PERIODS AND L-VALUES OF AUTOMORPHIC MOTIVES

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Abstract. A conjecture of Deligne predicts a relation between motivic L-functions and geometric periods. In this synopsis, we will explain an approach towards this conjecture for automorphic motives. This is a joint work with Harald Grobner and Michael Harris.

Contents

Introduction

The goal of this synopsis is to introduce a conjecture of Deligne on special values of L-functions and its automorphic variant. We first look at the most basic example of the Deligne conjecture.

Recall that the Riemann zeta function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

for $s \in \mathbb{C}$ with $Re(s) > 1$.

Theorem 0.1. *If m is a positive even integer, then $\zeta(m) \in (2\pi i)^m \mathbb{Q}$.*

For example, the values $\zeta(2) = \dfrac{\pi^2}{6}$ and $\zeta(4) = \dfrac{\pi^4}{90}$.

2010 *Mathematics Subject Classification.* 11F67 (Primary) 11F70, 11G18, 11R39, 22E55 (Secondary).
The author is supported by the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 290766 (AAMOT).

Proof. We briefly explain the proof given by Riemann in [Rie59] here.

The Gamma function $\Gamma(s)$, defined by the integral $\int_0^\infty t^s e^{-t}\frac{dt}{t}$, is a meromorphic function on the whole complex plane. It has no zeros everywhere, and has simple poles with rational residues at non-positive integers.

The product $\Gamma(s)\zeta(s)$ has integral intepretation $\int_0^\infty t^s \frac{1}{e^t-1}\frac{dt}{t}$. The integral defines a meromorphic function on the whole complex plane. At a non-positive integer, it is either holomorphic, or has a simple pole with rational residue.

In particular, $\zeta(s)$ is a ratio of two meromorphic functions, and hence has analytic continuation to the whole complex plane. Moreover, it is holomorphic and takes rational values at negative integers.

Let $\zeta_\infty(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. The Riemann zeta function satisfies the functional equation:

$$\zeta_\infty(s)\zeta(s) = \zeta_\infty(1-s)\zeta(1-s).$$

Since m is a positive even integer, both $\zeta_\infty(m)$ and $\zeta_\infty(1-m)$ are holomorphic at $s = m$. In this case, we say m is **critical**. One deduce easily that $\zeta(m) = \frac{\zeta_\infty(1-m)}{\zeta_\infty(m)}\zeta(1-m) \in (2\pi i)^m\mathbb{Q}$ since $\zeta(1-m)$ is a rational number.

□

Remark 0.2. For a positive odd integer $m \geqslant 3$, $\zeta_\infty(1-s)$ has a pole at $s = m$. This implies that $\zeta(1-m) = 0$ and $\zeta(m) = \frac{Res_{s=m}\zeta_\infty(1-s)}{\zeta_\infty(m)}\zeta'(1-m)$. In this case, it is much more difficult to calculate $\zeta(m)$.

In [Del79], Deligne generalized largely the above theorem as a conjecture. More precisely, he constructed two periods and predicted a precise relation between critical values and his periods.

In this synopsis, we first introduce Deligne's conjecture in Section 1. We define some other motivic periods and interpret Deligne's periods in terms of these newly defined periods in Section 2. The latter have natural automorphic analogues which are introduced in Section 3. We reformulate the Deligne conjecture for tensor products of automorphic motives and gives a variant of the Deligne conjecture in Section 4. We finally summarise the known results and discuss some possible generalizations in Section 5.

At the end of the introduction, we want to warn the readers that since there is not enough space in the synopsis, some definitions and statements are not very precise. We refer to the references for more details.

1. A conjecture of Deligne

Let M be a motive over $\mathbb{Q}$ with coefficients in a number field E and pure of weight w (c.f. [Del79]). For simplicity, we fix an embedding $E \hookrightarrow \mathbb{C}$. The motive M has several realizations as follows:

1.0.1. *The Betti realization:* M_B is a finite dimensional vector space over E endowed with:

- an E-linear action of the infinite Frobenius F_∞;
- a Hodge decomposition $M_B \otimes_E \mathbb{C} = \bigoplus_{p+q=w} M^{p,q}$ where each $M^{p,q}$ is a vector space over $\mathbb{C}$. The action of F_∞ on M_B extends naturally to an action on $M_B \otimes_E \mathbb{C}$ which exchanges $M^{p,q}$ and $M^{q,p}$.

1.0.2. *The de Rham realization:* M_{DR} is also a finite dimensional vector space over E endowed with an E-rational Hodge filtration: $M_{DR} \supset \cdots \supset F^i M \supset F^{i+1} M \supset \cdots$.

1.0.3. *The comparison isomorphism:* $I_\infty : M_B \otimes_E \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes_E \mathbb{C}$ is compatible with the Hodge structures on the two sides. More preciesly, for all integer p_0, we have:

$$I_\infty(\bigoplus_{p \geqslant p_0} M^{p,w-p}) = F^{p_0} M \otimes \mathbb{C}.$$

1.0.4. *The λ-adic realizations:* M_λ is a finite dimensional vector space over E_λ endowed with an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ for each finite place λ of E. The family $\{M_\lambda\}_\lambda$ forms a *compatible system* of Galois representations.

More precisely, for each finite place p of $\mathbb{Q}$, let I_p be the inertia subgroup of a decomposition group at p. Let F_p be the geometric Frobenius of this decomposition group. For a *compatible system* we mean that for any $\lambda \nmid p$, the polynomial $det(1 - F_p X \mid M_\lambda^{I_p})$ has coefficients in $E \subset E_\lambda$, and is independent of the choice of λ.

We can hence define the local L-factor $L_p(s, M) := det(1 - p^{-s} F_p | (M_\lambda)^{I_p})^{-1}$ as an element in $\mathbb{C}(p^{-s})$ by taking whatever $\lambda \nmid p$ (recall that we have fixed an embedding of E in $\mathbb{C}$). We define the L-function for the motive M as the Euler product:

$$L(s, M) = \prod_p L_p(s, M).$$

It converges absolutely for $Re(s) >> 0$.

As for the Riemann zeta function, one can define an archimedean factor $L_\infty(s, M)$ explicitly, determined by the Hodge type of the motive M, and the ϵ-factor $\epsilon(s, M)$ as in [Del79]. It is expected that:

Conjecture 1.1. *The motivic L-function $L(s, M)$ has an analytic continuation to the whole complex plane, and satisfies a functional equation:*

$$L_\infty(s, M) L(s, M) = \epsilon(s, M) L_\infty(1 - s, \check{M}) L(1 - s, \check{M})$$

where $\check{M}$ is the dual motive of M.

Definition 1.2. We say an integer m is **critical** for the motive M if both $L_\infty(s,M)$ and $L_\infty(1-s,\check{M})$ are holomorphic at $s=m$.

Deligne formulated his conjecture under the assumption that F_∞ acts as a scalar on $M^{w/2,w/2}$. We assume a stronger assumption $M^{w/2,w/2}=0$. In particular, we see that $\dim_E M_B = \dim_E M_{DR}$ is even.

Remark 1.3. When the motive M is restricted from a quadratic imaginary field, the condition $M^{w/2,w/2}=0$ is equivalent to that M has critical points. This is the case that we are going to consider in the next sections.

We define $M_B^{\pm} = M_B^{\pm F_\infty} \subset M_B$, $F^{\pm}M := F^{w/2}M \subset M_{DR}$. They are all vector spaces over E of dimension $\dim_E M_B/2$.

The comparison isomorphism then induces $\mathbb{C}$-linear maps:

$$I_\infty^{\pm} : M_B^{\pm} \otimes_E \mathbb{C} \hookrightarrow M_B \otimes_E \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes_E \mathbb{C} \to (M_{DR}/F^{\pm}M) \otimes_E \mathbb{C}. \tag{1.4}$$

It is easy to see that these maps are injective. Comparing the dimensions on the two sides, one deduces easily that I_∞^+ and I_∞^- are both isomorphisms.

We take any E-bases of $M_B^{\pm}$ and $M_{DR}/F^{\pm}M$. They can be considered as $\mathbb{C}$-bases of $M_B^{\pm} \otimes_E \mathbb{C}$ and $(M_{DR}/F^{\pm}M) \otimes_E \mathbb{C}$.

Definition 1.5. The **Deligne periods** $c^{\pm}(M)$ are defined as the determinants of $I_\infty^{\pm}$ with respect to these bases. They are non-zero complex numbers and are well-defined up to multiplication by elements in $E^\times$.

Conjecture 1.6 (Deligne 79). *Let $d^{\pm} := dim_E M_B^{\pm}$. If $m \in \mathbb{Z}$ is critical for M, then*

$$L(m,M) \in (2\pi i)^{d^{\epsilon}m} c^{\epsilon}(M) E$$

where ϵ is the sign of $(-1)^m$.

Remark 1.7. A positive integer is critical for the Riemann zeta function if and only if it is even. In this case, Theorem 0.1 affirms the Deligne conjecture for the Riemann zeta function.

2. Motivic periods

In general, we know very few about the L-function of a motive unless it is the same as the L-function of an automorphic representation after proper normalizations. In this case, we say the motive is attached to the automorphic representation. Roughly speaking, the Langlands correspondence asserts that any motive is attached to an automorphic representation and vice versa.

In the automorphic setting, we shall consider the Rankin-Selberg L-function for a pair of automorphic representations over a quadratic imaginary field K. This corresponds to the tensor product of two motives over K in the motivic setting.

We fix an embedding of K in $\mathbb{C}$. Let $\mathcal{M}$ (resp. $\mathcal{M}'$) be a regular pure motive over K with coefficients in E and of rank n (resp. n') with Hodge types $p_1 > p_2 > \cdots > p_n$ at the fixed embedding.

Let $M := Res_{K/\mathbb{Q}}\mathcal{M} \otimes \mathcal{M}'$ be a motive over $\mathbb{Q}$. One can show that $c^+(M)/c^-(M)$ is an algebraic number. We refer to [Har-Lin16] and [Har13b] for more details..

In the following, we write $x \sim y$ if $[x : y] \in \mathbb{P}^1(\overline{\mathbb{Q}})$. For simplicity, we only consider the Deligne conjecture up to multiplication by elements in $\overline{\mathbb{Q}}^\times$. We can henceforth forget $c^-(M)$. The Deligne conjecture predicts that if m is critical then

$$L(m, Res_{K/\mathbb{Q}}\mathcal{M} \otimes \mathcal{M}') \sim (2\pi i)^{d^+ m} c^+(Res_{K/\mathbb{Q}}\mathcal{M} \otimes \mathcal{M}')) \tag{2.1}$$

In order to separate $\mathcal{M}$ and $\mathcal{M}'$ in the term $c^+(Res_{K/\mathbb{Q}}\mathcal{M} \otimes \mathcal{M}'))$, we define motivic periods $Q_i(\mathcal{M})$, $1 \leqslant i \leqslant n$, $Q_j(\mathcal{M}')$, $1 \leqslant j \leqslant n'$ as in [Har-Lin16]. When the motive $\mathcal{M}$ is polarised, then $Q_i(\mathcal{M}) \sim < \omega_i, F_\infty \omega_i >$ where ω_i is a non-zero element in $(F^{p_i}\mathcal{M} + F^{p_{i-1}}\mathcal{M} \otimes_E \mathbb{C}) \cap I_\infty M^{p_i, w-p_i}$ and $<,>$ is the inner product given by the polarisation.

We also define $Q_0(\mathcal{M})$ (resp. $Q_0(\mathcal{M}')$) to be the determinant of the comparison isomorphism of $\mathcal{M}$ (resp. $\mathcal{M}'$) at the fixed embedding of K.

In the automorphic setting, the vector ω_i corresponds to a non-holomorphic automorphic form in general. In order to calculate L-values, we need to construct some periods which are related to holomorphic automorphic forms.

In fact, for each $0 \leqslant s \leqslant n$, $Q^{(s)}(\mathcal{M}) := \prod\limits_{i=0}^{s} Q_i(\mathcal{M})$ can be related to holomorphic forms. Similarly, we define $Q^{(t)}(\mathcal{M}') := \prod\limits_{j=0}^{t} Q_j(\mathcal{M})$ for each $0 \leqslant t \leqslant n'$.

Remark 2.2. Throughout the paper, the upper script $^{(*)}$ always indicates that a representation is holomorphic or a period is related to a holomorphic form.

Proposition 2.3 ([Har-Lin16])**.** *There exists an explicit monomial $F_{\mathcal{M},\mathcal{M}'}$, depending on the Hodge types of $\mathcal{M}$ and $\mathcal{M}'$, such that:*

$$c^+(Res_{K/\mathbb{Q}}\mathcal{M} \otimes \mathcal{M}') \sim F_{\mathcal{M},\mathcal{M}'}(Q^{(s)}(\mathcal{M}), 0 \leqslant s \leqslant n, Q^{(t)}(\mathcal{M}'), 0 \leqslant t \leqslant n').$$

3. Automorphic periods

We now consider motives coming from automorphic representations. c We want to define a motive attached to Π. Recall that in the classical case, modular forms can be lifted to cohomological classes on modular curves. In general, we hope to define a motive using cohomologies of Shimura varieties. Since there is no Shimura variety attached to GL_n when $n \geqslant 3$, we look at unitary groups instead.

Let U be any unitary group of rank n over $\mathbb{Q}$ with respect to the quadratic extension $K/\mathbb{Q}$. Over K we have $U_K \cong GL_{n,K}$. Hence Π can be viewed as a representation of $U(\mathbb{A}_K)$.

Let G be a reductive group over $\mathbb{Q}$. The theory of base change with respect to $K/\mathbb{Q}$ predicts a map from certain (packets of) automorphic representations of $G(\mathbb{A}_Q)$ to certain (packets of) automorphic representations of $G(\mathbb{A}_K)$. This map can be described more easily in the motivic setting. For example, if an automorphic representation of $G(\mathbb{A}_Q)$ is attached to a motive $H^*(X)$ where X is a projective smooth variety over $\mathbb{Q}$, then the automorphic representation of $G(\mathbb{A}_K)$ attached to $H^*(X \otimes_{\mathbb{Q}} K)$, whose existence is predicted by the Langlands correspondence, is the image of the $G(\mathbb{A}_Q)$-representation by base change.

One can also define base change in terms of Langlands parameters without referring to motives (c.f. [Art03]). The theory of base change for unitary groups is well-understood especially for cohomological representations (c.f. [Har-Lab04], [Lab11], [Shi14], [Mok14] and [KMSW14]).

Recall that Π is an automorphic representation of $U(\mathbb{A}_K)$. A necessary condition for Π to be in the image of base change is the conjugate self-duality. We henceforth assume that Π is conjugate self-dual. When n is even, we assume moreover that Π is a discrete series representation at a finite place of $\mathbb{Q}$ which is inert in K. Then there exists a unitary group $U(n-1,1)$ over $\mathbb{Q}$ of rank n with respect to the extension $K/\mathbb{Q}$ of signature $(n-1,1)$ at infinity such that Π is the base change of $\pi^{(1)}$, a holomorphic cohomological automorphic representation of $U(n-1,1)(\mathbb{A}_Q)$. We also say that Π descends to $\pi^{(1)}$ by base change.

The representation $\pi^{(1)}$ contributes in the interior coherent cohomologies of a Shimura variety attached to $U(n-1,1)$, and hence defines (the realizations of) a motive $\mathcal{M}(\Pi)$ over K attached to the representation Π (c.f. [Har97], [Lin15b] and [Gro-Har-Lin18]).

The representation Π descends to not only one representation on the unitary group. At the infinite place, Π_∞ descends to n inequivalent discrete series representations which can be distinguished by the parabolic sub-Lie algebra cohomological degree (c.f. [Har14]). In particular, Π descends to at least n automorphic representations of $U(n-1,1)(\mathbb{A}_Q)$, denoted by π_i for $1 \leqslant i \leqslant n$ where $i-1$ refers to the cohomological degree. One can take π_1 to be $\pi^{(1)}$, the holomorphic representation.

All the representations π_i, $1 \leqslant i \leqslant n$, contribute into the coherent cohomologies of the Shimura variety and inherit rational structures from the de Rham cohomologies, called the de Rham rational structures (c.f. [Har13a] and also [Clo90]). For each i, let $0 \neq \omega_i(\Pi)$ be a de Rham rational automorphic form in π_i. We define an automorphic period $Q_i(\Pi)$ as the Petersson norm of $\omega_i(\Pi)$ which does not depend on the choice of $\omega_i(\Pi)$ up to multiplication by elements in $\overline{\mathbb{Q}}^\times$. We have $Q_i(\mathcal{M}(\Pi)) \sim Q_i(\Pi)$ for all i.

We can define $Q_0(\Pi)$ as a CM period attached to the central character of Π (c.f. [Lin17]). As in the motivic setting, we let $Q^{(s)}(\Pi) := \prod_{i=0}^{n} Q_s(\Pi)$ be an automorphic period for $0 \leqslant s \leqslant n$. Hence $Q^{(s)}(\mathcal{M}(\Pi)) \sim Q^{(s)}(\Pi)$ for all s.

Let Π' be a cuspidal cohomological conjugate self-dual representation of $GL_{n'}(\mathbb{A}_K)$. Similarly, we assume that it is a discrete series representation at a finite inert place if n' is even. One can define a motive $\mathcal{M}(\Pi')$ as well as periods $Q^{(t)}(\Pi)$ for $0 \leqslant t \leqslant n'$ similarly.

4. The Deligne conjecture for automorphic motives

Combining Proposition 2.3 and the constructions in the previous section, the Deligne conjecture for $Res_{K/\mathbb{Q}}\mathcal{M}(\Pi) \otimes \mathcal{M}(\Pi')$ can be reformulated as follows:

Conjecture 4.1 (Deligne conjecture). *If $m \in \mathbb{Z}$ is critical for the motive $Res_{K/\mathbb{Q}}\mathcal{M}(\Pi) \otimes \mathcal{M}(\Pi')$, then*

$$L(m, Res_{K/\mathbb{Q}}\mathcal{M}(\Pi)\otimes\mathcal{M}(\Pi')) \sim (2\pi i)^{mnn'} F_{\mathcal{M}(\Pi),\mathcal{M}(\Pi')}(Q^{(s)}(\Pi), 0 \leqslant s \leqslant n, Q^{(t)}(\Pi'), 0 \leqslant t \leqslant n').$$

The motivic L-function $L(m, Res_{K/\mathbb{Q}}\mathcal{M}(\Pi) \otimes \mathcal{M}(\Pi'))$ is equal to the Rankin-Selberg L-function $L(m - \frac{n+n'-2}{2}, \Pi \times \Pi')$ (c.f. [Cog00]). It is easier to relate automorphic L-values to holomorphic automorphic forms than non-holomorphic ones. The periods $Q^{(s)}(\Pi)$, $1 \leqslant s \leqslant n$, are (expected to be) related to holomorphic automorphic forms not on $U(n-1,1)$, but on other unitary groups.

For each $1 \leqslant s \leqslant n$, there exists $U(n-s,s)$, a unitary group over $\mathbb{Q}$ with respect to $K/\mathbb{Q}$ of signature $(n-s,s)$ at infinity, such that Π descends to $\pi^{(s)}$, a holomorphic cohomological automorphic representation of $U(n-s,s)(\mathbb{A}_Q)$. Let $\omega^{(s)}(\Pi)$ be a de Rham rational holomorphic element in $\pi^{(s)}$. We define $P^{(s)}(\Pi)$ as the Petersson norm of $\omega^{(s)}(\Pi)$ which is well defined up to multiplication by elements in $\overline{\mathbb{Q}}^{\times}$.

The cohomologies of a Shimura variety attached to $U(n-s,s)$ also define a motive whose λ-adic realizations are isomorphic to those of $\Lambda^s\mathcal{M}(\Pi)$. This fact leads to the following prediction on automorphic periods:

Conjecture 4.2 (Factorization conjecture). *For each $0 \leqslant s \leqslant n$, we have*

$$P^{(s)}(\Pi) \sim Q^{(s)}(\Pi) = \prod_{i=0}^{s} Q_i(\Pi).$$

It is then natural to state the following variant of the Deligne conjecture:

Conjecture 4.3 (Automorphic Deligne conjecture). *If $m \in \mathbb{Z}$ is critical for the motive $Res_{K/\mathbb{Q}}\mathcal{M}(\Pi) \otimes \mathcal{M}(\Pi')$, then*

$$L(m, Res_{K/\mathbb{Q}}\mathcal{M}(\Pi)\otimes\mathcal{M}(\Pi')) \sim (2\pi i)^{mnn'} F_{\mathcal{M}(\Pi),\mathcal{M}(\Pi')}(P^{(s)}(\Pi), 0 \leqslant s \leqslant n, P^{(t)}(\Pi'), 0 \leqslant t \leqslant n').$$

It is clear that the above conjecture (Conjecture 4.3) and the factorization conjecture (Conjecture 4.2) together imply the Deligne conjecture for tensor products of automorphic motives (Conjecture 4.1).

5. Some known results

Theorem 5.1. *Conjecture 4.3 is true if either* (Π, Π') *is in good position (c.f. [Lin15b]), or if* $n \equiv n' \mod 2$ *and* $m = \frac{n+n'}{2}$.

This theorem is proved in [Lin15b] based on previous results in [Bla86], [Har97] and [Gro-Har15]. In the case where $n' = 1$ or $|n - n'| = 1$, the L-values have good integral interpretations and can be calculated as parings between cohomological classes. The other cases are proved based on these two cases and some auxiliary constructions.

Theorem 5.2 ([Gro-Har-Lin18])**.** *We assume the following hypotheses:*

(a) *The Ichino-Ikeda-N. Harris conjecture (c.f. [NHar14]) is true.*
(b) *Certain archimedean zeta integral is algebraic.*
(c) *For any integer* a*, there exists a Hecke character* χ *of infinity type* $z^a\overline{z}^{-a}$*, such that* $L(\frac{1}{2}, \Pi \otimes \chi) \neq 0$.

Then Conjecture 4.2 is true when Π *is very regular. In particular, the Deligne conjecture is true in the setting of Theorem 5.1.*

Remark 5.3. (1) The Ichino-Ikeda-N. Harris conjecture is known in many cases if Π is supercuspidal at one finite split place (c.f. [Zhang14], [Xue17], [Beu-Ple15] and [He17]). The general case should be a corollary to a work in progress by Chaudouard and Zydor.

(2) The second hypothesis is natural because its failure would contradict a conjecture of Tate for motives. It is also known to be true in the few cases where it can be checked. Methods are known for computing these integrals but they are not simple.

(3) The last hypothesis on non-vanishing of central value is expected to be true but seems very difficult to prove. In the last year, however, there has been significant progress in the cases $n = 3$ and $n = 4$, by two very different methods [Jia-Zha17, Blo-Li-Mil17].

We will prove another case of Theorem 5.1 in the future subsequent part of [Gro-Har-Lin18].

Theorem 5.4 (ongoing work of Grobner-Harris-Lin)**.** *Conjecture 4.3 is true if* $n \not\equiv n' \mod 2$ *and* $m = \frac{n+n'-1}{2}$ *under hypothesis (a) and hypothesis (c) above.*

Remark 5.5. The critical points for $Res_{K/\mathbb{Q}}\mathcal{M}(\Pi) \otimes \mathcal{M}(\Pi')$ form an interval of integers. The ratios of two successive critical values of a Rankin-Selberg L-function for $\mathrm{GL}(n) \times \mathrm{GL}(n')$ have been studied by Harder and Raghuram in [Har-Rag17] over totally real fields. If one can generalize their result to CM fields, then the general case of Conjecture 4.3 will follow from Theorem 5.1 and Theorem.5.4.

Remark 5.6. At the end, we explain the unnecessary conditions which can be removed in this synopsis.

(1) All the relations $\sim$ can be made over some number fields (not only over $\overline{\mathbb{Q}}$).

(2) The quadratic field K can be replaced by any CM field as in [Har-Lin16], [Lin15b], [Gue16] and [Gue-Lin16]. The key step is that the periods factorize as products of local periods over infinite places (c.f. [Lin17]). Hence one can reduce the general CM case to the quadratic imaginary case easily.

(3) One does not need to fix an embedding of E in $\mathbb{C}$. By varying the embeddings, we consider L-values and periods as families of complex numbers parametrized by $Aut(\mathbb{C})$. In fact, the Deligne conjecture was formulated in a $Aut(\mathbb{C})$-equivariant way, and all our results were also proved $Aut(\mathbb{C})$-equivariantly.

(4) The cuspidal condition on Π is also not necessary. Some results are already known for endoscopic representations. Assuming Hypothesis (a), one should be able to remove the cuspidal condition in most of the results above.

Acknowledgements. I would like to thank the organizers for the invitation. I also want to thank the other speakers for their wonderful talks. Finally, I am grateful to Simons Foundation and Meghan Fazzi for her help.

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