DEFORMATION OF PROPER ACTIONS ON REDUCTIVE HOMOGENEOUS SPACES

FANNY KASSEL

Abstract. Let G be a real reductive Lie group and H a closed reductive subgroup of G. We investigate the deformation of *standard* compact quotients of G/H, that is, of quotients of G/H by discrete groups Γ that are uniform lattices in some closed reductive subgroup L of G acting properly and cocompactly on G/H. For L of real rank 1, we prove that after a small deformation in G, such a group Γ keeps acting properly discontinuously and cocompactly on G/H. More generally, we prove that the properness of the action of any convex cocompact subgroup of L on G/H is preserved under small deformations, and we extend this result to reductive homogeneous spaces G/H over any local field. As an application, we obtain compact quotients of $\mathrm{SO}(2n,2)/\mathrm{U}(n,1)$ by Zariski-dense discrete subgroups of $\mathrm{SO}(2n,2)$ acting properly discontinuously.

1. Introduction

Let G be a real reductive linear Lie group and H a closed reductive subgroup of G. We are interested in the compact quotients of G/H by discrete subgroups Γ of G. We ask that the action of Γ on G/H be properly discontinuous in order for the quotient $\Gamma\backslash G/H$ to be Hausdorff. This imposes strong restrictions on Γ when H is noncompact. For instance, if $\mathrm{rank}_{\mathbb{R}}(G) = \mathrm{rank}_{\mathbb{R}}(H)$, then all discrete subgroups of G acting properly discontinuously on G/H are finite: this is the Calabi–Markus phenomenon [Ko1]. Usually the action of Γ on G/H is also required to be free, so that $\Gamma\backslash G/H$ be a manifold, but this is not very restrictive: if Γ acts properly discontinuously and cocompactly on G/H, then it is finitely generated, hence virtually torsion-free by Selberg's lemma [Sel]; thus $\Gamma\backslash G/H$ has a finite cover that is a manifold. Manifolds of the form $\Gamma\backslash G/H$ are sometimes called *Clifford–Klein forms* of G/H.

In this paper we investigate the deformation of compact Clifford–Klein forms $\Gamma\backslash G/H$ (which we simply call *compact quotients*) in the important case when Γ is *standard*, that is, when Γ is a uniform lattice in some closed reductive subgroup L of G acting properly and cocompactly on G/H. Most of our results hold for reductive homogeneous spaces over any local field, but in this introduction we first consider the real case.

1.1. **Deformation of compact quotients in the real case.** Let G be a real reductive linear Lie group and H a closed reductive subgroup of G. In all known examples, if G/H admits a compact quotient, then there is a closed reductive subgroup L of G that acts properly and cocompactly on G/H. For instance, $L = \mathrm{U}(n,1)$ acts properly and transitively on the

$(2n+1)$-dimensional anti-de Sitter space $G/H = \mathrm{SO}(2n,2)/\mathrm{SO}(2n,1)$ (see Section 6). Any torsion-free uniform lattice Γ of such a group L acts properly discontinuously, freely, and cocompactly on G/H; we will say that the corresponding compact quotient $\Gamma\backslash G/H$ is *standard*. Note that L always admits torsion-free uniform lattices by [Bo1]. Kobayashi and Yoshino conjectured that any reductive homogeneous space G/H admitting compact quotients admits standard ones ([KY], Conj. 3.3.10); this conjecture remains open.

Of course, nonstandard compact quotients may also exist: this is the case for instance for $G/H = (G_0 \times G_0)/\Delta_{G_0}$ where G_0 is locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ and Δ_{G_0} is the diagonal of $G_0 \times G_0$ (see [Ghy], [Gol], [Ko3], [Sal]). But in general we know only standard examples.

In order to construct nonstandard compact quotients of G/H, it is natural, given a reductive subgroup L of G acting properly and cocompactly on G/H, to slightly deform torsion-free uniform lattices Γ of L in G and to see whether their action on G/H remains properly discontinuous, free, and cocompact. When L has real rank ≥ 2 and Γ is irreducible, this is always the case: indeed, it then follows from Margulis's superrigidity theorem ([Mar], Cor. IX.5.9) that Γ is locally rigid in G, *i.e.* that all small deformations of Γ in G are obtained by conjugation. In this paper we prove that the action of Γ remains properly discontinuous, free, and cocompact when L has real rank 1 as well.

Theorem 1.1. *Let G be a real reductive linear Lie group, H and L two closed reductive subgroups of G. Assume that* $\mathrm{rank}_{\mathbb{R}}(L) = 1$ *and that L acts properly and cocompactly on G/H. For any torsion-free uniform lattice Γ of L, there is a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that if $\varphi \in \mathcal{U}$, then φ is injective, $\varphi(\Gamma)$ is discrete in G, and $\varphi(\Gamma)$ acts properly discontinuously and cocompactly on G/H.*

We denote by $\mathrm{Hom}(\Gamma, G)$ the set of group homomorphisms from Γ to G, endowed with the compact-open topology. In the real case, the fact that $\varphi(\Gamma)$ remains discrete in G for $\varphi \in \mathrm{Hom}(\Gamma, G)$ close to the natural inclusion is a general result of Guichard ([Gui], Th. 2).

The study of small deformations of properly discontinuous actions on real homogeneous spaces was initiated by Kobayashi [Ko3]. Theorem 1.1 improves [Ko3], Th. 2.4, which focused on homomorphisms of the form $\gamma \mapsto \gamma\psi(\gamma)$, where $\psi : \Gamma \to Z_G(L)$ is a homomorphism with values in the centralizer of L in G.

By [KY], Cor. 3.3.7, here are some triples (G, H, L) that Theorem 1.1 applies to:

(1) $(\mathrm{SO}(2n,2), \mathrm{SO}(2n,1), \mathrm{U}(n,1))$ for $n \geq 1$,
(2) $(\mathrm{SO}(2n,2), \mathrm{U}(n,1), \mathrm{SO}(2n,1))$ for $n \geq 1$,
(3) $(\mathrm{U}(2n,2), \mathrm{Sp}(n,1), \mathrm{U}(2n,1))$ for $n \geq 1$,
(4) $(\mathrm{SO}(8,8), \mathrm{SO}(8,7), \mathrm{Spin}(8,1))$,
(5) $(\mathrm{SO}(8,\mathbb{C}), \mathrm{SO}(7,\mathbb{C}), \mathrm{Spin}(7,1))$,
(6) $(\mathrm{SO}^*(8), \mathrm{U}(1,3), \mathrm{Spin}(6,1))$,
(7) $(\mathrm{SO}^*(8), \mathrm{Spin}(6,1), \mathrm{U}(1,3))$,
(8) $(\mathrm{SO}^*(8), \mathrm{SO}^*(6) \times \mathrm{SO}^*(2), \mathrm{Spin}(6,1))$,
(9) $(\mathrm{SO}(4,4), \mathrm{Spin}(4,3), \mathrm{SO}(4,1))$,
(10) $(\mathrm{SO}(4,3), \mathrm{G}_{2(2)}, \mathrm{SO}(4,1))$.

As mentioned above, we wish to deform standard compact quotients of G/H into nonstandard ones, which are in some sense more generic. The best that we may hope for is to obtain Zariski-dense discrete subgroups of G acting properly discontinuously, freely, and cocompactly on G/H. Of course, even when L has real rank 1, nontrivial deformations in G of uniform lattices Γ of L do not always exist. For instance, if L is semisimple, noncompact, with no quasisimple factor locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$, then the first cohomology group $H^1(\Gamma,\mathfrak{g})$ vanishes by [Rag], Th. 1, and so Γ is locally rigid in G by [Wei]. (Here $\mathfrak{g}$ denotes the Lie algebra of G.)

For $(G,H,L)=(\mathrm{SO}(2n,2),\mathrm{SO}(2n,1),\mathrm{U}(n,1))$ with $n\geq 2$, uniform lattices Γ of L are not locally rigid in G, but a small deformation of Γ will never provide a Zariski-dense subgroup of G. Indeed, by [Rag] and [Wei] there is a neighborhood of the natural inclusion in $\mathrm{Hom}(\Gamma,G)$ whose elements are all homomorphisms of the form $\gamma\mapsto\gamma\psi(\gamma)$, up to conjugation, where $\psi:\Gamma\to\mathrm{SO}(2n,2)$ is a homomorphism with values in the center of $\mathrm{U}(n,1)$.

On the other hand, for $(G,H,L)=(\mathrm{SO}(2n,2),\mathrm{U}(n,1),\mathrm{SO}(2n,1))$ with $n\geq 1$, there do exist small Zariski-dense deformations in G of certain uniform lattices of L (see Section 6): such deformations can be obtained by a bending construction due to Johnson and Millson [JM]. Theorem 1.1 therefore implies the following.

Corollary 1.2. *For any $n\geq 1$, there are Zariski-dense discrete subgroups of $\mathrm{SO}(2n,2)$ acting properly discontinuously, freely, and cocompactly on $\mathrm{SO}(2n,2)/\mathrm{U}(n,1)$.*

Note that by [KY], Prop. 3.2.7, the homogeneous space $\mathrm{SO}(2n,2)/\mathrm{U}(n,1)$ is a pseudo-Riemannian symmetric space of signature $(2n,n^2-1)$.

The existence of compact quotients of reductive homogeneous spaces G/H by Zariski-dense discrete subgroups of G was known so far only when H is compact or when $G/H=(G_0\times G_0)/\Delta_{G_0}$ for certain groups G_0.

1.2. **Deformation of properly discontinuous actions over a general local field.** We prove that the properness of the action is preserved under small deformations not only for real groups, but more generally for algebraic groups over any local field $\mathbf{k}$. By a local field we mean $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or the field $\mathbb{F}_q((t))$ of formal Laurent series over a finite field $\mathbb{F}_q$. Moreover we relax the assumption that Γ is a torsion-free uniform lattice of L, in the following way.

Theorem 1.3. *Let $\mathbf{k}$ be a local field, G the set of $\mathbf{k}$-points of a reductive algebraic $\mathbf{k}$-group $\mathbf{G}$, and H (resp. L) the set of $\mathbf{k}$-points of a closed reductive subgroup $\mathbf{H}$ (resp. $\mathbf{L}$) of $\mathbf{G}$. Assume that $\mathrm{rank}_{\mathbf{k}}(\mathbf{L})=1$ and that L acts properly on G/H. If $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$, let Γ be a torsion-free convex cocompact subgroup of L; if $\mathbf{k}$ is non-Archimedean, let Γ be any torsion-free finitely generated discrete subgroup of L. There is a neighborhood $\mathcal{U}\subset\mathrm{Hom}(\Gamma,G)$ of the natural inclusion such that if $\varphi\in\mathcal{U}$, then φ is injective, $\varphi(\Gamma)$ is discrete in G, and $\varphi(\Gamma)$ acts properly discontinuously on G/H.*

Recall that for $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$, a discrete subgroup of L is called *convex cocompact* if it acts cocompactly on the convex hull of its limit set in the symmetric space of L, this convex hull being nonempty. Convex cocompact groups include uniform lattices, but also discrete groups of infinite covolume such as Schottky groups. For non-Archimedean $\mathbf{k}$, every torsion-free finitely generated discrete subgroup of L satisfies an analogue of the convex cocompactness property in the *Bruhat–Tits tree* of L (see the proof of Proposition 4.1).

For $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, Theorem 1.1 follows from Theorem 1.3 and from a cohomological argument due to Kobayashi (see Section 5.4). This argument, which does not transpose to the non-Archimedean case, implies that the action of Γ on G/H remains cocompact whenever it remains properly discontinuous and the deformation is injective.

Note that in characteristic zero, every finitely generated subgroup of L is virtually torsion-free by Selberg's lemma ([Sel], Lem. 8), hence the "torsion-free" assumption in Theorem 1.3 may easily be removed in this case.

1.3. **Translation in terms of a Cartan projection.** Let $\mathbf{k}$ be a local field and G the set of $\mathbf{k}$-points of a connected reductive algebraic $\mathbf{k}$-group. Fix a Cartan projection $\mu : G \to E^+$ of G, where E^+ is a closed convex cone in a real finite-dimensional vector space E (see Section 2). For any closed subgroup H of G, the *properness criterion* of Benoist ([Ben], Cor. 5.2) and Kobayashi ([Ko2], Th. 1.1) translates the properness of the action on G/H of a subgroup Γ of G in terms of μ. Using this criterion (see Section 5.4), Theorem 1.3 is a consequence of the following result, where we fix a norm $\|\cdot\|$ on E.

Theorem 1.4. *Let $\mathbf{k}$ be a local field, G the set of $\mathbf{k}$-points of a connected reductive algebraic $\mathbf{k}$-group $\mathbf{G}$, and L the set of $\mathbf{k}$-points of a closed reductive subgroup $\mathbf{L}$ of $\mathbf{G}$ of $\mathbf{k}$-rank 1. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, let Γ be a torsion-free convex cocompact subgroup of L; if $\mathbf{k}$ is non-Archimedean, let Γ be any torsion-free finitely generated discrete subgroup of L. For any $\varepsilon > 0$, there is a neighborhood $\mathcal{U}_\varepsilon \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that*

$$\big\|\mu(\varphi(\gamma)) - \mu(\gamma)\big\| \leq \varepsilon \|\mu(\gamma)\|$$

for all $\varphi \in \mathcal{U}_\varepsilon$ and all $\gamma \in \Gamma$.

1.4. **Ideas of proofs.** The core of the paper is the proof of Theorem 1.4. We start by recalling, in Section 2, that certain linear forms ℓ on E are connected to representations (V, ρ) of G by relations of the form

$$\ell(\mu(g)) = \log \|\rho(g)\|_V$$

for all $g \in G$, where $\|\cdot\|_V$ is the operator norm on $\mathrm{End}(V)$ with respect to some fixed norm on V. We are thus led to bound ratios of the form $\|\rho(\varphi(\gamma))\|_V / \|\rho(\gamma)\|_V$ for $\gamma \in \Gamma \smallsetminus \{1\}$, where $\varphi \in \mathrm{Hom}(\Gamma, G)$ is close to the natural inclusion of Γ in G.

In order to bound these ratios we look at the dynamics of G acting on the projective space $\mathbb{P}(V)$, notably the dynamics of elements $g \in G$ that are *proximal* in $\mathbb{P}(V)$. By definition, such elements admit an attracting fixed point and a repelling projective hyperplane in $\mathbb{P}(V)$. In Section 3 we consider products $z_1 k_2 z_2 \dots k_n z_n$ of proximal elements z_i having a common attracting fixed point x_0^+ and a common repelling hyperplane X_0^-, with isometries k_i such that $k_i \cdot x_0^+$ remains bounded away from X_0^-. We describe the contraction properties of $z_1k_2z_2\dots k_nz_n$ on $\mathbb{P}(V) \smallsetminus X_0^-$ in terms of the contraction properties of the z_i.

In Section 4 we see how such dynamical considerations apply to the elements $\gamma \in \Gamma$ and their images $\varphi(\gamma)$ under a small deformation $\varphi \in \mathrm{Hom}(\Gamma, G)$. We use Guichard's idea [Gui] of writing every element $\gamma \in \Gamma$ as a product $\gamma_0 \dots \gamma_n$ of elements of a fixed finite subset F of Γ, where the norms $\|\mu(\gamma_i)\|$ and $\|\mu(\gamma_i\gamma_{i+1}) - \mu(\gamma_i) - \mu(\gamma_{i+1})\|$ are controlled for all i.

In Section 5 we combine the results of Sections 3 and 4 by carefully choosing the finite subset F of Γ in order to get a sharp control of the ratios $\|\rho(\varphi(\gamma))\|_V / \|\rho(\gamma)\|_V$, or equivalently of $\ell(\mu(\varphi(\gamma))-\mu(\gamma))$ for $\gamma \in \Gamma\smallsetminus\{1\}$. From this we deduce Theorem 1.4.

At the end of Section 5 we explain how Theorems 1.1 and 1.3 follow from Theorem 1.4. Finally, in Section 6 we establish Corollary 1.2 by relating Theorem 1.1 to Johnson and Millson's bending construction.

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2. Cartan projections, maximal parabolic subgroups, and representations

Throughout the paper, $\mathbf{k}$ denotes a local field, *i.e.* $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or the field $\mathbb{F}_q((t))$ of formal Laurent series over a finite field $\mathbb{F}_q$. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, we denote by $|\cdot|$ the usual absolute value on $\mathbf{k}$. If $\mathbf{k}$ is non-Archimedean, we denote by $\mathcal{O}$ the ring of integers of $\mathbf{k}$, by q the cardinal of its residue field, by π a uniformizer, by ω the (additive) valuation on $\mathbf{k}$ such that $\omega(\pi) = 1$, and by $|\cdot| = q^{-\omega(\cdot)}$ the corresponding (multiplicative) absolute value. If $\mathbf{G}$ is an algebraic group, we denote by G the set of its $\mathbf{k}$-points and by $\mathfrak{g}$ its Lie algebra.

In this section, we recall a few well-known facts on connected reductive algebraic $\mathbf{k}$-groups and their Cartan projections.

2.1. Weyl chambers.

Fix a connected reductive algebraic $\mathbf{k}$-group $\mathbf{G}$. The derived group $\mathbf{D}(\mathbf{G})$ is semisimple, the identity component $\mathbf{Z}(\mathbf{G})^\circ$ of the center of $\mathbf{G}$ is a torus, which is trivial if $\mathbf{G}$ is semisimple, and $\mathbf{G}$ is the almost product of $\mathbf{D}(\mathbf{G})$ and $\mathbf{Z}(\mathbf{G})^\circ$. Recall that the $\mathbf{k}$-split tori of $\mathbf{G}$ are all conjugate over $\mathbf{k}$. Fix such a torus $\mathbf{A}$ and let $\mathbf{N}$ (resp. $\mathbf{Z}$) denote its normalizer (resp. centralizer) in $\mathbf{G}$. The groups $X(\mathbf{A})$ of $\mathbf{k}$-characters and $Y(\mathbf{A})$ of $\mathbf{k}$-cocharacters of $\mathbf{A}$ are free $\mathbb{Z}$-modules whose rank is by definition the $\mathbf{k}$-rank of $\mathbf{G}$, and there is a perfect pairing

$$\langle \cdot , \cdot \rangle : X(\mathbf{A}) \times Y(\mathbf{A}) \longrightarrow \mathbb{Z}$$

given by $\langle \chi, \psi \rangle = \chi \circ \psi \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$. This pairing extends to a nondegenerate bilinear form $\langle \cdot, \cdot \rangle : (X(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R}) \times (Y(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$.

Note that $\mathbf{A}$ is the almost product of $(\mathbf{A}\cap\mathbf{D}(\mathbf{G}))^\circ$ and $(\mathbf{A}\cap\mathbf{Z}(\mathbf{G}))^\circ$, hence $X(\mathbf{A})\otimes_{\mathbb{Z}}\mathbb{R}$ is the direct sum of $X((\mathbf{A}\cap\mathbf{D}(\mathbf{G}))^\circ)\otimes_{\mathbb{Z}}\mathbb{R}$ and $X((\mathbf{A}\cap\mathbf{Z}(\mathbf{G}))^\circ)\otimes_{\mathbb{Z}}\mathbb{R}$. The set $\Phi = \Phi(\mathbf{A}, \mathbf{G})$ of restricted roots of $\mathbf{A}$ in $\mathbf{G}$, *i.e.* the set of nontrivial weights of $\mathbf{A}$ in the adjoint representation of $\mathbf{G}$, is a root system of $X((\mathbf{A} \cap \mathbf{D}(\mathbf{G}))^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$. For $\alpha \in \Phi$, let $\check{\alpha}$ be the corresponding coroot: by definition, $\langle \alpha, \check{\alpha} \rangle = 2$ and $s_\alpha : x \mapsto x - \langle x, \check{\alpha} \rangle\, \alpha$ is the unique reflection of

$X(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\Phi) = \Phi$. The group $W = N/Z$ is finite and identifies with the Weyl group of Φ, generated by the reflections s_α.

Similarly, $E = Y(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ is the direct sum of $E_D = Y((\mathbf{A} \cap \mathbf{D}(\mathbf{G}))^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$ and $E_Z = Y((\mathbf{A} \cap \mathbf{Z}(\mathbf{G}))^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$. The group $W = N/Z$ acts trivially on E_Z and identifies with the Weyl group of the root system $\check{\Phi} = \{\check{\alpha},\ \alpha \in \Phi\}$ of E_D, generated by the reflections $s_{\check{\alpha}} : y \mapsto y - \langle \alpha, y \rangle\, \check{\alpha}$. We refer to [BoT] and [Se1], Chap. V, for proofs and more detail.

If $\mathbf{k}$ is non-Archimedean, set $A^\circ = A$; if $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, set

$$A^\circ = \big\{ a \in A\,, \quad \chi(a) \in]0, +\infty[\quad \forall \chi \in X(\mathbf{A}) \big\}.$$

Choose a basis Δ of Φ and let

$$\begin{array}{rrcl} & A^+ & = & \big\{ a \in A^\circ, \quad |\alpha(a)| \geq 1 \quad \forall \alpha \in \Delta \big\} \\ \big(\text{resp.} & E^+ & = & \big\{ y \in E, \quad \langle \alpha, y \rangle \geq 0 \quad \forall \alpha \in \Delta \big\}\big) \end{array}$$

denote the corresponding closed positive Weyl chamber in A° (resp. in E). The set E^+ is a closed convex cone in the real vector space E. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, then E^+ identifies with $\log A^+ \subset \mathfrak{a}$.

2.2. **Cartan decompositions and Cartan projections.** If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, there is a maximal compact subgroup K of G such that the Cartan decomposition $G = KA^+K$ holds: for any $g \in G$, there are elements $k_g, \ell_g \in K$ and a unique $a_g \in A^+$ such that $g = k_g a_g \ell_g$ ([Hel], Chap. 9, Th. 1.1). Setting $\mu(g) = \log a_g$ defines a map $\mu : G \to E^+ \simeq \log A^+$, which is continuous, proper, and surjective. It is called the *Cartan projection* with respect to the Cartan decomposition $G = KA^+K$.

If $\mathbf{k}$ is non-Archimedean, let $\mathrm{Res} : X(\mathbf{Z}) \to X(\mathbf{A})$ denote the restriction homomorphism, where $X(\mathbf{Z})$ is the group of $\mathbf{k}$-characters of $\mathbf{Z}$. There is a unique group homomorphism $\nu : Z \to E$ such that

$$\langle \mathrm{Res}(\chi), \nu(z) \rangle = -\,\omega(\chi(z))$$

for all $\chi \in X(\mathbf{Z})$ and $z \in Z$. Let $Z^+ \subset Z$ denote the inverse image of E^+ under ν. There is a maximal compact subgroup K of G such that the *Cartan decomposition* $G = KZ^+K$ holds: for any $g \in G$, there are elements $k_g, \ell_g \in K$ and $z_g \in Z^+$ such that $g = k_g z_g \ell_g$, and $\nu(z_g)$ is uniquely defined. Setting $\mu(g) = \nu(z_g)$ defines a map $\mu : G \to E^+$, which is continuous and proper, and whose image $\mu(G)$ is the intersection of E^+ with a lattice of E. It is called the *Cartan projection* with respect to the Cartan decomposition $G = KZ^+K$. For proofs and more detail we refer to the original articles [BT1] and [BT2]; the reader may also find [Rou] a useful reference.

Note that for non-Archimedean $\mathbf{k}$, the decomposition $G = KA^+K$ does not hold in general. To unify notation, for $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ as well, we will denote by Z^+ the set of elements $z \in Z$ such that $|\chi(z)| \geq 1$ for all $\chi \in X(\mathbf{Z})$ satisfying $|\chi(a)| \geq 1$ for all $a \in A^+$.

2.3. **A geometric interpretation.** If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, let $X = G/K$ denote the Riemannian symmetric space of G. If $\mathbf{k}$ is non-Archimedean, let X denote the *Bruhat–Tits building* of G: it is a metric space on which G acts properly by isometries with a compact fundamental domain (see [BT1] or [Rou]); when $\mathrm{rank}_{\mathbf{k}}(\mathbf{G}) = 1$, it is a bipartite simplicial tree (see [Se2], § II.1, for $G = \mathrm{SL}_2(\mathbf{k})$). In both cases (Archimedean or not), the group K is the stabilizer of some point $x_0 \in X$, and there is a W-invariant Euclidean norm $\|\cdot\|$ on E such that $\|\mu(z)\| = d(x_0, z\cdot x_0)$ for all $z \in Z$, where d is the metric on X. Since G acts on X by isometries and K fixes x_0,

$$\|\mu(g)\| = d(x_0, g\cdot x_0) \tag{2.1}$$

for all $g \in G$. In particular,

$$\|\mu(gg')\| \ \leq\ \|\mu(g)\| + \|\mu(g')\| \tag{2.2}$$

for all $g, g' \in G$. In fact, the following stronger inequalities hold (see for instance [Ka1], Lem. 2.3): for all $g, g' \in G$,

$$\begin{cases} \|\mu(gg') - \mu(g')\| \ \leq\ \|\mu(g)\|, \\ \|\mu(gg') - \mu(g)\| \ \leq\ \|\mu(g')\|. \end{cases} \tag{2.3}$$

The following observation will be useful in Section 5.1.

Remark 2.1. If $\mu' : G \to E'^{+}$ is another Cartan projection, then there exist a linear isomorphism $i : E' \to E$ mapping E'^{+} to E^{+} and a constant $C > 0$ such that

$$\|i \circ \mu'(g) - \mu(g)\| \leq C$$

for all $g \in G$.

Indeed, let $\mu' : G \to E'^{+}$ be the Cartan projection associated with a Cartan decomposition $G = K'A'^{+}K'$ or $G = K'Z'^{+}K'$, where K' is a maximal compact subgroup of G and $\mathbf{A}'$ a maximal $\mathbf{k}$-split torus of $\mathbf{G}$ with centralizer $\mathbf{Z}'$. By [BoT], Th. 4.21, we have $\mathbf{A}' = g_0\mathbf{A}g_0^{-1}$ for some $g_0 \in G$. The map $X(\mathbf{A}') \to X(\mathbf{A})$ sending χ to $\chi(g_0 \cdot g_0^{-1})$ identifies the set Φ' of restricted roots of $\mathbf{A}'$ in $\mathbf{G}$ with Φ; the inverse image of Δ is the basis Δ' of Φ' defining A'^{+}. The map $Y(\mathbf{A}') \to Y(\mathbf{A})$ sending ψ to $g_0^{-1}\psi g_0$ induces a linear isomorphism i between $E' = Y(\mathbf{A}') \otimes_{\mathbb{Z}} \mathbb{R}$ and $E = Y(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R}$, such that $i(E'^{+}) = E^{+}$. By construction,

$$i \circ \mu'\big(k'(g_0 z g_0^{-1})\ell'\big) = i \circ \mu'(g_0 z g_0^{-1}) = \mu(z)$$

for all $k', \ell' \in K'$ and $z \in Z$. By (2.3), this implies

$$\|i \circ \mu'(g) - \mu(g)\| \leq \|\mu(g_0)\| + \|\mu(g_0^{-1})\| + 2 \max_{k' \in K'} \|\mu(k')\|$$

for all $g \in G$.

2.4. **Maximal parabolic subgroups.** For $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ denote the corresponding unipotent subgroup of $\mathbf{G}$, with Lie algebra $\mathfrak{u}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, where

$$\mathfrak{g}_{i\alpha} = \big\{X \in \mathfrak{g}, \quad \mathrm{Ad}(a)(X) = \alpha(a)^i X \quad \forall a \in A\big\}$$

for $i = 1, 2$. For any subset θ of Δ, let $\mathbf{P}_\theta$ denote the corresponding *standard* parabolic subgroup of $\mathbf{G}$, with Lie algebra

$$\mathfrak{p}_\theta = \mathfrak{z} \oplus \Big(\bigoplus_{\beta \in \Phi^+} \mathfrak{u}_\beta\Big) \oplus \Big(\bigoplus_{\beta \in \mathbb{N}(\Delta \smallsetminus \theta) \cap \Phi} \mathfrak{u}_{-\beta}\Big),$$

where $\Phi^+ \subset \Phi$ is the set of positive roots and $\mathbb{N}(\Delta \smallsetminus \theta)$ the set of linear combinations of elements of $\Delta \smallsetminus \theta$ with nonnegative integer coefficients. Every parabolic $\mathbf{k}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ is conjugate over $\mathbf{k}$ to a unique standard one. In particular, the maximal proper parabolic $\mathbf{k}$-subgroups of $\mathbf{G}$ are the conjugates of the groups $\mathbf{P}_\alpha = \mathbf{P}_{\{\alpha\}}$, where $\alpha \in \Delta$.

Fix $\alpha \in \Delta$. Since $\mathbf{P}_\alpha$ is its own normalizer in $\mathbf{G}$, the *flag variety* $\mathbf{G}/\mathbf{P}_\alpha$ parametrizes the set of parabolic $\mathbf{k}$-subgroups that are conjugate to $\mathbf{P}_\alpha$. It is a projective variety, defined over $\mathbf{k}$. Let $\mathbf{N}_\alpha^-$ denote the unipotent subgroup of $\mathbf{G}$ generated by the groups $\mathbf{U}_{-\beta}$ for $\beta \in (\alpha + \mathbb{N}\Delta) \cap \Phi$, with Lie algebra

$$\mathfrak{n}_\alpha^- = \bigoplus_{\beta \in (\alpha+\mathbb{N}\Delta)\cap\Phi} \mathfrak{u}_{-\beta}.$$

The variety $\mathbf{G}/\mathbf{P}_\alpha$ is the disjoint union of its subvarieties $\mathbf{N}_\alpha^- w \mathbf{P}_\alpha$, where $w \in W$; the "big cell" $\mathbf{N}_\alpha^- \mathbf{P}_\alpha$ is the only one with codimension zero. We refer to [BoT] for proofs and more detail.

2.5. **Representations of G.** For $\alpha \in \Delta$, let $\omega_\alpha \in X(\mathbf{A})$ denote the corresponding fundamental weight: by definition, $\langle \omega_\alpha, \check{\alpha} \rangle = 1$ and $\langle \omega_\alpha, \check{\beta} \rangle = 0$ for all $\beta \in \Delta \smallsetminus \alpha$. By [Tit], Th. 7.2, there is an irreducible $\mathbf{k}$-representation (ρ_α, V_α) of $\mathbf{G}$ whose highest weight χ_α is a positive multiple of ω_α and whose highest weight space x_α^+ is a line. The point $x_\alpha^+ \in \mathbb{P}(V_\alpha)$ is the unique fixed point of P_α in $\mathbb{P}(V_\alpha)$. The map from $\mathbf{G}/\mathbf{P}_\alpha$ to $\mathbb{P}(V_\alpha)$ sending $g\mathbf{P}_\alpha$ to $\rho_\alpha(g)(x_\alpha^+)$ is a closed immersion. We denote the set of restricted weights of (ρ_α, V_α) by Λ_α and, for every $\lambda \in \Lambda_\alpha$, the weight space of λ by $(V_\alpha)_\lambda$.

If $\mathbf{k} = \mathbb{R}$ (resp. if $\mathbf{k} = \mathbb{C}$), then the weight spaces are orthogonal with respect to some K-invariant Euclidean (resp. Hermitian) norm $\|\cdot\|_\alpha$ on V_α. The corresponding operator norm $\|\cdot\|_\alpha$ on $\mathrm{End}(V_\alpha)$ satisfies

$$\|\rho_\alpha(g)\|_\alpha = e^{\langle \chi_\alpha, \mu(g) \rangle} \tag{2.4}$$

for all $g \in G$. If $\mathbf{k}$ is non-Archimedean, then there is a K-invariant ultrametric norm $\|\cdot\|_\alpha$ on V_α such that

$$\Big\| \sum_{\lambda \in \Lambda_\alpha} v_\lambda \Big\|_\alpha = \max_{\lambda \in \Lambda_\alpha} \|v_\lambda\|_\alpha$$

for all $(v_\lambda) \in \prod_{\lambda \in \Lambda_\alpha} (V_\alpha)_\lambda$ and such that the restriction of $\rho_\alpha(z)$ to $(V_\alpha)_\lambda$ is a dilation of factor $q^{\langle \lambda, \nu(z) \rangle}$ for all $z \in Z$ and all $\lambda \in \Lambda_\alpha$ ([Qui], Th. 6.1). The corresponding operator norm $\|\cdot\|_\alpha$ on $\mathrm{End}(V_\alpha)$ satisfies

$$\|\rho_\alpha(g)\|_\alpha = q^{\langle \chi_\alpha, \mu(g) \rangle} \tag{2.5}$$

for all $g \in G$.

2.6. **The example of $\mathbf{GL}_n$.** Let $\mathbf{G} = \mathbf{GL}_n$ for some integer $n \geq 2$. It is the almost product of its derived group $\mathbf{D}(\mathbf{G}) = \mathbf{SL}_n$ and of its center $\mathbf{Z}(\mathbf{G})$, which is the group of invertible scalar matrices. The full group $\mathbf{A}$ of invertible diagonal matrices is a maximal $\mathbf{k}$-split torus of $\mathbf{G}$, which is its own centralizer, *i.e.* $\mathbf{Z} = \mathbf{A}$. The corresponding root system Φ is the set of linear forms $\varepsilon_i - \varepsilon_j$, $1 \leq i \neq j \leq n$, where

$$\varepsilon_i\big(\mathrm{diag}(a_1, \dots, a_n)\big) = a_i.$$

The roots $\varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq n-1$, form a basis Δ of Φ. If $\mathbf{k}$ is Archimedean (resp. non-Archimedean), the corresponding positive Weyl chamber is

$$\begin{array}{rrcl} & A^+ & = & \big\{\mathrm{diag}(a_1, \dots, a_n) \in A,\ \ a_i \in]0, +\infty[\ \forall i \ \text{ and } a_1 \geq \dots \geq a_n\big\} \\ \big(\text{resp.} & A^+ & = & \big\{\mathrm{diag}(a_1, \dots, a_n) \in A,\ |a_1| \geq \dots \geq |a_n|\big\}\big). \end{array}$$

Set $K = \mathrm{O}(n)$ (resp. $K = \mathrm{U}(n)$, resp. $K = \mathrm{GL}_n(\mathcal{O})$) if $\mathbf{k} = \mathbb{R}$ (resp. if $\mathbf{k} = \mathbb{C}$, resp. if $\mathbf{k}$ is non-Archimedean). The Cartan decomposition $G = KA^+K$ holds. If $\mathbf{k} = \mathbb{R}$ (resp. if $\mathbf{k} = \mathbb{C}$) it follows from the polar decomposition in $\mathrm{GL}_n(\mathbb{R})$ (resp. in $\mathrm{GL}_n(\mathbb{C})$) and from the reduction of symmetric (resp. Hermitian) matrices. If $\mathbf{k}$ is non-Archimedean, it follows from the structure theorem for finitely generated modules over a principal ideal domain. The real vector space $E = \mathbb{R}^n$, the subspaces

$$E_D = \big\{(y_1, \dots, y_n) \in E,\ y_1 + \ldots + y_n = 0\big\}$$

and $E_Z = \mathbb{R} \cdot (1, \dots, 1)$, and the closed convex cone

$$E^+ = \big\{(y_1, \dots, y_n) \in E,\ y_1 \geq \ldots \geq y_n\big\}$$

do not depend on $\mathbf{k}$. Let $\mu : G \to E^+$ denote the Cartan projection with respect to the Cartan decomposition $G = KA^+K$. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, then $\mu(g) = (\frac{1}{2} \log t_i)_{1 \leq i \leq n}$ where t_i is the i^{th} eigenvalue of ${}^t\overline{g}g$. If $\mathbf{k}$ is non-Archimedean and if m is any integer such that $\pi^m g \in \mathrm{M}_n(\mathcal{O})$, then $\mu(g) = (\omega(t_{m,i}) - m)_{1 \leq i \leq n}$ where $t_{m,i}$ is the i^{th} invariant factor of $\pi^m g$.

Fix a simple root $\alpha = \varepsilon_{i_0} - \varepsilon_{i_0+1} \in \Delta$. The parabolic group $\mathbf{P}_\alpha$ is defined by the vanishing of the (i,j)-matrix entries for $1 \leq j \leq i_0 < i \leq n$. The flag variety $\mathbf{G}/\mathbf{P}_\alpha$ is the Grassmannian $\mathcal{G}(i_0, n)$ of i_0-dimensional subspaces of the affine space $\mathbb{A}^n$. The Lie algebra $\mathfrak{n}_\alpha^-$ is defined by the vanishing of the (i,j)-matrix entries for $1 \leq i \leq i_0$ and for $i_0 + 1 \leq i, j \leq n$. The decomposition of $\mathbf{G}/\mathbf{P}_\alpha$ as a disjoint union of subvarieties $\mathbf{N}_\alpha^- w \mathbf{P}_\alpha$, where $w \in W$, is the decomposition of the Grassmannian $\mathcal{G}(i_0, n)$ into Schubert cells. The representation (ρ_α, V_α) is the natural representation of $\mathbf{GL}_n$ in the wedge product $\Lambda^{i_0} \mathbb{A}^n$. Its highest weight is the fundamental weight

$$\omega_\alpha = \varepsilon_1 + \ldots + \varepsilon_{i_0}$$

associated with α. The embedding of the Grassmannian $\mathcal{G}(i_0, n)$ into the projective space $\mathbb{P}(V_\alpha) = \mathbb{P}(\Lambda^{i_0} \mathbb{A}^n)$ is the Plücker embedding.

3. Dynamics in projective spaces

In this section we look at the dynamics of certain endomorphisms of $\mathbf{k}$-vector spaces V in the corresponding projective spaces $\mathbb{P}(V)$, where $\mathbf{k}$ is a local field. In Section 3.1 we start by recalling the notion of proximality. We then consider products of the form $z_1 k_2 z_2 \ldots k_n z_n$, where the z_i are proximal elements with a common attracting fixed point $x_0^+ \in \mathbb{P}(V)$ and a common repelling hyperplane $X_0^- \subset \mathbb{P}(V)$, and the k_i are isometries with $k_i \cdot x_0^+$ bounded away from X_0^-. We estimate the norm of such a product in terms of the norms of the z_i. In Section 3.2 we consider a connected reductive algebraic $\mathbf{k}$-group $\mathbf{G}$ and apply the result of Section 3.1 to the representations (V_α, ρ_α) introduced in Section 2.5. From (2.4) and (2.5) we get an upper bound for $|\langle \chi_\alpha, \mu(g_1 \ldots g_n) - \mu(g_1) - \ldots - \mu(g_n) \rangle|$ for elements $g_1, \dots, g_n \in G$ satisfying certain contractivity and transversality conditions.

3.1. **Proximality in projective spaces and norm estimates.** Let $\mathbf{k}$ be a local field and V a finite-dimensional vector space over $\mathbf{k}$. If $\mathbf{k}$ is non-Archimedean, we fix a basis $(v_1,\dots,v_n)$ of V and endow V with the norm $\|\cdot\|_V$ defined by

$$\Big\| \sum_{1\leq j\leq n} t_j\, v_j \Big\|_V = \sup_{1\leq j\leq n} |t_j|\,;$$

if $\mathbf{k}=\mathbb{R}$ or $\mathbb{C}$, we endow V with any norm $\|\cdot\|_V$. We keep the notation $\|\cdot\|_V$ for the corresponding operator norm on $\mathrm{End}(V)$. We endow the projective space $\mathbb{P}(V)$ with the metric d defined by

$$d(x_1,x_2) = \inf\big\{\|v_1'-v_2'\|_V,\ v_i'\in x_i \text{ and } \|v_i'\|_V = 1\ \forall i=1,2\big\}.$$

Recall that an element $g\in\mathrm{End}(V)\smallsetminus\{0\}$ is called *proximal* if it has a unique eigenvalue of maximal absolute value and if this eigenvalue has multiplicity 1. (The eigenvalues of g belong to a finite extension $\mathbf{k}_g$ of $\mathbf{k}$ and we consider the unique extension to $\mathbf{k}_g$ of the absolute value $|\cdot|$ on $\mathbf{k}$.) If g is proximal, then its maximal eigenvalue belongs to $\mathbf{k}$; we denote by $x_g^+\in\mathbb{P}(V)$ the corresponding eigenline and by X_g^- the image in $\mathbb{P}(V)$ of the unique g-invariant complementary subspace of x_g^+ in V. Note that g acts on $\mathbb{P}(V)$ by contracting $\mathbb{P}(V)\smallsetminus X_g^-$ towards x_g^+. For $\varepsilon>0$, we will say that g is *ε-proximal* if it satisfies the two following additional conditions:

(1) $d(x_g^+,X_g^-)\geq 2\varepsilon$,
(2) for any $x\in\mathbb{P}(V)$, if $d(x,X_g^-)\geq\varepsilon$, then $d(g\cdot x,x_g^+)\leq\varepsilon$.

We will need the following lemma.

Lemma 3.1. *Let X_0^- be a projective hyperplane of $\mathbb{P}(V)$, let $x_0^+\in\mathbb{P}(V)\smallsetminus X_0^-$, and let $\varepsilon>0$ such that $d(x_0^+,X_0^-)\geq 2\varepsilon$. There exists $r_\varepsilon>0$ with the following property: for any isometries $k_2,\dots,k_n\in\mathrm{End}(V)$ with $d(k_i\cdot x_0^+,X_0^-)\geq 2\varepsilon$, and any ε-proximal endomorphisms $z_1,\dots,z_n\in\mathrm{End}(V)$ such that*

- $x_{z_i}^+ = x_0^+$,
- $X_{z_i}^- = X_0^-$,
- *the restriction of z_i to the line x_0^+ is a dilation of factor $\|z_i\|_V$,*

we have

$$e^{-(n-1)\,r_\varepsilon}\cdot\prod_{i=1}^n \|z_i\|_V \ \leq\ \|z_1k_2z_2\dots k_nz_n\|_V\ \leq\ \prod_{i=1}^n\|z_i\|_V.$$

Proof. The right-hand inequality follows from the submultiplicativity of the operator norm $\|\cdot\|_V$ and from the fact that k_i is an isometry of V for all i. Let us prove the left-hand inequality. Let $v_0\in V\smallsetminus\{0\}$ satisfy $x_0^+=\mathbf{k}v_0$ and let V_0 be the hyperplane of V such that $X_0^-=\mathbb{P}(V_0)$. Set

$$\begin{aligned} b_\varepsilon &= \{x\in\mathbb{P}(V),\ d(x,x_0^+)\leq\varepsilon\}\\ \text{and}\quad B_\varepsilon &= \{x\in\mathbb{P}(V),\ d(x,X_0^-)\geq\varepsilon\}.\end{aligned}$$

Note that the map sending $v\in V$ to the element $t\in\mathbf{k}$ such that $v\in tv_0+V_0$ is continuous and that the set of unitary vectors $v\in V$ with $\mathbf{k}v\in B_\varepsilon$ is compact, hence there exists $r_\varepsilon>0$ such that any $v\in V\smallsetminus\{0\}$ with $\mathbf{k}v\in B_\varepsilon$ belongs to tv_0+V_0 for some $t\in\mathbf{k}$ with

$$e^{-\frac{r_\varepsilon}{2}}\|v\|\ \leq\ |t|\ \leq\ e^{\frac{r_\varepsilon}{2}}\|v\|. \tag{3.1}$$

For $1 \leq j \leq n$, set

$$h_j = z_j k_{j+1} z_{j+1} \dots k_n z_n \in \mathrm{End}(V).$$

We claim that $h_j \cdot B_\varepsilon \subset b_\varepsilon$ and

$$\|h_j \cdot v_0\|_V \ \geq\ e^{-(n-j)\, r_\varepsilon} \cdot \prod_{i=j}^{n} \|z_i\|_V \tag{3.2}$$

for all j. This follows from an easy descending induction on j. Indeed, for all i we have $k_i \cdot b_\varepsilon \subset B_\varepsilon$ since k_i is an isometry of V and $d(k_i \cdot x_0^+, X_0^-) \geq 2\varepsilon$, and $z_i \cdot B_\varepsilon \subset b_\varepsilon$ since z_i is ε-proximal with $x_{z_i}^+ = x_0^+$ and $X_{z_i}^- = X_0^-$. By (3.1), we have $k_{j+1} h_{j+1} \cdot v_0 \in t_j v_0 + V_0$ for some $t_j \in \mathbf{k}$ with

$$|t_j| \ \geq\ e^{-\frac{r_\varepsilon}{2}} \, \|k_{j+1} h_{j+1} \cdot v_0\|_V \ =\ e^{-\frac{r_\varepsilon}{2}} \, \|h_{j+1} \cdot v_0\|_V .$$

Using the inductive assumption, we obtain

$$|t_j| \ \geq\ e^{-(n-j-\frac{1}{2})\, r_\varepsilon} \cdot \prod_{i=j+1}^{n} \|z_i\|_V .$$

By hypothesis, z_j preserves V_0 and induces on the line x_0^+ a dilation of factor $\|z_j\|_V$, hence $h_j \cdot v_0 = z_j k_{j+1} h_{j+1} \cdot v_0 \in t'_j v_0 + V_0$ for some $t'_j \in \mathbf{k}$ with

$$|t'_j| = \|z_j\|_V \, |t_j| \ \geq\ e^{-(n-j-\frac{1}{2})\, r_\varepsilon} \cdot \prod_{i=j}^{n} \|z_i\|_V .$$

Inequality (3.2) follows, using (3.1) again. □

3.2. **Cartan projection along the fundamental weights.** Lemma 3.1 implies the following result.

Proposition 3.2. *Let $\mathbf{k}$ be a local field and G the set of $\mathbf{k}$-points of a connected reductive algebraic $\mathbf{k}$-group $\mathbf{G}$. Let $G = KA^+K$ or $G = KZ^+K$ be a Cartan decomposition and $\mu : G \to E^+$ the corresponding Cartan projection. Fix $\alpha \in \Delta$ and let $\mathcal{C}_\alpha$ be a compact subset of N_α^-. There exist $r_\alpha, R_\alpha > 0$ such that if $g_1, \dots, g_n \in G$ satisfy $\langle \alpha, \mu(g_i) \rangle \geq R_\alpha$ and $g_i = k_{g_i} z_{g_i} \ell_{g_i}$ for some $k_{g_i}, \ell_{g_i} \in K$ and $z_{g_i} \in Z^+$ with $\ell_{g_i} k_{g_{i+1}} \in \mathcal{C}_\alpha P_\alpha$ for all i, then*

$$\Big| \Big\langle \chi_\alpha, \mu(g_1 \dots g_n) - \sum_{i=1}^{n} \mu(g_i) \Big\rangle \Big| \leq n r_\alpha .$$

We keep the notation of Section 2. In particular, given a simple root $\alpha \in \Delta$, we denote by $\mathbf{N}_\alpha^-$ (resp. by $\mathbf{P}_\alpha$) the unipotent (resp. parabolic) subgroup of $\mathbf{G}$ introduced in Section 2.4, and by χ_α the highest weight of the representation (V_α, ρ_α) introduced in Section 2.5. We endow V_α with a norm $\| \cdot \|_\alpha$ as in Section 2.5, and $\mathbb{P}(V_\alpha)$ with a metric d as in Section 3.1. Proposition 3.2 follows from Lemma 3.1, from (2.4) and (2.5), and from the following lemma.

Lemma 3.3. *Let $x_\alpha^+ \in \mathbb{P}(V_\alpha)$ be the highest weight line $(V_\alpha)_{\chi_\alpha}$ and let X_α^- be the image in $\mathbb{P}(V_\alpha)$ of the sum of the weight spaces $(V_\alpha)_\lambda$ for $\lambda \in \Lambda_\alpha \smallsetminus \{\chi_\alpha\}$.*

(1) *Given $\varepsilon > 0$ with $d(x_\alpha^+, X_\alpha^-) \geq 2\varepsilon$, there exists $R_\alpha > 0$ such that for any $z \in Z^+$ with $\langle \alpha, \mu(z) \rangle \geq R_\alpha$, the element $\rho_\alpha(z)$ is ε-proximal in $\mathbb{P}(V_\alpha)$ with $x_{\rho_\alpha(z)}^+ = x_\alpha^+$ and $X_{\rho_\alpha(z)}^- = X_\alpha^-$.*

(2) *We have* $\rho_\alpha(N_\alpha^-)(x_\alpha^+) \cap X_\alpha^- = \emptyset$.

Proof. (1) It is sufficient to see that every restricted weight of (ρ_α, V_α) except χ_α belongs to $\chi_\alpha - \alpha - \mathbb{N}\Delta$. Consider the subgroup W_α of W generated by the reflections $s_\beta : x \mapsto x - \langle x, \check{\beta}\rangle\, \beta$ for $\beta \in \Delta \smallsetminus \{\alpha\}$. It fixes χ_α since χ_α is a multiple of ω_α and $\langle \omega_\alpha, \check{\beta}\rangle = 0$ for all $\beta \in \Delta \smallsetminus \{\alpha\}$. It is the Weyl group of the root subsystem of Φ generated by $\Delta \smallsetminus \{\alpha\}$, hence the "longest" element w of W_α exchanges $\mathbb{N}(\Delta \smallsetminus \{\alpha\})$ and $-\mathbb{N}(\Delta \smallsetminus \{\alpha\})$. Thus for every weight $\lambda \in \chi_\alpha - \mathbb{N}(\Delta \smallsetminus \{\alpha\})$ we have $w \cdot \lambda \in \chi_\alpha + \mathbb{N}\Delta$, which implies that $\lambda = \chi_\alpha$.

(2) For $n \in N_\alpha^-$, the identity element $1 \in G$ belongs to the closure of the conjugacy class $\{znz^{-1},\ z \in Z\}$, hence x_α^+ belongs to the closure of the orbit $\rho_\alpha(Zn)(x_\alpha^+)$ in $\mathbb{P}(V_\alpha)$. But X_α^- is closed in $\mathbb{P}(V_\alpha)$, stable under Z, and does not contain x_α^+. □

Proof of Proposition 3.2. The point $x_\alpha^+ \in \mathbb{P}(V_\alpha)$ is fixed by P_α. Moreover, $\rho_\alpha(N_\alpha^-)(x_\alpha^+) \cap X_\alpha^- = \emptyset$ by Lemma 3.3, hence there exists $\varepsilon > 0$ such that

$$d\big(\rho_\alpha(\mathcal{C}_\alpha P_\alpha)(x_\alpha^+), X_\alpha^-\big) \geq 2\varepsilon.$$

Let $r_\varepsilon > 0$ be given by Lemma 3.1 with respect to this ε and to $(V, X_0^-, x_0^+) = (V_\alpha, X_\alpha^-, x_\alpha^+)$, and let $r_\alpha = r_\varepsilon / \log q$, where $q = e$ if $\mathbf{k}$ is Archimedean and q is the cardinal of the residue field of $\mathcal{O}$ otherwise. Let $R_\alpha > 0$ be given by Lemma 3.3. We claim that r_α and R_α satisfy the conclusions of Proposition 3.2. Indeed, let $g_1, \dots, g_n \in G$ satisfy $\langle \alpha, \mu(g_i)\rangle \geq R_\alpha$ and $g_i = k_{g_i} z_{g_i} \ell_{g_i}$ for some $k_{g_i}, \ell_{g_i} \in K$ and $z_{g_i} \in Z^+$ with $\ell_{g_i} k_{g_{i+1}} \in \mathcal{C}_\alpha P_\alpha$ for all i. By Lemma 3.3, the element $\rho_\alpha(z_{g_i})$ is ε-proximal in $\mathbb{P}(V_\alpha)$ with $x^+_{\rho_\alpha(z_{g_i})} = x_\alpha^+$ and $X^-_{\rho_\alpha(z_{g_i})} = X_\alpha^-$. Moreover, the restriction of $\rho_\alpha(z_{g_i})$ to the line x_α^+ is a dilation of factor $\|\rho_\alpha(z_{g_i})\|_\alpha$. By Lemma 3.1,

$$q^{-nr_\alpha} \cdot \prod_{i=1}^n \|\rho_\alpha(z_{g_i})\|_\alpha \ \leq\ \|\rho_\alpha(g_1 \dots g_n)\|_\alpha \ \leq\ \prod_{i=1}^n \|\rho_\alpha(z_{g_i})\|_\alpha.$$

Using (2.4) and (2.5), we get

$$\Big\langle \chi_\alpha, \sum_{i=1}^n \mu(g_i)\Big\rangle - nr_\alpha \ \leq\ \langle \chi_\alpha, \mu(g_1 \dots g_n)\rangle \ \leq\ \Big\langle \chi_\alpha, \sum_{i=1}^n \mu(g_i)\Big\rangle. \quad \square$$

4. Transverse products

In this section we explain how, under the assumptions of Theorem 1.4, Proposition 3.2 applies to the elements $\gamma \in \Gamma$ and their images $\varphi(\gamma)$ under a small deformation $\varphi \in \mathrm{Hom}(\Gamma, G)$. We use Guichard's idea [Gui] of writing every element $\gamma \in \Gamma$ as a "transverse product" $\gamma_0 \dots \gamma_n$ of elements of a fixed finite subset F of Γ. The terminology "transverse product" is explained in Section 4.2.

4.1. **Transversality in L.** Let $\mathbf{k}$ be a local field and $\mathbf{L}$ a connected reductive algebraic $\mathbf{k}$-group of $\mathbf{k}$-rank 1. Fix a Cartan decomposition $L = K_L A_L^+ K_L$ or $L = K_L Z_L^+ K_L$, where K_L is a maximal compact subgroup of L and $\mathbf{A_L}$ a maximal $\mathbf{k}$-split torus of $\mathbf{L}$, with centralizer $\mathbf{Z_L}$ in $\mathbf{L}$. Let $\mu_L : L \to E_L^+$ denote the corresponding Cartan projection, where $E_L = Y(\mathbf{A_L}) \otimes_{\mathbb{Z}} \mathbb{R}$. Since $\mathbf{L}$ has $\mathbf{k}$-rank 1, the vector space E_L is a line, and any isomorphism from E_L to $\mathbb{R}$ gives a Cartan projection $\mu_L^{\mathbb{R}} : L \to \mathbb{R}$.

If $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 1, then $\mu_L^{\mathbb{R}}$ takes only nonnegative or only nonpositive values. We denote by α_L the indivisible positive restricted root of $\mathbf{A_L}$ in $\mathbf{L}$, by $\mathbf{P_L} = \mathbf{P}_{\alpha_L}$ the proper parabolic subgroup of $\mathbf{L}$ associated with α_L, and by $\mathbf{N_L^-} = \mathbf{U}_{-\alpha_L}$ the unipotent subgroup associated with $-\alpha_L$.

If $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 0, then $\mathbf{A_L}$ is central in $\mathbf{L}$, hence $\mathbf{Z_L} = \mathbf{L}$. In this case $\mu_L^{\mathbb{R}}$ is a group homomorphism from L to $\mathbb{R}$, thus taking both positive and negative values. We set $\mathbf{P_L} = \mathbf{Z_L} = \mathbf{L}$ and $\mathbf{N_L^-} = \{1\}$.

For the reader's convenience, we give a proof of the following result, which is due to Guichard in the real semisimple case ([Gui], Lem. 7 & 9). We consider the more general situation of a reductive algebraic group over a local field.

Proposition 4.1 (Guichard)**.** *Let $\mathbf{k}$ be a local field, L the set of $\mathbf{k}$-points of a connected reductive algebraic $\mathbf{k}$-group $\mathbf{L}$ of $\mathbf{k}$-rank 1, and $\mu_L^{\mathbb{R}} : L \to \mathbb{R}$ a Cartan projection. If $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, let Γ be a convex cocompact subgroup of L; if $\mathbf{k}$ is non-Archimedean, let Γ be any finitely generated discrete subgroup of L. There exist $D > 0$ and a compact subset $\mathcal{C}_L$ of N_L^- such that for $R \geq 2D$, any $\gamma \in \Gamma$ may be written as $\gamma = \gamma_0 \dots \gamma_n$ for some $\gamma_0, \dots, \gamma_n \in \Gamma$ satisfying the following conditions:*

1. *$|\mu_L^{\mathbb{R}}(\gamma_0)| \leq R + D$ and $R - D \leq |\mu_L^{\mathbb{R}}(\gamma_i)| \leq R + D$ for all $1 \leq i \leq n$,*
2. *if $\gamma_i = k_{\gamma_i} z_{\gamma_i} \ell_{\gamma_i}$ for some $k_{\gamma_i}, \ell_{\gamma_i} \in K_L$ and $z_{\gamma_i} \in Z_L^+$, then $\ell_{\gamma_i} k_{\gamma_{i+1}} \in \mathcal{C}_L P_L$ for all $1 \leq i \leq n-1$,*
3. *$\mu_L^{\mathbb{R}}(\gamma_1), \dots, \mu_L^{\mathbb{R}}(\gamma_n)$ all have the same sign as $\mu_L^{\mathbb{R}}(\gamma)$ if $n \geq 2$.*

To prove Proposition 4.1 we use the following lemma, which translates Condition (2) in terms of $\mu_L^{\mathbb{R}}$.

Lemma 4.2. *Under the assumptions of Proposition 4.1, there exists $D_0 \geq 0$ with the following property: given any $D \geq D_0$, there is a compact subset $\mathcal{C}_L$ of N_L^- such that for $k \in K_L$, if*

$$|\mu_L^{\mathbb{R}}(z_1 k z_2)| \geq |\mu_L^{\mathbb{R}}(z_1)| + |\mu_L^{\mathbb{R}}(z_2)| - D$$

for some $z_1, z_2 \in Z_L^+$ with $|\mu_L^{\mathbb{R}}(z_1)|, |\mu_L^{\mathbb{R}}(z_2)| \geq D$, then $k \in \mathcal{C}_L P_L$.

Note that Lemma 4.2 is meaningful only when $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 1. In this case, Proposition 3.2 implies some kind of converse to Lemma 4.2: for any compact subset $\mathcal{C}_L$ of N_L^-, there exists $D \geq 0$ such that for all $k \in K_L \cap \mathcal{C}_L P_L$ and all $z_1, z_2 \in Z^+$,

$$|\mu_L^{\mathbb{R}}(z_1 k z_2)| \geq |\mu_L^{\mathbb{R}}(z_1)| + |\mu_L^{\mathbb{R}}(z_2)| - D.$$

Proof of Lemma 4.2. We may assume that $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 1. Then L/P_L is the disjoint union of $N_L^- \cdot P_L$ and $\{w \cdot P_L\}$, where w denotes the nontrivial element of the (restricted) Weyl group of L. It is therefore sufficient to prove the existence of a neighborhood $\mathcal{U}$ of $w \cdot P_L$ in L/P_L such that for all $k \in K_L$ with $k \cdot P_L \in \mathcal{U}$ and all $z_1, z_2 \in Z_L^+$ with $|\mu_L^{\mathbb{R}}(z_1)|, |\mu_L^{\mathbb{R}}(z_2)| \geq D$,

$$|\mu_L^{\mathbb{R}}(z_1 k z_2)| < |\mu_L^{\mathbb{R}}(z_1)| + |\mu_L^{\mathbb{R}}(z_2)| - D.$$

Let X_L denote either the Riemannian symmetric space or the Bruhat–Tits tree of L, depending on whether $\mathbf{k}$ is Archimedean or not. Let x_0 be the point of X_L whose stabilizer is K_L. By (2.1), we may assume that $|\mu_L^{\mathbb{R}}(g)| = d(x_0, g \cdot x_0)$ for all $g \in L$, where d is the metric on X_L. The space X_L is Gromov-hyperbolic and we may identify L/P_L with the boundary at infinity ∂X_L of X_L, *i.e.* with the set of equivalence classes $[\mathcal{R}]$ of geodesic rays $\mathcal{R} : [0, +\infty[\rightarrow X_L$ for the equivalence relation "to stay at bounded distance". The point $P_L \in L/P_L$ (resp. $w \cdot P_L \in L/P_L$) corresponds to the equivalence class $[\mathcal{R}^+]$ (resp. $[\mathcal{R}^-]$) of the geodesic ray $\mathcal{R}^+ : [0, +\infty[\rightarrow X_L$ (resp. $\mathcal{R}^- : [0, +\infty[\rightarrow X_L$) whose image is the convex hull of $Z_L^+ \cdot x_0$ (resp. $(w \cdot Z_L^+) \cdot x_0$). By the "shadow lemma" (see [Bou], Lem. 1.6.2, for instance), there is a constant $D_0 > 0$ such that the open sets

$$\mathcal{U}_t = \Big\{ [\mathcal{R}] \in \partial X_L, \quad \mathcal{R}(0) = x_0 \text{ and } d\big(\mathcal{R}(t), \mathcal{R}^-(t)\big) < D_0 \Big\},$$

for $t \in [0, +\infty[$, form a basis of neighborhoods of $[\mathcal{R}^-]$ in ∂X_L. Fix $D \geq D_0$. For all $k \in K_L$ and $z_1, z_2 \in Z_L^+$ with $t_1 := |\mu_L^{\mathbb{R}}(z_1)| \geq D$ and $t_2 := |\mu_L^{\mathbb{R}}(z_2)| \geq D$, we have

$$\begin{aligned}
|\mu_L^{\mathbb{R}}(z_1 k z_2)| &= d(x_0, z_1 k z_2 \cdot x_0)\\
&= d(z_1^{-1} \cdot x_0, k z_2 \cdot x_0)\\
&= d\big(\mathcal{R}^-(t_1), k \cdot \mathcal{R}^+(t_2)\big)\\
&\leq d\big(\mathcal{R}^-(t_1), \mathcal{R}^-(D)\big) + d\big(\mathcal{R}^-(D), k \cdot \mathcal{R}^+(D)\big)\\
&\qquad + d\big(k \cdot \mathcal{R}^+(D), k \cdot \mathcal{R}^+(t_2)\big)\\
&= t_1 - D + d\big(\mathcal{R}^-(D), k \cdot \mathcal{R}^+(D)\big) + t_2 - D\\
&= |\mu_L^{\mathbb{R}}(z_1)| + |\mu_L^{\mathbb{R}}(z_2)| - 2D + d\big(\mathcal{R}^-(D), k \cdot \mathcal{R}^+(D)\big).
\end{aligned}$$

Therefore, if $[k \cdot \mathcal{R}^+] \in \mathcal{U}_D$ then $|\mu_L^{\mathbb{R}}(z_1 k z_2)| < |\mu_L^{\mathbb{R}}(z_1)| + |\mu_L^{\mathbb{R}}(z_2)| - D$. This completes the proof of Lemma 4.2. □

Proof of Proposition 4.1. As in the proof of Lemma 4.2, let X_L denote either the Riemannian symmetric space or the Bruhat–Tits tree of L, depending on whether $\mathbf{k}$ is Archimedean or not. Let x_0 be the point of X_L whose stabilizer is K_L. By (2.1), we may assume that $|\mu_L^{\mathbb{R}}(g)| = d(x_0, g \cdot x_0)$ for all $g \in L$, where d is the metric on X_L. We note that there exists a closed Γ-invariant convex subset $X'_L \neq \emptyset$ of X_L on which Γ acts cocompactly: indeed, if $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ this is the convex cocompactness assumption; if $\mathbf{k}$ is non-Archimedean it follows from [Bas], Prop. 7.9. Fix a compact fundamental domain $\mathcal{D}$ of X'_L for the action of Γ and a point x'_0 in the interior of $\mathcal{D}$. Let $\mathrm{d}_{\mathcal{D}}$ be the diameter of $\mathcal{D}$ and D_0 the constant given by Lemma 4.2. Let

$$D = \max\big(D_0, 6\,\mathrm{d}_{\mathcal{D}} + 6\,d(x_0, x'_0)\big) \ \geq D_0$$

and let $\mathcal{C}_L$ be the corresponding compact subset of N_L^- given by Lemma 4.2. We claim that D and $\mathcal{C}_L$ satisfy the conclusions of Proposition 4.1. Indeed, let $R \geq 2D$. Fix $\gamma \in \Gamma$ and let I be the geodesic segment of X'_L with endpoints x'_0 and $\gamma^{-1} \cdot x'_0$. Let $n \in \mathbb{N}$ such that

$$nR \,\leq\, d(x'_0, \gamma^{-1} \cdot x'_0) \,<\, (n+1)R.$$

For all $1 \leq i \leq n$, let $x'_i \in I$ satisfy $d(x'_i, x'_0) = iR$. We have $x'_i \in \lambda_i \cdot \mathcal{D}$ for some $\lambda_i \in \Gamma$. Let $\gamma_0 = \gamma\lambda_n \in \Gamma$ and $\gamma_i = \lambda_{n-i+1}^{-1}\lambda_{n-i} \in \Gamma$ for $i \geq 1$ (where $\lambda_0 = 1$), so that $\gamma = \gamma_0 \dots \gamma_n$. For all $1 \leq i \leq n$,

$$\begin{aligned}
\big| |\mu_L^{\mathbb{R}}(\gamma_i)| - d(x'_{n-i}, x'_{n-i+1}) \big| \;&=\; \big| d(\lambda_{n-i} \cdot x_0, \lambda_{n-i+1} \cdot x_0) - d(x'_{n-i}, x'_{n-i+1}) \big| \\
&\leq\; d(\lambda_{n-i} \cdot x_0, \lambda_{n-i} \cdot x'_0) + d(\lambda_{n-i} \cdot x'_0, x'_{n-i}) \\
&\qquad + d(x'_{n-i+1}, \lambda_{n-i+1} \cdot x'_0) + d(\lambda_{n-i+1} \cdot x'_0, \lambda_{n-i+1} \cdot x_0) \\
&\leq\; 2\,\mathrm{d}_{\mathcal{D}} + 2\,d(x_0, x'_0).
\end{aligned}$$

Since $d(x'_{n-i}, x'_{n-i+1}) = R$, we have $\big| |\mu_L^{\mathbb{R}}(\gamma_i)| - R \big| \leq 2\,\mathrm{d}_{\mathcal{D}} + 2\,d(x_0, x'_0) \leq D$. Similarly,

$$\big| |\mu_L^{\mathbb{R}}(\gamma_0)| - d(x'_n, \gamma^{-1} \cdot x'_0) \big| \leq 2\,\mathrm{d}_{\mathcal{D}} + 2\,d(x_0, x'_0),$$

hence $|\mu_L^{\mathbb{R}}(\gamma_0)| \leq R + 2\,\mathrm{d}_{\mathcal{D}} + 2\,d(x_0, x'_0) \leq R + D$. For $1 \leq i \leq n-1$, the same reasoning shows that

$$\begin{aligned}
|\mu_L^{\mathbb{R}}(\gamma_i\gamma_{i+1})| \;&\geq\; d(x_{n-i-1}, x_{n-i+1}) - 2\,\mathrm{d}_{\mathcal{D}} - 2\,d(x_0, x'_0) \\
&=\; 2R - 2\,\mathrm{d}_{\mathcal{D}} - 2\,d(x_0, x'_0) \\
&\geq\; |\mu_L^{\mathbb{R}}(\gamma_i)| + |\mu_L^{\mathbb{R}}(\gamma_{i+1})| - 6\,\mathrm{d}_{\mathcal{D}} - 6\,d(x_0, x'_0) \\
&\geq\; |\mu_L^{\mathbb{R}}(\gamma_i)| + |\mu_L^{\mathbb{R}}(\gamma_{i+1})| - D.
\end{aligned} \tag{4.1}$$

Choose Cartan decompositions $\gamma_i = k_{\gamma_i} z_{\gamma_i} \ell_{\gamma_i}$ for all i, where $k_{\gamma_i}, \ell_{\gamma_i} \in K_L$ and $z_{\gamma_i} \in Z_L^+$. By Lemma 4.2, we have $\ell_{\gamma_i} k_{\gamma_{i+1}} \in \mathcal{C}_L P_L$ for all $1 \leq i \leq n-1$.

We claim that $\mu_L^{\mathbb{R}}(\gamma_1), \dots, \mu_L^{\mathbb{R}}(\gamma_n) \in \mathbb{R}$ all have the same sign. Indeed, we may assume that $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 0, in which case $\mu_L^{\mathbb{R}} : L \to \mathbb{R}$ is a group homomorphism. If $\mu_L^{\mathbb{R}}(\gamma_i)$ and $\mu_L^{\mathbb{R}}(\gamma_{i+1})$ had different signs for some $1 \leq i \leq n-1$, then (4.1) would imply that

$$\min\big(|\mu_L^{\mathbb{R}}(\gamma_i)|, |\mu_L^{\mathbb{R}}(\gamma_{i+1})|\big) \leq \frac{D}{2},$$

which would contradict the fact that

$$\begin{aligned}
|\mu_L^{\mathbb{R}}(\gamma_i)|, |\mu_L^{\mathbb{R}}(\gamma_{i+1})| \;&\geq\; R - 2\,\mathrm{d}_{\mathcal{D}} - 2\,d(x_0, x'_0) \\
&\geq\; D - 2\,\mathrm{d}_{\mathcal{D}} - 2\,d(x_0, x'_0) \;>\; \frac{D}{2}.
\end{aligned}$$

Thus $\mu_L^{\mathbb{R}}(\gamma_1), \dots, \mu_L^{\mathbb{R}}(\gamma_n)$ all have the same sign. If $n \geq 2$, then

$$\begin{aligned}
\big| \mu_L^{\mathbb{R}}(\gamma) - \mu_L^{\mathbb{R}}(\gamma_1 \dots \gamma_n) \big| \;&\leq\; |\mu_L^{\mathbb{R}}(\gamma_0)| \\
&\leq\; R + 2\,\mathrm{d}_{\mathcal{D}} + 2\,d(x_0, x'_0) \\
&\leq\; n\big(R - 2\,\mathrm{d}_{\mathcal{D}} - 2\,d(x_0, x'_0)\big) \\
&\leq\; \sum_{i=1}^{n} |\mu_L^{\mathbb{R}}(\gamma_i)| = |\mu_L^{\mathbb{R}}(\gamma_1 \dots \gamma_n)|,
\end{aligned}$$

hence the sign of $\mu_L^{\mathbb{R}}(\gamma_1 \dots \gamma_n)$ is the same as that of $\mu_L^{\mathbb{R}}(\gamma)$, and so is that of $\mu_L^{\mathbb{R}}(\gamma_1), \dots, \mu_L^{\mathbb{R}}(\gamma_n)$. $\square$

4.2. **Interpretation in the boundary at infinity of X_L.** We now briefly explain the terminology "transverse product"; this paragraph is not needed in the rest of the paper. As before, let X_L denote either the Riemannian symmetric space or the Bruhat–Tits tree of L, depending on whether $\mathbf{k}$ is Archimedean or not. Let x_0 be the point of X_L whose stabilizer is K_L. By (2.1), we may assume that $|\mu_L^{\mathbb{R}}(g)| = d(x_0, g\cdot x_0)$ for all $g\in L$, where d is the metric on X_L. We endow the boundary at infinity ∂X_L of X_L with a K_L-invariant metric d_∞. For any $g\in L$ we write $g = k_g z_g \ell_g$ with $k_g, \ell_g \in K_L$ and $z_g \in Z_L^+$.

Assume that $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 1. Then ∂X_L naturally identifies with $L/P_L = N_L^- \cdot P_L \sqcup \{w\cdot P_L\}$, where w is the nontrivial element of the Weyl group of L. Condition (2) of Proposition 4.1 expresses that for all $1\leq i\leq n-1$ the distance between $k_{\gamma_{i+1}}\cdot P_L \in \partial X_L$ and $\ell_{\gamma_i}^{-1}\cdot wP_L \in \partial X_L$ is bounded away from 0. We now make the following observation (for a more precise statement in the non-Archimedean case, see [Ka2], Lem. 3.2).

Lemma 4.3. *For any $\varepsilon > 0$, there is a constant $R_\varepsilon > 0$ such that all hyperbolic elements $g\in L$ with $d(x_0, g\cdot x_0)\geq R_\varepsilon$ satisfy*

$$d_\infty(k_g\cdot P_L, \zeta_g^+) \leq \varepsilon \quad\text{and}\quad d_\infty(\ell_g^{-1}\cdot wP_L, \zeta_g^-)\leq\varepsilon,$$

where ζ_g^+ (resp. ζ_g^-) is the attracting (resp. repelling) fixed point of g in ∂X_L.

Recall that an element $g\in L$ is said to be *hyperbolic* if it has both an attracting fixed point and a repelling fixed point in ∂X_L; equivalently, g preserves a geodesic line $\mathcal{A}_g$ in X_L and acts on it by a nontrivial translation.

Proof. For $t>0$, let $x_t\in X_L$ be the point in the convex hull of $Z_L^+\cdot x_0$ such that $d(x_0,x_t)=t$, let

$$\mathcal{H}_t = \{x\in X_L,\ d(x,x_t)\leq d(x,x_0)\},$$

and let $\overline{\mathcal{H}_t}$ be the closure of $\mathcal{H}_t$ in the compactification $\overline{X_L} = X_L\sqcup\partial X_L$. We note that $\overline{\mathcal{H}_{t'}}\subset\overline{\mathcal{H}_t}$ for all $t'\geq t>0$ and that $\bigcap_{t>0}\overline{\mathcal{H}_t} = \{P_L\}$, hence for any $\varepsilon>0$ there is a constant $R_\varepsilon>0$ such that if $t\geq R_\varepsilon$, then $d(\zeta, P_L)\leq\varepsilon$ for all $\zeta\in\overline{\mathcal{H}_t}\cap\partial X_L$.

Let $g\in L$ be a hyperbolic element. We first assume that g has a Cartan decomposition of the form $g = z_g\ell_g$ with $z_g\in Z_L^+$ and $\ell_g\in K_L$. Then $g\cdot x_0 = x_t$, where $t = d(x_0, g\cdot x_0)$. We claim that $\zeta_g^+\in\overline{\mathcal{H}_t}$. Indeed, if y is the orthogonal projection of x_0 to $\mathcal{A}_g$, then

$$d(y,x_t) = d(y, g\cdot x_0) = d(g^{-1}\cdot y, x_0) > d(y,x_0)$$

and

$$d(g\cdot y, x_t) = d(g\cdot y, g\cdot x_0) = d(y,x_0) < d(g\cdot y, x_0),$$

hence $y\in\mathcal{A}_g\smallsetminus\mathcal{H}_t$ and $g\cdot y\in\mathcal{H}_t\cap\mathcal{A}_g$. This implies that $\mathcal{H}_t\cap\mathcal{A}_g$ is a geodesic ray with endpoint ζ_g^+ at infinity, hence $\zeta_g^+\in\overline{\mathcal{H}_t}$. Therefore, for any $\varepsilon>0$, if $t = d(x_0, g\cdot x_0)\geq R_\varepsilon$, then $d_\infty(\zeta_g^+, P_L)\leq\varepsilon$. In the general case, we write $g = k_g z_g\ell_g$ with $k_g,\ell_g\in K_L$ and $z_g\in Z_L^+$. Since d_∞ is K_L-invariant,

$$d_\infty(k_g\cdot P_L, \zeta_g^+) = d_\infty(P_L, \zeta_{g'}^+),$$

where $g' = k_g^{-1} g k_g = z_g(\ell_g k_g) \in Z_L^+ K_L$. As we have just seen, this distance is $\leq \varepsilon$ whenever $d(x_0, g' \cdot x_0) = d(x_0, g \cdot x_0) \geq R_\varepsilon$. To obtain the second inequality, we write the Cartan decomposition

$$g^{-1} = (\ell_g^{-1} n)(n^{-1} z_g^{-1} n)(n^{-1} k_g^{-1}),$$

where $n \in K_L$ is any element in the normalizer but not in the centralizer of $\mathbf{A_L}$. Since $n^{-1} z_g^{-1} n \in Z_L^+$, for any $\varepsilon > 0$ we have

$$d_\infty(\ell_g^{-1} n \cdot P_L, \zeta_{g^{-1}}^+) = d_\infty(\ell_g^{-1} \cdot wP_L, \zeta_g^-) \leq \varepsilon$$

whenever $d(x_0, g^{-1} \cdot x_0) = d(x_0, g \cdot x_0) \geq R_\varepsilon$. □

Let Γ be a discrete subgroup of L as in Proposition 4.1, and assume that it is torsion-free. Then all nontrivial elements of Γ are hyperbolic. With the notation of Proposition 4.1, let $\varepsilon = d_\infty(\mathcal{C}_L \cdot P_L, wP_L)/3 > 0$, and let R_ε be the corresponding constant given by Lemma 4.3. For $R \geq R_\varepsilon + 2D$, Conditions (1) and (2) imply that

$$d_\infty(\zeta_{\gamma_i}^+, \zeta_{\gamma_{i+1}}^-) \geq \varepsilon \tag{4.2}$$

for all $1 \leq i \leq n-1$. In other words, "the attracting direction of γ_i in X_L is transverse to the repelling direction of γ_{i+1}".

If $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 0, then X_L is a line and Conditions (1) and (3) imply that all elements γ_i, $1 \leq i \leq n$, have a common attracting point and a common repelling point in ∂X_L; in particular, (4.2) is still satisfied for $\varepsilon = d_\infty(\zeta, \zeta') > 0$, where $\partial X_L = \{\zeta, \zeta'\}$.

4.3. **Transversality in G.** Let $\mathbf{k}$ be a local field, $\mathbf{G}$ a connected reductive algebraic $\mathbf{k}$-group, and $\mathbf{L}$ a closed connected reductive subgroup of $\mathbf{G}$ of $\mathbf{k}$-rank 1. Fix a Cartan decomposition $L = K_L A_L^+ K_L$ or $L = K_L Z_L^+ K_L$, where K_L is a maximal compact subgroup of L and $\mathbf{A_L}$ a maximal $\mathbf{k}$-split torus of $\mathbf{L}$, with centralizer $\mathbf{Z_L}$ in $\mathbf{L}$. We can find a maximal $\mathbf{k}$-split torus $\mathbf{A}$ of $\mathbf{G}$ containing $\mathbf{A_L}$, together with a maximal compact subgroup K of G containing K_L, such that $G = KAK$ or $G = KZK$: this was proved by Mostow [Mos] and Karpelevich [Kar] in the Archimedean case, and follows from the work of Landvogt [Lan] in the non-Archimedean case. An appropriate choice of a basis Δ of the restricted root system Φ of $\mathbf{A}$ in $\mathbf{G}$ then defines subsets A^+ of A and Z^+ of Z, as in Section 2.2, such that $A_L^+ \cap A^+$ is noncompact and $G = KA^+K$ or $G = KZ^+K$. The following lemma provides a link between Propositions 3.2 and 4.1. We use the notation of Sections 2.4 and 4.1.

Lemma 4.4. *If the restriction of $\alpha \in \Delta$ to $\mathbf{A_L}$ is nontrivial, then $P_L \subset P_\alpha$ and $N_L^- \subset N_\alpha^- P_\alpha$.*

Proof. Fix $\alpha \in \Delta$ whose restriction to $\mathbf{A_L}$ is nontrivial, and let $a \in A_L^+ \cap A^+$ be such that $|\alpha(a)| > 1$. Note that $\mathfrak{g} = \mathfrak{n}_\alpha^- \oplus \mathfrak{p}_\alpha$ and $\mathfrak{p}_\alpha = \mathfrak{p}_\emptyset \oplus \mathfrak{n}_{\alpha^c}^-$, where

$$\begin{aligned} \mathfrak{n}_\alpha^- &= \bigoplus_{\beta \in (\alpha + \mathbb{N}\Delta) \cap \Phi} \mathfrak{u}_{-\beta}, \\ \mathfrak{p}_\emptyset &= \mathfrak{z} \oplus \bigoplus_{\beta \in \Phi^+} \mathfrak{u}_\beta, \\ \text{and} \qquad \mathfrak{n}_{\alpha^c}^- &= \bigoplus_{\beta \in \mathbb{N}(\Delta \smallsetminus \{\alpha\}) \cap \Phi} \mathfrak{u}_{-\beta} \end{aligned}$$

are all direct sums of eigenspaces of $\mathrm{Ad}(a)$, with eigenvalues of absolute value < 1 on $\mathfrak{n}_\alpha^-$ and ≥ 1 on $\mathfrak{p}_\emptyset$. Since $\mathfrak{p}_L$ is a sum of eigenspaces of $\mathrm{Ad}(a)$ for eigenvalues of absolute value ≥ 1, we have $\mathfrak{p}_L \subset \mathfrak{p}_\alpha$. Given that $\mathbf{P_L}$ and $\mathbf{P}_\alpha$ are connected, this implies that $P_L \subset P_\alpha$. Since $\mathfrak{n}_L^-$ is a sum of eigenspaces of $\mathrm{Ad}(a)$ for eigenvalues of absolute value < 1, we have $\mathfrak{n}_L^- \subset \mathfrak{n}_\alpha^- \oplus \mathfrak{n}_{\alpha^c}^-$. Note that $[\mathfrak{n}_\alpha^-, \mathfrak{n}_{\alpha^c}^-] \subset \mathfrak{n}_\alpha^-$, hence N_α^- is normalized by the group $N_{\alpha^c}^-$ generated by the groups $U_{-\beta}$ for $\beta \in \mathbb{N}(\Delta \smallsetminus \{\alpha\}) \cap \Phi$. This implies that

$$N_L^- \subset N_\alpha^- N_{\alpha^c}^- \subset N_\alpha^- P_\alpha. \quad \square$$

5. Cartan projection and deformation

The goal of this section is to prove Theorem 1.4, from which we deduce Theorems 1.1 and 1.3. Using Propositions 3.2 and 4.1, we establish the following result.

Proposition 5.1. *Let* $\mathbf{k}$ *be a local field,* G *the set of* $\mathbf{k}$*-points of a connected reductive algebraic* $\mathbf{k}$*-group* $\mathbf{G}$*, and* L *the set of* $\mathbf{k}$*-points of a closed reductive subgroup* $\mathbf{L}$ *of* $\mathbf{G}$ *of* $\mathbf{k}$*-rank* 1*. Fix a Cartan projection* $\mu : G \to E^+$ *and a norm* $\|\cdot\|$ *on* E*. If* $\mathbf{k} = \mathbb{R}$ *or* $\mathbb{C}$*, let* Γ *be a convex cocompact subgroup of* L*; if* $\mathbf{k}$ *is non-Archimedean, let* Γ *be any finitely generated discrete subgroup of* L*. Then for any* $\varepsilon > 0$*, there exist a finite subset* F_ε *of* Γ *and a neighborhood* $\mathcal{U}_\varepsilon \subset \mathrm{Hom}(\Gamma, G)$ *of the natural inclusion such that any* $\gamma \in \Gamma$ *may be written as* $\gamma = \gamma_0 \ldots \gamma_n$ *for some* $\gamma_0, \ldots, \gamma_n \in F_\varepsilon$ *with*

(*i*) $n \leq \varepsilon \, \|\mu(\gamma)\|$,
(*ii*) $\|\mu(\varphi(\gamma_i)) - \mu(\gamma_i)\| \leq 1$ *for all* $\varphi \in \mathcal{U}_\varepsilon$ *and* $0 \leq i \leq n$,
(*iii*) *for all* $\varphi \in \mathcal{U}_\varepsilon$,

$$\Big\| \mu(\varphi(\gamma)) - \sum_{i=0}^{n} \mu(\varphi(\gamma_i)) \Big\| \leq \varepsilon \|\mu(\gamma)\|.$$

Let us briefly explain how Proposition 5.1 implies Theorem 1.4. Let $\varepsilon > 0$ and let F_ε and $\mathcal{U}_\varepsilon$ be given by Proposition 5.1. For any $\eta > 0$, the set

$$\mathcal{U}_{\varepsilon,\eta} = \big\{ \varphi \in \mathcal{U}_\varepsilon, \ \|\mu(\varphi(g)) - \mu(g)\| \leq \eta \quad \forall g \in F_\varepsilon \big\}$$

is a neighborhood of the natural inclusion in $\mathrm{Hom}(\Gamma, G)$. Conditions (*i*), (*ii*), (*iii*) and the triangular inequality imply that

$$\|\mu(\varphi(\gamma)) - \mu(\gamma)\| \leq 3\varepsilon \|\mu(\gamma)\| + \eta$$

for all $\varphi \in \mathcal{U}_{\varepsilon,\eta}$ and $\gamma \in \Gamma$. Since Γ is discrete in G and μ is a proper map, the set $\Gamma \cap K$ of elements $\gamma \in \Gamma$ such that $\mu(\gamma) = 0$ is a finite subgroup of Γ and

$$r := \inf \big\{ \|\mu(\gamma)\|,\ \gamma \in \Gamma \text{ and } \mu(\gamma) \neq 0 \big\} > 0.$$

Let $\eta = \varepsilon r$. For all $\varphi \in \mathcal{U}_{\varepsilon,\eta}$ and all $\gamma \in \Gamma$ with $\mu(\gamma) \neq 0$,

$$\|\mu(\varphi(\gamma)) - \mu(\gamma)\| \leq 4\varepsilon \|\mu(\gamma)\|. \tag{5.1}$$

If Γ is torsion-free, then $\Gamma \cap K = \{1\}$ and (5.1) holds for all $\gamma \in \Gamma$. This implies Theorem 1.4.

Before we give the proof of Proposition 5.1 (in Section 5.3), let us first introduce some notation and make preliminary remarks.

5.1. **Norms on E and its subspaces.** By (2.3), in order to prove Proposition 5.1 we may assume that $\mathbf{L}$ is connected. Fix a Cartan decomposition $L = K_L A_L^+ K_L$ or $L = K_L Z_L^+ K_L$, with corresponding Cartan projection $\mu_L : L \to E_L^+$. As in Section 4.3, we can find a Cartan decomposition $G = KA^+K$ or $G = KZ^+K$ such that $K_L \subset K$ and $\mathbf{A_L} \subset \mathbf{A}$, with $A_L^+ \cap A^+$ noncompact. By Remark 2.1, we may assume that $\mu : G \to E^+$ is the Cartan projection associated with this Cartan decomposition of G. We now use the notation of Section 2. Since all norms on E are equivalent, we may assume that $\|\cdot\|$ is the W-invariant Euclidean norm introduced in Section 2.3.

We naturally see E_L as a line in E. If $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 1, then E_L^+ is a half-line in E^+; we set $L^+ = L$. If $\mathbf{L}$ has semisimple $\mathbf{k}$-rank 0, then $E_L^+ = E_L$ is a line in E; we choose a half-line E_L^{++} in $E_L^+ \cap E^+$ and set $L^+ = \{g \in L,\ \mu_L(g) \in E_L^{++}\}$. By composing μ_L with some isomorphism from E_L to $\mathbb{R}$, we get a Cartan projection $\mu_L^{\mathbb{R}} : L \to \mathbb{R}$ such that

$$\mu_L^{\mathbb{R}}(g) = \|\mu(g)\| \geq 0$$

for all $g \in L^+$. We note that for all $g \in L \smallsetminus L^+$ we have $g^{-1} \in L^+$. Moreover, the opposition involution $\iota : \mu(G) \to \mu(G)$, which maps $\mu(g)$ to $\mu(g^{-1})$ for all $g \in G$, is an isometry with respect to $\|\cdot\|$; indeed, $\iota(\mu(g)) = w \cdot (-\mu(g))$ where w is the "longest" element of the Weyl group W. Therefore we only need to prove Proposition 5.1 for elements $\gamma \in \Gamma$ that belong to L^+.

For any subspace E' of E we define a seminorm $|\cdot|_{E'}$ on E by

$$|y|_{E'} = \|\operatorname{pr}_{E'}(y)\|,$$

where $\operatorname{pr}_{E'} : E \to E'$ is the orthogonal projection onto E'. We will use the three seminorms $|\cdot|_{E_{\Delta_L}}$, $|\cdot|_{E_Z}$, and $|\cdot|_{E_{\Delta_L} \oplus E_Z}$, where E_Z is the subspace of E introduced in Section 2.1 and E_{Δ_L} the subspace defined as follows. For every $\alpha \in \Delta$ there is a constant $t_\alpha \geq 0$ such that

$$\langle \alpha, \mu(g) \rangle = t_\alpha \, \mu_L^{\mathbb{R}}(g) \tag{5.2}$$

for all $g \in L^+$. We let $\Delta_L = \{\alpha \in \Delta,\ t_\alpha > 0\}$ denote the set of simple roots of $\mathbf{A}$ in $\mathbf{G}$ whose restriction to $\mathbf{A_L}$ is nontrivial and define

$$E_{\Delta_L} = \big\{ y \in E_D,\ \langle \beta, y \rangle = 0 \quad \forall \beta \in \Delta \smallsetminus \Delta_L \big\}.$$

Note that $E_L \subset E_{\Delta_L} \oplus E_Z$. We also observe that E_{Δ_L} and E_Z are orthogonal with respect to the Euclidean norm $\|\cdot\|$. Indeed, $\|\cdot\|$ is W-invariant, the group W is generated by the reflections $s_{\check{\alpha}}$, and for all $\alpha \in \Phi$ we have

$s_{\check{\alpha}}(\check{\alpha}) = -\check{\alpha}$ and $s_{\check{\alpha}}(y) = y$ for all $y \in E_Z$, hence E_Z is orthogonal to $E_D = \bigoplus_{\alpha\in\Delta} \mathbb{R}\check{\alpha}$. Therefore

$$|y|^2_{E_{\Delta_L}\oplus E_Z} = |y|^2_{E_{\Delta_L}} + |y|^2_{E_Z} \tag{5.3}$$

for all $y \in E$.

For $\alpha \in \Delta$, let χ_α denote the highest weight of the representation (ρ_α, V_α) of $\mathbf{G}$ introduced in Section 2.5. Recall that $\langle \chi_\alpha, \check{\alpha}\rangle \neq 0$ and $\langle \chi_\alpha, \check{\beta}\rangle = 0$ for all $\beta \in \Delta \smallsetminus \{\alpha\}$.

Remark 5.2. The set $\{\langle \chi_\alpha, \cdot\rangle,\ \alpha \in \Delta_L\}$ is a basis of the dual of E_{Δ_L}.

Indeed, since $\dim E_{\Delta_L} = \#\Delta_L$, it is sufficient to see that the elements $\langle \chi_\alpha, \cdot\rangle$, for $\alpha \in \Delta_L$, are linearly independent as linear forms on E_{Δ_L}. By definition of E_{Δ_L}, it is sufficient to see that

$$\mathrm{span}\{\chi_\alpha,\ \alpha \in \Delta_L\} \cap \mathrm{span}(\Delta \smallsetminus \Delta_L) = \{0\}. \tag{5.4}$$

Let $(\cdot,\cdot)$ be any W-invariant scalar product on $X(\mathbf{A}) \otimes_Z \mathbb{R}$. For all $\beta \in \Delta$ and $x \in X(\mathbf{A}) \otimes_Z \mathbb{R}$,

$$s_\beta(x) = x - \langle x, \check{\beta}\rangle\, \beta = x - \frac{2(x,\beta)}{(\beta,\beta)}\, \beta,$$

hence $(\chi_\alpha, \beta) = \frac{(\beta,\beta)}{2} \cdot \langle \chi_\alpha, \check{\beta}\rangle = 0$ for all $\alpha \neq \beta$ in Δ, which implies (5.4).

By Remark 5.2, the function $y \longmapsto \sum_{\alpha\in\Delta_L} |\langle \chi_\alpha, y\rangle|$ is a norm on E_{Δ_L}. Since all norms on E_{Δ_L} are equivalent, there is a constant $c \geq 1$ such that

$$c^{-1} \cdot \sum_{\alpha\in\Delta_L} |\langle \chi_\alpha, y\rangle| \ \leq\ |y|_{E_{\Delta_L}} \ \leq\ c \cdot \sum_{\alpha\in\Delta_L} |\langle \chi_\alpha, y\rangle| \tag{5.5}$$

for all $y \in E$.

5.2. **Norm of the projection onto E_{Δ_L}.** The main step in the proof of Proposition 5.1 consists of the following proposition, which gives an upper bound for the seminorm $|\cdot|_{E_{\Delta_L}}$.

Proposition 5.3. *Under the assumptions of Proposition 5.1, for any $\delta > 0$ there exist a finite subset F'_δ of Γ and a neighborhood $\mathcal{U}'_\delta \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that any $\gamma \in \Gamma \cap L^+$ may be written as $\gamma = \gamma_0 \dots \gamma_n$ for some $\gamma_0, \dots, \gamma_n \in F'_\delta$ with*

1. $n \leq \delta \sum_{i=1}^n \|\mu(\gamma_i)\|$,
2. $\sum_{i=1}^n \|\mu(\gamma_i)\| = \|\sum_{i=1}^n \mu(\gamma_i)\|$,
3. $\|\mu(\varphi(\gamma_i)) - \mu(\gamma_i)\| \leq 1$ *for all* $\varphi \in \mathcal{U}'_\delta$ *and* $0 \leq i \leq n$,
4. *for all* $\varphi \in \mathcal{U}'_\delta$,

$$\Big|\mu(\varphi(\gamma_1 \dots \gamma_n)) - \sum_{i=1}^n \mu(\varphi(\gamma_i))\Big|_{E_{\Delta_L}} \leq \delta \Big(\sum_{i=1}^n \|\mu(\gamma_i)\|\Big).$$

To prove Proposition 5.3, we use Propositions 3.2 and 4.1, together with Lemma 4.4.

Proof. Let $D > 0$ be the constant and $\mathcal{C}_L$ the compact subset of N_L^- given by Proposition 4.1. By Lemma 4.4, for any $\alpha \in \Delta_L$, the set $\mathcal{C}_L$ is contained in $\mathrm{Int}(\mathcal{C}_\alpha)P_\alpha$ for some compact subset $\mathcal{C}_\alpha$ of N_α^-, where $\mathrm{Int}(\mathcal{C}_\alpha)$ denotes the interior of $\mathcal{C}_\alpha$. After replacing $\mathcal{C}_\alpha$ by some larger compact subset of N_α^-, we may assume that it is preserved under conjugation by $K\cap Z$. Let $r_\alpha, R_\alpha > 0$ be the constants given by Proposition 3.2 with respect to $\mathcal{C}_\alpha$. Fix $\delta > 0$ and choose $R \geq 2D$ large enough so that $\frac{1}{R-D} \leq \delta$ and $t_\alpha(R-D)-1 \ \geq\ R_\alpha$ for all $\alpha \in \Delta_L$, where t_α is defined by (5.2). Let F'_δ be the set of elements $\gamma \in \Gamma$ such that $|\mu_L^{\mathbb{R}}(\gamma)| \leq R+D$, and F''_δ the subset of elements $\gamma \in F'_\delta$ such that $|\mu_L^{\mathbb{R}}(\gamma)| \geq R-D$. Note that F'_δ et F''_δ are finite since $\mu_L^{\mathbb{R}}$ is a proper map and Γ is discrete in L. For every $\gamma \in F'_\delta$ we choose a Cartan decomposition $\gamma = k_\gamma z_\gamma \ell_\gamma$, where $k_\gamma, \ell_\gamma \in K_L$ and $z_\gamma \in Z_L^+$. Let $\mathcal{U}'_\delta$ be the set of elements $\varphi \in \mathrm{Hom}(\Gamma, G)$ satisfying the following two conditions:

- $\|\mu(\varphi(\gamma)) - \mu(\gamma)\| \leq 1$ and $|\langle\alpha, \mu(\varphi(\gamma)) - \mu(\gamma)\rangle| \leq 1$ for all $\gamma \in F'_\delta$ and all $\alpha \in \Delta_L$,
- for every $\gamma \in F''_\delta$ we can write $\varphi(\gamma) = k_{\varphi(\gamma)} z_{\varphi(\gamma)} \ell_{\varphi(\gamma)}$ for some $k_{\varphi(\gamma)}$, $\ell_{\varphi(\gamma)} \in K$ and $z_{\varphi(\gamma)} \in Z^+$ so that $\ell_{\varphi(\gamma)} k_{\varphi(\gamma')} \in \mathcal{C}_\alpha P_\alpha$ for all $\gamma, \gamma' \in F''_\delta$ with $\ell_\gamma k_{\gamma'} \in \mathcal{C}_L P_L$ and all $\alpha \in \Delta_L$.

We note that $\mathcal{U}'_\delta$ is a neighborhood of the natural inclusion in $\mathrm{Hom}(\Gamma, G)$. Indeed, if $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, this follows from the fact that if $g = k_g a_g \ell_g = k'_g a_g \ell'_g$ with $k_g, \ell_g, k'_g, \ell'_g \in K$ and $a_g \in A^+$, then $k'_g \in k_g(K\cap Z)$ and $\ell'_g \in (K\cap Z)\ell_g$ ([Hel], Chap. 9, Cor. 1.2). If $\mathbf{k}$ is non-Archimedean, it follows from the fact that K is a neighborhood of 1 in G, which implies that if $g = k_g z_g \ell_g$, where $k_g, \ell_g \in K$ and $z_g \in Z^+$, then any $g' \in G$ sufficiently close to g belongs to gK and Kg and admits the Cartan decompositions $g' = k_g z_g(\ell_g g^{-1} g')$ and $g' = (g' g^{-1} k_g) z_g \ell_g$.

We claim that F'_δ and $\mathcal{U}'_\delta$ satisfy the conclusions of Proposition 5.3. Indeed, let $\gamma \in \Gamma \cap L^+$. By Proposition 4.1, we may write $\gamma = \gamma_0 \dots \gamma_n$ for some $\gamma_0 \in F'_\delta$ and $\gamma_1, \dots, \gamma_n \in F''_\delta$ such that

- $\ell_{\gamma_i} k_{\gamma_{i+1}} \in \mathcal{C}_L P_L$ for all $1 \leq i \leq n-1$,
- $\mu_L^{\mathbb{R}}(\gamma_1), \dots, \mu_L^{\mathbb{R}}(\gamma_n)$ are all ≥ 0 if $n \geq 2$.

This last condition implies that $\mu(\gamma_1), \dots, \mu(\gamma_n)$ all belong to the same half-line in E^+, hence Condition (2) is satisfied. Moreover, since $\gamma_1, \dots, \gamma_n \in F''_\delta$, we have

$$n \ \leq\ \frac{1}{R-D} \sum_{i=1}^n |\mu_L^{\mathbb{R}}(\gamma_i)| \ \leq\ \delta \sum_{i=1}^n \|\mu(\gamma_i)\|,$$

i.e. Condition (1) is satisfied. To prove Condition (4), we may assume that $n \geq 2$. Let $\varphi \in \mathcal{U}'_\delta$. According to (5.5), in order to prove Condition (4) it is sufficient to bound

$$\left|\Big\langle \chi_\alpha, \mu(\varphi(\gamma_1 \dots \gamma_n)) - \sum_{i=1}^n \mu(\varphi(\gamma_i))\Big\rangle\right|$$

for all $\alpha \in \Delta_L$. Since $n \geq 2$, we have $\mu_L^{\mathbb{R}}(\gamma_i) \geq 0$ for all $1 \leq i \leq n$, *i.e.* $\gamma_i \in L^+$. By definition of $\mathcal{U}'_\delta$ and t_α we obtain

$$\begin{aligned} \langle\alpha, \mu(\varphi(\gamma_i))\rangle \ &\geq\ \langle\alpha, \mu(\gamma_i)\rangle - 1 \\ &\geq\ t_\alpha\, \mu_L^{\mathbb{R}}(\gamma_i) - 1 \\ &\geq\ t_\alpha(R-D) - 1 \ \geq\ R_\alpha \end{aligned}$$

for all $\alpha \in \Delta_L$. Moreover, by definition of $\mathcal{U}'_\delta$ we can write $\varphi(\gamma_i) = k_{\varphi(\gamma_i)} z_{\varphi(\gamma_i)} \ell_{\varphi(\gamma_i)}$ for some $k_{\varphi(\gamma_i)}$, $\ell_{\varphi(\gamma_i)} \in K$ and $z_{\varphi(\gamma_i)} \in Z^+$ so that $\ell_{\varphi(\gamma_i)} k_{\varphi(\gamma_{i+1})} \in \mathcal{C}_\alpha P_\alpha$ for all $1 \leq i \leq n-1$. Proposition 3.2 thus implies that

$$\begin{aligned} \Big|\Big\langle \chi_\alpha, \mu(\varphi(\gamma_1 \dots \gamma_n)) - \sum_{i=1}^n \mu(\varphi(\gamma_i))\Big\rangle\Big| &\leq n r_\alpha \\ &\leq \frac{r_\alpha}{R-D} \sum_{i=1}^n \|\mu(\gamma_i)\|. \end{aligned}$$

By (5.5), this implies Condition (4) whenever

$$\sum_{\alpha \in \Delta_L} \frac{c\, r_\alpha}{R-D} \leq \delta,$$

which holds for R large enough. □

5.3. **Proof of Proposition 5.1.** Proposition 5.1 follows from Proposition 5.3 and from the following general observation.

Lemma 5.4. *Let $(E, \|\cdot\|)$ be a Euclidean space and E' a subspace of E. For any $y \in E$, let $|y|_{E'} = \|\operatorname{pr}_{E'}(y)\|$, where $\operatorname{pr}_{E'} : E \to E'$ is the orthogonal projection onto E'. For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $y, y' \in E$ satisfying*

- *$y' \in E'$,*
- *$|y - y'|_{E'} \leq 2\delta\, \|y'\|$,*
- *$\|y\| \leq (1+\delta)\, \|y'\|$,*

we have $\|y - y'\| \leq \frac{\varepsilon}{8}\, \|y'\|$.

Proof. By the Cauchy–Schwarz inequality, $|y - y'|_{E'} \leq 2\delta\, \|y'\|$ implies $\langle y, y'\rangle \geq (1-2\delta)\, \|y'\|^2$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product associated with $\|\cdot\|$. Therefore, we only need to observe that for $y'_1 \in E$ with $\|y'_1\| = 1$, the diameter of the set

$$\big\{y_1 \in E, \quad 1 - 2\delta \leq \langle y_1, y'_1\rangle \leq \|y_1\| \leq 1 + \delta\big\}$$

tends to 0 with δ, independently of y'_1. □

Proof of Proposition 5.1. Fix $\varepsilon \in]0,1]$. Let $\delta \in]0, \frac{\varepsilon}{8}]$ satisfy the conclusions of Lemma 5.4, let F'_δ and $\mathcal{U}'_\delta$ be given by Proposition 5.3, and let

$$C'_\delta = \max_{g \in F'_\delta} \|\mu(g)\| + 1.$$

We claim that we may take

$$F_\varepsilon = F'_\delta \cup {F'}_\delta^{-1} \cup \Big\{g \in \Gamma,\ \|\mu(g)\| < \frac{6C'_\delta}{\varepsilon}\Big\}$$

and

$$\mathcal{U}_\varepsilon = \big\{\varphi \in \mathcal{U}'_\delta,\ \|\mu(\varphi(g)) - \mu(g)\| \leq 1 \quad \forall g \in F_\varepsilon\big\}.$$

Indeed, let us prove that any $\gamma \in \Gamma$ may be written as $\gamma = \gamma_0 \dots \gamma_n$ for some $\gamma_0, \dots, \gamma_n \in F_\varepsilon$ satisfying Conditions (i), (ii), (iii) of Proposition 5.1. We first note that we may restrict to elements γ such that $\|\mu(\gamma)\| \geq \frac{6C'_\delta}{\varepsilon}$. Furthermore, as we already noticed in Section 5.1, since $g^{-1} \in L^+$ for all $g \in L \smallsetminus L^+$ and since the opposition involution is an isometry, we may restrict to elements γ that belong to L^+. Let $\gamma \in \Gamma \cap L^+$ such that $\|\mu(\gamma)\| \geq \frac{6C'_\delta}{\varepsilon}$.

By Proposition 5.3, we may write $\gamma = \gamma_0 \dots \gamma_n$ for some $\gamma_0, \dots, \gamma_n \in F'_\delta$ such that

(1) $n \leq \delta \sum_{i=1}^n \|\mu(\gamma_i)\|$,
(2) $\sum_{i=1}^n \|\mu(\gamma_i)\| = \| \sum_{i=1}^n \mu(\gamma_i)\|$,
(3) $\|\mu(\varphi(\gamma_i)) - \mu(\gamma_i)\| \leq 1$ for all $\varphi \in \mathcal{U}'_\delta$ and $0 \leq i \leq n$,
(4) for all $\varphi \in \mathcal{U}'_\delta$,

$$\Big|\mu(\varphi(\gamma_1 \dots \gamma_n)) - \sum_{i=1}^n \mu(\varphi(\gamma_i))\Big|_{E_{\Delta_L}} \leq \delta \Big(\sum_{i=1}^n \|\mu(\gamma_i)\|\Big).$$

We note that

$$\mathrm{pr}_{E_Z}(\mu(gg')) = \mathrm{pr}_{E_Z}(\mu(g)) + \mathrm{pr}_{E_Z}(\mu(g'))$$

for all $g, g' \in G$, hence

$$\Big|\mu(\varphi(\gamma_1 \dots \gamma_n)) - \sum_{i=1}^n \mu(\varphi(\gamma_i))\Big|_{E_Z} = 0.$$

By (5.3), Condition (4) remains true after replacing $|\cdot|_{E_{\Delta_L}}$ by $|\cdot|_{E_{\Delta_L} \oplus E_Z}$. Now Conditions (1), (2), (3), and (4), together with (2.2) and the triangular inequality, imply that for all $\varphi \in \mathcal{U}'_\delta$,

$$\begin{aligned} &\Big|\mu(\varphi(\gamma_1 \dots \gamma_n)) - \sum_{i=1}^n \mu(\gamma_i)\Big|_{E_{\Delta_L} \oplus E_Z} \\ \leq\ &\Big|\mu(\varphi(\gamma_1 \dots \gamma_n)) - \sum_{i=1}^n \mu(\varphi(\gamma_i))\Big|_{E_{\Delta_L} \oplus E_Z} + \sum_{i=1}^n \|\mu(\varphi(\gamma_i)) - \mu(\gamma_i)\| \\ \leq\ & 2\delta \Big(\sum_{i=1}^n \|\mu(\gamma_i)\|\Big) \ = \ 2\delta \Big\| \sum_{i=1}^n \mu(\gamma_i)\Big\| \end{aligned}$$

and

$$\begin{aligned} \|\mu(\varphi(\gamma_1 \dots \gamma_n))\| &\leq \sum_{i=1}^n \|\mu(\varphi(\gamma_i))\| \\ &\leq \sum_{i=1}^n \|\mu(\gamma_i)\| + \sum_{i=1}^n \|\mu(\varphi(\gamma_i)) - \mu(\gamma_i)\| \\ &\leq (1+\delta) \Big(\sum_{i=1}^n \|\mu(\gamma_i)\|\Big) \ = \ (1+\delta) \Big\| \sum_{i=1}^n \mu(\gamma_i)\Big\|. \end{aligned}$$

Therefore, for $\varphi \in \mathcal{U}'_\delta$, Lemma 5.4 applies to

$$(E', y, y') \ = \ \Big(E_{\Delta_L} \oplus E_Z\, ,\, \mu\big(\varphi(\gamma_1 \dots \gamma_n)\big)\, ,\, \sum_{i=1}^n \mu(\gamma_i)\Big),$$

yielding

$$\Big\|\mu(\varphi(\gamma_1 \dots \gamma_n)) - \sum_{i=1}^n \mu(\gamma_i)\Big\| \ \leq\ \frac{\varepsilon}{8}\ \Big\| \sum_{i=1}^n \mu(\gamma_i)\Big\| \tag{5.6}$$

for all $\varphi \in \mathcal{U}'_\delta$. Moreover, Conditions (1), (2), (3), together with the triangular inequality, imply that

$$\Big\|\Big(\sum_{i=1}^n \mu(\varphi(\gamma_i))\Big) - \Big(\sum_{i=1}^n \mu(\gamma_i)\Big)\Big\| \leq \delta \Big(\sum_{i=1}^n \|\mu(\gamma_i)\|\Big) \leq \frac{\varepsilon}{8} \Big\|\sum_{i=1}^n \mu(\gamma_i)\Big\|,$$

hence

$$\Big\|\mu(\varphi(\gamma_1\dots\gamma_n)) - \sum_{i=1}^n \mu(\varphi(\gamma_i))\Big\| \leq \frac{\varepsilon}{4} \Big\|\sum_{i=1}^n \mu(\gamma_i)\Big\|$$

for all $\varphi \in \mathcal{U}'_\delta$. Since

$$\big\|\mu(\varphi(\gamma)) - \mu(\varphi(\gamma_1\dots\gamma_n))\big\| \leq \|\mu(\varphi(\gamma_0))\| \leq \|\mu(\gamma_0)\| + 1 \leq C'_\delta,$$

the triangular inequality yields

$$\Big\|\mu(\varphi(\gamma)) - \sum_{i=1}^n \mu(\varphi(\gamma_i))\Big\| \leq \frac{\varepsilon}{4} \Big\|\sum_{i=1}^n \mu(\gamma_i)\Big\| + C'_\delta.$$

In particular, taking φ to be the natural inclusion of Γ in G, we obtain

$$\Big\|\sum_{i=1}^n \mu(\gamma_i)\Big\| \leq \frac{1}{1-\frac{\varepsilon}{4}} \big(\|\mu(\gamma)\| + C'_\delta\big) \leq 2\big(\|\mu(\gamma)\| + C'_\delta\big), \tag{5.7}$$

hence

$$\Big\|\mu(\varphi(\gamma)) - \sum_{i=0}^n \mu(\varphi(\gamma_i))\Big\| \leq \frac{\varepsilon}{2} \|\mu(\gamma)\| + 3\,C'_\delta \leq \varepsilon\,\|\mu(\gamma)\|$$

for all $\varphi \in \mathcal{U}'_\delta$, where we use the fact that $\|\mu(\gamma)\| \geq \frac{6C'_\delta}{\varepsilon}$. Finally, Conditions (1) and (2), together with (5.7), imply that

$$n \leq 2\delta\big(\|\mu(\gamma)\| + C'_\delta\big) \leq \varepsilon\,\|\mu(\gamma)\|,$$

where we use the fact that $\|\mu(\gamma)\| \geq \frac{6C'_\delta}{\varepsilon}$ and $\delta \leq \frac{\varepsilon}{4} \leq 1$. This completes the proof of the claim, hence of Proposition 5.1. □

5.4. **Properness and deformation.** Let us now briefly explain how to deduce Theorems 1.1 and 1.3 from Theorem 1.4.

Theorem 1.3 follows from Theorem 1.4 and from the *properness criterion* of Benoist ([Ben], Cor. 5.2) and Kobayashi ([Ko2], Th. 1.1). Under the assumptions of Theorem 1.3, this criterion states that a closed subgroup Γ of G acts properly on G/H if and only if the set $\mu(\Gamma)\cap(\mu(H)+\mathcal{C})$ is bounded for any compact subset $\mathcal{C}$ of E. This condition means that the set $\mu(\Gamma)$ "goes away from $\mu(H)$ at infinity".

Proof of Theorem 1.3. We may assume that $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{L}$ are all connected. Let $G = KA^+K$ or $G = KZ^+K$ be a Cartan decomposition of G, and let $\mu : G \to E^+$ be the corresponding Cartan projection. Endow E with a W-invariant norm $\|\cdot\|$ as in Section 2.3. We claim that $\mu(L)$ is at finite Hausdorff distance from a union U_L of two half-lines in E^+. Indeed, let $L = K_L A_L^+ K_L$ or $L = K_L Z_L^+ K_L$ be a Cartan decomposition of L, with corresponding Cartan projection $\mu_L : L \to E_L^+$. As in Section 4.3, we can find a Cartan decomposition $G = K'A'^+K'$ or $G = K'Z'^+K'$ of G such that

$K_L \subset K'$ and $\mathbf{A_L} \subset \mathbf{A}'$. If $\mu' : G \to E'^+$ denotes the corresponding Cartan projection, then E_L is naturally seen as a line in E' and

$$\mu'(g) = E'^+ \cap (W' \cdot \mu'_L(g))$$

for all $g \in L$, where W' is the Weyl group of the restricted root system of $\mathbf{A}'$ in $\mathbf{G}$; therefore $\mu'(L)$ is contained in the union of two half-lines in E'^+. By Remark 2.1, $\mu(L)$ is at finite Hausdorff distance from a union U_L of two half-lines in E^+. Similarly, $\mu(H)$ is at finite Hausdorff distance from a finite union U_H of subspaces of E intersected with E^+. By the properness criterion, $U_L \cap U_H = \{0\}$, hence there are constants $\varepsilon, C > 0$ such that

$$d(\mu(g), \mu(H)) \geq 2\varepsilon \, \|\mu(g)\| - C$$

for all $g \in L$. By Theorem 1.4, there is a neighborhood $\mathcal{U}_\varepsilon \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that

$$\|\mu(\varphi(\gamma)) - \mu(\gamma)\| \leq \varepsilon \, \|\mu(\gamma)\|$$

for all $\varphi \in \mathcal{U}_\varepsilon$ and $\gamma \in \Gamma$. Fix $\varphi \in \mathcal{U}_\varepsilon$. For all $\gamma \in \Gamma$,

$$\begin{array}{rcl} d(\mu(\varphi(\gamma)), \mu(H)) & \geq & d(\mu(\gamma), \mu(H)) - \|\mu(\varphi(\gamma)) - \mu(\gamma)\| \\ & \geq & \varepsilon \, \|\mu(\gamma)\| - C. \end{array}$$

Therefore, using the fact that Γ is discrete in G and μ is a proper map, we obtain that $\mu(\varphi(\Gamma)) \cap (\mu(H) + \mathcal{C})$ is finite for any compact subset $\mathcal{C}$ of E. By the properness criterion, this implies that $\varphi(\Gamma)$ acts properly on G/H. It also implies that $\varphi(\Gamma)$ is discrete in G and that the kernel of φ is finite. Since Γ is torsion-free, φ is injective. □

Proof of Theorem 1.1. We may assume that G, H, and L are connected. By results of Chevalley ([Che], Chap. 2, Th. 14 & 15), G is the identity component (for the real topology) of the set of $\mathbb{R}$-points of some connected reductive algebraic $\mathbb{R}$-group $\mathbf{G}$ and H (resp. L) is the identity component of the set of $\mathbb{R}$-points of some closed connected reductive subgroup $\mathbf{H}$ (resp. $\mathbf{L}$) of $\mathbf{G}$. By Theorem 1.3, there is a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that if $\varphi \in \mathcal{U}$, then φ is injective, $\varphi(\Gamma)$ is discrete in G, and $\varphi(\Gamma)$ acts properly discontinuously on G/H. Since φ is injective, $\varphi(\Gamma)$ has the same cohomological dimension as Γ. We conclude by using the following fact, due to Kobayashi ([Ko1], Cor. 5.5): when a torsion-free discrete subgroup of G acts properly discontinuously on G/H, it acts cocompactly on G/H if and only if its cohomological dimension is $d(G) - d(H)$, where $d(G)$ (resp. $d(H)$) denotes the dimension of the Riemannian symmetric space of G (resp. of H). (We define Riemannian symmetric spaces as in Section 2.3.) □

6. Application to the compact quotients of $\mathrm{SO}(2n, 2)/\mathrm{U}(n, 1)$

Fix an integer $n \geq 1$. Note that $\mathrm{U}(n, 1)$ naturally embeds into $\mathrm{SO}(2n, 2)$ by identifying the Hermitian form $|z_1|^2 + \ldots + |z_n|^2 - |z_{n+1}|^2$ on $\mathbb{C}^{n+1}$ with the quadratic form $x_1^2 + \ldots + x_{2n}^2 - x_{2n+1}^2 - x_{2n+2}^2$ on $\mathbb{R}^{2n+2}$. As Kulkarni [Kul] pointed out, the group $\mathrm{U}(n, 1)$, seen as a subgroup of $\mathrm{SO}(2n, 2)$, acts transitively on the anti-de Sitter space

$$\begin{aligned}\mathrm{AdS}^{2n+1} &= \big\{(x_1,\dots,x_{2n+2})\in\mathbb{R}^{2n+2},\ x_1^2+\dots+x_{2n}^2-x_{2n+1}^2-x_{2n+2}^2=-1\big\}\\ &\simeq \mathrm{SO}(2n,2)/\mathrm{SO}(2n,1).\end{aligned}$$

The stabilizer of $(0,\dots,0,1)$ is the compact subgroup $\mathrm{U}(n)$, hence AdS^{2n+1} identifies with $\mathrm{U}(n,1)/\mathrm{U}(n)$ and the action of $\mathrm{U}(n,1)$ on $\mathrm{SO}(2n,2)/\mathrm{SO}(2n,1)$ is proper. By duality, the action of $\mathrm{SO}(2n,1)$ on $\mathrm{SO}(2n,2)/\mathrm{U}(n,1)$ is proper and transitive. In particular, any torsion-free uniform lattice Γ of $\mathrm{SO}(2n,1)$ provides a standard compact quotient $\Gamma\backslash\mathrm{SO}(2n,2)/\mathrm{U}(n,1)$ of $\mathrm{SO}(2n,2)/\mathrm{U}(n,1)$.

Corollary 1.2 follows from Theorem 1.1 and from the existence of small Zariski-dense deformations of certain uniform lattices of $\mathrm{SO}(m,1)$ into $\mathrm{SO}(m,2)$. For $m=2$, such deformations can be obtained by making the following observation: the identity component $\mathrm{SO}(2,2)^\circ$ of $\mathrm{SO}(2,2)$ (for the real topology) admits a two-fold covering by $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$ (induced by the action of $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{M}_2(\mathbb{R})\simeq\mathbb{R}^4$ by left and right multiplication, which preserves the determinant) and the preimage of $\mathrm{SO}(1,2)^\circ$ is the diagonal of $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$. Therefore it is sufficient to prove that for any uniform lattice Γ_0 of $\mathrm{SL}_2(\mathbb{R})$ and any neighborhood $\mathcal{U}\subset\mathrm{Hom}(\Gamma_0,\mathrm{SL}_2(\mathbb{R}))$ of the natural inclusion, there is an element $\varphi\in\mathcal{U}$ such that the group $\Gamma_0^\varphi=\{(\gamma,\varphi(\gamma)),\ \gamma\in\Gamma_0\}$ is Zariski-dense in $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$. This is a consequence of the following remark, which easily follows from the simplicity of $\mathrm{PSL}_2(\mathbb{R})$ and from Goursat's lemma, applied to the Zariski closure of Γ_0^φ.

Remark 6.1. Any subgroup of $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ whose projection to each factor of $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ is surjective is either conjugate to the diagonal of $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ or equal to $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$.

For $m\geq 3$, small Zariski-dense deformations of certain uniform lattices of $\mathrm{SO}(m,1)$ into $\mathrm{SO}(m,2)$ can be obtained by a bending construction due to Johnson and Millson. This construction was originally introduced in [JM] for deformations into $\mathrm{SO}(m+1,1)$ or $\mathrm{PGL}_{m+1}(\mathbb{R})$. For the reader's convenience, we shall describe it for deformations into $\mathrm{SO}(m,2)$, and check Zariski density in this case.

From now on we use Gothic letters to denote the Lie algebras of real Lie groups (*e.g.* $\mathfrak{g}$ for G).

6.1. **Uniform arithmetic lattices of** $\mathrm{SO}(m,1)$**.** Fix $m\geq 3$. The uniform lattices of $\mathrm{SO}(m,1)$ considered by Johnson and Millson are obtained in the following classical way. Fix a square-free integer $r\geq 2$ and identify $\mathrm{SO}(m,1)$ with the special orthogonal group of the quadratic form

$$x_1^2+\dots+x_m^2-\sqrt{r}x_{m+1}^2$$

on $\mathbb{R}^{m+1}$. Let $\mathcal{O}_r$ denote the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{r})$. The group $\Gamma=\mathrm{SO}(m,1)\cap\mathrm{M}_{m+1}(\mathcal{O}_r)$ is a uniform lattice in $\mathrm{SO}(m,1)$ (see [Bo1] for instance). For any ideal I of $\mathcal{O}_r$, the congruence subgroup $\Gamma\cap(1+\mathrm{M}_{m+1}(I))$ has finite index in Γ, hence is a uniform lattice in $\mathrm{SO}(m,1)$. By [MR], after replacing Γ by such a congruence subgroup, we may assume that it is torsion-free. Then $M=\Gamma\backslash\mathbb{H}^m$ is a m-dimensional compact hyperbolic manifold whose fundamental group identifies with Γ. By [JM], Lem. 7.1 & Th. 7.2, after possibly replacing Γ again by some congruence subgroup, we may assume that $N = \Gamma_0\backslash\mathbb{H}^{m-1}$ is a connected, orientable, totally geodesic hypersurface of M, where

$$\Gamma_0 = \Gamma \cap \mathrm{SO}(m-1,1)$$

and where

$$\mathbb{H}^{m-1} \simeq \big\{(x_2,\dots,x_{m+1}) \in \mathbb{R}^m,\ x_2^2+\ldots+x_m^2-\sqrt{r}x_{m+1}^2 = -1 \text{ and } x_{m+1} > 0\big\}$$

is embedded in

$$\mathbb{H}^{m} \simeq \big\{(x_1,\dots,x_{m+1}) \in \mathbb{R}^{m+1},\ x_1^2+\ldots+x_m^2-\sqrt{r}x_{m+1}^2 = -1 \text{ and } x_{m+1} > 0\big\}$$

in the natural way. We now embed $\mathrm{SO}(m,1)$ into $\mathrm{SO}(m,2)$. Since the centralizer of Γ_0 in $\mathrm{SO}(m,2)$ contains a subgroup isomorphic to $\mathrm{SO}(1,1) \simeq \mathbb{R}^*$, the idea of the bending construction is to deform Γ "along this centralizer", as we shall now explain.

6.2. **Deformations in the separating case.** Assume that N separates M into two components M_1 and M_2, and let Γ_1 (resp. Γ_2) denote the fundamental group of M_1 (resp. of M_2). By van Kampen's theorem, Γ is the amalgamated product $\Gamma_1 *_{\Gamma_0} \Gamma_2$. Fix an element $Y \in \mathfrak{so}(m,2) \smallsetminus \mathfrak{so}(m,1)$ that belongs to the Lie algebra of the centralizer of Γ_0 in $\mathrm{SO}(m,2)$. Following Johnson and Millson, we consider the deformations of Γ in $\mathrm{SO}(m,2)$ that are given, for $t \in \mathbb{R}$, by

$$\varphi_t(\gamma) = \begin{cases} \gamma & \text{for } \gamma \in \Gamma_1, \\ e^{tY}\gamma e^{-tY} & \text{for } \gamma \in \Gamma_2. \end{cases}$$

Note that $\varphi_t : \Gamma \to \mathrm{SO}(m,2)$ is well-defined since e^{tY} centralizes Γ_0. We now check Zariski density.

Lemma 6.2. *For $t \neq 0$ small enough, $\varphi_t(\Gamma)$ is Zariski-dense in* $\mathrm{SO}(m,2)$.

We need the following remark.

Remark 6.3. For $m \geq 3$, the only Lie subalgebra of $\mathfrak{so}(m,2)$ that strictly contains $\mathfrak{so}(m,1)$ is $\mathfrak{so}(m,2)$.

Indeed, $\mathfrak{so}(m,2)$ decomposes uniquely into a direct sum $\mathfrak{so}(m,1) \oplus \mathbb{R}^{m+1}$ of irreducible $\mathrm{SO}(m,1)$-modules, where $\mathrm{SO}(m,1)$ acts on $\mathfrak{so}(m,1)$ (resp. on $\mathbb{R}^{m+1}$) by the adjoint (resp. natural) action.

Proof of Lemma 6.2. Since $\mathrm{SO}(m,2)$ is Zariski-connected, it is sufficient to prove that for $t \neq 0$ small enough, the Zariski closure $\overline{\varphi_t(\Gamma)}$ of $\varphi_t(\Gamma)$ in $\mathrm{SO}(m,2)$ has Lie algebra $\mathfrak{so}(m,2)$.

By [JM], Lem. 5.9, the groups Γ_1 and Γ_2 are Zariski-dense in $\mathrm{SO}(m,1)$. By [JM], Cor. 5.3, and [Se2], § I.5.2, Cor. 1, they naturally embed into Γ. Therefore $\overline{\varphi_t(\Gamma)}$ contains both $\mathrm{SO}(m,1)$ and $e^{tY}\mathrm{SO}(m,1)e^{-tY}$, and the Lie algebra of $\overline{\varphi_t(\Gamma)}$ contains both $\mathfrak{so}(m,1)$ and the Lie algebra of $e^{tY}\mathrm{SO}(m,1)e^{-tY}$. By Remark 6.3, in order to prove that $\varphi_t(\Gamma)$ is Zariski-dense in $\mathrm{SO}(m,2)$, it is sufficient to prove that the Lie algebra of $e^{tY}\mathrm{SO}(m,1)e^{-tY}$ is not $\mathfrak{so}(m,1)$.

But if the Lie algebra of $e^{tY}\mathrm{SO}(m,1)e^{-tY}$ were $\mathfrak{so}(m,1)$, then we would have $e^{tY}\mathrm{SO}(m,1)^\circ e^{tY} = \mathrm{SO}(m,1)^\circ$; in other words, e^{tY} would belong to the normalizer $N_{\mathrm{SO}(m,2)}(\mathrm{SO}(m,1)^\circ)$ of the identity component $\mathrm{SO}(m,1)^\circ$ of $\mathrm{SO}(m,1)$. Recall that the exponential map induces a diffeomorphism between a neighborhood $\mathcal{U}$ of 0 in $\mathfrak{so}(m,2)$ and a neighborhood $\mathcal{V}$ of 1 in $\mathrm{SO}(m,2)$, and this diffeomorphism itself induces a one-to-one correspondence between $\mathcal{U}\cap\mathfrak{n}_{\mathfrak{so}(m,2)}(\mathfrak{so}(m,1))$ and $\mathcal{V}\cap N_{\mathrm{SO}(m,2)}(\mathrm{SO}(m,1)^\circ)$. Therefore, if we had $e^{tY} \in N_{\mathrm{SO}(m,2)}(\mathrm{SO}(m,1)^\circ)$ for some $t \neq 0$ small enough, then we would have

$$Y \in \mathfrak{n}_{\mathfrak{so}(m,2)}(\mathfrak{so}(m,1)) = \big\{X \in \mathfrak{so}(m,2),\ \mathrm{ad}(X)(\mathfrak{so}(m,1)) = \mathfrak{so}(m,1)\big\}.$$

But Remark 6.3 implies that $\mathfrak{n}_{\mathfrak{so}(m,2)}(\mathfrak{so}(m,1))$ is equal to $\mathfrak{so}(m,1)$, since it contains $\mathfrak{so}(m,1)$ and is different from $\mathfrak{so}(m,2)$. Thus we would have $Y \in \mathfrak{so}(m,1)$, which would contradict our choice of Y. □

6.3. **Deformations in the nonseparating case.** We now assume that $S = M \smallsetminus N$ is connected. Let $j_1 : \Gamma_0 \to \pi_1(S)$ and $j_2 : \Gamma_0 \to \pi_1(S)$ denote the inclusions in $\pi_1(S)$ of the fundamental groups of the two sides of N. The group Γ is a HNN extension of $\pi_1(S)$, *i.e.* it is generated by $\pi_1(S)$ and by some element $\nu \in \Gamma$ such that

$$\nu\, j_1(\gamma)\, \nu^{-1} = j_2(\gamma)$$

for all $\gamma \in \Gamma_0$. Fix an element $Y \in \mathfrak{so}(m,2) \smallsetminus \mathfrak{so}(m,1)$ that belongs to the Lie algebra of the centralizer of $j_1(\Gamma_0)$ in $\mathrm{SO}(m,2)$. Following Johnson and Millson, we consider the deformations of Γ in $\mathrm{SO}(m,2)$ that are given, for $t \in \mathbb{R}$, by

$$\begin{cases} \varphi_t(\gamma) = \gamma & \text{for } \gamma \in \pi_1(S), \\ \varphi_t(\nu) = \nu e^{tY}. \end{cases}$$

Note that $\varphi_t : \Gamma \to \mathrm{SO}(m,2)$ is well-defined since e^{tY} centralizes $j_1(\Gamma_0)$.

Lemma 6.4. *For $t \neq 0$ small enough, $\varphi_t(\Gamma)$ is Zariski-dense in* $\mathrm{SO}(m,2)$.

Proof. Let $\overline{\varphi_t(\Gamma)}$ denote the Zariski closure of $\varphi_t(\Gamma)$ in $\mathrm{SO}(m,2)$. By [JM], Lem. 5.9, the group $\pi_1(S)$ is Zariski-dense in $\mathrm{SO}(m,1)$, hence $\overline{\varphi_t(\Gamma)}$ contains both $\mathrm{SO}(m,1)$ and νe^{tY}. But $\nu \in \mathrm{SO}(m,1)$, hence $e^{tY} \in \overline{\varphi_t(\Gamma)}$. Therefore $\overline{\varphi_t(\Gamma)}$ contains both $\mathrm{SO}(m,1)$ and $e^{tY}\mathrm{SO}(m,1)e^{-tY}$, and we may conclude as in the proof of Lemma 6.2. □

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CNRS - Laboratoire Paul Painlevé, Université Lille 1, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex, France

E-mail address: `fanny.kassel@math.univ-lille1.fr`