Period relations for automorphic induction and applications

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Introduction

In [Har97], M. Harris has defined complex invariants, called automorphic periods, for certain automorphic representations of GL_n over quadratic imaginary field. It is defined as the Petersson inner product of a rational element in the coherent cohomology of certain Shimura variety associated to unitary groups. We believe that these periods are functorial. In this poster, we show this for cyclic automorphic induction of Hecke characters over CM fields.

These relations allow us to simplify a formula obtained by Grobner and Harris on the critical values for the Rankin-Selberg L-function of certain pair of automorphic representations for $GL_n \times GL_{n-1}$ over a quadratic imaginary field. This completes the proof of an automorphic version of Deligne's conjecture in certain case.

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Notation and conventions

We fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Let $K \subset \mathbb{C}$ be a quadratic imaginary field.

Fix n an integer at least 2. For simplicity of notation, we assume that n is odd.

Let F^+ be a totally real field of degree n over $\mathbb{Q}$ such that the CM field $F = KF^+$ is cyclic over K.

Let χ be a regular algebraic Hecke character of F. Let Ψ be a CM type of F. We define $\check{\chi} := \chi^{-1,c}$. A **CM period** $p(\check{\chi}, \Psi)$ is a complex number defined up to multiplication by an element in certain number field associated to χ.

Let Π (resp. Π') be a cuspidal representation of $GL_n(\mathbb{A}_K)$ (resp. $GL_{n-1}(\mathbb{A}_K)$) which is regular, cohomological and conjugate self-dual. Harris has defined **automorphic periods** $P^{(s)}(\Pi)$ for certain integers $0 \leq s \leq n$ which are complex numbers defined up to multiplication by an element in certain number field associated to Π. With some extra local conditions on Π, these automorphic periods can be defined for all $0 \leq s \leq n$.

For two complex numbers x, y, we say $x \sim y$ if $y \neq 0$ and x/y is an algebraic number.

Main result

Theorem 1:
Let χ be a regular conjugate self-dual algebraic Hecke character of F satisfying some technical conditions on any two finite places of $\mathbb{Q}$ which split in K. Let $\Pi(\chi)$ be the automorphic induction of χ from $GL_1(F)$ to $GL_n(K)$. There exists $\Phi_{s,\chi}$, a CM type of F depending only on s and χ, such that:

$$P^{(s)}(\Pi) \sim p(\check{\chi}, \Phi_{s,\chi}).$$

Application

The above theorem allows us to simplify Theorem 6.10 in [GH] and we get:

Theorem 2: *Let Π and Π' be cuspidal representations of $GL_n(\mathbb{A}_K)$ and $GL_{n-1}(\mathbb{A}_K)$ respectively which are very regular, cohomological, conjugate self-dual, supercuspidal over at least two places of $\mathbb{Q}$ that split in K. Let $m \geq 0$ be an integer such that $m+n-1$ is critical for $M(\Pi) \otimes M(\Pi')$ in the sense of Deligne where $M(\Pi)$ and $M(\Pi')$ are pure motives for absolute Hodge cycles associated to Π and Π' realized by Harris in [Har97]. We then have the following equation:*

$$L(m+\frac{1}{2}, \Pi \times \Pi') \sim (2\pi i)^{(m+\frac{1}{2})n(n-1)} \prod_{j=1}^{n-1} P^{(j)}(\Pi) \prod_{k=1}^{n-2} P^{(k)}(\Pi').$$

Remarks:
1. The above theorem is compatible with the Deligne conjecture and Harris' calculation on the Deligne period.
2. The equations in both theorems are equivariant under $Gal(\overline{\mathbb{Q}}/K)$-actions.

References

[GH] H. Grobner and M. Harris. Whittaker periods, motivic periods, and special values of tensor product of L-functions, preprint available at Harris' home page.

[Har97] M. Harris. L-functions and periods of polarized regular motives. *J. Reine Angew. Math*, (483):75–161, 1997.