SPECIAL VALUES OF RANKIN-SELBERG L-FUNCTIONS OVER CM FIELDS

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Abstract. In this paper, we prove an automorphic variant of the Deligne conjecture on critical values for Rankin-Selberg L-functions over CM fields. This automorphic variant is known to be equivalent to the original conjecture for automorphic motives under certain hypotheses.

Contents

Introduction

Deligne's conjecture is one of the most important and beautiful conjectures in the algebraic theory of L-values. It generalizes the well-known fact that the special values of the Riemann zeta function satisfy that $\zeta(2n) \sim (2\pi i)^{2n}$ for all positive integer n where two complex numbers x and y are equivalent, written as $x \sim y$, if $y = 0$, or if $y \neq 0$ and x/y is an algebraic

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number.

More precisely, let $\mathcal{M}$ be a motive over $\mathbb{Q}$ with coefficients in a number field (c.f. [Del79]). If a condition on the Hodge type of $\mathcal{M}$ is satisfied, Deligne constructed two complex numbers $c^+(\mathcal{M})$ et $c^-(\mathcal{M})$, defined as determinants of certain comparison isomorphisms between the de Rham realizations and the Betti realizations. He also defined two explicit integers n^+ and n^- as dimensions of certain cohomological spaces.

Conjecture 0.1. ***(of Deligne)*** *If an integer m is critical for $\mathcal{M}$, i.e., if both $L_\infty(s,\mathcal{M})$ and $L_\infty(1-s,\check{\mathcal{M}})$ are holomorphic at $s=m$, then*

$$L(m,\mathcal{M}) \sim (2\pi i)^{mn^\epsilon} c^\epsilon(\mathcal{M}) \tag{0.1}$$

where ϵ is the sign of $(-1)^m$.

The aim of this paper is to prove an automorphic variant of Deligne's conjecture [Del79] on critical L-values. In [HGL21], we proved that this automorphic variant is equivalent to Deligne's conjecture for certain automorphic motives under certain hypotheses (c.f. Conjecture 2.10 and Conjecture 4.15 of the *loc. cit*). Consequently, we can deduce Deligne's conjecture in some cases from our results under these hypotheses.

Let K be a CM field and M be a motive over K. We would like to study the Deligne conjecture for $Res_{K/\mathbb{Q}}M$, a motive over $\mathbb{Q}$. In [Lin17] and [HL17], we defined the arithmetic automorphic Q-periods for motives over CM fields and calculated Deligne's periods $c^\pm(Res_{K/\mathbb{Q}}M)$ in terms of these Q-periods. When the motive M is attached to a pair of automorphic representations, there $\mathbb{Q}$-periods are expected to be related to the arithmetic automorphic P-periods defined in [Har97], [Lin15b], [GL16] and [HGL21]. Hence we may formulate an automorphic variant of the Deligne conjecture by replacing the motivic Q-periods in terms of the automorphic P-periods.

More precisely, we have:

Conjecture 0.2. ***(automorphic variant of the Deligne conjecture)*** *Let Π (resp. Π') be a cohomological conjugate self-dual cuspidal automorphic representation of $G_n(\mathbb{A}_K)$ (resp. $G_{n'}(\mathbb{A}_K)$). Let $s_0 \in \mathbb{Z}+\frac{n+n'}{2}$ be a critical point of $L(s,\Pi\times\Pi')$, and let S be a fixed finite set of places of F, containing all infinite places. Then the critical value*

$$L^S(s_0,\Pi\otimes\Pi') \sim_{E(\Pi)E(\Pi')} (2\pi i)^{nn's_0} \prod_{\imath\in\Sigma}[\prod_{0\leqslant i\leqslant n} P^{(i)}(\Pi,\imath)^{sp(i,\Pi;\Pi',\imath)} \prod_{0\leqslant j\leqslant n'} P^{(j)}(\Pi',\imath)^{sp(j,\Pi';\Pi,\imath)}]. \tag{0.2}$$

Besides the basic case where $n'=1$ proved in [Har97] and [GL16], this automorphic variant is proved when $n\neq n'(mod 2)$ under certain hypotheses and relatively stronger regularity condition in [HGL21]. In this paper, we prove the other half of this automorphic variant where $n\equiv n'(mod 2)$ without these hypotheses. We also weaken significantly the regularity condition. If the pair (Π,Π') satisfies an interlacing property, called *piano* in Definition

1.6), we can remove the regularity condition completely.

In the contest, we first treat the piano case, and then the general case. The idea is to construct auxiliary representations and calculate critical values for representations of larger groups. The main ideas are the same for the two cases, but we need different constructions.

There are three ingredients in the proof:

(1) Conjecture 0.2 for the case $n' = 1$ proved in [Har97] and [GL16];
(2) Theorem 5.2 of [GL21] which says that if Π is a cuspidal representation of $GL_N(\mathbb{A}_K)$ and $\Pi^\flat$ is an Eisenstein representation of $GL_{N-1}(\mathbb{A}_K)$ such that $(\Pi, \Pi^\flat)$ is piano, then the critical values for $\Pi \times \Pi^\flat$ is related to the Whittaker periods, defined as ratios of certain algebraic models;
(3) An calculation of the Whittaker period of an Eisenstein representation which involves the near central L-values (Theorem 2.5 of [GL21]).

We now sketch the main idea of the proof for the two cases.

In the piano case, necessarily we have $n > n'$. We extend Π' to an Eisenstein representation $\Pi^\flat$ of GL_{n-1} by adding some Hecke characters. The critical values for $\Pi \times \Pi'$ is then a ratio of critical values of $\Pi \times \Pi^\flat$ and critical values of Π twisted by Hecke characters. The former is known by the second ingredient, and the latter is known by the first ingredient.

In the general case, we consider the Eisenstein representation $\Pi^\# := \Pi \bigoplus \Pi'^c$. The near central value of $\Pi \otimes \Pi'$ is critical and appears in the Whittaker period of $\Pi^\flat$ by the last ingredient. We then construct a cuspidal representation of $GL_{n+n'+1}$, called $\Pi^\#$, by automorphic induction of a Hecke character over a large CM field, such that $(\Pi^\#, \Pi^\flat)$ is piano (we need a mild regularity condition here). The Whittaker period of $\Pi^\flat$ is then related to the critical values of $\Pi^\# \times \Pi^\flat$ by the second ingredient. Note that the representation $\Pi^\#$ is constructed from a Hecke character. The L-function for $\Pi^\# \times \Pi^\flat$ is then decomposed as L-functions of Π and Π' twisted by characters which are again known by the first ingredient. We can finally pass from the near central value to the other critical values thanks to a result of Raghuram ([Rag21]) on the ratio of adjacent critical L-values.

We finally remark that all the calculations are local and can be done at each individual place. So in the contest, we will only consider the case where K is quadratic imaginary.

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1. Preliminaries

1.1. **Basic notation.** We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. Let K be a quadratic imaginary field. We fix an embedding of K in $\overline{\mathbb{Q}}$ and hence consider K as a subfield of $\overline{\mathbb{Q}}$. We denote by ι the complex conjugation of the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}$.

Let S be a finite set of places of K which contains the infinite place and all the ramified places for any date appeared in this paper.

We denote by c the complex conjugation on $\mathbb{C}$. Via the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, it can be considered as an element in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Throughout the text, let ϕ be an algebraic Hecke character of K with infinity type $z^1\overline{z}^0$ such that $\phi\phi^c = ||\cdot||_{\mathbb{A}_K}$ (see Lemma 4.1.4 of [CHT08] for its existence). It is easy to see that the restriction of $||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi$ to $\mathbb{A}_{\mathbb{Q}}^{\times}$ is the quadratic character associated to the extension $K/\mathbb{Q}$ by the class field theory. In particular, the Hecke character $||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi$ is similar to the Hecke character η constructed in section 2.1.1 of [GH15].

Let χ be a Hecke character of K. We define $\check{\chi} := \chi^{-1,c}$ and $\widetilde{\chi} := \chi/\chi^c$ two Hecke characters of K.

Let n and n' be two positive integers. We may assume that $n \geqslant n'$.

Let Π be an automorphic representation of $GL_n(\mathbb{A}_K)$. We denote the infinity type of Π by $\{z^{a_i}\overline{z}^{a'_i}\}_{1\leqslant i\leqslant n}$. We may assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. We denote by $\Pi^\vee$ the contragredient representation of Π.

Definition 1.1. *Let N be an integer. Let G_∞ be the group of real points of $Res_{K/\mathbb{Q}}GL_n$. The representation Π will be called:*

(1) ***pure of weight*** $\omega(\Pi)$ *if $a_i + a'_i = -\omega(\Pi)$ for all $1 \leqslant i \leqslant n$;*
(2) ***algebraic*** *if $a_i, a'_i \in \mathbb{Z} + \frac{n-1}{2}$ for all $1 \leqslant i \leqslant n$;*
(3) ***cohomological*** *if there exists W an irreducible algebraic finite dimensional representation of $GL_n(F\otimes_{\mathbb{Q}}\mathbb{R})$ such that $H^*(\mathfrak{g}_\infty, K_\infty; \Pi\otimes W) \neq 0$ where $\mathfrak{g}_\infty = Lie(G_\infty)$ and K_∞ is the product of a maximal compact group of G_∞ and the center of G_∞.*
(4) ***regular*** *if it is pure and $a_i - a_{i+1} \geqslant 1$ for all $1 \leqslant i \leqslant n-1$.*
(5) N***-regular*** *if it is pure and $a_i - a_{i+1} \geqslant N$ for all $1 \leqslant i \leqslant n-1$.*

We recall a definition from [GL21] (c.f. Definition 1.16 of *loc.cit*).

Definition 1.2. *Let E be a number field. Let $x = \{x(\sigma)\}_{\sigma\in \mathrm{Aut}(\mathbb{C})}$ and $y = \{y(\sigma)\}_{\sigma\in \mathrm{Aut}(\mathbb{C})}$ be two families of complex numbers. We say $x \sim_E y$ (and this relation) is equivariant under $Aut(\mathbb{C}/K)$, if either $y(\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, or if $y(\sigma) \neq 0$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ and the following two conditions are verified:*

(1) $\dfrac{x(\sigma)}{y(\sigma)} \in \sigma(E)$ *for all* $\sigma \in \mathrm{Aut}(\mathbb{C})$*;*
(2) $\sigma\left(\dfrac{x(\tau)}{y(\tau)}\right) = \dfrac{x(\sigma\tau)}{y(\sigma\tau)}$ *for all $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and all $\tau \in \mathrm{Aut}(\mathbb{C})$.*

It is easy to see that this relation is symmetric but not transitive unless some non-zero condition is guaranteed.

Remark 1.1. *We assume that $E \supset K$ and the families $y(\sigma) \neq 0$ for all σ in the previous definition. We assume moreover that the numbers $x(\sigma)$ and $y(\sigma)$ depend only on the restriction of σ to E.*

(1) Then the second condition of Definition 1.2 implies the first one. We refer to Lemma 1.18 *of* [GL21] *for the proof.*

(2) Let J_E be the set of complex embeddings of E in $\mathbb{C}$. We identify the two algebras $E\otimes_{\mathbb{Q}}\mathbb{C}$ and $\mathbb{C}^{J_E}$ by sending $e\otimes z$ to $(\rho(e)z)_{\rho\in J_E}$ where $e\in E$ and $z\in\mathbb{C}$. By the assumption, we may define $x(\rho)$ for $\rho\in J_E$ as $x(\sigma)$ where σ is any lifting of ρ in $\mathrm{Aut}\mathbb{C}$. We may consider $x=(x(\rho))_{\rho\in J_E}$ as an element in $E\otimes_{\mathbb{Q}}\mathbb{C}$. Similarly, we may consider y as an inversible element in $E\otimes_{\mathbb{Q}}\mathbb{C}$. In this sense, it is easy to see that $x\sim_E y$ (and this relation) is equivariant under $Aut(\mathbb{C}/K)$ if and only if $xy^{-1}\in E\otimes_{\mathbb{Q}}K\subset E\otimes_{\mathbb{Q}}\mathbb{C}$. In particular, our definition is compatible with that in page 2 of [Lin17].

1.2. **Split index and piano-condition for automorphic pairs.** Throughout the paper, let Π be a cuspidal cohomological representation of $GL_n(\mathbb{A}_K)$ with infinity type $\{z^{a_i}\overline{z}^{a'_i}\}_{1\leqslant i\leqslant n}$ such that $a_1>a_2>\cdots>a_n$. In particular, Π is algebraic and hence $a_i,a'_i\in\mathbb{Z}+\frac{n-1}{2}$. Moreover, Lemma 4.9 of [Clo90] implies that Π is pure. In other words, there exists an integer $\omega(\Pi)$ such that $a_i+a'_i=-\omega(\Pi)$ for all i. We say Π is **very regular** if $a_i-a_{i+1}\geqslant 3$ for all i.

Similarly, let Π' be a cuspidal cohomological representation of $GL_{n'}(\mathbb{A}_K)$ with infinity type $\{z^{b_j}\overline{z}^{b'_j}\}_{1\leqslant j\leqslant n}$ such that $b_1>b_2>\cdots>b_{n'}$. We have $b_j,\ b'_j\in\mathbb{Z}+\frac{n'-1}{2}$ for all j, and there exists an integer $\omega(\Pi')$ such that $b_j+b'_j=\omega(\Pi')$ for all j.

We assume that

$$\text{for all } 1\leqslant i\leqslant n, 1\leqslant j\leqslant n', a_i+b_j\neq-\frac{\omega(\Pi)+\omega(\Pi')}{2} \tag{1.1}$$

Definition 1.3. ***(Split Index)***

We split the sequence $(a_1>a_2>\cdots>a_n)$ with the numbers

$$-\frac{\omega(\Pi)+\omega(\Pi')}{2}-b_{n'}>-\frac{\omega(\Pi)+\omega(\Pi')}{2}-b_{n'-1}>\cdots>-\frac{\omega(\Pi)+\omega(\Pi')}{2}-b_1.$$

The sequence is split into $n'+1$ parts. We denote the length of each part by

$$sp(0,\Pi';\Pi),sp(1,\Pi';\Pi),\cdots,sp(n',\Pi';\Pi),$$

*and call them the **split indices**.*

Remark 1.2. *It is easy to see that $sp(i,\Pi;\Pi',v)$ is equal to the cardinal of the set $\{1\leqslant j\leqslant n'\mid-\frac{\omega(\Pi)+\omega(\Pi')}{2}-a_{n+1-i}>b_j>-\frac{\omega(\Pi)+\omega(\Pi')}{2}-a_{n-i}\}$. Here we put $a_{v,0}=+\infty$ and $a_{v,n+1}=-\infty$.*

Lemma 1.1. *Let η be an algebraic Hecke character. The split indices have the following properties:*

(1) $\sum\limits_{i=0}^{n'}sp(i,\Pi';\Pi)=n$.

(2) For any $0\leqslant j\leqslant n'$, $sp(j,\Pi';\Pi)=sp(n'-j,\Pi'^c;\Pi^c)=sp(n'-j,\Pi'^\vee;\Pi^\vee)$.

(3) For any $t,s\in\mathbb{Z}$, $sp(j,\Pi'\otimes||\cdot||^t_{\mathbb{A}_K};\Pi)=sp(j,\Pi';\Pi\otimes||\cdot||^s_{\mathbb{A}_K})=sp(j,\Pi';\Pi)$.

(4) *For any* $0 \leqslant j \leqslant n'$, $sp(j, \Pi' \otimes \eta; \Pi) = sp(j, \Pi'; \Pi \otimes \eta)$ *and* $sp(j, \Pi' \otimes \eta^c; \Pi) = sp(j, \Pi' \otimes \eta^{-1}; \Pi)$. *Similarly,* $sp(j, \Pi'; \Pi \otimes \eta^c) = sp(j, \Pi'; \Pi \otimes \eta^{-1})$.

The first two points of the above lemma are direct. For the remaining, we only need to notice that calculating the split index is nothing but comparing $a_i + b_j$ with $-\frac{\omega(\Pi)+\omega(\Pi')}{2}$.

Definition 1.4. *If* $n > n'$, *we say that the pair* (Π, Π') *is* ***piano*** *if the following equivalent conditions are verified:*

(1) *The* n' *numbers*

$$-\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_{n'} > -\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_{n'-1} > \cdots > -\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_1.$$

lie in different gaps between the n *numbers* $a_1 > a_2 > \cdots > a_n$.

(2) *For all* $0 \leqslant j \leqslant n'$, *we have* $sp(j, \Pi'; \Pi) \neq 0$.

(3) *For all* $0 \leqslant i \leqslant n$, *we have* $sp(i, \Pi; \Pi') \leqslant 1$ *and moreover* $sp(0, \Pi; \Pi') = sp(n, \Pi; \Pi') = 1$.

Remark 1.3. *The condition "piano" was called "good position" in my thesis. M. Harris first used the name "piano" in a talk at CIRM in spring 2016 since the numbers* $\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_j$, $1 \leqslant j \leqslant n'$ *insert into the gaps between the numbers* a_i, $1 \leqslant i \leqslant n$, *like the black keys insert into the gaps between the white keys of a piano.*

Definition 1.5. *Let* $m \in \mathbb{Z} + \frac{n+n'-2}{2}$. *We say* m *is* ***critical*** *for the pair* (Π, Π') *if*

(1.2)

$$1 - \min_{i,j}\{|a_i + b_j + \frac{\omega(\Pi)+\omega(\Pi')}{2}|\} \leqslant m - \frac{\omega(\Pi)+\omega(\Pi')}{2} \leqslant \min_{i,j}\{|a_i + b_j + \frac{\omega(\Pi)+\omega(\Pi')}{2}|\}.$$

This definition is compatible with section 1.7.1 of [GL21], also see equations (2.33) and (2.34) of [Rag16] .

Remark 1.4. *For* $1 \leqslant i \leqslant n$, *if there exists an number* $-\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_j$ *between* a_i *and* a_{i+1}, *then we have* $a_i - a_{i+1} \geqslant 2m - \omega(\Pi) - \omega(\Pi')$ *if* $m > \frac{\omega(\Pi)+\omega(\Pi')}{2}$, *and* $a_i - a_{i+1} \geqslant 2 - 2m + \omega(\Pi) + \omega(\Pi')$ *if* $m \leqslant \frac{\omega(\Pi)+\omega(\Pi')}{2}$.

The following hypothesis will be made for our main theorem in the piano case:

Hypothesis 1.1. *(regular hypothesis) If* $m > \frac{\omega(\Pi)+\omega(\Pi')}{2}$ *then* Π *is* $2m - \omega(\Pi) - \omega(\Pi')$ *regular; If* $m \leqslant \frac{\omega(\Pi)+\omega(\Pi')}{2}$ *then* Π *is* $1 - 2m + \omega(\Pi) + \omega(\Pi')$-*regular.*

1.3. **Rationality field and automorphic periods.** In this section, we shall introduce three types of automorphic periods. We will see that they are all related to special values of automorphic L-functions.

Recall that Π_f is an admissible irreducible representation of $GL_n(\mathbb{A}_{K,f})$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define ${}^\sigma\Pi_f$ as $\Pi_f \otimes_{\mathbb{C},\sigma^{-1}} \mathbb{C}$. It is the finite part of an algebraic cuspidal representation ${}^\sigma\Pi$ of $GL_n(\mathbb{A}_K)$ (c.f. Théorème 3.13 of [Clo90]). We define $\mathbb{Q}(\Pi_f)$, the **rationality field** of Π_f, by the subfield of $\mathbb{C}$ fixed by the automorphisms σ such that ${}^\sigma\Pi_f \cong \Pi_f$.

Let V be a representation over $\mathbb{C}$ of a general group G. Let E be a subfield of $\mathbb{C}$. We say that an E-vector space $V_E \subset V$ is an **E-rational structure** of V if V_E is stable by G and $V_E \otimes_E \mathbb{C} \cong V$ as G-module.

1.3.1. *Whittaker periods.* Since Π is cohomological and hence algebraic and regular, Théorème 3.13 of [Clo90] implies that $\mathbb{Q}(\Pi_f)$ is a number field and moreover, Π_f has a $\mathbb{Q}(\Pi_f)$-rational structure given by the Lie algebra cohomology.

On the other hand, since Π is cuspidal and hence generic, its Whittaker period also provides a $\mathbb{Q}(\Pi_f)$-rational structure. The two rational structures differ by a non-zero ratio $p(\Pi)$, called the **the Whittaker period**. It is well defined up to multiplication by a number in $\mathbb{Q}(\Pi_f)^\times$. We refer to Definition/Proposition 3.3 of [RS08] for detailed definition when the representation is cuspidal.

The definition has been generalized to a large family of non-cuspidal representations: fully-induced isobaric sums of distinct unitary cuspidal representations (c.f. Proposition 3.1 of [GH15] and Corollary 1.12 of [GL21]).

Proposition 1.1. *(Proposition 1.4, Lemma 1.6, Lemma 1.8.3 of [Har93]) Let χ_1, χ_2 be two algebraic Hecke characters of K. We have:*

$$\begin{array}{rcl} p(\chi_1\chi_2, 1) & \sim_{E(\chi_1)E(\chi_2)} & p(\chi_1, 1)p(\chi_2, 1) \\ p(\chi_1, \iota) & \sim_{E(\chi_1)} & p(\chi_1^c, 1) \\ p(||\cdot||, 1) & \sim_K & (2\pi i)^{-1} \end{array}$$

and are all equivariant under the action of $Aut(\mathbb{C}/K)$.

1.3.2. *CM periods.* Let χ be an algebraic Hecke character of K. We may associate two non-zero complex numbers $p(\chi, 1)$ and $p(\chi, \iota)$ where 1 indicates the fixed embedding $K \hookrightarrow \mathbb{C}$. They are defined as ratios of different rational structures for cohomology spaces associated to CM Shimura varieties and are well-defined up to multiplication by elements in a number field $E(\chi)$. We refer to appendix of [HK91] and section 1 of [Har93] for more details.

1.3.3. *Arithmetic automorphic periods.* For each integer $0 \leqslant s \leqslant n$, let U_s be a unitary group over $\mathbb{Q}$ with respect to the quadratic extension $K/\mathbb{Q}$ of infinity sign $(n-s, s)$. If n is odd, or n is even and $s \equiv n/2 \mod 2$, we may assume that U_s is quasi-split at each finite places. In this case, if Π is **conjugate self-dual**, then Π descends to a cohomological representation of $U_s(\mathbb{A}_\mathbb{Q})$ by base change.

If n is even and $s \not\equiv n/2 \mod 2$, we may assume that the unitary group U_s is quasi-split outside one finite space which is split in K. We assume moreover that Π is a discrete series representation at this split place and hence descends to a cohomological representation of $U_s(\mathbb{A}_\mathbb{Q})$ by base change. For base change of cohomological representations, we refer to [Clo91], [HL04] and [KMSW14] for more details.

We can then define an **arithmetic automorphic period** $P^{(s)}(\Pi)$ as in equation (4.28) of [GH15]. It is defined as Petersson inner product of a holomorphic automorphic form related to the descending of Π to U_s. We also refer to [Har97] for original constructions.

We say Π is essentially conjugate self-dual if there exists an algebraic Hecke character χ such that $\Pi\otimes\chi$ is conjugate self-dual. We assume that $\Pi\otimes\chi$ is a discrete series representation at a split place if n is even. We can then define the generalized **arithmetic automorphic period** $P^{s}(\Pi) := P^{(s)}(\Pi\otimes\chi)p(\breve{\chi},1)^{-s}p(\breve{\chi},\iota)^{-n+s}$. In this case (where Π is cuspidal, cohomological, essentially conjugate self-dual and discrete series at a split place), we say that Π has **definable arithmetic automorphic periods**.

The arithmetic automorphic periods are well defined up to multiplication by elements in a number field $E(\Pi)$ and does not depend on the choice of χ by Theorem 1.1. For simplicity, we assume that $E(\Pi)$ and $E(\chi)$ contain K.

We remark that if Π is conjugate self-dual, one can show from construction of automorphic periods that $P^{(0)}(\Pi)P^{(n)}(\Pi)\sim_{E(\Pi)} 1$ and is equivariant under the action of $Aut(\mathbb{C}/K)$. We refer to Lemma 5.2.1 of [Lin15b] for more details.

1.4. Special values of automorphic L-functions: theorems revisited.

We emphasize that whenever we talk about periods or L-values in the following, we consider them as $Aut(\mathbb{C})$-families. For example, when we talk about $L^{S}(s,\Pi)$, we consider it as the family $(L^{S}(s,{}^{\sigma}\Pi))_{\sigma\in Aut(\mathbb{C})}$.

All the periods in the previous section can be chosen in a $Aut(\mathbb{C})$-equivariant way such that the following theorems hold.

Theorem 1.1. *(Theorem 3.5.13 of [Har97] or Theorem 4.29 of [GH15]) Let Π be a conjugate self-dual of $GL_{n}(\mathbb{A}_{K})$ which has definable arithmetic automorphic periods. We write its infinity type as $\{z^{a_i}\overline{z}^{-a_i}\}_{1\leqslant i\leqslant n}$ where $a_1>a_2>\cdots>a_n$.*

Let χ be an algebraic Hecke character of K with infinity type $z^{a}\bar{z}^{b}$ such that $\chi*\chi^{c}=||\cdot||^{a+b}$. We assume that $a-b+2a_i\neq 0$ for any i. Write s for the integer $s=s(\Pi,\eta):=\#\{i\mid a-b+2a_i<0\}$.

If $m\in\mathbb{Z}+\frac{n-1}{2}$ is critical for the pair (Π,χ), then

$$L^{S}(m,\Pi\otimes\chi)\sim_{E(\Pi)E(\chi)}(2\pi i)^{mn}P^{(s)}(\Pi)p(\breve{\chi},1)^{s}p(\breve{\chi},\iota)^{n-s}$$

and is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/K)$.

Remark 1.5.
(1) We didn't state the CM periods as in Theorem 3.5.13 of [Har97]. Instead, the current form appears in middle steps of the proof for Theorem 3.5.13 (c.f. equation (2.9.12) or the third line in page 138 of the loc.cit). We refer to Corollary 4.1 of [Lin15a] for detailed calculation.

(2) In the original theorem, there is a term on the Gauss sum of the quadratic character associated to $K/\mathbb{Q}$ by the class field theory which is an element in K. We can remove it since we have assumed $E(\Pi)$ and $E(\chi)$ contain K.

(3) The original theorem was stated for the critical values larger or equal to $\frac{1-a-b}{2}$. We add the condition $\chi*\chi^{c}=||\cdot||^{a+b}$ and then the other critical values can be deduced easily from the functional equation. More precisely, let $m<\frac{1-a-b}{2}$ be a critical value, then by the functional equation, we get (c.f. [Cog])

$$L(m,\Pi\otimes\chi)\sim_{E(\Pi)E(\chi)}(2\pi i)^{2m+a+b-1}L(1-m,\Pi^{\vee}\otimes\chi^{-1}).$$

Since Π *is conjugate self-dual, we have*

$$\begin{aligned} L(1-m,\Pi^\vee\otimes\chi^{-1}) &= L(1-m,\Pi^c\otimes\chi^{-1}) = L(1-a-b-m,\Pi\otimes\chi^{-1,c}) \\ &= L(1-m,\Pi\otimes\chi||\cdot||^{-a-b}) = L(1-a-b-m,\Pi\otimes\chi). \end{aligned}$$

Then the theorem for $m<\frac{1-a-b}{2}$ *follows from that for* $1-a-b-m\geqslant\frac{1-a-b}{2}$.

Theorem 1.2. *(Theorem* 1.9 *of* [Gro17], *Theorem* 5.2 *of* [GL21]*) Let* Π *be a conjugate self-dual cohomological cuspidal representation of* $GL_n(\mathbb{A}_K)$. *Let* $\Pi^\#$ *be a conjugate self-dual cohomological automorphic representation of* $GL_{n-1}(\mathbb{A}_K)$ *which is fully induced from different conjugate self-dual cuspidal representations. We assume that the pair* $(\Pi,\Pi^\#)$ *is piano. Then if* $m\in\mathbb{Z}$ *such that* $m+\frac{1}{2}$ *is critical for* $(\Pi,\Pi^\#)$, *then there exists an archimedean factor* $p(m,\Pi_\infty,\Pi^\#_\infty)\in\mathbb{C}^\times$ *such that*

$$L^S(m+\frac{1}{2},\Pi\times\Pi^\#)\sim_{E(\Pi)E(\Pi^\#)} p(m,\Pi_\infty,\Pi^\#_\infty)p(\Pi)p(\Pi^\#).$$

Moreover, if $m\neq 0$, *or if* $m=0$ *and certain central values do not vanish, then*

$$p(m,\Pi_\infty,\Pi^\#_\infty)\sim(2\pi i)^{mn(n-1)-\frac{1}{2}(n-1)(n-2)}.$$

All the relations above are equivariant under the action of $\mathrm{Aut}(\mathbb{C}/K)$.

Remark 1.6.
1. *Since* $\Pi^\#$ *is conjugate self-dual, we can remove the Gauss sum in Theorem* 1.9 *of* [Gro17] *by Remark 1.31 of* [GL21].
2. *The piano condition is defined by highest weight in* [GL21]. *By the relation of the highest weight and the infinity type given in section* 1.4.1 *of the loc.cit, one can show easily that our definition here is equivalent to that in* [GL21].

Theorem 1.3. *(Theorem* 6.7 *of* [GH15]*) If* Π *is* 3*-regular, conjugate self-dual and has definable arithmetic automorphic period, then there exists a non-zero complex number* $Z(\Pi_\infty)$ *such that*

$$p(\Pi)\sim_{E(\Pi)} Z(\Pi_\infty)\prod_{i=1}^{n-1}P^{(i)}(\Pi)$$

and is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/K)$.

Theorem 1.4. *(Corollary* 5.7 *of* [GH15], *Corollary* 2.12 *of* [GL21]*) Let* $\Pi^\#=\Pi_1\boxplus\Pi_2\boxplus\cdots\Pi_k$ *be the fully induced isobaric sum of different conjugate self-dual cuspidal representation* Π_i *of* $GL_{n_i}(\mathbb{A}_K)$. *For each* i, *if* $n\equiv\sum_{i=1}^{k}n_i \bmod 2$ *then let* $\Pi_i^{alg}:=\Pi_i$; *otherwise let* $\Pi_i^{alg}:=\Pi_i\otimes\phi||\cdot||^{-1/2}$. *We assume that* Π_i^{alg} *is cohomological for each* i. *This implies that* $\Pi^\#$ *is also cohomological. We have:*

$$p(\Pi^\#)\sim_{E(\Pi^\#)E(\phi)}\prod_{1\leqslant i\leqslant k}p(\Pi_i^{alg})\prod_{1\leqslant i<j\leqslant k}L^S(1,\Pi_i\times\Pi_j^\vee)$$

and is equivariant under the action of $\mathrm{Aut}(\mathbb{C}/K)$.

Theorem 1.5. *(of Balsius, stated as Prop. 1.8.1 of* [Har93] *(and the attached erratum* [Har97]*, p. 82)) Let χ be an algebraic Hecke character of K with infinity type $z^a\overline{z}^b$ such that $a \neq b$. Let $m \in \mathbb{Z}$ be a critical point for χ. If $a < b$, then $L(m,\chi) \sim (2\pi i)^m p(\check{\chi},1)$. If $a > b$, then $L(m,\chi) \sim (2\pi i)^m p(\check{\chi},\iota)$.*

In both cases the relation is equivalent under the action of $Aut(\mathbb{C}/K)$.

1.5. **An automorphic version of a conjecture of Deligne.** We now state the conjecture and claim our results.

Conjecture 1.1. *(c.f. Conjecture 2.2 of* [Lin17]*) Let Π and Π' be cuspidal representations of $GL_n(\mathbb{A}_K)$ and $GL_{n'}(\mathbb{A}_K)$ respectively which have definable arithmetic automorphic periods. Let $m \in \mathbb{Z} + \frac{n+n'-2}{2}$ be critical for $\Pi \otimes \Pi'$. We assume that their infinity types satisfy equation (1.1). We predict that:*

$$L^S(m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi')} (2\pi i)^{nn'm} \prod_{u=0}^{n} P^{(u)}(\Pi)^{sp(u,\Pi;\Pi')} \prod_{v=0}^{n'} P^{(v)}(\Pi')^{sp(v,\Pi';\Pi)}.$$

and is equivariant under the action of $Gal(\overline{\mathbb{Q}}/K)$.

Remark 1.7. *If $n' = 1$, the above conjecture is known by Theorem 1.1.*

We shall prove the conjecture in the following two cases:

Theorem 1.6. *(piano case)*

Let $n > n'$ and the pair (Π,Π') be piano (see Definition 1.4). If $n \not\equiv n' \mod 2$, we assume that Π, Π' are conjugate self-dual. Otherwise we assume that Π and $\Pi' \otimes \phi^{-1}$ are conjugate self-dual, then Conjecture 1.1 is true.

Theorem 1.7. *(general case)*

Let (Π,Π') be very regular.

If $n \equiv n' \mod 2$, we assume that Π_1 and Π_2 are conjugate self-dual. In this case, the integer 1 is critical and Conjecture 1.1 is true for any critical values.

If $n \not\equiv n' \mod 2$, we assume that Π_1 and $\Pi_2 \otimes \phi^{-1}$ are conjugate self-dual. In this case, the half integer $\frac{1}{2}$ is critical and Conjecture 1.1 is true for $m = \frac{1}{2}$. If this central value is moreover non-vanishing, then Conjecture 1.1 is true for any critical values.

Remark 1.8. *In* [Rag21]*, Raghuram determined the ratio of adjacent critical values explicitly. Note that the set of critical values form an interval, we only need to prove the previous theorem for $m = 1$ if $n \equiv n' \mod 2$, and $m = 1/2$ if $n \not\equiv n' \mod 2$.*

2. Proof for the main result: the piano case

2.1. **The simplest case.** Let Π and Π' be as in Theorem 1.6.

We assume n is even and n' is odd at first. In this case, both Π and Π' are conjugate self-dual and hence $\omega(\Pi) = \omega(\Pi') = 0$. Moreover, the critical point m is a half integer, the numbers $a_i, 1 \leqslant i \leqslant n$ are half integers and the numbers $b_j, 1 \leqslant j \leqslant n'$ are integers.

There are $n-1$ gaps between the numbers a_i, $1 \leqslant i \leqslant n$. Since the pair (Π,Π') is **piano**, the n' numbers $-b_j$, $1 \leqslant j \leqslant n'$ lie in n' different gaps. Let

$l = n - n' - 1$ and $-k_l > -k_{l+1} > \cdots > -k_1$ be integers lie in the other l gaps between the half integers a_i, $1 \leqslant i \leqslant i$. For each $1 \leqslant t \leqslant l$, let χ_t be a conjugate self-dual algebraic Hecke character with infinity type $z^{k_t}\overline{z}^{-k_t}$ (see Lemma 4.1.4 of [CHT08] for its existence).

Recall that Π satisfies the regular hypothesis, namely, $a_i - a_{i+1} \geqslant 2m$ if $m > 0$ and $a_i - a_{i+1} \geqslant 2 - 2m$ if $m \geqslant 0$. For fixed t, we may choose k_t such that for any i, $1 - |a_i + k_t| \leqslant m \leqslant |a_i + k_t|$. In particular, the half integer m is critical for $\Pi \otimes \chi_t$.

Let $\Pi^{\#}$ be the isobaric sum of Π' and χ_t, $1 \leqslant t \leqslant l$. It is then a cohomological and conjugate self-dual automorphic representation of $GL_{n-1}(\mathbb{A}_K)$. The isobaric sum is fully induced at non-archimedean places since each summand is tempered (for the temperedness of Π', see [Clo12] and [Car12]) and hence the fully induced representation is already irreducible. It is fully induced at the archimedean place since the isobaric summands have disjoint Langlands parameters.

Moreover, the pair $(\Pi, \Pi^{\#})$ is trivially piano by the construction. We may then apply Theorem 1.2 to this pair and get:

$$L^S(m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#})} p(m - \tfrac{1}{2}, \Pi_\infty, \Pi^{\#}_\infty)p(\Pi)p(\Pi^{\#}). \tag{2.1}$$

2.1.1. *Calculate the left hand side of equation (2.1).* The left hand side is equal to $L^S(m, \Pi \times \Pi') \times \prod\limits_{1 \leqslant t \leqslant k} L^S(m, \Pi \otimes \chi_t)$.

By Theorem 1.1, we have:

$$L^S(m, \Pi \otimes \chi_t) \sim_{E(\Pi)E(\chi_t)} (2\pi i)^{mn} P^{(s(\Pi,\chi_t))} p(\check{\chi}_t, 1)^{s(\Pi,\chi_t)} p(\check{\chi}_t, \iota)^{n-s(\Pi,\chi_t)}.$$

Hence the left hand side of equation (2.1) is equivalent to:

$$L^S(m, \Pi \times \Pi') \times (2\pi i)^{mnl} \prod_{1 \leqslant t \leqslant l} \left[P^{(s(\Pi,\chi_t))} p(\check{\chi}_t, 1)^{s(\Pi,\chi_t)} p(\check{\chi}_t, \iota)^{n-s(\Pi,\chi_t)}\right]. \tag{2.2}$$

2.1.2. *Calculate the right hand side of equation (2.1).* By Theorem 1.4, we know:

$$p(\Pi^{\#}) \sim_{E(\Pi^{\#})E(\phi)} p(\Pi') \prod_{1 \leqslant t \leqslant l} L^S(1, \Pi' \otimes \chi_t^{-1}) \prod_{1 \leqslant t < s \leqslant l} L^S(1, \chi_t \chi_s^{-1}).$$

We have used the fact that $p(\chi_t)$ is equivalent to the Gauss sum of χ_t and hence the Gauss sum of $\chi_t \mid_{\mathbb{A}_\mathbb{Q}}$ (c.f. Remark 1.3.1 of [GL21]). Since χ_t is algebraic and conjugate self-dual, it is easy to show that $\chi_t \mid_{\mathbb{A}_\mathbb{Q}}$ is trivial. Hence we may remove the terms $p(\chi_t)$, $1 \leqslant t \leqslant l$ in the above equation.

Let E be the compositum of $E(\chi_t)$, $1 \leqslant t \leqslant l$. By Theorem 1.1 and Proposition 1.1, we know

$$\begin{aligned} L^S(1, \Pi' \otimes \chi_t^{-1}) \quad &\sim_{E(\Pi')E} \quad (2\pi i)^{n'} P^{(s(\Pi',\chi_t^{-1}))} p(\chi_t^c, 1)^{s(\Pi',\chi_t^{-1})} p(\chi_t^c, \iota)^{n'-s(\Pi',\chi_t^{-1})} \\ &\sim_{E(\Pi')E} \quad (2\pi i)^{n'} P^{(s(\Pi',\chi_t^{-1}))} p(\chi_t, 1)^{n'-s(\Pi',\chi_t^{-1})} p(\chi_t, \iota)^{s(\Pi',\chi_t^{-1})} \end{aligned}$$

The infinity type of $\chi_t\chi_s^{-1} = \chi_t\chi_s^c$ is $z^{k_t-k_s}\overline{z}^{-k_t+k_s}$. Since $s < t$, we have $k_t - k_s > 0$. By Theorem 1.5 and Proposition 1.1, we know

$$L^S(1, \chi_t\chi_s^{-1}) \sim_E (2\pi i)p(\chi_t\chi_s^c, \iota) \sim_E (2\pi i)p(\chi_t, \iota)p(\chi_s, 1).$$

Hence $\prod\limits_{1 \leqslant t < s \leqslant l} L^S(1, \chi_t\chi_s^{-1}) \sim_E \prod\limits_{1 \leqslant t \leqslant l} p(\chi_t, 1)^{t-1} p(\chi_t, \iota)^{l-t}$.

We deduce that the right hand side of equation (2.1) is equivalent to:

$$(2\pi i)^{n'l+\frac{l(l-1)}{2}}p(m-\tfrac{1}{2},\Pi_\infty,\Pi_\infty^{\#})p(\Pi)p(\Pi')\prod_{1\leqslant t\leqslant l}\left[P^{(s(\Pi',\chi_t^{-1}))}p(\chi_t,1)^{n'-s(\Pi',\chi_t^{-1})+t-1}p(\chi_t,\iota)^{s(\Pi',\chi_t^{-1})+l-t}\right] \tag{2.3}$$

2.1.3. *Compare both sides of equation (2.1).*

Lemma 2.1. *For each t, we have $s(\Pi,\chi_t)=n'-s(\Pi',\chi_t^{-1})+t$ and hence $n-s(\Pi,\chi_t)=n-n'+s(\Pi',\chi_t^{-1})-t=s(\Pi',\chi_t^{-1})+l-t+1$.*

Proof. By definition we have $s(\Pi,\chi_t)=\#\{i\mid k_t<-a_i\}$ and $s(\Pi',\chi_t^{-1})=\#\{j\mid k_t>b_j\}$. Moreover, we know $l-t=\#\{s\mid k_t>k_s\}$. Recall that the $n-1$ numbers k_s, $1\leqslant s\leqslant l$, b_j, $1\leqslant j\leqslant n'$ lie in different gaps between $-a_i$, $1\leqslant i\leqslant n$ by construction. It is easy to see that $\#\{i\mid k_t<-a_i\}+\#\{j\mid k_t>b_j\}+\#\{s\mid k_t>k_s\}=n-1$. Therefore $s(\Pi,\chi_t)=n-1-s(\Pi',\chi_t^{-1})-l+t=n'-s(\Pi',\chi_t^{-1})+t$.

□

By the above lemma, we deduce that:

$$\begin{aligned}
&\prod_{1\leqslant t\leqslant l}\left[p(\chi_t,1)^{n'-s(\Pi',\chi_t^{-1})+t-1}p(\chi_t,\iota)^{s(\Pi',\chi_t^{-1})+l-t}\right]\\
=&\prod_{1\leqslant t\leqslant l}\left[p(\chi_t,1)^{s(\Pi,\chi_t)-1}p(\chi_t,\iota)^{n-s(\Pi,\chi_t)-1}\right]\\
=&\prod_{1\leqslant t\leqslant l}\left[p(\chi_t,1)^{s(\Pi,\chi_t)}p(\chi_t,\iota)^{n-s(\Pi,\chi_t)}\right]
\end{aligned} \tag{2.4}$$

where the last equation is due to the fact that $p(\chi_t,1)p(\chi_t,\iota)\sim_E p(\chi_t,1)p(\chi_t^c,1)\sim_E p(\chi_t\chi_t^c,1)\sim_E 1$ by Proposition 1.1 and the fact that χ_t is conjugate self-dual.

We compare equations (2.2), (2.3) and (2.4), we obtain that:

$$L(m,\Pi\times\Pi')\times(2\pi i)^{mnl}\prod_{1\leqslant t\leqslant l}P^{(s(\Pi,\chi_t))} \tag{2.5}$$

$$\sim_{E(\Pi)E(\Pi')}\quad(2\pi i)^{n'l+\frac{l(l-1)}{2}}p(m-\tfrac{1}{2},\Pi_\infty,\Pi_\infty^{\#})p(\Pi)p(\Pi')\prod_{1\leqslant t\leqslant l}P^{(s(\Pi',\chi_t^{-1}))}\quad.$$

We have dropped the number field E since both sides are well defined up to $(E(\Pi)E(\Pi'))^\times$ (see Remark 1.1).

By Theorem 1.3 and the fact that $P^{(0)}(\Pi')P^{(n')}(\Pi')\sim_{E(\Pi')}1$, we know $p(\Pi)\sim_{E(\Pi)}Z(\Pi_\infty)\prod\limits_{1\leqslant i\leqslant n-1}P^{(i)}(\Pi)$ and $p(\Pi')\sim_{E(\Pi')}Z(\Pi'_\infty)\prod\limits_{0\leqslant j\leqslant n'}P^{(j)}(\Pi')$. We can then interpret $L(m,\Pi\times\Pi')$ in terms of archimedean factors and arithmetic automorphic periods. We now prove that the power for each arithmetic automorphic period is indeed the split index.

Lemma 2.2. *Let $0\leqslant u\leqslant n$ and $0\leqslant v\leqslant n'$.*

(1) If $u=0$ or n, then $sp(u,\Pi;\Pi')=0=-\#\{t\mid s(\Pi,\chi_t)=u\}$.
(2) If $u\neq 0,n$, then $sp(u,\Pi;\Pi')=1-\#\{t\mid s(\Pi,\chi_t)=u\}$.
(3) For all v, we have $sp(v,\Pi';\Pi)=\#\{t\mid s(\Pi',\chi_t^{-1})\}+1$.

Proof.

(1) If $u = 0$ or n, since (Π, Π') is piano, it is easy to see that $sp(u, \Pi; \Pi') = 0$.

Recall that $s(\Pi, \chi_t) = \#\{i \mid k_t < -a_i\}$. Since $-a_n > k_t > -a_0$, this number is neither 0 or n.

(2) Let $u \neq 0, n$. We first observe that $s(\Pi, \chi_t) = \#\{i \mid k_t < -a_i\} = u$ if and only if $-a_{n+1-u} > k_t > -a_{n-u}$. If there is a number among b_j $1 \leqslant j \leqslant n'$ lying in the gap between $-a_{n+1-u}$ and $-a_{n-u}$, then $sp(u, \Pi; \Pi') = \#\{j \mid -a_{n+1-u} > b_j > -a_{n-u}\} = 1$. Moreover, there is no t such that k_t is in this gap. Hence $\#\{t \mid s(\Pi, \chi_t) = u\} = 0$.

If there is no number among b_j $1 \leqslant j \leqslant n'$ lying in the gap between $-a_{n+1-u}$ and $-a_{n-u}$, then there is exact one t such that k_t is in this gap. Hence $sp(u, \Pi; \Pi') = 0$ and $\#\{t \mid s(\Pi, \chi_t) = u\} = 1$.

(3) We denote $b_0 = +\infty$ and $b_{n'+1} = -\infty$. For each $0 \leqslant v \leqslant n'$, $\{t \mid s(\Pi', \chi_t) = v\} = \{t \mid -b_{n'-v} < k_t < -b_{n'-v+1}\}$. It is clear that $\#\{t \mid -b_{n'-v} < k_t < -b_{n'-v+1}\} + 1 = \#\{i \mid -b_{n'-v} < a_i < -b_{n'-v+1}\} = sp(v, \Pi'; \Pi)$.

$\square$

Let $a(m, \Pi_\infty, \Pi'_\infty) := (2\pi i)^{n'l + \frac{l(l-1)}{2} - mnl} p(m - \frac{1}{2}, \Pi_\infty, \Pi'^{\#}_\infty) Z(\Pi_\infty) Z(\Pi'_\infty)$. Equation 2.5 and Lemma 2.2 then imply that:

$$L^S(m, \Pi \times \Pi') \sim_{E(\Pi)E(\Pi')} a(m, \Pi_\infty, \Pi'_\infty) \prod_{u=0}^{n} P^{(u)}(\Pi)^{sp(u,\Pi;\Pi')} \prod_{v=0}^{n'} P^{(v)}(\Pi')^{sp(v,\Pi';\Pi)}.$$

All the relations above are equivariant under the action of $\mathrm{Aut}(\mathbb{C}/K)$.

2.2. **Settings, the general cases.** Let $n > r$ be arbitrary integers. We still want to apply the previous strategy to get special values of L-function for $\Pi \times \Pi'$. But if we take $\Pi^{\#}$ to be Langlands sum of Π' and some algebraic Hecke characters, it may be no longer algebraic. For example, if $n - 1 \not\equiv n' (\mathrm{mod}\ 2)$, we know the Langlands parameters of Π' are in $\mathbb{Z} + \frac{n'-1}{2}$. But the Langlands parameters of an algebraic representation of GL_{n-1} should be in $\mathbb{Z} + \frac{n-1}{2} = \mathbb{Z} + \frac{n'}{2}$. In order to fix this, we will tensor the character $||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}} \phi$, a Hecke character of infinity type $(\frac{1}{2}, -\frac{1}{2})$, when necessary.

When $n - 1 \equiv r (\mathrm{mod}\ 2)$, we write $T_1 = 0$ and we will expand Π' to an algebraic representation of GL_{n-1} as previously. When $n-1 \not\equiv r (\mathrm{mod}\ 2)$, we write $T_1 = \frac{1}{2}$ and we will expand $\Pi' \times ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}} \phi$ to an algebraic representation of GL_{n-1}. In both cases, we assume the pair (Π, Π') is **piano**, namely,

$$\text{each } b_i + T_1 \text{ are included in one of the intervals }]-a_{j+1}, -a_j[, 1 \leqslant j \leqslant n-1 \text{ and each such interval contains at most one } b_i. \tag{2.6}$$

Let $w(1) > w(2) > \cdots > w(n)$ be the integers such that

$$-a_{n+1-w(i)} > b_{n'+1-i} + T_1 > -a_{n-w(i)} \tag{2.7}$$

for all $1 \leqslant i \leqslant n'$.

Let $\chi_1, \chi_2, \cdots, \chi_l$ be conjugate self-dual algebraic Hecke characters of $\mathbb{A}_K$ of infinity type $z^{k_1}\overline{z}^{-k_1}, z^{k_2}\overline{z}^{-k_2}, \cdots z^{k_l}\overline{z}^{-k_l}$ respectively. These characters will help us expand Π' or $\Pi' \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi$ to an algebraic representation of GL_{n-1}. Similarly, we will tensor them by $||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi$ if $n \not\equiv 0 (\mathrm{mod}\ 2)$ to settle the parity issue. We write $T_2 = \frac{1}{2}$ in this case and 0 otherwise.

We assume that $k_1 + T_2 > k_2 + T_2 > \cdots > k_l + T_2$ and lie in different intervals $]-a_{j+1}, -a_j[$ which doesn't contain any of $b_i + T_1$.

More precisely, we have

$$\begin{aligned} k_1 + T_2 > k_2 + T_2 > \cdots > k_{w(n')-1} + T_2 > \quad & b_1 + T_1 \quad > \\ > k_{w(n')} + T_2 > k_{w(n')+1} + T_2 > \cdots > k_{w(n'-1)-2} + T_2 > \quad & b_2 + T_1 \quad > \\ \cdots \quad & \\ k_{w(n'+2-i)-i+2} + T_2 > k_{w(n'+2-i)-i+3} + T_2 > \cdots > k_{w(n'+1-i)-i} + T_2 > \quad & b_i + T_1 \quad > \\ \cdots \quad & \\ k_{w(2)-n'+2} + T_2 > k_{w(2)-n'+3} + T_2 > \cdots k_{w(1)-n'} + T_2 > \quad & b_{n'} + T_1 \quad > \\ k_{w(1)-n'+1} + T_2 > k_{w(1)-n'+2} + T_2 > \cdots > k_l + T_2 \quad & \end{aligned} \tag{2.8}$$

and the above $l + k = n - 1$ numbers lie in the gaps between the n numbers $-a_n > -a_{n-1} > \cdots > -a_1$. Note the above $n-1$ numbers are in $\mathbb{Z} + \dfrac{n}{2}$ when $a_i \in \mathbb{Z} + \dfrac{n-1}{2}$ for all $1 \leqslant i \leqslant n$.

There are four cases:

(A) n is even and n' is odd, then $T_1 = 0$ and $T_2 = 0$. We set $\Pi^{\#} = \Pi' \boxplus \chi_1 \boxplus \chi_2 \boxplus \cdots \boxplus \chi_l$ as in previous subsections.

(B) n is even and n' is even, then $T_1 = \frac{1}{2}$ and $T_2 = 0$. We set $\Pi^{\#} = (\Pi' \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi) \boxplus \chi_1 \boxplus \chi_2 \boxplus \cdots \boxplus \chi_l$.

(C) n is odd and n' is even, then $T_1 = 0$ and $T_2 = \frac{1}{2}$. We set $\Pi^{\#} = \Pi' \boxplus (\chi_1 \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi) \boxplus (\chi_2 \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi) \boxplus \cdots \boxplus (\chi_l \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)$.

(D) n is odd and n' is odd, then $T_1 = \frac{1}{2}$ and $T_2 = \frac{1}{2}$. We set $\Pi^{\#} = (\Pi' \boxplus \chi_1 \boxplus \chi_2 \boxplus \cdots \boxplus \chi_l) \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi$.

In all cases, $\Pi^{\#}$ is a generic cohomological conjugate self-dual automorphic representation of $GL_{n-1}(\mathbb{A}_K)$ and Theorem 5.2 of [GL21] gives us that if $m + \frac{1}{2}$ is critical for $\Pi \times \Pi^{\#}$, then

$$L^S(\frac{1}{2} + m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#})} p(\Pi)p(\Pi^{\#})p(m, \Pi_\infty, \Pi_\infty^{\#}). \tag{2.9}$$

Again, we shall simplify both sides of this equation.

2.3. **Simplify the left hand side, general cases.**

For the left hand side of equation (2.9), we know by construction that:

(A) $L^S(\frac{1}{2} + m, \Pi \times \Pi^{\#}) = L^S(\dfrac{1}{2} + m, \Pi \times \Pi') \prod\limits_{j=1}^{l} L^S(\frac{1}{2} + m, \Pi \otimes \chi_j)$

(B)
$$\begin{aligned} L^S(\tfrac{1}{2}+m,\Pi\times\Pi^{\#}) &= L^S(\frac{1}{2}+m,\Pi\times(\Pi'\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi))\prod_{j=1}^{l}L^S(\tfrac{1}{2}+m,\Pi\otimes\chi_j) \\ &= L^S(m,\Pi\times(\Pi'\otimes\phi))\prod_{j=1}^{l}L^S(\tfrac{1}{2}+m,\Pi\otimes\chi_j)\end{aligned}$$

(C)
$$\begin{aligned} L^S(\tfrac{1}{2}+m,\Pi\times\Pi^{\#}) &= L^S(\frac{1}{2}+m,\Pi\times\Pi')\prod_{j=1}^{l}L^S(\tfrac{1}{2}+m,\Pi\otimes(\chi_j\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)) \\ &= L^S(\frac{1}{2}+m,\Pi\times\Pi')\prod_{j=1}^{l}L^S(m,\Pi\otimes(\chi_j\otimes\phi))\end{aligned}$$

(D) $L^S(\frac{1}{2}+m,\Pi\times\Pi^{\#}) = L^S(\frac{1}{2},\Pi\times(\Pi'\otimes\phi))\prod_{j=1}^{l}L^S(m,\Pi\otimes(\chi_j\otimes\phi))$

We set $s_j=\#\{1\leqslant i\leqslant n\mid k_j+T_2<-a_i\}=j+\#\{1\leqslant i\leqslant n'\mid b_i+T_1>k_j+T_2\}$ and $t_j=\#\{1\leqslant i\leqslant n'\mid (b_i+T_1)-(k_j+T_2)<0\}$ as before. Recall that $s_j+t_j=n'+j$ for all $1\leqslant j\leqslant l$.

If n is even (case (A) and (B)), we have for all $1\leqslant j\leqslant l$:

$$L^S(\frac{1}{2}+m,\Pi\otimes\chi_j)\sim_{E(\Pi)E(\chi_j)}(2\pi i)^{(m+\frac{1}{2})n}P^{(s_j)}(\Pi)p(\widecheck{\chi_j},1)^{2s_j-n}.$$

If n is odd (case (C) and (D)), we have for all $1\leqslant j\leqslant l$:

$$L^S(m,\Pi\otimes(\chi_j)\otimes\phi)\sim_{E(\Pi)E(\chi_j)}(2\pi i)^{mn}P^{(s_j)}(\Pi)p(\widecheck{\chi_j},1)^{2s_j-n}p(\check{\phi},1)^{s_j}p(\check{\phi},\iota)^{n-s_j}.$$

Therefore for cases (A) and (B), we have

$$\prod_{j=1}^{l}L^S(\frac{1}{2}+m,\Pi\otimes\chi_j)\sim_{E(\Pi)E}(2\pi i)^{(m+\frac{1}{2})nl}\prod_{k=1}^{n-1}P^{(k)}(\Pi)\prod_{k=1}^{n'}P^{(w(k))}(\Pi)^{-1}\prod_{j=1}^{l}p(\widecheck{\chi_j},1)^{2s_j-n}.$$

For cases (C) and (D), we put $s:=\sum_{j=1}^{l}s_j$ and then we have:

$$\prod_{j=1}^{l}L^S(m,\Pi\otimes(\chi_j\otimes\phi))\sim_{E(\Pi)EE(\phi)}$$

$$(2\pi i)^{mnl}\times\prod_{k=1}^{n-1}P^{(k)}(\Pi)\prod_{k=1}^{n'}P^{(w(k))}(\Pi)^{-1}\prod_{j=1}^{l}p(\widecheck{\chi_j},1)^{2s_j-n}p(\check{\phi},1)^{s}p(\check{\phi},\iota)^{nl-s}$$

2.4. **Simplify the right hand side, general cases.**
Calculate $p(\Pi^{\#})$: Apply Theorem 2.6 of [GL21] we get

(A) $p(\Pi^{\#})\sim_{E(\Pi^{\#})}\Omega(\Pi^{\#}_\infty)p(\Pi')\Omega(\Pi'_\infty)^{-1}\prod_{1\leqslant j\leqslant l}L^S(1,\Pi'\otimes\chi_j^c)\prod_{1\leqslant i<j\leqslant l}L^S(1,\chi_i\otimes\chi_j^c)$

(B) $p(\Pi^{\#})\sim_{E(\Pi^{\#})}\Omega(\Pi^{\#}_\infty)p(\Pi')\Omega(\Pi'_\infty)^{-1}\prod_{1\leqslant j\leqslant l}L^S(1,(\Pi'\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)\otimes\chi_j^c)\prod_{1\leqslant i<j\leqslant l}L^S(1,\chi_i\otimes\chi_j^c)$

(C) $p(\Pi^{\#})\sim_{E(\Pi^{\#})}\Omega(\Pi^{\#}_\infty)p(\Pi')\Omega(\Pi'_\infty)^{-1}\prod_{1\leqslant j\leqslant l}L^S(1,\Pi'\otimes(\chi_j\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)^c)\prod_{1\leqslant i<j\leqslant l}L^S(1,\chi_i\otimes\chi_j^c)$

(D) $p(\Pi^{\#}) \sim_{E(\Pi^{\#})} \Omega(\Pi^{\#}_{\infty}) p(\Pi')\Omega(\Pi'_{\infty})^{-1} \prod\limits_{1\leqslant j\leqslant l} L^S(1,\Pi'\otimes\chi_j^c) \prod\limits_{1\leqslant i<j\leqslant l} L^S(1,\chi_i\otimes \chi_j^c)$

Here we have used that:

Lemma 2.3. *If η is a conjugate self-dual Hecke character then:*

$$\frac{p(\Pi'\otimes\eta)}{\Omega((\Pi'\otimes\eta)_{\infty})} \sim_{E(\Pi')E(\eta)} \frac{p(\Pi')}{\Omega(\Pi'_{\infty})}.$$

Proof. By Theorem 2.6 of [GL21], we have:

$$p(\Pi'\otimes\eta) \sim_{E(\Pi')E(\eta)} Z((\Pi'\otimes\eta)_{\infty}) \prod_{1\leqslant i\leqslant n'-1} P^{(i)}(\Pi'\otimes\eta). \tag{2.10}$$

By the definition of arithmetic automorphic period, we know $P^{(i)}(\Pi'\otimes \eta) \sim_{E(\Pi')E(\eta)} p(\check{\eta},1)^i p(\check{\eta},\iota)^{n-i}$. The latter is equivalent to $p(\check{\eta},1)^{2i-n}$ since η is conjugate self-dual.

We see that:

$$\prod_{1\leqslant i\leqslant n} P^{(i)}(\Pi'\otimes\eta) \sim_{E(\Pi')E(\eta)} \prod_{1\leqslant i\leqslant n}[P^{(i)}(\Pi')p(\check{\eta},1)^{2i-n}] \sim_{E(\Pi')E(\eta)} \prod_{1\leqslant i\leqslant n} P^{(i)}(\Pi'). \tag{2.11}$$

By Theorem 2.6 of [GL21], This will imply that:

$$\frac{p(\Pi'\otimes\eta)}{Z((\Pi'\otimes\eta)_{\infty})} \sim_{E(\Pi')E(\eta)} \frac{p(\Pi')}{Z(\Pi'_{\infty})}.$$

Since $Z(\Pi'_{\infty}) \sim_{E(\Pi'_{\infty};K)} (2\pi i)^{\frac{n'(n'-1)}{2}}\Omega(\Pi'_{\infty})$ and a similar formula for $(\Pi'\otimes \eta)_{\infty}$, we get the lemma.

□

By the known case $n'=1$ of Conjecture 0.2, for all $1\leqslant j\leqslant l$, we have

$$L^S(1,\Pi'\otimes\chi_j^c) \sim_{E(\Pi')E(\chi_j)} (2\pi i)^{n'} P^{(t_j)}(\Pi')p(\widecheck{\chi_j},1)^{n'-2t_j}.$$

Similarly, we have

$$L^S(1,(\Pi'\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)\otimes\chi_j^c) = L^S(\frac{1}{2},\Pi'\otimes(\phi\chi_j^c))$$
$$\sim_{E(\Pi')E(\chi_j)E(\phi)} (2\pi i)^{\frac{n'}{2}} P^{(t_j)}(\Pi')p(\widecheck{\chi_j},1)^{n'-2t_j}p(\check{\phi},1)^{t_j}p(\check{\phi},\iota)^{n'-t_j};$$

and
$$L^S(1,\Pi'\otimes(\chi_i\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)^c) = L^S(\frac{1}{2},\Pi'\otimes(\chi_i\otimes\phi)^c)$$
$$\sim_{E(\Pi')E(\chi_j)E(\phi)} (2\pi i)^{\frac{n'}{2}} P^{(t_j)}(\Pi')p(\widecheck{\chi_j},1)^{n'-2t_j}p(\check{\phi},1)^{n'-t_j}p(\check{\phi},\iota)^{t_j}.$$

Along with equation (**??**), we get

(A) and (D): $p(\Pi^{\#}) \sim_{E(\Pi')EE(\phi)} \Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}(2\pi i)^{n'l+\frac{l(l-1)}{2}} \times$

$$\prod_{j=1}^{l} P^{(t_j)}(\Pi') \prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n}$$

(B) : $p(\Pi^{\#}) \sim_{E(\Pi')EE(\phi)} \Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}(2\pi i)^{\frac{n'l}{2}+\frac{l(l-1)}{2}}\times$

$$\prod_{j=1}^{l} P^{(t_j)}(\Pi')\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n}p(\check{\phi},1)^{t}p(\check{\phi},\iota)^{n'l-t}$$

(C) : $p(\Pi^{\#}) \sim_{E(\Pi')EE(\phi)} \Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}(2\pi i)^{\frac{n'l}{2}+\frac{l(l-1)}{2}}\times$

$$\prod_{j=1}^{l} P^{(t_j)}(\Pi')\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n}p(\check{\phi},1)^{n'l-t}p(\check{\phi},\iota)^{t}$$

where $t=\sum\limits_{j=1}^{l} t_j=\sum\limits_{j=1}^{l}(n'+j-s_j)=n'l+\dfrac{l(l+1)}{2}-s$.

We then apply equations (**??**), (**??**) and Lemma **??**, Corollary **??** to get:

(A) $p(\Pi)p(\Pi^{\#})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}) \sim_{E(\Pi)E(\Pi')E} (2\pi i)^{n(n-1)(m+\frac{1}{2})}\times$

$$\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

(B) $p(\Pi)p(\Pi^{\#})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}) \sim_{E(\Pi)E(\Pi')EE(\phi)} (2\pi i)^{n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}}\times$

$$\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n}p(\check{\phi},1)^{t}p(\check{\phi},\iota)^{n'l-t}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\phi;\Pi)}.$$

(C) $p(\Pi)p(\Pi^{\#})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}) \sim_{E(\Pi)E(\Pi')EE(\phi)} (2\pi i)^{n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}}\times$

$$\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n}p(\check{\phi},1)^{n'l-t}p(\check{\phi},\iota)^{t}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\phi;\Pi)}.$$

(D) $p(\Pi)p(\Pi^{\#})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}) \sim_{E(\Pi)E(\Pi')EE(\phi)} (2\pi i)^{n(n-1)(m+\frac{1}{2})}\times$

$$\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

2.5. **Compare both sides, general cases.** At first, observe that

$$p(\check{\phi},1)p(\check{\phi},\iota)\sim_{E(\phi)} p(\check{\phi},1)p(\check{\phi}^{c},1)\sim_{E(\phi)} p(\widecheck{\phi\phi^{c}},1)\sim_{E(\phi)} p(||\cdot||_{\mathbb{A}_K}^{-1},1)\sim_{E(\phi)} 2\pi i.$$

We can then conclude:

(A) $L^{S}(\dfrac{1}{2}+m,\Pi\times\Pi')\sim_{E(\Pi)E(\Pi')}$

$$(2\pi i)^{(m+\frac{1}{2})nn'}\prod_{k=1}^{n'}P^{(w(k))}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}$$

(B) Since $n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}-(m+\frac{1}{2})nl=(m+\frac{1}{2})n(n-1-l)-\frac{n'l}{2}=(m+\frac{1}{2})nn'-\frac{n'l}{2}=mnn'+\frac{nn'}{2}-\frac{n'l}{2}$, we have

$$L^S(m,\Pi\times(\Pi'\otimes\phi))\sim_{E(\Pi)E(\Pi')E(\phi)}(2\pi i)^{mnn'+\frac{nn'}{2}-\frac{n'l}{2}}\times \prod_{k=1}^{n'}P^{(w(k))}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\phi;\Pi)}p(\check{\phi},1)^{t}p(\check{\phi},\iota)^{n'l-t} \tag{2.12}$$

Since $(2\pi i)^{\frac{nn'}{2}-\frac{n'l}{2}}\sim_{E(\phi)}p(\check{\phi},1)^{\frac{nn'}{2}-\frac{n'l}{2}}p(\check{\phi},\iota)^{\frac{nn'}{2}-\frac{n'l}{2}}$, and

$$\begin{aligned}\frac{nn'}{2}-\frac{n'l}{2}+t &= \frac{nn'}{2}-\frac{n'l}{2}+(n'l+\frac{l(l+1)}{2}-s)=\frac{nn'}{2}+\frac{n'l}{2}+\frac{l(l+1)}{2}-s\\ &= \frac{nn'}{2}+\frac{(n'+l+1)l}{2}-s=\frac{nn'}{2}+\frac{nl}{2}-s\\ &= \frac{n(n'+l)}{2}-s=\frac{n(n-1)}{2}-s;\end{aligned}$$

$$\begin{aligned}\frac{nn'}{2}-\frac{n'l}{2}+n'l-t &= \frac{nn'}{2}-\frac{n'l}{2}+n'l-(n'l+\frac{l(l+1)}{2}-s)=s+\frac{nn'}{2}-\frac{(n'+l+1)l}{2}\\ &= s+nn'-\frac{nn'}{2}-\frac{nl}{2}=s+nn'-\frac{n(n-1)}{2}\end{aligned}$$

We get $L^S(m,\Pi\times(\Pi'\otimes\phi))\sim_{E(\Pi)E(\Pi')E(\phi)}(2\pi i)^{mnn'}\times$

$$\prod_{k=1}^{n'}P^{(w(k))}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}p(\check{\phi},1)^{\frac{n(n-1)}{2}-s}p(\check{\phi},\iota)^{s+nn'-\frac{n(n-1)}{2}}$$

(C) Since $n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}-mnl=n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}-(m+\frac{1}{2})nl+\frac{nl}{2}=(m+\frac{1}{2})nn'+\frac{nl}{2}-\frac{n'l}{2}$, we have

$$L^S(\frac{1}{2}+m,\Pi\times\Pi')\sim_{E(\Pi)E(\Pi')E(\phi)}(2\pi i)^{(m+\frac{1}{2})nn'+\frac{nl}{2}-\frac{n'l}{2}}\times \prod_{k=1}^{n'}P^{(w(k))}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}p(\check{\phi},1)^{n'l-t-s}p(\check{\phi},\iota)^{t+s-nl} \tag{2.13}$$

Moreover, we know $t+s=n'l+\frac{l(l+1)}{2}$, we have $2(t+s)=2n'l+(l+1)l=n'l+(n'+l+1)l=n'l+nl$. Thus $n'l-t-s=t+s-nl=\frac{n'l}{2}-\frac{nl}{2}$. We then get $p(\check{\phi},1)^{n'l-t-s}p(\check{\phi},\iota)^{t+s-nl}=p(\widecheck{\phi\otimes\phi^c},1)^{\frac{n'l}{2}-\frac{nl}{2}}=(2\pi i)^{\frac{n'l}{2}-\frac{nl}{2}}$.

Therefore:

$$L^S(\frac{1}{2}+m,\Pi\times\Pi')\sim_{E(\Pi)E(\Pi')}(2\pi i)^{(m+\frac{1}{2})nn'}\prod_{k=1}^{n'}P^{(w(k))}(\Pi)\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

(D) Similarly, since $n(n-1)(m+\frac{1}{2})-mnl = n(n-1)m+\frac{n(n-1)}{2}-mnl = mnn'+\frac{n(n-1)}{2}$, we have

$$L^S(m,\Pi\times(\Pi'\otimes\phi)) \sim_{E(\Pi)E(\Pi')E(\phi)} (2\pi i)^{mnn'} \times \prod_{k=1}^{n'} P^{(w(k))}(\Pi)\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\phi;\Pi)}p(\check{\phi},1)^{\frac{n(n-1)}{2}-s}p(\check{\phi},\iota)^{s+nn'-\frac{n(n-1)}{2}}. \tag{2.14}$$

It is easy to verify that $s-nl+\frac{n(n-1)}{2} = s-nl+n(n-1)-\frac{n(n-1)}{2} = s+nn'-\frac{n(n-1)}{2}$.

2.6. **Final conclusion: general cases.** Before concluding, we notice that in case (B) or (D),

$$s=\sum_{1\leqslant j\leqslant n-1} s_j = \sum_{j=1}^{n-1} j - \sum_{j=1}^{n'} w(j) = \frac{n(n-1)}{2} - \sum_{j=1}^{n'} w(j).$$

Recall that $w(j)=\sum\limits_{k=j}^{n'} sp(k,\Pi'\otimes\phi;\Pi)$ for all $1\leqslant k\leqslant n'$ by (**??**). Therefore:

$$\begin{aligned}\frac{n(n-1)}{2}-s=\sum_{j=1}^{n'} w(j) &= \sum_{j=1}^{n'}\sum_{j\leqslant k\leqslant n'} sp(j,\Pi'\otimes\phi;\Pi) = \sum_{k=1}^{n'} k*sp(k,\Pi'\otimes\phi;\Pi)\\ &= \sum_{k=0}^{n'} k*sp(k,\Pi'\otimes\phi;\Pi); \end{aligned} \tag{2.15}$$

$$\begin{aligned}\text{and } s+nn'-\frac{n(n-1)}{2} &= nn'-\sum_{k=0}^{n'} k*sp(k,\Pi'\otimes\phi;\Pi)\\ &= r\sum_{k=0}^{n'} sp(k,\Pi'\otimes\phi;\Pi) - \sum_{k=0}^{n'} j*sp(k,\Pi'\otimes\phi;\Pi)\\ &= \sum_{k=0}^{n'}(n'-k)sp(k,\Pi'\otimes\phi;\Pi)\end{aligned}$$

by Lemma 1.1 which says that $\sum\limits_{k=0}^{n'} sp(k,\Pi'\otimes\phi;\Pi)=n$.

Therefore, we get

$$\begin{aligned}&\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\phi;\Pi)}p(\check{\phi},1)^{\frac{n(n-1)}{2}-s}p(\check{\phi},\iota)^{s+nn'-\frac{n(n-1)}{2}}\\ \sim_{E(\Pi')E(\phi)} &\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\phi;\Pi)}p(\check{\phi},1)^{\sum\limits_{k=0}^{n'} k*sp(k,\Pi'\otimes\phi;\Pi)}p(\check{\phi},\iota)^{\sum\limits_{k=0}^{n'}(n'-k)sp(k,\Pi'\otimes\phi;\Pi)}\\ \sim_{E(\Pi')E(\phi)} &\prod_{k=0}^{n'}\left(P^{(k)}(\Pi')p(\check{\phi},1)^k p(\check{\phi},\iota)^{n'-k}\right)^{sp(k,\Pi'\otimes\phi;\Pi)}.\end{aligned}$$

Recall that $P^{(k)}(\Pi'\otimes\phi):=P^{(k)}(\Pi')p(\check{\phi},1)^{k}p(\check{\phi},\iota)^{n'-k}$ by definition, we obtain that:

Theorem 2.1. *Let $n>n'$ be two positive integers. Let K be a quadratic imaginary field. Let Π and Π' be cuspidal representations of GL_n and $GL_{n'}$ respectively which are very regular, cohomological, conjugate self-dual and supercuspidal at at least two finite split places. We assume that (Π,Π') is piano in the sense of Definition 1.4.*

(i) If $n\not\equiv n'(\text{mod } 2)$, then for any critical value $m+\frac{1}{2}$ for $\Pi\otimes\Pi'$ such that $m\geqslant 1$, or $m\geqslant 0$ along with a non-vanishing condition, we have:

$$L^{S}(\frac{1}{2}+m,\Pi\times\Pi')\sim_{E(\Pi)E(\Pi')}(2\pi i)^{(m+\frac{1}{2})nn'}\prod_{i=0}^{n}P^{(i)}(\Pi)^{sp(i,\Pi;\Pi')}\prod_{k=0}^{n'}P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

(ii) If $n\equiv n'(\text{mod } 2)$, then for any critical value m for $\Pi\otimes\Pi'$ such that $m\geqslant 1$, or $m\geqslant 0$ along with a non-vanishing condition, we have:

$$L^{S}(m,\Pi\times(\Pi'\otimes\phi))\sim_{E(\Pi)E(\Pi')E(\phi)}(2\pi i)^{mnn'}\prod_{i=0}^{n}P^{(i)}(\Pi)^{sp(i,\Pi;\Pi'\otimes\phi)}\prod_{k=0}^{n'}P^{(k)}(\Pi'\otimes\phi)^{sp(k,\Pi'\otimes\phi;\Pi)}.$$

3. Proof for the main result: central or near-central case

3.1. Settings.

Let r_1 and r_2 be two positive integers.

Let Π_1 and Π_2 be two cuspidal representations of $GL_{r_1}(\mathbb{A}_K)$ and $GL_{r_2}(\mathbb{A}_K)$ respectively which has definable arithmetic automorphic periods. We assume they are also conjugate self-dual.

We write the infinity type of Π_1 (resp. Π_2) by $(z^{b_j}\overline{z}^{-b_j})_{1\leqslant j\leqslant r_1}$ (resp. $(z^{c_k}\overline{z}^{-c_k})_{1\leqslant k\leqslant r_2}$). We see that $b_j\in\mathbb{Z}+\frac{r_1-1}{2}$ for all $1\leqslant j\leqslant r_1$ (resp. $c_k\in\mathbb{Z}+\frac{r_2-1}{2}$ for all $1\leqslant k\leqslant r_2$).

(A) If $r_1\equiv r_2\equiv 0(\text{ mod } 2)$, we write $\Pi^{\#}=\Pi_1\boxplus\Pi_2^c$. We define $T_3=T_4=0$.
(B) If $r_1\equiv r_2\equiv 1(\text{ mod } 2)$, we write $\Pi^{\#}=(\Pi_1\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)\boxplus(\Pi_2^c\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)$. We define $T_3=T_4=\frac{1}{2}$.
(C) If $r_1\not\equiv r_2(\text{mod } 2)$, we may assume that r_1 is even and r_2 is odd. We write $\Pi^{\#}=(\Pi_1\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\phi)\boxplus\Pi_2^c$. We define $T_3=\frac{1}{2}$ and $T_4=0$.

It is easy to see that $\Pi^{\#}$ is an algebraic generic representation of $GL_{r_1+r_2}(\mathbb{A}_K)$ with infinity type $(z^{b_j+T_3}\overline{z}^{-b_j-T_3},z^{-c_k+T_4}\overline{z}^{c_k-T_4})_{1\leqslant j\leqslant r_1,1\leqslant k\leqslant r_2}$.

We assume that $\Pi^{\#}$ is regular, i.e. for any $1\leqslant j\leqslant r_1$ and any $1\leqslant k\leqslant r_2$, we have $b_j+T_3\neq -c_k+T_4$.

Set $n=r_1+r_2+1$. We see that $\{b_j+T_3\mid 1\leqslant j\leqslant r_1\}\cup\{-c_k+T_4\mid 1\leqslant k\leqslant r_2\}$ are $n-1$ different numbers in $\mathbb{Z}+\frac{n-2}{2}$. We take $a_1>a_2>\cdots>a_n\in\mathbb{Z}+\frac{n-1}{2}$ such that the $n-1$ numbers above are in different gaps between $\{a_i \mid 1 \leqslant i \leqslant n\}$. Let Π be a cuspidal conjugate self-dual representation of $GL_n(\mathbb{A}_K)$ which has arithmetic automorphic periods and infinity type $(z^{a_i}\overline{z}^{-a_i})$.

Our method also requires Π to be 3-regular. To guarantee this, we assume that

$$|(b_j + T_3) - (-c_k + T_4)| \geqslant 3 \text{ for all } 1 \leqslant j \leqslant r_1, 1 \leqslant k \leqslant r_2. \tag{3.1}$$

In this case, we say the pair (Π_1, Π_2) is **very regular**. We can then take a_i as above such that $1 + \frac{1}{2}$ is critical for $\Pi \otimes \Pi^{\#}$. Moreover, results in [Har07] show the existence of Π as above, such that $L^S(1 + \frac{1}{2}, \Pi \otimes \Pi^{\#}) \neq 0$.

We fix such Π and $m = 1$, then $m + \frac{1}{2}$ is critical for $\Pi \times \Pi^{\#}$ and moreover

$$L^S(\frac{1}{2} + m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#})} p(\Pi)p(\Pi^{\#})p(m, \Pi_\infty, \Pi^{\#}_\infty) \tag{3.2}$$

with both sides non zero.

In the end of this subsection, let us show some simple facts on the split index. We can read from the construction of a_i that

$$sp(j, \Pi_1 \otimes \phi^{2T_3}; \Pi) = sp(j, \Pi_1 \otimes \phi^{2T_3}; \Pi_2 \otimes (\phi)^{2T_4}) + 1 \text{ for all } 0 \leqslant j \leqslant r_1$$

and similarly,

$$\begin{aligned} sp(j, \Pi_2^c \otimes \phi^{2T_4}; \Pi) &= sp(j, (\Pi_2 \otimes (\phi^c)^{2T_4})^c; (\Pi_1 \otimes \phi^{2T_3})^c) + 1 \\ &= sp(r_2 - j, \Pi_2 \otimes (\phi^c)^{2T_4}); \Pi_1 \otimes \phi^{2T_3}) + 1 \text{ for all } 0 \leqslant j \leqslant r_2 \end{aligned}$$

Here we have used Lemma 1.1.

Moreover, for each $1 \leqslant i \leqslant n-1$, one of $sp(i, \Pi; \Pi_1 \otimes (\phi^c)^{2T_3})$ and $sp(i, \Pi; \Pi_2^c \otimes \phi^{2T_4})$ is 1 and another is 0. We also know that $sp(0, \Pi; \Pi_1 \otimes \phi^{2T_3}) = sp(0, \Pi; \Pi_2^c \otimes \phi^{2T_4}) = 0$ and $sp(n, \Pi; \Pi_1 \otimes \phi^{2T_3}) = sp(n, \Pi; \Pi_2^c \otimes \phi^{2T_3}) = 0$.

3.2. **Simplify the left hand side.** We are going to simply the left hand side of equation (3.2) now.

(A) In this case we have $L^S(m + \frac{1}{2}, \Pi \times \Pi^{\#}) = L^S(m + \frac{1}{2}, \Pi \times \Pi_1) \times L^S(m + \frac{1}{2}, \Pi \times \Pi_2^c)$.

By Theorem 2.1, we know that

$$L^S(\frac{1}{2} + m, \Pi \times \Pi_1) \sim_{E(\Pi)E(\Pi_1)} (2\pi i)^{(m+\frac{1}{2})nr_1} \prod_{i=0}^{n} P^{(i)}(\Pi)^{sp(i,\Pi;\Pi_1)} \prod_{j=0}^{r_1} P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi)} \tag{3.3}$$

and similarly

$$L^S(\frac{1}{2} + m, \Pi \times \Pi_2^c) \sim_{E(\Pi)E(\Pi_2)} (2\pi i)^{(m+\frac{1}{2})nr_2} \prod_{i=0}^{n} P^{(i)}(\Pi)^{sp(i,\Pi;\Pi_2^c)} \prod_{k=0}^{r_2} P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi)} \quad .$$

Therefore, since $sp(i,\Pi;\Pi_1)+sp(i,\Pi;\Pi_2^c)=1$ for all $1\leqslant i\leqslant n-1$, we obtain that

$$L^S(m+\frac{1}{2},\Pi\times\Pi^{\#}) \tag{3.4}$$

$$\sim_{E(\Pi)E(\Pi)E(\Pi_2)} (2\pi i)^{(m+\frac{1}{2})n(n-1)}\prod_{i=0}^{n}P^{(i)}(\Pi)^{sp(i,\Pi;\Pi_1)+sp(i,\Pi;\Pi_2^c)}$$

$$\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2^c;\Pi)}$$

$$\sim_{E(\Pi)E(\Pi)E(\Pi_2)} (2\pi i)^{(m+\frac{1}{2})n(n-1)}\prod_{i=1}^{n-1}P^{(i)}(\Pi)$$

$$\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi)}.$$

(B) In this case, we have $L^S(m+\frac{1}{2},\Pi\times\Pi^{\#})=L^S(m,\Pi\times(\Pi_1\otimes\phi))\times L^S(m,\Pi\times(\Pi_2^c\otimes\phi))$.

Applying the second part of Theorem 2.1, we have

$$L^S(m+\tfrac{1}{2},\Pi\times\Pi^{\#})\sim_{E(\Pi)E(\Pi)E(\Pi_2)} \tag{3.5}$$

$$(2\pi i)^{mn(n-1)}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\phi;\Pi)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c\otimes\phi;\Pi)}\times$$

$$p(\check{\phi},1)^{\sum\limits_{j=0}^{r_1}j*sp(j,\Pi_1\otimes\phi;\Pi)+\sum\limits_{k=0}^{r_2}k*sp(k,\Pi_2^c\otimes\phi;\Pi)}p(\check{\phi},\iota)^{\sum\limits_{j=0}^{n'}(r_1-j)*sp(j,\Pi_1\otimes\phi;\Pi)+\sum\limits_{k=0}^{r_2}(r_2-k)*sp(k,\Pi_2^c\otimes\phi;\Pi)}.$$

Lemma 3.1. *We have:*

$$\sum_{j=0}^{r_1}j*sp(j,\Pi_1\otimes\phi;\Pi)+\sum_{k=0}^{r_2}k*sp(k,\Pi_2^c\otimes\phi;\Pi)$$

$$=\sum_{j=0}^{n'}(r_1-j)*sp(j,\Pi_1\otimes\phi;\Pi)+\sum_{k=0}^{r_2}(r_2-k)*sp(k,\Pi_2^c\otimes\phi;\Pi)$$

$$=\frac{n(n-1)}{2}$$

Proof. We set $w(j,\Pi_1\otimes\phi;\Pi)$, $1\leqslant j\leqslant r_1$ (resp. $w(k,\Pi_2^c\otimes\phi;\Pi)$, $1\leqslant k\leqslant r_2$) to be the index $w(j)$ for the pair $(\Pi,\Pi_1\otimes\phi)$ (resp. $(\Pi,\Pi_2^c\otimes\phi))$) as in (**??**). We see from (2.15) that $\sum\limits_{j=0}^{r_1}j*sp(j,\Pi_1\otimes\phi;\Pi)=\sum\limits_{j=1}^{r_1}w(j,\Pi_1\otimes\phi;\Pi)$ and $\sum\limits_{k=0}^{r_2}k*sp(k,\Pi'\otimes\phi;\Pi)=\sum\limits_{k=1}^{r_1}w(k,\Pi_2^c\otimes\phi;\Pi)$.

Recall that $w(j,\Pi_1\otimes\phi;\Pi)$ (resp. $w(k,\Pi_2^c\otimes\phi;\Pi)$) is the position of the infinity type of $\Pi_1\otimes\phi$ (resp. $\Pi_2^c\otimes\phi$) in the gaps of the infinity type of Π. It is easy to see that the $n-1$ numbers $w(j,\Pi_1\otimes\phi;\Pi)$, $w(k,\Pi_2^c\otimes\phi;\Pi)$ for $1\leqslant j\leqslant r_1$ and $1\leqslant k\leqslant r_2$ runs over $1,2,\cdots,n-1$. We then deduce the first formula of the lemma.

The second follows easily from the first one.

$\square$

From the lemma we see that

$$p(\check{\phi},1)^{\sum\limits_{j=0}^{r_1} j*sp(j,\Pi_1\otimes\phi;\Pi)+\sum\limits_{k=0}^{r_2} k*sp(k,\Pi_2^c\otimes\phi;\Pi)} p(\check{\phi},\iota)^{\sum\limits_{j=0}^{n'} (r_1-j)*sp(j,\Pi_1\otimes\phi;\Pi)+\sum\limits_{k=0}^{r_2} (r_2-k)*sp(k,\Pi_2^c\otimes\phi;\Pi)}$$
$$\sim_{E(\phi)} (2\pi i)^{\frac{n(n-1)}{2}}. \tag{3.6}$$

We thus obtain that

$$L^S(m+\tfrac{1}{2},\Pi\times\Pi^{\#}) \sim_{E(\Pi)E(\Pi)E(\Pi_2)} (2\pi i)^{(m+\frac{1}{2})n(n-1)} \prod_{i=1}^{n-1} P^{(i)}(\Pi)$$
$$\prod_{j=0}^{r_1} P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\phi;\Pi)} \prod_{k=0}^{r_2} P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c\otimes\phi;\Pi)}. \tag{3.7}$$

(C) In this case, we have $L^S(m+\frac{1}{2},\Pi\times\Pi^{\#}) = L^S(m,\Pi\times(\Pi_1\otimes\phi))\times L^S(m+\frac{1}{2},\Pi\times\Pi_2^c)$.
Similarly, we get:

$$L^S(m+\tfrac{1}{2},\Pi\times\Pi^{\#}) \sim_{E(\Pi)E(\Pi)E(\Pi_2)} (2\pi i)^{(m+\frac{1}{2})n(n-1)-\frac{nr_1}{2}} \prod_{i=1}^{n-1} P^{(i)}(\Pi)$$
$$\prod_{j=0}^{r_1} P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\phi;\Pi)} \prod_{k=0}^{r_2} P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi)} p(\check{\phi},1)^{\sum\limits_{j=0}^{r_1} j*sp(j,\Pi_1\otimes\phi;\Pi)} p(\check{\phi},\iota)^{\sum\limits_{j=0}^{n'} (r_1-j)*sp(j,\Pi_1\otimes\phi;\Pi)}. \tag{3.8}$$

3.3. **Simplify the right hand side.** By Corollary **??** and Corollary **??**, for cases (A) and (B), we have:

$$\begin{aligned} p(\Pi^{\#}) &\sim_{E(\Pi^{\#})} \Omega(\Pi^{\#}_\infty)p(\Pi_1)\Omega(\Pi_{1,\infty})^{-1}p(\Pi_2)\Omega(\Pi_{2,\infty})^{-1}L^S(1,\Pi_1\times\Pi_2) \\ &\sim_{E(\Pi^{\#})} \Omega(\Pi^{\#}_\infty)Z(\Pi_{1,\infty})\Omega(\Pi_{1,\infty})^{-1}Z(\Pi_{2,\infty})\Omega(\Pi_{2,\infty})^{-1}L^S(1,\Pi_1\times\Pi_2)\times \\ &\qquad \prod_{j=1}^{r_1-1} P^{(j)}(\Pi_1) \prod_{k=1}^{r_2-1} P^{(k)}(\Pi_2^c) \\ &\sim_{E(\Pi^{\#})} (2\pi i)^{\frac{(r_1-1)r_1}{2}+\frac{(r_2-1)r_2}{2}} \Omega(\Pi^{\#}_\infty) L^S(1,\Pi_1\times\Pi_2) \prod_{j=1}^{r_1-1} P^{(j)}(\Pi_1) \prod_{k=1}^{r_2-1} P^{(k)}(\Pi_2^c). \end{aligned}$$

Therefore, for cases (A) and (B), we obtain that:

$$
\begin{aligned}
& p(\Pi)p(\Pi^{\#})p(m,\Pi_\infty,\Pi_\infty^{\#}) \\
\sim_{E(\Pi^{\#})}\ & (2\pi i)^{\frac{(r_1-1)r_1}{2}+\frac{(r_2-1)r_2}{2}}\Omega(\Pi_\infty^{\#})Z(\Pi_\infty)p(m,\Pi_\infty,\Pi_\infty^{\#})\times \\
& L^S(1,\Pi_1\times\Pi_2)\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{j=1}^{r_1-1}P^{(j)}(\Pi_1)\prod_{k=1}^{r_2-1}P^{(k)}(\Pi_2^c) \\
\sim_{E(\Pi^{\#})}\ & (2\pi i)^{n(n-1)(m+\frac{1}{2})-\frac{n(n-1)}{2}+\frac{(r_1-1)r_1}{2}+\frac{(r_2-1)r_2}{2}}L^S(1,\Pi_1\times\Pi_2)\times \\
& \prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{j=1}^{r_1-1}P^{(j)}(\Pi_1)\prod_{k=1}^{r_2-1}P^{(k)}(\Pi_2^c) \\
\sim_{E(\Pi^{\#})}\ & (2\pi i)^{n(n-1)(m+\frac{1}{2})-r_1r_2}L^S(1,\Pi_1\times\Pi_2)\prod_{i=1}^{n-1}P^{(i)}(\Pi)\times \\
& \prod_{j=0}^{r_1}P^{(j)}(\Pi_1)\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)
\end{aligned}
\tag{3.9}
$$

We have used Lemma **??**, the fact that $\binom{n-1}{2}=\binom{r_1+r_2}{2}=\binom{r_1}{2}+\binom{r_2}{2}+r_1r_2$ and also the fact that $P^{(0)}(\Pi_1)P^{(r_1)}(\Pi_1)\sim_{E(\Pi_1)}1$, $P^{(0)}(\Pi_2^c)P^{(r_2)}(\Pi_2^c)\sim_{E(\Pi_2)}1$.

For case (C), we only need to change $L^S(1,\Pi_1\times\Pi_2)$ to $L^S(\frac{1}{2},(\Pi_1\otimes\phi)\times\Pi_2)$ in the above formula.

3.4. **Final conclusion.** Comparing (3.4) and (3.9), we get for case (A):

$$
\begin{aligned}
L^S(1,\Pi_1\times\Pi_2)\ &\sim_{E(\Pi_1)E(\Pi_2)}\ (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi)-1}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi)-1} \\
&\sim_{E(\Pi_1)E(\Pi_2)}\ (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi_1^c)} \\
&\sim_{E(\Pi_1)E(\Pi_2)}\ (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(r_2-k)}(\Pi_2)^{sp(r_2-k,\Pi_2;\Pi_1)} \\
&\sim_{E(\Pi_1)E(\Pi_2)}\ (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1)}.
\end{aligned}
$$

Comparing (3.7) and (3.9), we get for case (B):

$$
\begin{aligned}
L^S(1,\Pi_1\times\Pi_2)\ &\sim_{E(\Pi_1)E(\Pi_2)}\ (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\phi;\Pi_2\otimes\phi^c)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2\otimes\phi^c;\Pi_1\otimes\phi)} \\
&\sim_{E(\Pi_1)E(\Pi_2)}\ (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1)}.
\end{aligned}
$$

Here we have used that $sp(j,\Pi_1\otimes\phi;\Pi_2\otimes\phi^c)=sp(j,\Pi_1\otimes\phi;\Pi_2\otimes\phi^{-1})=sp(j,\Pi_1;\Pi_2)$ by Lemma 1.1.

Similarly, for case (C), comparing (3.8) and (3.9), we obtain that:

$$\begin{aligned}
& L^S(\frac{1}{2},(\Pi_1\otimes\phi)\times\Pi_2) \\
\sim_{E(\Pi_1)E(\Pi_2)E(\phi)} \quad & (2\pi i)^{r_1r_2-\frac{nr_1}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\phi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\phi)}\times \\
& p(\check{\phi},1)^{\sum_{j=0}^{r_1} j*(sp(j,\Pi_1\otimes\phi;\Pi_2)+1)}\, p(\check{\phi},\iota)^{\sum_{j=0}^{r_1}(r_1-j)*(sp(j,\Pi_1\otimes\phi;\Pi_2+1))} \\
\sim_{E(\Pi_1)E(\Pi_2)E(\phi)} \quad & (2\pi i)^{\frac{r_1r_2}{2}-\frac{r_1(r_1+1)}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\phi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\phi)}\times \\
& p(\check{\phi},1)^{\sum_{j=0}^{r_1} j*sp(j,\Pi_1\otimes\phi;\Pi_2)+\frac{r_1(r_1+1)}{2}}\, p(\check{\phi},\iota)^{\sum_{j=0}^{r_1}(r_1-j)*sp(j,\Pi_1\otimes\phi;\Pi_2)+\frac{r_1(r_1+1)}{2}} \\
\sim_{E(\Pi_1)E(\Pi_2)E(\phi)} \quad & (2\pi i)^{\frac{r_1r_2}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\phi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\phi)}\times \\
& p(\check{\phi},1)^{\sum_{j=0}^{r_1} j*sp(j,\Pi_1\otimes\phi;\Pi_2)}\, p(\check{\phi},\iota)^{\sum_{j=0}^{r_1}(r_1-j)*sp(j,\Pi_1\otimes\phi;\Pi_2)} \\
\sim_{E(\Pi_1)E(\Pi_2)E(\phi)} \quad & (2\pi i)^{\frac{r_1r_2}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1\otimes\phi)^{sp(j,\Pi_1\otimes\phi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\phi)}.
\end{aligned}$$

The last step is deduced by definition of $P^{(*)}(\Pi_1\otimes\phi)$.

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