Sc(X,g)(x)$ which satisfies $\bullet_1$-$\bullet_4$ exists and unique*;

$\star_2$ *The unit spheres and the hyperbolic spaces with sect.curv = -1 satisfy*

$$Sc(S^n(1)) = n(n-1) \text{ and } Sc(\mathbf{H}^n_{-1}) = -n(n-1).$$

Thus,

$$Sc(S^n(1) \times \mathbf{H}^n_{-1}) = 0 = Sc(\mathbb{R}^n),$$

which implies that the volumes of the small balls in $S^n(1) \times \mathbf{H}^n_{-1}$ are "very close" to the volumes of the Euclidean $2n$-balls.

Also it is elementary to show that the definition of the scalar curvature via volumes of balls agrees with the traditional $Sc = \sum \kappa_{ij}$, where the definition via volumes seem to have an advantage of being geometrically more usable.

But this is an illusion:

THERE IS NO SINGLE KNOWN (ARE THERE UNKNOWN?)
GEOMETRIC ARGUMENT WHICH WOULD MAKE USE OF THIS DEFINITION.

The immediate reason for this is *the infinitesimal* nature of the volume comparison property: it *doesn't integrate* to the corresponding property of balls of specified, let them be small, radii $r \leq \varepsilon > 0$. [4]

Standard Examples of Manifolds with Positive Scalar Curvatures. Since *compact symmetric spaces* X have non-negative sectional curvatures κ, they satisfy $Sc(X) \geq 0$, where the equality holds only for flat tori.

Since the inequality $\kappa \geq 0$ is preserved under dividing spaces by isometry groups, *all compact homogeneous spaces* G/H carry such metrics, since the bi-variant metrics on Lie groups have $\kappa \geq 0$.

Since the scalar curvature is additive, *fibrations with compact non-flat homogeneous fiberes* carry metrics with $Sc > 0$. (See §1 in [G(positive) 1996] for a bit more about it.)

Since *convex hypersurfaces* in $\mathbb{R}^n$ and in general spaces with $\kappa \geq 0$ have their $\kappa \geq 0$, their scalar curvatures are also non-negative.

2 Useful Formulas.

The logic of most (all?) arguments concerning the global geometry of manifolds X with *scalar curvatures bounded from below* is, in general terms, as follows.

Firstly, one uses (or proves) the *existence theorems for solutions Φ of certain partial differential equations*, where the existence of these Φ and their properties depend on global, topological and/or geometric assumptions $\mathcal{A}$ on X, which are, a priori, unrelated to scalar curvature.

[4] An attractive conjecture to the contrary appears in *Volumes of balls in large Riemannian manifolds* by Larry Guth in Annals of Mathematics173(2011), 51-76.

Secondly, one concocts some *algebraic-differential expressions* $\mathcal{E}(\Phi, Sc(X))$, where the crucial role is played by certain *algebraic formulae* and issuing inequalities satisfied by $\mathcal{E}(\Phi, Sc(X))$ under assumptions $\mathcal{A}$.

Then one arrives at *a contradiction*, by showing that

if $Sc(X) \geq \sigma$, then the implied properties, e.g. the sign, of $\mathcal{E}(\Phi, Sc(X))$ are

opposite to those satisfied under assumption(s) $\mathcal{A}$.

[I] Historically the first Φ in this story were *harmonic spinors*, that are solutions s of $\mathcal{D}(s) = 0$, where $\mathcal{D}$ is the *Dirac operator.*[5]

The existence of harmonic spinors on certain manifolds X follows by *the Atiyah-Singer index theorem* of 1963, while the relevant formula involving $Sc(X)$ is the following algebraic identity between the squared *Dirac operator* and the, *a priori positive,* (coarse) *Bochner-Laplace operator* $\nabla^*\nabla$ also denoted ∇^2.

Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) Formula[6] Lichnerowicz

$$\mathcal{D}^2 = \nabla^2 + \frac{1}{4} Sc.$$

(see section 3.4).

Confronting these, André Lichnerowicz [Lich 1963] arrived at examples of

closed $4k$-dimensional manifolds X, which admit no metrics with $Sc > 0$.[7]

[II] Next Φ to come [SY(structure) 1979] were *smooth* stable minimal hypersurfaces in X for $n = dim(X) \leq 7$, the existence of which was proved by Federer [Fed 1970] relying on the *regularity* of volume minimizing cones of dimensions ≤ 6 proved by Simons [Sim 1968], while the key algebraic identity employed by Schoen and Yau in [SY(structure) 1979] was a suitably rewritten *Gauss formula*, that led, in particular, to

non-existence theorem of metrics with $Sc > 0$ on the 3-torus.

[III] The third kind of Φ are solutions to the 4-dimensional *Seiberg-Witten equation* of 1994, that is the Dirac equation coupled with a certain non-linear equation and where the relevant formula is essentially the same as in [I].

Using these, LeBrun [LeB 1999] established a non-trivial (as well as sharp)

lower bound on $\int_X Sc(X,x)^2 dx$ for Riemannian manifolds X diffeomorphic to algebraic surfaces of general type.

[IV] The *Hamilton Ricci flow* $\Phi = g(t)$ of Riemannin metrics on a manifold X, that is defined by a *parabolic* system of equations, also delivers a geometric information on the scalar curvature, where the main algebraic identity for

[5] All you have to know at this stage about $\mathcal{D}$ is that $\mathcal{D}$ is a certain first order differential operator on sections of some bundle over X associated with the tangent bundle $T(X)$.

[6] All natural selfadjoint geometric second order operators differ from the Bochner Laplacians by *zero order* terms, i.e. (curvature related) endomorphisms of the corresponding vector bundles, but it is remarkable that this operator in the case of $\mathcal{D}^2$ reduces to *multiplication by a scalar function*, which happens to be equal to $\frac{1}{4}Sc_X(x)$. See section 3.2 for more about it.

[7] Prior to 1963, no such manifold was known and no *simply connected* manifold that would admit *no metric with positive sectional curvature* was known either. But Lichnerowicz' theorem, saying, in fact, that

if X is spin, then $Sc(X) > 0 \Rightarrow \hat{A}[X] = 0$

delivered lots of *simply connected manifolds* that admitted *no metrics with positive scalar curvatures, see section 3.4.*

$Sc(t) = Sc(g(t))$ reads

$$\frac{dSc(t)}{dt} = \Delta_{g(t)} Sc(t) + 2Ricci(t)^2 \geq \Delta_{g(t)} Sc(t) + \frac{2}{3} Sc(t)^2,$$

.which implies by the maximum principle that the minimum of the scalar curvature grows with time as follows:

$$Sc_{\min}(t) \geq \frac{Sc_{\min}(0)}{1 - \frac{2tSc_{min}(0)}{3}}.$$

If $X = (X, \underline{g})$ is a closed 3-manifold of constant sectional curvature -1, then, using the Ricci flow, Pereleman proved that

all Riemannin metrics g on X with $Sc(g) \geq -6 = Sc(\underline{g})$ satisfy $Vol(X, g) \geq Vol(X, \underline{g})$.[8]

(The logic of the Ricci flow, at least on the surface of things, is quite different from how it goes in the above three cases that rely on *elliptic* equations:

the quantities Φ in the former are kind of residues of certain geometric or topological *complexity* of underlying manifolds X, that is necessary for the very existence of these Φ, while the Ricci flow, as a road roller, leaves a flat terrain behind itself as it crawls along erasing all kinds of complexity.)

In what follows in this section, we enlist classical formulae involved with [II] and indicate their (more or less) immediate applications.

2.1 Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) **Riemannian Variation Formula.** Let h_t, $t \in [0, \varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold Y and let us incorporate h_t to the metric $g = h_t + dt^2$ on $Y \times [0, \varepsilon]$.

Notice that an arbitrary Riemannin metric on an n-manifold X admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The t-derivative of h_t is equal to *twice the second fundamental form* of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0, \varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_\nu h = 2A^*,$$

where

[8] I recall this from reading Perelman's papers time ago, but today, I can't locate this statement in the sources available on the net, where it is buried in plethora of technical lemmas on Ricci flow.

ν is the unit normal field to Y defined as $\nu = \frac{d}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the *principal values* $\alpha_i^*(y)$, $i = 1, ..., n-1$, of the quadratic form A_t^* on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal vectors $\tau_i^* \in T_i(Y)$, which *diagonalize* A^*, are called *the principal curvatures* of Y, and that the sum of these is called *the mean curvature* of Y,

$$mean.curv(Y, y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = trace(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames τ_i in $T_y(Y)$ by the Pythagorean theorem.

Also observe that A^* changes sign under reversion of the t-direction. Accordingly the sign of the quadratic form $A^*(Y)$ depends on the coorientation of Y in X, where our convention is such that

the boundaries of *convex* domains have *positive definite* second fundamental forms A^*, hence, positive mean curvatures.

(2.1.B) First Variation Formula. This concerns the t-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$[\circ_U] \qquad \partial_\nu vol_{n-1}(U) = \frac{dh_t}{dt} vol_{n-1}(U_t) = \int_{U_t} mean.curv(U_t) dy_t$$

where dy_t is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi\nu = \psi \cdot \nu$ for bounded Borel functions $\psi = \psi(y)$ as follows

$$[\circ_\psi] \qquad \partial_{\psi\nu} vol_{n-1}(Y_t) = \int_{Y_t} \psi(y) mean.curv(Y_t) dy_t$$

Now comes the first formula with the Riemannin curvature in it.

2.2 Gauss' Theorema Egregium

Let $Y \subset X$ be a smooth hypersurface in a Riemannin manifold X. Then the sectional curvatures of Y and X on a tangent 2-plane $\tau = \tau^2 \subset T_y(Y) \subset T)y(X)$ $y \in Y$, satisfy

$$\kappa(Y, \tau) = \kappa(X, \tau) + \wedge^2 A^*(\tau),$$

where $\wedge^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form form $A^* = A^*(Y) \subset X$ restricted to the plane τ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

This, with the definition the scalar curvature by the formula $Sc = \sum \kappa_{ij}$, implies that

$$Sc(Y, y) = Sc(X, y) + \sum_{i \neq j} \alpha_i^*(y) \alpha_j^*(y) - \sum_i \kappa_{\nu,i},$$

where:

• $\alpha_i^*(y)$, $i = 1, ..., n-1$ are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors τ_i in $T_y(Y)$;

• α^*-sum is taken over all ordered pairs (i,j) with $j \neq i$;

• $\kappa_{\nu,i}$ are the sectional curvatures of X on the bivectors (ν, τ_i) for ν being a unit (defined up to ±-sign) normal vector to Y;

• the sum of $\kappa_{\nu,i}$ is equal to the value of the Ricci curvature of X at ν,

$$\sum_i \kappa_{\nu,i} = Ricci_X(\nu,\nu).$$

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of Y and that in the case of $Y = S^{n-1} \subset \mathbb{R}^n = X$ this gives the correct value $Sc(S^{n-1}) = (n-1)(n-2)$.

Also observe that

$$\sum_{i\neq j} \alpha_i\alpha_j = \left(\sum_i \alpha_i\right)^2 - \sum_i \alpha_i^2,$$

which shows that

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu).$$

In particular, if $Sc(X) \geq 0$ and Y is *minimal*, that is $mean.curv(Y) = 0$, then

(Sc ≥ −2Ric) $$Sc(Y) \geq -2Ricci(\nu,\nu).$$

Example. The scalar curvature of a hypersurface $Y \subset \mathbb{R}^n$ is expressed in terms of the mean curvature of Y, the (point-wise) L_2-norm of the second fundamental form of Y as follows.

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2$$

for $||A^*(Y)||^2 = \sum_i(\alpha_i^*)^2$, while $Y \subset S^n$ satisfy

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n\max_i(c_i^*)^2.$$

It follows that *minimal* hypersurfaces Y in $\mathbb{R}^n$, i.e. these with $mean.curv(Y) = 0$, have *negative scalar curvatures*, while hypersurfaces in the n-spheres with all principal values $\leq \sqrt{n-2}$ have $Sc(Y) > 0$.

Let $A = A(Y)$ denote *the shape operator* that is the symmetric operator on $T(Y)$ associated with A^* via the Riemannin scalar product g restricted from $T(X)$ to $T(Y)$,

$$A^*(\tau,\tau) = \langle A(\tau), \tau\rangle_g \text{ for all } \tau \in T(Y).$$

2.3 Variation of the Curvature of Equidistant Hypersurfaces

(2.3.A) *The Second Main Formula of Riemannian Geometry.*[9] Let Y_t be a family of hypersurfaces t-equidistant to a given $Y = Y_0 \subset X$. Then the shape operators $A_t = A(Y_t)$ satisfy:

[9]The first main formula is *Gauss' Theorema Egregium.*

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

where B_t is the symmetric operator associated with the quadratic differential form B^* on Y_t, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the *sectional curvature* of g at (the 2-planes spanned by) the bivectors $\left(\tau, \nu = \frac{d}{dt}\right)$.

Remark. Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannin geometry. [10]

Tracing this formula yields

(2.3.B) *Hermann Weyl's Tube Formula.*

$$trace\left(\frac{dA_t}{dt}\right) = -||A^*||^2 - Ricci_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$trace(\partial_\nu A) = \partial_\nu trace(A) = -||A^*||^2 - Ricci(\nu, \nu),$$

where

$$||A^*||^2 = ||A||^2 = trace(A^2),$$

where, observe,

$$trace(A) = trace(A^*) = mean.curv = \sum_i \alpha_i^*$$

and where *Ricci* is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_x(X)$ is equal to the trace of the above B^*-form (or of the operator B) on the normal hyperplane $\nu^\perp \subset T_x(X)$ (where $\nu^\perp = T_x(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$Sc(X) = trace(Ricci)$$

and that the above formula $Sc(Y, y) = Sc(X, y) + \sum_{i\neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$Ricci(\nu, \nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i\neq j} \alpha_i^* \cdot \alpha_j^*\right) =$$

$$= \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + ||A^*||^2\right)$$

where, recall, $\alpha_i^* = \alpha_i^*(y)$, $y \in Y$, $i = 1, ..., n-1$, are the principal curvatures of $Y \subset X$, where $mean.curv(Y) = \sum_i \alpha_i^*$ and where $||A^*||^2 = \sum_i (\alpha_i^*)^2$.

[10] Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl's Tube Formula", appears in [Darm 1927] with the reference to Darboux' textbook of 1897.

2.4 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface $Y \subset X$ is called umbilic if all principal curvatures of Y are mutually equal at all points in Y.

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres $S^n_{\kappa>0}$, Euclidean spaces $\mathbb{R}^n$ and hyperbolic spaces $\mathbf{H}^n_{\kappa<0}$) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let $\varphi = \varphi(y) > 0$ be a smooth positive function on a Riemannian (n-1)-manifold $Y = (Y, h)$, and let $g = h_t + dt^2 = \varphi^2 h + dt^2$ be the corresponding metric on $X = Y \times [0, \varepsilon]$.

Then the hypersurfaces $Y_t = Y \times \{t\} \subset X$ are umbilic with the principal curvatures of Y_t equal to $\alpha_i^*(t) = \frac{\varphi'}{\varphi}$, $i = 1, ..., n-1$ for

$$A_t^* = \frac{\varphi'}{\varphi} h_t \text{ for } \varphi' = \frac{d\varphi}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}\ .$$

The Weyl formula reads in this case as follows.

$$(n-1)\left(\frac{\varphi'}{\varphi}\right)' = -(n-1)^2\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{1}{2}\left(Sc(g) - Sc(h_t) - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2\right).$$

Therefore,

$$Sc(g) = \frac{1}{\varphi^2}Sc(h) - 2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2 =$$

$$(\star) \qquad = \frac{1}{\varphi^2}Sc(h) - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

where, recall, $n = dim(X) = dim(Y) + 1$ and the mean curvature of Y_t is

$$mean.curv(Y_t \subset X) = (n-1)\frac{\varphi'(t)}{\varphi(t)}.$$

Examples. (a) If $Y = (Y, h) = S^{n-1}$ is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

which for $\varphi = t^2$ makes the expected $Sc(g) = 0$, since $g = dt^2 + t^2 h$, $t \geq 0$, is the Euclidean metric in the polar coordinates.

If $g = dt^2 + \sin t^2 h$, $-\pi/2 \leq t \leq \pi/2$, then $Sc(g) = n(n-1)$ where this g is the spherical metric on S^n.

(b) If h is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}^n_{-1}).$$

What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2 h + dt^2$, where h is flat, is *constant positive*, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation[11]

Higher Warped Products. Let Y and S be Riemannian manifolds with the metrics denoted dy^2 (which now play the role of the above dt^2) and ds^2 (instead of h), let $\varphi > 0$ be a smooth function on Y, and let

$$g = \varphi^2(y)ds^2 + dy^2$$

be the corresponding warped metric on $Y \times S$,

Then

($\star\star$)

$$Sc(g)(y,s) = Sc(Y)(y) + \frac{1}{\varphi(y)^2}Sc(S)(s) - \frac{m(m-1)}{\varphi^2(y)}||\nabla\varphi(y)||^2 - \frac{2m}{\varphi(y)}\Delta\varphi(y),$$

where $m = dim(S)$ and $\Delta = \sum \nabla_{i,i}$ is the Laplace operator on Y.

To prove this, apply the above ($\star$) to $l \times S$ for naturally parametrised geodesics $l \subset Y$ passing trough y and then average over the space of these l, that is the unit tangent sphere of Y at y.

The most relevant example of ($\star\star$) is where S is the real line $\mathbb{R}$ or the circle S^1 also denoted $\mathbb{T}^1$ and where it reduces to

$$(\star\star)_1 \qquad Sc(g)(y,s) = Sc(Y)(y) - \frac{2}{\varphi}\Delta\varphi(y).$$

(The roles of Y and $S = \mathbb{R}$ and notationally reversed here with respect to those in ($\star$).)

The basic feature of the metrics $\varphi^2(y)ds^2 + dy^2$ on $Y \times \mathbb{R}$ is that they are $\mathbb{R}$-*invariant*, where the quotients $(Y \times \mathbb{R})/\mathbb{Z} = Y \times \mathbb{T}^1$ carry the corresponding $\mathbb{T}^1$-*invariant* metrics, while *the* $\mathbb{R}$-*quotients are isometric to* Y.

Besides $\mathbb{R}$-invariance, a characteristic feature of warped product metrics is *integrability* of the tangent hyperplane field normal to the $\mathbb{R}$-orbits, where $Y \times \{0\} \subset Y \times \mathbb{R}$, being normal to these orbits, serves as an integral variety for this field.

Also notice that $Y = Y \times \{0\} \subset Y \times \mathbb{R}$ is totally geodesic with respect to the metric $\varphi^2(y)ds^2 + dy^2$, while the ($\mathbb{R}$-invariant) *curvature* (vector field) *of the* $\mathbb{R}$-*orbits* is equal to the *gradient field* $\nabla\varphi$ extended from Y to $Y \times \mathbb{R}$.

In what follows, we emphasize $\mathbb{R}$-invariance and interchangeably speak of $\mathbb{R}$-invariant metrics on $Y \times \mathbb{R}$ and metrics warped with factors φ^2 over Y.

2.5 Second Variation Formula

The Weyl formula also yields the following formula for the *second derivative* of the $(n-1)$-volume of a cooriented hypersurface $Y \subset X$ under a normal deformation of Y in X, where the scalar curvature of X plays an essential role.

The deformations we have in mind are by vector fields directed by geodesic normal to Y, where in the simplest case the norm of his field equals one.

In this case we have an equidistant motion $Y \mapsto Y_t$ as earlier and the second derivative of $vol_{n-1}(Y_t)$, denoted here $Vol = Vol_t$, is expressed in terms of of

[11] See §12 in [GL 1983].

the shape operator $A_t = A(Y_t)$ of Y_t and the Ricci curvature of X, where, recall $trace(A_t) = mean.curv(Y_t)$ and

$$\partial_\nu Vol = \int_Y mean.curv(Y)dy$$

by the first variation formula.

Then, by Leibniz' rule,

$$\partial_\nu^2 Vol = \partial_\nu \int_Y trace(A(y))dy = \int_Y trace^2(A(y))dy + \int_Y trace(\partial_\nu A(y))dy,$$

and where, by Weyl's formula,

$$trace(\partial_\nu A) = -trace(A^2) - Ricci(\nu,\nu)$$

for the normal unit field ν.

Thus,

$$\partial_\nu^2 Vol = \int_Y (mean.curv)^2 - trace(A^2) - Ricci(\nu,\nu),$$

which, combining this with the above expression

$$Ricci(\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + \|A^*\|^2\right),$$

shows that

$$\partial_\nu^2 Vol = \int \frac{1}{2}\left(Sc(Y) - Sc(X) + mean.curv^2 - \|A^*\|^2\right).$$

In particular, if $Sc(X) \geq 0$ and Y is minimal, then,

$$(\textstyle\int \mathrm{Sc} \geq 2\partial^2 \mathrm{V}) \qquad \int_Y Sc(Y,y)dy \geq 2\partial_\nu^2$$

(compare with the $(Sc \geq -2Ric)$ in 2.2).

Warning. Unless Y is minimal and despite the notation ∂_ν^2, this derivative depends on how the normal filed on $Y \subset X$ is extended to a vector filed on (a neighbourhood of Y in) X.

Illuminative Exercise. Check up this formula for concentric spheres of radii t in the spaces with constant sectional curvatures that are S^n, $\mathbb{R}^n$ and $\mathbf{H}^n$.

Now, let us allow a non-constant geodesic field normal to Y, call it $\psi\nu$, where $\psi(y)$ is a smooth function on Y and write down the full second variation formula as follows:

$$\partial_{\psi\nu}^2 vol_{n-1}(Y) = \int_Y \|d\psi(y)\|^2 dy + R(y)\psi^2(y)dy$$

for

$$[\circ\circ] \qquad R(y) = \frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - \|A^*(Y)\|^2\right),$$

where $M(y)$ stands for the mean curvature of Y at $y \in Y$ and $\|A^*(Y)\|^2 = \sum_i (\alpha^*)^2$, $i = 1,...,n-1$.

Notice, that the "new" term $\int_Y \|d\psi(y)\|^2 dy$ depends only on the normal field itself, while the R-term depends on the extension of $\psi\nu$ to X, unless

Y is minimal, where $[\circ\circ]$ *reduces to*

$$[**]\qquad \partial^2_{\psi\nu} vol_{n-1}(Y) = \int_Y ||d\psi||^2 + \frac{1}{2}\left(Sc(Y) - Sc(X) - ||A^*||^2\right)\psi^2.$$

Furthermore, if Y is volume minimizing in its neighbourhood, then $\partial^2_{\psi\nu} vol_{n-1}(Y) \geq 0$; therefore,

$$[\star\star]\qquad \int_Y (||d\psi||^2 + \frac{1}{2}(Sc(Y))\psi^2 \geq \frac{1}{2}\int_Y (Sc(X,y) + ||A^*(Y)||^2)\psi^2 dy$$

for all non-zero functions $\psi = \psi(y)$.

Then, if we recall that

$$\int_Y ||d\psi||^2 dy = \int_Y \langle -\Delta\psi, \psi\rangle dy,$$

we will see that $[\star\star]$ says that

the operator $\psi \mapsto -\Delta\psi + \frac{1}{2}Sc(Y)\psi$ *is greater than*[12] $\psi \mapsto \frac{1}{2}(Sc(X,y) + ||A^*(Y)||^2)\psi$.

Consequently,

if $Sc(X) > 0$, then the operator $-\Delta + \frac{1}{2}Sc(Y)$ on Y is positive.

Justification of the $||d\psi||^2$ Term. Let $X = Y \times \mathbb{R}$ with the product metric and let $Y = Y_0 = Y \times \{0\}$ and $Y_{\varepsilon\psi} \subset X$ be the graph of the function $\varepsilon\psi$ on Y. Then

$$vol_{n-1}(Y_{\varepsilon\psi}) = \int_Y \sqrt{1 + \varepsilon^2 ||d\psi||^2} dy = vol_{n-1}(Y) + \frac{1}{2}\int_Y \varepsilon^2 ||d\psi||^2 + o(\varepsilon^2)$$

by the Pythagorean theorem

and

$$\frac{d^2 vol_{n-1}(Y_{\varepsilon\psi})}{d^2\varepsilon} = ||d\psi||^2 + o(1).$$

by the binomial formula.

This proves $[\circ\circ]$ for product manifolds and the general case follows by *linearity/naturality/functoriality* of the formula $[\circ\circ]$.

Naturality Problem. All "true formulas" in the Riemannin geometry should be derived with minimal, if any, amount of calculation – only on the basis of their "naturality" and/or of their validity in simple examples, where these formulas are obvious.

Unfortunately, this "naturality principle" is absent from the textbooks on differential geometry, but, I guess, it may be found in some algebraic articles (books?).

Exercise. Derive the second main formula (above (IV) by pure thought from its manifestations in the examples in (VI).[13]

[12] $A \geq B$ for selfadjoint operators signifies that $A - B$ is positive semidefinite.

[13] I haven't myself solved this exercise.

2.6 Conformal Modification of Scalar Curvature.

Let (X_0, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and let $\varphi = \varphi(x)$ be a smooth positive function on X.

Then, by a straightforward calculation,[14]

$$◐ \qquad Sc(\varphi^2 g_0) = \gamma_n^{-1} \varphi^{-\frac{n+2}{2}} L(\varphi^{\frac{n-2}{2}}),$$

where L is the *conformal Laplace operator* on (X_0, g_0)

$$L(f(x)) = -\Delta f(x) + \gamma_n Sc(g_0, x) f(x)$$

for the ordinary Laplace (Beltrami) $\Delta f = \Delta_{g_0} f = \sum_i \partial_{ii} f$ and $\gamma_n = \frac{n-2}{4(n-1)}$.

Thus, we conclude to the following.

Kazdan-Warner Conformal Change Theorem [KW 1975]. Let $X = (X, g_0)$ be a closed Riemannin manifold, such the the conformal Laplace operator L is positive.

Then X admits a Riemannin metric g (conformal to g_0) for which $Sc(g) > 0$.

Proof. Since L is positive, its first eigenfunction, say $f(x)$ is positive and since $L(f) = \lambda f$, $\lambda > 0$,

$$Sc\left(f^{\frac{4}{n-2}} g_0\right) = \gamma_n^{-1} L(f) f^{-\frac{n+2}{n-2}} = \gamma_n^{-1} f^{\frac{2n}{n-2}} > 0.$$

Example: the Schwarzschild metric. If (X_0, g_0) is the Euclidean 3-space, and $f = f(x)$ is positive function, then

the sign of $Sc(f^4 g_0)$ is equal to that of $-\Delta f$.

In particular, since the function $\frac{1}{r} = (x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}$, is harmonic, *the Schwarzschild metric $g_{Sw} = \left(1 + \frac{1}{r}\right)^4 g_0$ has zero scalar curvature.*

Question. What is the geometric/topological significance of positivity of the operator $-\Delta_X + \gamma Sc(X)$ for particular numbers γ, e.g, for those smaller than the above $\gamma_n = \frac{n-2}{4(n-1)}$?

For instance, do, for a given $\gamma < \gamma_n$, all n-manifolds X admit Riemannin metrics g with positive operators $-\Delta_g + \gamma Sc_g$?

(It is easy to see that all closed n-manifolds, $n \geq 2$, admit Riemannin metrics g with positive operators $-\Delta_g + \gamma Sc_g$ for all $\gamma < \frac{1}{n^{10n}}$.)

2.7 Schoen-Yau's Proofs of $[Sc \ngtr 0]$-Theorems via Minimal Surfaces and Hypersurfaces

Let X be a three dimensional Riemannian manifold with $Sc(X) > 0$ and $Y \subset X$ be a cooriented surface with minimal area in its homology class.

Then the inequality $(\int Sc \geq 2\partial^2 V)$ from section 2.5, which says in the present case that

$$\int_Y Sc(Y, y) dy > 2\partial_\nu^2 area(Y),$$

implies that

Y must be a topological sphere.

[14] There must be a better argument.

In fact, minimality of Y makes $\partial_\nu^2 area(Y) \geq 0$, hence $\int_Y Sc(Y,y)dy > 0$, and the sphericity of Y follows by the Gauss-Bonnet theorem.

And since all integer homology classes in closed orientable Riemannin 3-manifolds admit area minimizing representatives by the geometric measure theory developed by Federer, Fleming and Almgren, we arrive at the following conclusion.

★$_3$ *Schoen-Yau 3d-Theorem. All integer 2D homology classes in closed Riemannian 3-manifolds with $Sc > 0$ are spherical.*

For instance, the 3-torus admits no metric with $Sc > 0$.

The above argument appears in Schoen-Yau's 15-page paper [SY(incompressible) 1979], most of which is occupied by an independent proof of the existence and regularity of minimal Y.

In fact, the existence of minimal surfaces and their regularity needed for the above argument has been known since late (early?) 60s[15] but, what was, probably, missing prior to the Schoen-Yau paper was the innocuously looking corollary of Gauss' formula in 2.2,

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu)$$

and the issuing inequality

$$Sc(Y) > -2Ricci(\nu,\nu)$$

for minimal Y in manifolds X with $Sc(X) > 0$.

For example, Burago and Toponogov, come close to the above argument in [BT 1973], where, they bound from below the injectivity radius of Riemannian 3-manifolds X with $sect.curv(X) \leq 1$ and $Ricci(X) \geq \rho > 0$ by

$$inj.rad(X) \geq 6e^{-\frac{6}{\rho}},$$

where this is done by carefully analysing minimal surfaces $Y \subset X$ bounded by, a priori very short, closed geodesics in X, and where an essential step in the proof is the lower bound on the first eigenvalue of Y by $\sqrt{Ricci(X)}$.

Exercises. Let X be homeomorphic to $Y \times S^1$, where Y is a closed orientable surface with the Euler number χ.

(a) Let $\chi > 0$, $Sc(X) \geq 2$ and show that there exists a surface $Y_o \subset X$ homologous to $Y \times \{s_0\}$, such that $area(Y_o) \leq 4\pi$.

(b) Let $\chi < 0$, $Sc(X) \geq -2$ and show that all surfaces $Y_* \in X$ homologous to $Y \times \{s_0\}$ have $area(Y_*) \geq -2\pi\chi$.

(c) Show that (a) remains valid for complete manifolds X homeomorphic to $Y \times \mathbb{R}$.[16]

Schoen-Yau Codimension 1 Descent Theorem. [SY(structure) 1979] . Let X be a compact orientable n-manifold with $Sc > 0$.

If $n \leq 7$, then all integer homology classes $h \in H_{n-1}(X)$ are representable by compact oriented $(n-1)$-submanifolds Y in X, which admit a metrics with $Sc > 0$.

[15] Regularity of volume minimizing hypersurfaces in manifolds X of dimension $n \leq 7$, as we mentioned earlier, was proved by Herbert Federer in [Fed 1970], by reducing the general case of the problem to that of minimal cones resolved by Jim Simons in [Sim 1968].

[16] I haven't solved this exercise.

Proof. Let Y be a volume minimizing hypersurface representing h, the existence and regularity of which is guaranteed by [Fed 1970] and recall that by $[\star\star]$ in 2.5 the operator $-\Delta + \frac{1}{2}Sc(Y)$ is positive. Hence, the conformal Laplace operator $-\Delta + \gamma_n Sc(Y)$ is also positive for $\gamma_n = \frac{n-2}{4n-1} \leq \frac{1}{2}$ and the proof follows by Kazdan-Warner conformal change theorem.

$\bigstar_n$ *Corollary.* If a closed orientable n-manifold X admits a map to the torus $\mathbb{T}^n$ with *non-zero degree,* then X admits *no metric with* $Sc > 0$.

Indeed, if a closed submanifold Y^{n-1} is *non-homologous to zero* in this X then it (obviously) admits a map to $\mathbb{T}^{n-1}$ with non-zero degree. Thus, the above allows an inductive reduction of the problem to the case of $n = 2$, where the Gauss-Bonnet theorem applies.

Remarks. (a) The original argument by Schoen and Yau yields the following stronger topological constraints on X.

Call a closed orientable n-manifold *Schoen-Yau-Schick* if it admits a smooth map $f : X \to \mathbb{T}^{n-2}$, such that the homology class of the pullback of a generic point,

$$h = [f^{-1}(t)] \in H_2(X)$$

is *non-spherical,* i.e. it is not in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$.

What the above argument actually shows, is that

$\bigstar\bigstar_n$ *SYS-manifolds of dimensions $n \leq 7$ admit no metrics with $Sc > 0$.*

(b) *Exercise.* Construct examples of SYS manifolds of dimension $n \geq 4$, where all maps $X \to \mathbb{T}^n$ have zero degrees.

Hint: apply surgery to $\mathbb{T}^n$.

(c) The limitation $n \leq 7$ of the above argument is due a presence of singularities of minimal subvarieties in X for $dim(X) \geq 8$.

If $n = 8$, these singularities were proven to be unstable, (see [Smale 1993] and section 5.2), which improve $n \leq 7$ to $n \leq 8$ in $\bigstar\bigstar_n$

More recently, the dimension restriction was fully removed in [SY(singularities) 2017] and in [Loh(smoothing) 2018]; the arguments in both papers are difficult and I have not mastered them.

On the other hand, there are several short and technically simple (modulo standard index theorems) proofs of $\bigstar_n$ (but not of $\bigstar\bigstar_n$) for *spin* [17] manifolds X, e.g. for X homeomorphic to $\mathbb{T}^n$. (see section 3.2).

Also notice, that besides being short, the Dirac operator arguments deliver in some cases obstructions to $Sc > 0$ that lie fully beyond the range of the minimal surface techniques. For instance (see [G (positive) 1996] and [G(inequalities), 2018])

$\otimes_{\wedge\omega}$ Quasisymplectic Non-Existence Theorem. *If a closed orientable manifold of dimension $dim(X) = 2k$ carries a closed 2-form ω (e.g. a symplectic one), such that $\int_X \omega^k \neq 0$, and if the universal cover $\tilde{X}$ is contractible,*[18] *then X admits no metric with $Sc > 0$.*

[17] A smooth connected n-manifolds X is spin if the frame bundle over X admits a double cover extending the natural double cover of a fiber, where such a fiber is equal to the linear group, (each of the two connected components of) which admits a a unique non-trivial double cover $\tilde{G}L(n) \to GL(n)$. For instance, all manifolds X with $H^2(X;\mathbb{Z}_2) = 0$ are spin.

[18] It's enough to have $\tilde{X}$ spin and the lift $\tilde{\omega}$ to $\tilde{X}$ exact.

This applies, for instance, to *even dimensional tori*, to *aspherical* 4-*manifolds with* $H^2(X,\mathbb{R}) \neq 0$ and to *products* of such manifolds.

3 Topics, Results, Problems

We present in this section a (very) brief overview of what is known and what is unknown about scalar curvature, where we illustrate general results by their simplest instances. The general formulations and the proofs will appear in the sections to come.

3.1 C^0-Closures of Spaces of Metrics g with $Sc(g) \gtrless \sigma$

Let X be a smooth Riemannian manifold, let $G = G^2(X)$ the space of C^2-smooth Riemannin metrics g on X and let $G_{Sc\geq\sigma} \subset G$ and $G_{Sc\leq\sigma} \subset G$, $-\infty < \sigma < \infty$, be the subsets of metrics g with $Sc(g) \geq \sigma$ and with $Sc(g) \leq \sigma$ respectively.

Then:

A: C^0-Closure Theorem. The subset $G_{Sc\geq\sigma} \subset G$ is closed in G with respect to C^0-topology:

uniform limits $g = \lim g_i$ *of metric* g_i *with* $Sc(g_i) \geq \sigma$ *have* $Sc \geq \sigma$, *provided these* g *are* C^2*-smooth in order to have their scalar curvature defined.*

B: C^0-Density Theorem. The subset $G_{Sc\leq\sigma} \subset G$ is dense in G with respect to C^0-topology.

Moreover, *all* $g \in G$ *admit fine (which is stronger than uniform for non-compact* X*) approximations by metrics with scalar curvatures* $\leq \sigma$.

There are *two proofs* of A. The first one in [G(billiards) 2014] depends on *non-existence of metrics with* $Sc > 0$ *on tori* and the second one in [Bamler 2016] uses *Ricci flow.*

In fact, the proof in [G(billiards) 2014] delivers the following geometric interpretation of $Sc(g) \geq \sigma$, which for $\sigma = 0$ reads as follows.

■-*Criterion.* *A Riemannian* n*-manifold (not assumed compact or complete)* X *has*

$$Sc(X, x_0) < 0 \text{ at some point } x_0 \in X,$$

if and only if there exists is a domain $■^n \subset X$ *diffeomorphic to the cube* $[-1,1]^n$, *such that all codimension one faces* $F_i \subset ■^n$ *have positive mean curvatures and all dihedral angles between (the tangent spaces of) these faces along the "edges" of* $■^n$, *that are codimension two faces, satisfy*

$$\angle_{ij} = \angle(F_i, F_j) < \frac{\pi}{2}.$$

(This, when applied to manifolds X multiplied by surfaces with scalar curvatures $-\sigma$, yields a geometric criterion for $Sc(X) \geq \sigma \neq 0$.)

Exercise. Prove ■-*criterion.* the for $n = 2$.

What is essential here, is that, unlike the "small volume of the ball"-definition from section 1, the characterisation of $Sc(X) \geq 0$ by *non-existence* of cubes $■^n \subset X$ with *mean.curv*$(F_i) > 0$ and $\angle_{ij} < \frac{\pi}{2}$ *doesn't require these cubes* *to be small*, which makes this criterion *stable under C^0-limits of metrics* (see [G(billiards) 2014]).

The proof of B is achieved by a (more or less) direct and elaborate geometric construction in [Lohkamp 1994], where it is, in fact, shown that the metics with *Ricci* < 0 are C^0-dense as well.

Observe that if contrary to A the space of metrics with $Sc \geq 0$ were dense, there would be no hope for a non-trivial geometry of such metrics, while A leads us to the following.

C^0-Limit Problem. Study *continuous* Riemannian metrics which are C^0-limits of smooth g_i, such that $\liminf_{i\to\infty} Sc(g_i) \geq 0$.

Notice that the experience with a similar problem concerning *C^0-limits of symplectic diffeomorphisms* offers little expectations on geometry of such limits, but *stability* (see below) of basic geometric inequalities with $Sc \geq 0$ (we shall meet these later on) points toward a more optimistic solution.

Remark. Non-existence of cubes with

$$Sc(\blacksquare^n) \geq 0,\ mean.curv(F_i) > 0 \text{ and } \angle_{ij} < \frac{\pi}{2},$$

which is derived in [G(billiards) 2014] from non-existence of metric with $Sc > 0$ on tori, also follows from the $\times\blacktriangle^i$ *-Inequality* in section 3.16, which is a corollary of the *Goette-Semmelmann theorem* applied to 1-*Lipschitz and area contracting* maps from manifolds X with $Sc(X,x) \geq \sigma(x)$ to convex hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ (see $[X^{\to\bigcirc}]$ in sections 3.5 and 4).

This suggests a more direct application of this theorem, similarly to how *Llarull's inequality* for area contracting maps $X \to S^n$ (see sections 3.5 and 4) was used in [G(positive 1996)] for the proof of the following special case of A:

Riemannian metrics with *constant* sectional curvatures κ on n-dimensional manifolds can't be C^0-approximated by metrics with $Sc \geq n(n-1)\kappa + \varepsilon_0$, for $\varepsilon_0 > 0$.

3.2 $\mathbb{T}^n_{Sc\not>0}$: No Metrics with $Sc > 0$ on Tori

We have already explained (see section 2.7) Schoen-Yau's proof from [SY(structure) 1979] by an *inductive descent argument with minimal hypersurfaces* of the fact that

The tori $\mathbb{T}^n$, $n \leq 7$, admit no metrics with $Sc > 0$,

Schoen and Yau also show that

Riemannin metrics on these tori with $Sc \geq 0$ are Riemannin flat: the universal coverings of these tori are isometric to $\mathbb{R}^n$. (We shall explain this in section 5.7)

And as we mentioned earlier, the condition $n \geq 7$ was removed in the difficult papers [SY(singularities) 2017] and [Loh(smoothing) 2018].

An alternative proof of $\mathbb{T}^n_{Sc\not>0}$, albeit very short and simple but lacking the geometric transparency of the Schoen-Yau argument, was given in [GL(fundamental group) 1980] for all n with a use of *twisted Dirac operators*[19] *$\mathcal{D}$ on $\mathbb{T}^n$.*

[19]The "untwisted" Dirac operator acts on the spin bundle $\mathbb{S}(X)$ and a "twisted" one operates on the tensor product of $\mathbb{S}(X)$ with some vector bundle L over X, see section 4.1.

At the present moment there are (at least) five such proofs which rely on different versions of the *Atiyah-Singer index theorem* which guarantees the existence of *non-zero harmonic* representatives in various spaces of sections of *twisted spinors* on $\mathbb{T}^n$ (or on $\mathbb{R}^n$ which cover $\mathbb{T}^n$) with *arbitrary* metrics .

Then non-existence of a metric on $\mathbb{T}^n$ with $Sc > 0$ (eventually) follows from *Schroedinger-Lichnerowicz-Weitzenboeck* algebraic identity

$$\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$$

for a positive (*coarse Bochner Laplace*) operator ∇^2,

that implies that *no non-zero harmonic spinor exists if* $Sc > 0$. (see §4 for more about it).

∇^2 Versus $||\nabla||^2$. Here and everywhere in our lectures, ∇^2 is an abbreviation for $\nabla^*\nabla = -\sum_i \nabla_i \nabla_i$, where ∇ is the covariant differentiation operator in a Euclidean vector bundle $L \to X$ with an orthogonal connection, that is

$$\nabla : C^\infty(L) \to C^\infty(L \otimes T^*(X)),$$

where the cotangent bundle $T^*(X))$ is customary interchanged for the tangent one via the Riemannin metric on X and where positivity of ∇^2 is seen via the identity $\int_X \langle \nabla^2\psi, \psi\rangle dx = \int_X ||\nabla\psi||^2 dx$ for the sections ψ of L.

Notice that this identity, unlike $\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$, is it by no means algebraic: $\langle \nabla^2\psi_1, \psi_2\rangle$ is *not equal* to $||\nabla\psi||^2$ *pointwise.*

In fact, it follows by the *analytic* Green's-Stokes' formula applied to the following *algebraic* one:

$$\langle \nabla^*\nabla\psi_1, \psi_2\rangle - \langle \nabla\psi_1, \nabla\psi_2\rangle = -div\langle \nabla\psi_1, \psi_2\rangle_{T(X)}$$

(here $T^*(X)$ is interchanged for $T(X)$), where $div : C^\infty(T(X)) \to C^\infty(X)$ is the divergence operator and where $\langle ..., ...\rangle_{T(X)}$ is the obvious $T(X)$-valued coupling $(L \otimes T(X)) \otimes L = T(X) \otimes (L \otimes L) \to T(X)$ associated with the scalar product in L.

From Even to Odd. The index theorem involved in the above argument delivers non-zero twisted harmonic spinors on manifolds X homeomorphic to $\mathbb{T}^n$ *only for even* n, where the case of n odd follows by taking $X \times \mathbb{T}^{2k+1}$ (e.g. with $k = 0$) or $X \times X$ instead of X.

Albeit *logically simple*, not to say naive, these passages $X \rightsquigarrow X \times \mathbb{T}^{2k+1}$ and $X \rightsquigarrow X \times X$, if you stop to think of it, strike you as *mathematically nontrivial* and *conceptually unsatisfactory.*

To appreciate *the depth of mathematical structures* behind these, think of (twisted) Dirac operators on $X \times \mathbb{T}^m$ for $m \to \infty$, even better for $m = \infty$ and/or of such operators on $\underbrace{X \times ... \times X}_m$ for large (infinite?) m.

And this is *unsatisfactory*, since it artificially and non-canonically shifts a K_1-theoretic situation to K_0, where this kind of "shift" in more complicated cases, besides getting more elaborate, may lead to weakening of the final results.

Toric Stability Problem for $[Sc \geq -\varepsilon]$. Let a metric g on the torus have $Sc(g) > -\varepsilon$. Find additional conditions on g that would make it close to a flat metric.

The simplest expected result of this kind would be as follows:

if a sequence of smooth metrics g_i with $Sc(g_i) \geq -\varepsilon_i \underset{i\to\infty}{\to} 0$ uniformly converges to a continuous metric g, then this g is Riemannian flat.

(Compare with the C^0-Limit Problem stated in the previous section and see section 5.7 for a possible approach to the proof.)

3.3 Asymptotically Flat Spaces with $Sc \geq 0$

It was conjectured by Geroch for $n = 3$ [Ger 1975] that

The Euclidean metric on $\mathbb{R}^n$ admits no compactly supported perturbations with increase of the scalar curvature. Moreover,

If a metric g on $\mathbb{R}^n$ with $Sc(g \geq 0$ is equal to g_{Eucl} outside a compact subset in $\mathbb{R}^n$, then $(\mathbb{R}^n, g)$ is isometric to $(\mathbb{R}^n, g_{Eucl})$.

This, of course, trivially follows from the above $\mathbb{T}^n_{Sc \not< 0}$, since compactly supported perturbations of the flat metric on $\mathbb{R}^n$ yields similar perturbations of flat metrics on tori.

In fact, a more general version of this was originally proven by Schoen and Yau in [SY(positive mass) 1979] for a class of metrics g on 3-manifolds asymptotic to g_{Eu} under the name of *positive mass/energy theorem* (see sections 3.13) with a use of minimal surfaces.

Then Witten in [Witten 1981] (also see [Bartnik 1986]) suggested a proof with a use of a perturbation argument in the space of invariant (non-twisted) harmonic spinors on $\mathbb{R}^n$ and Min-Oo [Min(hyperbolic) 1989] adapted Witten's argument to *the hyperbolic spaces* $\mathbf{H}^n$ (see section 3.10).

Later, Lohkamp [Loh(hammocks) 1999] found a (relatively) simple reduction of the general, and technically more challenging, case of the positive mass theorem to that of compactly supported perturbations, that in turn, (trivially) reduces to $\mathbb{T}^n_{Sc \not< 0}$.

Also notice that the *doubling property* for *mean convex manifolds* with boundaries (see [GL(fundamental group) 1980]) allows a reduction of the Geroch Conjecture and of similar more general results to the *Goette-Semmelmann theorem* [GS 2002] concerning *extremality/rigidity of the metrics g with positive curvature operators* (see $[X^{\to}\bigcirc]$ in section 3.5).

Problems. What are other (homogeneous?) Riemannian spaces that admit no (somehow) localised deformations with increase of the scalar curvatures?

What are most general asymptotic (or boundary) conditions on such deformations that would allow their localization?

Here is a definite result along these lines due to Michael Eichmair, Pengzi Miao and Xiadong Wang, [EMW 2009] generalizing an earlier result by Yuguang Shi and Luen-Fai Tam[ST 2002]

STEMW Rigidity Theorem. Let $\underline{X} \subset \mathbb{R}^n$ be a star convex domain, e.g. a convex one, such as the unit ball, for example, and let X be a compact Riemannin manifold the boundary $Y = \partial X$ of which is isometric to the boundary $\underline{Y} = \partial \underline{X}$.

If $Sc(X) \geq 0$ and if the total scalar curvature of Y is bounded from below by that of $\underline{Y}$,

$$\int_Y mean.curv(Y, y)dy \geq \int_{\underline{Y}} mean.curv(\underline{Y}, \underline{y})d\underline{y},$$

then X is isometric to $\underline{X}$.

Remark. Originally, this was proven for $n \leq 7$ but this restriction can be now removed in view of [SY(singularities) 2017] and/or of [Loh(smoothing) 2018].

Conjecture. Let X be a compact Riemannin manifold with $Sc \geq \sigma$. Then the integral mean curvature of the boundary $Y = \partial X$ is bounded by

$$\int_Y mean.curv(Y, y)dy \leq const,$$

where this $const$ depends on σ and on the (intrinsic) Riemannian metric on Y induced from that of $X \supset Y$.

(See section 3.6 for description of some results in this direction.)

3.4 Simply Connected Manifolds with and without $Sc > 0$

As we already stated earlier, according to Lichnerowicz [Lich 1963], the *Atiyah-Singer index theorem* for the Dirac operator $\mathcal{D}$ and the identity $\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$, imply that

there are smooth *closed simply connected* manifolds X of all dimensions $n = 4k$, $k = 1, 2, ...$, that admit no metrics with $Sc > 0$.

The simplest example of these for $n = 4$ is *the Kummer surface* X_{Ku} given by the equation

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$$

in the complex projective space $\mathbb{C}P^3$.

Also by Lichnerowicz' theorem, other complex surfaces of *even degrees* $d \geq 4$ as well as their Cartesian products, e.g $X_{\mathsf{Ku}} \times ... \times X_{\mathsf{Ku}}$ admit no metrics with $Sc > 0$.

A decade later, using a more general index theorem by Atiyah and Singer, Hitchin [Hit 1974] pointed out that

there exist manifolds Σ *homeomorphic (but no diffeomorphic!) to the spheres* S^n*, for all* $n = 8k + 1, 8k + 2$*,* $k = 1, 2, 3...$*, which admit no metrics with* $Sc > 0$.

Notice that, by *Yau's solution of the Calabi conjecture,* the Kummer surface admits a metric with $Sc = 0$, even with $Ricci = 0$, but, probably, (I guess this must be known) there is no metrics with $Sc = 0$ on these Σ.

The actual Lichnerowicz-Hitchin theorem says that if a certain topological invariant $\hat{\alpha}(X)$ *doesn't vanish*, then X *admits no metric with* $Sc > 0$, since, by the Atiyah and Singer index formulae,[20]

$$\hat{\alpha}(X) \neq 0 \Rightarrow Ind(\mathcal{D}_{|X}) \neq 0 \Rightarrow \exists \text{ harmonic spinor} \neq 0 \text{ on } X.$$

Conversely,

if X is a *simply connected* manifold of *dimension* $n \neq 4$, and if $\hat{\alpha}(X) = 0$ then it admits a metric with positive scalar curvature [GL(classification) 1980], [Stolz 1992].

Thus, for instance

[20]The Dirac operator is defined only on *spin* manifolds and to avoid entering into this at the present moment we postulate $\hat{\alpha}(X) = 0$ for non-spin manifolds X.

Also notice that in the Lichnerowicz' case, where $n = dim(X) = 4k$, this $\hat{\alpha}(X)$ is a certain linear combination of the Pontryagin numbers of X, called $\hat{A}$*-genus* and denoted $\hat{A}[X]$.

all simply connected manifolds of dimension $n \neq 0,1,2,4 \mod 8$ *admit metrics with* $Sc > 0$, since $\hat{\alpha}(X) = 0$ is known to vanish for these n.[21]

A Few Words on $n = 4$ *and on* $\pi_1 \neq 0$. (See sections 3.11, 3.14) more about it.) If $n = 4$ then, besides vanishing of the $\hat{\alpha}$-invariant (which is equal to a non-zero multiple of first Pontryagin number for $n = 4$), positivity of the scalar curvature also implies the vanishing of the *Seiberg-Witten invariants* (See lecture notes by Dietmar Salamon, [Sal 1999]; also we say more about it in section 3.14).

If X is a closed spin manifold of dimension $n \geq 5$ with the fundamental group $\pi_1(X) = \Pi$, then

the existence/non-existence of a metric g *on* X *with* $Sc(g) > 0$ *is an invariant of the spin bordism class* $[X]_{sp} \in bord_{sp}(\mathsf{B}\Pi)$ in the classifying space $\mathsf{B}\Pi$,

where, recall, that (by definition of "classifying") the universal covering of $\mathsf{B}\Pi$ is contractible and $\pi_1(\mathsf{B}\Pi) = \Pi$. (See lecture notes [Stolz(survey) 2001].)

There is an avalanche of papers, most of them coming under the heading of "Novikov Conjecture", with various criteria for the class $[X]_{sp}$, and/or for the corresponding homology class $[X] \in H_n\ (\mathsf{B}\Pi)$ (not) to admit g with $Sc(g) > 0$ on manifolds in this class, where these criteria usually (always?) linked to generalized index theorems for twisted Dirac operators on X with several levels of sophistication in arranging this "twisting". Yet, despite a significant progress in this direction, the following remains unsettled for $n \geq 4$.

(Naive?) Conjecture. No closed *aspherical*[22] manifold X admits a metric with $Sc > 0$.

Moreover,

if a closed oriented n-manifold X admits a continuous map to an aspherical space, that is $\mathsf{B}\Pi$ for some group Π, such that the image of the rational fundamental homology class of $[X]_{\mathbb{Q}}$ in the rational homology[23] homology $(\mathsf{B}\Pi;\mathbb{Q})$ doesn't vanish, then X admits no metic g with $Sc(g) > 0$.

(We shall briefly describe the status of this conjecture in section 3.11.)

3.5 1-Lipschitz and Area Contracting Maps, Hyperspherical Radii Rad_{S^n} and $Rad^{\wedge^2}_{S^n}$, Extremality, Rigidity.

The inequality $Sc(X) \geq \sigma > 0$, as it becomes a *positive curvature* condition, imposes an *upper bound* on the size of X, where an instance of this can be expressed in terms of the *hyperspherical radius* $Rad_{S^n}(X)$, defined for *closed* Riemannian n-manifolds X as

the supremum of the radii $R > 0$ *of* n*-spheres, such that* X *admits a non-contractible 1-Lipschitz, i.e. distance non-increasing, map* $f : X \to S^n(R)$.

More generally, if X is an *open* manifold, this definition still make sense if it exclusively applies maps $f : X \to S^n$ which are *locally constant* outside compact

[21]As far as the exotic spheres Σ are concerned, these Σ admit metrics with $Sc > 0$ if and only if $\hat{\alpha}(\Sigma) = 0$, i.e. if Σ bound spin manifolds, which directly follows by the codimension 3 surgery of manifolds with $Sc > 0$ described in [SY(structure) 1979] and in [GL(classification) 1980]. Moreover, many of these Σ, e.g. all 7-dimensional ones, admits metrics with *non-negative sectional curvatures* but the full extent of "curvature positivity" for exotic spheres remains problematic (see [JW(exotic) 2008] and references therein.

[22]*Aspherical* means that the universal covering is contractible.

[23]Bernhard Hanke pointed out to me that the role of homology with finite coefficients in prohibiting $Sc > 0$, especially for finite groups Π, remains obscure even on the level of conjectures.

subsets in X, i.e. constant at all ends of X. Similarly, if X allowed a boundary, then f should be constant on all components of this boundary.

Notice that a (locally constant at infinity if X is open) map f from an *orientable* n-manifold X to the sphere S^n is *contractible* (in the space of locally constant at infinity maps in the open case) if and only if f has *zero degree.*

Examples. Let X be a complete simply connected manifold of dimension n with non-positive sectional curvature. Then

$\bullet_B$ the balls $B(R) \subset X$ of radii R satisfy

$$Rad_{S^n}(B(R)) = \frac{R}{\pi};$$

$\bullet_\partial$ the spheres $\partial B(R)$ endowed with the induced Riemannian metric satisfy

$$Rad_{S^{n-1}}(\partial B(R)) \geq R;$$

$\bullet_{-1}$ if the sectional curvature of X satisfy $\kappa(X) \leq -1$, then

$$Rad_{S^{n-1}}(\partial B(R)) \geq \frac{e^R - 1}{2\pi}.$$

The existence of

a non-trivial bound,

$$Rad_{S^n}(X) \leq \frac{const_n}{\sqrt{\sigma}},\ \sigma = \inf_{x \in X} Sc(X, x),$$

for orientable spin[24] *manifolds X of even dimensions n* [25]

follows by confronting the index theorem with a "twisted version" of the formula $\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc$ for the Dirac operator on X twisted with the f-pullback of a suitable vector bundle L over S^n [GL(spin) 1980], where

the optimal constant $const_n = \sqrt{n(n-1)} = \sqrt{Sc(S^n)}$ is achieved with L being the (complexified) positive spin bundle over S^n, (see [Llarull 1998] and section 4.2)

This sharp inequality, says, in particular, that one can't enlarge the spherical metric g_{sphr} on S^n without making the scalar curvature smaller at some point. That is if a metric g on S^n satisfies

$$g \geq g_{sphr} \text{ and } Sc(g) \geq n(n-1) = Sc(g_{sphr})$$

then, necessarily, $Sc(g) = n(n-1)$, which we express by saying that *spheres are extremal.*

[24] All surfaces are spin and an orientable manifold X of dimension $n \geq 3$ is spin if and only if the restriction of the tangent bundle $T(X)$ to all surfaces $Y^2 \subset X$ are trivial, e.g. if $H^2(X; \mathbb{Z}_2) = 0$. The simplest examples or spin n-manifolds are smooth hypersurfaces in $\mathbb{R}^{n+1}$, such as product of spheres.

More interesting in this respect are complex projective spaces $\mathbb{C}P^m$ and smooth complex hypersurfaces $X \subset \mathbb{C}P^m$ of degree d: these X are spin if and only if $m + d$ is odd, as it the case for the Kummer surface, for instance.

[25] A trivial (and ungraceful) reduction to the even dimensional one follows taking X times the circle, but there is a better way of doing it.

In fact, Llarull's argument (we say a few words about it in section 4.2) shows that spheres are *rigid*:

$[g \geq g_{sph}]\&[Sc(g) \geq Sc(g_{sph})]$ implies that $g = g_{sph}$.

This extremality/rigidity property of spheres was generalised by Goette and Semmelmann to manifolds $\underline{X}$ with *positive curvature operators*, where the examples of such manifolds we are concerned with now are smooth *locally convex hypersurfaces* in Riemannin flat $(n+1)$-manifolds, e.g. products of convex hypersurfaces in $\mathbb{R}^{m+1}$ by the flat tori $\mathbb{T}^{n-m}$.

The (proof of the) main result in [GS 2002] implies in this case the following.

$[X^{\to\bigcirc}]$: *Corollary to Goette-Semmelmann's Theorem.* Let X be a compact connected orientable Riemannian n-manifold without boundary, let $\underline{X} \subset \mathbb{R}^{n+1}$ be a smooth closed locally convex hypersurface in a Riemannin flat $(n+1)$-manifold and let $f : X \to \underline{X}$ be a smooth map.

Let the norm of the differential of f and the scalar curvatures of X and $\underline{X}$ be related by the inequality

$$Sc(X,x) \geq Sc(\underline{X}, f(x)) \cdot ||df(x)||^2, \; x \in X.$$

If X is orientable and the degree of f is non-zero, then, provided X is spin, this inequality becomes an equality:

$$Sc(X,x) = Sc(\underline{X}, f(x)) \cdot ||df(x)||^2,$$

at all points $x \in X$.

Notice that the above mentioned sharp Llarull's inequality for metrics g on S^n with $Sc(g) \geq n(n-1)$, as well as non-existence of metrics with $Sc > 0$ on tori are special cases of $[X^{\to\bigcirc}]$.

Questions. What are further examples of extremal/rigid manifolds $\underline{X}$ with $Sc(\underline{X}) > 0$? (We shall meet a few later on.)

For instance, are biinvarinat metrics on compact Lie groups extremal?

(This is already problematic for $SO(5)$.)

Can products of spaces of positive and of negative curvatures, e.g. of spheres and hyperbolic spaces, be extremal/rigid in some sense?

Is there a meaningful combination of the above with the Ono-Davaux spectral inequality (see section 3.10)?

How do *non-compact* semisimple Lie groups G fare in this regard?

(There may be a non-trivial interplay between *positivity of the scalar curvatures* of the maximal compact subgroups $H \subset G$ and *negativity of the sectional curvatures* κ of G/H, where negativity of κ serves as an obstruction to $Sc > 0$, e.g. via Mishchenko's construction of the Fredholm-Bott class in the K-theory of spaces with $\kappa \leq 0$.)

Do all closed manifolds which admit metrics with $Ricci > 0$ admit extremal/rigid metrics with $Sc > 0$?

(The products X of negatively and positively curved manifolds a manifestly non-extremal, since one can indefinitely enlarge their negative factors and increase their scalar curvatures at the same time, but it is unclear that no metric on such an X is extremal.)

$\wedge^2$-Hyperspherical Radius $Rad^{\wedge^2}_{S^n}$. This is defined similarly to Rad_{S^n} with "*area non-increasing*" instead of "*distance non-increasing*":

$Rad^{\wedge^2}_{S^n}(X)$ is the supremum of the radii $R > 0$ of n-spheres, such that X admits a *non-contractible C^1-smooth*[26] *map $f : X \to S^n(R)$, which doesn't increase the areas* of smooth (rectifiable if you wish) surfaces in X.

Examples. (a) Orientable surfaces S satisfy

$$Rad^{\wedge^2}_{S^2}(X) = \sqrt{\frac{area(S)}{4\pi}}.$$

(b) The rectangular solids

$$X = \bigtimes_{i=1}^{n} [0, l_i]$$

satisfy

$$\frac{\min_{i \neq j} \sqrt{l_i l_j}}{2\pi} \leq Rad^{\wedge^2}_{S^n}(X) \leq \min_{i \neq j} \sqrt{\frac{l_i l_j}{4\pi}}.$$

(The lower bound is obvious and the upper one follows from the *waist inequality for S^n*, see [Guth(waist) 2014].)

(c) The R-balls in the n-spaces X with $\kappa(X) \leq 0$ from the above examples $\bullet_B$, $\bullet_\partial$, $\bullet_{-1}$ satisfy:

$\bullet^{\wedge^2}_B$ $Rad^{\wedge^2}_{S^n}(B(R)) \geq \frac{R}{2}$;

$\bullet^{\wedge^2}_{-1}$ If $\kappa(X) \leq -1$, then

$$Rad^{\wedge^2}_{S^n}(B(R)) \geq \frac{e^R - 1}{4\pi},$$

that is much greater, than $Rad_{S^n}(B(R)) = \frac{R}{\pi}$ for large R;

$\bullet^{\wedge^2}_{rank \geq 2}$ the balls $B(R)$ in the symmetric spaces X (of non-compact type) with $rank(X) \geq 2$ satisfy

$$Rad^{\wedge^2}_{S^n}(B(R)) = \frac{R}{2} \approx \frac{R}{\pi} = Rad_{S^n}(B(R));$$

$\bullet^{\wedge^2}_{\partial, rank=2}$ the R-spheres in the symmetric spaces X with $rank(X) = 2$ have

$$Rad^{\wedge^2}_{S^n}(\partial B(R)) \geq \epsilon_X \cdot (e^{\delta_X \cdot R} - 1)$$

for (small) positive constants $\epsilon_X, \delta_X > 0$;

$\bullet^{\wedge^2}_{\partial, rank \geq 3}$ the symmetric spaces X with $rank(X) \geq 3$ satisfy

$$Rad^{\wedge^2}_{S^n}(\partial B(R)) = R.$$

[26] It is unclear what happens if C^1-smooth is replaced by continuous or by Lipschitz.

Albeit, typically, $Rad^{\wedge^2}_{S^n}(X) > Rad_{S^n}(X)$ and often $Rad^{\wedge^2}_{S^n}(X) >> Rad_{S^n}(X)$ (obviously, $Rad^{\wedge^2}_{S^n}(X) \geq Rad_{S^n}(X)$ for all X) the above bounds on $Rad_{S^n}(X)$ by the scalar curvature of X, remain valid for $Rad^{\wedge^2}_{S^n}(X)$. For instance,

$$Rad^{\wedge^2}_{S^n}(X) \leq \sqrt{\frac{n(n-1)}{\inf_{x\in X} Sc(X,x)}}$$

for all complete orientable spin n-manifolds X with $Sc(X) > 0$ [Llarull 1998].

In fact, the proofs of these bounds via twisted Dirac operators depend on the Schroedinger-Lichnerowicz-Weitzenboeck formula that expresses the zero order terms in these operators in terms of the *curvatures* of certain auxiliary vector bundles $L \to X$, where these curvatures, being 2-*forms* on X, are related to *areas of surfaces* rather than to distances recorded by lengths of curves X.

But no bounds on $Rad^{\wedge^2}_{S^n}(X)$ is known for *non-spin* manifolds, where the available methods, that rely on minimal hypersurfaces (and/or on stable μ-bubbles, see section 5.1), do depend on distances, but even then they deliver only a non-sharp bound, namely,

$$Rad_{S^n}(X) \leq const_n \sqrt{\frac{n(n-1)}{\inf_{x\in X} Sc(X,x)}}$$

with $const_n > 1$ for $n \geq 5$.

Remarks, (a) The sharp bound on $Rad_{S^n}(X)$ for 4-manifolds X follows by reduction to $n = 3$ with a use of stable μ-bubbles in section 5.5. where we prove this inequality/extremality for punctured spheres.

(b) "Area sizes" of higher dimensional non-spin manifolds X are also bounded by their scalar curvatures but in a limited way. Namely,

let $\underline{X}$ and X be closed orientable Riemannian n-manifolds and $f : X \to \underline{X}$ a smooth *spin*[27] map of *non-zero* degree, Then

there exists a smooth surface $S \subset X$ such that

$$area(f(S)) \geq \underline{R}^2 \cdot \frac{\inf_{x\in X} Sc(X,x)}{n(n-1)} \cdot area(S),$$

where $\underline{R} = \underline{R}(\underline{X})$ is a positive constant that depends only on $\underline{X}$.

In fact, since f is spin, (the total space of) the f-pullback to X of the unit sphere bundle of $\underline{X}$ is a spin manifold, to which the twisted Dirac operator method applies. (See §$5\frac{3}{4}$ in [G(positive) 1996] and section 10 in [G(101) 2017] for details and for another proof.)

Notice that this tells you nothing new in the most interesting case of $\underline{X} = S^n$, where it does't even give a realistic lower bound on $\underline{R}$. (We know that $\underline{R}(S^n) = 1$ by Llarull's inequality.)

[27] A continuous map between smooth manifolds, say $f : X \to \underline{X}$, is *spin* if the *second Stiefel-Whitney class* of $\underline{X}$ goes to that of X:

$$f^*(w_2(\underline{X})) = w_2(X).$$

For instance,

- a homotopy equivalences, e.g. homeomorphisms, f between manifolds are spin;
- if $\underline{X}$ is spin, e.g. $\underline{X} = S^n$, then $f : X \to \underline{X}$ is spin if and only if the manifold X is spin.

3.6 Doubles of Manifolds with Boundaries and Bounds on $Rad_{S^{n-1}}$ of Mean Convex Hypersurfaces

The above Dirac theoretic arguments extend to *non-compact complete* Riemannian manifolds, where the existence of (twisted) harmonic spinors follows in the relevant cases from the *relative index theorem(s)* (see [GL 1983], [Roe 1996], [Roe 2012]).

It is unclear what should be a suitable version of this theorem for non-complete manifolds and/or for manifolds with boundaries, but the original Atiyah-Singer theorem, when applied to the *double* $\mathbb{D}(X)$ *of a compact manifold* X with boundary does deliver a non-trivial geometric information on X as well as on the boundary $Y = \partial X$.

For instance, if $mean.curv(Y) > 0$,[28] then the natural, a priori continuous, metric g on $\mathbb{D}(X)$ can be approximated by C^2-metrics g' by smoothing g along the "Y-edge" *without a decrease of the scalar curvature* (see [GL(spin) 1980][29]), where a particular such smoothing described below (which is similar to the the one in ([GL(spin) 1980]) leads to the following

sharp purely Euclidean inequality accompanied by rigidity.

❂ Mean Curvature Rigidity Theorem. Let $Y \subset \mathbb{R}^n$ be a smooth closed hypersurface with the mean curvature bounded from below by $\mu > 0$.

Then the hyperspherical radius of Y is bounded by

$$Rad_{S^{n-1}}(Y) \leq \frac{\mu}{n-1},$$

that is (locally) 1-Lipschitz maps $Y \to S^n(r)$, where "Lipschitz" is understood with respect to the intrinsic, *i.e.* induced Riemannian, *metric in Y, are contractible for all $r > \frac{\mu}{n-1}$.*

Moreover, if $Rad_{S^{n-1}}(Y) = \frac{\mu}{n-1}$, i.e. Y admits a smooth[30] *non-contractible 1-Lipschitz map $Y \to S^n(r)$ for $r = \frac{\mu}{n-1}$, then $mean.curv(Y, y) = \mu$ for all $y \in Y$,* which, by a theorem of A.D. Alexandrov, implies that

Y is the sphere of radius $\frac{\mu}{n-1}$.

This is shown (see section 4.3) by applying $[X^{\to\bigcirc}]$ from the previous section to a *smoothed double* $\mathbb{D}_\varepsilon(X)$ defined as follows.

Let $\mu = n - 1$ and let

$$X_{1/2} \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$$

be the (closed) domain in $\subset \mathbb{R}^n$ bounded by Y and let $X_\varepsilon = \mathbb{D}_\varepsilon(X) \subset \mathbb{R}^{n+1}$ be a (more or less) naturally/canonically C^2-smoothed boundary of the ε-neighbourhood (which is only C^1-smooth) of $X_{1/2} \subset \mathbb{R}^{n+1}$.

Then let $\underline{X}_{1/2} \subset \mathbb{R}^{n+1}$ be the unit n-ball $B^n \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$ and let, accordingly, $\mathbb{D}_\varepsilon(\underline{X}) = \underline{X}_\varepsilon \subset \mathbb{R}^{n+1}$ be a (more or less) naturally/canonically C^2-smoothed boundary of its ε-neighbourhood.

Then maps $f : Y \to \underline{S}^{n-1}$ define maps

$$F_\varepsilon : X_\varepsilon \to \underline{X}_\varepsilon,$$

[28] Some results, where the inequality $mean.curv(\partial X) > 0$ is not, at least not immediately, available are indicated in section 4.6.

[29] Similar smoothing with a control on the scalar curvature is possible for isometric gluing X_1 to X_2 by isometries $\partial X_1 \leftrightarrow \partial X_2$ if $mean.curv(\partial X_1, x_1) + mean.curv(\partial X_2, x_2).0$ for $x_1 \leftrightarrow \partial x_2$, see [EMW 2009, [G(billiard) 2014] and references therein.

[30] This "smooth" is, probably, redundant.

to which $[X^{\rightarrow\bigcirc}]$ applies and, when $\varepsilon \to 0$, it yields the inequality $Rad_{S^{n-1}}(Y) \leq \frac{1}{n-1}$. (See [G(boundary) 2019] and section 4.3.)

Questions. Is there a direct proof of this inequality?

What exactly happens in the limit when $\varepsilon \to 0$ to the Dirac operator used in the proof of $[X^{\rightarrow\bigcirc}]$?

Exercise + Problem. Let $Y_0 \subset \mathbb{R}^n$ be a smooth compact cooriented submanifold with boundary $Z = \partial Y_0$.

If the mean curvature of Y_0 with respect to its coorientation satisfies

$$mean.curv(Y) \geq n-1 = mean.curv(S^{n-1}),$$

then every distance decreasing map

$$f : Z \to S^{n-2} \subset \mathbb{R}^{n-1}$$

is contractible, where "distance decreasing"refers to the distance functions on $Z \subset \mathbb{R}^n$ and on $S^{n-2} \subset \mathbb{R}^{n-1}$ coming from the ambient Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$.

Hint. Observe that the maximum of the principal curvatures of Y_0 is ≥ 1 and show that *the filling radius* of $Z \subset \mathbb{R}^n$ is ≤ 1.[31]

Question. Does contractibility of f remains valid if the distance decreasing property of f is defined with the (intrinsic) spherical distance in S^{n-2} and with the distance in $Z \subset Y_0$ associated with the *intrinsic metric* in $Y_0 \supset Z$, where $dist_{Y_0}(y_1, y_2)$ is defined as the infimum of length of curves in Y_0 between y_1 and y_2?

Bringing Scalar Curvature into the Open. Our proof of the inequality

$$\inf_{y \in Y} mean.curv(Y, y) \leq \frac{1}{Rad_{S^{n-1}}(Y)}$$

applies not only to hypersurfaces in $\mathbb{R}^n$ but to

the boundaries $Y = \partial X$ of all compact Riemannin spin manifolds X with $Sc(X) \geq 0$.

This, suggests the following version of the conjecture following STEMW Rigidity Theorem in section 3.3.

Let the above $Y = \partial X$ be λ-bi-Lipschitz homeomorphic to the unit sphere S^n. Then, conjecturally,

$$\int_Y mean.curv(Y, y) dy \leq C(\lambda)(n-1) vol(S^n),$$

where – this might follows from the STEMW proof – $C(\lambda) \to 1$ for $\lambda \to 1$.

3.7 Widths of Riemannian Bands X with $Sc(X) \geq Sc(S^n)$

Bands, sometime we call them *capacitors*, are manifolds X with two distinguished disjoint non-empty subsets in the boundary $\partial(X)$, denoted

$$\partial_- = \partial_- X \subset \partial X \text{ and } \partial_+ = \partial_+ X \subset \partial X.$$

[31] This means that Z is homologous to zero in its 1-neighbourhood.

A band is called *proper* if $\partial_\pm$ are unions of connected components of ∂X and

$$\partial_- \cup \partial_+ = \partial X.$$

The basic instance of such a band is the segment $[-1,1]$, where $\pm\partial = \{\pm 1\}$.

Furthermore, *cylinders* $X = X_0 \times [-1,1]$ are also bands with $\pm\partial = X_0 \times \{\pm 1\}$, where such a band is proper if X_0 has no boundary.

Riemannian bands are those endowed with Riemannin metrics and

the width of a Riemannin band $X = (X, \partial_\pm)$ is defined as

$$width(X) = dist(\partial_-, \partial_+),$$

where this distance is understood as the infimum of length of curves in V between ∂_- and ∂_+.

We are concerned at this point with *proper compact Riemannin bands X of dimension n*, such that

no closed hypersurface $Y \subset X$, which separates ∂_- from ∂_+, admits a metric with strictly positive scalar curvature.

Simplest Examples of such bands are (we prove this in section 5.2)

$\bullet_{\mathbb{T}^{n-1}}$ *toric bands* which are homeomorphic to $X = \mathbb{T}^{n-1} \times [-1,1]$;

$\bullet_{\hat{\alpha}}$ these, called $\hat{\alpha}$ bands, are diffeomorphic to $Y_{-1} \times \times [-1,1]$, where the Y_{-1} is a closed spin $(n-1)$-manifold with non-vanishing $\hat{\alpha}$-invariant (see the IV above);

$\bullet_{\mathbb{T}^{n-1}\times\hat{\alpha}}$ these are bands *diffeomorphic to products* $X_{n-k} \times \mathbb{T}^k$, where $\hat{\alpha}(X_{n-k}) \neq 0$.

$\frac{2\pi}{n}$-Inequality. Let X be a *proper compact Riemannin bands X of dimension n with $Sc(X) \geq n(n-1) = Sc(S^n)$.*

If no closed hypersurface in X which separates ∂_- from ∂_+ admits a metric with positive scalar curvature, then

$$\left[\circledcirc_\pm \leq \tfrac{2\pi}{n}\right] \qquad width(V) \leq \frac{2\pi}{n}.$$

Moreover, the equality holds only for warped products $X = Y \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$[32] *with metrics $\varphi^2 h + dt^2$, where the metric h on Y has $Sc(h) = 0$ and where*

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

as in section 2.4.

Corollary. Let Y be a closed manifold of dimension $\neq 4$ (see 3.15 below about $n = 4$). Then the following three conditions are equivalent.

1: *the open cylinder $Y \times \mathbb{R}$ admits a complete metric g_1 with uniformly positive scalar curvature, i.e. with $\inf_{x\in X} Sc(g_1, x) > 0$;*

2: the open cylinder $Y \times \mathbb{R}$ admits a complete metric g_2 with positive scalar curvature which decays subquadratically:

$$\liminf_{x\to\infty} Sc(g_2, x) \cdot dist(x, x_0)^2 = \infty.$$

[32]Here, since X is non-compact, the width is understood as the distance between the two ends of X.

3: *the closed cylinder $Y \times [-1,1]$ admits a metric g_3 with $Sc(g_2) \geq n(n-1)$ and such that*

$$dist_{g_3}(Y \times \{-1\}, Y \times \{1\}) \geq \frac{2\pi}{n}.$$

There are two somewhat different proofs (see section 5.2) of $[\circledcirc_{\pm} \leq \frac{2\pi}{n}]$ which use the calculus of variation but advance along slightly different routes.

The first route follows an inductive descent with minimal hypersurfaces á la Schoen-Yau adapted to manifolds with boundaries similarly to that in [GL 1983]. This applies only to the toric and to similar bands, but not to $\hat{\alpha}$-bands. (See [G(inequalities) 2018].)

The second route proceeds with a use of *stable μ-bubbles* which are closed hypersurfaces in X with (prescribed) mean curvature μ, where $\mu = \mu(x)$ is a signed measure on X as in §$5\frac{5}{6}$ of [G(positive) 1996].

This applies to all bands and it also improves certain results from [G(inequalities) 2018] obtained with the first proof.

Both proof, when it comes to $dim(X) = n \geq 9$ have to face the problem of (possibly) stable singularities of minimal (and minimal-like) hypersurface in X.

I feel more comfortable in this respect with the first proof, where a direct application of theorem 4.6 from the recent Schoen-Yau paper [SY(singularities) 2017], (also see [Sch 2017]) is possible.

And as far the second proof for $n \geq 9$ is concerned, the argument from [Loh(smoothing) 2018] seems to be applicable to our case, but this seems harder than the analysis in [SY(singularities) 2017] (which, honestly, I haven't carefully studied, either).

3.8 Bound on Widths of Riemannian Cubes

Let g be a Riemannin metric on the cube $X = [-1,1]^n$ and let d_i, $i = 1, 2, ..., n$, denote the g-distances between the pairs of the opposite faces denoted $\partial_{i\pm} = \partial_{i\pm}(X)$ in this cube X, that are the length of the shortest curves between ∂_{i-} and ∂_{i+} in X.

$\square^n$*-Inequality. If $Sc(g) \geq n(n-1) = Sc(S^n)$, then*

$$\square_{\Sigma} \qquad \sum_{i=1}^{n} \frac{1}{d_i^2} \geq \frac{n^2}{4\pi^2}$$

In particular,

$$\square_{\min} \qquad \min_i dist(\partial_{i-}, \partial_{i+}) \leq \frac{2\pi}{\sqrt{n}}.$$

About the Proof. On the surface of things, this inequality is purely geometric with no topological strings attached. But in truth, the combinatorics of the cube fully reflects toric topology in it.

The proof of $\square_{\Sigma}$ indicated in section 5.4 proceeds along the above *second route* which, in fact, applies to more general "cube-like" manifolds X, such as $Y_{-m} \times [-1,1]^{n-m}$ and yields inequalities mediating between the above $\left[\circledcirc_{\pm} \leq \frac{2\pi}{n}\right]$ and $\square_{\Sigma}$.

But the proof of $\square_\Sigma$ as it stands for $m = n$ is also possible closely following t *the first route*, where the argument from [SY(singularities)] seems easily adaptable.

This makes the proof of $\square_\Sigma$ for $n \geq 9$ more tractable.

Corollary. Let X be a Riemannin manifold with $Sc(X) \geq n(n-1) = Sc(S^n)$, which admits a λ_n-Lipschitz[33] homeomorphism onto the hemisphere S^n_+,

$$f : X \to S^n_+.$$

Then

$$\lambda_n \geq \frac{\arcsin \beta_n}{\pi \beta_n} > \frac{1}{\pi} \text{ for } \beta_n = \frac{1}{\sqrt{n}}.$$

Proof. The hemisphere S^n_+ admits an obvious cubic decomposition with the (geodesic) edge length $2\arcsin\frac{1}{\sqrt{n}}$ and $\square_{\min}$ applies to the pairs of the f-pullbacks of the faces of this decomposition.

Remarks. (a) This lower bound on λ_n improves those in §12 of [GL 1983] and in §3 of [G(inequalities) 2018].

Moreover the *sharp* inequality for Lipschitz maps to the punctured sphere stated in the next section implies that $\lambda_n \geq \frac{1}{2}$ for all n.

But it remains *problematic* if, in fact, $\lambda \geq 1$.

Exercise. Show that $\lambda_2 \geq 1$.

(b) The proof of the inequality $\square_\Sigma$ in section 5.4 applies to *proper* ((boundary$\to$ boundary) λ-*Lipschitz* maps with *non-zero degrees* from all compact connected orientable manifolds X to S^n_+, while the proof via punctured spheres needs X to be spin.

Additional Exercises. (i) Show that the Riemannin metrics with sectional curvatures ≥ 1 on the square $[-1,1]^2$ satisfy

$\square^2_{\min}$. $$\min_{i=1,2} dist(\partial_{i-}, \partial_{i+}) \leq \pi.$$

(ii) Construct iterated warped product metrics g_n on the n-cubes $[-1,1]^n$ with $Sc(g_n) = n(n-1)$, where, for $n = 2$, both d_i, $i = 1,2$, are equal to π and such that

$$d_i > 2\arcsin\frac{1}{\sqrt{n}}, \ i = 1, ..., n, \text{ for all } n = 3, 4, ..., \ .$$

(iii) Show, that $\square_{\min}$ is equivalent to the *over-torical* case of $\frac{2\pi}{n}$*-Inequality.* modulo constants. Namely,

A. If a Riemannin n-cube X has $\min_i dist(\partial_{i-}, \partial_{i+}) \geq d$, then it contains an n-dimensional Riemannin band $X_\circ \subset X$, where $dist(\partial_- X_\circ, \partial_+ X_\circ) \geq \varepsilon_n \cdot d$, $\varepsilon_n > 0$, and where $X_\circ$ admits a continuous map to the $(n-1)$ torus, $f_\circ : X_\circ \to \mathbb{T}^{n-1}$, such that all closed hypersurfaces $Y_\circ \subset X_\circ$ which separate $\partial_- X_\circ$ from $\partial_+ X_\circ$ are sent by $f_\circ$ to $\mathbb{T}^{n-1}$ with *non-zero degrees*.

B. Conversely, let X_o be a band, where $dist(\partial_- X_o, \partial_+ X_o) \geq d)$ and which admits a continuous map to the $(n-1)$ torus, such that the hypersurfaces $Y_o \subset X_o$, which separate $\partial_- X_o$ from $\partial_- X_o$, are sent to this torus with *non-zero degrees*.

[33] A map f between metric spaces is λ-Lipschitz if $dist(f(x)f(y)) \leq dist(x,y)$.

Then there is a (finite if you wish) covering $\tilde{X}_o$ of X_o, which contains a domain $X_\square \subset \tilde{X}_o$, where this domain admits a continuous proper map of degree one onto the d-cube $f_\square : X_\square \to (0,d)^n$, such that the n coordinate projections of this map, $(f_\square)_i : X_\square \to (0,d)$, are distance decreasing.

3.9 Extremality and Rigidity of Punctured Spheres

Let $(\underline{X}, \underline{g})$ be the unit sphere S^n minus two opposite points with the spherical Riemannin metric $\underline{g} = g_{sphe}$.

Double Puncture Rigidity Theorem. *If a smooth metric g on X satisfies*

$$g \geq \underline{g} \text{ and } Sc(g) \geq n(n-1) = Sc(\underline{g}),$$

then $g = \underline{g}$.

About the Proof. By following the above second route, one can reduce this to (a version of) Llarull's theorem (see section 5.5, where again I can fully vouch only for $n \leq 8$.

Remark. It follows by Llarull's argument *for all n* that

no complete metric g on the n-sphere minus a finite subset Σ satisfies the inequalities $g \geq \underline{g}$ and $Sc(g) \geq n(n-1)$ at the same time.

Moreover, this applies to *piecewise smooth 1-dimensional subsets* (graphs) $\Sigma \subset S^n$, such that the *monodromy transformations* of the principal tangent $Spin(n)$-bundle (that is double cover of the orthonormal tangent frame-bundle over all closed curves in Σ are *trivial* (e.g. Σ is contractible).

But if one makes *no completeness assumption*, our result is limited to Σ being either empty, or consisting of a single point or of a pair of opposite points.

Exercise. Prove with the above that no metric g on the hemisphere $(S^n_+, \underline{g})$ can satisfy the inequalities $g \geq 4\underline{g}$ and $Sc(g) > n(n-1)$. Then directly show that if $n = 2$ then the inequality $g \geq \underline{g}$ and $Sc(g) \geq 2$ imply that $g = \underline{g}$.

Question. Does the implication

$$[g \geq \underline{g}] \& [Sc(g) \geq n(n-1)] \Rightarrow g = \underline{g}$$

ever hold for $S^n \setminus \Sigma$ apart from the above cases?

3.10 Manifolds with Negative Scalar Curvature Bounded from Below

If a "topologically complicated" closed Riemannin manifolds X, e.g. an aspherical one with a *hyperbolic fundamental group*, has $Sc(X) \geq \sigma$ for $\sigma < 0$, then a certain "growth" of the universal covering $\tilde{X}$ of X is expected to be *bounded from above* by $const\sqrt{-\sigma}$ and accordingly, the "geometric size" – ideally $\sqrt[n]{vol(X)}$– must be *bounded from below* by $const'/\sqrt{-\sigma}$.

If $n = 3$ this kind of lower bound are easily available for areas of stable minimal surfaces of large genera via Gauss Bonnet theorem by the Schoen-Yau argument from [SY(incompressible) 1979].

Also Perelman's proof of the geometrization conjecture delivers a sharp bound of this kind for manifolds X with hyperbolic $\pi_1(X)$ and similar results for $n = 4$ are possible with the Seiberg-Witten theory for $n = 4$ (see section 3.14).

No such estimate has been established yet for $n \geq 5$ but the following results are available.

Min-Oo Hyperbolic Rigidity Theorem [Min(hyperbolic) 1989]. Let X be a complete Riemannin manifold, which is isometric at infinity (i.e. outside a compact subset in X) to the hyperbolic space $\mathbf{H}^n_{-1}$.

If $Sc(X) \geq -n(n-1) = Sc(\mathbf{H}^n_{-1})$, then X is isometric to $\mathbf{H}^n_{-1}$.

About the Proof. The original argument by Min-Oo, which generalizes Dirac-theoretic Witten's proof of the positive mass/energy theorem for asymptotically Euclidean (rather than hyperbolic) spaces, needs X to be *spin*.

But granted spin, Min-Oo's proof allows more general asymptotic (in some sense) agreement between X and $\mathbf{H}^n_{-1}$ at infinity.

In order to get rid of spin, one may use here either minimal hypersurfaces with boundaries (as in [G(inequalities) 2018]) or stable μ-bubbles (as in [G(positive) 1996]).

To accomplish this it is convenient, here as in the flat case, to pass to a quotient space $\mathbf{H}^n_{-1}/\Gamma$, where, instead of letting $\Gamma = \mathbb{Z}^n$ that allows a reduction of the rigidity of $\mathbb{R}^n$ to that of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, one takes a *parabolic* isometry group isomorphic to $\mathbb{Z}^{n-1}$ for Γ, for which the quotient $\mathbf{H}^n_{-1}/\Gamma$ is the *hyperbolic cusp-space*, that is $\mathbb{T}^{n-1} \times \mathbb{R}$ with the metric $e^{2r}dt^2 + dr^2$. (Here as earlier, when it comes to $n \geq 9$, I feel more comfortable with minimal hypersurfaces to which Schoen-Yau's theorem 4.6 from [SY(singularities) 2017] directly applies.)

Finally, a derivation of the *hyperbolic positive mass theorem* from the rigidity theorem follows by an extension of the Euclidean Lohkamp's argument from [Loh(hammocks) 1999] to the hyperbolic spaces (see [AndMinGal 2007]).

Ono-Davaux *Spectral Inequality* [Ono 1988], [Dav 2002]. Let X be a closed Riemannian manifold and let all smooth functions $f(\tilde{x})$ with compact supports on $\tilde{X}$ satisfy

$$\int_{\tilde{X}} f(\tilde{x})^2 d\tilde{x} \leq \frac{1}{\tilde{\lambda}_0^2} \int_{\tilde{X}} ||df(\tilde{x})||^2 d\tilde{x}.$$

(The maximal such $\tilde{\lambda}_0 \geq 0$ serves as the lower bound on the spectrum of the Laplace operator on the universal covering $\tilde{X}$ of X).

If $\tilde{X}$ is spin and if one of the following two conditions (A) or (B) is satisfied, then

$$[Sc/\tilde{\lambda}_0] \qquad \inf_{x \in X} Sc(X, x) \leq \frac{-4n\tilde{\lambda}_0}{n-1}.$$

Condition (A). The dimension of X is $n = 4k$ and the $\hat{\alpha}$-invariant from section 3.4 (that is a certain linear combination of Pontryagin numbers called *$\hat{A}$-genus*) doesn't vanish. *Condition (B).* The manifold X is *enlargeable*: there exists a covering $\tilde{X}'$ of X, which admits a *proper distance decreasing* map $\tilde{X}' \to \mathbb{R}^n$ of *non-zero* degree.

Remarks. (a) The inequality $[Sc/\tilde{\lambda}_0]$ is sharp: if X has constant negative curvature -1, then

$$-n(n-1) = Sc(X) = \tfrac{-4n\tilde{\lambda}_0}{n-1}$$

for $\tilde{\lambda}_0 = \frac{(n-1)^2}{4}$, that is the bottom of the spectrum of $\mathbf{H}^n_{-1} = \tilde{X}$.

(b) The rigidity sharpening of $[Sc/\tilde{\lambda}_0]$ is proved in [Dav 2002] in the case A and it seems that a minor readjustment of the argument from [Dav 2002] would work in the case B as well. If so it would yield yet another proof of Min-Oo rigidity theorem in the spin case.

Question. Can one put the index theoretic and associated Dirac-spectral considerations on equal footing with Witten's and Min-Oo's kind of arguments on stability of harmonic spinors with a given asymptotic behavior under deformation/modifications of manifolds away from infinity?

THREE CONJECTURES

$[\#_{-n(n-1)}]$ *Let X be a closed orientable Riemannin manifold of dimension n with $Sc(X) \geq -n(n-1)$.*

Then the following topological invariants of X must be bounded by the volume of X, and, even more optimistically, (and less realistically), where the constants are such that the equalities are achieved for compact hyperbolic manifolds with sectional curvatures -1.

1. *Simplicial Volume Conjecture.* There exist orientable n-dimensional *pseudomanifolds* $\mathsf{X}_i^{\triangle}$ and continuous maps $f_i^{\triangle} : \mathsf{X}_i^{\triangle} \to X$ with degrees

$$deg(f_i^{\triangle}) \underset{i\to\infty}{\to} \infty,$$

such that the numbers N_i of *simplices* in the triangulations of $\mathsf{X}_i^{\triangle}$ and the degrees $deg(f_i^{\triangle})$ are related to the volume of X by the following inequality:

$$N_i \leq C_n^{\triangle} \cdot deg(f_i^{\triangle}) \cdot vol(X).$$

2. *The L-Rank Norm Conjecture:* There exist, for all sufficiently large $i \geq i_0 = i_0(X)$, smooth orientable n-dimensional *manifolds* X_i° and continuous maps $f_i : \mathsf{X}_i^{\circ} \to X$, with degrees

$$deg(f_i^{\triangle}) \underset{i\to\infty}{\to} \infty,$$

such that the minimal possible numbers N_i of *the cells* in the cellular decompositions of X_i° and the degrees of the maps $f_i^{\triangle}$ are related to the volume of X by the following inequality:

$$N_i \leq C_n^{\circ} \cdot deg(f_i^{\circ}) \cdot vol(X).$$

3. *Characteristic Numbers Conjecture.* if, additionally to **$[\#_{-n(n-1)}]$**, the manifold X is *aspherical*, then *the Euler characteristic* $\chi(X)$ and *the Pontryagin numbers p_I of X are bounded by*

$$|\chi(X)|, |p_I(X)| \leq C_n^{\square} \cdot vol(X).$$

Remarks. (i) Conjecture 1 makes sense for an X, in so far as X has *non-vanishing* simplicial volume $\|X\|_{\triangle}$, e.g. if X admits a metric with *negative sectional curvature* or a locally symmetric metric with *negative Ricci curvature* [LS(simplicial) 2017]. (See the monograph [Frigero(Bounded Cohomology) 2016] for the definition and basic properties of the simplicial volume.)

(ii) The L-rank norm $||[X]_L||$ is defined in §$8\frac{1}{2}$ of [G(positive) 1996] via the Witt-Wall L-groups of the fundamental group of X.

This $||[X]_L||$ is known to be *non-zero* for compact locally symmetric spaces with non-zero Euler characteristic as it follows from [Lusztig(cohomology) 1996].[34]

In fact, all *known* manifolds X with $||[X]_L|| \neq 0$ admit maps of non-zero degrees to locally symmetric spaces with non-zero Euler characteristics.

And nothing is known about zero/non-zero possibility for the values of the L-rank norm for manifolds with negative sectional curvatures of odd dimensions > 3.

(Vanishing of $||[X]_L||$ for all 3-manifolds X trivially follows from the Agol-Wise theorem on virtual fibration of hyperbolic 3-manifolds over S^1.)

Questions. What are realtions between the $||X||_\triangle$ and $||[X]_L||$? Are there natural invariants mediating between the two?

(It is tempting to suggest that $||X||_\triangle \leq ||[X]_L||$, but it is unlikely to be true in general.)

[$\#_\int$] *Integral Strengthening of the Three Conjectures.* The above conjectural inequalities 1,2,3, for the three topological invariants, call them here inv_i, $i = 1,2,3$, may, for all we know, hold (with no a priori assumption $Sc(X) \geq -n(n-1)$) in the following integral form,

$$inv_i \leq const_i \cdot \int_X |Sc_-(X,x)|^{\frac{n}{2}} dx,$$

where $Sc_-(x) = \min(Sc(x), 0)$, but no lower bound on this integral is anywhere in sight for $n \geq 5$. [35] (See section 3.14. for what is known for $n = 4$.)

3.11 Positive Scalar Curvature, Index Theorems and the Novikov Conjecture

Given a proper (infinity goes to to infinity) smooth map between smooth oriented manifolds, $f : X \mapsto \underline{X}$ of dimensions $n = dim(X) = 4k + \underline{n}$ for $\underline{n} = dim(\underline{X})$, let $sign(f)$ denote the signature of the pullback $Y^{4k}_{\underline{x}} = f^{-1}(\underline{x})$ of a generic point $\underline{x} \in \underline{X}$, that is the signature of the (quadratic) intersection form on the homology $H_2(Y^{4k}_{\underline{x}}; \mathbb{R})$, where observe orientations of X and $\underline{X}$ define an orientation of $Y^{4k}_{\underline{x}}$ which is needed for the definition of the intersection index.

Since the f-pullbacks of generic (curved) segments $[\underline{x}_1, \underline{x}_2] \subset \underline{X}$ are manifolds with boundaries $Y^{4k}_{\underline{x}_1} - Y^{4k}_{\underline{x}_2}$, (the minus sign means the reversed orientation),

$$sign(Y^{4k}_{\underline{x}_1}) = sign(Y^{4k}_{\underline{x}_2}),$$

---

[34] In the simplest case, where X is the product of k closed surfaces $S_1, S_2, ..., S_k$ with negative Euler characteristics, non-vanishing of $||[X]_L||$ is proven in [G(positive 1996]:

If a manifold X° admits a map of degree d to such an X, then X° can't be decomposed into less than

$$N = const_k \cdot d \cdot |\chi(S_1)| \cdot |\chi(S_2) \cdot ... \cdot |\chi(S_k)|, \ const_k > 0,$$

cells.

[35] One doesn't even know if there are such bounds for $||X||_\triangle$ and/or $||[X]_L||$ in terms of the *full Riemannian curvature tensor* $R(X,x)$, namely the bounds

$$||X||_\triangle, ||[X]_L|| \leq const_n \int_X ||R(X,x)||^{\frac{n}{2}} dx.$$

as it follows from the Poincaré duality for manifolds with boundary by a two-line argument. Similarly, one sees that $sign(f)$ depends only on the proper homotopy class $[f]_{hom}$ of f.

Thus, granted $\underline{X}$ and a proper homotopy class of maps f, the signature $sign[f]_{hom}$ serves as a *smooth invariant* denoted $sign_{[f]}(X)$, (which is actually equal to the value of some polynomial in Pontryagin classes of X at the homology class of $Y^{4k}_{\underline{x}_2}$ in the group $H_{4k}(X)$).

If X and $\underline{X}$ are closed manifolds, where $dim(X) > dim(\underline{X}) > 0$, and if $\underline{X}$, is *simply connected*, then, by the Browder-Novikov theory, as one varies the smooth structure of X in a given homotopy class $[X]_{hom}$ of X, the values of $sign_{[f]}(X)$ run through *all integers* $i = sign_{[f]}(X) \mod 100n!$ (we exaggerate for safety's sake), provided $dim(\underline{X}) > 0$ and $Y^{4k}_{\underline{x}} \subset X$ is non-homologous to zero.

However, according to the (illuminating special case of the) *Novikov conjecture*,

if $\underline{X}$ is a *closed aspherical* manifold[36] then this $sign_{[f]}(X)$ *depends only on the homotopy type of* X. [37]

Originally, in 1966, Novikov proved this, by an an elaborated surgery argument, for the torus $\underline{X} = \mathbb{T}^{\underline{n}}$, where $X = Y \times \mathbb{T}^{\underline{n}}$ and f is the projection $Y \times \mathbb{T}^{\underline{n}} \to \mathbb{T}^{\underline{n}}$.

Then in 1971, Gheorghe Lusztig [Lusztig(Novikov) 1972] found a proof for general X and maps $f : X \to T^{\underline{n}}$ based on *the Atiyah-Singer index theorem for families of differential operators D_p parametrised by topological spaces P,* where the index takes values not in $\mathbb{Z}$ anymore but in the K-theory of P, namely, this index is defined as the K-class of the (virtual) vector bundle over P with the fibers $ker(D_p) - coker(D_p)$, $p \in P$, (Since the operators D_p are Fredholm, this makes sense despite possible non-constancy of the ranks of $ker(D_p)$ and $coker(D_p)$.)

The family P in Lusztig's proof is composed of the *signature operators* on X twisted with complex line bundles L_p, $p = P$, over X, where these L are induced by a map $f : X \to T^{\underline{n}}$ from *flat* complex unitary line bundles $\underline{L}_p$ over $T^{\underline{n}}$ parametrised by P (which is the $\underline{n}$-torus of homomorphism $\pi_1(T^{\underline{n}}) = \mathbb{Z}^{\underline{n}} \to \mathbb{T}$).

Using the the Atiyah-Singer index formula, Lusztig computes the index of this operator, shows that it is equal to $sign(f)$ and deduce from this the homotopy invariance of $sign_{[f]}(X)$.

What is relevant for our purpose is that Lusztig's computation equally applies to the Dirac operator twisted with L_p and shows the following.

Let X be a closed orientable spin manifolds of even dimension $\underline{n}$ and $f : X \to \mathbb{T}^{\underline{n}}$ be continuous map of non-zero degree. Then

$$ind(\mathcal{D}_{\otimes\{L_p\}}) \neq 0.$$

Therefore, there exits a point $p \in P$, such that X carries a harmonic L_p-twisted spinor

[36] *Aspherical* means that the universal cover of $\underline{X}$ is contractible

[37] Our topological formulation, which is motivated by the history of the Novikov conjecture, is deceptive: in truth, Novikov conjecture is 90% about infinite groups, 9% about geometry and only 1% about manifolds.

But if $Sc(X) > 0$, this is incompatible with the the Schroedinger-Lichnerowicz-Weitzenboeck formula (see sections 3.4, 4.1) which says for *flat* L_p that

$$\mathcal{D}_{\otimes L_p} = \nabla^2_{\otimes L_p} + \frac{1}{4} Sc(X).$$

Thus, the existence of a map $f : X \to T^{\underline{n}}$ with $deg(f) \neq 0$ implies that X carries no metric with $Sc > 0$.

Moreover, Lusztig's computation applies to manifolds X of all dimensions $n = \underline{n} + 4k$, shows that if a generic pullback manifold $Y^4_p = f^{-1}(p) \subset X$ (here f is smooth) has *non-vanishing* $\hat{\alpha}$*-invariant* defined in section 3.4 (that is the $\hat{A}$-genus for $4k$-dimensional manifolds), then the index $ind(\mathcal{D}_{\otimes\{L_p\}})$ doesn't vanish either and, assuming X is spin, it *can't carry metrics with* $Sc > 0$.

Remark on $X = (X, g_0) = \mathbb{T}^{\underline{n}}$. If (X, g_0) is isometric to the torus, then the only g_0-harmonic L_p-twisted spinors on X are parallel ones, which allows a direct computation of the index of $\mathcal{D}_{\otimes\{L_p\}}$. Then the result of this computation extends to all Riemannin metrics g on $T^{\underline{n}}$ by the invariance of the index of $\mathcal{D}_{\otimes\{L_p\}}$ under deformations of $\mathcal{D}$, where the essential point is that, albeit the harmonic spinors of the (untwisted) $\mathcal{D}$ may (and typically do) disappear under a deformation $\mathcal{D}_{g_0} \rightsquigarrow \mathcal{D}_g$, they re-emerge as harmonic spinors of $\mathcal{D}_g$ twisted with a non-trivial flat bundle L_p.

The index theorem for families can be reformulated with P being replaced by the algebra $cont(P)$ of all continuous functions on P, where in Lusztig's case the algebra $cont(T^{\underline{n}})$ is Fourier isomorphic to the algebra $C^*(\mathbb{Z}^{\underline{n}})$ of bounded linear operators on the Hilbert space space $l_2(\mathbb{Z}^{\underline{n}})$ of square-summarable functions on the group $\mathbb{Z}^{\underline{n}}$, which commute with the action of $\mathbb{Z}^{\underline{n}}$ on this space.

A remarkable fact is that a significant portion of Lusztig's argument generalizes to all discrete groups Π instead of $\mathbb{Z}^{\underline{n}}$, where the algebra $C^*(\Pi)$ of bounded operators on $l_2(\Pi)$ regarded as algebra of functions on a "non-commutative space" dual to Π (that is the actual space, namely that of of homomorphisms $\Pi \to \mathbb{T}$ for commutative Π.)

This allows a formulation of what is called in [Ros 1984] the *strong Novikov Conjecture*, the relevant for us special case of which reads as follows.

$\mathcal{D}_{\otimes C^*}$-Conjecture. If a smooth closed orientable Riemannin spin n-manifold X for n even admits a continuous map F to the classifying space $\mathsf{B}\Pi$ of a group Π, such that the homology homomorphism F_* sends the fundamental homology class $[X] \in H_n(X; \mathbb{R})$ to *non-zero* element $h \in H_n(\mathsf{B}\Pi; \mathbb{R})$, then

the Dirac operator on X twisted with some flat unitary Hilbert bundle over X has non-zero kernel.

(Here "unitary" means that the monodromy action of $\pi_1(X)$ on the Hilbert fiber $\mathcal{H}$ of this bundle is unitary and where an essential structure in this $\mathcal{H}$ is the action of the algebra $C^*(\Pi)$, which commute with the action of $\pi_1(X)$.)

This, if true, would imply, according to the Schroedinger-Lichnerowicz-Weitzenboeck formula, the spin case of the conjecture stated in section 3.4. saying that

X admits no metric with $Sc > 0$.

Also "Strong Novikov" would imply, as it was proved by Rosenberg, the validity of the

Zero in the Dirac Spectrum Conjecture. Let $\tilde{X}$ be a complete *contractible* Riemannin manifold the quotient of which under the action of the isometry group

$iso(\tilde{X})$ is *compact.*

Then the spectrum of the Dirac operator $\tilde{\mathcal{D}}$ *on* $\tilde{X}$ *contains zero, that is, for all* $\varepsilon > 0$, *there exist* L_2*-spinors* $\tilde{s}$ *on* $\tilde{X}$, *such that*

$$||\tilde{\mathcal{D}}(\tilde{s}|| \le \varepsilon||\tilde{s}||.$$

This, confronted with the Schroedinger-Lichnerowicz-Weitzenboeck formula, would show that $\tilde{X}$ can't have $Sc > 0$.

Are we to Believe in these Conjectures. A version of the *Strong Novikov conjecture* for a rather general class of groups, namely those which *admit discrete isometric actions on spaces with non-positive sectional curvatures*, was proven by Alexander Mishchenko in 1974.

Albeit this has been generalized since 1974 to many other cases groups Π and/or representatives $h \in H_n(\mathsf{B}\Pi;\mathbb{R})$, the sad truth is that one has a poor understanding of what these classes actually are, how much they overlap and what part of the world of groups they fairly represent.

At the moment, there is no basis for believing in this conjecture and there is no idea where to look for a counterexample either.[38]

3.11.1 Almost Flat Bundles, $\otimes_\varepsilon$-Twist Principle and the Relative index theorem on Complete Manifolds

Let X be a Riemannin manifold and $L = (L, \nabla)$ be a complex vector bundle L with unitary connection. If the curvature of L is ε-close to zero,

$$||\mathcal{R}_L|| \le \varepsilon,$$

then, locally, L looks, approximately as the flat bundle $X \times \mathbb{C}^r$, $r = rank_{\mathbb{C}}(L)$, and the Dirac operator twisted with L, denoted $\mathcal{D}_{\otimes L}$, that acts on the spinors with values in L, is locally approximately equal to the direct sum $\underbrace{\mathcal{D} \oplus ... \oplus \mathcal{D}}_{r}$.

It follows that if $Sc(X) \ge \sigma > 0$ and if ε is much smaller than σ, then by the (obvious) continuity of the Schroedinger-Lichnerowicz-Weitzenboeck formula, this twisted Dirac operator has trivial kernel, $ker(\mathcal{D}_{\otimes L}) = 0$ and, accordingly,

$$ind(\mathcal{D}^+_{\otimes L}) = 0,$$ [39]

where, by the Atiyah-Singer index theorem, this index is equal to a certain topological invariant

$$ind(\mathcal{D}^+_{\otimes L}) = \hat{\alpha}(X, L).$$

For instance, if X is an even dimensional topological torus, and if the top Chern class of L doesn't vanish, $c_m(L) \ne 0$ for $m = \frac{dim(X)}{2}$, then $\alpha(X, L) \ne 0$ as well.

On the other hand, given a Riemannin metric g on the torus $\mathbb{T}^n$, $n = 2m$, and $\varepsilon > 0$,

[38] Geometrically most complicated groups are those which represent one way or another universal Turing machines; a group, the k-dimensional homology (L-theory?) of which, say for $k = 3$, models such a "random" machine, would be a good candidate for a counterexample.

[39] Here we assume that $n = dim(X)$ is *even*, which makes $\mathcal{D}$ *split* as $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$, such that $ind(\mathcal{D}^+) = ind(\mathcal{D}^-)$, see section 4.1.

there exists a finite covering $\tilde{\mathbb{T}}^n$ of the torus, which admits an ε-flat vector bundle $\tilde{L} \to \tilde{\mathbb{T}}^n$ of $\mathbb{C}$-rank $r = m = \frac{n}{2}$ with $c_m(L) \neq 0$,

where the "flatness" of $\tilde{L}$, that is the norm of the curvature $\mathcal{R}_{\tilde{L}}$ regarded as a 2-form with the values in the Lie algebra of the unitary group $U(r)$, $r = rank_{\mathbb{C}}(\tilde{L})$, is measured with the lift $\tilde{g}$ of the metric g to $\tilde{\mathbb{T}}^n$.

Indeed, let $\hat{L} \to \mathbb{R}^n$, $n = 2m$, be a vector bundle with a unitary connection, such that $\hat{L}$ is isomorphic (together with it connection) at infinity to the trivial bundle and such that $c_m(\hat{L}) \neq 0$, where such an $\hat{L}$ may be induced by a map $\mathbb{R}^n \to S^n$, which is constant at infinity and has degree one, from a bundle $\underline{L} \to S^n$ with $c_m(\underline{L}) \neq 0$.

Let $\hat{L}_\varepsilon$ be the bundle induced from $\hat{L}$ by the scaling map $x \mapsto \varepsilon x$, $x \in \mathbb{R}^n$. Clearly, the curvature of $\hat{L}_\varepsilon$ tends to 0 as $\varepsilon \to 0$.

Since the finite coverings $\tilde{\mathbb{T}}^n$ of the torus converge to the universal covering $\mathbb{R}^n \to \mathbb{T}^n$ this $\hat{L}_\varepsilon$ can be transplanted to a bundle $\tilde{L}_\varepsilon \to \tilde{\mathbb{T}}^n$ over a sufficiently large finite covering $\tilde{\mathbb{T}}^n$ of the torus, where the top Chern number remains unchanged and where the curvature of $\tilde{L}$ with respect to the flat metric on $\tilde{\mathbb{T}}^n$ can be assumed as small as you wish, say $\leq \epsilon$.

But then this very curvature with respect to the lift $\tilde{g}$ of a given Riemannin metric g on $\mathbb{T}^n$ also will be small, namely $\leq const_g \epsilon$ and our claim follows.[40]

Thus, we obtain

yet another proof of nonexistence of metrics g with $Sc(g) > 0$ on tori.

Seemingly Technical Conceptual Remark. The above rough qualitative argument admits a finer quantitative version, which depends on the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula (see 4.1),

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \frac{1}{4}Sc(X) + \mathcal{R}_{\otimes L},$$

where $\mathcal{R}_{\otimes L}$ is an operator on twisted spinors, i.e. on the bundle $\mathbb{S}\otimes L$, associated with the curvature of L and where an essential feature of $\mathcal{R}_{\otimes L}$ is a bound on its norm by the it operator norm $\|\mathcal{R}_L\|$, with a constant *independent* of the rank of L.

Thus, for instance the above proof of nonexistence of metrics g with $Sc(g) > 0$ on tori, that was performed with the twisted Dirac operator $\mathcal{D}_{\otimes \tilde{L}}$ over a *finite covering* $\tilde{X}$ of our torical X, can be brought *back to* X by pushing forward $\tilde{L}$ from the $\tilde{X}$ to X, where this push forward bundle $(\tilde{L})_* \to X$ has

$$rank(\tilde{L})_* = N \cdot rank(\tilde{L})$$

for N being the number of sheets of the covering.

(The lift of $(\tilde{L})_*$ to $\tilde{X}$ is the Whitney's sum of N-bundles obtained from $\tilde{L}$ by the deck transformations of $\tilde{L}$.)

This property of $\mathcal{R}_{\otimes L}$, in conjunction with the shape of the Atiyah-Singer index formula, fo Dirac operator twisted with Whitney's N-multiples

[40] Why do we need twelve lines to express, not even fully at that, so an obvious idea? Is it due to an imperfection of our mathematical language or it is something about our mind that makes instantaneous images of structurally protracted objects? Probably both, where the latter depends on the *parallel processing* in the human *subliminal* mind, which can't be well represented by any sequentially structured language that follows our *conscious* mind and where besides "*parallel*" there are many other properties of "*subliminal*" hidden from our conscious mind eye.

$$L \oplus \ldots \oplus L = \underbrace{L \oplus \ldots \oplus L}_{N},$$

which implies that in the relevant cases

$$ind(\mathcal{D}^+_{\otimes(L\oplus\ldots\oplus L)}) = \alpha(X, L \oplus \ldots \oplus L) = N \cdot \hat{\alpha}(X, L) + O(1),$$

allows $N \to \infty$ and even $N = \infty$ in a suitable sense, e.g. in the context of infinite coverings (see section 4.1.1) and/or of C^*-algebras as was mentioned in the previous section.

What is also crucial, is that twisting with almost flat bundles is a *functorial* operation, where this functoriality yields the following.

$\otimes_\varepsilon$-Twist Principle. *All* (known) *arguments with Dirac operators for non-existence of metrics with $Sc \geq \sigma > 0$ under specific topological conditions on X can be* (more or less) *automatically transformed to inequalities between σ and certain geometric invariants of X defined via ε-flat bundles over X.*

From Compact to Complete Via the Relative Index Theorem

Most (probably, not all) bounds on the scalar curvature of *closed* Riemannian manifolds derived with twisted Dirac operators $\mathcal{D}_{\otimes L}$ have their counterparts for *complete* manifolds X, where one uses a relative version of the Atiyah-Singer theorem for *pairs of Dirac operators which agree at infinity* (see [GL 1983], [Bunke 1992], [Roe 1996]), where the simplest and the most relevant case of this theorem applies to vector bundles $L \to X$ with unitary connections which are *flat trivial at infinity.*

In this case the pair in question is $(\mathcal{D}_{\otimes\mathcal{L}}, \mathcal{D}_{\otimes|\mathcal{L}|})$, where $|L|$ denotes the trivial flat bundle $X \times \mathbb{C}^k \to X$ for $k = rank_{\mathbb{C}}(L)$, which comes along with an isometric connection preserving isomorphism between L and $|L|$ outside a compact subset in X.

f^*-Example. Let $f : X \to S^n$ be a smooth map which is locally constant at infinity (i.e. outside a compact subset) and let $\underline{L} \to S^n$ be a bundle with a unitary connection on S^n.

Then the pullback bundle $f^*(\underline{L}) \to X$ is an instance of such an L.

The relative index theorem, similarly to its absolute counterpart implies that if the scalar curvature of X is *uniformly positive (i.e. $Sc \geq \sigma > 0$) at infinity* and if

a certain topological invariant, call it $\hat{\alpha}(X, L)$,[41] *doesn't vanish, then either X admits a non-zero (untwisted) harmonic L_2-spinor s on X, that is a solution of $\mathcal{D}(s) = 0$, or there is a non-zero L-twisted harmonic L_2-spinor on X.*[42]

For instance,

if $L = f^*(\underline{L})$ as in the above example, where $n = dim(X)$ is even, the bundle $\underline{L}$ has a non-zero top Chern class (e.g. $\underline{L}$ is the bundle of spinors on the sphere, $\underline{L} = \mathbb{S}(S^n)$) and if the map $f : X \to S^n$ has non-zero degree, then $\hat{\alpha}(X, L) \neq 0$.

[41] See section for the definition of this invariant.

[42] If we don't assume that $Sc(X)$ is uniformly positive at infinity, then one can only claim the existence of either non-zero untwisted or non-zero twisted *almost harmonic* L_2-spinors, i.e. satisfying $\int_X \mathcal{D}^2(s)dx \leq \varepsilon \int_X \|s(x)\|^2$ or $\int_X \mathcal{D}^2_{\otimes L}(s)dx \leq \varepsilon \int_X \|s(x)\|^2$, for arbitrarily small $\varepsilon > 0$.

Finally, since the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula (obviously) applies to L_2-spinors, one obtains, for example, as an application of the $\otimes_\varepsilon$-Twist Principle the following relative version of the Lichnerowicz' theorem for k-dimensional manifolds from section 3.4, that, let us remind it, says that

$$\hat{A}[X] \neq 0 \Rightarrow Sc(X) \not> 0 \textit{ for closed spin manifolds } X.$$

If a complete Riemannian orientable spin manifolds X (of dimension $n+4k$) admits a proper λ -Lipschitz map $f: X \to \mathbb{R}^n$ for some $\lambda < \infty$, then the pullbacks of generic points $y \in \mathbb{R}^n$ satisfy $\hat{A}[f^{-1}(y)] = 0$.

This, in the case $dim(X) = n$, shows that

the existence of proper Lipschitz map $X \to \mathbb{R}^n$ implies that $\inf_x Sc(X,x) \leq 0$.

This rules out, in particular, metrics with $Sc > 0$ on tori.

(See [GL(spin) 1980], [GL 1983], [Roe 2012] for further examples and references.)

Remark. Recent regularization theorems by Lohkamp and Schoen-Yau allows proof by means of minimal hypersurfaces with the advantage of dropping the spin condition in certain cases. But analytic technicalities behind these proofs are significantly more complicated than what is needed for the Dirac operator proofs.

Besides – this is non-automatic and, geometrically, most amusing – a linear-algebraic analysis of the L-curvature term $\mathcal{R}_{\otimes L}$ in the above twisted Schroedinger-Lichnerowicz-Weitzenboeck formula, may lead to *sharp* geometric inequalities for Riemannin manifolds X with $Sc \geq \sigma$, such as Llarull's, Min-Oo's and Goette-Semmelmann's inequalities.

For instance, a certain *sharp version* of $\otimes_\varepsilon$-Twisting Principle turns Lichnerowicz theorem to the following (see [Llarull 1998]).

If a complete Riemannian orientable spin manifolds X (of dimension $n+4k$) with $Sc(X) > n(n-1)$ admis a locally constant at infinity 1 -Lipschitz map $f: X \to S^n$, then the pullbacks of generic points $y \in \mathbb{S}^n$ satisfy $\hat{A}[f^{-1}(y)] = 0$.

(See sections 3.5, 4.5 and also 4.1.1 where we outline a functorial perspective on such inequalities).

Notice that the spin condition is essential here if $dim(X) > n$, but it is, probably, redundant in many cases, e.g. for equidimensional maps f.

What is even more annoying is *compactness or, at least, completeness* of manifolds, where Dirac operators reside. [43]

But at the present day, Llarull's and similar *sharp, and certain non-sharp, inequalities* with scalar curvature, remains beyond of what can be achieved with the geometric measure theory.

Remark/Question. Llarull's inequality in conjunction with the relative index theorem shows that

if a complete orientable spin n-manifold X admits an *area contracting* locally constant at infinity (i.e. outside a compact subset) map $f: X \to S^n$, which has *non-zero degree*, and if the scalar curvature of X on the *support of the differential* of f (where $df \neq 0$) is *bounded from below* by that of S^n,

$$\inf_{s \in supp(df)} Sc(X,x) \geq n(n-1),$$

[43] Albeit indirectly, Dirac operators do apply to scalar curvature problems on *manifolds with boundaries*, as we shall see in the next section and in section 4.6.

then X can't have *uniformly positive* scalar curvature,

$$\inf_{x\in X} Sc(X,x) \leq 0.$$

However, since the relative index theorem needs uniformly positive scalar curvature at infinity,

it remains unclear if, in fact, X can't even have non-negative scalar curvature, that is if

$$\inf_{s\in X} Sc(X,x) < 0.$$

(Possibly, the argument from [Cecchini 2018] may be useful here, also see section 4.6)

$\otimes_\varepsilon$-Problem. Can one turn $\otimes_\varepsilon$-Twisting Principle to a $\otimes_\varepsilon$-theorem?

At the present moment, an application of the $\otimes_\varepsilon$-principle necessitates tracking *step by step*, let it be in a purely mechanical/algorithmic fashion, a particular Dirac theoretic argument, rather than a direct application of this principle to *the conclusion* of such an argument.

What, apparently, happens here is that the true outcomes of Dirac operator proofs are *not* the geometric theorems they assert, but certain linearized/hilbertized generalization(s) of these, possibly, in the spirit of Connes' non-commutative geometry.

To understand what goes on, one needs, for example, to reformulate (reprove?) Llarull's, Min-Oo's and Goette-Semmelmann's inequalities in such a "linearized" manner.

Twists with non-Unitary Bundles. Available (rather limited) results concerning scalar curvature geometry of manifolds X, which support almost flat non-unitary bundles and of (global spaces of possibly) non-linear fibrations with almost flat connections over X, are discussed in section 6.4.

Flat or Almost Flat? Lusztig's approach to the Novikov conjecture via the signature operators twisted with (families of) *finite dimensional non-unitary flat* bundles was superseded, starting with the work by Mishchenko and Kasparov, by more general index theorems, for *infinite dimensional flat unitary* bundles.

Then it was observed in [GL(spin) 1980] and proven in a general form in [Ros 1984]) that all these results can be transformed to the corresponding statements about Dirac operators on spin manifolds, thus providing obstructions to $Sc > 0$ essentially for the same kind of manifolds X, where the generalized signature theorems were established.

Besides following topology, the geometry of the scalar curvature suggested a quantitive version of these topological theorems by allowing twisted Dirac and signature operators with *non-flat vector bundles* with *controllably small curvatures*, thus providing geometric information on X with $Sc \geq \sigma > 0$, which complements the information on pure topology of X.

At the present moment, there are two groups of papers on twisted (sometimes untwisted) Dirac operators on manifolds with $Sc > \sigma$.

The first and a most abundant one goes along with the work on the Novikov conjecture, where it is framed into the KK-theoretic formalism.

A notable achievement of this is

Alain Connes' topological obstruction for leaf-wise metrics with $Sc > 0$ on foliations,

where

a geometric shortcut through the KK-formalism of Connes' proof is unavailable at the present moment.

Another direction is a geometrically oriented one, where we are not so much concerned with the K-theory of the C^*-algebras of fundamental groups $\pi_1(X)$, but with geometric constraints on X implied by the inequality $Sc(X) \geq \sigma$.

This goes close to what happens in the papers inspired by the general relativity, where one is concerned with specified (and rather special, e.g. asymptotically flat) geometries at infinity of complete Riemannian manifolds and where one plays, following Witten and Min-Oo, with Dirac operators, which are asymptotically adapted at infinity to such geometries. (In this context, the Schoen-Yau and the related methods relying of the *mean curvature flows* are also used.)

In the present paper, we are primarily concerned with *geometry* of manifolds, while *topology* is confined to *an auxiliary,* let it be irreplaceable, role.

3.11.2 Roe's Translation Algebra and Dirac Operators on Complete Manifolds with Compact Boundaries

C^*-algebras bring forth the following interesting perspective on *coarse geometry* of non-compact spaces proposed by John Roe following Alain Connes' idea of non-commutative geometry of foliations.

Given a metric space Ξ, e.g. a discrete group with a word metric, let $\mathcal{T} = Tra(\Xi)$ be the semigroup of translations of M that are maps $\tau : \Xi \to \Xi$, such that

$$\sup_{\xi \in \Xi} dist(\xi, \tau(\xi)) < \infty.$$

The (reduced) Roe C^*-algebra $R^*(\Xi)$ is a certain completion of the semigroup algebra $\mathbb{C}[\mathcal{T}]$. For instance if Ξ is a group with a word metric for which, say the left action of Ξ on itself is isometric, then the right actions lie in $\mathcal{T}$ and $R^*(\Xi)$ is equal to the (reduced) algebra $C^*(\Xi)$.[44]

Using this algebra, Roe proves in [Roe 1996], (also see [Hig 1991], [Roe 2012]) a *partitioned index theorem*, which implies, for example, that.

✏ *the toric half cylinder manifold $X = \mathbb{T}^{n-1} \times \mathbb{R}_+$ admits no complete Riemannin metric with $Sc \geq \sigma > 0$.*[45]

The subtlety here is twofold:

(i) the presence of non-empty boundary which is poorly tolerated by Dirac operators,

(ii) the metric on this X may (can it if $Sc \geq sigma > 0$ at infinity?), similarly to the hyperbolic metric $dr^2 + e^{-2r}dt^2$, exponentially (even super-exponentially, if you wish) contracts at infinity.

Notice, that ✏ can be also proved with the techniques of minimal hypersurfaces and/or of stable μ-bubbles.

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[44] "Reduced" refers to a minor technicality not relevant at the moment. A more serious problem – this is not joke – is impossibility of definition of "right" and "left" without an appeal to violation of mirror symmetry by weak interactions.

[45] I must admit I haven't fully understood Roe's argument.

In fact, the $\frac{2\pi}{n}$ -inequality from section 3.7 implies that the scalar curvature on $\mathbb{T}^1 \times \mathbb{R}_+$

not only approaches zero for $r \to \infty$ but it must decay quadratically fast.

(Compare with corollary in section 3.7 and exercise (a) in 5.2.)

But the Dirac theoretic proof of ✏ reveals certain geometric features of the scalar curvature, which are non-detectable by minimal hypersurfaces and which are interesting in their own rights.

(Unlike C^*-algebraically and K-theoretically oriented papers on scalar curvature, which are focused on non-existence theorems for $Sc > 0$, these theorems serve only a preparatory purpose in the geometric picture we develop in these lectures.)

Also notice in this regard that if X is sufficiently "thick at infinity", then ✏ follows by a simple argument with twisted Dirac operators and the standard bound on the number of small eigenvalues in the spectrum of the Laplace (or directly of the Dirac) operator in vicinity of ∂X, which applies to all manifolds with boundaries and which yields, in particular, (see sections 4.5, 4.6?????) the following.

✐ Let X be a complete oriented Riemannin spin n-manifold *with compact boundary*, such that

there exists a sequence of smooth area decreasing maps $f_i : X \to S^n$, which are constant in a (fixed) neighbourhood $V \subset X$ of the boundary ∂X as well as away from compact subsets $W_i \subset V$, and such that

$$deg(f_i) \underset{i\to\infty}{\to} \infty.$$

Then the scalar curvature of X satisfies

$$\inf_{x \in X} Sc(X, x) \le n(n-1).$$

Conclude by formulating the following.

Coarse $\mathcal{D}$-Spectrum Conjecture. Let $\hat{X}$ be a complete *uniformly contractible* Riemannian manifold, i.e. there exists a function $R(r) \ge r$, such that the ball $B_{\hat{x}}(r) \subset \hat{X}$, $x \in X$, of radius r is contractible in the concentric ball $B_{\hat{x}}(R(r))$ for all $\hat{x} \in \hat{X}$ and all radii $r > 0$.

Then the spectrum of the Dirac operator on $\hat{X}$ contains zero.

This conjecture, as it stands, must be, in view of [DRW 2003], *false*, but finding a counterexample becomes harder if we require the bounds $vol(B_{\hat{x}}(r)) \le \exp r$ for all $\hat{x} \in \hat{X}$ and $r > 0$.

3.12 Foliations With Positive Scalar Curvature

According to the philosophy (supported by a score of theorems) of Alain Connes much of the geometry and topology of manifolds with discrete group actions, notably, those concerned with index theorems for Galois actions of fundamental groups on universal coverings of compact manifolds, can be extended to foliations.

In particular, Connes shows in [Con 1986] that compact manifolds X which carry foliations $\mathscr{L}$ with leaf-wise Riemannin metrics with positive scalar curvatures behave in many respects as manifold which themselves admit such metrics.

For instance,

★ *if $\mathscr{L}$ is spin, i.e. the tangent (sub)bundle $T(\mathscr{L}) \subset T(X)$ of such an $\mathscr{L}$ is spin,then,*

by Connes' theorem, $\hat{A}[X] = 0$.

This generalises Lichnerowicz' theorem from section 3.4. for oriented spin manifolds of dimensions $n = 4k$, where, recall, $\hat{A}[X]$ is the value of a certain rational polynomial $\hat{A}(p_i)$ in the Pontryagin classes $p_i \in H^{4i}(X : \mathbb{Z})$ (see section 4.1) on the fundamental homology class $[X] \in H_n(X)$.[46]

In fact, the full Connes' theorem implies among other things

vanishing of the $\smile$-products of the $\hat{A}$-genus $\hat{A}(p_i)$, $j = 0, 1, ..., k = \frac{n}{4}$, with all polynomials in the Pontryagin classes of the "normal" bundle $T^{\perp}(\mathscr{L}) = T(X)/T(\mathscr{L})$, in the case where $\mathscr{L}$ is spin.

Connes' argument, which relies on Connes-Scandalis *longitudinal index theorem for foliations*), delivers a non-zero almost harmonic spinor on some leaf of $\mathscr{L}$ and an alternative and simpler proof of the existence of such spinors under suitable conditions was given in [BM 2018], where $\mathscr{L}$, besides being spin, is required to have *Hausdorff homotopy groupoid*.[47]

Another simplified proof of (a part of) Connes' theorem was also suggested in [Zhang 2016], where the *manifold X itself*, rather than the tangent bundle $T(\mathscr{L})$ is assumed *spin* [48] and where the existence of almost harmonic spinor is proven on some auxiliary manifolds associated with X.

One can get more mileage from the index theoretic arguments in these papers by applying the $\otimes_\varepsilon$-Twisting Principle (see the previous section), but this needs honest checking all steps in the proofs in there. This was (partly) done in [BM 2018] and in [Zhang 2018] in the context of the index theorems used by the authors in their papers.

Also I recall going through Connes' paper for this purpose long time ago and deriving the following proposition by applying this principle to Connes' argument (see §$9\frac{2}{3}$ in [G(positive) 1996]).

Closed manifolds X with infinite K-waist$_2$ (called "K-area" in [G(positive) 1996]) , *e.g. tori $\mathbb{T}^n$, carry no spin foliations which admit leaf-wise Riemannin metric with $Sc > 0$.*

Since my memory is uncertain at this point, I wouldn't claim this as a proven result, but rather formulate a geometrically more attractive conjecture that also must follow from $\otimes_\varepsilon$-Twisting Principle applied to Connes' argument.

Sharp Foliated $\otimes_\varepsilon$-Twisting Conjecture. Let X be a complete oriented n-dimensional Riemannin manifold with a smooth m-dimensional, $2 \leq m \leq n$, spin foliation $\mathscr{L}$, such that the induced Riemannin metrics on the leaves of $\mathscr{L}$ have their scalar curvatures $> n(n-1)$.

Then X admits no smooth area decreasing locally constant at infinity map $f : X \to S^n$ with $deg(f) \neq 0$.

[46]By definition, the values of the p_i-monomials $P_d = \smile_j p_{i_j} \in H^d(X)$, $4\sum_i i_j = d$, on $[X]$ equals zero for all $d \neq n$.

[47]One finds a helpful explanation of the meaning this condition in [Con 1983] and in the lectures [Mein 2017].

[48]In the ambience of Connes' arguments [Con 1986], these two spin conditions reduce one to another.

All of the above, however, leaves the following question open.

[$\star$?] Let (X,g) be a (possibly non-complete) Riemannin n-manifold with a smooth foliation, such that scalar curvature of the induced metric on the leaves satisfies $Sc \geq \sigma > 0$.

Does the product of X by a Euclidean space, $X \times \mathbb{R}^N$, admit an $\mathbb{R}^N$-*invariant* Riemannin metric $\tilde{g}$, such that $Sc(\tilde{g}) \geq \sigma$ and the quotient map $(X \times \mathbb{R}^N, \tilde{g})/\mathbb{R}^N \to (X,g)$ is 1-Lipschitz, or, at least, $const_n$-Lipschitz?

(See §$1\frac{7}{8}$ in [G(positive) 1996] and section 6.3 for partial results in this direction based on the geometry of *Connes' fibrations*.)

Notice that even the complete (positive) resolution of [$\star$?] wouldn't yield the entire Connes' vanishing theorem from [Con 1986], nor would this fully reveal the geometry of foliated Riemannian manifolds X with scalar curvatures of the leaves bounded from below.

For instance,

do compact Riemannian n-manifolds with constant curvature -1 admit k-dimensional foliations, $2 \leq k \leq n-1$, such that the scalar curvatures of the induced Riemannian metrics in the leaves are bounded from below by $-\varepsilon$ for a given $\varepsilon > 0$?

3.13 Scalar Curvature in Dimension 3

If $n \geq 4$, then then all known bounds on the size of n-manifolds X with $Sc(X) \geq \sigma > 0$ are expressed by *non-existence* of "topologically complicated but geometrically simple" maps from these X to "standard manifolds" $\underline{X}$.

But if $n = 3$ the following two more satisfactory results are available.

Let X be a complete Riemannin 3-manifold with scalar curvature $\geq 6 = Sc(S^3)$.

Then

A. *There exists a continuous map $f : X \to P^1$, where P is a 1-dimensional polyhedral space (topological graph) such that the diameters of the pullback of all points are bounded by*

$$[width_{3-2}] \qquad diam(f^{-1}(p)) \leq 2\pi\sqrt{6}.$$

B. *If X is homeomorphic to S^3, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ or $S^2 \times S^1$ then there exists a map $\Phi : S^2 \times T \to X$, where, either $T = \mathbb{R}$ or $T = S^1$ of degree 1*[49] *and such that the areas (counted with multiplicities if you wish) of the images $\Phi(S^2 \times \{t\}$, $t \in S^1$ satisfy*

$$[waist_{3-2}] \qquad area(\Phi(S^2 \times \{t\}) \leq 4\pi.$$

Remarks and Conjectures. (a) The proof of A (corollary 10.11 in [GL 1983]) relies on stable minimal surfaces in X, while B follows from the Marques-Neves estimate in [MN 2011] on the areas of surfaces with *Morse index* 1.

(b) The inequality $[waist_{3-2}]$, unlike $[width_{3-2}]$, is sharp, with the equality for the unit sphere S^3.

[49] If $T = \mathbb{R}$ then "degree 1" here presupposes here that there are at most two points in X, such that if a compact subset $C \subset X$ doesn't contain either of these points, then the pullback $\Phi^{-1}(C) \subset S^2 \times T$ is compact.

(c) The factor $\sqrt{6}$ in $[width_{3-2}]$ is, probably redundant, but even without this factor it wouldn't look as pretty as $[waist_{3-2}]$.[50]

(d) Proposition A, as it stands, (obviously) fails to be true for compact manifolds X with non-empty boundaries but, by the argument in §10 from [GL 1983], it remains valid for the part of X within distance $d > 2\sqrt{6}\pi$ from the boundary .

(e) *Conjecturally,* all complete n-manifolds X with $Sc(X) \geq n(n-1)$ admit continuous maps to polyhedral spaces of dimension $n-2$, say, $F : X \to P^{n-2}$, such that

$$diam(F^{-1}(p)) \leq const_n \text{ and } vol_{n-2}(F^{-1}(p)) \leq const'_n \text{ for all } p \in P^{n-2}.$$

Probably, this can be shown for $n = 3$ by combining the arguments from [GL 1983] and [MN 2011].

For instance if X is a connected sum of copies of $S^2 \times S^1$, then it can't be "sliced" entirely by spheres but, this becomes possible if we allow singular slices homeomorphic to *joins* of spheres $S^2 \star S^2$. However, it is not obvious if $Sc(X) \geq 6$ would allow such a slicing with the areas of the slices bounded by 4π or some other constant for this matter.

(d) Let X be a complete Riemannin n-manifolds with a 3-dimensional foliation such that the scalar curvature of the induced leaf-wise metric is bounded from below by 6.

Does X admit a continuous map $F : X \to P^{n-2}$ with $diam(F^{-1}(p)) \leq const_n$, $p \in P^{n-2}$?

(If so, this would provide yet another geometric proof of ???? [$\star$] from the previous section for 3-dimensional foliations.)

Penrose Inequality. Start with recalling that

the (space slice of the) Schwarzschild metric with mass m

is defined on $\mathbb{R}^3$ minus the origin in polar coordinates as

$$g_{Sw_m} = g_{Sw} = \left(1 + \frac{\rho}{r}\right)^4 g_{Eucl}, \text{ for } \rho = \rho_m = \tfrac{m}{2},$$

and that the

scalar curvature of this metric is zero

by the conformal change formula from section 2.6.

Since the function $s(r) = r^2 \left(1 + \frac{\rho}{r}\right)^4$ is invariant under the transformation

$$r \mapsto \frac{\rho^2}{r},$$

this g_{Sw} is invariant under the (conformal) reflection of $\mathbb{R}^3$ around the sphere $S^2(R_m) \subset \mathbb{R}^3$ of radius $\rho = \frac{m}{2}$, that is

$$(s, r) \mapsto \left(s, \frac{\rho^2}{r}\right).$$

[50] The inequality $[width_{3-2}]$ says that X can be "sliced" by surfaces of small diameters, but it doesn't tell anything about topologies and/or areas and intrinsic diameters of thees surfaces.

Thus the sphere $S(\rho)$ is totally geodesic in geometry of g_{Sw} with area

$$area_{g_{Sw}}(S(\rho) = \pi\rho^2\left(1+\frac{\rho}{\rho}\right)^4 = 16\pi m^2.$$

In 1973 Penrose formulated in [Pen 1973] a conjecture concerning black holes in general relativity with an evidence in its favour, that would, in particular imply the following.

Special case of the Riemannian Penrose Inequality. Let X be complete Riemannin 3-manifolds with compact boundary $Y = \partial X$, such that

• X is isometric at infinity to the Schwarzschild space of mass m at one of its two ends at infinity;

• the scalar curvature of X is everywhere non-negative: $Sc(X) \geq 0$;

• the boundary Y of X has zero mean curvature;[51]

• no minimal surface in X separates a connected component of Y from infinity.

Then the area of $Y = \partial X$ is bounded by the mass of the Schwarzschild space as follows.[52]

$$area(Y) \leq 16\pi m^2.$$

.

This, in a greater generality was proven by Hubert Bray in [Bray 2009].

On Geometric Meaning of Mass. The Schwarzschild metric at infinity fast approaches the Euclidean metric, where the greater the mass the slower is the growth rate of this metric.

To get a rough idea, let is compare g_{Sw} with the conical metrics

$$g_a = a^2 \cdot r^2 ds^2 + dr^2.$$

If $a < 1$ these metrics have positive scalar curvatures (zero for $a = 1$) and if you compare them with g_{Sw} these have infinite masses, and would violate any kind of Penrose-like inequality.

But if $a > 1$, then these g_a have masses $-\infty$ and one can show, e.g. using the bound on Rad_{S^2} from section 3.6 for suitable surfaces at infinity, that such a fast growth rate of general Riemannian manifolds is incompatible with $Sc > 0$.

Moreover, the positive mass theorem says that even finite but negative mass of an asymptotically Euclidean metric needs a bit of negativity in its scalar curvature.

But I must admit I haven't thought through further the geometric meaning of what physicists call "mass" in general relativity.

3.14 Scalar Curvature in Dimension 4

The simplest examples of 4-manifolds where non-existence of metrics with $Sc >$ follows from non-vanishing of Seiberg-Witten invariants are complex algebraic surfaces X in $\mathbb{C}P^3$ of degrees $d \geq 3$. (If d is even and these X are spin, this also follows from Lichnerowicz' theorem from section 3.4.)

[51] It suffices to assume that the the boundary is *mean convex*, i.e. its mean curvature relative to the normal field pointing outward is positive.

[52] This version of the Penrose conjecture is taken from the modern literature. It is unclear, at least to the present author, when, where and by whom an influence of positivity of scalar curvature in 3D on geometry of surfaces, which was, probably, known to physicists since the early 1970s (1960s?) was explicitly formulated in mathematical terms for the first time.

In fact, it was shown by LeBrun (see [Sal 1999] and references therein) that

no minimal (no lines with self-intersections one) *Kähler surface X admits a Riemannin metric with $Sc > 0$, unless X is diffeomorphic to $\mathbb{C}P^2$ or to a ruled surface .*

Furthermore, LeBrun shows in [LeB 1997] that

if such an X has *Kodaira dimension* 2, which is the case, for instance, for the algebraic surfaces $X \subset \mathbb{C}P^3$ of degree $d \geq 5$, then

the total squared scalar curvature is bounded by the first Chern number of X,

$$\int_X Sc(X,x)^2 dx \geq 32\pi^2 c_2(X),$$

where, moreover this inequality is sharp.

One may only dream of this kind of a bound on $\int_X Sc(X,x)^{\frac{n}{2}} dx$ for a manifolds X of dimension $n > 4$.

In fact the ideal bound, would be on $\int_X |Sc_-(X,x)|^{\frac{n}{2}} dx$ for $Sc_-(X,x) \min(Sc(X,x)$.

Although one doesn't expect anything comparable to the Seiberg-Witten equations for $n = dim(x) > 4$, one wonders if some coupling between the twisted Dirac $\mathcal{D}_{\otimes L}$ and an energy like functional in the space of connections in L may be instrumental in the study of the scalar curvature of X.

For instance,

Let a closed orientable Riemannin n-manifold X admits a map of non-zero degree to a closed locally symmetric manifold $\underline{X}$ with negative Ricci curvature, e.g. with constant negative curvature.

Does then the scale invariant integral of the negative part of the scalar curvature is bounded from below as follows:

$$\int_X |Sc_-(g,x)|^{\frac{n}{2}} dx \geq \int_{\underline{X}} |Sc(\underline{X},\underline{x})|^{\frac{n}{2}} d\underline{x}?$$

(Three conjectures related to this one are formulated in section 3.10.)

3.15 Topology and Geometry of Spaces of Metics with $Sc \geq \sigma$.

Non-triviality of the homotopy types of metrics with positive scalar curvatures, which was first proven by Nigel Hitchin in [Hit 1974], starts with the following observation.[53]

Let a closed n-manifold X be decomposed as $X_- \cup X_+$ where X_- and X_+ are smooth domains (n-submanifolds) in X with a common boundary $Y = \partial X_- = \partial X_+$ and where $X_\mp$ are equal to regular neighbourhoods of disjoint polyhedral subsets $P_\mp \subset X$ of dimensions $n_\mp$ such that $n_- + n_+ = n - 1$.

If $n_\mp \leq n-2$, then, by an easy elementary argument, both manifolds X_- and X_+ admit Riemannin metrics, say $g_\mp$, such that

the restrictions of these $g_\mp$ to Y, call them $h_\mp$, both have *positive* scalar curvatures.

And if X admits no metric with positive scalar curvature, e.g. if X is homeomorphic to the n-torus or to product of two Kummer surfaces, then h_-

[53]Hitchin himself argued differently.

and h_+ can't be joined by a homotopy of metrics with positive scalar curvatures.

Indeed, such a homotopy, h_t, $t \in [-1,+1]$ could be easily transformed to a metric on the cylinder $Y \times [-1,+1]$ with positive scalar curvature and with relatively flat boundaries isometric to (Y,h_-) and (Y,h_+), which would then lead in obvious way to a metric on $X = X_- \cup Y \times [-1,+1] \cup X_+$ with $Sc > 0$ as well.

This kind of argument combined with surgery with $Sc > 0$ and empowered by index theorem(s) for Dirac operators leads, to the following results.

[HaSchSt 2014]. *If m is much greater than k then the kth homotopy group of the space of metrics with $Sc > 0$ on the sphere S^{4m-k-1} is infinite.*

[EbR-W 2017]. *There exists a compact Spin 6-manifold such that its space of positive scalar curvature metrics has each rational homotopy group infinite dimensional.* [54]

However, there is *no closed manifold* of dimension $n \geq 4$, which admits a metric with $Sc > 0$ and where the (rational) homotopy type, or even the set of connected components, of *the space of such metrics is fully determined.* [55]

Let us formulate two specific questions motivated by the following vague one:

What is the "*topology of the geometric shape*" of the (sub)space of metrics with $Sc \geq \sigma$?

Question 1. Given a Riemannian manifold $\underline{X}$ and a pair of numbers $(\lambda,\sigma) \in \mathbb{R}^2_+$, let $G(X;\underline{X},\lambda,\sigma)$ be the space of pairs (g,f) where g is a Riemannian metrics on a X with $Sc(g) \geq \sigma$ and $f : X \to \underline{X}$ is a λ-Lipschitz map.

What is the topology and geometry of this space and of the natural embeddings

$$G(X;\underline{X},\lambda_1,\sigma_1) \hookleftarrow G(X;\underline{X},\lambda_2,\sigma_2)$$

for $\lambda_2 \geq \lambda_1$ and $\sigma_2 \geq \sigma_1$.

Question 2. Let D be some natural distance function on the space G of smooth Riemannin metrics g on a closed manifold X. For instance $\mathsf{D}(g_1,g_2)$ may be defined as log of the infimum of $\lambda > 0$, such that

$$\lambda^{-1} g_1 \leq g_2 \leq \lambda g_1.$$

Let $\mathsf{D}_\sigma(\mathsf{g})$ denote the D-distance from $g \in G$ to the subspace of metrics with $Sc \geq \sigma$ and $\tilde{\mathsf{D}}_\sigma(g)$ be the D-distance from the $diff(X)$-orbit of g to this subspace.

What are topologies, e.g. homologies, of the *a-sublevels*, $a \geq 0$, of the functions $D_\sigma : G \to [0,\infty)$ and $\tilde{\mathsf{D}}_\sigma : G \to [0,\infty)$ and of the inclusions

$$D_\sigma^{-1}(0,a] \hookrightarrow D_\sigma^{-1}(0,b], \text{ and } \tilde{\mathsf{D}}_\sigma^{-1}(0,a] \hookrightarrow \tilde{\mathsf{D}}_\sigma^{-1}(0,b] \text{ for } b > a?$$

[54] It seems, judging by the references in [EbR-W 2017], that all published results in this direction depend on the Dirac operator techniques which do not cover the above example, if we take a *Schoen-Yau-Schick manifold* (see [G(inequalities) 2018]) for X.

[55] If $n = 3$ contractibility of this space, (if it is true) must follow from the known results on the Ricci flow á la Perelman.

3.16 Manifolds with Corners.

Most (all?) theorems concerning closed manifolds X with $Sc \geq \sigma$ and, more visibly, manifolds with smooth boundaries $Y = \partial X$, have (some proven, some conjectural) counterparts for Riemannin manifolds X with corners at the boundary,

where the mean curvature $mean.curv(\partial X)$ for the smooth part of ∂X plays the role of *singular/distributional* scalar curvature supported on ∂X and where the *dihedral angles* $\angle$ along the corners, or rather the complementary angles $\pi - \angle$, can be regarded as *singular/distributional* mean curvature supported on the corners.

Below are two examples illustrating this idea.

Let X be a compact n-dimensional manifold with *simple*, also called *cosimplicial*, corners. This means that X is locally diffeomorphic at all points $x \in X$ to the positive cone $\mathbb{R}^n_+$ at some points $x' \in \mathbb{R}^n_+$, where an example of such an X is the n-cube $[0,1]^n$.

Call such an X *semihyperbolic* if whenever three 1-faces of X pairwise meet then all three meet at some point in X.

Example. The n-cube is semihyperbolic but the n simplex, $n \geq 2$, is not.

Semitopological $\urcorner$-Reflection Rigidity Theorem. Let X be a semihyperbolic manifold X of dimension n with corners, assume for safety sake that all faces of X are contractible and let g be a Riemannin metric on X, such that

$\bullet_n$ the scalar curvature of g is non-negative: $Sc(g) \geq 0$;

$\bullet_{n-1}$ the mean curvatures of all $(n-1)$-faces F_i of X are also *non-negative* : $mean.curv_g(F_i) \geq 0$;

$\bullet_{n-2}$ The dihedral angles $\angle_{ij}$ of X at all points of all $(n-2)$-faces, that are intersection of certain $(n-1)$ faces F_i and F_j, satisfy $\angle_{ij} \leq \frac{\pi}{2}$.

Then

$Sc(X) = 0$, $mean.curv(Y_{reg}) = 0$, all $\alpha = \frac{\pi}{2}$ and X itself admits a homeomorphism onto the n-cube $[0,1]^n$, which sends the faces of X onto faces of the cube.

About the Proof. This is shown by reflecting X around its $(n-1)$-faces, smoothing around the edges and applying the corresponding result for closed manifolds as it was done in [G(billiard] 2014] for cubical X, and where the general case needs an intervention of arguments from [G(inequalities) 2018], where the (non-spin) case $n \geq 9$ relies on [SY(singularities) 2017]. (Also see section 5.6).

Remarks. (a) There is little doubt that $\urcorner$-geometric rigidity also holds for our X:

the conditions $\bullet_n$, $\bullet_{n-1}$, $\bullet_{n-2}$ should imply that X is isometric to a rectangular solid.

But there are several technical details (especially for $n \geq 4$) still to settle in the proof.

(b) The semitopological $\urcorner$-rigidity for cubical (i.e. topologically isomorphic to cubes) yields, by an elementary argument, the C^0-closeness of spaces of metrics with $Sc \geq \sigma$ stated in section 3.1.

There are two major limitation to our $\urcorner$:

☹$_1$ the semihypebolicity condition rules out many promising spaces X, e.g. those isomorphic to n-simplices;

☹$_2$ condition $\bullet_{n-2}$ is unrealistically strong, e.g. for such X as planar k-gons with $k \geq 5$.

Below is an instance of where ☹$_1$ is partly appeased.

$\times\blacktriangle^i$-*Inequality.* Let $X_0 \subset \mathbb{R}^n$ be a polytope, i.e. a convex compact polyhedron with non-empty interior, and let $X \subset \mathbb{R}^n$ be diffeomorphic to X_0.

Let all $(n-1)$-faces F_i of X have *positive mean curvatures*, e.g. the subset $X \subset \mathbb{R}^n$ is convex.

Let the dihedral angles between (the tangent spaces of) the faces F_i and F_j of X at all points in the $(n-2)$-faces where/if these faces meet, are bounded by the corresponding dihedral angles of X_0,

$$\angle_{ij}(X) \leq \angle_{ij}(X_0).$$

If all dihedral angles of X_0 are $\leq \frac{\pi}{2}$, then

$$\angle_{ij}(X) = \angle_{ij}(X_0).$$

This is shown, by doubling and smoothing X_0 and X and then applying $[X^{\to}\bigcirc]$ (see section 4.3 and 4.4).

Remarks/Exercises. (a) The only polytopes with $\angle_{ij} \leq \frac{\pi}{2}$ are products of simplices, such as the n-cubes $[0,1]^n$, for example.

(b) If both X_0 and X are affine n-simplices then the implication

$$\angle_{ij}(X) \leq \angle_{ij}(X_0) \Rightarrow \angle_{ij}(X) = \angle_{ij}(X_0)$$

follows from the Kirszbraun theorem with no need for the condition $\angle_{ji} \leq \pi/2$.

(c) There are cases where $\times\blacktriangle^i$-*inequality* is known to hold for certain polytopes e.g. for k-gonal prisms, where (some) dihedral angles may be $> \frac{\pi}{2}$ (an approach via minimal (hyper)surfaces is indicated in [G(billiards) 2014] and in [Li 2017]) but this remains problematic in general even for *simple* n-polytopes, where at most n faces of dimension $n-1$ may meet at the vertices.

(d) It is unknown which pairs of combinatorially equivalent polytopes P and P' (convex polyhedra) may have their corresponding dihedral angles satisfying $\angle_{ij} \geq \angle'_{ij}$ without all corresponding angles being mutually equal.[56]

Motivations for Corners. Besides opening avenues for generalisations of what is known for smooth manifolds, Riemannin manifolds with corners and $Sc \geq \sigma$ may do good to the following.

1. Suggesting new techniques, (calculus of variations, Dirac operator) for the study of Euclidean polyhedra.

2. Organising the totality of manifolds with $Sc \geq 0$ (or, more generally with $Sc \geq \sigma$) into a nice category (A_∞-category?) $\mathcal{P}^\square$, that would include, as objects manifolds Y with Riemannian metrics h and functions M on them and where

[56]Recently, Karim Adiprasito told me he proved that *no convex polytope admits an infinitesimal deformation simultaneously decreasing all its dihedral angles.*

morphisms are *(co)bordisms* (*h-cobordisms*?) (X, g), $\partial X = Y_0 \cup Y_1$, where g is a Riemannian metric on X with $Sc \geq 0$, which restricts to h_0 and to h_1 on Y_0 and Y_1 and where the the mean curvature of Y_0 with inward coorientation is equal to $-M_0$ while the mean curvature of Y_1 with the outward coorientation is equal to M_1.

Conceivably, the [SY]-variational techniques for "flags" of hypersurfaces or its generalisation(s), may have a meaningful interpretation in $\mathcal{P}^\square$, while a suitably adapted Dirac operator method may serve as a quantisation of $\mathcal{P}^\square$.

3.17 Who are you, Scalar Curvature?

There are two issues here.

1. What are most general geometric objects that display features similar to these of manifolds with positive and more generally, bounded from below, scalar curvatures?

2. Is there a direct link between Dirac operators and minimal varieties or their joint appearance in the ambience of scalar curvature is purely accidental?

Notice in this regards that there are two divergent branches of the growing tree of scalar curvature.

A. The first one is concerned with the effects of $Sc > 0$ on the *differential structure* of spin (or spin$^{\mathbb{C}}$) manifolds X, such as their $\hat{\alpha}$ and Seiberg-Witten invariants.

B. The second aspect is about coarse geometry and topology of X with $Sc(X) \geq \sigma$, the (known) properties of which are derived by means of minimal varieties and twisted Dirac operators; here the spin condition, even when it is present, must be redundant.

To better visualise separation between A and to B, think of possible *singular spaces X with $Sc(X) \geq 0$* corresponding to A and to B – these must be grossly different.

For instance, if X is an *Alexandrov space* with (generalised) sectional curvature $\geq \kappa > -\infty$ then the inequality $Sc \geq 0$ makes perfect sense and, probably most (all?) of B can be transplanted to these spaces. [57]

But nothing, from spin on, of what we know of A makes sense in singular Alexandrov spaces.

And if you start from the position of 2 you better go away from conventional spaces and start dreaming of geometric magic glass ball with ghosts of harmonic spinors and of minimal varieties dancing within.

In concrete terms one formulates two problems.

A. What is the largest class of spaces (singular, infinite dimensional ...) which display the basic features of manifolds with $Sc \geq 0$ and/or with $Sc \geq \sigma > -\infty$ and, more generally, of spaces X, where the properly understood operator $-\Delta + \frac{1}{2}Sc(X)$ is positive or, at least not too negative?

For instance, which (isolated) conical singularities and which singular volume minimising hypersurfaces belong to this class?

[57] It seems, much of the geometric measure theory extends to Alexandrov spaces but it is unclear what would correspond to twisted Dirac operators on these spaces.

B. Is there a partial differential equation, or something more general, the solutions of which would mediate between twisted harmonic spinors and minimal hypersurfaces (flags of hypersurfaces?) and which would be non-trivially linked to scalar curvature?

Could one, as it was suggested in section 3.14, non-trivially couple the twisted Dirac $\mathcal{D}_{\otimes L}$ with some equation $\mathcal{E}_{\mathcal{L}}$ on the connections in the bundle L the Dirac operator in the spirit of the Seiberg-Witten equation?[58]

4 Dirac Operator Bounds on the Size and Shape of Manifolds X with $Sc(X) \geq \sigma$

4.1 Spinors, Twisted Dirac Operators, and Area Decreasing maps.

The Dirac operator $\mathcal{D}$ on a Riemannin manifold X tells you by itself preciously little about the geometry of X, but the same $\mathcal{D}$ twisted with vector bundles L over X carries the following message:

manifolds with scalar curvature $Sc \geq \sigma > 0$
can't be too large area-wise.

Albeit the best possible result of this kind (due to Marques and Neves, see B in section 3.13), which is known for X homeomorphic to S^3, which says that

if $Sc(X) \geq 6 = Sc(S^3)$, *then* X *can be "swept over" by* 2*-spheres of areas* $\leq 4\pi$,

was proven by means of minimal surfaces, all known bounds on "areas" of Riemannin manifolds of dimensions ≥ 4 depend on Dirac operators $\mathcal{D}$ twisted (or "non-linearly coupled" for n=4) with complex vector bundles L over X with unitary connections in L, where, don't forget it, the very definition of $\mathcal{D}$ needs X to be spin.

Recall (compare with section 3.11.1) that the *twisted* Dirac operator, denoted

$$\mathcal{D}_{\otimes L} : C^{\infty}(\mathbb{S} \otimes L) \to C^{\infty}(\mathbb{S} \otimes L),$$

acts on the tensor product of the spinor bundle $\mathbb{S} \to X$ [59] with $L \to X$, where it is related to the (a priori, positive Bochner Laplace) operator $\nabla^2_{\otimes L} = \nabla^2_{\otimes L} = \nabla^*_{\otimes L}\nabla_{\otimes L}$ in the bundle $\mathbb{S} \otimes L$, by the *twisted* Schroedinger-Lichnerowicz-Weitzenboeck formula

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \tfrac{1}{4} Sc(X) + \mathcal{R}_{\otimes L},$$

where $\nabla_{\otimes L}$ denotes the covariant derivative operator in $\mathbb{S} \otimes L$ and $\mathcal{R}_{\otimes L}$ is a certain (zero order) operator which acts in the fibers of the twisted spin bundle $\mathbb{S} \otimes L$ and which is derived from the curvature of the connection in L.

[58]Natural candidates for $\mathcal{E}_{\mathcal{L}}$ are equations for critical points of energy-like functional on spaces of connections, where, observe, L-twisted harmonic spinors $s : X \to \mathbb{S} \otimes L$ themselves minimize $s \mapsto \int_X \langle \mathcal{D}_{\otimes L}(s(x)) \mathcal{D}_{\otimes L}(s(x)) \rangle \, dx$.

[59]All you have to know about $\mathbb{S}(X)$ is that it is a vector bundle associated with the tangent bundle $T(X)$, which can be defined for spin manifolds X, where "spin" is needed, since the structure group of $\mathbb{S}(X)$ is the double cover of the orthogonal group $O(n)$ rather than $O(n)$ itself.

If we are not concerned with the sharpness of constants, all we have to know is that $\mathcal{R}_{\otimes L}$ is controlled by

$$||\mathcal{R}_{\otimes L}|| \le const \cdot ||curv(L)||$$

for $const = const(n, rank(L))$, where a little thought (no computation is needed) shows that, in fact, this constant depends only on $n = dim(X)$. (The actual formula for $\mathcal{R}_{\otimes L}$ is written down in the next section, also see [MarMin 2012] for further details and references.)

We regard a closed orientable *even dimensional* Riemannin manifold X *area wise large*, if it carries a *homologically substantial* or *essential bundle* L over it with *small curvature*, where "homologically substantial" signifies that some Chern number of L doesn't vanish. It is easy in this case[60] that there exists an associated bundle $L^\wedge$, such that

$$|curv|(L^\wedge) \le const_n |curv|(L)$$

and such that the Chern character in the index formula guaranties non-vanishing of the cup product $\hat{A}(X) \smile Ch(L^\wedge)$ evaluated at $[X]$,

$$(\hat{A}(X) \smile Ch(L^\wedge))[X] \ne 0$$

and, thus, by Atiyah-Singer theorem, the presence of *non-zero harmonic twisted spinors*: sections s of the bundle $\mathbb{S} \otimes L^\wedge$ for which $\mathcal{D}_{\otimes L^\wedge}(s) = 0$.

If the dimension n of X is odd, the above applies to $X \times S^1$ for a sufficiently long circle S^1.

For instance, n-manifolds, which admit area decreasing non-contractible maps to spheres $S^n(R)$ of large radii R are area wise large, where the relevant bundles L are induced from non trivial bundles over the spheres. (One may take $L^\wedge = L$ for these L.)

But if the scalar curvature of X is $\ge \sigma$ for a *large* $\sigma > 0$, where this *"large"* properly matches the above *"small"*, then by the Schroedinger-Lichnerowicz-Weitzenboeck formula the operator $\mathcal{D}_{\otimes L^\wedge}$ is positive and no such harmonic twisted exists; therefore, a suitably defined "area"(X) must be bounded by $\frac{const}{\sigma}$. (See the next section for a definition of this "area".)

(Recall that

$$\hat{A}(X) = 1 - \frac{1}{24} p_1 + \frac{1}{5760}(-4p_2 + 7p_1^2) + ... \in H^*(X)$$

is a certain polynomial in Pontryagin classes $p_i \in H^{4i}(X)$ of X and

$$Ch(L) = rank_{\mathbb{C}}(L) + c_1(L) + \frac{1}{2}(c_1(L)^2 - 2c_2(L)) + ... \in H^*(X)$$

is a polynomial in Chern classes $c_i(L) \in H^{2i}(X)$ of L, while $[X] \in H_n(X)$ denotes the fundamental class of X.

If $n = dim(X)$ is even, the spin bundle $\mathbb{S}$ naturally splits, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, the operator $\mathcal{D}_{\otimes L}$ also splits: $\mathcal{D}_{\otimes L} = \mathcal{D}^+_{\otimes L} \oplus \mathcal{D}^-_{\otimes L}$, for

$$\mathcal{D}^{\pm}_{\otimes L} : C^\infty(\mathbb{S}^\pm \otimes L) \to C^\infty(\mathbb{S}^\mp \otimes L)$$

[60] See $(L^\wedge)$ in section 4.1.3 and references therein.

and the index formula reads:

$$ind(\mathcal{D}^{\pm}_{\otimes L}) = \pm(\hat{A}(X) \smile Ch(L))[X].)$$

Relative Index Theorem on Complete Manifolds. Let X be a *complete* Riemannin manifold the scalar curvature of which is *uniformly positive at infinity.* Then the Schroedinger-Lichnerowicz-Weitzenboeck formula implies that

the Dirac operator is positive at infinity, i.e. outside some compact subset $V \subset X$:

$$\int_X \langle \mathcal{D}^2 s(x), s(x)\rangle dx \geq \varepsilon \int_X ||s(x)||^2 dx$$

for some $\varepsilon = \varepsilon(X) > 0$ and all L_2-spinors s supported outside V. This (easily) implies, in turn, that the operators $\mathcal{D}^{\pm}$ are Fredholm but the indices of these operators depend on delicate information on geometry of X at infinity and no simple formula for $ind(\mathcal{D}^{\pm})$ is available.

However if there are two operators $\mathcal{D}_1$ and $\mathcal{D}_2$, which are *equal at infinity*, e.g. $\mathcal{D}^{+}_{\otimes L}$, and $\mathcal{D}^{+}_{\otimes |L|}$, where $L \to X$ is a bundle with a unitary connection, where $|L|$ is the trivial bundle of rank $k = rank_{\mathbb{C}} L$ over X and where L comes with an isometric connection preserving isomorphism with $|L|$ at infinity, as in section 3.11.1, then the difference of their indices – both are Fredholm for the same reason as $\mathcal{D}^{\pm}$– satisfy the Atiyah-Singer formula:

$$ind(\mathcal{D}^{+}_{\otimes L}) - ind(\mathcal{D}^{+}_{\otimes |L|}) = (\hat{A}(X) \smile (Ch(L) - Ch|L|))[X].$$

where,

$$Ch(L) - Ch|L| = c_1(L) + \frac{1}{2}(c_1(L)^2 - 2c_2(L)) + ...$$

is understood as a cohomology class with compact supports and $[X]$ is the fundamental homology class with infinite supports.

More generally, if $\mathcal{D}_i = \mathcal{D}_{\otimes L_i}$, $i = 1,2$, where L_1 is equated with L_2 at infinity, then

$$ind(\mathcal{D}^{+}_1) - ind(\mathcal{D}^{+}_2) = (\hat{A}(X) \smile (Ch(L_1) - Ch(L_2))[X],$$

where one needs the operators $\mathcal{D}_i$ be positive at infinity.

The proof of this can be obtained by adapting any version of the *local proof* of the compact Atiyah-Singer theorem (see (see [GL 1983], [Bunke 1992], [Roe 1996]).

Namely, the index is represented by the difference of the traces of families of auxiliary operators $K^{+}_{1,t} - K^{+}_{2,t}$ and $K^{-}_{1,t} - K^{-}_{2,t}$, $t > 0$, where

(i) these $K_{...,t}$-operators are given by continuous kernels $K_{...,t}(x,y)$ which are supported in the *t-neighbourhood of the diagonal* in $X \times X$, i.e. where $dist(x,y) \leq t$;

(ii) $K^{\pm}_{1,t}(x,y) = K^{\pm}_{2,t}(x,y)$ for x and y in the *complement of a compact subset* $V_t \subset X$, where $V_{t_1} \subset V_{t_2}$ for $t_2 > t_1$ and where $\bigcup_t V_t = X$;

(iii) $trace(K^{+}_{1,t} - K^{+}_{2,t}) - trace(K^{1}_{1,t} - K^{1}_{2,t}) = (\hat{A}(X) \smile (Ch(L_1) - Ch(L_2))[X]$;

for all $t > 0$;

(iv) the operators $K^{\pm}_{i,t}$, $i = 1,2$, *weakly converge*[61] for $t \to \infty$ to the projection

[61] The corresponding functions $K_{...,t}(x,y)$ uniformly converge on compact subsets in $X \times X$.

operators on the kernels of $\mathcal{D}\pm_i$.

The quickest way to get such $K_{\dots,t}$ is by taking suitable functions ψ_t of the corresponding Dirac operators, where the Fourier transforms of ψ_t have *compact supports*, and where (as in all arguments of this kind) the essential issue is the proof of *uniform bounds* on the traces of the operators $K^{\pm}_{1,t} - K^{\pm}_{2,t}$ for $t \to \infty$.[62]

4.1.1 Negative Sectional Curvature against Positive Scalar Curvature.

A characteristic topological corollary of the above is as follows.

$[\kappa \leq 0] \rightsquigarrow [Sc \not> 0]$: *If a closed orientable spin n-manifold X admits a map to a complete Riemannin manifold $\underline{X}$ with* $sect.curv(\underline{X}) \leq 0$,

$$f : X \to \underline{X},$$

such that the homology image $f_*[X] \in H_n(\underline{X}; \mathbb{Q})$ doesn't vanish, then X admits no metric with $Sc(X) > 0$.

Two Words about the Proof. All we need of $sect.curv \leq 0$ is the existence of distance decreasing maps from the universal covering of $\underline{X}$ to (large) spheres,

$$F_{\underline{x}} : \underline{X} \to S^{\underline{n}}(R),\ \underline{n} = dim(\underline{X}),\ \underline{x} \in \underline{X},$$

which can be (trivially) obtained with a use of inverse exponential maps

$$\exp^{-1}_{\underline{x}} : \tilde{\underline{X}} \to T_{\underline{x}}(\underline{X}),\ \underline{x} \in \underline{X}.$$

To make the idea clear, let $\underline{X}$ be compact, the fundamental group of $\underline{X}$ be residually finite, (e.g. $\underline{X}$ having constant sectional curvature or, more generally being a locally symmetric space) and X be embedded to $\underline{X}$.

Let $X^{\perp} \subset \underline{X}$ be a closed oriented submanifold of dimension $m = \underline{n} - n$ for $\underline{n} = dim(\underline{X})$, which has *non-zero intersection index* with $X \subset \underline{X}$.

Also assume that the restriction of the tangent bundle of $\underline{X}$ to $\underline{X}^{\perp} \subset \underline{X}$ is trivial.

Then – this is rather obvious – there exist finite covers $\tilde{\underline{X}}_i \to \underline{X}$, such that the products of the lifts (i.e. pull-backs) of X and of $X^{\perp}$ to $\tilde{\underline{X}}_i$, denoted $\tilde{X}_i \times \tilde{X}^{\perp}_i$, admit smooth maps to the spheres of radii R_i,

$$F_i : \tilde{X}_i \times X^{\perp}_i \to S^{\underline{n}}(R_i),$$

where

$\bullet_1$ $R_i \to \infty$,

$\bullet_2$ $deg(F_i) \neq 0$,

$\bullet$ the maps F_i are *distance decreasing on the fibers* $\tilde{X}_i \times x^{\perp}$ for all $x^{\perp} \in X^{\perp}_i$ for the Riemannian metric in these fibers induced by the embedding $\tilde{X}_i \times x^{\perp} = \tilde{X}_i \subset \tilde{\underline{X}}_i$.

It follows that for *arbitrary* Riemannin metrics g and $g^{\perp}$ on X and on $X^{\perp}$ there exists (large) constants λ and C independent of i, such that

[62] Specific bounds for particular $K_{\dots,t}$ are crucial for an (approximate) extension of the index theory to non-complete manifolds (e.g. needed for the problem discussed in sections 4.6) but these bounds are often buried in the K-theoretic formalism of the recent papers. Also, I must admit, this point was not explained (overlooked?) in the exposition of Roe's argument in my paper [G(positive) 1996].

the maps F_i are C-*Lipschitz* with respect to the sum of the lift of the metric g to $\tilde{X}_i$ and the lift of $\lambda \cdot g^\perp$ to $\tilde{X}_i^\perp$ that is the metric

$$\tilde{g}_i \oplus \lambda \cdot \tilde{g}_i^\perp \text{ on } \tilde{X}_i \times \tilde{X}_i^\perp.$$

If $Sc(g) \geq \sigma > 0$, then also $Sc(\tilde{g}_i \oplus \lambda \cdot \tilde{g}_i^\perp) \geq \sigma' > 0$ for all sufficiently large λ, which, for large R_i, rules out non-zero harmonic spinors on $\tilde{X}_i \times \tilde{X}_i^\perp$ twisted with the bundle $L^* = F_i^*(L)$ induced from any given bundle L on $S^{\underline{n}}$.

But if $\underline{n} = 2k$ and the Chern class $c_k(L)$ is non-zero, then non-vanishing of $deg(F_i)$ implies non-vanishing of of $ind(\mathcal{D}_{\otimes L})$ via the index formula and the resulting contradiction delivers the proof for even $\underline{n}$ and the odd case follows with $\underline{X} \times S^1$.

Remarks. This argument, which is rooted in Mishchenko's proof of Novikov conjecture for the fundamental group of the above $\underline{X}$, which was adapted to scalar curvature in [GL 1983]) and further [generalized/formalised in [CGM 1993], doesn't really need compactness of $\underline{X}$, residual finiteness of $\pi_1(\underline{X})$ and triviality of $T(\underline{X})|X^\perp$. Beside, the spin condition for X can be relaxed to that for the universal cover of X.

Moreover, since the bound on the size of $\tilde{X}_i \times \mathbb{T}^{\underline{n}-n}$ by $\frac{const}{\sqrt{\sigma}}$ can be obtained with the use of minimal hypersurfaces (see §12 in [GL 1983]), [G(inequalities) 2018] and section 5.4) the spin condition can be dropped altogether.

Question. Are there other topological non-spin obstructions to $Sc > 0$?

For instance, is the following true?

Conjecture. Let X be a closed orientable Riemannin n-manifold, such that no closed orientable n-manifold X' which admits a map $X' \to X$ with non-zero degree admits a metric with $Sc > 0$. Then there exists an integer m and a sequence of maps

$$F_i : \tilde{X} \times \mathbb{R}^m \to S^{n+m}(R_i),$$

where $\tilde{X}$ is some (possibly infinite) covering of X, such that

- the maps F_i are constant at infinity and they have non-zero degrees,
- $R_i \to \infty$,
- the maps F_i are distance decreasing on the fibers $\tilde{X} \times x^\perp$ for all $x^\perp \in \mathbb{R}^m$.

Apparently, there is no instance of a *specific* homotopy class $\mathcal{X}$ of closed manifolds X of dimension $n \geq 5$, where a Dirac theoretic proof of non existence of metrics with $Sc > 0$ on all $X \in \mathcal{X}$ couldn't be replaced by a proof via minimal hypersurfaces.

(This seems to disagree with what was said concerning $\otimes_{\wedge\omega}$ at the end of section 2.7.

In fact the *general* condition for $Sc \ngtr 0$ in $\otimes_{\wedge\omega}$, can't be treated, not as it stands, with minimal hypersurfaces, but this may be possible in all *specific examples*, where this condition was *proven to be* fulfilled.)

And it is conceivable when it comes to the Novikov conjecture, that its validity in all proven specific examples, can be derived by an elementary argument from the invariance of rational Pontryagin classes under ε-homeomorphisms.[63])

[63]The original proof of topological invariance of Pontryagin classes by Novikov, as well as simplified versions and modifications of his proof in [G(positive) 1996) automatically apply to ε-homeomorphisms and, sometimes, of homotopy equivalences

But even though the relevance of twisted Dirac theoretic methods is questionable as far as *topological* non-existence theorems are concerned, these methods seem irreplaceable when it comes to *geometry* of $Sc \geq \sigma$ as we shall see presently.

4.1.2 Global Negativity of the Sectional Curvature, Singular Spaces with $\kappa \leq 0$, and Bruhat-Tits Buildings.

The non-existence theorem $[\kappa \leq 0] \rightsquigarrow [Sc \not> 0]$ from the previous section generalizes to *singular spaces* $\underline{X}$ with $\kappa(\underline{X}) \leq 0$,roughly as follows.

In fact, an essential feature of complete simply connected (non-singular) manifolds $\underline{X}$ with $\kappa \leq 0$ [64] is as follows,

$[\circlearrowleft_\varepsilon]$ *Self-contraction Property.* $\underline{X}$ *admits a family of proper ε-Lipschitz selfmaps $\phi_\varepsilon : \underline{X} \to \underline{X}$, for all $\varepsilon > 0$, where these maps are properly homotopic to the identity map id.* [65] This property implies the existence of proper Lipschitz maps $\underline{X} \to \mathbb{R}^n$ of degree one, but unlike the latter it makes sense for singular spaces that are not topological manifolds or pseudomanifolds.

On the other hand, if a possibly singular, say finite dimensional polyhedral space X has this property $\circlearrowleft_\varepsilon$, then there exists a manifold $\underline{X}^+ \supset \underline{X}$, which also satisfies $\circlearrowleft_\varepsilon$, where the most transparent case is that of spaces $\underline{X}$ which come with free isometric actions by discrete groups Γ with compact quotients $\underline{X}$.

To derive $\underline{X}^+$ from $\underline{X}$ in this case, embed $\underline{X}/\Gamma \hookrightarrow \mathbb{R}^N$, take a small regular neighbourhood $U \subset \mathbb{R}^N$ of $\underline{X}^+/\Gamma \subset \mathbb{R}^N$ and let $\tilde{U} \to U$. be the universal covering of U.

Then this $\tilde{U}$ with a *suitably blown-up metric* serves for $\underline{X}^+$, where the simplest such blow up is achieved by multiplying the (locally Euclidean) metric in $\tilde{U}$ by the function $\frac{1}{dist(\tilde{u}, \partial U)}$.

In fact, what is truly needed for the non-existence argument, and what is satisfied by complete simply connected spaces $\underline{X}$ with $\kappa < 0$ is the following parametric version of $[\circlearrowleft_\varepsilon]$.

$[\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon]$. There exist a continuous map $\Phi_\varepsilon : \underline{X} \times \underline{X} \to \underline{X}$ with the following properties.

$\bullet_\varepsilon$ *the maps $\phi_{\varepsilon,\underline{x}_0} = \Phi_\varepsilon : \underline{X} = \underline{x}_0 \times \underline{X} \to \underline{X}$ are proper ε-Lipschitz for all $\underline{x}_0 \in \underline{X}$ and all $\varepsilon > 0$;*

$\bullet_n$ *the restrictions of these maps $\phi_{\varepsilon,\underline{x}_0} : \underline{X} \to \underline{X}$ to the n-skeleton $\underline{X}^{(n)} \subset \underline{X}$ are proper homotopic to the inclusions $\underline{X}^{(n)} \subset \underline{X}$;* [66]

$\bullet_\Gamma$ *the family $\phi_{\varepsilon,\underline{x}_0}$ is equivariant under the isometry group of $\underline{X}$:*

if $\gamma : \underline{X} \to \underline{X}$ is an isometry, then

$$\phi_{\varepsilon,\gamma(\underline{x}_0)} = \gamma \circ \phi_{\varepsilon,\underline{x}_0}.$$

The above argument combined with that in the previous section yields the following generalization of the non-existence theorem $[\kappa \leq 0] \rightsquigarrow [Sc \not> 0]$.

[64] The only geometric feature of the space $\underline{X}$ with $\kappa \leq 0$ (these often come under heading of CAT(0)-spaces) needed here is (something less than) strict geodesic convexity of the balls in $\underline{X}$.

[65] See [G(large) 1986] for more about such manifolds.

[66] Here we assume that $\underline{X}$ is triangulated and n denotes the dimension of a manifold X we are going to map to $\underline{X}$;

$[\kappa \leq 0]_{global} \rightsquigarrow [Sc \not> 0]$: *If a complete Riemannian spin manifold $\tilde{X}$ of dimension n with a discrete (not necessarily free) co-compact isometric action of a group Γ admits a proper Γ-equivariant map to an $\underline{X}$ which satisfies $\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon$, then $\inf_x(Sc(X,x) \leq 0$.*

Corollary. Let Γ be a finitely generated subgroup in the linear group $GL_N(\mathbb{C})$,[67] *let X be a compact oriented Riemannin spin n-manifold with $Sc(X) > 0$ and let $f : X \to \mathsf{B}(\Gamma)$ be a continuous map, where $\mathsf{B}(\Gamma)$ denotes the classifying (Eilenberg MacLane) space of Γ.*

Then the image

$$f_*[X]_{\mathbb{Q}} \in H_n(\mathsf{B}(\Gamma); \mathbb{Q})$$

of the rational fundamental class

$$[X]_{\mathbb{Q}} \in H_n(X; \mathbb{Q}) \text{ for } f_* : H_*(X; \mathbb{Q}) \to H_*(\mathsf{B}(\Gamma); \mathbb{Q})$$

is zero.

Proof. A finite index subgroup in Γ freely,[68] discretely and isometrically acts on the product $\underline{X}$ of Riemannian symmetric spaces and *Bruhat-Tits buildings*, where such products, according to Bruhat-Tits are

complete simply connected polyhedral space with $\kappa(X) \leq 0$.

Since $\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon$ apply to such spaces, the proof of the corollary follows.

Historical Remark. A.D. Alexandrov and H. Busemann, who suggested (two somewhat different) definitions of $\kappa \leq 0$ applicable to *singular* metric spaces, and their followers focused on essentially local geometric properties of these spaces X, and tried to *alleviate effects of singularities* by adding extra assumptions on X.

The theory of $\kappa \leq 0$ has acquired a global mathematical status with the discovery of Bruhat-Tits buildings. (Bruhat and Tits independently developed the local and global theory of their spaces being unaware of definitions of $\kappa \leq 0$ suggested by differential geometers.)

This has eventually led to the modern perspective on CAT(0)-spaces, i.e. those with $\kappa \leq 0$, the main interest in which is due to a multitude of significant examples of singular CAT(0)-spaces with interesting fundamental groups inspired by the ideas behind the construction(s) and applications of the Bruhat-Tits buildings

Hyperbolic Remark. "ε-Lipschitz" in the theorem $[\kappa \leq 0]_{global} \rightsquigarrow [Sc \not> 0]$ is only needed on the *large scale*, that is expressed by the inequality

$$dist(f_{\varepsilon,x_0}(x_1), f_{\varepsilon,x_0}(x_2)) \leq \varepsilon dist(x_1, x_2) + const.$$

Thus, for instance,

the non-existence conclusion for metrics with $Sc > 0$ on X applies, where $\underline{X}$ is the *Rib complex* of a *hyperbolic group*.

It follows, that the conclusion of the above corollary holds for hyperbolic groups Γ:

[67] One may place here any field instead of $\mathbb{C}$.
[68] Finite index was needed for his "freely"

Let X be a closed orientable Riemannian spin manifold with $Sc(X) > 0$ and let Γ be a hyperbolic group. Then the class $f_*[X]_{\mathbb{Q}} \in H_n(\mathsf{B}(\Gamma); \mathbb{Q})$ vanishes for all continuous maps $f : X \to \mathsf{B}(\Gamma)$.

"ε-Area" Remark. Instead of "ε-Lipschitz" one may require "ε-*area contracting*" or some large scale counterpart to this condition.

This may be significant, because the ε-area version of $[\kappa \le 0]_{global} \rightsquigarrow [Sc \not> 0]$ is not-approachable with the (known) techniques of minimal hypersurfaces and/or of stable μ-bubbles, while the above "ε-Lipschitz" $[\kappa \le 0]_{global} \rightsquigarrow [Sc \not> 0]$ can be proved in many, probably in all, cases with these techniques having an advantage of not requiring manifolds X to be spin.

On the other hand, for all I know, there is no example of an $\underline{X}$, say with a cocompact action of an isometry group Γ which satisfies a version of $\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon$ with the ε-contracting area property but not with the ε-Lipschitz one.[69]

4.1.3 Curvatures of Unitary Bundles, Virtual Bundles and Fredholm Bundles.

Let us try to formalise the concept of

"area", of a Riemannian manifold X, where this "area" is associated with curvatures of vector bundles over X and which has the property of being bounded by $const \cdot \frac{1}{\sigma}$, for $\sigma = \inf_x Sc(X,x) > 0$.

$||curv(L)||$. Given a vector bundle (L, ∇) with an orthogonal (unitary in the complex case) connection, over a Riemannian manifold X, let

$$||curv(L)||(x) = ||curv(\nabla)||(x) = ||curv(L,\nabla)||(x)$$

denote

the infimum of positive functions $C(x)$, such that the maximal rotation angles $\alpha \in [-\pi, \pi]$ of the parallel transports along the boundaries of smooth discs D in X satisfy

$$|\alpha| = |\alpha_D| \le \int_D C(d).^{70}$$

(The holonomy operator splits into the direct sum of rotations $z \mapsto \alpha_i z$, $z \in \mathbb{C}$, $\alpha_i \in \mathbb{T} \subset \mathbb{C}$, $i = 1, 2, ..., rank(L)$, and our $\alpha = \max_i \alpha_i$.)

For instance, if D is a geodesic digon in S^2 with the angles $\beta\pi$, $\beta \le 1$, then the holonomy of the tangent vectors around the boundary of D satisfies:

$$|\alpha_D| = 2\beta\pi = area(D),$$

which agrees with the equality $|curv|(T(S^2)) = 1$.

It follows the curvature of the tangent bundle (complexified if you wish) of the product of spheres, satisfies

$$\left|\left|curv\left(T\left(\bigtimes_i S^{n_j}(R_j)\right)\right)\right|\right| = \frac{1}{\min_j R_j^2}.$$

[69] Neither, it seems, there are examples of $\underline{X}$ with compact quotients $\underline{X}/\Gamma$, which satisfy $\circlearrowleft_\varepsilon$ but not $\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon$.

[70] This definition is adapted to vector bundles over rather general metric spaces, e.g. polyhedra with piecewise smooth metrics.

What is more amusing is that the even dimensional spheres S^n, $n = 2m$, support unitary bundles L with with twice smaller curvatures and *non-zero* top Chern classes,

$$|curv|(L) = \tfrac{1}{2} \text{ and } c_m(L) \neq 0.$$

For instance, if $n = 2$, then the Hopf bundle, that is the square root of the tangent bundle, has these properties and in general, the positive $\mathbb{C}$-spin bundle $\mathbb{S}^+$ can be taken for such an L.

This is the smallest curvature a non-trivial bundle over S^n may have:

Unitary vector bundles over S^n with $|curv| < \frac{1}{2}$ are trivial.

Proof. Follow the parallel transport of tangent vectors from the north to the south pole.

More generally

there are bundles L on the products of even dimensional spheres $\times_i S^{n_j}(R_j)$, which are induced by λ-Lipschitz maps to S^n, $n = \sum n_j$, $\lambda = \frac{1}{\min_j R_j^2}$, such that $|curv| \leq \frac{1}{2\min_j R_j^2}$ and such that *some Chern numbers of these L are non-zero,* and this is the best one can do.

In fact,

If a unitary vector bundle $L = (L, \nabla)$ over a product manifold $S^n \times Y$ has $|curv|(L) < \frac{1}{2}$, then all Chern numbers of L vanish. (see §13 in [G(101) 2017]).

The role of the Chern numbers here is motivated by the following observation (see [GL (spin) 1980, [G(positive) 1996]).

Let X be a closed orientable spin manifold of dimension $n = 2m$ and $L = (L, \nabla)$ a unitary vector bundle, such that *some Chern number of L doesn't vanish.* Then

$(L^\wedge)$ *there exists an associated bundle $L^\wedge$, which is a polynomial in the exteriors powers of L, such that*

$$ind(\mathcal{D}_{\otimes L^\wedge}) \neq 0$$

.

Since (it is easy to see) the degree and the coefficients of such a polynomial must be bounded by a constant depending only on n, the curvature of $L^\wedge$ satisfies

$$|curv|(L^\wedge) \leq const_n \| curv|(L);$$

Therefore,

● *if the scalar curvature of a closed orientable $2m$-dimensional spin manifold satisfies $Sc(X) \geq \sigma > 0$, then* – this is explained in the previous section – *non-vanishing $c_m(L) \neq 0$, implies the following lower bound on the curvature of the bundle L:*

$$|curv|(L) \geq \epsilon \cdot \sigma, \ \ \epsilon = \epsilon(n) > 0.$$

Open Problem. Prove ● without the spin condition.

The above suggest the definition of "area"(X) of a Riemannin manifold X as the supremum of $\frac{1}{|curv|(L)}$ over all unitary vector bundles $(L = L, \nabla)$ with non-zero Chern numbers.

However, the "area" terminology we introduced in [G(positive) 1996], despite several natural/functorial properties of this "area" (see [G(positive) 1996] and [G(101 2017]), seems inappropriate, since this "area" is *by no means additive.* A more adequate word , which we prefer to use from now on is *K-theoretic waist.*

Virtual Hilbert and Fredholm. To define this, we represent the *(Grothendieck) classes* $\mathbf{h}$ of vector bundles over X, which are also called *virtual (Fredholm) bundles*, by *Fredholm homomorphisms* between Hilbert bundles with unitary connections $\mathcal{L}_i = (\mathcal{L}_i, \nabla_i)$, $i = 1, 2$,

$$h : \mathcal{L}_1 \to \mathcal{L}_2,$$

where these h must *almost commute*, i.e. *commute modulo compact operators*, with the parallel transports in in $\mathcal{L}_1$ and $\mathcal{L}_2$ along smooth paths in X.

(This idea for flat bundles goes back to [Atiyah(global) 1969], [Kasp 1973], [Kasp 1975], [Mishch 1974] and where non-flat generalizations and applications are discussed in §$9\frac{1}{6}$ of [G(positive) 1996].)

(Such an $\mathbf{h}$ represents the finite dimensional virtual (not quite) bundle $ker(h) - coker(h)$.)

Define

$$|curv|(h) = \max(|curv|\|(\mathcal{L}_1), |curv|(\mathcal{L}_2))$$

and let

$$|curv|(\mathbf{h}) = \inf |curv|(h)$$

where the infimum is taken over all h in the class $\mathbf{h}$.

Why Hilbert? If one limits the choice of representatives of $\mathbf{h}$ to virtual *finite dimensional* bundles $L \to X$, then the resulting curvature function on $K^0(X)$ may only increase:

$$|curv|(\mathbf{h})_{fin.dim} \geq |curv|(\mathbf{h}).$$

Apparently, this must be standard, the Hilbert spaces in the definition of Fredholm bundles can be approximated by finite dimensional Euclidean ones, [71] that implies that

$$|curv|(\mathbf{h})_{fin.dim} = |curv|(\mathbf{h}),$$

but even so "Hilbert" allows greater flexibility of certain constructions, example of which we shall see below.

Naive (Strong Novikov) Conjecture. Let Y be a compact *aspherical*[72] Riemannian manifold, possibly with a boundary. Then

all (classes of complex vector bundles) $\mathbf{h} \in K^0(Y)$ *satisfy:*

$$\inf_N |curv|(N \cdot \mathbf{h}) = 0, \ \ N = 1, 2, 3, ..., \ .$$

Exercises. (a) Show that the equalities $|curv|(\mathbf{h}) = 0$ and $\inf_N |curv|(N \cdot \mathbf{h}) = 0$ are *homotopy invariants* of Y.

(b) Show that if Y satisfies this naive conjecture and X is a closed Riemannian orientable spin n-manifold with $Sc(X) > 0$, then all continuous maps $f : X \to Y$ send the fundamental rational homology class $[X]_{\mathbb{Q}} \in H_n(X, \mathbb{Q})$ to zero in $H_n(Y, \mathbb{Q})$.

[71] This is an exercise that the author delegates to the reader.

[72] The universal covering of X is contractible.

4.1.4 Area, Curvature and K-Waist.

K-Theoretic Waist. Given a Riemannin manifold Y (or a more general space, e.g. a polyhedral one with a piecewise smooth metric), define the *K-waist* on the homology classes $h_* \in H_*(Y)$, denoted K-$waist_2(h)$ [73] as the infimum of $|curv|(\mathbf{h})$ over all $\mathbf{h} \in K^0(Y)$, such that $\mathbf{h}(h_*) \neq 0$, where this equality serves as an abbreviation for the value of the Chern character of $\mathbf{h}$ on h_*,

$$\mathbf{h}(h_*) =_{def} Ch(\mathbf{h})(h_*).$$

In these terms the above • can be reformulated as follows.

K-Waist Inequality for Closed Manifolds. The K-waists of (the fundamental classes of) closed orientable $2m$-dimensional spin manifolds X with $Sc(X) \geq \sigma > 0$ satisfy:

$$K\text{-}waist_2[X] \leq \frac{const_m}{\sigma}.$$

Notice, that *conjecturally,* a similar inequality also holds for the *ordinary 2-waist*, (see [Guth(waist) 2014] for an exposition of this "waist") where it is confirmed for 3-manifold by the Marques-Neves theorem (see B in section 3.13)

Exercises. Show that the K-waist is bounded by the hyperspherical radius defined in section 3.5 as follows,

$$\text{K-}waist_2[X] \leq 4\pi Rad^2_{S^{2m}}(X)$$

(b) Show that K-$waist_2(S^n) = 4\pi$.

Almost Flat Bundles Over Open Manifolds. If X is a non-compact manifold, then we deal with the K-theory with compact support that is represented by Fredholm homomorphisms

$$h : \mathcal{L}_1 \to \mathcal{L}_2$$

which are *isometric and connection preserving isomorphisms at infinity*, i.e. away from compact subsets in X where the corresponding K-group is denoted $K^0(X/\infty)$. (If X is compact then $K^0(X/\infty) = K^0(X)$

Here the Hilbertian nature of "Fredholm" allows a painless (and obvious by deciphering terminology) definition of the *pushforward homomorphism* for possibly *infinitely* sheeted covering maps $F : X_1 \to X_2$,

$$F_\star : K^0(X_1/\infty) \to K^0(X_2/\infty),$$

where, clearly,

$$|curv|(F_\star(\mathbf{h})) \leq |curv|(\mathbf{h})$$

for all $\mathbf{h} \in K^0(X_1/\infty)$.

It follows that Therefore,

$$K\text{-}waist_2[X_1] \leq K\text{-}waist_2[X_1]$$

for coverings $X_1 \to X_2$ between orientable Riemannian manifolds.

[73] Subindex 2 is to remind that curvature of bundle L over Y is seen on restrictions of L to surfaces in Y.

On K-Wast Contravariance. The *compact support* property of (virtual) bundles $L \to X_2$ is preserved under pullbacks by *proper* maps $F : X_1 \to X_2$, e.g. by finite coverings, but it fails, for instance, for *infinitely sheeted* coverings $F : X_1 \to X_2$.

This makes the inequality

$$K\text{-}waist_2[X_1] \geq K\text{-}waist_2[X_1]$$

(that is obvious for *finitely sheeted* coverings) *problematic* for infinite covering maps $F : X_1 \to X_2$.

This should be compared with the *covariance problem* for *max-scalar curvature* which is defined in section 5.4.2 and which obviously lifts under covering maps,

$$Sc^{\mathsf{max}}_{prop}[X_1] \geq Sc^{\mathsf{max}}_{prop}[X_2],$$

while the opposite inequality causes a problem (see section 5.4.2).

Question. Can one match the covariance of Sc^{max} by a somehow generalized K-$waist_2$ that would be invariant under (finite and infinite) covering maps $F : X_1 \to X_2$?

Specifically, one looks for *almost flat* (virtual) *infinite dimensional* Hilbert bundles in a suitable K-theory, which would be compatible with the index theory and with the Schroedinger-Lichnerowicz-Weitzenboeck formula in the spirit of Roe's C^*-algebras.

Amenable Cutoff Subquestion. Let X_2 be a closed orientable Riemannin manifold of dimension $n = 2k$ and let $L \to X_2$ be a vector bundle induced by an ε-Lipschitz map $f : X_2 \to S^n$ from the positive spinor bundle $L = \mathbf{S}^+ = \mathbf{S}^+(S^n) \to S^n$.

Suppose that the fundamental group $\pi_1(X_2)$ is *amenable*, let $X_1 = \tilde{X}_2 \to X_2$ be the universal covering map and let

$$\tilde{L} = F^*(L) \to X_1$$

be the pullback of L.

When do there exist unitary bundles $\tilde{L}_i \to X_1$, $i = 1, 2, ...$, with unitary connections, such that

$\bullet_\infty$ the bundles $\tilde{L}_i$ are flat trivial at infinity;

$\bullet_{|\tilde{L}}$ there is an exhaustion of X_1 by compact *Følner subsets*

$$V_1 \subset ... \subset V_i \subset ... \subset X_1,$$

such that the restrictions of $\tilde{L}_i$ to V_i are equal to the restrictions of $\tilde{L}$,

$$(\tilde{L}_i)_{|V_i} = \tilde{L}_{|V_i};$$

$\bullet_\int$ the integrals of the k-th powers of the curvatures of L_i are dominated by such integrals for $\tilde{L}$ over V_i,

$$\frac{\int_{X_1} |curv|^k(\tilde{L}_I) dx_1}{\int_{V_i} |curv|^k(\tilde{L}) dx_1} \underset{i\to\infty}{\to} 0;$$

$\bullet_\epsilon$ the curvatures of all $\tilde{L}_i$ are bounded by

$$|curv|(\tilde{L}_i) \leq \epsilon,$$

where $\epsilon = \epsilon_n(\varepsilon) \to 0$ for $\varepsilon \to 0$.

(The Federer Fleming isoperimetric/filling inequality in the rendition of [MW 2018] may be useful here.)

Non-Amenable Cutoff Example. Let $X(= X_2)$ be a closed orientable Riemann surface of genus ≥ 0 and $L \to X$ a complex line bundle with a unitary connection, e.g. L is the tangent bundle $T(X)$, the Chern number of which $c_1(T(X))[X] = \chi(X)$ doesn't vanish for $genus(X) > 0$.

Let $\tilde{L} \to \tilde{X}$ be the lift (pullback) of L to the universal covering $\tilde{X}(= X_1)$ of X and observe that there exit disks $\tilde{D}^2(R) \subset \tilde{X}$, such that the parallel translates over the boundary circles $\tilde{S}^1(R) = \partial\tilde{D}^2(R)$ are a multiples of 2π and where the radii R of such disks can be arbitrary large.

Then the restriction of $\tilde{L} \to \tilde{X}$ to such a disk $\tilde{D}^2(R) \subset \tilde{X}$ *extends to a bundle*, call it $\tilde{L}_R \to \tilde{X}$, which is *trivial outside* $\tilde{D}^2(R)$ and such that

$$c_1(\tilde{L}_R/\tilde{S}^1(R)) \sim area(\tilde{D}^2(R)) \underset{R\to\infty}{\to} \infty,$$

provided the curvature of L (that is a closed 2-form on X) *doesn't vanish.*

Problem for $n > 2$. The main difficulty in similarly trivializing at infinity bundles over n-dimensional Riemannian manifolds X for $n = dim(X) \geq 3$ seems to be associated with the following *questions.*

Let $\mathcal{U}_b(k) = \mathcal{U}_b(k, X)$, $b \geq 0$, be the space of the unitary connections ∇ on a trivial bundle $L \to X$ of rank k, such that $|curv|(\nabla) \leq b$.

(a) *For which values b_1 and $b_2 > b_1$ are the connections from $\mathcal{U}_{b_1}(k)$ homotopic in $\mathcal{U}_{b_2}(k) \supset \mathcal{U}_{b_1}(k)$?*

(b) *When do the homomorphisms of the homotopy groups*

$$\pi_i(\mathcal{U}_{b_1}(k)) \to \pi_i(\mathcal{U}_{b_2}(k)),\ i \geq 1,$$

induced by the inclusions $\mathcal{U}_{b_1}(k) \hookrightarrow \mathcal{U}_{b_2}(k)$ vanish?

(c) *How do the Whitney sum homomorphisms*

$$\mathcal{U}_b(k_1) \times \mathcal{U}_b(k_2) \to \mathcal{U}_b(k_1 + k_2)$$

behave in this respect?

In particular, what happens to the homomorphisms $\pi_i(\mathcal{U}_{b_1}(k)) \to \pi_i(\mathcal{U}_{b_2}(k))$ under stabilization

$$\underbrace{\mathcal{U}_b(k) \times ... \times \mathcal{U}_b(k)}_{N} \rightsquigarrow (\mathcal{U}_b(Nk))$$

for $N \to \infty$?

Exercise. Let X be a complete orientable even dimensional Riemannin manifold with nonpositive sectional curvature. Show that there exists a K-class $\mathbf{h} \in K^0(X/\infty)$, such that

$$|curv|(\mathbf{h}) = 0 \text{ and } \mathbf{h}[X] \neq 0,$$

where $[X]$ denotes the fundamental homology class of X with *infinite supports.*

4.1.5 Twisted Schroedinger-Lichnerowicz-Weitzenboeck formula, Normalization of Curvature and Sharp Algebraic Inequalities.

Normalization of Curvature. In so far as the scalar curvature is concerned we are interested not in the curvature $|curv|(L)$ per se but rather in the norm of the endomorphism(operator)

$$\mathcal{R}_{\otimes L} : \mathbb{S} \otimes L \to \mathbb{S} \otimes L$$

in the Schroedinger-Lichnerowicz-Weitzenboeck formula for the twisted Dirac operator,

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \tfrac{1}{4}Sc(X) + \mathcal{R}_{\otimes L},$$

(see the previous section) where this $\mathcal{R}_{\otimes L}$ is as following linear/tensorial combination of the values of the curvature of L on the tangent bivectors in the manifold X, (see [GL(spin) 1980])

$$\mathcal{R}_{\otimes L}(s \otimes l) = \frac{1}{2}\sum_{i,j}(e_i \circ e_j \circ s) \otimes R^L_{e_i \wedge e_j}(l),$$

where

$e_i \in T_x(X)$, $i = 1, ...n = dim(X)$ is an orthonormal frame of tangent vectors at a point $x \in X$,

$s \in \mathbb{S}$, are spinors,

$l \in L$ vectors in the bundle L,

$R^L(e_i \wedge e_j) : L \to L$ is the curvature of L (written down as the operator valued 2-form on X)

and

"$\circ$" denotes the Clifford multiplication.

This suggest the definition of

$$\lambda_{\min}[curv]_{\otimes\mathbb{S}}(L)$$

as the smallest (usually negative) eigenvalue of the operator $||\mathcal{R}_{\otimes L}||$.

$\ast$ Example: Llarull's algebraic inequality. [Llarull 1996] *Let $f : X \to S^n$ be a smooth 1-Lipschitz, or more generally, an area non-increasing map and let $L \to X$ be the pullback the spinor bundle $\mathbb{S}(S^n)$. Then this minimal eigenvalue of the operator $\mathcal{R}_{\otimes L}$ satisfies:*

$$\lambda_{\min}[curv]_{\otimes\mathbb{S}}(L) = -\frac{1}{4}(n(n-1) = -\frac{1}{4}Sc(S^n).$$

(We return to this in next section,)

Using this $\lambda_{min}[curv]$ instead of the $|curv|$ one defines

$$\lambda_{min}[curv]_{\otimes\mathbb{S}}(\mathbf{h}),\ \mathbf{h} \in K^0(X),$$

as the *supremum* of $\lambda_{min}[curv]_{\otimes\mathbb{S}(L)}$ for all (virtual) bundles L in the class of $\mathbf{h}$,

Accordingly one modifies the above K-$waist_2(\mathbf{h})$ and define the corresponding *K-waist coupled with spinors*, denoted K-$waist_{\otimes\mathbb{S},2}(h_*)$, $h_* \in H_*(X)$, as the supremum of $\lambda_{min}[curv]_{\otimes\mathbb{S}}(\mathbf{h})$ over over all $\mathbf{h} \in K^0(Y)$, such that $\mathbf{h}(h_*) \neq 0$.

Then, for instance, the above $\bullet_{wst}$ for spin manifolds X takes more elegant form:

$$\text{K-}waist_{\otimes\mathbb{S},2}[X] \leq \frac{4}{\sigma} \text{ for } \sigma = \inf_x Sc(X,x) > 0.$$

Notice that this inequality, combined with the above $\ast$, implies Llarull's *geometric inequality* $Rad_{S^n}(X) \leq \sqrt{\frac{n(n-1)}{\sigma}}$, which we discuss at length in the next section.

Also this may give better formulae for K-waists of product of manifolds.

(See section 5.4.1 and also [G(positive) 1996] and [G(101) 2017] for other known and conjectural properties of $|curv|(\mathbf{h})$ formulated in these papers in the language of the K-area.)

4.2 Llarull's and Goette-Semmelmann's Sc-Normalised Estimates for Maps to Spheres and to Convex Surfaces .

Let us now look closer at the above

$$\mathcal{R}_{\otimes L}(s \otimes l) = \frac{1}{2}\sum_{i,j}(e_i \circ e_j \circ s) \otimes R_{ij}(l),$$

that is the endomorphism of (operator on) the bundle $\mathbb{S} \otimes L \to X$, which appears in the zero order term in the twisted Dirac operator

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \frac{1}{4}Sc(X) + \mathcal{R}_{\otimes L},$$

for

$$\mathcal{D}_{\otimes L} : C^\infty(\mathbb{S} \otimes L) \to C^\infty(\mathbb{S} \otimes L).$$

Example of $L = \mathbb{S}$ on S^n. Since the norm of the curvature operator of (the Levi-Civita connection on) the tangent bundle is one, the norm of the curvature operators $R_{ij} : \mathbb{S} \to \mathbb{S}$ are at most (in fact, are to) $\frac{1}{2}$,

$$\|R_{ij}(s)\| \leq \frac{1}{2},$$

since the spin bundle $\mathbb{S}(X)$ serves as the "square root" of the tangent bundle $T(X)$, where this is literally true for $n = dim(X) = 2$, that formally implies the inequality $\|R_{ij}(s)\| \leq \frac{1}{2}$ for all $n \geq 2$.

And since the Clifford multiplication operators $s \mapsto e_i \cdot e_j \cdot s$ are unitary,

$$\|\mathcal{R}_{\otimes L}(s \otimes l)\| \leq \frac{1}{4}n(n-1) = \frac{1}{4}Sc(S^n)$$

This doesn't, a priori, imply this inequality for all (non-pure) vectors v on the tensor product $\mathbb{S} \otimes L$ for $L = \mathbb{S}$, but, by diagonalising the Clifford multiplication operators in a suitable basis and by employing the *essential constancy*[74] of the curvature R_{ij} of S^n, Llarull [Ll 1998] shows that

$$\|\langle \mathcal{R}_{\otimes L}(\underline{\theta}), \underline{\theta}\rangle\| \geq -\frac{1}{4}n(n-1)$$

[74]Some eigenvalues of this operator are ±1 and some zero.

for all unit vectors $\underline{\theta} \in \mathbb{S}(S^n) \otimes \mathbb{S}(S^n)$.

This inequality for twisted spinors on S^n trivially yields the corresponding inequality on all manifolds X mapped to S^n, where the bundle $L \to X$ is the induced from the spin bundle $\mathbb{S}(S^n)$.

Namely, let $X = (X, g)$ be an n-dimensional Riemannin manifold, $f : X \to S^n$ be a smooth map, $L = f^*(\mathbb{S}(S^n))$, let $df : T(X) \to T(S^n)$ be the differential of f and

$$\wedge^2 df : \wedge^2 T(X) \to \wedge^2 T(S^n)$$

be the exterior square of df.[75]

Then the operator

$$\mathcal{R}_{\otimes L} : \mathbb{S}(X) \otimes L \to \mathbb{S}(X) \otimes L$$

satisfies

$$||\langle \mathcal{R}_{\otimes L}(\theta), \theta \rangle|| \geq -|| \wedge^2 df|| \frac{n(n-1)}{4}, \ L = f^*(\mathbb{S}(S^n)),$$

for all unit vectors $\theta \in \mathbb{S}(X) \otimes f^*(\mathbb{S}(S^n))$.

Moreover, – this is formula (4.6) in [Ll 1998]] –

$$||\langle \mathcal{R}_{\otimes L}(\theta), \theta \rangle|| \geq -\frac{1}{4} |trace \wedge^2 df|,$$

where $trace \wedge^2 df$ at a point $x \in X$ stands for

$$\sum_{i \neq j} \lambda_i \lambda_j,$$

for the differential $df : T_x(X) \to T_{f(x)}(S^n)$ diagonalised to the orthogonal sum of multiplications by λ_i.

This inequality, restricted to $L^+ = f^*(\mathbb{S}^+(S^n))$ together with the index formula, which says for this L_+ that

$$ind(\mathcal{D}_{\otimes L^+}) = \frac{|deg(f)|}{2} \chi(S^n),$$

provided X is a closed oriented spin manifold.

Thus we arrive at a formulation of Llarull's theorem suggested by Mario Listing in [List 2010] and in a coarser form in §$5\frac{4}{9}(D)$ of [G(positive) 1996].

★ *$trace \wedge^2 df$-Inequality.* Let X be a closed orientable Riemannian spin n-manifold and $f : X \to S^n$ a smooth map of nonzero degree.

If

$$Sc(X, x) \geq \frac{1}{4} |trace \wedge^2 df(x)|$$

at all points $x \in Xn$ then, in fact, $Sc(X) = \frac{1}{4}|trace \wedge^2 df|$ everywhere on X.

About the proof. If n is even and $\chi(S^n) = 2 \neq 0$, this follows from the above. And if n is odd, there are (at lest) three different reductions to the even dimensional case (see [Ll 1998], [List 2010], [G(inequalities) 2018]), all three being artificial and conceptually unsatisfactory as it was explained for $\mathbb{T}^n_{Sc \not> 0}$ in section 3.2.

[75] Recall that the norm $|| \wedge^2 df||$ measures by how f contracts/expands surfaces in X. For instance the inequality $|| \wedge^2 df|| 1$ signifies that f decreases the areas of the surfaces in X.

Also see see [Ll 1998] and [List 2010], for characterisation of maps f, where $Sc(X) = \frac{1}{4}|trace \wedge^2 df|$.

llarull's theorem, starting from his estimate for $\mathcal{R}_{\otimes f*(S(S^n)}$, was generalized by Goette and Semmelmann [GS 2002] to Riemannian manifolds $\underline{X}$ with *non-negative curvature operators* instead of S^n. We state below their result only the case of $\underline{X}$ homeomorphic to S^n, where our formulation follows that in [List 2010].

$\star\star$ $\wedge^2 df$*-Inequality.* Let $\underline{X} = (S^n, \underline{g})$ where $\underline{g}$ is a Riemannin metric with *non-negative curvature operator*, let X be a closed orientable Riemannin spin n-manifold and

$$f : X \to \underline{X}$$

smooth map of non-zero degree.

If

$$Sc(X, x) \geq || \wedge^2 df || Sc(\underline{g}, f(x))$$

at all $x \in X$, *then* $Sc(X, x) = || \wedge^2 df || Sc(\underline{g}, f(x))$.

See [GS 2002] and [List 2010] for the proof, where the authors also identify the extremal cases, where f is an isometry or close to an isometry.

Examples. (a) The induced metrics on convex hypersurfaces $\underline{X} \subset \mathbb{R}^{n+1}$ have non-negative positive curvature operators.

Thus, $[X^{\to\bigcirc}]$ from section 3.5 is a special case of $\star\star$.

(b) By a theorem of Alan Weinstein [Wein 1970], the above (a) remains true for submanifolds $\underline{X}^n \subset \mathbb{R}^{n+2}$ with non-negative sectional curvatures of the induced metrics.

In particular,

the induced Riemannin metrics on convex hypersurfaces in S^{n+1} and, more generally, on convex hypersurfaces $\underline{X}^n \subset \Sigma^{n+1}$, where Σ^{n+1} themselves are convex in $\mathbb{R}^{n+2}$, have non-negative curvature operators.

Accordingly, $[X^{\to\bigcirc}]$ generalizes to this case.

$\star\ \star\ \star$ *Products and Stabilisations.* We shall need (see section 5.5, 5.6) a generalization of theorem $\star\star$ to maps

$$f : X \to \underline{X} = (S^m, \underline{g}) \times \mathbb{T}^{n-m},$$

where $\underline{g}$ is a metric with non-negative curvature operator and $\mathbb{T}^{n-m}$ is the torus with a Riemannin flat metric.

This is achieved (compare $\S 5\frac{4}{9}$ in [G(positive) 1996]) by replacing the bundle $\underline{L} = \mathbb{S}^+(S^n)$ in $\star\star$ by $\mathbb{S}^+(S^m) \otimes L_p$, where L_p are flat line bundles L_p over $\mathbb{T}^{n-m}$ as in section 3.11.[76]

Furthermore, whenever this kind of argument applies to $\underline{X}_1$ and $\underline{X}_2$, it goes over to maps $X \to \underline{X}_1 \times \underline{X}_2$.

Almost Example. Let $\underline{X}_1 = (S^m, \underline{g})$, where $\underline{g}$ is a metric with non-negative curvature operator and let $\underline{X}_2$ be a manifold which admits a complete metric with $sect.curv(X_2) \leq 0$.

[76]Instead of flat family L_p one can use individual almost flat bundles over the universal cover $\mathbb{R}^{n-m}$ of $\mathbb{T}^{n-m}$ or any other, possibly infinite dimensional, flat or almost flat bundle used in some proof of non-existence of metrics with $Sc > 0$ on tori.

Let X be a closed orientable Riemannian spin n-manifold and let

$$f : X \to X \to \underline{X}_1 \times \underline{X}_2$$

be a smooth map, such that the image of the fundamental class of X,

$$f_*[X] \in H^n(\underline{X}_1 \times \underline{X}_2; \mathbb{Q}),\ \ n = dim(X),$$

doesn't vanish.

If the composition of f with the projection $\underline{X}_1 \times \underline{X}_2 \to \underline{X}_1$, that is $f_1 : X \to \underline{X}_1$, satisfies

$$Sc(X, x) \ge || \wedge^2 df_1 || Sc(\underline{g}, f_1(x))$$

at all $x \in X$, then $Sc(X, x) = || \wedge^2 df_1 || Sc(\underline{g}, f_1(x))$.

Non-Spin Remark. If X is not assumed spin, then, by the arguments from section 5, one can prove the following rough bound on the Lipschitz constant of f_1.

Let $\underline{X}_1$ is the standard m-sphere with the metric of constant sectional curvature 1 and let $Sc(X) \ge m(m-1)$.

Then the X_1-components $f_1 : X \to \underline{X}_1 = S^m$ of the maps $f : X \to \underline{X}_1 \times \underline{X}_2$, such that $f_*[X] \ne 0$, satisfy

$$|||df_1|| \ge \frac{1}{\pi}.$$

And if $n = 4$ then - this follows from 5.5 *– the maps $f : X \to S^4$ of non-zero degrees satisfy the sharp inequality*

$$||df_1|| \ge 1$$

(which is weaker than $|| \wedge^2 df_1 || \ge 1$) which holds in the spin case.

Category Theoretic Perspective on the Sc*-normalization of Riemannin metrics.* The above suggests that the geometry of Riemannian manifolds $X = (X, g)$, where $Sc(g) > 0$ is well depicted by the Sc*-normalised metric* $Sc(X) \cdot g$ and that maps, which are *1-Lipschitz with respect to the* Sc*-normalised metrics* can be taken for morphisms in the category of manifolds with $Sc > 0$.

4.2.1 Stabilized Hyperspherical Radius and Families of Dirac Operators

It is clear that

$$Rad_{S^{n+1}}(X \times \mathbb{R}) \ge Rad_{S^{n+1}}(X) \text{ as well as } Rad^{\wedge^2}_{S^{n+1}}(X \times \mathbb{R}) \ge Rad^{\wedge^2}_{S^{n+1}}(X)$$

(these radii are defined in section 3.5), since maps $f : X \to S^n$ suspends to (locally) constant at infinity maps $f^{\vee\mathbb{R}} : X \times \mathbb{R} \to S^{n+1}$, such that $deg(f^{\vee\mathbb{R}}) = deg(f)$ and which can be made with

$$||df^{\vee\mathbb{R}}|| \le ||df|| + \varepsilon \text{ and } || \wedge^2 df^{\vee\mathbb{R}}|| \le || \wedge^2 df|| + \varepsilon$$

for all $\varepsilon > 0$by shrinking these maps along the $\mathbb{R}$-direction.

It is also obvious that $Rad_{S^{1+N}}(X\times\mathbb{R}^N) = Rad_{S^1}(X$ for $n = dim(X) = 1$ and one knows that $Rad_{S^{2+N}}(X\times\mathbb{R}^N) = Rad_{S^{2+N}}(X)$ for $n = 2$.[77]

But apart from this dependence of $Rad_{S^{n+N}}(X\times\mathbb{R}^n)$ and of $Rad^{\wedge^2}_{S^{n+2}}(X\times\mathbb{R}^n)$ on n remains obscure.

Conceivably, one may have $Rad_{S^{n+N+1}}(X\times\mathbb{R}^{N+1})$ significantly greater than $Rad_{S^{n+N}}(X\times\mathbb{R}^N)$, even for large N, but I don't clearly see any example, where $Rad_{S^3}(X\times\mathbb{R}^1) > Rad_{S^2}(X))$ for $n = dim(X) = 2$.

Non-Example. A C^2-small symmetric perturbation $X\subset S^3$ of the equatorial $S^2\subset S^3$, possibly, may have $Rad_{S^3}(X\times\mathbb{R}^1) > 1$ and $Rad_{S^2}(X) < 1$).

Now, let $\underline{X}$ be a closed orientable manifold (or pseudomanifold) $\underline{X}$ of dimension n and let us indicated two Dirac theoretic arguments for bounds on the scalar curvature of closed orientable *spin* n-manifolds X, which admit

stabilized proper homologically substantial 1-Lipshitz maps to $\underline{X}$,

that are 1-Lipshitz maps of non-zero degrees:

$$f^{\times\mathbb{R}^N} : X\times\mathbb{R}^N \to \underline{X}\times\mathbb{R}^N,$$

(In the language of section 5.4.1 these are bounds on $Sc^{\max}_{prop,sp}(\underline{X}\times\mathbb{R}^N)$.)

1. *Dirac on the Products.* Since $Sc(X\times\mathbb{R}^N) = Sc(X)$,

$$Sc(X) \le \frac{(n+N)(n+N-1)}{Rad_{S^{n+N}}(\underline{X}\times\mathbb{R}^N)^2}$$

by Llarull's inequality applied to the Dirac operator on $X\times\mathbb{R}^N$ twisted with the (virtual)[78] vector bundle $L^*\to X\times\mathbb{R}^N$ with a unitary connection, which is induced by the composition of the the maps

$$X\times\mathbb{R}^N \to \underline{X}\times\mathbb{R}^N \to S^{n+N}$$

from the positive spinor bundle $\mathbb{S}^+(S^{n+N})$.

2. *Families of Dirac Operators.* Let $\mathcal{D}_p = \mathcal{D}_{\otimes L^*_p}$, $p\in\mathbb{R}^N$, be the family of Dirac operators on X twisted with the (virtual) vector bundles $L^*_p\to X$, that are the restrictions of the above bundle $L^*\to X\times\mathbb{R}^N$ to $X = X\times\{p\}\subset X\times\mathbb{R}^N$.

The Atiyah-Singer theorem for families (if one is uncomfortable with non-compact manifolds and prefer actual, rather than virtual bundles, one can replace $\mathbb{R}^N$ by a sphere $S^N(R)$ of very large radius R) yields non-vanishing of the index of this family.

Since the curvature of this bundle satisfies,

$$||curv||(L^*) \le ||curv||(\mathbb{S}^+(S^{n+N}) = \frac{1}{2}$$

(see section 4.1.1), the existence of an L_p-twisted harmonic spinor on X for some $t\mathbb{R}^N$ in conjunction with the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula formula shows that

$$\inf_{x\in X} Sc(X,x) \le const_n \frac{n(n-1)}{Rad_{S^{n+N}}(\underline{X}\times\mathbb{R}^N)^2}.$$

[77]According to the *waist inequality* $Rad^{\wedge^n}_{S^{n+N}}(X\times\mathbb{R}^N) = Rad^{\wedge^n}_{S^{n+N}}(X)$ for all n.

Two different proofs of this, a geometric and a topological one, are explained in [Guth (waist) 2014] and [G(101) 2017]. Also, if $n = 2$ there is an "analytic" proof with a use of families of Dirac operators on X as we shall see below.

[78]"Virtual" is needed, since $X\times\mathbb{R}^N$ is non-compact and L^*, which represent an element in $K_0(X\times\mathbb{R}^N)/\infty$, must be trivial at infinity.

Remarks and Questions. (a) Probably, Llarull's computation adapted to L_p^* would yield the above inequality with $const_n = 1$.

This is instantaneous for $m = 2$ and, in the case of 3-manifolds X, this follows from the bound

$$waist_2(X) \leq 4\pi \frac{6}{\inf_{x \in X} Sc(X,x)}$$

which, for X homeomorphic to S^3, is a corollary of the Marques-Neves theorem and which, probably holds true for all 3-manifolds X with $Sc(X) > 0$ by a similar argument (see section 3.13).

(b) *Non-product families.* Let X_p be a continuous family of Riemannian manifolds parametrized by a topological space $P \ni p$, that is represented by fibration $\mathcal{X} = \{X_p\} \to P$.

Let the manifolds X_p be complete for all $p \in P$ and let $Sc(X_p) \geq \sigma > 0$.

Let $f : \mathcal{X} \to S^{n+N}$, where $n = dim(X_p)$ and $N = dim(P)$, be a (locally) constant at infinity map of *non-zero degree*, the *restrictions* of which to all X_p are smooth *area non-decreasing*, e.g. 1-Lipschitz.

Notice that this makes sense if P is a psedomanifold (e.g. a manifold) and $\mathcal{X}$ is orientable (also a pseudomanifold).

$\mathbf{1_P}$. *If $\mathcal{X}$ is a smooth orientable* spin *manifold, then*

$$\sigma \leq (n+N)(n+N-1).$$

$\mathbf{2_P}$. *If the fiberwise tangent bundle $\{T(X_p)\}$ of $\mathcal{X}$ is* spin*, then*

$$\sigma \leq const_n \cdot n(n-1).$$

On Spin Discrepancy. Even if P is a smooth manifold and we ignore for constants, the propositions $\mathbf{1_P}$ and $\mathbf{2_P}$ differ by the locations of the spin conditions in them.

However, one can reduce one to another as follows.

$\ast_{1\Rightarrow 2}$ If $\{T(X_p)\}$ is spin but $\mathcal{X}$ is non-spin, replace P by the (total space of the) tangent bundle $T(P)$ and observe that the total space $\mathcal{Y} = \{X_t\}$, $t \in T(P)$, of the pullback of $T(P)$ to $\mathcal{X}$ is spin.

$\ast_{2\Rightarrow 1}$ Let P be orientable; if not, pass to its oriented double covering. If $\mathcal{X}$ is orientable spin but $\{T(X_p)\}$ is non-spin, replace the manifolds X_p by the total spaces Y_p of their tangent bundles $T(X_p)$. These Y_p are spin.

Two Words about Foliations. The above generalizes further to *Alain Connes' non-commutative parameter spaces P* e.g. (spaces of leaves of) foliations, where the counterparts of the above $\mathbf{1_P}$ and $\mathbf{2_P}$ remain valid (see sections 3.12, 6.3 and references therein).

(c) *Problem with Singularities.* Is there a meaningful version of the above for families X_p, where some X_p are singular, as it happens, for instance, for Morse functions $\mathcal{X} \to \mathbb{R}$?

Notice in this regard that Morse singularities, are, essentially, conical, where positivity of $Sc(X_p)$ for singular X_p in the sense of section 5.4.1 can be enforced by a choice of a Riemannin metric in $\mathcal{X}$. [79]

[79] These are cones over $S^k \times S^{n-k-1}$, $n = dimX_p$, where the scalar curvature of such a cone can be made positive, unless $k \leq 1$ and $n-k-1 \leq 1$.

Conversely, positivity of $Sc(X_p)$, for all X_p including the singular ones, probably, yields a smooth metric with $Sc > 0$ on $\mathcal{X}$.

And it must be more difficult (and more interesting) to decide if/when a manifolds with $Sc > 0$ admits a Morse function, where all, including singular, fibers have positive scalar curvatures or, at least, positive operators $-\Delta + \frac{1}{2}Sc$.

.

4.3 Bounds on Mean Convex Hypersurfaces

Recall that the *spherical radius* $Rad_{S^{n-1}}(Y)$ of a connected orientable Riemannin manifold of dimension $(n-1)$ is the supremum of the radii R of the spheres $S^{n-1}(R)$, such that X admits a distance decreasing map $f: Y \to S^{n-1}(R)$ of non-zero degree, where this f for non-compact Y this map is supposed to be constant at infinity.[80]

We already indicated in section 3.6 also see [G(boundary) 2019] that Goette-Semmlenann's theorem (above ★★), applied to smoothed doubles $\mathbb{D}X$ and $\mathbb{D}\underline{X}$ yields the following corollary.

$\bigcirc^{n-1}$*Let X be a compact orientable Riemannin manifold with boundary $Y = \partial X$.*

If $Sc(X) \geq 0$ and the mean curvature of Y is bounded from below by $mean.curv(Y) \geq \mu > 0$, then the hyperspherical radius of Y for the induced Riemannin metric is bounded by

$$Rad_{S^{n-1}}(Y) \leq \frac{n-1}{\mu}.$$

In fact, the proof of this indicated in section 3.6 (also see [G(boundary) 2019]) together with the above ★ ★ ★ yields the following more general theorem.

$\bullet^{n,n-1}$ Let X and $\underline{X}$ be compact connected orientable Riemannian n-manifolds with boundaries $Y = \partial X$ and $\underline{Y} = \partial \underline{X}$, and let $f: X \to \underline{X}$ be a smooth proper[81] map of *non-zero degree.*

Let $\underline{X}$ admit a locally convex isometric immersion to $\mathbb{T}^{n+1}$ and let the boundary $\underline{Y}$ of $\underline{X}$ be (geodesically) convex in $\underline{X}$.

If X is spin, if

SCAL $\qquad Sc(X,x) \geq ||\wedge^2 df|| Sc(\underline{X}, f(x))$ *for all* $x \in X$

and if

MEAN $\qquad mean.curv(Y,y) \geq ||df|| mean.curv(\underline{Y}, f(y))$ *for all* $y \in Y$,

then, in fact,

$$Sc(X,x) = ||\wedge^2 df|| Sc(\underline{X}, f(x))$$

and

$$mean.curv(Y,y) = ||df|| mean.curv(\underline{Y}, f(y)).$$

[80] Alternatively, one might require f to be *locally* constant at infinity, or more generally, to have the limit set of codimension≥ 2 in $S^{n-1}(R)$.

[81] Here, "proper" means boundary$\to$ boundary.

Remarks. (a) If $Sc(\underline{X}) = 0$, e.g. if $\underline{X}$ is a convex subset in $\mathbb{R}^{n+1}$, then the condition **SCAL** reduces to $Sc(X) \geq 0$.

(b) The above also yields some information on manifolds X with *negative scalar curvatures bounded from below.*

For instance, if $Sc(X) \geq -m(m-1)$, then $\bullet^{n,n-1}$ applies to maps from $f: X \times S^m$ to the $(m+n)$-balls $B^{m+n} \subset \mathbb{R}^{m+n}$ (see [G(boundary) 2019]).

However, the sharp inequalities for $Sc(X) < 0$, such, for instance, as *optimality* of the hyperspherical radii of the boundary spheres of balls $B^n(R)$ in the hyperbolic spaces $\mathbb{H}^n_{-1}$, remain *conjectural*.[82]

(c) It is unknown if the spin condition on X is necessary, but it can be relaxed by requiring the universal cover of X, rather than X itself, to be spin. In fact, $\bullet^{n,n-1}$ generalizes to non-compact complete manifolds with an extra attention to uniformity of the curvature inequalities involved.

And if one is content with a non-sharp bound

$$Rad_{S^{n-1}}(Y) \leq \frac{const_n}{\inf mean.curv(Y)},$$

then one and can prove this without the spin assumption by the "cubical type argument" from section 5.4.

4.4 Lower Bounds on the Dihedral angles of Curved Polyhedral Domains.

We want to generalise the above $\bullet^{n,n-1}$ to manifolds X with non-smooth boundaries with suitably defined mean curvatures bounded from below, where we limit ourself to manifolds with rather simple singularities at their boundaries.

Namely, let X and $\underline{X}$ be Riemannian n-manifolds *with corners*, which means that their boundaries $Y = \partial X$ and $\underline{Y} = \partial\underline{X}$ are decomposed into $(n-1)$-faces F_i and $\underline{F}_i$ correspondingly, where, locally, at all points $y \in Y$, and $\underline{y} \in \underline{Y}$ these decompositions are is diffeomorphic to such decomposition of the boundary of a convex n-dimensional polyhedron (polytope) in $\mathbb{R}^n$.

Let $f : X :\to \underline{X}$ be a smooth map, which is compatible with the corner structures in X and $\underline{X}$:

f sends the $(n-1)$-faces F_i of X to faces $\underline{F}_i$ of $\underline{X}$.

Assume as earlier that

$$[\geq]^{SCAL} \qquad Sc(X,x) \geq \|\wedge^2 df\| \cdot Sc(\underline{X}, f(x)) \text{ for all } x \in X$$

and replace **MEAN** by the corresponding condition applied to *for all faces* $F_i \subset Y$ individually,

$$[\geq]^{MEAN}_{\{i\}}, \qquad mean.curv(F_i, y) \geq \|df\| \cdot mean.curv(\underline{F}_i, f(y)) \text{ for all } y \in F_i\ .$$

Let $\angle_{i,j}(y)$ be the dihedral angle between the faces F_i and F_j at $y \in F_i \cap F_i$ and let us impose our main inequality between these $\angle_{i,j}(y)$ for all F_i and $F)j$

[82]This "optimality" means that if $Sc(X) \geq -n(n-1)$ and $mean.curv(\partial X) \geq mean.curv(\partial B^n(R))$ than $Rad_{S^{n-1}}(\partial X) \leq Rad_{S^{n-1}}(\partial B^n(R))$.

and the dihedral angles between the corresponding faces faces $\underline{F}_i$ and $\underline{F}_j$ at the points $f(y) \in \underline{F}_i \cap \underline{F}_j$:

$[\leq]^{\angle ij}$ $\qquad \angle_{i,j}(y) \leq \angle_{i,j}(f(y))$ for all F_i, F_j and $y \in F_i \cap F_j$.

Besides the above, we need to add the following condition the relevance of which remains unclear.

Call a point $y \in Y = \partial X$ *suspicious* if one of the following two conditions is satisfied

(i) the corner structure of X at y is *non-simple* (not cosimplicial), where simple means that a neighbourhood of y is diffeomorphic to a neighbourhood of a point in the n-cube, which is equivalent to transversality of the intersection of the $(n-1)$-faces which meet at y;

(ii) there are two $(n-1)$-faces in X which contain y, say $F_i \ni y$ and $F_j \ni y$, such that the dihedral angle $\angle_{ij} = \angle(F_i, F_j$ is $> \frac{\pi}{2}$;

Then out final condition says that

$[=]^{\angle ij}$ $$\angle_{i,j}(y) = \angle_{i,j}(f(y)).$$

for all suspicious points y.

$\blacklozenge_{\angle ij}$ *Theorem.* Let $f : X \to \underline{X}$ be a smooth map between connected orientable n-dimensional Riemannian manifolds with corners, where this map respects the corner structure and satisfies the above conditions $[\geq]^{SCAL}$, $[\geq]^{MEAN}_{\{i\}}$, $[\leq]^{\angle ij}$ and $[=]^{\angle ij}$.

If X is spin, $\underline{X}$ admits a locally convex isometric immersion to $\mathbb{T}^{n+1}$, the boundary of $\underline{X}$ is convex and the map f has non-zero degree, then f satisfies the equalities corresponding to the inequalities $[\geq]^{SCAL}$, $[\geq]^{MEAN}_{\{i\}}$ *and* $[\leq]^{\angle ij}$:

$$Sc(X, x) = \| \wedge^2 df \| \cdot Sc(\underline{X}, f(x)) \text{ for all } x \in X,$$

$$mean.curv(F_i, y) = \|df\| \cdot mean.curv(\underline{F}_i, f(y)) \text{ for all } y \in F_i,$$

$$\angle_{i,j}(y) = \angle_{i,j}(f(y)) \text{ for all } F_i, F_j \text{ and } y \in F_i \cap F_j.$$

About the Proof. This is shown by smoothing the boundaries of X and applying $\bullet_{n,n-1}$ from the previous section, where an essential feature of non-suspicious points follows from the following

Elementary Lemma. Let $\Delta \subset S^n$ be a spherical simplex with all edges of length $\geq l \geq \frac{\pi}{2}$. Then there exists a continuous family of simplices $\Delta_t \subset S^n$, $t \in [0,1]$ with the following properties.

- *$\Delta_0 = \Delta$ and Δ_1 is a regular simplex with the edge length l;*
- *all Δ_t have the edges of length $\geq l$;*
- *$\Delta_{t_2} \subset \Delta_{t_1}$ for $t_2 \geq t_1$;*
- *for each $t < 1$ there exists an $\varepsilon > 0$, such that n (out of $n+1$) vertices of $\Delta_{t+\varepsilon}$ coincide with those of Δ_t.*

The proof of the lemma is a high school exercise while *construction of adequate smoothing of X* with the help of this lemma, which is straightforward and boring, will be given elsewhere. $\blacklozenge_{\angle ij}$

Notice that the $\times\blacktriangle^i$-*Inequality* from section 3.10, which says that convex polyhedra $\underline{X} \subset \mathbb{R}^n$ with the dihedral angles $\leq \frac{\pi}{2}$ admit no deformations which would *decrease their dihedral angles* and simultaneously *increase the mean curvatures of their faces,*

is an immediate corollary of $\blacklozenge_{\angle ij}$.

But it remains unclear what is the *full class* of polyhedra which enjoy this property.

Fundamental Domains of Reflection Groups. What underlies the double $\mathbb{D}$-construction, $X \rightsquigarrow \mathbb{D}X$ in the proof of the $\blacklozenge_{\angle ij}$ theorem is the doubling $S^n = \mathbb{D}S^n_+$, which is associated with the reflection of S^n with respect to the equatorial subsphere.

With this in mind, one can generalise everything from this section to general reflection groups, including spherical, Euclidean, "abstract" (semi)hyperbolic ones, (such as what we met in weak $\neg$-reflection rigidity theorem in section 3.16.) and also products of these.

Example. Let X be a manifold with corners, where the (combinatorial) corner structure is isomorphic to that of the product of an $(n-m)$-simplex $\blacktriangle$ with the rectangular fundamental domain $\blacksquare$ (orbifold) of a reflection group in an aspherical m-manifold which is *non-diffeomorphic to* $\mathbb{R}^m$. (These exist for all $m \geq 4$ by Michael Davis 1983 theorem, see his lectures [Dav 2008] and references therein.)

X admits no Riemannian metric with $Sc \geq 0$, with all faces having $mean.curv \geq 0$ and with the dihedral angles smaller than those in the product of a regular Euclidean simplex $\blacktriangle$ by $\blacksquare$ with $\frac{\pi}{2}$ dihedral angles.

On Necessity of Condition $[=]^{\angle ij}$???

4.5 Stability of Geometric Inequalities with $Sc \geq \sigma$ and Spectra of Twisted Dirac Operators.

Sharp geometric inequalities beg for being accompanied by their nearest neighbours.

For instance, the Euclidean isoperimetric inequality for bounded domains $X \subset \mathbb{R}^n$, which says that

$$vol_n(X) \leq \gamma_n vol_{n-1}(\partial X)^{\frac{n}{n-1}} \text{ for } \gamma_n = \frac{vol(B_n)}{vol_{n-1}(S^{n-1})^{\frac{n}{n-1}}},$$

goes along with the following.

A. *Rigidity. If $vol_n(X) = \gamma_n vol_{n-1}(\partial X)^{\frac{n}{n-1}}$, then X is a ball.*

B. *Isoperimetric Stability.* Let $X \subset \mathbb{R}^n$ be a bounded domain with $vol_n(X) = vol_n(B^n)$ and $vol(\partial X) \leq vol_{n-1}(S^n) + \varepsilon$.

Then there exists a ball $B = B^n_x(1+\delta) \subset \mathbb{R}^n$ of radius δ with center $x \in X$, where $\delta \underset{\varepsilon \to 0}{\to} 0$, such that the volume of the difference satisfies

$$vol_n(X \setminus B) \leq \delta_1,$$

and, moreover,

$$vol_{n-1}(\partial B \cap X) \leq \delta_2, \text{ and } vol_{n-2}(\partial B \cap \partial X) \leq \delta_3,$$

where

$$\delta_1, \delta_2, \delta_3 \underset{\varepsilon\to 0}{\to} 0.$$

(Unless $n = 2$ and X is connected, there is no bound on the diameter of X, but the constants $\delta, \delta_1, \delta_2, \delta_3$ can be explicitly evaluated even for moderately large ε.)

Turning to scalar curvature, observe, following Llarull, Min-Oo and Goette-Semmelmann, that their proofs (see [Ll 1998], [Min(Hermitian) 1998], [GS 2002]) (more or less) automatically deliver rigidity. For instance,

★ *if a manifold $\underline{X}$ homeomorphic to S^n, besides having $curv.oper(\underline{X}) \geq 0$ has $Ricci(\underline{X}) > 0$ and if X is a closed orientable spin Riemannin manifold with $Sc(X) \geq n(n-1)$ then, all* smooth 1-Lipschitz *maps $X \to \underline{X}$ of non-zero degrees are* isometries. [83]

What we want to understand next is what happens if the inequality $Sc(X) \geq n(n-1)$ is relaxed to $Sc(X) \geq n(n-1) - \varepsilon$ for a small $\varepsilon > 0$, where one has to keep in mind the following.

Example. (Compare [GL(classification) 1980], [BDS 2018] and section 2, and 23 in [G(questions) 2017].) Let $\Sigma \subset S^n$ be a compact smooth submanifold of dimension $\leq n-3$. Then there exists an arbitrary small ε-neighbourhood $U_\varepsilon = U_\varepsilon(\Sigma) \subset S^n$ with a smooth boundary $\partial_\varepsilon = \partial U_\varepsilon$ and a family of smooth metrics $g_{\varepsilon,\epsilon}$ on the double

$$\mathbb{D}(S^n \smallsetminus U_\varepsilon) = (S^n \smallsetminus U_\varepsilon) \cup_{\partial_\varepsilon} (S^n \smallsetminus U_\varepsilon),$$

where $Sc(g_{\varepsilon,\epsilon}) \geq n(n-1) - \varepsilon - \epsilon$ and which, for $\epsilon \to 0$, *uniformly converge* to the natural continuous Riemannian metric on $\mathbb{D}(S^n \smallsetminus U_\varepsilon(\Sigma))$.

Moreover, if $\Sigma \subset S^n$ is contained in a hemisphere, then – this follows from the *spherical Kirszbraun theorem* – the (double) manifolds $\mathbb{D}(S^n \smallsetminus U_\varepsilon, g_{\varepsilon,\epsilon})$ admit 1-Lipschitz maps to the sphere S^n with degrees one, for all sufficiently small $\varepsilon > 0$ and , $\epsilon = \epsilon(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$.

For instance, if $n \geq 3$ and Σ consists of a single point, then $\mathbb{D}(S^n \smallsetminus U_\varepsilon)$, that is the connected sum $S^n \# S^n = S^n \#_{S^{n-1}(\varepsilon)} S^n$ of the sphere S^n with itself (where the ε-sphere $S^{n-1}(\varepsilon)$ serves as ∂_ε and $S^n \# S^n$ is homeomorphic to S^n), admits, for small ε, a 1-Lipschitz map to S^n with degree 2.

Furthermore, iteration of the connected sum construction, delivers manifolds (topologically spheres)

$$(S^n)^{k\#_\varepsilon} = \underbrace{S^n \#_{S^{n-1}(\varepsilon)} S^n \# \ldots \#_{S^{n-1}(\varepsilon)} S^n}_{k} m$$

which carry metrics with $Sc(S^n)^{k\#_\varepsilon} \geq n(n-1) - \varepsilon - \epsilon$ and, at the same time, admit maps to S^n of degree k, where these maps are 1-Lipschitz everywhere and which are locally isometric away from $\sqrt{\varepsilon}$-neighbourhoods of $k-1$ ε-spherical "necks" in $(S^n)^{k\#_\varepsilon}$.

(For general Σ and even k one has such maps f with $deg(f) = k/2$.

[83] Even if Ricci vanishes somewhere, one still may have a satisfactory description of the extremal cases. For instance, if $\underline{X} = (S^{n-m} \times \mathbb{R}^m)/\mathbb{Z}^m$, e.g. $\underline{X} = S^{n-m} \times \mathbb{T}^m$, then all (orientable spin) X with $Sc(X) \geq Sc(\underline{X}) = (n-m)(n-m-1)$, which admit maps $f : X \to \underline{X}$ with $deg(f) \neq 0$, are *locally isometric* to $\underline{X}$ (albeit the map f itself doesn't have to be a local isometry.

Conjecturally, this example faithfully represents possible geometries of closed Riemannian n-manifolds X with $Sc(X) \geq n(n-1) - \varepsilon$, which admit 1-Lipschitz maps to the unit sphere S^n, but only the following two, rather superficial, results of this kind are available.

1. Let $X = (X, g)$ be a closed oriented Riemannin spin n-manifold with $Sc(X) \geq n(n-1) - \varepsilon$ and let $f : X \to \underline{X} = S^n$ be a smooth 1-Lipshitz map of degree $d \neq 0$.

Denote by $\tilde{g}$ the (possibly singular) Riemannin metric on X induced by f from the spherical metric $\underline{g}$ on $\underline{X} = S^n$ and let $\underline{l}(f, x)$ be the minimum

$$\underline{l}_f(x) = \min_{\|\tau\|_g = 1} \|df(\tau)\|_{\underline{g}}, \ \tau \in T_x(X).$$

(Since f is 1-Lipschitz, $\underline{l}(f, x) \leq 1$ and $(\underline{l}(f, x))^{-1}$ measures the the distance from the differential $df(x) : T_x(X) \to T_{f(x)}(\underline{X})$ to an isometry.) Let

$$\tilde{V} = \widetilde{vol}(X) = vol_{\tilde{g}}(X) = \int_{\underline{X}} card(f^{-1}(\underline{x}))d\underline{x}$$

be the $\tilde{g}$-volume of X.

Then the $\widetilde{g}$-volume of the subset $X_{\leq\lambda} \subset X$, $\lambda < 1$, where $\underline{l}_f(x) \leq \lambda$ satisfies

[$|X_{\leq\lambda}| \leq$] $$\widetilde{vol}(X_{\leq\lambda}) \leq c_{\lambda, n, \tilde{V}}(\varepsilon) \underset{\varepsilon\to 0}{\to} 0.$$

Sketch of the Proof. Since the twisted Dirac operator $D_\otimes$ in Llarull's rigidity argument from [Ll 1998] has non-zero kernel, its square $D^2_\otimes$ is non-positive (we assume here that $n = dim(X) = dim(\underline{X})$ is even), and, by the Bochner-Schrödinger-Lichnerowicz-Weitzenböck formula (that is above $[D^2_\otimes]_f$), this implies non-positivity of

$$\nabla^2 + \frac{1}{4}Sc(X) + \mathcal{R}_\otimes.$$

Consequently, $-\Delta_g - \frac{1}{4}(\varepsilon + (1 - \underline{l}(x)))$, where Δ_g is an ordinary Laplace operator on $X = (X, g)$, also non-positive, since the coarse (Bochner) Laplacian ∇^2 is "more positive" than the (positive) Laplace(-Beltrami) operator $-\Delta$ as it follows from the *Kac-Feynman* formula and/or from the *Kato inequality.*

(In general, this applies in the context of the above rigidity theorem ★ and yields non-positivity of $-\Delta_g - \frac{1}{4}(\varepsilon + \underline{C}(1 - \underline{l}_f(x))$ with $\underline{C}$ depending on the smallest eigenvalue of $Ricci(\underline{X})$.)

In order to extract required geometric information concerning the metric $\tilde{g}$ from this property of the metric g, we observe that the essential part of X, that is the one, where we need to bound from below the L_2-norms of the g-gradients of functions $\phi(x)$ (to which the above Δ_g applies) is where

$$\lambda \geq \underline{l}_f(x) \geq \lambda_{\tilde{V}} > 0$$

for some $\lambda_{\tilde{V}} > 0$, and where the geometries of g and of $\tilde{g}$ are mutually $(\lambda_{\tilde{V}})^{-1}$-close.

Thus, the relevant lower g-gradient estimate for $\phi(x)$ comes from the isoperimetric inequality for $\tilde{g}$ which, in turn, follow from such an inequality in $\underline{X}$, that is the sphere in the present case. (Filling in the details is left to the reader.)

Remark. (a) The above example shows that the g-volume of $X_{\leq\lambda} \subset X$ can be large and that the bound on $\tilde{V}$ concerns not only the subset $X_{\leq\lambda}$ but its complement $X \setminus X_{\leq\lambda}$ as well.

Corollary + Question. (a) Let X be a closed orientable Riemannin spin n-manifold with $Sc(X) \geq n(n-1)$ and let $f : X \to S^n$ a (possibly non-smooth!) 1-Lipshitz map of degree $\neq 0$.

If the map Y is a homeomorphism, then it is an isometry.

(b) Is this remain true for *all* 1-Lipshitz maps?

The inequality [$|X_{\leq\lambda}| \leq$] doesn't take advantage of $deg(f)$ when this is large, but the following proposition does just that.

2. Let X be a compact oriented Riemannian spin n-manifold with a boundary $Y = \partial X$, such that $Sc(X) \geq n(n-1) + \varepsilon$, $\varepsilon > 0$.

Let $f : X \to S^n$ be a smooth map, which is constant on Y, which is *area contracting away from the a neighbourhood $U \subset X$ of $Y = \partial X \subset X$,*

$$|| \wedge^2 df(x)|| \leq 1 \textit{ for all } x \in X \setminus U,$$

and where

$$|| \wedge^2 df(x)|| \leq C_o \text{ for all } x \in X \setminus U \text{ and some constant } C_o > 0.$$

Then the degree of f is bounded by a constant d depending only on U and on C_o,

$$|deg(f)| \leq d = const_{U,C_o}.$$

Sketch of the Proof. (Compare with §§$5\frac{1}{2}$ and 6 in [G(positive) 1996].) Let $s(x)$ be the (Borel) function on X which equals to ε away from U and is equal to $E = -C_n \times C_o$ on U for some universal $C_n \approx n^n$.

Then arguing (essentially) as in the first part of the above proof, we conclude that the spectrum of the operator $-\Delta + s(x)$ on the (smoothed) double $\mathbb{D}(X)$ contains at least $d = deg(f)$ negative eigenvalues.

This an easy argument would deliver d eigenvalues λ_i of the operator $-\Delta$ on $\mathbb{D}(U)$, where the corresponding eigenfunctions vanish on the two copies of the boundary of U in X (but not, necessarily on Y), and such that $\lambda_i \lessapprox E$.

This would yield the required bound on d. (Here again, the details are left to the reader.)

Remark + Example + Two Problems. (a) If the boundary of $Y = \partial X$ admits an orientation reversing involution, then the constancy of f on Y can be relaxed to $dim(f(Y)) \leq n-2$, where the constant d will have to depend on the geometry of this involution and of the map $Y \to S^n$.

(It is unclear if the existence of such an involution is truly necessary.)

(b) This (a) apply, for instance, to coverings $X = \Sigma^2_{d,\delta}$ of the 2-sphere minus two δ-discs as well as to the products of these $\Sigma^2_{d,\delta}$ with the Euclidean ball $B^{n-1}(R)$ of radius $R > \pi$.

(c) What are the sharp and/or comprehensive versions of these **1** and **2**?

(d) Let Y be a homotopy sphere of dimension $4k-1$, which bounds a Riemannin manifold X with $Sc \geq \varepsilon > 0$. Give an *effective* bound on the $\hat{A}$-genus of X in terms of the geometry of Y and its second fundamental form $h = \mathrm{II}(Y \subset X)$ and study the resulting invariant

$$Inv_\varepsilon(Y,h) = \sup_X |\hat{A}(X)|, \text{ where } \partial X = Y,\ Sc(X) \geq \varepsilon,\ \mathrm{II}(Y \subset X) = h.$$

4.6 Dirac Operators on Manifolds with Boundaries

As far as the scalar curvature is concerned, all the index theorems are needed for is delivering *non-zero harmonic* or *approximately harmonic* (often twisted) spinors on Riemannian manifolds X under certain certain geometric/topological conditions on X, which, a priori, have nothing to do with the scalar curvature but which are eventually used to obtain upper bounds on $Sc(X)$ via the (usually twisted) Bochner-Schrödinger-Lichnerowicz-Weitzenböck formula.

Albeit all(?) known index theorems for (twisted) Dirac operators $\mathcal{D}$ used for this purpose directly apply only to *complete* Riemannin manifolds, these theorems can yield a non-trivial information on existence of approximately harmonic spinors on non-complete manifolds as well as on manifolds with boundaries, where the main issue, say for manifolds with boundaries, can be formulated as follows.

Spectral $\mathcal{D}^2$-Problem. Let X be a compact Riemannian spin manifold with a boundary and $L \to X$ be a (possibly infinite dimensional Hilbert) vector bundle with a unitary connection. Under which geometric/topological conditions does X support a smooth *non-zero* twisted spinor $s : X \to \mathbb{S}(X) \otimes L$, which *vanishes on the boundary* of X and such that

✂$_\lambda$
$$\int_X \langle \mathcal{D}^2_{\otimes L}(s), s\rangle \leq \lambda^2 \int_X ||s||^2 dx$$

for a given constant $\lambda \geq 0$?

Motivating Example. If X is obtained from a complete manifold $X_+ \supset X$ by cutting away $X_+ \smallsetminus X$, and if X_+ carries a non-vanishing (twisted) L_2-spinor s_+ delivered by applying the relative index theorem, then the cut-off spinor $s = \phi \cdot s_+$, for a "*slowly* decaying" positive function ϕ with supports in X satisfies ✂$_\lambda$ with "*rather small* " λ.

Potential Corollary. Since

$$\mathcal{D}^2_{\otimes L}(s) \geq \nabla^2_{\otimes L}(s) + \frac{1}{4} Sc(X)(s) - const'_n |curv|(L)$$

by the Bochner-Schrödinger-Lichnerowicz-Weitzenböck formula and since

$$\int \langle \nabla^2_{\otimes L}(s), s\rangle = \int_X \langle \nabla_{\otimes L}(s), \nabla_{\otimes L}(s)\rangle \geq 0$$

for $s_{|\partial X} = 0$, the inequality ✂$_\lambda$ implies

✂$_{Sc}$
$$\inf_x Sc(X,x) \leq \frac{4const_n}{\rho^2} + 4const'_n |curv|(\nabla).$$

for some universal positive constants $const_n$ and $const'_n$.

From a geometric perspective, the role of above is to advance the solution of the following.

Long Neck Problem. Let X be an orientable (spin?) Riemannian n-manifold with a boundary and $f : X \to S^n$ be a smooth area decreasing map.

What kind of a *lower bound on* $Sc(X,x)$ and a *lower bound on* the "length of the neck" of (X,f), that is

the distance between the support of the differential of f *and the boundary of* X, would make $deg(f) = 0$?

An instance of a desired result would be

$$[Sc(X) \geq n(n-1)]\&[dist(supp(df), \partial X) \geq const_n] \Rightarrow deg(f) = 0,$$

but it is more realistic to expect a weaker implication

$$[Sc(X) \geq n(n-1)]\&[dist(supp(df), \partial X) \geq const_n \cdot \sup_{x \in X} ||| df(x) ||] \Rightarrow deg(f) = 0.$$

In fact, Roe's proof of the partitioned index theorem as well as the proof of the relative index theorem, e.g. via the *finite propagation speed* argument, combined with Vaffa-Witten kind spectral estimates (see $6\frac{1}{2}$ in [G(positive) 1996]) suggest that

if a compact orientable Riemannian spin manifold of even dimension n with boundary admits a a smooth map $f : X \to S^n$, which is locally constant on the boundary of X and which has *non-zero degree*, then there exists a *non-zero spinor* s, twisted with the pullback bundle $L = f^*(\mathbb{S}(S^n))$ such that *s vanishes on the boundary* ∂X and which satisfies ✂$_\lambda$,

$$\int_X \langle \mathcal{D}^2_{\otimes L}(s), s\rangle \leq \lambda^2 \int_X ||s||^2 dx,$$

where

$$\lambda \leq const_n \frac{\sup_{x\in X} ||df(x)||}{dist(supp(f), \partial X)}.$$

This still remains problematic, but we prove in the sections below some inequalities in this regard for manifolds X with certain restrictions on their local geometries.

4.6.1 Bounds on Geometry and Riemannian Limits..

Some properties of manifolds X with boundaries trivially follow by a limit argument from the corresponding properties of complete manifolds as follows.

A sequence of manifolds X_i marked with distinguished points $\underline{x}_i \in X_i$ is said to *Lipschitz converge* to a marked Riemannin manifold $(X_\infty, \underline{x}_\infty)$, if

there exist $(1+\varepsilon_i)$-*bi-Lipschitz* maps [84] from the balls $B_{\underline{x}_i}(R_i) \subset X_j$ to the balls $B_{\underline{x}_\infty}(R_i) \subset X_\infty$, say

$$\alpha_i : B_{\underline{x}_i}(R_i) \to B_{\underline{x}_\infty}(R_i + 1),$$

[84] Here and below "λ-bi-Lipschitz" is understood as the λ-bound on the *norms of the differentials* of our maps and their inverse.

which send $\underline{x}_i \to \underline{x}_\infty$ and where

$$\varepsilon_i \to 0 \text{ for } i \to \infty.$$

Observe that if

$$dist(\underline{x}_i, \partial X_i) \to \infty \text{ for } i \to \infty,$$

then *the limit manifold X_∞ is complete.*

★ *Cheeger Convergence Theorem.* If the (local) C^k-geometries of Riemannian manifolds X_i at the points $x_i \in B_{\underline{x}_i}(R_i)$ for $R_i \to \infty$ are bounded (as defined below) by $c(dist(x_i, \underline{x}_i))$ for some continuous function $b(d)$, $d \geq 0$ independent of i, then some subsequence of X_i converges to a C^{k-1}-smooth Riemannian manifold X_∞.

See [Boileau 2005] for the proof and further references.

Definition of Bounded Geometry. The C^k-geometry of a smooth Riemannian n-manifold X is bounded by a constant $geq0$ at a point $x \in X$, if the ρ-ball $B_x(\rho) \subset X$ for $\rho = \frac{1}{b}$ admits a smooth $(1+b)^2$-bi-Lipschitz map $\beta : B_x(\rho) \to \mathbb{R}^n$, such that the norms of the kth covariant derivatives of β in $B_x(\rho)$ are bounded by b.

Notice that the *traditionally defined bound* on geometry in terms of the curvature and the injectivity radius of X, implies the above one:

if the norms of the curvature tensor of X and its kth-covariant derivatives are bounded by β^2 and there is no geodesic loop in X based at x of length $\leq \frac{1}{\beta}$, then (the proof is very easy) the C^{k+1}-geometry of X at x is bounded by $b(\beta)$ for some universal continuous function $b(\beta) = b_{n,k}(\beta)$.

✪ *Application of ★ to Scalar Curvature.* Let $b = b(d) \geq 0$, $d > 0$, be a continuous function and let $(X, \underline{x} \in X)$ be a marked compact Riemannian n-manifold with a boundary, such that the local geometry of X at $x \in X$ is bounded by $b(dist(x, \underline{x}))$ and let

$$R = dist(\underline{x}, \partial X).$$

Let d_0 be a positive number and let $f : X \to S^n$ be a smooth *area decreasing* map which is constant within distance $\geq d_0$ from $\underline{x} \in X$ and which has *non-zero* degree.

A. *If X is spin and $n = dim(X)$ is even, then there exists a spinor s on X twisted with the induced spinor bundle $L = f^*(\mathbb{S}(S^n)) \to X$, such that s vanishes on the boundary ∂X of X and such that*

$$\int_X \langle \mathcal{D}^2_{\otimes L}(s), s\rangle \leq \lambda(R)^2 \int_X ||s||^2 dx$$

where $\lambda = \lambda_{n,b,d_0}(R)$ is a certain universal function in R, which asymptotically vanishes at infinity,

$$\lambda(R) \underset{R\to\infty}{\to} 0.$$

B. *The scalar curvature of X is bounded by*

$$\inf_{x\in X} \leq n(n-1) + \lambda'_{n,b,d_0}(R),$$

where, similarly to the above λ, this $\lambda'(R) \to 0$ for $R \to \infty$.(One can actually arrange $\lambda' = \lambda$.)

Proof. According to Cheeger's theorem, if $R = dist(\underline{x}, \partial X)$ is sufficiently large , then X can be well approximated by a complete manifold X_∞, where such an X_∞ supports a non-zero L-twisted harmonic spinor s_∞ by the relative index theorem (we say more about it in section ???).

Then this s can be truncated to s_i by multiplying it with a slowly decaying function on X with compact support and then transporting it to the required spinor on X.

This takes care of A and B follows by Llarull's inequality.

Remarks. (a) The major drawback of ✪ is an excessive presence and non-effectiveness of the bounded geometry condition.

We *don't know what the true dependence of λ on the geometry of X is*, but we shall prove several inequalities in the following sections that suggest what one may expect in this regard.

(b) If the "area decreasing" property of the above map $f : X \to S^n$ is strengthened to "1-Lipschitz", then a version of B follows from the double puncture theorem (see sections 3.9 and 5.4), which needs neither spin nor the bounded geometry conditions.

4.6.2 Construction of Mean Convex Hypersurfaces and Applications to $Sc > 0$.

Since doubling of manifolds with mean convex boundaries preserves positivity of the scalar curvature (see section 3.6), some problems concerning $Sc > 0$ for manifolds X with boundaries can be reduced to the corresponding ones for closed manifolds by doubling *mean convex* domains $X_\bigcirc \subset X$ across their boundaries $\partial X_\bigcirc$.

To make use of this, we shall present below some a simple criterion for the existence of such $X_\bigcirc$ and apply this for establishing effective versions of the above B.

Let X be a compact n-dimensional *Riemannian band* (capacitor), that is the boundary of X is divided into two disjoint subsets, that are certain unions of boundary components of X,

$$\partial X = \partial_- \cup \partial_+$$

and let us give a condition for the existence of a domain $X_\bigcirc \subset X$ which *contains* ∂_- and the boundary of which is smooth and has *positive* mean curvature.

Lemma. Let the boundaries of all domains $U \subset X$, which *contain the d_0-neighbourhood of ∂X_-* for a given $d_0 < dist(\partial_-, \partial_+)$, satisfy

$$[*_1] \qquad vol_{n-1}(\partial U) > vol_{n-1}(\partial_-)$$

and let all *minimal*[85] *hypersurfaces* $Y \subset X$, the boundaries of which are contained in ∂_+ and which themselves contain points $y \in Y$ far away from ∂_+, namely, such that

$$dist(y, \partial_+) \geq dist(\partial_-, \partial_+) - d_0,$$

satisfy

$$[*_2] \qquad vol_{n-1}(Y) > vol_{n-1}(\partial_-).$$

[85] Here "minimal" means "volume minimizing" with a given boundary.

Then there exists a domain $X_{\bigcirc} \subset X$ which contains ∂_- and such that the boundary of which is smooth with positive mean curvature.

Proof. Let $X_0 \subset X$ minimises $vol_{n-1}(\partial X_0)$ among all domains in X which contain ∂_- and observe that, because of $[*_1]$, the boundary of X_0 contains a point $y \in \partial X_0$ with $dist(y, \partial_+) \geq dist(\partial_-, \partial_+) - d_0$ and, because of $[*_2]$, this X_0 doesn't intersect ∂_+.

Then, by an elementary argument (see [G(Plateau-Stein) 2014]) the hypersurface ∂X_0 can be smoothed and its mean curvature made everywhere positive.

$[\star\star]$ *Two Words about* $[*_2]$. There are several well known cases of manifolds where the lower bound on the volumes of minimal hypersurfaces $Y \subset X$, where $\partial Y \subset partial X$ and where $dist(y, \partial)X \geq R$ for some $y \in Y$, are available.

For instance if X is λ-bi-Lipschitz to the R-ball in the simply connected space X^n_κ with constant curvature κ, then the volume of Y is bounded from below in terms of the volume of the R-ball $B_0^{n-1}(R) \subset X_\kappa^{n-1}$ as follows.

Let $g = dr^2 + \phi^2(r) ds^2$, $r \in [0, R]$, be the metric in the ball $B(R) = B_0^{n-1}(R) \subset X_\kappa^{n-1}$ in the polar coordinates where ds^2 is the metric on the unit sphere S^{n-1} and let $g_\lambda = dr^2 + \phi_\lambda^2(r) ds^2$ be the metric (which is typically singular at $R = 0$), such that the volumes of the concentric balls and of their boundaries satisfy

$$[\star] \qquad \frac{vol_{g_\lambda, n-1} B(r)}{vol_{g_\lambda, n-2}(\partial B(r))} = \Psi_\lambda(r) = \lambda^{2n-3} \frac{vol_{g, n-1} B(r)}{vol_{g, n-2}(\partial B(r))}.$$

Then the standard relation between $vol(Y)$ and the filling volume bound in X says that,

the volume of the above Y is bounded by $vol_{g_\lambda, n-1}(B(R))$.[86]

Notice that $[\star]$ uniquely and rather explicitly defines the function ϕ_λ.

In fact, since

$$vol_{g_\lambda, n-2}(\partial B(r)) = \phi_\lambda^{n-2} \sigma_{n-2}$$

for $\sigma_{n-2} = vol(S^{n-2})$, and since

$$\frac{d vol_{g_\lambda, n-1}(B(r)}{dr} = vol_{g_\lambda, n-2}(\partial B(r))$$

this $[\star]$ can be written as the following differential equation on ϕ_λ

$$\phi_\lambda^{n-2} = \frac{d(\phi_\lambda^{n-2} \Psi_\lambda)}{dr},$$

where our ϕ_λ satisfies $\phi_\lambda(0) = 0$.

.

Examples of Corollaries.

A. Let X be a complete Riemannian n-manifold with *infinite $(n-1)$-volume at infinity*, which means that the boundaries of compact domains which exhaust X,

$$U_1 \subset U_2 \subset \ldots \subset U_i \subset \ldots \subset X,$$

[86] The quickest way to show this is with a use of *Almgren's sharp isoperimetric inequality*. But since this still remains unproved for $\kappa < 0$, one needs a slightly indirect argument in this case, which, possibly – I didn't check it carefully – gives a slightly weaker inequality, namely $Vol(Y) \geq c_n \cdot vol_{g_\lambda, n-1}(B(R))$ for some $c_n > 0$.

have $vol_{n-1}(U_i) \to \infty$.

If X contains no complete non-compact minimal hypersurface with finite $(n-1)$-volume, then X can be exhausted by compact smooth domains the boundaries of which have positive mean curvatures.

Notice that according to [⋆⋆],

no such minimal hypersurface exists in manifolds with uniformly bounded, or even, slowly growing, local geometries.

Also notice that

infinite non-virtually cyclic coverings $\tilde{X}$ of *compact* Riemannian manifolds X, besides having *uniformly bounded* local geometries, also have *infinite* $(n-1)$-*volumes at infinity*; hence they can be exhausted by compact smooth mean convex domains.

And even the *virtually cyclic coverings* $\tilde{X}$ admit such exhaustions unless they are isometric cylinders $Y \times \mathbb{R}$.

Also notice that if $\tilde{X}$ is a Galois (e.g. universal) covering with *non-amenable* deck transformation (Galois) group, then it can be exhausted by U_i with $mean.curv(\partial U_i) \geq \varepsilon > 0$. (See 1.5(C) in [G(Plateu-Stein) 2014].)

Exercises. (a) Show that if a complete connected non-compact Riemannin n-manifold X has uniformly bounded local geometry, then $X \times \mathbb{R}$ has infinite n-volume at infinity.

(b) Show that if X has $Ricci(X) > -(n-1)$, then $X \times \mathbf{H}^2_{-1}$ has infinite $(n+1)$-volume at infinity and that it can be exhausted by compact smooth mean convex domains.

B. Let A be λ-bi-Lipschitz to the annulus $\underline{A} = \underline{A}(r, r+R)$ between two concentric spheres of radii r and $r+R$ in the Euclidean space $\mathbb{R}^n$. [87]

If $R \geq 100\lambda r$, then A contains a hypersurface Y which separates the two boundary components of A and such that

$$mean.curv(Y) \geq \frac{100}{r}.$$

C. Let $\underline{X}$ be a complete simply connected n-dimensional manifold with non-positive sectional curvature and such that $Ricci(X) \leq -(n-1)$, e.g. an irreducible symmetric space with $Sc(X) = -n(n-1)$.

Let A be a compact Riemannin manifold which is λ-bi-Lipschitz to the annulus between two concentric balls $B(r)$ and $B(r+R)$ in $\underline{X}$.

There exists a (large) constant $const_n > 0$, such that if $R \geq const_n \cdot \log\lambda$, then there exists a smooth closed hypersurface $Y \subset A$, which separates the two boundary components in A and such that

$$mean.curv(Y) \geq \frac{n-1}{\lambda + const_n(\lambda - 1)}.$$ [88]

About the Proof. If $\kappa(X) \leq -1$ this follows from [⋆⋆], while the general case needs a minor generalization of this.

[87] This means the existence of a λ-Lipschitz homeomorphism from $\underline{A}$ onto A, the inverse of which $A \to \underline{A}$ is also λ-Lipschitz.

[88] The sign convention for the mean curvature is such that the *mean convex* part of V bounded by Y is the one which contains the boundary component *corresponding to the sphere* $\partial B(r)$ in $\underline{X}$.

First Application to Scalar Curvature. Since

$$Rad_{S^{n-1}}(Y) \geq \lambda^{-1} Rad_{S^{n-1}}(\partial B(r)) \gtrapprox \exp r,$$

the above inequality together with Remark (b) after $\bigcirc^{n-1}$ from section 4.3. yields the following.

If a Riemannin manifold X is λ-bi-Lipschitz to the ball $B(R) \subset \underline{X}$, where $R \geq const_n \log \lambda$, then the scalar curvature of X is bounded by:

$$\inf_{x \in X} Sc(X, x) \leq -\frac{1}{const_n \cdot \lambda^2}.$$

Second Application to Scalar Curvature. It may happen that a manifold X with $Sc(X) > 0$ itself contains no mean convex domain, but it may acquire such domains after a modification of its metric that doesn't change the sign of the scalar curvature. Below is an instance of this.

Let $X = (X, g)$ be a compact n-dimensional Riemannian band, as in the above ***Lemma***, where the boundary of a compact Riemannin manifold $X = (X, g)$ with $Sc(X) \geq 0$ is decomposed as earlier, $\partial X = \partial_- \cup \partial_+$.

Let $Sc(X) > 0$ and let us indicate possible modifications of the Riemannin metric g, that would enforce the conditions $[*_1]$ and $[*_2]$ in the ***Lemma***, while keeping the scalar curvature positive.

We will show below that this can be achieved in some cases by multiplying g by a positive function $e = e(x)$, which is equal one near $\partial_- \subset X$ and which is as large far from ∂_- as is needed for $[*_1]$ and where we also need the Laplacian of $e(x)$ to be bounded from above by $\varepsilon_n Sc(X, x)$ in order to keep $Sc > 0$ in agreement with the Kazdan-Warner conformal change formula from section 2.6.

The simplest case, where there is no need for any particular formula, is where the sectional curvatures of X are pinched between $\mp b^2$, no geodesic loop in X of length$< \frac{1}{b}$ exists, while the scalar curvature of X is bounded from below by $\sigma > 0$.

In this case, let

$$e_0(x) = c \frac{\sqrt{\sigma}}{b+1} dist_g(x, \partial\_0)$$

and observe that if $c = c_n > 0$ is sufficiently small, then $e_0(x)$ has a *small* (generalized) gradient $\nabla(e_0)$ and, because the the geometry of X is suitably bounded, the function e_0 can be approximated by a smooth function $e(x)$ with second derivatives significantly smaller than σ,

thus, *ensuring the inequality* $Sc(eg) > 0$.

On the other hand, if

$$dist(\partial_-, \partial_+) \geq C(b+1) \|\nabla(e)\|^{-1} vol(\partial_-)^{\frac{1}{n-1}},$$

for a large $C = C_n$, then

the *condition* $[*_1]$ *is satisfied*, say with $d_0 = \frac{1}{2} dist(\partial_-, \partial_+)$,

and, due to the bound on the geometry of X,

the *condition* $[*_2]$ *is satisfied* as well.

Now let us look closer at what kind $e(x)$ we need and observe the following

[1] The bound on the geometry of X is needed only, where the gradient of e doesn't vanish.

Thus, it suffices to have the geometry of X

bounded only in the $\frac{1}{b}$*-neighbourhoods of the boundaries of domains* U_i,

$$\partial_- \subset U_1 \subset ... \subset U_i \subset ... \subset U_k \subset X,$$

where $dist(U_i, \partial U_{i+1}) \geq \frac{1}{b}$ and where $\frac{k}{b}$ is sufficiently large.

[2] Since, the by the standard comparison theorem(s),

Laplacians of the distance-like functions are bounded from above in terms of the Ricci

curvature,

the b-bound on the full local geometry can be replaced by $Ricci(X, x) \geq -b^2 g$.

Summing up, this yields the following refinement of B in ✪ from the previous section.

Let $X = (X, g)$ be a, *possibly non-complete* Riemannian n-manifold, such that

$$Sc(X) \geq 0,$$

and let

$$f : X \to S^n$$

be an *area non-increasing map*, such that the support of the differential of f is compact and the scalar curvature of X in this support is bounded from below by that of S^n,

$$\inf_{x \in supp(df)} Sc(X, x) \geq n(n-1).$$

Let A_i be disjoint "bands" in X, that are a_i*-neighbourhoods of the boundaries* of compact domains U_i, such that

$$supp(df) \subset U_1 \subset ... \subset U_i ... \subset U_k \subset X.$$

Let us give an effective *criterion for vanishing of the degree of the map* f in terms of the geometries of A_i.

Proposition. Let the scalar and the Ricci curvatures of X in A_i for $i = 2, ...k-1$ be bounded from below by

$$Sc(A_i) \geq \sigma_i \text{ and } Ricci(A_i) \geq -b^2 g, \ 2 \leq i \leq k-1,$$

and set

$$\beta_i = \frac{\sqrt{\sigma_i}}{b_i}.$$

Let the *sectional curvatures* of U_k outside U_{k-1} be bounded from above by

$$\kappa(U_k \setminus U_{k-1}) \leq c^2, \ c > 0,$$

and let the complement $U_k \setminus U_{k-1}$ contains *no geodesic loop of length* $\leq \frac{1}{c}$.

If the following weighted sum of a_i (that are half-widths of the bands A_i) *is sufficiently large,*

$$\sum_{1 < i < k} \beta_i a_i \geq const_n \frac{(vol_{n-1}(\partial U_1))^{\frac{1}{n-1}}}{\frac{a_k}{c}},$$

and if X is orientable spin, then

$$deg(f) = 0.$$

Proof. Arguing as above, one finds a smooth function $e(x)$, the differential of which is supported in the union of A_i, $1 < i < k$, such that $Sc(e \cdot g)$ remains nonnegative (and even can be easily made everywhere positive) and such that U_k satisfy the assumptions $[*_1]$ and $[*_2]$ of the above ***Lemma***, that yields a subdomain

$$X_{\bigcirc} \subset U_k,$$

which is mean convex with respect to the metric eg and to a smoothed double of which compact Llarull's theorem applies.

Remarks. (a) Even in the case of *complete* manifolds X, this doesn't (seem to) directly follow from Llarull's theorem, since the latter, unlike the former, needs *uniformly positive* scalar curvature at infinity.

(b) The above proposition, as well construction of mean-convex hypersurfaces in general, doesn't advance, at least not directly, the solution of the *spectral $\mathcal{D}^2$-problem* formulated at the beginning of section 4.6.

4.6.3 Enhancing Positivity of the Scalar Curvature by Crossing with Spheres.

Let $X = (X, g)$ be a complete Riemannian n-manifold, let $f : X \to S^n$ be a smooth *area contracting* map the differential df of which has *compact* support.

Let

$$|d| = \sup_{x \in X} ||df(x)||$$

and

$$r = r(x) = dist(x, supp(df)).$$

Let the Ricci curvature of X outside $supp(df)$ be bounded from below by

$$Ricci(x) \geq -b(r(x))^2 g(x)$$

for some continuous function $b(r)$, $r \geq 0$.

If the function $b(r)$ grows sufficiently slowly for $r \to \infty$, e.g. $\sigma(r) \leq \sqrt[3]{r}$ for large r, then there is an effective lower bound

$$Sc(X, x) \geq \sigma(r(x)),$$

which implies that

the map f has zero degree,

where $\sigma(r)$, $r \geq 0$, is a certain "universal" function, which is "small negative" at infinity.

More precisely, there exists a universal effectively computable family of functions in r,

$$\sigma(r) = \sigma_{b,|d|,N,}(r),\ r \geq 0,\ N = 1, 2, ...,$$

with the following five properties

(i) the functions $\sigma(r)$ are *monotone decreasing* in $r \geq 0$,

(ii) $\sigma_{b,|d|,N,}(r)$ is *monotone decreasing* in N,
(iii) $\sigma_{b,|d|,N,}(r)$ *is monotone increasing in b and in $|d|$,*

(iv) $$\sigma(0) = N(N-1), \text{ while } \sigma(r) \underset{r\to\infty}{\to} -\infty$$

(v) $$\sigma_N(r) = \sigma_{b,|d|,N,}(r) \underset{N\to\infty}{\to} -\infty \text{ for fixed } b, \ |\mathrm{d}| \text{ and } r > 0,$$

such that

[$\times\bigcirc^{N-n}$] *if $Sc(X,x) \geq \sigma_{b,|d|,N,}(r(x))$ for all $x \in X$ and some $N \geq n+2$, then, assuming X is orientable and spin, the degree of f is zero.*[89]

Proof. The bound on $\Delta\varphi(x)$ for *Ricci* $\geq -b^2$ (compare with [2] from the previous section) shows that there exists $\sigma_{b,|d|,N,}(r)$ with the above properties (i)-(v) and a positive function $\varphi(x)$ on X, such that

(a) φ is equal to $|d|$ on the support $supp(df) \subset X$

and such that

$$(b)\ \ \sigma(r(x)) + \frac{m(m-1)}{\varphi(x)^2} - \frac{m(m-1)}{\varphi^2(x)}||\nabla\varphi(x)||^2 - \frac{2m}{\varphi(x)}\Delta\varphi(x) \geq \varepsilon > 0 \text{ for } r(x) > 0.$$

Therefore, by the formula ($\star\star$) from section 2.4 for the scalar curvature of the warped product metrics $g_\varphi = g + \varphi^2 ds^2$ on $X \times S^m$, $m = N - n$,

$$Sc(g_\varphi)(x,s) = Sc(g)(x) + \frac{m(m-1)}{\varphi(x)^2} - \frac{m(m-1)}{\varphi^2(x)}||\nabla\varphi(x)||^2 - \frac{2m}{\varphi(x)}\Delta\varphi(x),$$

the metric g_φ has uniformly positive curvature and because of (a) the map $f : X \to S^n$ suspends to an area decreasing map $(X \times S^m, g_\varphi) \to S^{n+m}$ of the same degree as f. Then Llarull's theorem applies and the proof follows.

On Manifolds with Boundaries. If X is a compact manifold with a boundary, the above can be applies to the smoothed double $X \cup_{\partial X} X$, where the scalar curvature of such a double near the smoothed boundary can be bounded from below by the geometry of X near the boundary and the (mean) curvature of the boundary $\partial X \subset X$.

Thus, the above yields a condition for $deg(f) = 0$ in terms of the lower bound on $Sc(X,x)$ and on $dist(x, supp(df))$, which is similar to, yet is different from such a condition from the previous section.

(A similar but somewhat different result follows by the argument in section 5.4)

4.6.4 Amenable Boundaries

If the *volume of the boundary* of a compact manifold X is *significantly smaller than the volume of X* and if it is additionally supposed that the manifold is *not very much curved near the boundary,* then we shall see in this section that

the index theorem applied to the double of such an X with a smoothed metric, yield geometric bounds on the area-wise size of X in terms of the lower bound on the scalar curvature of X.

[89] Compare with "inflating balloon" used in 7.36 of [GL 1983].

Elliptic Preliminaries. Let V be a (possibly non-compact) Riemannian manifold with a boundary, and let l be a section of a bundle $L \to V$ with a unitary connection ∇, such that l satisfy the following (elliptic) *Gårding* $(\delta_\circ, C_\circ)$-*inequality*: the C^1-norm of l at $v \in V$ is bounded at by the L_2-norm of l in the $\delta_\circ$-ball $B = B_v(\delta_\circ) \subset V$ as follows

$$||l(v)|| + ||\nabla l(v)|| \le C_\circ \sqrt{\int_B ||(l)||^2 dv}$$

for all points $v \in V$, where

$$dist(v, \partial V) \ge \delta_\circ.$$

Let

$$\rho(v) = dist(v, \partial V) \text{ and } \beta = \sup_{v \in V} vol(B_v(\delta_\circ))$$

Lemma. If l vanishes on an ε-net $Z \subset V$, then

$$||l(v)|| + ||\nabla l(v)|| \le (10 C_\circ \varepsilon \beta)^{\rho(x) - 2\delta_\circ} \sqrt{\int_V l^2(v) dv}$$

Moreover, if V can be covered by $2\delta_\circ$-balls with the multiplicity of the covering at most m, then the L_2-norms of l and ∇l on the subset $V_{-\rho} \subset V$ of the points ρ-far from the boundary, that is

$$V_{-\rho} = V \setminus U_\rho(\partial V) = \{v \in V\}_{dist(v, \partial V) \ge \rho},$$

satisfies

$$\sqrt{\int_{V_{-\rho}} ||l||^2(v) dv} \le \epsilon \sqrt{\int_V ||l||^2(v) dv}$$

for $\epsilon = m\,(10 C_\circ \varepsilon \beta)^{\rho(x) - 2\delta_\circ}$.

Proof. Combine Gårding's inequality with the following obvious one:

$$||l|| \le \varepsilon ||\nabla|| l$$

and iterate the resulting inequality i times insofar as $\rho - i\delta_\circ$ remains positive.

Remark. A single round of iterations suffices for our immediate applications.

Corollary. Let X be a complete orientable Riemannian manifold of dimension n with compact boundary (e.g. X is compact or homeomorphic to $X_0 \times \mathbb{R}_+$, where X_0 is a closed manifold), and let, y *for some $\rho > 0$ and $0 < \delta_\circ < \frac{1}{4}\rho$,*

the ρ-neighbourhood of the boundary of X, denoted $U = U_\rho(\partial X) \subset X$, has (local) geometry bounded by $\frac{1}{\delta_\circ}$,

where we succumb to tradition and define this bound on geometry as follows:

the sectional curvatures κ of U are pinched between $-\frac{1}{\delta_\circ^2}$ and $\frac{1}{\delta_\circ^2}$ and the injectivity radii are bounded from below by $\delta_\circ$ at all points $x \in U$, for which $dist(x, \partial X) \ge \delta_\circ$, that is, in formulas,

$$|\kappa(X, x)| \le \tfrac{1}{\delta_\circ^2} \text{ for } dist(x, \partial X) \le \rho \text{ and } injrad(X, x) \ge \delta_\circ \text{ for } \delta_\circ \le dist(x, \partial X) \le \rho.$$

Let the scalar curvature of X be non-negative $\frac{1}{2}\rho$-away from the boundary,

$$Sc(X, x) \ge 0 \text{ for } dist(x, \partial X) \ge \frac{1}{2}\rho.$$

Let $f : X \to S^n(R)$, where $S^n(R)$ is the sphere of radius R, be a smooth *area decreasing* map , which is constant on U_ρ, and, if X is non-compact, also locally constant at infinity.

Let the degree of this map be *bounded from below by the volume of* $U_\rho = U_\rho(\partial X)$ as follows.

$$d > Cvol(U_\rho) \text{ for some } C \geq 0.$$

If $\delta_\circ$, ρ and C are *sufficiently large*, then, provided X is *spin,* the scalar curvature of the complement

$$X_{-\rho} = X \setminus U_\rho = \{x \in X\}_{dist(x, \partial X > \rho}$$

can't be everywhere much greater than $Sc(S^n(R)) = \frac{n(n-1)}{R^2}$. Namely

$$[✒] \qquad \inf_{x \in X_{-\rho}} Sc(X, x) \leq \sigma_+ \frac{n(n-1)}{R^2} + \sigma,$$

where $\sigma = \sigma_n(\delta_\circ, \rho, C)$ is a positive function, which may be infinite for small $\delta_\circ$ and/or ρ and/or C and which has the following properties.

- *the function σ is monotone decreasing in $\delta_\circ$, ρ and C;*
- *$\sigma_n(\delta_\circ, \rho, C) \to 0$ for $C \to \infty$ and arbitrarily fixed $\delta_\circ > 0$ and $\rho > \delta_\circ$.*

Proof. Let $2X = \mathbb{D}X$ be a smoothed double of X and $L \to 2X$ the vector bundle induced from $\mathbb{S}^+(S^n)$ by f applied to a copy (both copies, if you wish) of $X \subset 2X$.

Assume $n = dim(X)$ is even, apply the index theorem and conclude that the dimension of the space of L-twisted harmonic spinors on $2X$ is $\geq d$.

Therefore, there exists such a non-zero spinor l that vanishes at given $d - 1$ points in $2X$.

Let such points make a ε-net on the subset $2U_{\rho_\circ} = \mathbb{D}U_{\rho_\circ} \subset 2X$ with a minimal possible ε.

If d is much larger then $vol(2U_\rho) \approx 2vol(U_\rho)$, then this ε becomes small and, consequently, ϵ in the above inequality [✒] also becomes small. Then, the inequality [✒] applied to the domain $2U_\rho \subset 2X$, shows that the integral

$$\int_{2U_\rho} ||l||^2(x)dx$$

is much smaller then the integral of $||l||^2$ over the complement $2X_0 = 2X \setminus 2U_\rho$.

Therefore, if σ_+ is large then the sign of the full integral

$$\int_{2X} Sc(X, x)||l||^2(x)dx = \int_{2X_\rho} Sc(X, x)||l||^2(x)dx + \int_{U_\rho} Sc(X, x)||l||^2(x)dx$$

is equal to the sign of $\int_{2X_\rho} Sc(X, x)||l||^2(x)dx$, which contradicts the Schroedinger-Lichnerowicz-Weitzenboeck formula for harmonic l.

Thus, modulo simple verifications and evaluations of constants left to the reader, the proof is completed.

Example 1. Let a complete non-compact orientable spin Riemannian n-manifold X with *compact boundary* admits smooth *area decreasing* maps $f_i : X \to S^n$ of *non-zero degrees*,[90] such that the "supports" of f_i, i.e. the subsets where these maps are *non-constant*, may lie arbitrarily far from the boundary of X,

$$\text{dist}\ (\text{"supp"} f_i, \partial X) \to \infty \text{ for } i \to \infty.$$

Then the scalar curvature of X can't be uniformly positive at infinity:

$$\liminf_{x\to\infty} Sc(X,x) \le 0.$$

Moreover, the same conclusion holds, if

there exist i-sheeted coverings $\tilde{X}_i \to X$, which admit smooth area decreasing maps $f_i : \tilde{X}_i \to S^n$, such that

$$\frac{deg(f_i)}{i} \to \infty \text{ for } i \to \infty.$$

Example 2. Let Y_k be a k-sheeted covering of the unit 2-sphere $S^2 = S^2(1)$ minus two opposite balls of radii $\frac{1}{k^m}$, for some $m \ge 1$.

Then the product manifold $X_0 = Y_k \times S^{n-2}(k)$ admits an area decreasing map $f : X_0 \to S^n(R)$ constant on the boundary and such that

$$deg(f) \ge \frac{k}{10d}$$

and it follows from the above corollary that the Riemannin metric on X_0 can't be extended to a larger manifold $X \supset X_0$, with bounded geometry and $Sc \ge 0$ without adding much volume to X_0, say in the case $m = n-1$, although $vol_{n-1}(\partial X_0)$ remains bounded for $R \to \infty$.

Melancholic Remarks. Rather than indicating the richness of the field, the diversity of the results in the above sections 4.6.1-4.6.4 is due to our inability to formulate and to prove the true general theorem(s).

4.6.5 Almost Harmonic Spinors on Locally Homogeneous and and Quasi-homogeneous Manifolds with Boundaries

Let X be a complete Riemannian manifold with a transitive isometric action of a group G, let $L \to X$ be a vector bundle with a unitary connection ∇ and let the action of G equivariantly lifts to an action on (L, ∇).

Let the L_2-index of the twisted Dirac operator $\mathcal{D}_{\otimes L}$ (see [Atiyah(L_2) and [ConMos 1982] 1976], be non zero. For instance, if X admits a free discrete isometry group $\Gamma \subset G$ with compact quotient, then this is equivalent to this index to be non-zero on X/Γ.

The main class of examples of such X are *symmetric spaces with non-vanishing "local Euler characteristics* (compare with [AtiyahSch 1977]) i.e. where the corresponding (G-Invariant) n-forms, $n = dim(X)$ don't vanish.

The simplest instances of these are hyperbolic spaces $\mathbf{H}^{2m}_{-1}$, where the indices of the Dirac operators twisted with the positive spinor bundles don't vanish. In

[90] Here as everywhere in this paper, when you you speak of $deg(f)$ the map f is supposed to be locally constant at infinity as well as on the boundary of X.

fact, such an index for a compact quotient manifold $\mathbf{H}^{2m}_{-1}/\Gamma$ is equal to ±one half of the Euler characteristics of this manifold by the Atiyah-Singer formula (compare [Min(K-Area) 2002]).

Let (X,L) be an above homogeneous pair with $ind(\mathcal{D}_{\otimes L})\neq 0$ and let $X_R\subset X$ be a ball of radius R. Then the restrictions of L_2-spinors on X (delivered by the L_2-index theorem) to X_R can be perturbed (by taking products with slowly decaying cut-off functions) to ε-*harmonic* spinors that *vanish on the boundary* of X_R, where $\varepsilon\to 0$ for $R\to\infty$ and where "ε-harmonic" means that

$$\int_{X_R}\langle\mathcal{D}^2_{\otimes L}(s),s\rangle\leq\varepsilon^2\int_{X_R}||s||^2dx$$

as in ✄$_\lambda$ in section 4.6.

In fact, it follows from the local proof of the L_2-index theorem in [Atiyah(L_2) 1976] or, even better, from its later version(s) relying on the finite propagation speed, that these ε-harmonic spinors can be constructed internally in X_R with no reference to the ambient $X\supset X_R$.

Moreover, a trivial perturbation (continuity) argument shows that

similar spinors exist on manifolds X'_R with these metrics close to these on X_R.

but it is unclear "how close" they should be. Here is a specific problem of this kind.

Let X_R be a compact Riemannian spin manifold with a boundary, such that

$$\sup_{x\in X} dist(x,\partial X_R)\geq R$$

and let the sectional curvatures of X are everywhere pinched between -1 and $-1-\delta$.

(A) *Under what conditions on R,δ and ε does X_R support a non-vanishing ε-harmonic spinor twisted with the spin bundle $\mathbb{S}(X_R)$?*

Besides, one wishes to have

(B) similar spinors on manifolds $\overline{X}$ mapped to X_R with *non-zero* degrees and with

controlled metric distorsions

in order to get *bounds on the scalar curvatures* of such $\overline{X}$

(See section 6.4.3 for continuation of this discussion to *fibrations* with quasi-homogeneous fibers.)

5 Variation of Minimal Bubbles and Modification of their Metrics

Given a a Borel measure μ on an n-dimensional Riemannian manifold X, μ-*bubbles* are critical points of the following functional on a topologically defined class of domains $U\subset X$ with boundaries called $Y=\partial U$:

$$(U,Y)\mapsto vol_{n-1}(Y)-\mu(U).$$

Observe that in our examples, $\mu(U)=\int_U\mu(x)dx$ for (not necessarily positive) continuous functions μ on X and that $\mu(U)$ can be regarded as a *closed*

1-form on the space of cooriented hypersurfaces $Y \subset X$. Then $vol_{n-1}(Y) - \mu(U)$ also comes as such an 1-form which we denote $vol^{[-\mu]}_{n-1}(Y)(+const)$.

The first and the second variations of $vol^{[-\mu]}_{n-1}(Y)(+const)$ are the sums of these for $Vol_{-1}(Y)$ and of $vol(U)$ where the former were already computed in section 2.5.

And turning to the latter, it is obvious that the first derivative/variation of $\mu(U)$ under $\psi\nu$, where ν is the outward looking unit normal normal field to Y and $\psi(y)$ is a function on Y, is

$$\partial_{\psi\nu}\int_U \mu(x)dx = \int_Y \mu(y)\psi(y)dy$$

and the second derivative/variation is

$$\partial^2_{\psi\nu}\int_U \mu(x)dx = \partial_{\psi\nu}\int_Y \mu(y)\psi(y)dy = \int_Y (\partial_\nu\mu(y) + M(y)\mu(y))\psi^2(y)dy,$$

where the field ν is extended along normal geodesics to Y, (compare section 2.5) and where $M(y)$ denotes the mean curvature of Y in the direction of ν.

It follows that μ-bubbles Y, (critical points of $vol^{[-\mu]}_{n-1}(Y) = vol_{n-1}(Y) - \mu(U)$) have

$$mean.curv(Y) = \mu(y)$$

and that

second variation of *locally minimal bubbles* $Y \subset X$,

$$\partial_{\psi\nu}(vol^{[-\mu]}_{n-1}(Y)) = \partial_{\psi\nu}\left(vol_{n-1}(Y) - \int_U \mu(x)dx\right),$$

is non-positive.

Then we recall, the formula [∘∘] from section 2.5

$$\partial^2_{\psi\nu}vol_{n-1}(Y) = \int_Y ||d\psi(y)||^2 dy + R_-(y)\psi^2(y)dy$$

for

$$R_-(y) = -\frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - \sum_{i=1}^{n-1}\alpha_i(y)^2\right),$$

where $\alpha_i(y)$ are the principal curvatures of Y at y, and where $\sum \alpha_i^2$ is related to the mean curvature $M = \alpha_1 + ... + \alpha_{n-1}$, by the inequality

$$\sum \alpha_i^2 \geq \frac{M^2}{n-1}.$$

Thus, summing up all of the above, observing that

$$\partial_\nu\mu(x) \geq -||d\mu(x)||$$

and letting

[$R_+ =$] $$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + Sc(X,x),$$

we conclude that

if Y locally minimises $vol_{n-1}^{[-\mu]}(Y)(=vol_{n-1}(Y)-\mu(U))$, then

$$\int ||d\psi||^2 dy + \left(\frac{1}{2}Sc(Y) - \frac{1}{2}R_+(y)\right)\psi^2(Y)dy \geq \partial_{\psi\nu} vol_{n-1}^{[-\mu]}(Y) \geq 0$$

for all functions ψ on Y.

Hence,

$\maltese_{\geq 0}$ *the operator $-\Delta + \frac{1}{2}Sc(Y,y) - \frac{1}{2}R_+(y)$, for $\Delta = \sum_i \partial_{ii}^2$ is positive on Y.*

Examples. (a) Let $X = \mathbb{R}^n$ and $\mu(x) = \frac{n-1}{r}$, that is the mean curvature of the sphere of radius r. Then

$$R_+(x) = \frac{n(n-1)}{R} - 2\frac{n-1}{r^2} + 0 = \frac{(n-1)(n-2)}{r^2} = Sc(S^{n-1}(r)).$$

(b) Let $X = \mathbb{R}^{n-1}\times\mathbb{R}$ be the hyperbolic space with the metric $g_{hyp} = e^{2r}g_{Eucl} + dr^2$ and let $\mu(x) = n-1$. Then

$$R_+(x) = n(n-1) - 0 + (-n(n-1)) = 0 = Sc(\mathbb{R}^n).$$

(c) Let $X = Y \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ with the metric $\varphi^2 h + dt^2$, where the metric h is a metric on Y and where

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}.$$

Then a simple computation shows that

$$R_+(x) = \frac{n(n-1)}{R} - 2\frac{n-1}{r^2} + 0 = \frac{(n-1)(n-2)}{r^2} = Sc(S^{n-1}(r)).$$

$$\frac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + n(n-1) = 0.$$

Furthermore, if $Sc(h) = 0$, than $Sc(X(= n(n-1)$ and $R_+ = 0$.

Two relevant corollaries to $\maltese_{\geq 0}$ are as follows.

Let X be a Riemannian manifold of dimension n, let $\mu(x)$ be a continuous function and Y be a smooth minimal μ-bubble in X.

$\maltese_{conf}$ *If*

$$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + Sc(X,x) > 0,$$

then by Kazdan-Warner conformal change theorem (see section 2.6) *Y admits a metric with $Sc > 0$.*

$\maltese_{warp}$ *There exists a metric $\hat{g}$ on the product $Y\times\mathbb{R}$ of the form $g_Y + \phi^2 dr^2$ for the metric g_Y on Y induced from X, such that*

$$Sc_{\hat{g}}(y,r) \geq R_+(y).$$

Proof. Let $\phi(y)$ be the first, necessarily positive eigenfunction of the operator $-\Delta + \frac{1}{2}Sc(g_Y, y) - R_+(y)$ and recall (see section 2.4) that $Sc(\hat{g}) = Sc(g_Y) - 2\frac{\Delta\phi}{\phi}$. Then

$$-\Delta\phi + \frac{1}{2}Sc(g_Y, y)\phi - \frac{1}{2}R_+(y)\phi = \lambda\phi, \ \lambda > 0,$$

$$\frac{\Delta\phi}{\phi} = -\lambda + \frac{1}{2}Sc(g_Y, y) - \frac{1}{2}R_+(y)$$

and

$$Sc(\hat{g}) = R_+ + 2\lambda,$$

which implies that $Sc_{\hat{g}}(y,r) \geq R_+(y)$, since $\lambda \geq 0$. QED.

5.1 On Existence and Regularity of Minimal Bubbles.

Let X be a compact connected Riemannian manifold of dimension n with boundary ∂X and let $\partial_- \subset \partial X$ and $\partial_+ \subset \partial X$ be disjoint compact domains in ∂X.

Example. Cylinders $Y \times [-1,1]$ naturally come with such a $\partial_\mp$-pair for $\partial_- = Y \times \{-1\}$ and $\partial_+ = Y \times \{1\}$, where, observe, $\partial_- \cup \partial_+ = \partial(Y \times [-1,1])$ if and only if Y is a manifold without boundary.

Let us agree that the mean curvature of ∂_- is evaluated with the incoming normal field and $mean.curv(\partial_+)$ is evaluated with the outbound field.

For instance, if the boundary of X is *concave*, as for instance for X equal to the sphere minus two small disjoint balls, t then $mean.curv(\partial_-) \geq 0$ and $mean.curv(\partial_+) \leq 0$.

Barrier $[\gtrless \mp mean]$*-Condition.* A continuous function $\mu(x)$ on X is said to satisfy $[\gtrless \mp mean]$-condition if

$[\gtrless \mp mean]$ $\qquad \mu(x) \geq mean.curv(\partial_-, v)$ and $\mu(x) \leq mean.curv(\partial_+, x)$

for all $x \in \partial_- \cup \partial_+$.

It follows by the maximum principle in the geometric measure theory that

$\bigstar$ *the* $[\gtrless \mp mean]$*-condition ensures the existence of a minimal* μ*-bubble* $Y_{min} \subset X$*. which separates* ∂_- *from* ∂_-+*.*

If this condition is *strict*, i.e. if $\mu(x) > mean.curv(\partial_-)$ and $\mu(x) < mean.curv(\partial_+)$ and if X has no boundary apart from $\partial_\mp$, then $Y_{min} \subset X$ doesn't intersect $\partial\mp$; in general, the intersections $Y_{min} \cap \partial_\mp$ are contained in the *side boundary* of X that is the closure of the complement $\partial X \setminus (\partial_- \cup \partial_-)$. (This, slightly reformulated, remains true for non-strict $[\gtrless \mp mean]$.)

If $dim(X) = n \leq 7$, then, (this well known and easy to see) Federer's regularity theorem(see section 2.7) applies to minimal bubbles as well as to minimal subvarieties and the same can be said about Nathan Smale's theorem on non-stability of singularities for $n = 8$. Thus, in what follows we may assume our minimal bubbles smooth for $n \leq 8$.

Then, by the stability of Y_{min} (see section 5.1 above),

$\bullet_{\varphi_\circ}$: *there exits a function* $\phi_\circ = \phi_\circ(y) > 0$ *defined in the interior* $^\circ Y$ *of* Y*, i.e. on* $Y \setminus \partial X$*, such that the metric*

$$g_{\varphi_\circ} = \varphi_\circ^2 g_Y + dt^2 \text{ on the cylinder } {}^\circ Y \times \mathbb{R},$$

where g_Y *is the Riemannin metric on* Y *induced from* X*, satisfies*

$$Sc_{g_{\varphi_\circ}}(y,t) \geq Sc(X,y) + \frac{n\mu(y)^2}{n-1} - 2\|d\mu(y)\|$$

for all $y \in {}^\circ Y$.[91]

WHAT IF $n \geq 9$?.

The overall logic of the proof indicated in [Loh(smoothing) 2018] leads one to believe that, assuming strict $[\gtreqless \mp mean]$, there always exists a smooth $Y_o \subset X$, which separates $\partial_\mp$ and and which admits a function $\phi_\circ$ with the property ❍.

The proof of this, probably, is automatic, granted a full understanding Lohkamp's arguments. But since I have not seriously studied these arguments, everything which follows in sections 5.3-5.7 should be regarded as *conjectural* for $n \geq 9$.

Barrier $[\gtreqless mean = \mp\infty]$*-Condition.* Let X be a non-compact, possibly non-complete, Riemannin manifold X and let the set of the ends of X is subdivided to $(\partial_\infty)_- = (\partial_\infty)_-(X)$ and $(\partial_\infty)_+ = (\partial_\infty)_+(X)$, where this can be accomplished, for instance, with a proper map from X to an open (finite or infinite) interval (a_-, a_+) where "convergence" $x_i \to (\partial_\infty)_\mp$, $x_i \in X$, is defined as $e(x_i) \to a_\mp$.

For example, if X is the open cylinder, $X = Y \times (a,b)$, where Y is a compact manifold, possibly with a boundary, this is done with the projection $Y \times (a_-, a_+) \to (a_-, a_+)$.

Obvious Useful Observation. If a function $\mu(x)$ satisfies

$$\mu(x_i) \to \pm\infty \text{ for } x_i \to (\partial_\infty)_\mp$$

then X can be exhausted by compact manifolds X_i with distinguished domains $(\partial_\mp)_i \subset \partial X_i$, such that

- these $(\partial_\mp)_i$ separate $(\partial_\infty)_-$ from $(\partial_\infty)_-$ for all i and

$$(\partial_\mp)_i \to (\partial_\infty)_\mp;$$

- restrictions of μ to $(X_i, (\partial_\mp)_i)$ satisfy the barrier $[\gtreqless \mp mean]$-condition.

This ensures the existence of locally minimising μ-bubbles in X which separate $(\partial_\infty)_-$ from $(\partial_\infty)_+$.

5.2 Bounds on Widths of Riemannin Bands.

Let us prove the following version of the $\frac{2\pi}{n}$-inequality from section 2.6.

$\frac{2\pi}{n}$*-Inequality*.* Let X be an open, possibly non-complete Riemannian manifold of dimension n and let

$$f : X \to (-l, l)$$

be a proper (i.e. infinity → infinity) smooth distance non-increasing map, such that the pullback $f^{-1}(t_o) \subset X$ of a generic point t_o the interval $(-l, l)$ is non-homologous to zero in X.

If $Sc(X) \geq n(n-1) = Sc(S^n)$ and if the following condition $\nparallel_{Sc \ngtr 0}$ is satisfied, then

$$l \leq \frac{\pi}{n}.$$

$\nparallel_{Sc \ngtr 0}$ No smooth closed cooriented hypersurface in X homologous to $f^{-1}(t_o)$ admits a metric with $Sc > 0$.

[91] Since the metric $g_{\varphi_\circ}$ is $\mathbb{R}$-*invariant* its scalar curvature is *constant* in $t \in \mathbb{R}$.

Proof. Assume $l > \frac{\pi}{n}$. and let $\underline{\mu}(t)$ denote the mean curvature of the hypersurface $\underline{Y} \times \{t\}$ in the warped product metric $\varphi^2 h + dt^2$. on $\underline{Y} \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ for

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}$$

as in example (c) from the previous section.

Since $\underline{\mu}(t) \to \pm\infty$ for $t \to \mp\frac{\pi}{n}$, the barrier [$\gtrless$ *mean* $= \mp\infty$]-condition from the section 5.2 guaranties the existence of a locally minimizing μ-bubble in X for μ being a slightly modified f-pullback of $\underline{\mu}$ to X.

Let us spell it out in detail.

Assume without loss of generality that the pullbacks $Y_{\mp} = f^{-1}\left(\mp\frac{\pi}{n}\right) \subset X$ are smooth, and let $\mu(x)$ be a smooth function on X with the following properties.

$\bullet_1$ $\mu(x)$ is constant on X on the complement of $f^{-1}\left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ for $\left(-\frac{\pi}{n}, \frac{\pi}{n}\right) \subset (-i, i)$;

$\bullet_2$ $\mu(x)$ is equal to $\underline{\mu} \circ f$ in the interval $\left(-\frac{\pi}{n} + \varepsilon, \frac{\pi}{n} - \varepsilon\right)$ for a given (small) $\varepsilon > 0$;

$\bullet_3$ the absolute values of the mean curvatures of the hypersurfaces $Y_{\mp}$ are everywhere smaller than the absolute values of μ;

$\bullet_4$ $\frac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + n(n-1) \geq 0$ at all points $x \in X$.

In fact, achieving $\bullet_3$ is possible, since $\underline{\mu}(t)$ is infinite at $\mp\frac{\pi}{n}$, while the mean curvatures of the hypersurfaces $Y_{\mp}$ and what is needed for $\bullet_4$ are the inequality $\|df\| \leq 1$ and the equality

$$\frac{n\underline{\mu}(t)^2}{n-1} - \left|\frac{d\underline{\mu}(t)}{dt}\right| + n(n-1) = 0$$

indicated in example(c) from section 5.1).

Because of $\bullet_3$, the submanifolds $Y_{\mp}$ serve as barriers for μ-bubbles (see the previous section) between them; this implies the existence of a minimal μ-bubble Y_{min} in the subset $f^{-1}\left(-\frac{\pi}{n}, \frac{\pi}{n}\right) \subset X$ homologous to Y_o. by $\bigstar$ in section 5.2.

Due to $\bullet_4$, the operator $\Delta + \frac{1}{2}Sc(Y)$ is positive by $\maltese_{\geq 0}$ from the section 5.1 .

Hence, by $\maltese_{conf}$ the manifold Y_{min} admits a metric with $Sc > 0$ and the inequality $l \leq \frac{\pi}{n}$ follows.

On Rigidity. A a close look at minimal μ-bubbles (see section 5.7) shows that

if $l = \frac{\pi}{n}$, then X is isometric to a warped product , $X = Y \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ with the metric $\varphi^2 h + dt^2$, where the metric h on Y has $Sc(h) = 0$ and where

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}.$$

Exercises. (a) Let X be an open manifolds with two ends, Show that if no closed hypersurface in X that separates the ends admits a metric with positive scalar curvature then X admits no metric with $Sc > 0$ either.[92]

(b) Let X be a complete Riemannian manifold, and let

$$S(R) = \min_{B(R)} Sc(X)$$

[92]This, for a class of spin manifolds X, was shown in [GL [1983] by applying a relative index theorem for suitably twisted Dirac operators on $X \times S^2(R)$.

denote the minimum of the scalar curvature (function) of X on the ball $B(R) = B_{x_0}(R) \subset X$ for some centre point $x_0 \in X$. Show that

if X is homeomorphic to $\mathbb{T}^{n-2} \times \mathbb{R}^2$, then there exists a constant $R_0 = R_0(X, x_0)$, such that

$[\asymp \frac{4\pi^2}{R^2}]$
$$S(R) \leq \frac{4\pi^2}{(R - R_0)^2} \text{ for all } R \geq R_0.$$ [93]

Hint. Since the bands between the concentric spheres of radii r and $r + R$, call them $X(r, r+R) = B(r+R) \setminus B(r)$, are, for large r, quite similar to the cylinders $\mathbf{T}^{N-1} \times [0, R]$, the $\frac{2\pi}{n}$*-Inequality** applies to them and says that their scalar curvatures satisfy

$$S(R) = \inf Sc_x(X(r, r+R), x) \leq \frac{4(n-1)\pi^2}{nR^2}.$$

5.3 Bounds on Distances Between Opposite Faces of Cubical Manifolds with $Sc > 0$

Let us see what kind of geometry Y_{min} may have if we drop the condition $\nparallel_{Sc \ngtr 0}$ from the previous section and allow $l > \frac{\pi}{n}$.

$\square$-*Lemma.* Let X be a compact connected Riemannian manifold of dimension n with boundary ∂X and let $\partial_- \subset \partial X$ and $\partial_+ \subset \partial X$ be disjoint compact domains in ∂X as in section 5.2.

Let

$$Sc(X) \geq \sigma + \sigma_1, \ ,$$

where $\sigma_1 > 0$ is related to the distance $d = dist_X(\partial_-, \partial_+)$ by the inequality

$$\sigma_1 d^2 > \frac{4(n-1)\pi^2}{n}.$$

(If scaled to $\sigma_1 = n(n-1)$, this becomes $d > \frac{2\pi}{n}$.)

Then there exists a smooth hypersurface $Y_{-1} \subset X$, which separates ∂_- from ∂_+, and a smooth positive function ϕ_{-1} on the interior of Y_{-1}, such that the scalar curvature of the metric $g_{-1} = g_{Y_{-1}} + \phi_{-1}^2 dt^2$ on $Y_{-1} \times \mathbb{R}$ is bounded from below by

$$Sc(g_{-1}) \geq \sigma.$$

Proof. The general case of this reduces to that of $\sigma = n(n-1)$ by on obvious scaling/rescaling argument and when $\sigma = n(n-1)$ we use the same μ as above associated with $\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan \frac{nt}{2} dt$, $-\frac{\pi}{n} < t < \frac{\pi}{n}$. Then, as earlier, since

$$Sc_{g_{\varphi_\circ}}(y, t) \geq Sc(X, y) + \frac{n\mu(y)^2}{n-1} - 2||d\mu(y)||$$

by $\bigcirc$ from the previous section, the above equality $\frac{n\mu(t)^2}{n-1} - |\frac{d\mu(t)}{dt}| + n(n-1) = 0$ implies the requited bound $Sc(g_o) \geq \sigma_1$. QED.

---

[93] We shall indicate in section ??? a Dirac operator proof of a rough version of this for a class of spin manifolds X.

Example. Let X be an *orientable spin* manifold, let $\partial_- \cup \partial_+ = \partial X$ and let $f : X \to S^{n-1} \times [-l, l]$ be a smooth map, such that $\partial_\mp \to S^{n-1} \times \{\mp l\}$.

Let the following conditions be satisfied.

• $deg(f) \neq 0$,

• the map $X \to S^{n-1}$, that is the composition of f with the projection $S^{n-1} \times [-l, l] \to S^{n-1}$, is *area decreasing*;

• $Sc(X) \geq (n-1)(n-2) + \sigma_1$ for some $\sigma_1 \geq 0$.

Then the above lemma in conjunction with the (stabilised) Llarull theorem shows that

$$dist(\partial_-, \partial_+) \leq \frac{2\pi}{n} \frac{n(n-1)}{\sqrt{\sigma_1}} = \frac{2\pi(n-1)}{\sqrt{\sigma_1}}.$$

Remark. This inequality if it looks sharp, then only for $\sigma_1 \to 0$, while sharp(er) inequality of this kind need different functions μ.

Equivariant $\square$-Lemma. Let X in the $\square$-Lemma be free isometrically acted upon by a unimodular Lie group G that preserves $\partial_\mp$.

Then there exists a submanifold $Y_{-1} \subset X$ and a function ϕ_{-1} on Y, which, besides enjoying all properties in the $\square$-Lemma, are also invariant under the action of G and the resulting metric on g_{-1} on $Y_{-1} \times \mathbb{R}$ is $G \times \mathbb{R}$-invariant.

In fact, the proof of the $\square$-Lemma applies to X/G.

Remark. This lemma may hold for all G, but what we need below is only the case of $G = \mathbb{R}^i$.

$\square^{n-m}$-*Theorem.* Let X be a compact connected orientable Riemannian manifold with boundary and let $\underline{X}_\bullet$ is a closed orientable manifold of dimension $n - m$, e.g. a single point $\bullet$ if $n = m$.

Let

$$f : X \to [-1, 1]^m \times \underline{X}_\bullet$$

be a continuous map, which sends the boundary of X to the boundary of $[-1, 1]^m \times \underline{X}_\bullet$ and which has *non-zero degree*.

Let $\partial_{i\pm} \subset X$, $i = 1, ..., m$, be the pullbacks of the pairs of the opposite faces of the cube $[-1, 1]^m$ under the composition of f with the projection $[-1, 1]^m \times \underline{X}_\bullet \to [-1, 1]^m$.

Let X satisfy the following condition:

$\pitchfork^m_{Sc \not> 0}$ No transversal intersection $Y_{-m\pitchfork} \subset X$ of m-hypersurfaces $Y_i \in X$ which separates ∂_{i-} from ∂_{i+}, admits a metric with $Sc > 0$; moreover, the products $Y_{-m\pitchfork} \times \mathbb{T}^m$ admit no metrics with $Sc > 0$ either.[94]

If $Sc(X) \geq n(n-1)$ that the distances $d_i = dist(\partial_{i-}, \partial_{i+})$ satisfy the following inequality (which generalise that from section 3.8).

$$\square_\Sigma \qquad \sum_{i=1}^{m} \frac{1}{d_i^2} \geq \frac{n^2}{4\pi^2}$$

Consequently

$$\square_{\min} \qquad min_i dist(\partial_{i-}, \partial_{i+}) \leq \sqrt{m}\frac{2\pi}{n}.$$

[94]This "moreover" is unnecessary, since the relevant for us case of stability of the $Sc \not> 0$ condition under multiplication by tori is more or less automatic. (The general case needs some effort.)

Proof. Let

$$\sigma_i' = \left(\frac{2\pi}{n}\right)^2 \frac{n(n-1)}{d^2} = \frac{4\pi^2(n-1)}{nd^2}$$

and rewrite $\square_\Sigma$ as

$$\sum_i \sigma_i' \geq n(n-1).$$

Assume $\sum_i \sigma_i' < n(n-1)$ and let $\sigma_i > \sigma_i'$ be such that $\sum_i \sigma_i < n(n-1)$.

Then, by induction on $i = 1, 2, ..., m$ and using $\mathbb{R}^{i-1}$-invariant $\square$-Lemma on the ith step, construct manifolds $X_{-i} = Y_{-i} \times \mathbb{R}^i$ with $\mathbb{R}^i$-invariant metrics g_{-i}, such that

$$Sc(X_{-i}) > n(n-1) - \sigma_1 - .. - \sigma_i.$$

The proof s concluded by observing that this for $i = m$ would contradict to $\#^m_{Sc\not>0}$.

Remarks. (a) As we mentioned earlier, this inequality is non-sharp starting from $m = 2$, where where the sharp inequality

$\square^2_{\min}$ $$min_{i=1,2} dist(\partial_{i-}, \partial_{i+}) \leq \pi.$$

for squares with Riemannin metrics on them with $Sc \geq 2$ follows by an elementary argument.

(b) One can show for all n that

$$min_i dist(\partial_{i-}, \partial_{i+}) \leq \sqrt{m}\frac{2\pi}{n} - \varepsilon_{m,n},$$

where $\varepsilon_{m,n} > 0$ for $m \geq 2$.

(c) A possible way for sharpening $\square_\Sigma$, say for the case $m = n$, is by using $n-2$ inductive steps instead of n and then generalizing the elementary proof of $\square^2_{\min}$ to $\mathbb{T}^{n-2}$-invariant metrics on $[-1,1]^2 \times \mathbb{T}^{n-2}$.

In fact, all theorems for surfaces X with positive (in general, bounded from below) sectional curvatures beg for their generalisations to $\mathbb{T}^{m-2}$-invariant metrics on $X \times \mathbb{T}^{m-2}$ with positive (and/or bounded from below) scalar curvatures.

5.3.1 Max-Scalar Curvature with and without Spin.

It remains a big open problem of making sense of the inequality $Sc(X) \geq \sigma$, e.g. for $\sigma = 0$, for *non-Riemannian* metric spaces, e.g. for piecewise smooth polyhedral spaces P.

But lower bounds on Lipschitz constants of homologically substantial maps $X \to P$ entailed by the inequality $Sc(X) \geq \sigma > 0$, that, for a fixed P, tell you something about the geometry of X, can be used the other way around for the definition of scalar curvature-like invariants of general metric spaces P as follows.

Given a metric space P [95] and a homology class $h \in H_n(P)$ define $Sc^{\max}(h)$ as the supremum of the numbers $\sigma \geq 0$, such that, there exists a closed orientable

[95]To be specific we assume that P is locally compact and locally contractible, e.g. it is locally triangulable space

Riemannian n-manifold X and a 1-Lipschitz map $f : X \to P$, such that the fundamental homology class $[X]$ goes to h,

$$f_*[X] = h.$$

Similarly, one defines $Sc^{\mathsf{max}}_{sp}(h)$ by allowing only *spin* manifolds X, where, for instance, the discussion in section 4.1.1 shows that

$$Sc^{\mathsf{max}}_{sp}(h) \leq const_n \cdot \text{K-}waist_2(h).$$

Below are a few observations concerning these definitions.

$\bullet_1$ $Sc^{\mathsf{max}}[X] \geq \inf_x Sc(X,x)$ for all closed *Riemannian* manifolds X, where the equality $Sc^{\mathsf{max}}[X] = Sc(X,x)$, $x \in X$, holds for what we call *extremal* manifolds X.

$\bullet_2$ More generally, the product homology class $h \otimes [X] \in H^{n+m}(P \times X)$, $m = dim(X)$, where $P \times X$ is endowed with the Pythagorean product metric, satisfies

$$Sc^{\mathsf{max}}(h \otimes [X]) \geq Sc^{\mathsf{max}}(h) + \inf_x Sc(X,x).$$

$\bullet_3$ *Possibly,*

$$Sc^{\mathsf{max}}(h \otimes [S^m]) = Sc^{\mathsf{max}}(h) + m(m-1),$$

but even the rough inequality

$$Sc^{\mathsf{max}}(h \otimes [S^m]) \leq Sc^{\mathsf{max}}(h) + const_m.$$

remains beyond splitting techniques from section 5.3. [96]

$\bullet_4$ If $F : X_1 \to X_2$ is a finitely sheeted covering between closed orientable Riemannian manifolds, then

$$Sc^{\mathsf{max}}_{sp}[X_1] \geq Sc^{\mathsf{max}}_{sp}[X_2] \text{ as well as } Sc^{\mathsf{max}}_{sp}[X_1] \geq Sc^{\mathsf{max}}_{sp}[X_2],$$

but the equality may fail to be true, e.g. for SYS-manifolds X_2 defined in section 2.7.

(It is less clear when/why this happens to *infinitely sheeted* coverings, where the problem can be related to possible failure of contravariance of K-$waist_2$, see section 4.1.1.)

Non-Compact Spaces and Sc^{max}_{prop}. The above definitions naturally extends to homology with infinite supports in non-compact spaces , e.g. to the fundamental classes $[P]$ of open manifolds and pseudomanifolds P, where the Riemannin manifolds X mapped to these spaces are now non-compact and not even complete.

Also we use the notation Sc^{max}_{prop} for fundamental classes of (psedo)manifolds P *with boundaries*, where *proper maps* $X \to P$ are those sending $\partial X \to \partial P$.

Stabilized max-Scalar Curvatures. These for a space P are defined as

$$stabSc^{max}_{\dots}(P) = Sc^{max}_{\dots}(P \times \mathbb{T}^N)$$

[96] These techniques deliver such an inequality for the stabilized max-scalar curvature: $Sc^{\mathsf{maxstab}}(h) = \lim_{m\to\infty}(h \otimes [\mathbb{T}^m])$, where one may additionally require the manifolds X mapped to $P \times \mathbb{T}^m$ to be isometrically acted upon by the m-tori

where $\mathbb{T}^N$ is flat torus that may be assumed arbitrarily large (this proves immaterial at the end of day), where N is also large and where the implied metric in the product is the Pythagorean one:

$$dist((p_1,t_1),(p_2,t_2)) = \sqrt{dist(p_1,p_2)^2 + dist(t_1,t_2)^2}.$$

Examples. (a) Llarull's and Goette-Semmelmann's inequalities from section 4.2 can be regarded as sharp bounds on $Sc^{\max}_{sp}$ for (the fundamental homology classes of) spheres and convex hypersurfaces.

(b) The $\square$-inequalities from the previous section provide similar bounds on stabilised $Sc^{\max}_{prop}(P)$ for the fundamental homology classes of the rectangular solids $P = \times_{i=1}^{n}[0,a_i]$.

(It seems, there are interesting examples in the spirit of SYS-spaces from section 2.7, where one needs to allow $f_*[X]_{\mathbb{Z}/l\mathbb{Z}} \neq 0$, at least for for odd l.

Also one may ask in this regard if $Sc^{\max}_{prop}$ of the universal covering of a closed orientable manifold X with a residually finite fundamental group is equal to the limit of $Sc^{\max}_{prop}$ of the finite coverings of X.)

(c) *Spaces with S-Conical Singularities and $Sc \geq \sigma$.* Let us define classes $\mathscr{S}^n_{\geq\sigma}$, $n = 2,3,...$ of piecewise Riemannian spaces with $Sc \geq \sigma > 0$ by induction on dimension $n \geq 2$ as follows.

Let $Y = Y^{n-1}$ from $\mathscr{S}^{n-1}_{\geq\sigma}$ be isometrically realized by a piecewise smooth $(n-1)$-dimensional subvariety in a $(N-1)$-dimensional sphere, $N >> n$, that serves as the boundary of the N-dimensional hemisphere,

$$Y \subset S^{N-1}(R) = \partial S^N_+(R),$$

where the radius of the sphere satisfies,

$$R \geq \sqrt{\frac{(n-1)(n-2)}{\sigma}}$$

and where "isometrically" means preservation of the lengths of piecewise smooth curves in Y.

Then the *spherical cone* of Y, that is the union of the geodesic segments which the center of the spherical n-ball $S^N_+ \subset S^N$ to all $y \in Y$ is, by definition, belongs to $\mathscr{S}^n_{\geq\sigma'}$ for

$$\sigma' = \sigma\frac{n}{n-2}$$

and, more generally, a piecewise smooth Y is in $\mathscr{S}^n_{\geq\sigma'}$ if its scalar curvature at all non-singular points is $\geq \sigma'$ and near singularities Y is isometric to a spherical cone over a space from $\mathscr{S}^{n-1}_{\geq\sigma}$.

To conclude the definition, we agree to start the induction with $n-1 = 1$, where our admissible spaces are circles of length $\leq 2\pi$ and, if we allow boundaries, segments of any length.

$Y \subset S^{N-1}$ be a closed submanifold of dimension $n-1 \geq 2$, and let $S(Y) \subset S^N \supset S^{N-1}$ be the *spherical suspension* of Y, that is the union of the geodesic segments which go from the north and the south poles of S^N to Y.

Notice that this $S(Y)$ with the induced Riemannian metric is smooth away from the poles, where it is singular unless the induced Riemannian metric in Y

has constant sectional curvature $+1$ and Y is simply connected (hence, isometric to S^{n-1}).

Let Y be a space from $\mathscr{S}^n_{\geq\sigma}$ with k isolated singular points $y_i \in Y$ where X is locally isometric to S-cones over $(n-1)$-manifolds, call them V_i, $i = 1, ...k$ such that every such V_i bounds a Riemannian manifold W_i, where $Sc(W_i) > 0$ and the mean curvature of $V_i = \partial W_i$ is positive. Then

$$Sc^{\max}_{prop}(Y) \geq \sigma.$$

Sketch of the Proof. Arguing as in [GL(classification) 1980], one can, for all $\varepsilon > 0$, deform the metric in X near singularities keeping $Sc \geq \sigma - \varepsilon$, such that the resulting metric on Y minus the singular points y_i becomes complete, where its k ends are isometric to the cylinders $\varepsilon V_i \times [\infty)$, where εV stands for an V with its Riemannin metric multiplied by ε^2.

This complete manifold, call it Y_ε, admits a locally constant at infinity 1-Lipschitz map $Y_\varepsilon \to Y$ of degree 1, and then the closed manifold $\bar{Y}_\varepsilon$, obtained from Y_ε by attaching εW_i to $\varepsilon V_i \times \{t_i\}$, for large $t_i \in [0, \infty]$ admits a required 1-Lipschitz map to Y as well. QED

Remark. Instead of filling V_i by W_i *individually* it is sufficient to fill in their (correctly oriented!) disjoint union $V = \bigsqcup_i V_i$ by W. For instance, if there are only two singular points, where V_1 and V_2 are isometric and admit orientation reversing isometries then $V_1 \sqcup -V_2$ bounds the cylinder W between them.

This kinds of "desingularization by surgery" also applies to Y, where the singular loci $\Sigma \subset Y$ have dimensions $dim(\Sigma) \geq 1$, similarly to how it is done to manifolds with corners (see section 1.1 in [G(billiard0 2014]) but the filling condition becomes less manageable.

In fact even if $dim(\Sigma) = 0$, it is unclear how essential our filling truly is, especially for evaluation $Sc^{\max}$ of a *multiple* of the fundamental class of an Y; yet, the spaces $Y \in \mathscr{S}^n_{\geq\sigma}$ with isolated singularities seem to enjoy the same metric properties as smooth manifolds with $Sc \geq \sigma$ filling or no filling.

For instance, if the non-singular locus of such an Y is spin then the hyperspherical radius Y is bounded in the same way as it is for smooth manifolds:

$$Rad_{S^n}(Y) \leq \sqrt{\frac{n(n-1)}{\sigma}},$$

as it follows from Llarull's theorem for complete manifolds.

In fact, the construction from [GL(classification) 1980] for connected sums of manifolds with $Sc > 0$, when applied to $Y \setminus \Sigma$, achieves a blow-up of the metric g of Y on $Y \setminus \Sigma$ to a complete one, say g_+, such that $g_+ \geq g$ and $\inf_x Sc(g_+, x) \geq \inf_x Sc(g, x) - \varepsilon$ for an arbitrarily small $\varepsilon > 0$.

Also mean convex cubical domains U in Y with none of the singular $y_i \in Y$ lying on the boundary ∂U satisfy the constraints on the dihedral angles similar to those for smooth Riemannin manifolds with $Sc \geq \sigma$

But the picture becomes less transparent for $dim(\Sigma) > 0$, as it is exemplified by the following.

Question. Does the inequality $Rad^2_{S^n}(Y) \leq const_n \frac{n(n-1)}{\sigma}$ hold true for all $Y \in \mathscr{S}^n_{\geq\sigma}$?

Perspective. In view of [Ch 1983], [GSh 1993] and [AlbGell 2017], it is tempting to use the Dirac operator on the non-singular locus $Y \setminus \Sigma$ with a controlled behavior for $y \to \Sigma$, but it remains unclear if one can actually make this work for $dim(\Sigma) > 0$.

The only realistic approach at the present moment is offered by the method of minimal hypersurfaces (and/or of stable μ-bubbles), which may be additionally aided by surgery desingularization, such as multi-doubling similar to that described in [G(billiards) 2014] for manifolds with corners.

Max-Scalar Curvature Defined via Sc-*Normalized Manifolds .* Given a Riemannin manifold $X = (X, g)$ with positive scalar curvature, let $g_\sim = Sc(g) \cdot g$, consider Lipschitz maps f of closed oriented Riemannian manifolds $X = (X, g)$ with $Sc(X) > 0$ to P, such that $f_*[X] = h$, for a given $h \in H_n(P)$, let $\lambda_\sim^{min}$ be the infimum of the Lipschitz constants of these maps with respect to the metrics $g_\sim$ and let

$$Sc_\sim^{\max}(h) = \frac{1}{(\lambda_\sim^{min})^2}.$$

And if P is a *a piecewise smooth polyhedral space* (e.g. a Riemannian manifold), define $Sc_{\wedge_\sim^2}^{\max}(h)$ by taking the infimum $\inf_f \sup_{x \in X} \| \wedge^2 df(x)\|$ instead of the $\lambda_\sim^{min}$ (as in $\wedge^2$-inequality from section 4.2[97]):

$$Sc_{\wedge_\sim^2}^{\max}(h) = \frac{1}{\inf_f \sup_{x \in X} \| \wedge^2 df(x)\|}.$$

Clearly,

$$Sc^{\max} \leq Sc_\sim^{\max} \leq Sc_{\wedge_\sim^2}^{\max}.$$

(Similar inequalities are satisfied by the spin and by proper versions of $Sc^{\max}$), where *most bounds on* $Sc^{\max}$ we prove and/or conjecture below *can be* more or less automatically *sharpened to their* $Sc_\sim^{\max}$ *and* $Sc_{\wedge_\sim^2}^{\max}$ (as well as to their spin and proper) *counterparts.*)

Problem. Evaluate $Sc_{prop}^{\max}$ of (the fundamental classes of) "simple" metric space, such as products of m_i-dimensional balls of radii a_i where $\sum_i m_i = n$ and the product distance is l_p, i.e. $dist_{l_p}((x_i), (y_i)) = \sqrt[p]{\sum_i dist(x_i, y_i)^p}$, e.g. for $p = 2$.

This is related to the problem of a general nature of evaluating $Sc^{\max}(h_1 \otimes h_2)$ of $h_1 \otimes h_2 \in H_{n_1+n_2}(P_1 \times P_2)$ in terms of $Sc^{\max}(h_1) \in H_{n_1}(P_1)$ and $Sc^{\max}(h_2) \in H_{n_2}(P_2)$.

It follows from the additivity of the scalar curvature (see section 1) that

$$Sc^{\max}(h_1 \otimes h_2) \geq Sc^{\max}(h_1) + Sc^{\max}(h_2),$$

but it is unrealistic (?) to expect that, in general

$$Sc^{\max}(h_1 \otimes h_2) \leq const_{n_1+n_2} \cdot (Sc^{\max}(h_1) + Sc^{\max}(h_2)),$$

[97]The definition of $\| \wedge^2 df(x)\|$ makes sense for Lipschitz maps (at almost all x) but the arguments with Dirac operators need smoothness of the maps. But it may be interesting to go beyond smooth manifolds and maps to general continues maps with *bounded area dilations*, where, probably, the most adequate definition of "area" in non-smooth metric spaces P is the Hilbertian one in the sense of [G(Hilbert) 2012].

albeit the geometric method from the section 5.4 does deliver non-trivial bounds on $Sc^{\max}_{prop}$ of *product spaces* whenever lower bounds on the *hyperspherical radii of the factors* are available.[98]

5.4 Extremality and Rigidity of log-Concave Warped products.

The inequalities proven in section 5.3 say, in effect, that the metric

$$g_\phi = \phi^2 g_{flat} + dt^2 \text{ on } \mathbb{T}^{n-1}\times\mathbb{R} \text{ for } \phi(t) = \exp\int_{-\pi/n}^{t} -\tan\tfrac{nt}{2}dt$$

is *extremal*: *one can't increase g_ϕ without decreasing its scalar curvature,*[99] where the essential feature of ϕ (implicitly) used for this purpose was *log-concavity of ϕ*:

$$\frac{d^2\log\phi(t)}{dt^2} < 0.$$

We show in this section that the same kind of extremality (accompanied by rigidity) holds for other log-concave functions, notably for $\varphi(t) = t^2$, $\varphi(t) = \sin t$ and $\varphi(t) = \sinh t$ which results in

rigidity of punctured Euclidean, spherical and hyperbolic spaces.

More generally, let $X = Y\times\mathbb{R}$ comes with the warped product metric $g_\phi = \phi^2 dg_y + dt^2$. Then the mean curvatures of the hypersurfaces $Y_t = Y\times\{t\}$, $t\in\mathbb{R}$, satisfy (see 2.4)

$$mean.curv(Y_t) = \mu(t) = (n-1)\frac{d\log\phi(t)}{dt} = \frac{\phi'(t)}{\phi(t)},$$

and, obviously, are these $Y_t\subset X$ are locally (non-strictly) minimizing μ-bubbles.[100]

Now, clearly, *ϕ is log-concave, if and only if*

$$\frac{d\mu}{dt} = -\left|\frac{d\mu}{dt}\right|.$$

Thus, R_+ defined (see section 5) as

$$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + Sc(X,x)$$

is equal in the present case to

$$\frac{n\mu(t)^2}{n-1} + 2\mu'(t) + Sc(g_\phi(t)) = \frac{2(n-1)\phi''(t)}{\phi(t)} + (n-1)(n-2)\left(\frac{\phi'}{\phi}\right)^2 + Sc(g_\phi(t))$$

[98]One may define $Rad_{S^n}(h)$, $h\in H^n(P)$, as the suprema of the radii R of the n-spheres, for which P admits a 1-Lipschitz map $f: P\to S^n(R)$, such that $f_*(h)\neq 0$.

[99]To be precise, one should say that

one can't modify the metric, such that the *scalar curvature increases* but the metric itself *doesn't decrease.*

The relevance of this formulation is seen in the example of $X = S^n\times S^1$, where one can stretch the obvious product metric g in the S^1-direction without changing the scalar curvature, but one *can't increase* the scalar curvature by deformations that increase g.

[100]If Y is non-compact, the minimization is understood here for variations with compact supports.

which implies (see section 5) that

$$(R_+)_{Y_t} = \frac{1}{\phi^2} Sc(g_{Y_t}) = Sc(g_{Y_t}) \text{ for } g_{Y_t} = \phi^2 g_Y.$$

Thus our operators $-\Delta_{Y_t} + \frac{1}{2} Sc(g_{Y_t}) - (R_+)_{Y_t}$ equal $-\Delta_{Y_t}$, the lowest eigenvalue of which are *zero* with *constant* corresponding eigenfunctions and the corresponding (S^1-invariant warped product) metrics on $Y_t \times S^1$ are (non-warped) $g_{Y_t} + ds^2$ for $Y_t = Y \times \{t\} \subset X = Y \times \mathbb{R}$ and all $t \in \mathbb{R}$.

(We "warp" with the circle S^1 rather than with $\mathbb{R}$ to avoid a confusion between two different $\mathbb{R}$.)

This computation together with ✤$_{warp}$ in section 5 yield the following.

Comparison Lemma. Let $\underline{X} = \underline{Y} \times [a, b]$ be an $\underline{n}$-dimensional warped product manifold with the metric

$$g_{\underline{X}} = g_{\underline{\phi}} = \underline{\phi}^2 g_{\underline{Y}} + dt^2, \; t \in [a, b],$$

where $\underline{\phi}(t)$ is a smooth positive log-*concave* function on the segment $[a, b]$.

Let X be an n-dimensional Riemannian manifold, with a smooth function $\mu(x)$ on it and let $Y_\circ \subset X$ be a stable, e.g. locally minimising μ-bubble in X.

Let $g_\circ = g_{\phi_\circ} = \phi_\circ^2 g_{Y_\circ} + ds^2$ be the metric on $Y_\circ \times S^1$ where g_Y is the metric on Y induced from X, and where $\phi_\circ$ is the first eigenfunction of the operator

$$-\Delta + \tfrac{1}{2} Sc(g_Y, y) - R_+(y) \text{ for } R_+(x) = \tfrac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + Sc(X, x)$$

(where $\phi_\circ$ is not assumed positive at this point).

Let $f : X \to \underline{X}$ be a smooth map let $f_{\underline{Y}} : X \to \underline{Y}$ denote the $\underline{Y}$-component of f, that is the composition of f with the projection $\underline{X} = \underline{Y} \times [a, b] \to \underline{Y}$.

Let

$$f_{[a,b]} : X \to [a, b]$$

be the $[a, b]$-component of f, let

$$\underline{\mu}^*(x) = \underline{\mu} \circ f_{[a,b]}(x) \text{ for } \underline{\mu}(t) = (\underline{n} - 1) \frac{d \log \underline{\phi}(t)}{dt} = mean.curv(\underline{Y}_t), \; t = f_{[a,b]}(x)$$

and let

$$\underline{\mu}'^* = \underline{\mu}' \circ f_{[a,b]}(x) \text{ where } \mu' = \mu'(t) = \frac{d\underline{\mu}(t)}{dt}.$$

Let

$$\underline{R}_+^*(x) = \frac{n\underline{\mu}^*(x)^2}{\underline{n} - 1} - 2\|d\underline{\mu}^*(x)\| + Sc(\underline{X}, f(x))$$

If

$$R_+(x) \geq \underline{R}_+^*(x),$$

then the function $\phi_\circ$ is positive and the scalar curvature of the metric $g_\circ = g_{\phi_\circ}$ on $Y_\circ \times S^1$ satisfies

$$Sc_{g_\circ}(y_\circ, s) \geq \frac{1}{\|df_{[a,b]}(y_\circ)\|^2} Sc(\underline{Y}, f_{\underline{Y}}(y_\circ)) = Sc(\underline{Y}_t, f(y_\circ)) \textit{ for } Y_t \ni f(y_\circ).$$

The main case of this lemma, which we use below, is where

$\bullet_{df_{[a,b]}}$ *the function* $f_{[a,b]} : X \to [a,b]$ *is 1-Lipschitz, i.e.* $||df_{[a,b]}|| \le 1$,

and

$\bullet_{\mu}$ $\mu(x) = \underline{\mu} \circ f_{[a,b]}$, *that is* $\mu(x) = mean.curv(\underline{Y}_t, f(x))$ *for* $\underline{Y}_t \ni f(x)$

and where the conclusion reads:

$$[Sc \ge]. \qquad Sc_{g_\phi}(y,s) \ge \frac{1}{(f_{[a,b]}(y))^2} Sc(\underline{Y}, f_{\underline{Y}}(y)) + Sc(X,y) - Sc(\underline{X}, f(y)).$$

Corollary. Let $X_\circ$ denote the above Riemannian (warped product) manifold $(Y_\circ \times S^1, g_\circ = g_{\phi_\circ})$ and let $f_\circ : X_\circ \to \underline{Y}$ be defined by $(y_\circ, s) \mapsto f_{\underline{Y}}(y_\circ)$.

If besides $\bullet_{df_{[a,b]}}$ *and* $\bullet_\mu$,

$$|| \wedge^2 df|| \le 1, \ \ e.g. \ ||df|| \le 1$$

and if

$$Sc(X,y) \ge Sc(\underline{X}, f(y)),$$

then the map $f_\circ$ *satisfies*

$$Sc(X_\circ, x_\circ) \ge ||df_\circ||^2 Sc(\underline{Y}, f_\circ(x_\circ)) \ge || \wedge^2 df_\circ|| Sc(\underline{Y}, f_\circ(x_\circ)).$$

Now, the existence of minimal bubbles under the barrier $[\gtrless mean = \mp\infty]$-condition (see section 5.2) and a combination of the above with the Llarull *trace* $\wedge^2 df$*-inequality* from section 4.2 yields the following.

$\odot_{S^n}$. *Extremality of Doubly Punctured Spheres.* Let X be an oriented Riemannian spin n-manifold, let $\underline{X}$ be the n-sphere with two opposite points removed and let $f : X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \ge n(n-1) = Sc(\underline{X}) = Sc(S^n)$, *then*

(A) the scalar curvature of X *is constant* $= n(n-1)$;

(B) the map f *is an isometry.*

Proof. The spherical metric on $\underline{X} = S^n \setminus \{s, -s\}$ is the warped product $S^{n-1} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ where the warping factor $\underline{\phi}(t) = \cos t$ which is logarithmically concave, where $\underline{\mu}(t) = \frac{d \log \underline{\phi}(t)}{dt} \to \pm\infty$ for $t \to \mp\frac{\pi}{2}$. [101]

This implies (A) while (B) needs a little extra (rigidity) argument indicated in section 5.7.

1-*Lipschitz Remark.* As it is clear from the proof, the 1-Lipshitz condition can be relaxed to the following one.

The radial component $f_{[-\frac{\pi}{2}, \frac{\pi}{2}]} : X \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of f, which corresponds to the signed distance function from the equator in $S^n \setminus \{s, -s\}$ is 1-Lipschitz and (the exterior square of) the differential of the S^{n-1} component $f_{S^{n-1}} : X \to S^{n-1}$ satisfies

$$df_{S^{n-1}} \wedge^2 df(x) \le \left(\sin f_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(x)\right)^2.$$

[101] If a log-concave function ϕ on the segment $[-l, l]$ is positive for $-l < t < l$ and it vanishes at $-l$, then the logarithmic derivative of ϕ goes to ∞ for $t \to -l$; similarly,

$$\frac{\phi'}{\phi} \underset{t \to l}{\to} -\infty,$$

if ϕ vanishes at $t = l$.

correct to ???

$$\wedge^2 df_{S^{n-1}}(x) \leq \frac{1}{\left(\cos f_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x)\right)^2}.$$

Non-Spin Remark. If $n=4$, one can drop the spin condition, since μ-bubbles $Y \in X$, being 3-manifolds, are spin.

Similarly to $\odot_{S^n}$ one shows the following.

$\odot_{\mathbb{R}^n}$. Let Let X be as above, let $\underline{X}$ be $\mathbb{R}^n$ with a point removed and let $f: X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \geq n(n-1) \geq 0$ and if X is an isometry at infinity, then

(A) $Sc(X) = 0$;

(B) the map f is an isometry.

$\odot_{\mathbf{H}^n}$. Let Let X be as above, let $\underline{X}$ be the hyperbolic space with a point removed and let $f: X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \geq -n(n-1)$ and if X is an isometry at infinity, then

(A) $Sc(X) = -n(n-1)$;

(B) the map f is an isometry.

Question. Let $d_0(\underline{x}) = dist(\underline{x}, \underline{x}_0)$ be the distance function in $\underline{X}$ (used in $\odot_{\mathbb{R}^n}$ and/or in $\odot_{\mathbf{H}^n}$) to the point $\underline{x}_0$, which was removed from $\mathbb{R}^n$ or from $\mathbf{H}^n$, and let $d_f(x) = d_0(f(x))$.

Can one relax the 1-Lipschitz condition in the propositions $\odot_{\mathbb{R}^n}$ and in $\odot_{\mathbf{H}^n}$ by requiring that not f but only the function $d_f(x)$ is 1-Lipschitz?

5.5 On Extremality of Warped Products of Manifolds with Boundaries and with Corners.

We explained in section 4.4 how reflection+ smoothing allows an extension of the Llarull and Goette-Semmelmann theorems from section 4.2 to manifolds with smooth boundaries and to a class of manifolds with corners. This, combined with the above, enlarges the class of manifolds with corners to which the conclusion of the extremality $\blacklozenge_{\angle ij}$ *theorem* applies.

Here is an example.

Let $\triangle^{n-1} \subset S^{n-1}$ be the regular spherical simplex with flat faces and the dihedral angles $\frac{\pi}{2}$ and let $\mathsf{S}^*_* \triangle^{n-1} \subset S^n \subset S^{n-1}$ be the spherical suspension of $\triangle^{n-1}$ and let $\underline{X} = \mathsf{S}^b_a(\triangle^{n-1}) \subset \mathsf{S}^*_* \triangle^{n-1}$, $a, b \in (-\frac{\pi}{2}, \frac{\pi}{2})$, be the region of $\mathsf{S}^*_* \triangle^{\mathsf{n}-1}$ between a pair of $(n-1)$-spheres concentric to our equatorial $S^{n-1} \subset S^n$.

Let X be an n-dimensional orientable Riemannin spin manifold with corners and let $f: X \to \underline{X}$ be a smooth 1-Lipschitz map which respects to the corner structure and which has non-zero degree.

Spherical $\mathsf{S}^b_a(\triangle)$-Inequality. If $Sc(X) \geq Sc(\underline{X}) = n(n-1)$, if all $(n-1)$-faces $F_i \subset \partial X$ have their mean curvatures bounded from below by those of the corresponding faces in $\underline{X}$, [102]

$$mean.curv(F_i) \geq mean.curv(\underline{F}_i),$$

[102] All these but two have zero mean curvatures.

and if all dihedral angle of X are bounded by the corresponding ones of $\underline{X}$,

$$\angle_{ij} \leq \underline{\angle}_{ij} = \frac{\pi}{2},$$

then

$$Sc(X) = n(n-1),$$

$$mean.curv(F_i) = mean.curv(\underline{F}_i)$$

and

$$\angle_{ij} = \frac{\pi}{2}.$$

Exercise. Formulate and prove the Euclidean and the hyperbolic versions of the $\mathsf{S}_a^b(\triangle)$-inequality.

Question. Do the counterparts to the $\mathsf{S}_a^b(\triangle)$-inequality hold for other simplices and polyhedra?

5.6 Disconcerting Problem with Boundaries of non-Spin Manifolds

Typically, μ-bubbles serve as well if not better than Dirac operators for manifolds with boundaries, but something goes wrong with a natural (naive?) approach to geometric bounds on $Y = \partial X$, where $Sc(X) \geq 0$ and $mean.curv(Y) \geq M > 0$, via μ-bubbles for non-spin manifolds X.

Albeit the existence and regularity theorems from section 5.1 extend to manifolds with boundaries, the second variation formula turns out a disappointment.

To see what happens, let X be a compact Riemannian manifold with a boundary Y, let $\mu(y)$ be a continuous function $\mu : Y \to (-1, 1)$ and let $\mathcal{Z}$ be the set of cooriented hypersurfaces $Z \subset X$ with boundaries $\Omega = \partial Z \subset Y = \partial X$, where the (unit normal) field ν , which defines the coorientation is called the *upward* field.

Then such a Z is called a *μ-bubble* (compare 5.1), if it is extremal for

$$Z \mapsto vol_{n-1}^{[-\mu]}(Z) =_{def} vol_{n-1}(Z) - \int_{Y_-} \mu(y) dy,$$

in the class $\mathcal{Z}$, where $Y_- \subset Y$ the region in Y "below" $\Omega = \partial Z \subset Y$ and where our direction/coorientation/sign/angle convention is dictated by the following.

Encouraging Example. Let $X = B^n \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ be the unit ball, $Y = \partial B^n = S^{n-1}$ and let $Z_\theta = Z_\theta^{n-1} \subset B^n$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, where θ is the *latitude parameter* on the sphere $Y = S^{n-1} \supset \partial Z_\theta$, be the horizontal discs, that are the intersections

$$Z_\theta = B^n \cap \mathbb{R}^{n-1} \times \{t\},\ t = \sin\theta \in (-1, 1) \subset \mathbb{R}$$

and where – this is a matter of convention– the latitude parameter $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is related to the dihedral angle between the hypersurfaces Z_θ and Y along their intersection

$$\Omega_\theta = \partial Z_\theta = Z_\theta \cap Y,\ Y = \partial X = S^{n-1},\ \text{for } X = B^n,$$

by

$$\angle_\theta = \angle_{\Omega_\theta}(Z_\theta, Y) = \theta + \frac{\pi}{2}.$$

Next, let $\mu(y)$ for $y = (x,t) \in Y \subset \mathbb{R}^{n-1} \times \mathbb{R}$ be equal to the minus height t, i.e. $\mu(x,\theta) = -t = -\sin\theta$.

Then the *normal derivative* $\partial_\nu = \frac{d}{dt}$ of the volume of $Z_t = Z_{\sin\theta}$ is expressed in terms of

$$|\Omega_t| = vol_{n-2}(\Omega_t),\ t = \sin\theta, \text{ and the angle } \angle_\theta \in (0,\pi)$$

as follows

$$\partial_\nu vol_{n-1}(Z_t) = -|\Omega_t|\tan\theta = |\Omega_t|\cot\angle_\theta \text{ for } \theta = \arcsin t,$$

while the derivative of the μ-measure of the region $(Y_-)_t \subset Y$ below Z_t for the above $\mu(\theta) = -\sin\theta = -t = cos\angle_\theta$ is

$$\partial_\nu \mu((Y_-)_t) = \frac{|\Omega_t|\mu(t)}{\sin\angle_\theta} = |\Omega_t|\cot\angle_\theta.$$

Thus, the hypersurfaces $Z_\theta \subset X$ serve as μ-bubbles for this μ, and since they come in a "parallel" family they are *locally minimizing* ones.

Let us return to the general Riemannin manifold X with boundary $Y = \partial X$, a hypersurface $Z \subset X$, such that $\Omega = \partial Z \subset Y = \partial X$ and a function $\mu(y)$ on Y and observe the following.

First Variation Formula for $vol_{n-1}^{[-\mu]}(Z)$. . Let $\angle_\omega \in (0,\pi)$ denote the angle between Z and Y at $\omega \in \Omega = \partial Z = Z \cap Y$ and let us use the following abbreviations

$$\csc_\omega = \frac{1}{\sin\angle_\omega} \text{ and } \cos_\omega = \cos\angle_\omega.$$

Then

$$\partial_\nu vol_{n-1}(Z) = \int_Z mean.curv(Z,z)dz + \int_\Omega scs_\omega cos_\omega d\omega,$$

and

$$\partial_\nu \mu(Y_-) = \int_\Omega \csc_\omega \mu(\omega)d\omega.$$

and, since $vol_{n-1}^{[-\mu]}(Z) = vol_{n-1}(Z) - \mu(Y_-)$,

Z is a (stationary) μ-bubble, i.e. $\partial_{\psi\nu} vol_{n-1}^{[-\mu]}(Z) = 0$ for all smooth functions $\psi(z)$, if and only if

$$mean.curv(Z) = 0 \text{ and } \mu(\omega) = \cos_\omega.$$

Second Variation Formula for $vol_{n-1}^{[-\mu]}(Z)$. If Z is stationary then the ω contribution to the second variation/derivative $\partial^2_{\psi\nu} vol_{n-1}^{[-\mu]}(Z)$ is as follows

$$\partial_{\psi\nu}\int_\Omega \psi(\omega)(scs_\omega cos_\omega d\omega - \csc_\omega \mu(\omega))d\omega = \int_\Omega \psi^2(\omega)(-\csc_\omega(\varrho(\omega) - \partial_\nu\mu(\omega)))d\omega,$$

where $\varrho(\omega)$ is the curvature of $Y \subset Z$, i.e. the value of the second fundamental form of $Y \subset X$, on the unit tangent vector $\tau \in T_\omega(Y)$ normal to $T_\omega(\Omega) \subset T_\omega(Y)$.

(Our sign convention is such that this ϱ is positive for convex $Y = \partial X$ and negative for concave ones.)

Let $M_Y(\omega) = M(\Omega \subset Y, \omega)$ denote the mean curvature of $\Omega \subset Y$ and observe that ϱ equals the difference between the mean curvature of $Y \subset X$ and the values of the mean curvature (second fundamental form) of Ω on the unit normal bundle of $Y \subset X$, denotes $M(\Omega, T^\perp(Y \subset X))$,

$$\varrho = M(Y \subset X) - M(\Omega, T^\perp(Y \subset X)),$$

and that

$$M(\Omega, T^\perp(Y \subset X)) = \csc \cdot M(\Omega \subset Z) + \cos \cdot \csc \cdot M(\Omega \subset Y).$$

The essential problem, as I see it here, is that the mean curvature $M(\Omega \subset Y = \partial X)$ may (may not?) be uncontrollably ±large and, unless $\mu = 0$, the positivity of the second variation operator doesn't yield a significant information on the intrinsic geometry of $Z \subset X$ at the boundary $\Omega = \partial Z$. (Am I missing something obvious?)

5.7 On Rigidity of Extremal Warped Products.

Let us explain, as a matter of example, that

doubly punctured sphere $\underline{X} = S^n \setminus \{\pm s\}$ *is rigid.*

This means (see (B) in $\odot_{S^n}$ of section 5.4) that

if an oriented Riemannin spin n-manifold X with $Sc(X) \geq n(n-1) = Sc(\underline{X} = Sc(S^n)$ admits a smooth proper 1-Lipschitz map $f : X \to \underline{X}$ such that $deg(f) \neq 0$, then, in fact, such an f is an isometry.

Proof. We know (see the the proof of $\odot_{S^n}$) that X contains a minimal μ-bubble Y, which separates the two (union of) ends of X, where $\mu(x)$ is the f-pullback of the mean curvature function of the concentric $(n-1)$-spheres in $\underline{X} = S^n \setminus \{\pm s\}$ between the two punctures and that this m-bubble must be umbilic, where we assume at this point that Y is non-singular, e.g. $n \leq 7$.

What we want to prove now is that these bubbles *foliate all of* X, namely they come in a continuous family of mutually disjoint minimal μ-bubbles Y_t, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which together cover X.

Indeed, if the *maximal* such family Y_t wouldn't cover X, then the would exists a small perturbation $\mu'(x)$ of $\mu(x)$ in the gap between two Y_t in the maximal family, such that $|\mu'| > |\mu|$ in this gap, while $||d\mu'|| = ||d\mu||$ in there and such that there would exist a minimal μ'-bubble Y' in this gap.

But then, by calculation in section 5.4, the resulting warped product metric on $Y' \times S^1$ would be $> n(n-1)$, thus proving "no gap property" by contradiction.

Therefore, X itself is the warped product, $X = Y \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with the metric $dt^2 = (\sin t)^2 g_Y$, where $Sc(g_Y) = n(n-1)$ and which by Llarull's rigidity theorem, has constant sectional curvature. QED.

Remarks (a) On the positive side, this argument is quite robust, which makes it compatible with approximation of bubble and metrics. For instance it nicely works for $n = 8$ in conjunction with Smale's generic regularity theorem and, probably, for all n with Lohkamp's smoothing theorem.

But it is not quite clear how to make this work for non-smooth limits of smooth metrics.

For instance (this was already formulated in section 3.2),

let g_i be a sequence of Riemannian metrics on the torus $\mathbb{T}^n$, such that

$$Sc(g_i) \geq -\varepsilon_i \underset{i\to\infty}{\to} 0$$

and such that g_i uniformly converge to a continuous metric g.

Is this g, say for $n \leq 7$, Riemannian flat?

(The above argument shows that, given an indivisible $(n-1)$-homology class in $\mathbb{T}^n$, there exists a foliation of $\mathbb{T}^n$ by g-minimal submanifolds from this class. But it is not immediately clear how to show that these submanifolds are totally geodesic.)

6 Problems, Generalisations, Speculations.

The most tantalizing aspect of scalar curvature is that it serves as a meeting point between two different branches of analysis: the index theory and the geometric measure theory,

Each of the this theories, has its own domain of applicability to the scalar curvature problems (summarized in the section 6.1 below) with a significant overlaps and distinctions between the two domains.

This suggests, on the one hand,

a possible unification of these two theories

and, on the other hand,

a radical generalization, or several such generalizations,

of the concept of a space with the scalar curvature bounded from below.

This is a dream. In what follows, we indicate what seems realistic, something lying within the reach of the currently used techniques and ideas.

6.1 Dirac Operators versus Minimal Hypersurfaces

Let us briefly outline the relative borders of the domains of applicability of the two methods.

1. Spin/non-Spin. There is no single instance of *topological obstruction* for a metric with $Sc > 0$ on a closed manifold X, the *universal coverings* $\tilde{X}$ of which is *non-spin*[103] that is obtainable by the (known) Dirac operator methods.[104]

But the minimal hypersurface method delivers such obstructions for a class manifolds X, which admits continuous maps f to *aspherical spaces* $\underline{X}$, such that such an f doesn't annihilate the fundamental class $[X] \in H_n(\underline{X}, n = dim(X)$, i.e. where the image $f_*[X] \in H_n(\underline{X}$ doesn't vanish. *Example.* The connected sum $X = \mathbb{T}^n \# \Sigma$, where Σ is a simply connected non-spin manifold are instance of such X with the universal coverings $\tilde{X}$ being non-spin.)

2. Homotopy/Smooth Invariants. The minimal hypersurface method alone can only deliver *homotopy theoretic* obstructions for the existence of metrics with $Sc > 0$ on X.

[103] The condition "X is spin" can be weakened to "$\tilde{X}$ is spin" with (a version of) the Atiyah L_2-index theorem from [Atiyah(L_2) 1976], as it is explained in §§$9\frac{1}{9}, 9\frac{1}{8}$ of [G(positive) 1996].

[104] Never mind Seiberg-Witten equation for $n = 4$

But $\hat{\alpha}(X)$, non-vanishing of which obstructs $Sc > 0$ according to the results by Lichnerowicz and Hitchin proven with *untwisted* Dirac operators is not homotopy invariant. (Non-vanishing of $\hat{\alpha}$ is *the only* obstruction for $Sc > 0$ for simply connected manifolds of dimension ≥ 5, see section 3.4.)

Here, observe, the spin condition is essential, but when it comes to twisted Dirac operators, those obstructions for the existence of metrics with $Sc > 0$, which are *essentially due to twisting* are also *homotopy invariant*, and, for all we know, the spin condition is redundant there.

Furthermore, minimal hypersurfaces can be applied together with that Dirac operators.

For example the product manifold $X = X_1 \times X_2$, where $\hat{\alpha}(X_1) \neq 0$ and $X_2 = \mathbb{T}^n \# \Sigma$, doesn't carry metrics with $Sc > -0$, which for $dim(X) \leq 8$ follows from Schoen-Yau's [SY(structure) 1979] (with a use Nathan Smale's generic non-singularity theorem for $n = 8$), while the general case needs Lohkamp's [Loh(smoothing) 2018].

Notice that the twisted Dirac operator method also applies to these, $X = X_1 \times X_2$, provided that Σ is spin, or at least, the universal covering $\tilde{\Sigma}$ is spin.

3. SYS-Manifolds. The most challenging for the Dirac operator methods is Schoen-Yau's proof of non-existence of metrics with $Sc > 0$ on Schoen-Yau-Schick manifolds (see section 2.7), where the known Dirac operator methods, even in the spin case, don't apply.

And as far as the topological non-existence theorems go, the minimal hypersurface method remains silent on the issue of metrics with $Sc > 0$ on quasisymplectic manifolds X as in section 2.7, (e.g. closed aspherical 4-manifolds X with $H^2(X;\mathbb{Q}) \neq 0$.) And we can't rule out metrics with $Sc > 0$ on the connected sums $X \# \Sigma$ with any one of the present day methods, if the universal coverings $\tilde{\Sigma}$ are non-spin.

4. Area Inequalities. The main advantage of the twisted Dirac over minimal hypersurfaces is that geometric application of the latter to $Sc > 0$ depend on lower bounds on the sizes of Riemannin manifolds X, where these sizes are expressed in terms of the *distance functions* on X, while the twisted Dirac relies on the *area-wise lower bounds* on X.

The simplest (very rough) result in this regard says that every (possibly non-spin) smooth manifold X admits a Riemannin metric g_0, such that every *complete*[105] metric g on X, for which

$$area_g(S) \geq area_{g_0}(S)$$

for all smooth surfaces $S \subset X$, satisfies:

$$\inf_{x \in X} Sc(g, x) \leq 0$$

(see section 11 in [G(101) 2017]). More interestingly, there are better, some of them sharp, bounds on the area-wise size of manifolds with $Sc \geq \sigma > 0$, as we saw in section 4.

5. Scalar Curvature in Families. Individual index formulas typically (always?) extends to families of operators and deliver harmonic spinors on

[105] "Complete" is essential as it is seen already for $dim(X) = 2$. But if $area_g(S) \geq area_{g_0}(S)$ is strengthened to $g \geq g_0$ one can drop "complete", where the available proof goes via minimal hypersurfaces and where there is a realistic possibility of a Dirac operator proof as well.

members of appropriate families. But there is no (apparent?) counterpart of this for minimal hypersurfaces and/or for stable μ-bubbles.

Thus, for instance, the distance-kind inequalities indicated in section 4.2.1 as well topological and geometric obstruction for $Sc > \sigma$ on foliations escape the embrace of minimal hypersurfaces.

6. Non-Completeness and Boundaries. The major drawback of the Dirac operator methods is its essential reliance on completeness of manifolds X it applies to,[106] while minimal hypersurfaces and especially stable μ-bubbles in conjunction with twisted Dirac operators, fare well in non-complete manifolds and/or in manifolds with boundaries as it is demonstrated in section 5 of this paper.

7. $Sc \geq \sigma$ for $\sigma < 0$. Both methods have more limited applications here than for $\sigma \geq 0$, where the most impressive performance of the Dirac operator is in the proof of the Ono-Davaux spectral inequality (stated in section 3. 10), which also may be seen from a more geometric perspective of stable μ-bubbles, as it is suggested by the *Maz'ya-Cheeger inequality.* j

6.2 Logic of Propositions about the Scalar Curvature

Propositions/properties $\mathcal{P}Sc$ concerning the scalar curvatures of Riemannin manifolds or related invariants, makes a kind of an "algebra" , where pairs of $\mathcal{P}Sc$ concerning X and Y, can be coupled to corresponding propositions, let them be only conjectural, concerning

the *Riemannian products* $X \times Y$.

Then these hybridised propositions can be developed/generalized to statements on

fibrations over Y with X-like fibers

and then further to

foliations with X-leaves, where a properly understood (non-commutative?) space of leaves is taken for Y.

Conjectural Example: Lichnerowicz × Llarull × Min-Oo. Let $\underline{X}$ be the product of the the hyperbolic space by the unit sphere,

$$\underline{X} = \mathbf{H}^n \times S^n.$$

Let X be a complete orientable spin Riemannian manifold, such that $Sc(X) \geq 0$. Let $f : X \to \underline{X}$ be a smooth proper map with the following two properties.

$\bullet_{S^n}$ The S^n-component $f_{S^n} : X \to S^n$ of f, that is the composition of f with the projection $\underline{X} = \mathbf{H}^n \times S^n \to S^n$, is an *area contracting*, e.g. 1-*Lipschitz* map.

$\bullet_{\mathbf{H}^n}$ The $\mathbf{H}^n$-component of f is a *Riemannian submersion at infinity*:

the map $f_{\mathbf{H}^n} : X \to \mathbf{H}^n$ is a *submersion* outside a compact subset in X, where the differential $df_{\mathbf{H}^n} : T(X) \to T(\mathbf{H}^n)$ is *isometric* on the orthogonal complement to the kernel of $df_{\mathbf{H}^n}$.

Then either $Sc(X) = 0$, or the $\hat{A}$-genera of the pullbacks $f^{-1}(\underline{x}) \subset X$ of generic points $\underline{x} \in \underline{X}$ vanish.

In particular, if $dim(X) = 2n$ and $Sc(X, x_0) > 0$ at some $x_0 \in X$, then $deg(f) = 0$.

[106]We tried to alleviate this in section 4.6, but the situation remains unsatisfactory.

Suggestion to the Reader. Hybridize in a similar way other theorems/inequalities from the previous as well as of the following sections.

6.3 Almost flat Fibrations, K-waist and *max*-Scalar Curvature

Let let P and Q be Riemannian manifolds, let $F : P \to Q$ be a smooth fibration. and let $\underline{\nabla}$ be the connection defined by the *horizontal tangent (sub) bundle* on P that is the orthogonal complement to the *vertical* subbundle of $T(P)$, where "vertical" means "tangent to the fibers" called $S_q = F^{-1}(q) \subset P$, $q \in Q$.

Problem. Find relations between the K-waists$_2$ and between *max*-scalar curvatures of P, Q and the fibers $F^{-1}(q)$ for fibrations with "small" curvatures $|curv|(\underline{\nabla})$.[107]

We already know in this regard the following

(A) If $P \to Q$ is a *unitary vector bundle with a non-trivial Chern number*, then, by its very definition, *K-waist*$_2(Q)$ *is bounded from below* by $\frac{const_n}{|curv|(\underline{\nabla})}$.

(B) There is a fair bound on $Sc^{\max}$ of product spaces $P = Q \times S$, such as the rectangular solids, for instance, as is shown by methods of minimal hypersurfaces and of stable μ-bubbles in section 5.4.

In what follows, we say a few words about (A) for non-unitary bundles in the next section and then turn to several extensions of (B) to non-trivial fibrations.

6.3.1 Unitarization of Flat and Almost Flat Bundles.

Let Q be a closed oriented manifold and start with the case where $L \to Q$ is a *flat* vector bundle with a structure group G, e.g. the orthogonal group $O(N_1, N_2)$.

Let some characteristic number of L be non-zero, which means that the classifying map $f : Q \to \mathsf{B}(G)$ sends the fundamental class $[P]_{\mathbb{Q}}$ to a non-zero element in $H_n(\mathsf{B}(G); \mathbb{Q})$.[108]

Then X admits no metric with $Sc > 0$.

First Proof. Let $\Gamma \subset G$ be the monodromy group of L and recall (see section 4.1.2) that Γ properly and discretely acts on a product $\underline{X}$ of Bruhat-Tits building. Since this $\underline{X}$ is $CAT(0)$ and $Sc(P) > 0$, the homology homomorphism $H_n(P; \mathbb{Q}) \to H_n(\mathsf{B}(\Gamma); \mathbb{Q})$ induced by the classifying map $f_\Gamma : P \to \mathsf{B}\Gamma$ is zero (see section 4.1.2).

Since the classifying map $f : Q \to \mathsf{B}(G)$ factors through $f_\Gamma : P \to \mathsf{B}\Gamma$ via the embedding $\Gamma \hookrightarrow G$, the homomorphism $H_n(P; \mathbb{Q}) \to H_n(\mathsf{B}(G); \mathbb{Q})$ is zero as well and the proof follows.

Second Proof? Let $K \subset G$ be the maximal compact subgroup and let S be the quotient space, $S = G/K$ endowed with a G-invariant Riemannin metric.

[107] Recall that the K-waists$_2$ defined in section 4.1.4 measure area-wise sizes of spaces, e.g. K-$waist_2(S) = area(S)$ for simply connected surfaces and K-$waist_2(S^n) = 4\pi$, while *max*-scalar curvature of a metric space P defined in section 5.3.1 P is the supremum of scalar curvatures of Riemannin manifolds X that are in a certain sense are greater than P.

[108] If G is compact, or if $G = GL_N(C)$, then $H_n(\mathsf{B}(G); \mathbb{Q})$, then the homology homomorphism $f_* : H_i(Q, \mathbb{Q}) \to H_i(\mathsf{B}(G); \mathbb{Q})$, $i > 0$, for *flat* bundles L, but it is not so, for instance, if $G = O(N_1, N_2)$ with $N_1, N_2 > 0$.

Let $\mathcal{S}_*$ be the space of L_2-spinors on S twisted with some bundle $L_* \to S$ associated with the tangent bundle of S and let $\mathscr{S}_* \to Q$ be the corresponding Hilbert bundle over Q with the fiber $\mathcal{S}_*$.

Apparently, an argument by Kasparov (see below) implies that, at least under favorable conditions on G, a certain generalized *index of the Dirac operator* on Q twisted with $\mathscr{S}_* \to Q$ is *non-zero*; hence, Q carries a *non-zero harmonic* (possibly almost harmonic) *spinor* and the proof follows by revoking the Schroedinger-Lichnerowicz-Weitzenboeck formula.

Kasparov KK-Construction. Let G be semisimple, and observe that the quotient space $S = G/K$ carries a G-invariant metric with non-positive sectional curvature.

Take a point $s_0 \in S$ and let $\tau_0(s) = \tau_{s_0}(s)$ be the gradient of the distance function $s \mapsto dist(s, s_0)$ on S regularized at r_0 by smoothly interpolating between $r \mapsto dist(s, s_0)^2$ in a small ball around s_0 with $dist(s, s_0)$ outside such a ball.

Let $\tau_0^\bullet : \mathcal{S}_* \to \mathcal{S}_*$ be the Clifford multiplication by $\tau_0(r)$, that is $\tau_0^\bullet : s \mapsto \tau_0(r) \bullet s$, $s \in \mathcal{S}_*$.

Discreetness Assumption. Let the monodromy subgroup $\Gamma \subset G$ be discrete and let us restrict the space $\mathcal{S}_*$ and the operator $\tau_0^\bullet$ to a Γ orbit $\Gamma(s) \subset S$ for a point $r \in R$ different from r_0

Then, according to an *observation by Mishchenko* [Mishch 1974] the resulting operator on the space of spinors restricted to $\Gamma(s)$,

$$\tau^\bullet_{s_0,\Gamma} = \tau^\bullet_{s_0|\Gamma(s)} : \mathcal{S}_{*|\Gamma(s)} \to \mathcal{S}_{*|\Gamma(s)},$$

has the following properties:

$(\star)$ *$\tau^\bullet_{s_0,\Gamma}$ is Fredholm;*

$(\star\star)$ *$\tau^\bullet_{s_0,\Gamma}$ commutes with the action of Γ modulo compact operators in the following sense: the operators*

$$\tau^\bullet_{\gamma(s_0),\Gamma} - \tau^\bullet_{s_0,\Gamma} : \mathcal{S}_{*|\Gamma(s)} \to \mathcal{S}_{*|\Gamma(s)}$$

are compact for all $r \notin \Gamma(s_0)$ and all $\gamma \in \Gamma$.

These properties and the contractibility of S, show, by an elementary extension by skeleta argument [Mishch 1974], that

$(\star\star\star)$ *the (graded) Hilbert bundle $\mathcal{S}_{*|\Gamma} \to Q$ admits a Fredholm endomorphism homotopically compatible with $\tau^\bullet_{s_0,\Gamma}$.*

Finally, a *K-theoretic index computation* in [Kasp 1973], [Kasp 1975] and/or in [[Mishch 1974] yields

$(\star\star\star\star)$ *non-vanishing of the index of the Dirac operator on Q twisted with $\mathscr{S}_{*|\Gamma}$* in relevant cases (which delivers non-zero harmonic spinors on Q and the issuing $Sc(Q) \not>)$ conclusion in our case). [109]

[109] The properties $(\star)$ and $(\star\star)$, however simple, establish the key link between geometry and the index theory. These were discovered and used by Mishchenko in the ambience of the Novikov higher signatures conjecture and the Hodge, rather than the Dirac operator, on general manifolds with non-positive curvatures.

It seems, no essentially new geometry-analysis connection has be been discover since, while $(\star\star\star\star)$ grew into a fast field of the KK-theory of C^*-algebras in the realm of the non-commutative geometry.

Now, let us *drop the discreetness assumption* and make the above (Γ-equivariant) construction(s) fully G-equivariant.

The (unrestricted to an orbit $\Gamma(s) \subset S$) operator $\tau_0^\bullet : \mathcal{S}_* \to \mathcal{S}_*$ seems at the first sight no good for tis purpose:

the properties $(\star)$ and $(\star\star)$ fails to be true for it, since the space $\mathcal{S}_*$ of L_2-spinors on S is too large and "flabby".

On the positive side, the space $\mathcal{S}_*$ may contain a G-invariant subspace, roughly as large as $\mathcal{S}_{*|\Gamma}$, namely the subspace of *harmonic* spinors in it. But the operator $\tau_0^\bullet$ doesn't, not even approximately, keeps this space invariant. However – this is an idea of Kasparov, I presume, – one can go around this problem by invoking the full Dirac operator $\mathcal{D} : \mathcal{S}_* \to \mathcal{S}_*$, rather than its kernel alone.

Namely, we add the following extra structure to $\mathcal{S}_*$:

(A) the action of the Dirac operator $\mathcal{D}$ or rather of the technically more convenient first order operator

$$\mathcal{E} = \mathcal{D}(1 - \mathcal{D}^2)^{\frac{1}{2}} : \mathcal{S}_* \to \mathcal{S}_*$$

:

(B) the action of continuous functions ϕ with compact supports in S.

These functions $\phi(s)$ act on spinors by multiplication, where this action, besides *commuting with the action by* G,

commute with $\mathcal{E}$ *modulo compact operators.*

Now, because of (A) and(B), a suitably generalized index theorem applies, I guess, and, under suitable topological conditions, yields non-zero (almost) harmonic spinors on Q.[110]

Problem. Does the above (assuming it is correct) generalises to non-flat bundles $L \to Q$?

Namely,

is there a natural Hilbert bundle $\mathscr{S} \to Q$ associated with L and having its curvature bounded in terms of that of L and such that $\mathscr{S}$ carries an additional structure, such as a (graded) Fredholm endomorphism, that would yield, under some topological conditions, *non-zero harmonic* (or almost harmonic) $\mathscr{S}$-twisted spinors on Q via a suitable index theorem?[111]

Generalized Problem. Does the above generalizes further to fibrations with variable fibers with nonpositive curvatures?

Namely, let $F : P \to Q$ be a smooth fibrations between complete Riemannin manifolds, where the fibers $S_q = f^{-1}(q) \subset P$ are simply connected and the induced metrics in which have non-positive sectional curvatures.

Let a connection in this fibration be given by a horizontal subbundle $T^{hor} \subset T(P)$, that is the orthogonal complement to the vertical bundle – the kernel of the differential $dF : T(P) \to T(Q)$.

[110] I couldn't find any explicit statement of this kind in the literature, but it must be buried somewhere under several layers of KK-theoretic formalism, which fills pages of the books and articles I looked into.

(In my article [G(positive) 1996]), §$8\frac{1}{2}$, I mistakenly use a simplified argument of composing $\tau_0^\bullet$ with a projection on $ker(\mathcal{D})$)

[111] "Almost flat" generalizations of the "flat" Lusztig signature theorem are given in §§$8\frac{3}{4}, 8\frac{8}{9}$ of [G(positive) 1996].

Let $[q,q'] \subset Q$ be a (short) geodesic segment between $q,q' \in Q$ and let $[p,p']^\sim \subset P$ be a horizontal lift of $[q,q']$.

We don't assume that the holonomy transformations $S_q \to R_{q'}$ are isometric and let

(1) $maxdil_p(\varepsilon)$ be the supremum of the norm of the differentials of the transformations $S_q \to S_{q'}$ at $p \in S_q$ for all horizontal path $[p,p']^\sim \subset P$ of length $\leq \varepsilon$ issuing from $p \in P$;

and

(2) $maxhol_p(\varepsilon,\delta)$ be the supremum of $dist(p,p')$ for all horizontal paths $[p,p']^\sim$ of length$\leq \varepsilon$, where p' lies in the fiber of p, i.e. $F(p') = F(p) = q$ and where there is a smooth surface $S \subset P$ the boundary of which is contained in the union of the path $[p,p']^\sim$ and the fiber F_q which contains p and p' and such that $area((S) \leq \delta^2$.

Can one bound $\inf_q Sc(Q,q)$, or, more generally, $max\text{-}Sc(Q)$ in terms of bounds on the functions $\log maxdil_p(\varepsilon)$ and $maxhol_p(\varepsilon,\delta)$, for all (small) $\varepsilon,\delta > 0$ and all $p \in P$?

6.3.2 Comparison between Hyperspherical Radii and K-waists of Fibered Spaces.

A. The methods of minimal hypersurfaces and of stable μ-bubbles from section 5.3 that deliver fair bounds on $Sc^{\max}$ of product spaces P, such as the rectangular solids, for instance, dramatically fail (unless I miss something obvious) for fibrations with *non-flat connections* because of the following.

Distortion Phenomenon. What may happen, even for (the total spaces of) *unit m-sphere bundles* P with orthogonal connections $\underline{\nabla}$ over closed Riemannin manifolds Q, where *the hyperspherical radius is large, and the curvature is small*, say

$$Rad_{S^n}(Q) = 1,\ n = dim(Q), \text{ and } |curv|(\underline{\nabla}) \leq \varepsilon,$$

is that, at the same time,

$$Rad_{S^{m+n}}(P) \leq \delta,\ m+n = dim(P),$$

where $\varepsilon > 0$ and $\delta > 0$ can be *arbitrarily small.*[112]

This possibility is due to the fact that, in general, P admits *no Lipschitz controlled retractions* to the spherical fibers of our fibration, even if the fibration is topologically trivial and continuous retractions (with uncontrollably large Lipschitz constants) do exits, where

non-triviality of monodromy, say at $q \in Q$ can make the distance function $dist_P$ on the fiber $S^m_q \subset P$ *significantly smaller* than the (intrinsic) spherical metric.

Example. Let Q be obtained from the unit sphere S^2 by adding ε-small handles at finitely many points which are together ε-dense in S^2 and such that Q goes to S^2 by a 1-Lipshitz map of degree one.[113]

Let $P \to Q$ be a topologically trivial flat unit circle bundle, such that the monodromy rotations $\alpha \in \mathbb{T}^1$ of he fiber $S_q = S^1$ around the loops at $q \in Q$ of length $\leq \delta$ are δ-dense in the group $\mathbb{T}^1$ for all $q \in Q$.

[112]This doesn't happen if the action of the structure group on the fiber of our fibration has bounded displacement, see (2) in section 6.3.7.

[113]E.g. let the handles lie outside (the ball bounded by) the sphere $S^2 \subset \mathbb{R}^3$ and let our map be the normal projection $Q \to S^2$.

Then, clearly, $Rad_{S^3}(P) \leq 10\delta$, where δ can be made arbitrarily small for $\varepsilon \to 0$, whilst the trivial fibration has large hyperspherical radius, namely, $Rad_{S^3}(Q \times S^1) = 1$.

B. Metric distortion of the fibers of the fibration $P \to Q$ has, however, little effect on the K-waist of P, that can be used, instead of the hyperspherical radius, as a measure of the size of P and that allows non-trivial bounds on $Sc^{\mathsf{max}}(P)$ for *spin* manifolds P with a use of twisted Dirac operators.

In practice, to make this work, one needs vector bundles with unitary connections over the base Q and over the manifold S isometric to the fibers $S_q \subset P$, call these bundles $L_Q \to Q$ and $L_S \to S = S_q$, where the following properties of these bundles are essential.

$\bullet_{\mathrm{I}}$ *Monodromy Invariance of* L_S. The bundle $L_S \to S$, where S is isometric to the fibers S_q of the fibration $P \to Q$, must be *equivariant* under the action of the monodromy group G of the connection $\underline{\nabla}$ on the fibers S_q of the fibration $P \to Q$.

(Recall that an equivariance structure on a bundle L over a G space S is an *equivariant lift* of the action of G on S to an action of G on L.)

If a bundle $L_S \to L$ is G-equivariant, it extends fiberwise to a bundle over P, call it $L_{\updownarrow} \to P$.

(An archetypical example of this is the tangent bundle $T(S)$ which extends to what is called call the vertical tangent bundle for all fibration with S-fibers. But, in general, actions of groups G on S do not lift to vector bundles $L \to S$. However, such lifts may become possible for suitably modified spaces S and/or bundles over them.)

$\bullet_{\mathrm{II}}$ *Homologically Substantiality of the two Vector Bundles.* Some Chern numbers. of the bundles L_S and L_Q must be *non-zero.*

$\bullet_{\mathrm{III}}$ *Non-vanishing of* $F^*[Q]^\circ_{\mathbb{Q}} \in H^n(P;\mathbb{Q})$. The image of the fundamental cohomology class $[Q]^\circ \in H^n(Q)$, $n = dim(Q)$, under the rational cohomology homomorphism induced by $F : P \to Q$ doesn't vanish,

$$F^*[Q]^\circ \neq 0.$$

(This is satisfied, for instance, if the fibration $P \to Q$ admits a section $Q \to P$.)

Granted $\bullet_{\mathrm{I}}$-$\bullet_{\mathrm{II}}$-$\bullet_{\mathrm{III}}$, there exists a vector bundle $L_{\rtimes} \to P$, which is equal to a tensor product of exterior powers of the "vertical bundle" $L_{\updownarrow} \to P$ and $F^*(L_Q) \to P$ (that is F-pull back of L_Q) and such that a *suitable* Chern number of $L_{\rtimes}$ doesn't vanish.

Here "suitable" is what ensures *non-vanishing of the index* of the twisted Dirac operators $\mathcal{D}_{\otimes f^*(L_{\rtimes})}$ on manifolds X mapped to P by maps $f : X \to P$ with *non-zero degrees.* (Compare with $5\frac{1}{4}$ in [G(positive) 2016].)

Then bounds on curvatures of the bundles L_S and L_Q together with such a bound for $\underline{\nabla}$ and also a bound on *parallel displacement of the* G *action on* S (see below) yield a bound on $|curv|(L_{\rtimes})$, which implies a bound on $Sc^{\mathsf{max}}(P)$ according to the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula applied to the operators $\mathcal{D}_{\otimes f^*(L_{\rtimes})}$ on manifolds X mapped to P by (smoothed) *1-Lipschitz maps* $f : X \to P$, used in the definition of $Sc^{\max}(P)$.

Parallel Displacement. The geometry of a G-equivariant unitary bundle $L = (L, \nabla)$ over a Riemannian G-space S is characterized, besides the (norm of the) curvature of ∇, by the difference between the parallel transform and transformations by small $g \in G$.

To define this, fix a norm in the Lie algebra of G and let $|g|$, $g \in G$ denote the distance from g to the identity in the corresponding "left" invariant Riemannin metric in G.

Then, given a transformation $g : S \to S$ and a lift $\hat{g} : L \to L$ of it to L, compose it with the parallel translate of it back to L along shortest curves (geodesics for complete S) between all pairs $s, g(s) \in S$. Denote by $\hat{g} \div \nabla : L \to L$ the resulting endomorphism and let

$$|\hat{G} \div \nabla| = \limsup_{|g| \to 0} \frac{||(\hat{g} \div \nabla) - \mathbf{1}||}{|g|},$$

where $\mathbf{1} : L \to L$ is the identity endomorphism (operator) and $||...||$ denotes the operator norm.

Notice at this point that the curvature of the connection $\underline{\nabla}$ takes values in the Lie algebra of G and the norm $|curv|(\underline{\nabla})$, similarly to the above "parallel displacement", depends on a choice of the norm in this Lie algebra.

If S is compact, we agree to use the norm equal to the sup-norms of the corresponding vector fields on S, but one must be careful in the case of non-compact S. (Compare with (2) in section 6.3.1 and also see section 6.3.7.)

6.3.3 $Sc^{\max}$ and $Sc^{\max}_{sp}$ for Fibrations with Flat Connections

Let P and Q be closed orientable Riemannin manifolds and let us observe that what happens to the non-spin and spin *max*-scalar curvatures and of the K-waists[114] of fibrations $P \to Q$ with flat connections, follows from what we know for trivial fibrations over covering spaces $\tilde{Q} \to Q$. [115]

(A) *If the monodromy group of a flat fibration of) $F : P \to Q$ is* finite *and the map F is* 1-Lipschitz, *then*

$$\star_{waist_2} \qquad Sc^{\max}_{sp}(P) \leq const_{m+n} \cdot \max\left(\frac{1}{K\text{-}waist_2(Q)}, \frac{1}{K\text{-}wast_2(S)}\right),$$

$$\star_{Rad^2} \qquad Sc^{\max}(P) \leq const'_{m+n} \cdot \max\left(\frac{1}{Rad^2_{S^n}(Q)}, \frac{1}{Rad^2_{S^m}(S)}\right)$$

[114] K-waist$_2(P)$ is the reciprocal of the *infimum of the norms of the curvatures* of unitary bundles over P with *non-zero* Chern numbers (see section 4.1.4).

$Sc^{max}(P)$ is the *supremum* of σ, such that P admits an equidimensional *1-Lipschitz* map with *non-zero* degree from a closed Riemannian manifold X with $Sc \geq \sigma$, and where X in the definition of Sc^{max}_{sp} must be *spin* (see section 5.3.1).

The hyperspherical radius $Rad_{S^N}(P)$, $N = dimP$, the *supremum* R_{max} of *radii of the spheres* $S^N(R)$, which receive *1-Lipshitz* maps from P of *non-zero* degree, $P \to S^N$. (see section 3.5).

It is (almost) 100% obvious that $Rad_{S^N}(S^N) = 1$, it is not hard to show (see section 4.1.3) that K-waist$_2(P)$ is 4π, the equality $Sc^{max}_{sp}(S^N) = Sc(S^N) = N(N-1)$ follows from Llarull's' inequality for twisted Dirac operators and it remains unknown if $Sc^{max}_{sp}(S^N) = Sc(S^N) = N(N-1)$ for $N \geq 5$ (see section 5.4 for $N = 4$).

[115] *A flat structure (connection)* in a fibration $F : P \to Q$ with S-fibers is defined for arbitrary topological spaces Q, S and P, as a Γ-equivariant splitting $\tilde{F} : \tilde{P} = \tilde{Q} \times S \to \tilde{Q}$ for some Γ-covering $\tilde{Q} \to Q$ and the induced covering $\tilde{P} \to P$.

In the present case we assume that our Q and S, hence P, are compact *orientable pseudo-manifolds* with piecewise smooth Riemannin metrics, where $\tilde{P} = \tilde{Q} \times S$ carries the (piecewise) Riemannin product metric and the action of Γ on $\tilde{P}$ is isometric.

and

$$\star_{sp,Rad^2} \qquad Sc^{\max}_{sp}(P) \le (m+n)(m+n-1)\cdot \max\left(\frac{1}{Rad^2_{S^n}(Q)}, \frac{1}{Rad^2_{S^m}(S)}\right)$$

for $n = dim(Q)$ *and* $m = dim(S)$, where S is the fiber of our fibration $P \to Q$.

In fact, these reduce to the corresponding inequalities for the product $\tilde{P} = \tilde{Q} \times S$ for the finite(!) covering $\tilde{P}$ of P, induced from the monodromy covering $\tilde{Q} \to Q$, where

• in the case $\star_{waist_2}$, one uses the tensor product of the relevant vector bundles over $\tilde{Q}$ and S and where the $\otimes$-product bundle can be pushed forward from $\tilde{P}$ back to P, if one wishes so;

• in the case $\star_{Rad^2}$, the (obvious) inequalities

$$Rad_{S^{n+m}}(\tilde{P}) \ge Rad_{S^{n+m}}(P)$$

– the finiteness of monodromy is crucial in this one – and

$$Rad_{S^{n+m}}(\tilde{Q} \times S) \ge min(Rad_{S^n}(\tilde{Q}), Rad_{S^m}(S))$$

allows a use of the "cubical bounds" from the previous section, which need no spin condition, while the corresponding sharp inequality $\star_{sp,Rad^2}$ for spin manifolds P follows from Llarull's theorem.

(B) *If the monodromy group* Γ *of the fibration* $P \to Q$ *is* infinite, *then the above argument yields the following modifications of the inequalities* $\star_{sp,Rad^2}$, $\star_{sp,Rad^2}$ *and* $\star_{waist_2}$.

$\star^{\infty}_{Rad^2}$ The two Rad^2 inequalities $\star_{Rad^2}$ and $\star_{sp,Rad^2}$ for spin manifolds P remain valid for infinite monodromy, if $Rad_{S^n}(Q)$ *is replaced in these inequalities by* $Rad_{S^n}(\tilde{Q})$ *for a (now infinite)* Γ*-covering* $\tilde{Q}$ *of* Q.

(The universal covering of Q serves this purpose but the monodromy covering gives an a priori sharper result.)

$\star^{\infty}_{waist_2}$ *One keeps* $\star_{waist_2}$ *valid for infinite* $\underline{\nabla}$*-monodromy by replacing K-*$waist_2(Q)$ *by K-*$waist_2(\tilde{Q})$.[116]

Remarks. (a) *Sharpening the of Constants.* Our argument allows improvements of the above inequalities as we shall see, at least for $\star_{sp,Rad^2}$, in the following sections.

(b) *On Displacement and Distortion.* None of the above inequalities contains corrections terms for parallel displacement defined earlier in section 6.3, albeit it may result in a decrease of the hyperspherical radii of P due to distortion of the fibers $S \subset P$ as the example in section 6.3.2 shows.

Notice at this point that the presence of large distortion is inevitable for fibrations with non-compact fibers, where the monodromy along short loops has unbounded displacement.

Example. Let Q be a surface and $P \to Q$ an $\mathbb{R}^2$-bundle with an orthogonal connection, the curvature form of which doesn't vanish, and let g be a Riemannin metric on P which agrees with the Euclidean metrics in the $\mathbb{R}^2$-fibers and such that the map $P \to Q$ is a Riemannin fibration, i.e. it is isometric on the horizontal subbundle in $T(P)$ corresponding to the connection.

[116] It is known [BH 2009] that the hyperspherical radius can drastically decrease under infinite coverings but the situation with K-$waist_2$ remains unclear.

The the Euclidean distance between points in the fibers,

$$p_1, p_2 \in \mathbb{R}^2_q \subset P,\ q \in Q$$

is related to the g-distance in P as follows

$$dist_{\mathbb{R}^2}(p_1,p_2) \sim (dist_P(p_1,p_2))^2 \text{ for } dist_{\mathbb{R}^2}(p_1,p_2) \to \infty.$$

(This is the same phenomenon as the distortion of central subgroups in two-step nilpotent groups.)

6.3.4 Even and Odd Dimensional Sphere Bundles

$Sc^{\max}_{sp}$-Bound for Sphere Bundles. Let P and Q be closed orientable spin manifolds, where P serves as the total space of a unit m-sphere bundle $F : P \to Q$ with an orthogonal connection $\underline{\nabla}$.

If the map $F : P \to Q$ is 1-Lipschitz[117] *and if the cohomology class*

$$F^*[Q]^\circ_{\mathbb{Q}} \in H^n(P;\mathbb{Q}),\ n = dim(Q),$$

doesn't vanish (as in $\bullet_{\text{III}}$ in section 6.3.2), then the spin max-scalar curvature of P (defines with spin manifolds X mapped to P) is bounded in terms of the hyperspherical radius $R = Rad_{S^n}(Q)$ and of the norm of the curvature of $\underline{\nabla}$ as follows:

$$[\rtimes S^m] \qquad Sc^{\max}_{sp}[P] \le const \cdot (1 + \underline{\epsilon}) \cdot (Sc(S^n(R)) + Sc(S^m)),$$

where, recall, $Sc(S^n(R)) = \frac{n(n-1)}{R^2}$, $Sc(S^m) = m(m-1)$, where $const = const_{m+n}$ is a universal constant (specified later) and where *$\underline{\epsilon}$ is a certain positive function $\underline{\epsilon} = \underline{\epsilon}_{m+n}(\underline{c})$, for $\underline{c} = |curv|(\underline{\nabla})$, such that*

$$\underline{\epsilon}_{m+n}(\underline{c}) \to 0 \ for \ \underline{c} \to 0.$$

Proof. Start by observing that if either $m = 0$ or $n = 0$, then $[\rtimes S^m]$ with $const = 1$ reduces to Llarull's inequality (sections 3.5, 4.2) which says in these terms, e.g. for Q, that

$$Sc^{\max}(Q) \le \frac{n(n-1)}{Rad^2_{S^n}(Q)} = Sc(S^n(R)).$$

What we need in the general case if we want $const = 1$ is a complex vector bundle $L \to P$ with non-zero top Chern number and such that the normalised curvature (defined in section 4.1.1.) satisfies

$$|curv|_{\otimes\mathbb{S}}(L) \le \frac{Sc(S^n(R)) + Sc(S^m(1)}{4} + const' \cdot \underline{\epsilon}.$$

Now, let $m = dim(S = S^m)$ and $n = dim(Q)$ be even and observe that the non-vanishing condition $F^*[Q]^\circ_{\mathbb{Q}} \ne 0$ always holds for *even dimensional* sphere bundles.

[117]The role of this "1-Lipschitz" is seen by looking at the trivial fibrations $P = Q \times S \to Q$ and also at *Riemannian* fibrations $F : P \to Q$ (the differentials of) which are *isometric* on the horizontal (sub)bundle. In general, when the metrics in the horizontal tangent spaces may vary, estimates on $Sc^{\max}(P)$ should incorporate along with , besides $curv(\underline{\nabla})$, (a certain function of) these metrics. (Observe, that the scalar curvature of P itself is influenced by the first and second "logarithmic derivatives" of these metrics.)

Also observe that S^m and Q support bundles needed for our purpose, call them L_S and L_Q, where L_S is the positive spinor bundle $\mathbb{S}^+(S^m) \to S = S^m$ and $L_Q \to Q$ is induced from the spinor bundle $\mathbb{S}^+(S^n(R))$ by a 1-Lipschitz map $Q \to S^n(R)$ with non-zero degree.

One knows that the top Chern numbers of these bundle don't vanish and, according to Llarull's calculation,

$$|curv|_{\otimes\mathbb{S}}(L_S) = \frac{1}{4}Sc(S^m) = \frac{1}{4}m(m-1)$$

and

$$|curv|_{\otimes\mathbb{S}}(L_Q) \le \frac{1}{4}(Sc(S^n(R)) = \frac{n(n-1)}{4R^2}.$$

Since the (unitary) bundle $L_S \to S^m$ is *invariant* under the action of the spin group, that is the double covering of $SO(m)$, [118] it defines a bundle $L_\updownarrow \to P$, the curvature of which satisfies

$$|curv|(L_\updownarrow) = |curv|(L_S) + O(\underline{\epsilon}).$$

Then all one needs to show is that the tensor product of

$$L = L_\rtimes = L_\updownarrow \otimes F^*(L_Q),$$

satisfies

$$|curv|_{\otimes\mathbb{S}}(L) \le \frac{Sc(S^n(R)) + Sc(S^m(1)}{4} + const' \cdot \underline{\epsilon}.$$

This follows by a multilinear-algebraic computation similar to what goes on in the paper by Llarull, where, I admit, I didn't carefully check this computation.

But if one doesn't care for sharpness of *const*, then a direct appeal to the $\otimes_\varepsilon$-Twisting Principle formulated in section 3.11.1 suffices.

Remark. Even the non-sharp version of $[\rtimes S^m]$, unlike how it is with a non-sharp bound $Rad_{S^n}(X) \le const_n(\inf_x Sc(X,x))^{-\frac{1}{2}}$, $n = dim(X)$, *can't be proved* at the present moment *without Dirac operators*, which necessitate spin as well as compactness (sometimes completeness) of our manifolds.

Odd Dimensions. If $n = dim(Q)$ is odd, multiply P and Q by a long circle, and then either of the three arguments, used in the odd case of Llarull's theorem which are mentioned in section 4.2 and referred to [Ll 1998], [List 2010] and [G(inequalities) 2018], applies here.

Now let n be even and the dimension m of the fiber be odd. Here we multiply the fiber S, and thus P by $\mathbb{R}$, and endow the new fiber, call it $S' = S^m \times \mathbb{R}$ with the bundle $L_{S'}$ over it, which is induced by an $O(m+1)$-*equivariant* 1-Lipschitz map $S^m \times \mathbb{R} \to S^{m+1}$, which is *locally constant at infinity.* Since the curvature of the new fibration $P' = P \times \mathbb{R} \to Q$ is equal to that of the original one of $\underline{\nabla}$ in $P \to Q$, the proof follows via the relative index theorem.

Remarks/Questions. (a) Is there an alternative argument, where, instead of $\mathbb{R}$, one multiplies the fiber S with the circle $\mathbb{T}$, and uses, in the spirit of Lusztig's argument, the obvious $\mathbb{T}$-family of flat connection in it.

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[118] This bundle is *not* $SO(m)$ -invariant, but I am not certain if this is truly relevant.

(b) Is there a version of the inequality $[\rtimes S^m]$, which is sharp for $|curv|(\underline{\nabla})$ far from zero?

(c) What are $Sc^{\max}_{sp}$ of the Stiefel manifolds of orthonormal 2-frames in the Euclidean $\mathbb{R}^n$, Hermitian $\mathbb{C}^n$ and quaternion $\mathbb{H}^n$?[119]

6.3.5 K-Waist and $Sc^{\max}$ of Iterated Sphere Bundles, of Compact Lie Groups and of Fibrations with Compact Fibers.

Classical compact Lie groups are equivariantly homeomorphic to iterated sphere bundles.

For instance, $U(k)$ is equal to the complex Stiefel manifold of Hermitian orthonormal k-frames $St_k(\mathbb{C}^k)$, where $St_i(\mathbb{C}^n)$ fibers over $St_{i-1}(\mathbb{C}^n)$ with fibres $S^{2(k-i)-1}$ for all $i = 1, ..., k$.

Since the rational cohomology of $U(k)$ is the same as of the product $S^1 \times S^3 \times ... \times S^{2k-1}$, these fibrations satisfy the above non-vanishing condition $\bullet_{\mathrm{III}}$, which implies by the above $[\rtimes S^m]$ that

the product $U(k)\times\mathbb{R}^k$ carries a $U(k)$-invariant bundle, which is trivialized at infinity, such that the top Chern number of it is non-zero.

This, by the argument from the previous section, delivers

complex vector bundles with *curvature controlled* unitary connections and *non-vanishing* top Chern classes over total spaces P of principal $U(k)$-fibrations $F : P \to Q$, *provided* $F^*[Q]^\circ_{\mathbb{Q}} \neq 0$ (that is the above $\bullet_{\mathrm{III}}$).

This yields

a lower bound on the K-waist of $P \times \mathbb{T}^k$,

which, in turn, implies, the following.

Corollary 1. Let $F : P \to Q$ be a principal $U(k)$-fibration with a unitary connection $\underline{\nabla}$, where the map F is 1-Lipschitz and $F^*[Q]^\circ_{\mathbb{Q}} \neq 0$.[120]

Then

$$[\rtimes U(k)], \qquad Sc^{\max}_{sp}[P] \le const_{m+k}\cdot(1+\underline{\epsilon})\cdot\left(\frac{n(n-1)}{Rad_{S^n}(Q)^2} + const_k\right),$$

where $\underline{\epsilon}$ is a certain positive function $\underline{\epsilon} = \underline{\epsilon}_{k+n}(\underline{c})$, for $\underline{c} = |curv|(\underline{\nabla})$, such that

$$\underline{\epsilon}_{k+n}(\underline{c}) \to 0 \textit{ for } \underline{c} \to 0.$$

Now let us state and prove a similar inequality for topologically trivial fibrations with *arbitrary compact holonomy groups G.*

Corollary 2. Let S and Q be compact connected orientable Riemannian manifolds of dimensions $m = dim(S)$ and $n = dim(Q)$ and let G be a compact isometry group of S endowed with a biinvariant Riemannin metric.[121]

[119] Notice that $St_2(\mathbb{C}^2) = S^3$ and $St_2(\mathbb{H}^2) = S^7$, but not all invariant metrics on Stiefel manifolds are symmetric.

Also notice that the corresponding (Hopf) fibrations $F : P = S^3 \to Q = S^2$ and $F : P = S^7 \to Q = S^4$ have $F^*[Q]^\circ = 0$ in disagreement with the above condition $\bullet_{\mathrm{III}}$; this makes one wonder whether this condition is essential.

[120] For a principal fibration, this is a very strong condition, saying, in effect, that the fibration is "rationally trivial".

[121] If G is disconnected "Riemannin" refers to the connected components of G.

Let $F_{pr} : P_{pr} \to Q$ be a principal G-fibration with a G-connection $\underline{\nabla}$ and with a Riemannian metric on P_{pr}, which agrees with our metric on the G-fibers, for which the action of G is isometric and for which the differential of the map F_{pr} is isometric on the $\underline{\nabla}$-horisontal tangent bundle $T_{hor}(P_{pr}) \subset T(P_{pr})$.

Let $F : P \to Q$ be an associated S-fibration that is

$$P = (P_{pr} \times S)/G$$

where the quotient is taken for the diagonal action of G.

Endow P with with the Riemannin quotient metric.

$[\rtimes S_G]$ *Let* $F_{pr} : P_{pr} \to Q$ *be a topologically* (but not, in general geometrically) *trivial fibration* (i.e. $P_{pr} = Q \times G$ with the obvious action by G).

There exists a positive constant $\underline{c}_0$ *and a function* $\underline{\varepsilon} = \underline{\varepsilon}_{m+n}(\underline{c})$, $0 \leq \underline{c} \leq \underline{c}_0$, *where* $\underline{\varepsilon} \to 0$ *for* $\underline{c} \to 0$, *and such that if* $|curv|(\underline{\nabla}) = \underline{c} \leq \underline{c}_0$, *then the spin max-scalar curvature of* P *is bounded by*

$$Sc^{\max}_{sp}[P] \leq const_* \cdot (1 + \underline{\epsilon}) \cdot \left(\frac{n(n-1)}{Rad_{S^n}(Q)^2} + \frac{m(m-1)}{Rad_{S^m}(S)^2} + const_G \right).$$[122]

Proof. Embed G to a unitary group $U(k)$ and let $F_U : P_U \to Q$ be the fibration with the fiber $U = U(k)$ associated to $F_{pr} : P_{pr} \to Q$.

Let $P^U \to Q$ be the fibration with the fibers $S_q \times U_q$, $q \in Q$ and observe that this P^U fibers over P with U-fibers and over P_U with S-fibers, where the latter is a *trivial fibration.*

To show this it is enough to consider the case, where P is the principal fibration P_{pr} for which $P^U = P_{pr} \times U$ and P_U is the quotient space, $P_U = (P_{pr} \times U)/G$ for the diagonal action of G.

Then the triviality of the principal G-fibration $P^U \to P_U$ is seen with the map $P^U \to U = U(k)$ for $\{G_q \times U_q\} \mapsto U_q = U$ which sends the diagonal G-orbits from all $G_q \times U_q$ to $G \subset U(k) = U$.

Thus, assuming $m = dim(S)$ is even (the odd case is handled by multiplying by the circle as earlier) we obtain an *upper bound* on spin max-scalar curvature of $P^U = P_U \times S$ in terms of the K-waist of P_U and $Rad_{S^m}(S)$.

On the other hand, if the fibration $P \to Q$ has curvature bounded by $\underline{c}$, the same applies to the induced fibration $P^U \to P$ with U-fibers, and since the (biinvariant metric in the) unitary group $U = U(k)$ has positive scalar curvature, the max-scalar curvature of P^U is *bounded from below by one half of that for* P for all sufficiently small $\underline{c}$ and when $\underline{c} \to 0$ these estimate converge to what happens to Riemannian product $P = Q \times S$.

Confronting these upper and lower bounds yields a qualitative version of $[\rtimes S_G]$, while completing the proof of the full quantitative statement is left to the reader.

About the Constants. A Llarull's kind of computation seems to show that the above inequalities hold with $const_{m+n} = const_* = 1$.

[122] I apologise for the length of this statement that is due to so many, probably redundant, conditions needed for the proof.

6.4 K-Waist and Max-Scalar Curvature for Fibration with Non-compact Fibers.

Let $P \to Q$ be a Riemannin fibration where the fiber S is a complete contractible manifold with non-positive curvature and such that the monodromy of the natural connection $\underline{\nabla}$ in this fibration (defined by the horizontal tangent subbundle $T^{hor} \subset T(P)$) *isometrically* acts on S.

Problem. (Compare with "Generalized Problem" in section 6.3.1.) Is there a lower bound on the K-$waist_2(P)$ in terms of such a bound on K-$waist_2(Q)$ and on an upper bound on the norm of the curvature of $\underline{\nabla}$ that can be represented by the function $maxhol_p(\varepsilon, \delta)$ as in (2) of section 6.3.1?

6.4.1 Stable Harmonic Spinors and Index Theorems.

Our primarily interest in such a lower bound is that it would yield an *upper bound* on the *proper spin max-scalar* curvature of P, [123] where, following recipes $\bullet_{\mathrm{I}}, \bullet_{\mathrm{II}}, \bullet_{\mathrm{III}}$, from **B** in section 6.3.2 one has to construct a (finite or infinite dimensional graded) with a unitary connection vector bundle $\mathcal{L} \to S$, which is

$\star_{\mathrm{I}}$ *invariant (modulo compact operators?) under isometries of S* (compare with $\bullet_{\mathrm{I}}$ in section 6.3.2

and

$\star_{\mathrm{II}}$ *homologically substantial*, where this substantiality must generalize that of $\bullet_{\mathrm{II}}$ by properly *incorporating the action of the isometry group G of S*. (An inviting possibility is the above $L^{\otimes_N}$.)

What one eventually needs is not such a bundle $\mathcal{L} \to S$ per se, but rather some Hilbert space of sections for a class of related bundles over P, where

(i) a suitable *index theorem*, e.g. in the spirit of our the second "proof" in section 6.3.1 (with a *Hilbert C^*-module $\mathscr{H}$* over the *reduced C^*-algebra* of the group G being utilized),

and where

(ii) *the Schroedinger-Lichnerowicz-Weitzenboeck formula* applies to twisted harmonic L_2-spinors delivered by such a theorem and provides a bound on the scalar curvature of P.

Who is Stable? Harmonic spinors delivered by index theorems (and also spinors with a given asymptotic behaviour as in Witten's and Min-Oo's arguments) are stable under certain deformations (and some discontinuous modifications, such as surgeries) of the metrics and bundles in questions, albeit the exact range of these perturbation on non-compact manifolds is not fully understood.

But the Schroedinger-Lichnerowicz-Weitzenboeck formula doesn't use, at least not in a visible way, this stability, which is unlike how it is with stable minimal hypersurfaces and stable μ-bubbles.

One wonders, however,

whether there is a common ground for these two stabilities in our context.

6.4.2 Euclidean Fibrations

Let us indicate an elementary approach to the above *problem* in the case where the fiberes S of the fibration $F : P \to Q$ are isometric to the *Euclidean space.*

[123] This "proper spin max-scalar" is defined via proper 1-Lipschitz maps of open spin manifolds X to P, see in section 5.3.1.

(1) Start with the case where the (isometric!) action of the (structure) group G on the fiber S of the fibration $P \to Q$ has a fixed point, then assume $m = dim(S)$ is even and observe that radial maps $S \to S^m$, which are constant at infinity and have degrees one, induce homologically substantial G-invariant bundles $L = L_S$ bundles on S.

Since $S = \mathbb{R}^m$, such maps can be chosen with arbitrarily small Lipschitz constants, thus making the curvatures of these bundles arbitrarily small, namely, (this is obvious) with the supports in the R-balls $B_{s_0}(R) \subset S$, around the fixed point $s_0 \in S$ for the G-action and with curvatures of our (induced from $\mathbb{S}(S^m)$) bundles $L_S = L_{S,s_0,R} \to S$ bounded by $\frac{1}{R^2}$.[124]

Then we see as earlier that in the limit for $R \to \infty$, the curvature of the bundle $L_\updownarrow \to P$, which is on the fibers $S = S_q \subset P$ is equal to $L_S \to S$, (see $\bullet_{\mathrm{I}}$ in **B** of section 6.3.2) will be bounded by the curvature of the connection $\underline{\nabla}$ on $P \to Q$, provided the map $P \to Q$ is 1-Lipschitz.[125]

Consequently,

the K-waist$_2$ of P is bounded from below by the minimum of the K-waist$_2$ of Q and the reciprocal of the curvature $|curv|(\underline{\nabla})$

(2) Next, let us deal with the opposite case, where the structure group $G = \mathbb{R}^m$, i.e. the Euclidean space $\mathbb{R}^m$ acts on itself by parallel translations.

Then, topologically speaking, the fibration $F : P \to Q$ is trivial, but the above doesn't, apply since this $P \to Q$ typically admits *no parallel section.*

But since the $\underline{\nabla}$-monodromy transformations, that are parallel translations on the fiber $S = \mathbb{R}^m$, have bounded displacements, there exists a continuous *trivialization map*

$$G : P \to Q \times \mathbb{R}^n,$$

which, assuming Q is compact, (obviously) has the following properties.

(i) The fibers $\mathbb{R}^m_q \subset P$ are *isometrically* sent by G to $\mathbb{R}^m = \{q\} \times \mathbb{R}^m \subset Q \times \mathbb{R}^m$ for all $q \in Q$.

(ii) The composition of G with the projection $Q \times \mathbb{R}^m \to \mathbb{R}^m$, call it

$$G_{\mathbb{R}^m} : P \to \mathbb{R}^M$$

is *1-Lipshitz on the large scale*,

$$dist(G^m_{\mathbb{R}}(q_1, q_2)) \le dist(q_1, q_2) = cost.$$

It follows by a standard *Lipschitz extension* argument, that, for an arbitrary $\varepsilon > 0$, there exists a smooth map

$$G'_\varepsilon : P \to Q \times \mathbb{R}^m, \ \varepsilon > 0,$$

which is properly homotopic to G and such that the corresponding map

$$G'_{\varepsilon, \mathbb{R}^m} : P \to \mathbb{R}^m$$

is λ-Lipschitz for $\lambda \le m + n + \varepsilon$, where, recall, $m + n = dim(P)$

[124]It suffices to have the universal covering $\tilde{S}$ of S isometric to $\mathbb{R}^m$, where radial bundles on $\tilde{S}$ can be pushed forward to Fredholm bundles on S.

[125]The parallel displacement contribution to the curvature of $L_\updownarrow$ (see **B** of section 6.3.2)) cancels away by an easy argument.

Now, the concern expressed in **A** of section 6.3.2 notwithstanding, the μ-bubble splitting argument from section 5.3 applies and shows that

(a) *the stabilized max-scalar curvature of* P (defined in section 5.3.1 via products of P with flat tori) *is bounded, up to a multiplicative constant, by that of* Q.

Besides, the existence of fiberwise contracting scalings of P, which fix a given section $Q \to P$, show that

(b) *if* Q *is compact and if* m *is even, then the K-waist*$_2$ *of* P *is bounded from below, by that of* Q.

Notice here, that

unlike most previous occasions, neither a bound on the curvature of the fibration $P \to Q$ is required, nor the manifold X in the definition of the max-scalar curvature mapped to P need to be spin.

And besides dispensing of the spin condition, one may allow here

non-complete manifolds Q and X and/or manifolds in (a) and compact manifolds Q with boundaries in (b).

(3) Finally, let us turn to the general case where the structure group of a fibration $P \to Q$ with the fiber $S = \mathbb{R}^m$ is the full isometry group G of the Euclidean space $\mathbb{R}^m$.

Recall that G is a the semidirect product, $G = O(m) \rtimes \mathbb{R}^m$, let $P_G \to Q$ be the principal bundle with fiber G associated with $P \to Q$ and let $P_O \to P$ be the associated $O(m)$ bundle. Let

$$P_O \leftarrow P_G \to P$$

be the obvious fibrations.

Now, granted a bound on the Lipschitz constant of $F : P \to Q$ and the curvature of this fibration, we obtain

(i) *a bound on the max-scalar curvature of the space* P_G *in terms of such a bound on* P

In fact, the curvature of the fibration $P_G \to P$ as well as its Lipschitz constant are bounded by those of $F : P \to Q$ and our bound (i) follows from non-negativity of the scalar curvature of the fiber $O(m)$ of this fibration by the (obvious) argument used in section 6.3.5.

Then we look at the fibrations $P_G \to P_O \to Q$ and observe that

(ii) the fibration $P_O \to Q$ has $O(m)$-fibers and, thus the K-$waist_2(P_O)$ is bounded from below by that of Q as it was shown in section 6.3.5;

(iii) the fibration $P_G \to P_O$ has $\mathbb{R}^m$-fibers and the structure group $\mathbb{R}^m$ and, by the above (2), the K-$waist_2$ of P_G is bounded from below by that of Q; hence

K-waist$_2(P_G)$ *of* P_G *is bounded by K-waist*$_2(Q)$.

We recall at this point the basic bound on $Sc^m_{sp}ax(P_G)$ by the reciprocal of the K-$waist_2(P_G)$, confront (i) with (iii) and conclude (similarly to how it was done in section 6.3.5) to the final result of this section.

✎✐ Let $F : P \to Q$ be a smooth fibration between Riemannin manifolds with fibers $S_q = \mathbb{R}^m$ and a connection $\underline{\nabla}$, the monodromy of which isometrically acts on the fibers. If the map F is 1-Lipschitz, then

the proper spin max-scalar curvature of P *is bounded in terms of the curvature* $|curv|(\underline{\nabla})$ *and the reciprocal to K-waist*$_2(Q)$.

Corollary. Let Q admit a constant at infinity area decreasing map to S^n, $n = dim(Q)$, of non-zero degree.

Let the norm of the curvature of (the connection $\underline{\nabla}$ on) a bundle $P \to Q$ with $\mathbb{R}^m$-fibers is bounded by $\underline{c}$.

Let a complete orientable Riemannian spin manifold X of dimension $m + n$ admit a proper area decreasing map to P.

Then

$$\inf_{x \in X} Sc(X, x) \leq \Psi(\underline{c}),$$

where, $\Psi = \Psi_{m+n}$ is an effectively describable positive function; in fact, the above proof of ✎✐ *shows that one may take*

$$\Psi(\underline{c}) = (m + n)(m + n - 1) + const_m \underline{c}$$

and where, probably, $(m + n)(m + n - 1)$ can be replaced by $n(n - 1)$.

6.4.3 Spin Harmonic Area of Fibrations With Riemannian Symmetric Fibers.

Let S be a complete Riemannian manifold with a transitive isometric action of a group G which equivariantly lifts to a vector bundle $\mathbf{L}_S \to S$ with a unitary connection, such that the L_2-index of the twisted Dirac operator $\mathcal{D}_{\otimes L_S}$ is non zero.[126]

Example: Hyperbolic and Hermitian Symmetric spaces.

(a) If $S = \mathbf{H}^{2m}_{-1}$ then $S = \mathbf{H}^{2m}_{-1}$ and $L_S = \mathbb{S}^+(\mathbf{H}^{2m}_{-1})$ (compare with section 4.6.5).

(b) If S is a Hernitian Symmetric space, e.g. a product of hyperbolic planes or the quotient space of the symplectic group $Sp(2k, \mathbb{R})$ by $U(k) \subset Sp(2k, \mathbb{R})$, then the canonical bundle (or, possibly, its tensorial power) can be taken for L_S.

Let $F : P \to Q$ be a fibration with the fiber S and the structure group G, let P be endowed with a complete Riemannin metric and let $L_\updownarrow \to P$ be the natural extension of the (G-equivariant!) bundle L_S to P (compare with section 6.3.2).

Let $L_Q \to Q$ be a vector bundle with a unitary connection. and let

$$L_\rtimes = F^*(L_Q) \otimes L_\updownarrow \to P$$

.

Conceivably there must exist (already exists) an index theorem for the Dirac operator on P twisted with the bundle $L_\rtimes$ that would ensure the existence of non-zero twisted harmonic L_2-spinors on P under favorable topological and geometric conditions.

For instance, if Q is a complete Riemannian of *even* dimension n, if the bundle L_Q is induced from the spin bundle $\mathbb{S}^+(S^n)$ by a smooth constant at infinity map $Q \to S^n$ of positive degree, if P is spin and if the map $F : P \to Q$ is isometric on the horizontal subbundle in $T(P)$, then, *conjecturally*,

[126] As we have already mentioned in section 4.6.5, if S admits a free discrete cocomapct isometric action of a group Γ, this is equivalent to the non-vanishing of the index of the corresponding operator on S/Γ [Atiyah (L2) 1976]; in general, this index is defined by Connes and Moscovici in [ConMos 1982].

the manifold P supports a non-zero $L_{\rtimes}$-twisted harmonic L_2-spinor.

In fact this easy if the fibration is flat, e.g. if the fibration $P = Q \times S$ and, if the curvature of this fibration is (very) small, then a trivial perturbation argument as in section 4.6.5 yields almost harmonic spinors on large domains $P_R \subset P$.

But what we truly wish is the solutions of the following counterparts to (A) and (B) from section 4.6.5.

Let $F : P_R \to Q_R$ be a submersion between compact Riemannin manifolds with boundaries, where

$$R = \sup_{p \in P} dist(p, \partial P)$$

and where the local geometries of the fibers are δ-close (in a reasonable sense) to the geometry of an above homogeneous S and let $L_{\rtimes,R} \to P_R$ be a vector bundle, also δ-close (in a reasonable sense) to an above $L_{\rtimes}$.

(A_F) When does P_R support a ε-harmonic $L_{\rtimes,R}$-twisted spinor which vanishes on the boundary of P?

(B_F) When does a similar spinor exist on a manifold $\overline{P}_R$, which admits a map to P_R with non-zero degree and with a controlled metric distorsion.?

6.5 Scalar Curvature of Foliations

Let X be a smooth n-dimensional manifold and $\mathscr{L}$ a smooth foliation of X that is a smooth partition of X into $(n-k)$-dimensional leaves, denoted $\mathcal{L}$.

Let $T(\mathscr{L}) \subset T(X)$ denote the tangent bundle of $\mathscr{L}$ and Recall that the *transversal (quotient) bundle* $T(X)/T(\mathscr{L})$ carries a natural *leaf-wise flat affine connection* denoted $\nabla^{\perp}_{\mathcal{L}}$, where the parallel transport is called *monodromy.*

This $\nabla^{\perp}_{\mathcal{L}}$ can be (obviously but non-uniquely) extended to an actual (non-flat) connection on the bundle $T(X)/T(\mathscr{L}) \to X$, which is called *Bott connection.*

Two Examples (1) Let $\mathscr{L}$ admit a transversal k dimensional foliation, say $\mathscr{K}$ and observe that the bundle $T(X)/T(\mathscr{L}) \to X$ is canonically (and obviously) isomorphic to the tangent bundle $T(\mathscr{K})$.

Thus, every $\mathscr{K}$-leaf-wise connection in the tangent bundle $T(\mathscr{K})$, e.g. the Levi-Civita connection for a leaf-wise Riemannian metric in $\mathscr{K}$, defines a $\mathscr{K}$-leaf-wise connection, say $\nabla_{\mathscr{K}}$ of $T(X)/T(\mathscr{L})$

Then there is a unique connection on the bundle $T(X)/T(\mathscr{L}) \to X$, which agrees with $\nabla^{\perp}_{\mathcal{L}}$ on the $\mathscr{L}$-leaves and with $\nabla_{\mathscr{K}}$ on the $\mathscr{K}$-leaves, that is the Bott connection.

(2) Let the bundle $T(X)/T(\mathscr{L}) \to X$ be topologically trivial and let $\partial_i : X \to T(X)$, $i = 1, ..., k$, be linearly independent vector fields transversal to $\mathscr{L}$. Then there exists a unique Bott connection, for which the projection of ∂_i to $T(X)/T(\mathscr{L})$ is parallel for the translations along the orbits of the field ∂_i for all $i = 1, ..., k$.

In what follows, we choose a Bott connection on the bundle $T(X)/T(\mathscr{L}) \to X$ and denote it $\nabla^{\perp}_X$.

Also we choose a subbundle $T^{\perp} \subset T(X)$ complementary to $T(\mathscr{L})$, which, observe, is canonically isomorphic to $T(X)/T(\mathscr{L})$, where this isomorphism is implemented by the quotient homomorphism $T^{\perp} \subset T(X) \to T(X)/T(\mathscr{L})$.

With this isomorphism, we transport the connections $\nabla^{\perp}_{\mathcal{L}}$ and $\nabla^{\perp}_{X}$ from $T(X)/T(\mathscr{L}$ to $\nabla^{\perp}_{X}$ keeping the notations unchanged. (Hopefully, this will bring no confusion.)

6.5.1 Blow-up of Transversal Metrics on Foliations

Let $g = g_{\mathscr{L}}$ be a leaf-wise Riemannin metric on the foliation $\mathscr{L}$, that is a positive quadratic form on the bundle $T(\mathscr{L})$, let $g^{\perp}$ be such a form on $T^{\perp}$ and observe that the sum of the two $g^{\oplus} = g \oplus g^{\perp}$ makes a Riemannin metric on the manifold X.

This metric itself doesn't tell you much about our foliation $\mathscr{L}$, but the family

$$g^{\oplus}_{e} = g \oplus e^2 g^{\perp},\ \ e > 0,$$

is more informative in this respect, especially for $e \to \infty$. For instance,

[a] if the metric $g = g_{\mathscr{L}}$ has *strictly positive scalar curvature*, i.e. $Sc_g(\mathcal{L}) > 0$ for all leaves $\mathcal{L}$ of $\mathscr{L}$, and, this is essential, if the metric $g^{\perp}$ is *invariant under the monodromy along the leaves* $\mathcal{L}$ – foliations which comes with such a $g^{\perp}$ are called *transversally Riemannian*, – then, assuming X is *compact*,

$$Sc(g^{\oplus}_{e}) > 0$$

for all sufficiently large $e > 0$.

Proof of [a]. Let $x_0 \in X$, let $\mathcal{L}_0 = \mathcal{L}_{x_0} \subset X$ be the leaf which contains x_0 and observe that the pairs pointed Riemannian manifolds $(X_e, \mathcal{L}_0 \ni x_0)$ for $X_e = (X, g^{\oplus}_{e})$ converge to the (total space of the) Euclidean vector bundle $T^{\perp}$ restricted to $\mathcal{L}_0$ with the metric

$$[\oplus] \qquad g_{\lim} = g_{\mathcal{L}_0} \oplus g^{\perp}_{Eu},$$

where $g_{\mathcal{L}} = g_{\mathscr{L}}|\mathcal{L}_0$, where $g^{\perp}_{Eu} = g^{\perp}_{Eu}(l)$, $l \in \mathcal{L}_0$, is the a family of the Euclidean metrics in the fibers of the bundle $T^{\perp}|\mathcal{L}_0$ corresponding to $g^{\perp}$ on $\mathcal{L}_0$, and where "$\oplus$" refers to the local splitting of this bundle via the (flat!) connection $\nabla^{\perp}_{\mathscr{L}}|\mathcal{L}_0$.[127] The scalar curvature of the metric $g_{\mathcal{L}_0} \oplus g^{\perp}_{Eu}$ is determined by

> the scalar curvature of the leaf $\mathcal{L}_0$ and the first and second (covariant) logarithmic derivatives of $g^{\perp}_{Eu}(l)$,

where $g^{\perp}_{Eu}(l)$ is regarded as a function on $\mathcal{L}_0$ with values in the space of (positive) quadratic forms on $\mathbb{R}^k$, which in the case $g^{\perp}_{Eu}(l) = \varphi(l)^2 g_0$ reduces to the "higher warped product formula" from section 2.4:

$$(\star\star_{\mathcal{L}}) \qquad Sc(\varphi(l)^2 g_0)(l,r) = Sc(\mathcal{L}_0)(l) - \frac{k(k-1)}{\varphi^2(l)}||\nabla\varphi(l)||^2 - \frac{2k}{\varphi(l)}\Delta\varphi(l),$$

where $(l,r) \in \mathcal{L}_0 \times \mathbb{R}^k$ and $\Delta = \sum \nabla_{i,i}$ is the Laplace operator on $\mathcal{L}_0$.

Since, in general, these "logarithmic derivatives" denoted $g^{\perp}_{Eu}(l)'/g^{\perp}_{Eu}(l)$ and $g^{\perp}_{Eu}(l)''/g^{\perp}_{Eu}(l)$ are the same as of the original (prelimit) metric $g^{\perp}(l)$, it follows, that

$(\star\star_{Sc})$

$$Sc(g_{\mathcal{L}_0} \oplus g^{\perp}_{Eu}) \geq Sc(g_{\mathcal{L}_0}) - const_n \left(||(g^{\perp}(l)'/g^{\perp}(l))^2|| + ||g^{\perp}(l)''/g^{\perp}(l)|| \right).$$

[127] The limit space $(T^{\perp}, g_{lim})$ can be regarded as the *tangent cone of* X *at* $\mathcal{L}_0 \subset X$, where the characteristic feature of this cone is its scale invariance under multiplication of the metric g_{lim} normally to $\mathcal{L}_0$ by constants.

In particular, if $g^\perp$ is constant with respect to $\nabla^\perp_{\mathscr{L}}|\mathcal{L}_0$, then the limit metric $g_{\lim}$ locally is the Riemannian product $(\mathscr{L}, g_{\mathscr{L}}) \times \mathbb{R}^k$ with the scalar curvature equal to that of $\mathscr{L}$. QED.

However obvious, this immediately implies

[a1] *vanishing of the $\hat{A}$-genus as well as of its products with the Pontryagin classes of* $T^\perp$ for *transversally Riemannian* foliations on closed spin manifolds X, where the "product part" of this claim follows from the *twisted* Schroedinger-Lichnerowicz-Weitzenboeck formula for the Dirac operator $\mathcal{D}_{\otimes T^\perp}$, since the *curvature of the (Bott connection in the) bundle $T^\perp \to X$ converges to zero for $e \to \infty$.*

(This is not *formally* covered by Connes' theorem stated in section 3.12, where the spin condition must be satisfied by $\mathscr{L}$ rather than X itself as it is required here; but it can be easily derived from Connes' theorem.)

Another equally obvious corollary of [$\oplus$] is as follows.

[a2] *If $Sc(\mathscr{L}) > n(n-1)$ and if X is closed orientable spin, then X admits no map $f: X \to S^n$, such that $deg(f) \neq 0$ and such that the restrictions of f to the leaves of $\mathscr{L}$ are 1-Lipschitz.*

But this is not fully satisfactory, since it it *remains unclear*

if one truly needs the inequality $Sc(\mathscr{L}) > n(n-1)$ or $Sc(\mathscr{L}) > (n-k)(n-k-1)$ for $n-k = dim(\mathscr{L})$ will suffice?

Exercise. Show that $Sc(\mathscr{L}) > 2$ does suffice for 2-dimensional foliations.

Flags of Foliations. Let

$$\mathscr{L} = \mathscr{L}_0 \prec \mathscr{L}_1 \prec \ldots \prec \mathscr{L}_j,$$

where the relation $\mathscr{L}_{i-1} \prec \mathscr{L}_i$ signifies that $\mathscr{L}_i$ *refines* $\mathscr{L}_{i-1}$, which means the inclusions between their leaves,

$$\mathcal{L}_i \subset \mathcal{L}_{i-1},$$

and where $\mathscr{L}_0$ is the bottom foliation with a single leaf equal X.

Let $T_i^\perp = T_i^\perp \subset T(\mathscr{L}_{i-1})$, $i = 1, 2, \ldots, j$ be transversal subbundles isomorphic to $T(\mathscr{L}_{i-1})/T(\mathscr{L}_i)$, let $g_j = g_{\mathscr{L}_j}$ be a $\mathscr{L}_j$-leaf-wise Riemannian metric, let $g_i^\perp$, $i = 1 \ldots j$,, be Riemannin metrics on $T_i^\perp$ and let

$$g^\oplus_{e_1,\ldots,e_j} = g_0 \oplus e_1^2 g_1^\perp \oplus \ldots \oplus e_j^2 g_j^\perp.$$

[b] *If the metrics in the quotient bundles $T(\mathscr{L}_{i-1})/T(\mathscr{L}_i)$, $i = 1, \ldots, j$, which corresponds to $g_i^\perp$, are invariant under holonomies along the leaves of $\mathscr{L}_j$, if $e_i \to \infty$, then*

$$Sc(g^\oplus_{e_1,\ldots,e_j}) \to Sc(g_j),$$

where this convergence is uniform on compact subsets in X.

Proof. Since

the logarithmic derivatives of maps from Riemannian manifolds to the Euclidean spaces tend to zero as the metrics in these manifolds are scaled by constants $\to \infty$,

the above $(\star\star_{Sc})$ implies the following.

$[\mathbf{b}_{lim}]$ The pair of pointed Riemannian manifolds $(X_{e_1,\dots,e_j}, \mathcal{L}_j \ni x_j)$, for all leaves $\mathcal{L}_j$ of $\mathscr{L}_j$ and all $x_j \in L_j$, converges to the (total space of the) flat Euclidean vector bundle $T_1^\perp \oplus \dots \oplus T_j^\perp \to \mathcal{L}_j$, where the limit metric on (the total space of) $T_1^\perp \oplus \dots \oplus T_j^\perp$ locally splits as

$[\oplus_i \perp]$, $$g_{\lim} = g_{\mathcal{L}_j} \oplus g_{Eu} \otimes g_{Eu,1(l)},$$

where g_{Eu} is the Euclidean metric on $\mathbb{R}^{k_2+\dots k_i+\dots+k_j}$ for $k_i = rank(T_i^\perp)$ and $g_{Eu,1(l)}$, $l \in \mathcal{L}_j$ is a family of Euclidean metrics in the fibers of the bundle $T_1^\perp \to X$ restricted to $\mathcal{L}_j$, where the logarithmic derivatives of these metrics are equal these for the original (prelimit) metrics in the bundle $T_1^\perp$ over $\mathcal{L}_j$.

Now, we see, as earlier, that $[\mathbf{b}_{lim}] \Rightarrow [\mathbf{b}]$ and the proof follows.

Thus, the above [a1] and [a2] generalize to transversally Riemannian flags of foliations

6.5.2 Connes' Fibration.

Let the "normal" bundle $T^\perp \to X$ to a foliation $\mathscr{L}$ on X admits a smooth G-structure for a subgroup G of the linear group $GL(k)$, $k = codim(\mathscr{L})$, which (essentially) means that the monodromy transformation for the above canonical flat leaf-wise connection $\nabla^\perp_{\mathscr{L}}$ are contained in G.

For instance, being Riemannian for a foliation is the same as to admit $G = O(k)$ and $G = GL(k)$ serves all foliation.

Let G *isometrically* act on a Riemannin manifold S and let $P \to X$ be a fibration associated to $T^\perp \to X$.

Then the monodromy of $\nabla^\perp_{\mathscr{L}}$ is isometric on the fibers $S_x \subset P$.

Principal Example.[Con 1986] Let

$$G = GL(k) \text{ and } S = GL(k)/O(k)$$

and let us identify the fiber S_x, for all $x \in X$, with the space of Euclidean structures, i.e. of positive definite quadratic forms, in the linear space $T_x^\perp$.

Clearly, this S canonically splits as

$$S = R \times \mathbb{R} \text{ for } R = SL(k)/SO(k),$$

where, observe, R carries a unique up to scaling $SO(k)$-invariant Riemannian (symmetric) metrics with non-positive sectional curvature and where the $\mathbb{R}$-factor is the logarithm of the central multiplicative subgroup $\mathbb{R}_+^\times \subset GL(k)$.

Thus, $S = R \times \mathbb{R}$ carries an invariant Riemannian product metric, call it g_S, which is unique up-to scaling of the factors.

Next, observe that the tangent bundle $T(P)$ splits as usual

$$T(P) = T^{vert} \oplus T^{hor}$$

where T^{vert} consists of the vectors tangent to the fibers $S_x \subset P$, $x \in X$, and where T^{hor} is the horizontal subbundle corresponding to the Bott connection, and where the splitting $T(X) = T(\mathscr{L}) \oplus T^\perp$ lifts to a splitting of T^{hor}, denoted

$$T^{hor} = \tilde{T}(\mathscr{L}) \oplus \tilde{T}^\perp.$$

Thus, the tangent bundle $T(P)$ splits into sum of three bundles,

$$T(P) = T^{vert} \oplus \tilde{T}(\mathscr{L}) \oplus \tilde{T}^{\perp},$$

where, to keep track of things, recall that

$$rank(\tilde{T}(\mathscr{L})) = dim(\mathscr{L}) = n - k,\ rank(\tilde{T}^{\perp}) = codim(\mathscr{L}) = k$$

and

$$rank(T^{vert}) = dim(GL(k)/O(k)) = \frac{k(k+1)}{2}.$$

Let us record the essential features of these three bundles and their roles in the geometry of the space P (see [Con 1986] and compare with §$1\frac{7}{8}$ in [G(positive) 1996]).

(1) *Metric $\tilde{g}^{\perp}$ in $\tilde{T}^{\perp}$.* The (sub)bundle $\tilde{T}^{\perp} \subset T(P)$ carries a *tautological metric* call it $\tilde{g}^{\perp}$, which, in the fiber $\tilde{T}^{\perp}_p \subset \tilde{T}^{\perp}$ for $p \in P$ over $x \in X$, is equal to this very $p \in P_x$ regarded as a metric in $T^{\perp}_x \subset T^{\perp} \to X$.

(2) *Foliation $\mathscr{L}^+$ of P.* The leaves $\mathcal{L}^+ \subset P$ of this foliations are the pullbacks of the leaves $\mathcal{L}$ of $\mathscr{L}$ under the map $P \to X$. These $\mathcal{L}^+$ have dimensions $n - k + \frac{k(k+1)}{2}$ and the tangent bundle $T(\mathscr{L}^+)$ is canonically isomorphic to $\tilde{T}(\mathscr{L}) \oplus T^{vert}$.

(3) Foliation $\tilde{\mathscr{L}}$ of P. This is the natural lift of the original foliation $\mathscr{L}$ of X:

the leaf $\tilde{\mathcal{L}}_p$ of $\tilde{\mathscr{L}}$ through a given point $p \in P$ over an $x \in X$ is equal to the set of the Euclidean metrics in the fibers $T^{\perp}_l \subset T^{\perp} \to X$ for all $l \in \mathcal{L}_x \subset X$, which are obtained from p, regarded as such a metric in $T^{\perp}_x \subset T^{\perp} \to X$, by the monodromy along the leaf $\mathcal{L}_x$ of the foliation $\mathscr{L}$ of X.

This foliation can be equivalently defined via its tangent (sub)bundle, that is

$$T(\tilde{\mathscr{L}}) = \tilde{T}(\mathscr{L}) \subset T(P).$$

Also observe that this $\tilde{\mathscr{L}}$ refines $\mathscr{L}$, written as $\tilde{\mathscr{L}} \succ \mathscr{L}^+$, where, in fact, the leaves of $\mathscr{L}^+$ are products of the monodromy covers of the leaves of $\mathscr{L}$ by S.

(4) $\tilde{\mathscr{L}}$-Monodromy Invariance of the Metric $\tilde{g}^{\perp}$. The bundle $\tilde{T}^{\perp} \subset T(P)$, where the metric $\tilde{g}^{\perp}$ resides, is naturally isomorphic to the "normal" bundle $T(P)/T(\mathscr{L}^+)$, but this metric is *not invariant* under the monodromy of the foliation $\mathscr{L}^+$.

However, $\tilde{g}^{\perp}$ is *invariant under the monodromy of the sub-foliation* $\tilde{\mathscr{L}} \succ \mathscr{L}^+$ with the leaves $\tilde{\mathcal{L}} \subset \mathcal{L}^+$ as it follows from the above description of the leaves $\tilde{\mathcal{L}}_p$ of $\tilde{\mathscr{L}}$.

(5) $\tilde{\mathscr{L}}$-Monodromy Invariance of $\tilde{g}_S$ in the Bundle T^{vert}. Since the fibration $P \to X$ with the fiber $S = GL(k)/O(k)$ is associated with $T^{\perp} \to X$, every $GL(k)$ metric g_S on S gives rise to a *monodromy invariant metric in this fibration* which we keep denoting g_S.

Then the natural lift of g_S to the bundle $T^{vert} == T(\mathscr{L}^+)/T(\tilde{\mathcal{L}})$, *denoted* $\tilde{g}_S$, is, obviously, $\tilde{\mathscr{L}}$*-monodromy invariant.*

(6) *Scalar Curvature under Blow-up of Metrics in $T(P)$.* Let $g = g_{\mathscr{L}}$ be a Riemannin metric in the tangent bundle $T(\mathscr{L}) \subset T(X)$ of a foliation $\mathscr{L}$ of X as earlier and let $\tilde{g}$ be its lift to the bundle $\tilde{T}(\mathscr{L}) = T(\tilde{\mathscr{L}}) \subset T(P)$.

Let $\tilde{g}_{e^+,e^\perp}$ be the Riemannin metric on the manifold P that is the metric in the bundle

$$T(P) = \tilde{T}(\mathscr{L}) \oplus T^{vert} \oplus \tilde{T}^\perp.$$

where our $\tilde{g}_{e^+,e^\perp}$ is split into the sum of the metrics from th above (5) and (4). which are taken here with (large) positive e-weights as follows.

$$\tilde{g}_{e^+,e^\perp} = \tilde{g} \oplus e_+^2 \tilde{g}_R^+ \oplus e_\perp^2 \tilde{g}^\perp.$$

Then it follows from the above **[b]**, that if

$$e_+^2, e_\perp^2 \to \infty,$$

then

$[\Uparrow_{Sc}]$ *the scalar curvature of the metric $\tilde{g}_{\varepsilon^+,\varepsilon^\perp}$ at $p \in P^+$ over $x \in X$ converges to that of g on the leaf $\mathcal{L}_x \ni x$ at x, where this convergence is uniform on the compact subsets in P^+.*[128]

Generalizations. Much of the above (1) - (6) applies to foliations with monodromy groups G not necessarily equal to $GL(k)$ and with fibrations with the fibers that may be different from G/K, which we will approach in the following section on the case-by-case basis.

6.5.3 Foliations with Abelian Monodromies

Let a foliation $\mathscr{L}$ of an orientable n-dimensional Riemannian manifold X admit a smooth G-structure invariant under the monodromy, where the group G is Abelian and let the scalar curvatures of the leaves with the indices Riemannin metrics are bounded from below by $\sigma > n(n-1)$.

$\square\bigcirc$. *The hyperspherical radius of X is bounded by one,*

$$Rad_{S^n}(X) \leq 1.$$

That is, if $R > 1$, then

X admits no 1-Lipschitz map to the sphere $S^n(R)$, which is constant at infinity and which has non-zero degree.

Prior to turning to the proof, that is an easy corollary of what we discussed about $\mathbb{R}^k$-fibration in section 6.4.2, we'll clarify a couple of points.

1. We don't assume here that the manifold X is compact or complete, nor do we require it is being spin.

2. We don't know if our Abelian assumption on G is essential. It is conceivable that

$\square\bigcirc$ holds for all foliations, i.e. for $G = GL(k)$, $k = codim(\mathscr{L})$, and, moreover, with the bound $Sc(\mathscr{L}) \geq (n-k)(n-k-1)$.

2. Examples of foliations with Abelian G, include:

[128] This convergence property is implicit in [Con 1986] and it is articulated explicitly in §1$\frac{7}{8}$ of [G(positive) 1996], where it is additionally required that $e_\perp^2/e_+^2 \to \infty$. This, as I see it now, only serves a psychological purpose: when it comes to applications one starts with scaling $\tilde{g}^+$ with a large e_+ and then let $e_\perp^2 \to \infty$. Also this convergence appears in "adiabatic" terms in Proposition 1.4 of [Zhang 2016].

foliations with transversal conformal structure, e.g. (orientable) foliations of codimension one, where G is the multiplicative group $\mathbb{R}^\times$;

flags of codimension one foliations (where $G = (\mathbb{R}^\times)^k$) and/or of foliations with transversal conformal structures.

Proof of $\square\bigcirc$. Let $P \to X$ be the principal fibration associated with the bundle $T(X)/T(\mathscr{L})$ and by blowing up the metric of P transversally to the lift $\tilde{\mathscr{L}}$ to P as in the previous section, make the scalar curvature of P on a given compact domain $P_\varepsilon \subset P$ greater than $n(n-1) - \varepsilon$ for a given $\varepsilon > 0$.

Also with this blow-up, make the Lipschitz constant of the map $P \to X$ as small as you want.

(A possibility of this formally follows from the above (1) - (6) for foliations of codimension one, while the proof in general case amounts to replaying (1) - (6) word-for-word in the present case.)

Next, let $G = \mathbb{R}^m$, observe as in (2) in section 6.4.2 that P_ε admits a $(1+\varepsilon)$-Lipschitz map of degree one from P_0 to $X \times [0, L]^k$ for an arbitrary large L and apply the maximality/extremality theorem for punctured spheres from sections 3.9 and 5.4.

This concludes the proof for $G = \mathbb{R}^m$ and the case of the general Abelian G follows by passing to the quotient of G by the maximal compact subgroup.

To get an idea why one can control the geometry of the blow-up only on compact subsets in P, look at the following.

Geometric Example. Let (Y, g) be a Riemannian manifold and let $P_Y \to Y$ be the fibration, with the fibers S_y, $y \in Y$, equal to the spaces of quadratic forms in the tangent spaces $T_y(Y)$ of the form $c \cdot g_y$, $c > 0$. Thus, $P_Y = Y \times \mathbb{R}$, for $\mathbb{R} = \log \mathbb{R}_+^\times$ with the metric $e^{2r} dy^2 + dr^2$.

When $r \to +\infty$ and the curvature of $e^{2r} g$ tends to zero, then the metric $e^{2r} dy^2 + dr^2$ converges to the hyperbolic one with constant curvature -1, but when $r \to -\infty$, then the curvatures of $e^{2r} g$ and of $e^{2r} dy^2 + dr^2$ blow up at all points $y \in Y$, where the curvature of g doesn't vanish.

And if apply this to the fibration $P = P_Y \times \mathcal{L} \to X = Y \times \mathcal{L}$ with the same $\mathbb{R}$-fibers, then we see that the convergence of the scalar curvatures of the blown-up P to those of $\mathcal{L}$ is by no means uniform.

6.5.4 Hermitian Connes' Fibration

Let $\mathscr{L}$ be a foliation on X of codimension k as earlier with a transversal (sub)bundle $T^\perp \subset T(X)$ and a Bott connection in it. Let $T^\bowtie$ be the sum of $T^\perp$ with its dual bundle and endow $T^\bowtie$ with the natural, hence monodromy invariant, symplectic structure.

Let S_x denote the space of Hermitian structures in the space $T_x^\bowtie$, for all $x \in X$, and let $P \to X$ be the corresponding fibration, that is the fibration associated with $T_x^\bowtie$ with the fiber $S = Sp(2k, \mathbb{R})/U(k)$.

Equivalently, this fibration $P \to X$ is associated to $T^\perp \to X$ via the action of $GL(k)$ on $S = Sp(2k, \mathbb{R})/U(k)$ for the natural embedding of the linear group $GL(k)$ to the symplectic $Sp(2k, \mathbb{R})$

Besides sharing the properties (1)-(6) of the original Connes' bundle formulated in section 6.5.2, this new $P \to X$ has, a lovely additional feature:

S is a *Hermitian* (irreducible) symmetric space, which implies (see section 6,4.3) non-vanishing of the index of some twisted Dirac operator on S that is invariant under the isometry group (that is $Sp(2k,\mathbb{R})$) of S.

Let us formulate several conjectures the (positive) solution of which besides being interesting in its own right, might simplify the proof by Connes in [Con 1986] as well as the version of Connes' proof from [Zhang 2016].

L et $Y = (Y,\omega)$ be a closed symplectic manifold of dimension $2k$ and let $F : P_Y \to Y$ be the fibration associated with the tangent bundle $T(Y)$ with the fiber $S = Sp(2k,\mathbb{R})/U(k)$.

Observe that the quotient bundle $T(P_Y)/T^{vert}$ carries a tautological Hermitian metric $g_{\bowtie}$, and a granted $Sp(2k,\mathbb{R})$-connection in the tangent bundle $T(Y)$, that is a horizontal subbundle $T^{hor} \subset T(P_Y)$, one obtains a Riemannin metric g_P in the tangent bundle $T(P_Y) = T^{vert} \otimes T^{hor}$ that is

$$g_P = g_S \otimes + g_{\bowtie}$$

where g_S is a $Sp(2k,\mathbb{R})$-invariant Hermitian metric in S, which is unique up to scaling.

Let the symplectic form ω be integer and thus serves as the curvature of a unitary line bundle $L \to Y$.

Conjecture 1 *The bundle of spinors on P_Y twisted with some tensorial power of the bundle $F^*(L) \to P^Y$ admits a non-zero harmonic L_2-section on P^Y.*

Remarks and Examples. (a) The geometry of this P, unlike of what we met in section 6.4.3, is as far from being a product as in P_Y from the geometric example. in section 6.5.3.

(b) The simplest instance of Y is that of an even dimensional torus $\mathbb{T}^{2k}$ with an invariant symplectic form ω and trivial flat symplectic connection.

In this case, the universal covering $\tilde{P}_Y$ of the manifold P_Y is Riemannian homogeneous; moreover, the (local) index integrant is homogeneous as well. It is probable, that a version of the Connes-Moscovici theorem applies in this case and yields twisted harmonic L_2-spinors on $\tilde{P}_Y$ and, eventually, on P_Y.

(c) It would be most amusing if twisted Dirac operators on P_Y were non-trivially influenced by the symplectic geometry of (Y,ω).

Let us generalize the above conjecture to make it applicable to foliations.

to be continued.

6.5.5 Scalar Curvature and Dynamics of Foliations

to be continued.

6.6 Moduli Spaces Everywhere

All topological and geometric constraints on metrics with $Sc \geq \sigma$ are accompanied by non-trivial homotopy theoretic properties of spaces of such metrics.

A manifestation of this principle is seen in how topological obstructions for the existence of metrics with $Sc > 0$ on closed manifolds X of dimension $n \geq 5$ give rise to

pairs (h_0, h_1) of metrics with $Sc \geq \sigma > 0$ on closed hypersurfaces $Y \subset X$ which *can't be joined by homotopies h_t with $Sc(h_t) > 0$.*

The elementary argument used for the proof of this (see section 3.15) also shows that (known) constraints on *geometry*, not only on topology, of manifolds with $Sc \geq \sigma$ play a similar role.

For instance, assuming for notational simplicity, $\sigma = n(n-1)$, and recalling the $\frac{2\pi}{n}$-inequality from sections 3.7, 5.3, we see that

(a) *if $l \geq \frac{2\pi}{n}$, then the pairs of metrics $h_0 \oplus dt^2$ and $h_1 \oplus dt^2$ on the cylinder $Y \times [-l, l]$, for the above Y and $l \geq \frac{2\pi}{n}$, can't be joined by homotopies of metrics h_t with $Sc(h_t) \geq n(n-1)$ and with $dist_{h_t}(Y \times \{-l\}, Y \times \{l\}) \geq \frac{2\pi}{n}$.*

This phenomenon is also observed for manifolds with *controlled mean curvatures of their boundaries*, e.g. for Riemannian bands X with $mean.curv(\partial_\mp X) \geq \mu_\mp$ and with $Sc(X) \geq \sigma$, whenever these inequalities imply that $dist(\partial_- X, \partial_+ X) \leq d = d(n, \sigma, \mu_\mp)$. (One may have $\sigma < 0$ here in some cases.)

Namely,

(b) *certain sub-bands $Y \subset X$ of codimension one with $\partial_\mp(Y) \subset \partial_\mp(X)$ admit pairs of metrics (h_0, h_1), such that $mean.curv_{h_0,h_1}(\partial_\mp Y) \geq \mu_\mp$ and $Sc_{h_0,h_1}(Y) \geq \sigma$ while $dist_{h_0,h_1}(\partial_-, \partial_+) \geq D$ for a given $D \geq d$. But these metrics can't be joined by homotopies h_t , which would keep these inequalities on the scalar and on the mean curvatures and have $dist_{h_t}(\partial_-, \partial_+) \geq d$ for all $t \in [0,1]$.*

(c) This seems to persist (I haven't carefully checked it) for manifolds with corners, e.g. for cube-shaped manifolds X: these, apparently contain hypersurfaces $Y \subset X$, the boundaries of which $\partial Y \subset \partial X$ inherit the corner structure from that in X, and which admit pairs of "large" metrics h_0, h_1, which also have "large" scalar curvatures, "large" mean curvatures of the codimension one faces F_i in Y and "large" complementary $(\pi - \angle_{ij})$ dihedral angles along the codimension two faces F_{ij}, but where these h_0, h_1 can't be joint by homotopies of metrics h_t with comparable "largeness" properties.

It is unclear, in general, how to extend the π_0-non-triviality (disconnectedness) of our spaces of metrics to the higher homotopy groups, since the techniques currently used for this purpose rely entirely on the Dirac theoretic techniques (see [EbR-W 2017] and references therein), which are poorly adapted to manifolds with boundaries. But some of this is possible for closed manifolds.

For instance, let Y be a smooth closed spin manifold, and h_p, $p \in P$, be a homotopically non-trivial family of metrics with $Sc(h_p) \geq \sigma > 0$, where, for instance, P can be a k-dimensional sphere and non-triviality means non-contractibility.

Let $\mathcal{S}^m_\sigma(S^m \times Y)$ denote the space of pairs (g, f), where g is a Riemannian metric on $S^m \times Y$ with $Sc(g) \geq \sigma$ and $f : (S^m \times Y, g) \to S^m$ is a distance decreasing map homotopic to the projection $f_o : S^m \times Y \to S^m$.

If non-contractibility of the family h_p follows from non-vanishing of the index of some Dirac operator, then (the proof of) Llarull's theorem suggests that the corresponding family $(h_p + ds^2, f_o) \in \mathcal{S}^m_{\sigma_+}(S^m \times Y)$ for $\sigma_+ = \sigma + m(m-1)$ is non-contractible in the space

$$\mathcal{S}^m_{m(m-1)}(S^m \times Y) \supset \mathcal{S}^m_{\sigma_+}(S^m \times Y).$$

This is quite transparent in many cases, e.g. if $h_p = \{h_0, h_1\}$ is an above kind of pair of metrics with $Sc > 0$, say an embedded codimension one sphere in a Hitchin's homotopy sphere.

Remarks. (i) If *"distance decreasing"* of f is strengthened to *"ε_n-Lipschitz"* for a sufficiently small $\varepsilon_n > 0$, then the above disconnectedness of the space of pairs (g, f) follows for all X with a use of minimal hypersurfaces instead of Dirac operators.

(ii) The above definition of the space $\mathcal{S}^m_\sigma$ makes sense for all manifolds X instead of $S^m \times Y$, where one may allow $dim(X) < m$ as well as $> m$.

However, the following remains problematic in most cases.

For which closed manifolds X and numbers m, σ_1 and $\sigma_2 > \sigma_1 > 0$ is the inclusion $\mathcal{S}^m_{\sigma_2}(X) \le \mathcal{S}^m_{\sigma_1}(X)$ homotopy equivalence?

Suggestion to the Reader. Browse through all theorems/inequalities in the previous as well as in the following sections, formulate their possible homotopy parametric versions and try to prove some of them.

6.7 Corners, Categories and Classifying Spaces.

It seems (I may be mistaken) that all known results concerning the homotopies of spaces with metrics $Sc > 0$ are about the *iterated (co)bordisms* of manifolds with $Sc > 0$ rather than about spaces of metrics per se.

To explain this, start with thinking of morphisms $a \to b$ in a category as members of class of *labeled* (directed) *edges/arrows* [0,1] with the 0-ends labeled by a and the 1-end labeled by b.

Then define a *cubical category* $\mathcal{C}$ (I guess there is a standard term but I don't know it) as a class of *labeled combinatorial cubes* of all dimensions, $[0,1]^i$, $i = 1, 2, ...$, where all faces are labeled by members of some class and which satisfied the obvious generalisations of the axioms of the ordinary categories: associativity and the presence of the identity morphisms.

Example. Let $\mathcal{C} = A^\square$ consist of continuous maps from cubes to a topological space A, e.g. to the space $A = G_+ = G_+(X)$ of metrics with positive curvature on a given manifold X, where these maps are regarded as labels on the cubes they apply to.

If we glue all such cubes along faces with equal labels, we obtain a cubical complex, call it $|\mathcal{C}|$, which is (weekly) homotopy equivalent to A, where possible degeneration of cubes.e.g. gluing two faces of the same cube, is offset by possibility of unlimited subdivision of cubes by means of cubical identity morphisms.

Next, given a smooth closed manifold X, consider "all" Riemannian manifolds of the form $(X \times [0,1]^i, g)$, $i = 0, 1, 2, ...$, such that $Sc(g) > 0$, and such that the metrics g in small neighbourhoods of all "X-faces" $X \times F_j$, where F_j is are ($(i-1)$-cubical) codimension one faces in the cube $[0,1]^i$), split as Riemannin products: $g = g_{X \times F_j} \otimes dt^2$. Denote the resulting cubical category by $XG_+^\square$ and observe that there is a natural cubical map

$$\Xi : |G_+(X)^\square| \hookrightarrow |XG_+^\square|.$$

Now we can express the above "iterated cobordism" statement by saying that the only part of the homotopy invariants of $G_+(X)$ (which is homotopy equivalent to $G_+(X)$), e.g of its homotopy groups, which is detectable by the present methods is what remains non-zero in $|XG_+^\square|$ under Ξ.

Similarly one can enlarge other spaces of Riemannin metrics on non-closed manifolds from the previous section with lower bounds on their curvatures and their sizes, where the latter can be expressed with maps $f:(X,g)\to\underline{X}$, with controlled Lipschitz constants with respect to g, or with respect to the Sc-*normalised metric* $Sc(X)\cdot g$ defined as in section 4.2.

There is yet another way of enlarging the cubical category $XG_+^\square$, namely by $B^*G^\square(\mathsf{D})$, where D is topological, e.g. metric space and where

$\bullet_0$ closed oriented Riemannian manifolds X of all dimensions n along with continuous maps $X\to\mathsf{D}$ stand for 0-cubes - "vertices",

$\bullet_1$ "edges" ; i.e 1-cubes are cobordisms W^{n+1} between X_0,X_1, with Riemannin metrics split near their boundaries $\partial W^{n+1}=X_0\sqcup -X_1$, and continuous maps to D extending those from $X_0 and X_1$,

$\bullet_2$ "squares", are (rectangularly cornered $(n+2)$-dimensional) cobordisms between W-cobordisms with maps to D, etc.

The actual cubical subcategory of $B^*G^\square(\mathsf{D})$, which is relevant for the study of the space $|XG_+^\square|$ (that is, essentially, the space of metrics with $Sc>0$ on X) is where all manifolds in the picture are spin, the scalar curvatures of their metrics are positive, D is the classifying space of a group Π and where one may assume the fundamental groups of all X to be coherently (with inclusion homomorphisms) to be isomorphic Π [129] (compare [EbR-W 2017], [HaSchSt 2014],

Question. What are possible generalizations of the above to manifolds with corners, which are far from being either cubical or rectangular?

For instance, prior to speaking of spaces of metrics and of categories of cobordisms, let X be an individual manifold with corners, say a (smoothly) topological n-simplex or a dodecahedron, let $(\infty<\sigma<\infty)$, let $(\infty<\mu_i<\infty)$ be numbers assigned to the codimension one faces F_i of X and $0<\beta_i j<\pi$ be assigned to the codimension two faces of the kind $F_i\cap F_j$.

When does X admit a Riemannian metric g such that

$$Sc_g\geq\sigma,\ mean.curv_g(F_i)\geq\mu_i \text{ and } \angle_g(F_1,F_j)\leq\pi-\beta_{ij}?$$

Let moreover, $D\subset\mathbb{R}_+^N$, where the N Euclidean coordinates are associated with the faces F_i of X, be a closed convex subset, introduce the following additional condition on g:

the N- vector of distances $\{d_i(x)=dist_g(x,F_i)\}$ is in D for all $x\in X$.

We ask when does there exist a g with this additional condition and also

what is the homotopy type of the space of metrics g on X, such that

$$Sc_g\geq\sigma,\ mean.curv_g(F_i)\geq\mu_i,\ \angle_g(F_1,F_j)\leq\pi-\beta_{ij} \text{ and } \{d_i(x)\}\in D?$$

(For instance, if X is a topological n-simplex, then an "interesting" D is defined by $\sum_i d_i(x)\geq const.$)

One may also try to generalize the concept of cubical category by allowing all kinds of combinatorial types of manifolds X with corners and of attachments

[129] This "assume" relies on the codimension two surgery of manifolds with $Sc>0$, which is possible for making the fundamental groups of n-manifolds isomorphic to Π if $n\geq4$ and where more serious topological conclusions need $n\geq5$.

of X to X' along isometric codimension one faces $X \supset F \leftrightarrow F' \subset X'$, where the isometries $F \leftrightarrow F'$, must match the mean curvatures of the faces:

$mean, curv(F') = -mean.curv(F)$ which is equivalent to the natural metric on

$$X \bigcup_{F \leftrightarrow F'} X$$

being C^1-smooth.

Is there a coherent category-style theory along these lines of thought?

6.8 Limits and Singularities.

.... to be continued.

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