#### Pierre Le Bris

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Motivation Propagation of chaos Results

#### II. Proof

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# Uniform in time propagation of chaos for the 2D vortex model and other singular stochastic systems.

Pierre Le Bris Joint work with : Arnaud Guillin (LMBP), Pierre Monmarché (LJLL)

LJLL, Sorbonne Université - Paris

Journées EFI 2021 13/10/2021

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## Idea

In a system of N interacting particles, as N increases, two particles become more and more statistically independent.

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# Formal limit of SDE

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### *N*-particle system on the torus $\mathbb{T}^d$

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N}\sum_{j=1}^N K(X_t^j - X_t^j)dt.$$

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### Limit as N tends to infinity?

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Limit as N tends to infinity? Formally

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(NL)

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# Liouville equations

### For the particle system

$$dX_t^i = \sqrt{2}dB_t^i + \frac{1}{N}\sum_{j=1}^N K(X_t^j - X_t^j)dt$$

$$\partial_t \rho_t^N = -\sum_{i=1}^N \nabla_{x_i} \cdot \left( \left( \frac{1}{N} \sum_{j=1}^N \mathcal{K}(x_i - x_j) \right) \rho_t^N \right) + \sum_{i=1}^N \Delta_{x_i} \rho_t^N.$$

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For the non linear equation

 $\leftrightarrow$ 

$$\begin{cases} d\bar{X}_t = \sqrt{2}dB_t + K * \bar{\rho}_t(\bar{X}_t)dt, \\ \bar{\rho}_t = \mathsf{Law}(\bar{X}_t). \end{cases} \longleftrightarrow \quad \partial_t \bar{\rho}_t = -\nabla \cdot (\bar{\rho}_t(K * \bar{\rho}_t)) + \Delta \bar{\rho}_t.$$

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# Main example : 2D vortex model

The Biot-Savart kernel, defined in  $\mathbb{R}^2$  by

$$K(x) = rac{1}{2\pi} rac{x^{\perp}}{|x|^2} = rac{1}{2\pi} \left( -rac{x_2}{|x|^2}, rac{x_1}{|x|^2} 
ight).$$

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Consider the 2D incompressible Navier-Stokes system on  $u \in \mathbb{R}^2$ 

$$\partial_t u = - u \cdot \nabla u - \nabla p + \Delta u$$
  
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where *p* is the local pressure.

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**Goal** : Obtain a limit " $\rho_t^N \to \bar{\rho}_t$ " as *N* tends to infinity for this Biot-Savart kernel.

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In a system of N interacting particles, as N increases, two particles become more and more statistically independent.

To quantify this "more and more", we compare the law of any subset of k particles within the N particles system to the law of k independent non-linear particles.

We denote, for any  $k \leq N$ 

$$\rho_t^{k,N}(x_1,..,x_k) = \int_{\mathbb{T}^{(N-k)d}} \rho_t^N(x_1,..,x_N) dx_{k+1}...dx_N$$
$$\bar{\rho}_t^k = \bar{\rho}_t^{\otimes k}$$

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# (Rescaled) relative entropy

### Definition

Let  $\mu$  and  $\nu$  be two probability measures on  $\mathbb{T}^{dN}$ . We consider the rescaled relative entropy

$$\mathcal{H}_{N}(\nu,\mu) = \begin{cases} \frac{1}{N} \mathbb{E}_{\mu} \left( \frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu} \right) & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

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### Theorem (Guillin-LB-Monmarché ('21))

Under some assumptions (satisfied by the Biot-Savart kernel) there are constants  $C_1$ ,  $C_2$  and  $C_3$  such that for all  $N \in \mathbb{N}$ , all exchangeable probability density  $\rho_0^N$  and all  $t \ge 0$ 

$$\mathcal{H}_{N}(\rho_{t}^{N},\bar{\rho}_{t}^{N}) \leq C_{1}e^{-C_{2}t}\mathcal{H}_{N}(\rho_{0}^{N},\bar{\rho}_{0}^{N}) + \frac{C_{3}}{N}$$

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## Various distances

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For  $\mathbf{x} = (x_i)_{i \in [\![1,N]\!]} \in \mathbb{T}^{dN}$ , we write  $\pi(\mathbf{x}) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$  the associated empirical measure.

### Corollary

Under some assumptions (satisfied by the Biot-Savart kernel), assuming moreover that  $\rho_0^N = \overline{\rho}_0^N$ , there is a constant *C* such that for all  $k \le N \in \mathbb{N}$  and all  $t \ge 0$ ,

$$\|\rho_t^{k,N} - \bar{\rho}_t^k\|_{L^1} + \mathcal{W}_2\left(\rho_t^{k,N}, \bar{\rho}_t^k\right) \le C\left(\left\lfloor\frac{N}{k}\right\rfloor\right)^{-\frac{1}{2}}$$

and

$$\mathbb{E}_{\rho_t^N}\left(\mathcal{W}_2(\pi(\boldsymbol{X}),\bar{\rho}_t)\right)\leqslant \boldsymbol{C}\alpha(\boldsymbol{N})$$

where  $\alpha(N) = N^{-1/2} \ln(1 + N)$  if d = 2 and  $\alpha(N) = N^{-1/d}$  if d > 2.

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# Step one : Time evolution of the relative entropy

# $\mathcal{H}_{N}(t) = \mathcal{H}_{N}(\rho_{t}^{N}, \bar{\rho}_{t}^{N}), \quad \mathcal{I}_{N}(t) = \frac{1}{N} \sum_{i} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left| \nabla_{x_{i}} \log \frac{\rho_{t}^{N}}{\bar{\rho}_{t}^{N}} \right|^{2} d\mathbf{X}^{N}.$

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# Step one : Time evolution of the relative entropy

### We write

$$\mathcal{H}_{N}(t) = \mathcal{H}_{N}(\rho_{t}^{N}, \bar{\rho}_{t}^{N}), \quad \mathcal{I}_{N}(t) = \frac{1}{N} \sum_{i} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left| \nabla_{\mathbf{x}_{i}} \log \frac{\rho_{t}^{N}}{\bar{\rho}_{t}^{N}} \right|^{2} d\mathbf{X}^{N}.$$

It has been shown, by Jabin-Wang, that

$$\begin{split} \frac{d}{dt} \mathcal{H}_{N}(t) &\leq -\mathcal{I}_{N}(t) \\ &- \frac{1}{N^{2}} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left( \mathcal{K}(x_{i} - x_{j}) - \mathcal{K} * \rho(x_{i}) \right) \cdot \nabla_{x_{i}} \log \bar{\rho}_{t}^{N} d\mathbf{X}^{N} \\ &- \frac{1}{N^{2}} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left( \operatorname{div} \, \mathcal{K}(x_{i} - x_{j}) - \operatorname{div} \, \mathcal{K} * \bar{\rho}_{t}(x_{i}) \right) d\mathbf{X}^{N}. \end{split}$$

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**Goal**: 
$$K(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2} = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$$

Justifying the calculations

• 
$$\bar{\rho} \in \mathcal{C}^{\infty}(\mathbb{R}^+ \times \mathbb{T}^d)$$

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### Justifying the calculations

•  $\bar{\rho} \in \mathcal{C}^{\infty}(\mathbb{R}^+ \times \mathbb{T}^d)$  and there is  $\lambda > 1$ , s.t  $\frac{1}{\lambda} \leq \bar{\rho} \leq \lambda$ 

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### Justifying the calculations

• 
$$\bar{\rho} \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^d)$$

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$$K(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2} = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$$

### Justifying the calculations

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Goal: 
$$K(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2} = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$$

### Justifying the calculations

• There is  $\lambda > 1$  such that  $\bar{\rho}_0 \in C^{\infty}_{\lambda}(\mathbb{T}^d)$  $\implies \bar{\rho} \in C^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^d)$  (Ben-Artzi ('94))

• 
$$ho^{\mathsf{N}} \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{R}^+ imes \mathbb{T}^{\mathsf{Nd}})$$
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Goal: 
$$K(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2} = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$$

### Justifying the calculations

There is λ > 1 such that ρ
<sub>0</sub> ∈ C<sup>∞</sup><sub>λ</sub>(T<sup>d</sup>) ⇒ ρ
 ∈ C<sup>∞</sup><sub>λ</sub>(ℝ<sup>+</sup> × T<sup>d</sup>) (Ben-Artzi ('94))
 ρ<sup>N</sup> ∈ C<sup>∞</sup><sub>λ</sub>(ℝ<sup>+</sup> × T<sup>Nd</sup>) (???)

Dealing with the terms

• In the sense of distributions,  $\nabla \cdot K = 0$ .

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# Step one : Time evolution of the relative entropy

### We write

$$\mathcal{H}_{N}(t) = \mathcal{H}_{N}(\rho_{t}^{N}, \bar{\rho}_{t}^{N}), \quad \mathcal{I}_{N}(t) = \frac{1}{N} \sum_{i} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left| \nabla_{x_{i}} \log \frac{\rho_{t}^{N}}{\bar{\rho}_{t}^{N}} \right|^{2} d\mathbf{X}^{N}.$$

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$$\begin{split} \frac{d}{dt} \mathcal{H}_{N}(t) &\leq -\mathcal{I}_{N}(t) \\ &- \frac{1}{N^{2}} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left( \mathcal{K}(x_{i} - x_{j}) - \mathcal{K} * \rho(x_{i}) \right) \cdot \nabla_{x_{i}} \log \bar{\rho}_{t}^{N} d\mathbf{X}^{N} \\ &- \frac{1}{N^{2}} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left( \operatorname{div} \, \mathcal{K}(x_{i} - x_{j}) - \operatorname{div} \, \mathcal{K} * \bar{\rho}_{t}(x_{i}) \right) d\mathbf{X}^{N}. \end{split}$$

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# Step one : Time evolution of the relative entropy

### We write

$$\mathcal{H}_{N}(t) = \mathcal{H}_{N}(\rho_{t}^{N}, \bar{\rho}_{t}^{N}), \quad \mathcal{I}_{N}(t) = \frac{1}{N} \sum_{i} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left| \nabla_{\mathbf{x}_{i}} \log \frac{\rho_{t}^{N}}{\bar{\rho}_{t}^{N}} \right|^{2} d\mathbf{X}^{N}.$$

### It has been shown, by Jabin-Wang, that

$$egin{aligned} rac{d}{dt}\mathcal{H}_{N}(t) &\leq -\mathcal{I}_{N}(t) \ &-rac{1}{N^{2}}\sum_{i,j}\int_{\mathbb{T}^{dN}}
ho_{l}^{N}\left(\mathcal{K}(x_{i}-x_{j})-\mathcal{K}*
ho(x_{i})
ight)\cdot
abla_{x_{i}}\logar{
ho}_{l}^{N}d\mathbf{X}^{N} \end{aligned}$$

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# Step two : Integration by part

### We are left with

 $egin{aligned} &rac{d}{dt}\mathcal{H}_{N}(t)\leq &-\mathcal{I}_{N}(t)\ &-rac{1}{N^{2}}\sum_{i,j}\int_{\mathbb{T}^{dN}}
ho_{t}^{N}\left(\mathcal{K}(x_{i}-x_{j})-\mathcal{K}*
ho(x_{i})
ight)\cdot
abla_{x_{i}}\logar{
ho}_{t}^{N}d\mathbf{X}^{N}. \end{aligned}$ 

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### **Idea** : Use the regularity of $\bar{\rho}$ to deal with the singularity of K

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# Step two : Integration by part

### We are left with

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$$egin{aligned} rac{d}{dt}\mathcal{H}_{N}(t) &\leq -\mathcal{I}_{N}(t) \ &-rac{1}{N^{2}}\sum_{i,j}\int_{\mathbb{T}^{dN}}
ho_{t}^{N}\left(\mathcal{K}(x_{i}-x_{j})-\mathcal{K}*
ho(x_{i})
ight)\cdot
abla_{x_{i}}\logar{
ho}_{t}^{N}d\mathbf{X}^{N}. \end{aligned}$$

**Idea** : Use the regularity of  $\bar{\rho}$  to deal with the singularity of K **Remark :** Notice that, for the Biot-Savart kernel on the whole space  $\mathbb{R}^2$ 

$$\tilde{K}(x)=\frac{1}{2\pi}\frac{x^{\perp}}{|x|^2},$$

we have  $\tilde{K} = \nabla \cdot \tilde{V}$  with

$$ilde{V}(x) = rac{1}{2\pi} \left( egin{array}{c} -\arctan\left(rac{x_1}{x_2}
ight) & 0 \\ 0 & \arctan\left(rac{x_2}{x_1}
ight) \end{array} 
ight).$$

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# Assumptions?

**Goal :** 
$$K(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2} = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)$$

### Justifying the calculations

• There is  $\lambda > 1$  such that  $\bar{\rho}_0 \in C^{\infty}_{\lambda}(\mathbb{T}^d)$  $\implies \bar{\rho} \in C^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^d)$  (Ben-Artzi '94)

• 
$$ho^{\sf N}\in \mathcal{C}^\infty_\lambda(\mathbb{R}^+ imes \mathbb{T}^{\sf Nd})$$
 (???

### Dealing with the terms

- In the sense of distributions,  $\nabla \cdot K = 0$ .
- There is a matrix field  $V \in L^{\infty}$  such that  $K = \nabla \cdot V$ , i.e for  $1 \leq \alpha \leq d$ ,  $K_{\alpha} = \sum_{\beta=1}^{d} \partial_{\beta} V_{\alpha,\beta}$  (Phuc-Torres '08).

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# Step two : Integration by part

$$\frac{d}{dt}\mathcal{H}_N(t) \leq A_N(t) + \frac{1}{2}B_N(t) - \frac{1}{2}\mathcal{I}_N(t),$$

### with

For all  $t \ge 0$ ,

$$\begin{split} A_{N}(t) &:= \frac{1}{N^{2}} \sum_{i,j} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \left( V(x_{i} - x_{j}) - V * \bar{\rho}(x_{i}) \right) : \frac{\nabla_{x_{i}}^{2} \bar{\rho}_{t}^{N}}{\bar{\rho}_{t}^{N}} d\mathbf{X}^{N} \\ B_{N}(t) &:= \frac{1}{N} \sum_{i} \int_{\mathbb{T}^{dN}} \rho_{t}^{N} \frac{\left| \nabla_{x_{i}} \bar{\rho}_{t}^{N} \right|^{2}}{\left| \bar{\rho}_{t}^{N} \right|^{2}} \left| \frac{1}{N} \sum_{j} V(x_{i} - x_{j}) - V * \bar{\rho}(x_{i}) \right|^{2} d\mathbf{X}^{N}. \end{split}$$

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### Lemma

For two probability densities  $\mu$  and  $\nu$  on a set  $\Omega$ , and any  $\Phi \in L^{\infty}(\Omega)$ ,  $\eta > 0$  and  $N \in \mathbb{N}$ ,

$$\mathbb{E}^{\mu} \Phi \leq \eta \mathcal{H}_{\mathsf{N}}(\mu, 
u) + rac{\eta}{\mathsf{N}} \log \mathbb{E}^{
u} e^{\mathsf{N} \Phi / \eta}$$

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# Large deviation estimates -1

### Theorem (Jabin-Wang '18)

Consider any probability measure  $\mu$  on  $\mathbb{T}^d$ ,  $\epsilon > 0$  and a scalar function  $\psi \in L^{\infty}(\mathbb{T}^d \times \mathbb{T}^d)$  with  $\|\psi\|_{L^{\infty}} < \frac{1}{2\epsilon}$  and such that for all  $z \in \mathbb{T}^d$ ,  $\int_{\mathbb{T}^d} \psi(z, x)\mu(dx) = 0$ . Then there exists a constant C such that

$$\int_{\mathbb{T}^{dN}} \exp\Big(\frac{1}{N}\sum_{j_1,j_2=1}^N \psi(x_1,x_{j_1})\psi(x_1,x_{j_2})\Big)\mu^{\otimes N} d\mathbf{X}^N \leq C,$$

where C depends on

$$lpha = (\epsilon \|\psi\|_{L^{\infty}})^4 < 1$$
 ,  $\beta = \left(\sqrt{2\epsilon} \|\psi\|_{L^{\infty}}\right)^4 < 1$ .

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# Large deviation estimates -2

### Theorem (Jabin-Wang '18)

Consider any probability measure  $\mu$  on  $\mathbb{T}^d$  and  $\phi \in L^{\infty}(\mathbb{T}^d \times \mathbb{T}^d)$  with

$$\gamma := \left(1600^2 + 36e^4\right) \left(\sup_{p \ge 1} \frac{\|\sup_{z} |\phi(\cdot, z)|\|_{L^p(\mu))}}{p}\right)^2 < 1.$$

Assume that  $\phi$  satisfies the following cancellations

$$\forall z \in \mathbb{T}^d, \quad \int_{\mathbb{T}^d} \phi(x, z) \mu(dx) = 0 = \int_{\mathbb{T}^d} \phi(z, x) \mu(dx).$$

Then, for all  $N \in \mathbb{N}$ ,

$$\int_{\mathbb{T}^{dN}} \exp\Big(\frac{1}{N}\sum_{i,j=1}^{N}\phi(x_i,x_j)\Big)\mu^{\otimes N}d\boldsymbol{X}^N \leq \frac{2}{1-\gamma} < \infty.$$

## Conclusion

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#### Unif. in time Prop. of Chaos for the 2D vortex model

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### For all $t \ge 0$ ,

$$\frac{d}{dt}\mathcal{H}_N(t) \leq C\left(\mathcal{H}_N(t) + \frac{1}{N}\right) - \frac{1}{2}\mathcal{I}_N(t),$$

with

$$\boldsymbol{C} = \hat{\boldsymbol{C}}_1 \|\nabla^2 \bar{\rho}_t\|_{L^{\infty}} \|\boldsymbol{V}\|_{L^{\infty}} \lambda + \hat{\boldsymbol{C}}_2 \|\boldsymbol{V}\|_{L^{\infty}}^2 \lambda^2 \boldsymbol{d}^2 \|\nabla \bar{\rho}_t\|_{L^{\infty}}^2$$

where  $\hat{C}_1, \hat{C}_2$  are universal constants.

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# Step four : Uniform bounds and logarithmic Sobolev inequality

Two goals :

• A logarithmic Sobolev inequality for  $\bar{\rho}^N$  :  $\mathcal{H}_N(t) \leq C \mathcal{I}_N(t)$ 

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# Step four : Uniform bounds and logarithmic Sobolev inequality

### Two goals :

• A logarithmic Sobolev inequality for  $\bar{\rho}^N$  :  $\mathcal{H}_N(t) \leq C \mathcal{I}_N(t)$ 

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• Uniform in time bounds on  $\|\nabla \bar{\rho}_t\|_{L^{\infty}}$  and  $\|\nabla^2 \bar{\rho}_t\|_{L^{\infty}}$ 

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# A logarithmic Sobolev inequality

### Lemma (Tensorization)

If  $\nu$  is a probability measure on  $\mathbb{T}^d$  satisfying a LSI with constant  $C_{\nu}^{LS}$ , then for all  $N \geq 0$ ,  $\nu^{\otimes N}$  satisfies a LSI with constant  $C_{\nu}^{LS}$ 

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# A logarithmic Sobolev inequality

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### Lemma (Perturbation)

If  $\nu$  is a probability measure on  $\mathbb{T}^d$  satisfying a LSI with constant  $C_{\nu}^{LS}$ , and  $\mu$  is a probability measure with density h with respect to  $\nu$  such that, for some constant  $\lambda > 0$ ,  $\frac{1}{\lambda} \le h \le \lambda$ , then  $\mu$  satisfies a LSI with constant  $C_{\mu}^{LS} = \lambda^2 C_{\nu}^{LS}$ .

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# A logarithmic Sobolev inequality

### Lemma (Tensorization)

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### Lemma (LSI for the uniform distribution)

The uniform distribution u on  $\mathbb{T}^d$  satisfies a LSI with constant  $\frac{1}{8\pi^2}$ .

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# A logarithmic Sobolev inequality

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### Lemma (LSI for the uniform distribution)

The uniform distribution u on  $\mathbb{T}^d$  satisfies a LSI with constant  $\frac{1}{8\pi^2}$ .

For all  $N \in \mathbb{N}$ ,  $t \ge 0$  and all probability density  $\mu_N \in \mathcal{C}^{\infty}_{>0}(\mathbb{T}^{dN})$ ,

$$\mathcal{H}_{N}\left(\mu_{N}, \bar{\rho}_{t}^{N}\right) \leq \frac{\lambda^{2}}{8\pi^{2}} \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{T}^{d}} \mu_{N} \left| \nabla_{x_{i}} \log \frac{\mu_{N}}{\bar{\rho}_{t}^{N}} \right|^{2} d\mathbf{X}^{N}$$

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# Uniform in time bounds on the derivatives

### Lemma

For all  $n \ge 1$  and  $\alpha_1, ..., \alpha_n \in \llbracket 1, d \rrbracket$ , there exist  $C_n^u, C_n^\infty > 0$  such that for all  $t \ge 0$ ,

$$\|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_t\|_{L^{\infty}} \leq C_n^u \quad and \quad \int_0^t \|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_s\|_{L^{\infty}}^2 ds \leq C_n^{\infty}$$

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# Uniform in time bounds on the derivatives

### Lemma

For all  $n \ge 1$  and  $\alpha_1, ..., \alpha_n \in [\![1, d]\!]$ , there exist  $C_n^u, C_n^\infty > 0$  such that for all  $t \ge 0$ ,

$$\|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_t\|_{L^{\infty}} \leq C_n^u \quad \text{and} \quad \int_0^t \|\partial_{\alpha_1,\ldots,\alpha_n}\bar{\rho}_s\|_{L^{\infty}}^2 ds \leq C_n^{\infty}$$

Thanks to Morrey's inequality and Sobolev embeddings, it is sufficient to prove such bounds in the Sobolev space  $H^m$  for all *m*, i.e in  $L^2$ 

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# Uniform in time bounds on the derivatives-2

By induction on the order of the derivative

$$\frac{1}{2}\frac{d}{dt}\|\bar{\rho}_t\|_{L^2}^2+\|\nabla\bar{\rho}_t\|_{L^2}^2=0,$$

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# Uniform in time bounds on the derivatives-2

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$$\frac{1}{2}\frac{d}{dt}\|\partial_{\alpha_1}\bar{\rho}_t\|_{L^2}^2 + \frac{1}{2}\sum_{\alpha_2}\|\partial_{\alpha_1,\alpha_2}\bar{\rho}_t\|_{L^2}^2 \leq \frac{1}{2}\|K\|_{L^1}^2\|\bar{\rho}_t\|_{L^\infty}^2\|\nabla\bar{\rho}_t\|_{L^2}^2,$$

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$$\frac{1}{2} \frac{d}{dt} \|\partial_{\alpha_1,\alpha_2} \bar{\rho}_t\|_{L^2}^2 + \frac{1}{2} \sum_{\alpha_3} \|\partial_{\alpha_1,\alpha_2,\alpha_3} \bar{\rho}_t\|_{L^2}^2 \le \|V\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2 + \|K\|_{L^1}^2 \|\bar{\rho}_t\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2$$

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$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_{\alpha_1, \alpha_2} \bar{\rho}_t\|_{L^2}^2 + \frac{1}{2} \sum_{\alpha_3} \|\partial_{\alpha_1, \alpha_2, \alpha_3} \bar{\rho}_t\|_{L^2}^2 \leq \|V\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2 \|\nabla \bar{\rho}_t\|_{L^2}^2 \\ + \|K\|_{L^1}^2 \|\bar{\rho}_t\|_{L^\infty}^2 \|\partial_{\alpha_1} \nabla \bar{\rho}_t\|_{L^2}^2, \end{aligned}$$

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# Assumptions?

Goal : 
$$K(x) = rac{1}{2\pi} rac{x^{\perp}}{|x|^2} = rac{1}{2\pi} \left( -rac{x_2}{|x|^2}, rac{x_1}{|x|^2} 
ight)$$

### Justifying the calculations

1

There is λ > 1 such that ρ
<sub>0</sub> ∈ C<sup>∞</sup><sub>λ</sub>(T<sup>d</sup>) ⇒ ρ
 ∈ C<sup>∞</sup><sub>λ</sub>(ℝ<sup>+</sup> × T<sup>d</sup>) (Ben-Artzi '94)
 ρ<sup>N</sup> ∈ C<sup>∞</sup><sub>λ</sub>(ℝ<sup>+</sup> × T<sup>Nd</sup>) (???)

### Dealing with the terms

- In the sense of distributions,  $\nabla \cdot K = 0$ .
- There is a matrix field  $V \in L^{\infty}$  such that  $K = \nabla \cdot V$ , i.e for  $1 \leq \alpha \leq d$ ,  $K_{\alpha} = \sum_{\beta=1}^{d} \partial_{\beta} V_{\alpha,\beta}$  (Phuc-Torres '08).

### Uniformity in time

- For all  $n \geq 1, \ C_n^0 := \| \nabla^n \bar{\rho}_0 \|_{L^\infty} < \infty$
- $\|K\|_{L^1} < \infty$  (also used to show regularity).

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# Step five : Conclusion

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There are constants  $C_1, C_2^{\infty}, C_3 > 0$  and a function  $t \mapsto C_2(t) > 0$  with  $\int_0^t C_2(s) ds \le C_2^{\infty}$  for all  $t \ge 0$  such that for all  $t \ge 0$ 

$$\frac{d}{dt}\mathcal{H}_N(t) \leq -(C_1 - C_2(t))\mathcal{H}_N(t) + \frac{C_3}{N}$$

Multiplying by  $\exp(C_1 t - \int_0^t C_2(s) ds)$  and integrating in time we get

$$egin{aligned} \mathcal{H}_{N}(t) &\leq e^{-C_{1}t+\int_{0}^{t}C_{2}(s)ds}\mathcal{H}_{N}(0)+rac{C_{3}}{N}\int_{0}^{t}e^{C_{1}(s-t)+\int_{s}^{t}C_{2}(u)du}ds\ &\leq e^{C_{2}^{\infty}-C_{1}t}\mathcal{H}_{N}(t)+rac{C_{3}}{C_{1}N}e^{C_{2}^{\infty}}\,, \end{aligned}$$

which concludes.

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#### On the assumptions

# On $\rho^{\mathsf{N}} \in \mathcal{C}^{\infty}_{\lambda}(\mathbb{R}^+ \times \mathbb{T}^{\mathsf{Nd}})$

Everything works for regularized kernels  $K^{\epsilon}$ , and the final result is independent of  $\epsilon$ .

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### On the initial condition

- There is  $\lambda > 1$  such that  $\bar{\rho}_0 \in \mathcal{C}^\infty_\lambda(\mathbb{T}^d)$
- For all  $n \geq 1$ ,  $C_n^0 := \|\nabla^n \bar{\rho}_0\|_{L^\infty} < \infty$

### On the potential K

- $\|K\|_{L^1} < \infty$ .
- In the sense of distributions,  $\nabla \cdot K = 0$ ,
- There is a matrix field  $V \in L^{\infty}$  such that  $K = \nabla \cdot V$ , i.e for  $1 \leq \alpha \leq d$ ,  $K_{\alpha} = \sum_{\beta=1}^{d} \partial_{\beta} V_{\alpha,\beta}$ .

# Assumptions

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### Thank you

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