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Special values of automorphic L-functions for $GL_n \times GL_{n'}$ over CM fields, factorization and functoriality of arithmetic automorphic periods

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This thesis is dedicated to my family.

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Abstract

Résumé

Michael HARRIS a défini les périodes arithmétiques automorphes pour certaines représentations cuspidales de GL_n sur corps quadratiques imaginaires en 1997. Il a aussi montré que les valeurs critiques de fonctions L automorphes pour $GL_n \times GL_1$ peuvent être interprétées en termes de ces périodes. Dans la thèse, ses travaux sont généralisés sous deux aspects. D'abord, les périodes arithmétiques automorphes ont été définies pour tous corps CM. On montre aussi que ces périodes factorisent comme produits des périodes locales sur les places infinies. De plus, on montre que les valeurs critiques de fonctions L automorphes pour $GL_n \times GL_{n'}$ peuvent être interprétées en termes de ces périodes dans beaucoup de cas. Par conséquent on montre que les périodes sont fonctorielles pour l'induction automorphe et changement de base cyclique.

On aussi définit des périodes motiviques si le motif est restreint d'un corps CM au corps des nombres rationnels. On peut calculer la période de Deligne pour le produit tensoriel de deux tels motifs. On voit directement que nos résultats automorphes sont compatibles avec la conjecture de Deligne pour les motifs.

Mots-clefs

fonction L automorphe, fonctorialité de Langlands, la conjecture de Deligne, périodes automorphes, périodes motiviques

Abstract

Michael HARRIS defined the arithmetic automorphic periods for certain cuspidal representations of GL_n over quadratic imaginary fields in his Crelle paper 1997. He also showed that critical values of automorphic L-functions for $GL_n \times GL_1$ can be interpreted in terms of these arithmetic automorphic periods. In the thesis, we generalize his results in two ways. Firstly, the arithmetic automorphic periods have been defined over general CM fields. We also prove that these periods factorize as products of local periods over infinity places. Secondly, we show that critical values of automorphic L functions for $GL_n \times GL_{n'}$ can be interpreted in terms of these automorphic periods in many situations. Consequently we show that the automorphic periods are functorial for automorphic induction and cyclic base change.

We also define certain motivic periods if the motive is restricted from a CM field to the field of rational numbers. We can calculate Deligne's period for tensor product of two such motives. We see directly that our automorphic results are compatible with Deligne's conjecture for motives.

Keywords

automorphic L-function, Langlands functoriality, Deligne conjecture, automorphic periods, motivic periods

Contents

Introduction

Special values of L-functions play an important role in the Langlands program. Numerous conjectures predict that special values of L-functions reflect arithmetic properties of geometric objects. Most of these conjectures are still open and difficult to attack.

At the same time, concrete results on the special values of L-functions appear more and more in automorphic settings. For example, in [13], M. Harris constructed complex invariants called arithmetic automorphic periods and showed that the special values of automorphic L-function for $GL_n * GL_1$ could be interpreted in terms of these invariants.

We generalize his results in two ways. Firstly, the arithmetic automorphic periods have been defined over general CM fields. Secondly, we show that special values of arithmetic automorphic periods for $GL_n * GL_{n'}$ can be interpreted in terms of these arithmetic automorphic periods in many situations. In fact, we have found a concise formula for such critical values. This is our first main automorphic result. One possible application is to construct p-adic L-functions.

We remark that we have not finished the proof for $GL_n * GL_1$ over general CM fields in the current article. We shall do it later. We have assumed Conjecture 5.1.1 throughout the text. This is one important ingredient for automorphic results over general CM fields.

The results over quadratic imaginary field follow from the ideas in [8] and some technical calculation. Over general CM fields, one can still follow such arguments and get formulas for critical values in terms of arithmetic automorphic periods. But these formulas are ugly and complicated. In fact, we don't know how to write down a formula adapted to most cases. However, if one can show that the arithmetic automorphic periods can be factorized as products of local periods over infinite places, then the generalization to CM fields is straight forward.

The factorization of arithmetic automorphic periods was actually a conjecture of Shimura (c.f. [28], [29]). One possible way to show this is to define local periods geometrically and prove that special values of L-functions can be interpreted in terms of local periods. This was done by M. Harris for Hilbert modular forms in [11]. But it is extremely difficult to generalize his arguments to GL_n. Instead, we show that there are relations between arithmetic automorphic periods. These relations lead to a factorization which is our second main automorphic result.

We remark that the factorization is not unique. We show that there is a natural way to factorize such that the local periods are functorial for automorphic induction and base change. This is our third main automorphic result. We believe that local periods are also functorial for endoscopic transfer. We will try to prove this in the near future.

Although our local periods are not defined geometrically, they must have geometric meanings. This may be done by defining certain geometric invariants and show that they are related to our local periods with the help of special values of L-functions. It is likely to show that our local periods are equal to the geometric invariants defined in [11] for Hilbert modular forms in this way.

On the other hand, Deligne's conjecture related critical values for motives over $\mathbb{Q}$ and Deligne's period (c.f. [7]). When the motive is the restriction to $\mathbb{Q}$ of the tensor product of two motives over a CM field, we may calculate Deligne's period in terms of motivic periods defined in [16]. The formula was first given in [16] when the motives are self-dual. We have dropped the self-dual condition here.

If the two motives are associated to automorphic representations of GL_n and $GL_{n'}$ respectively, we may define motivic periods which are analogues of the arithmetic automorphic periods. We get a formula of Deligne's period in terms of these motivic periods. Our main motivic result says that our formula for automorphic L-functions are at least formally compatible with Deligne's conjecture.

Theorems:

Let K be a quadratic imaginary field and $F \supset K$ be a CM field of degree d over K. We fix an embedding $K \hookrightarrow \mathbb{C}$. Let $\Sigma_{F;K}$ be the set of embeddings $\sigma : F \hookrightarrow \mathbb{C}$ such that $\sigma\mid_K$ is the fixed embedding.

Let E be a number field. Let $\{a(\sigma)\}_{\sigma\in Aut(\mathbb{C}/K)}$, $\{b(\sigma)\}_{\sigma\in Aut(\mathbb{C}/K)}$ be two families of complex numbers. Roughly speaking, we say $a \sim_{E;K} b$ if $a = b$ up to multiplication by elements in $E^\times$ and equivariant under G_K-action.

Let Π be a cuspidal cohomological representation of $GL_n(\mathbb{A}_F)$ which has definable arithmetic automorphic periods (c.f. Definition 5.3.2). In particular, we know that Π_f is defined over a number field $E(\Pi)$. For any $I : \Sigma_{F;K} \to \{0, 1, \cdots, n\}$, we may define the arithmetic automorphic periods $P^{(I)}(\Pi)$ as the Petersson inner product of a rational vector in a certain cohomology space associated to a unitary group of infinity sign I. It is a non zero complex number well defined up to multiplication by elements in $E(\Pi)^\times$.

We assume that Conjecture 5.1.1 is true. Our second main automorphic result mentioned above is as follows (c.f. Theorem 7.6.1):

Theorem 0.0.1. *If conditions in Theorem 7.5.1 are satisfied, in particular, if Π is regular enough, then there exists some complex numbers $P^{(s)}(\Pi,\sigma)$ unique up to multiplication by elements in $(E(\Pi))^\times$ such that the following two conditions are satisfied:*

1. $P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod\limits_{\sigma\in\Sigma_{F;K}} P^{(I(\sigma))}(\Pi,\sigma)$ *for all* $I = (I(\sigma))_{\sigma\in\Sigma_{F;K}} \in \{0, 1, \cdots, n\}^{\Sigma_{F;K}}$
2. *and* $P^{(0)}(\Pi,\sigma) \sim_{E(\Pi);K} p(\breve{\xi_\Pi}, \overline{\sigma})$

where ξ_Π *is the central character of* Π, $\breve{\xi_\Pi} := \xi_\Pi^{-1,c}$ *and* $p(\breve{\xi_\Pi}, \overline{\sigma})$ *is the CM period (c.f. Section 4.1).*

We now introduce our first main automorphic result. Let Π' be a cuspidal cohomological representation of $GL_{n'}(\mathbb{A}_F)$ which has definable arithmetic automorphic periods. For any $\sigma \in \Sigma_{F;K}$, we may define the split indices $sp(j,\Pi;\Pi',\sigma)$ and $sp(k,\Pi';\Pi,\sigma)$ for $0 \leqslant j \leqslant n$ and $0 \leqslant k \leqslant n'$ (c.f. Definition 1.2.1). Roughly speaking, we have:

Theorem 0.0.2. *If* $m \in \mathbb{Z} + \frac{n+n'}{2}$ *is critical for* $\Pi \times \Pi'$ *then*

$$L(m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{nn'md} \prod_{\sigma\in\Sigma_{F;K}} (\prod_{j=0}^{n} P^{(j)}(\Pi,\sigma)^{sp(j,\Pi;\Pi',\sigma)} \prod_{k=0}^{n'} P^{(k)}(\Pi',\sigma)^{sp(k,\Pi';\Pi,\sigma)})$$

in the following cases:

1. $n' = 1$ *and* m *is bigger than the central value.*
2. $n > n'$ *and* $m \geqslant 1/2$, *both* Π *and* Π' *are conjugate self-dual and the pair* (Π,Π') *is in good position (c.f. Definition 1.2.2).*
3. $m = 1$, *both* Π *and* Π' *are conjugate self-dual and the pair* (Π,Π') *is regular enough.*

Our third main automorphic result says that the periods are functorial for automorphic induction and base change. Roughly speaking, we have:

Theorem 0.0.3. *(a) Let* $\mathcal{F}/F$ *be a cyclic extension of CM fields of degree* l *and* $\Pi_{\mathcal{F}}$ *be a cuspidal representation of* $GL_n(\mathbb{A}_{\mathcal{F}})$. *We write* $AI(\Pi_{\mathcal{F}})$ *for the automorphic induction of* $\Pi_{\mathcal{F}}$. *We assume both* $AI(\Pi_{\mathcal{F}})$ *and* $\Pi_{\mathcal{F}}$ *have definable arithmetic automorphic periods.*

Let $I_F \in \{0,1,\cdots,nl\}^{\Sigma_{F;K}}$. *We may define* $I_{\mathcal{F}} \in \{0,1,\cdots,n\}^{\Sigma_{\mathcal{F};K}}$ *as in Lemma 8.2.1. Or locally let* $0 \leqslant s \leqslant nl$ *be an integer and* $s(\cdot)$ *be as in Definition 8.3.1. We have:*

$$P^{(I_F)}(AI(\Pi_{\mathcal{F}})) \sim_{E(\Pi_{\mathcal{F}});K} P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}})$$
$$\text{or locally } P^{(s)}(AI(\Pi_{\mathcal{F}},\tau) \sim_{E(\Pi_{\mathcal{F}});K} \prod_{\sigma|\tau} P^{(s(\sigma))}(\Pi_{\mathcal{F}},\sigma).$$

(b) Let π_F *be a cuspidal representation of* $GL_n(\mathbb{A}_F)$. *We write* $BC(\pi_F)$ *for its strong base change to* $\mathcal{F}$. *We assume that both* π_F *and* $BC(\pi_F)$ *have definable arithmetic automorphic periods.*

Let $I_F \in \{0,1,\cdots,n\}^{\Sigma_{F;K}}$. *We write* $I_{\mathcal{F}}$ *the composition of* I_F *and the restriction of complex embeddings of* $\mathcal{F}$ *to* F.

We then have:

$$P^{(I_{\mathcal{F}})}(BC(\pi_F)) \sim_{E(\pi_F);K} p^{I_F}(\pi_F)^l$$
$$\text{or locally} \quad P^{(s)}(BC(\pi_F),\sigma)^l \sim_{E(\pi_F);K} P^{(s)}(\pi_F,\sigma\mid_F)^l.$$

Consequently, we know

$$P^{(s)}(BC(\pi_F),\sigma) \sim_{E(\pi_F)} \lambda^{(s)}(\pi_F,\sigma)P^{(s)}(\pi_F,\sigma\mid_F).$$

where $\lambda^{(s)}(\pi_F,\sigma)$ *is an algebraic number whose* l*-th power is in* $E(\pi_F)^\times$.

We now introduce the motivic results. Let M, M' be motives over F with coefficients in E and E' of rank n and n' respectively. We assume that $M\otimes M'$ has no $(\omega/2,\omega/2)$-class. We may define motivic periods $Q^{(t)}(M,\sigma)$ for $0\leqslant t\leqslant n$ and $\sigma\in\Sigma_{F;K}$. We can calculate Deligne's period of $Res_{F/\mathbb{Q}}(M\otimes M')$ in terms of these periods. If M and M' are motives associated to Π and Π', Deligne's conjecture is equivalent to the following conjecture:

Conjecture 0.0.1. *If $m\in\mathbb{Z}+\frac{n+n'}{2}$ is critical for $\Pi\times\Pi'$ then*

$$L(m,\Pi\times\Pi')=L(m+\tfrac{n+n'-2}{2},M\otimes M')$$
$$\sim_{E(\Pi)E(\Pi');K}(2\pi i)^{mnn'd}\prod_{\sigma\in\Sigma_{F;K}}(\prod_{j=0}^{n}Q^{(j)}(M,\sigma)^{sp(j,\Pi;\Pi',\sigma)}\prod_{k=0}^{n'}Q^{(k)}(M',\sigma)^{sp(k,\Pi';\Pi,\sigma)})$$

We see that it is compatible with Theorem 0.0.2. The main point of the proof is to fix proper basis. Deligne's period is defined by rational basis. The basis that we have fixed are not rational. But they are rational up to unipotent transformation matrices. We can still use such basis to calculate determinant.

Idea of the proof for automorphic results:

Blasius has shown that special values of L-functions for Hecke character are related to CM periods. The proof of our automorphic results involve this fact and the following three main ingredients:

Ingredient A is Theorem 5.2.1. If follows from Conjecture 5.1.1. It says that if χ is a Hecke characters then critical values $L(m,\Pi\otimes\chi)$ can be written in terms of the arithmetic automorphic periods of Π and CM periods of χ.

Ingredient B is Theorem 3.9 of [8]. It says that if $\Pi^{\#}$ is a certain automorphic representation of $GL_{n-1}(\mathbb{A}_F)$ such that $(\Pi,\Pi^{\#})$ is in good position then critical values $L(m,\Pi\otimes\Pi')$ are products of the Whittaker period $p(\Pi)$, $p(\Pi^{\#})$ and an archimedean factor. The advantage of the results in [8] is that we don't need $\Pi^{\#}$ to be cuspidal. This gives us large freedom to choose $\Pi^{\#}$.

Ingredient C is a calculation of Whittaker period $p(\Pi^{\#})$ when $\Pi^{\#}$ is the Langlands sum of cuspidal representations $\Pi_1,\cdots,\Pi_l$. Following the idea in [23] and [8], we know $p(\Pi^{\#})$ equals to product of $p(\Pi_i)$ and the value at identity of a certain Whittaker function. Shahidi's calculation in [27] shows that the latter is related to $\prod\limits_{1\leqslant i<j\leqslant l}L(1,\Pi_i\times\Pi_j^c)$.

The proof of the case (a) in Theorem 0.0.3 is relatively simple. It is enough to take suitable algebraic Hecke character η of F and calculate $L(m,AI(\Pi_{\mathcal{F}})\otimes\eta)=L(m,\Pi_{\mathcal{F}}\otimes\eta\circ N_{\mathbb{A}_{\mathcal{F}}^{\times}/\mathbb{A}_F^{\times}})$ by ingredient A.

The idea for the case (b) is similar. But we have to show that the arithmetic automorphic periods of $BC(\pi_F)$ are $Gal_{\mathcal{F}/F}$-invariant. This is due to the fact that $BC(\pi_F)$ itself is $Gal_{\mathcal{F}/F}$-invariant.

We now explain the proof for Theorem 0.0.1 and Theorem 0.0.2.

Step 0: determine when a function can factorize through each factor.

For example, let X, Y be two sets and f be a map from $X \times Y$ to $\mathbb{C}^\times$. Then there exists functions $g : X \to \mathbb{C}^\times$ and $h : Y \to \mathbb{C}^\times$ such that $f(x,y) = g(x)h(y)$ for any $x \in X$ and $y \in Y$ if and only if $f(x,y)f(x',y') = f(x,y')f(x',y)$ for any $x, x' \in X$ and $y, y' \in Y$. Therefore, to show that the arithmetic automorphic periods factorize is equivalent to show that there are certain relations between these periods.

Step 1: interpret $p(\Pi)$ in terms of arithmetic automorphic periods.

The idea is the same as in [8]. We take $\Pi^{\#}$ to be the Langlands sum of Hecke characters $\chi_1, \cdots, \chi_{n-1}$. We have $L(m, \Pi \times \Pi^{\#}) = \prod\limits_{1 \leqslant i \leqslant n-1} L(m, \Pi \otimes \chi_i)$.

Ingredient B says that the left hand side equals to the product of $p(\Pi)$, $p(\Pi^{\#})$ and an archimedean factor. Ingredient C tells us that $p(\Pi^{\#})$ is almost $\prod\limits_{1 \leqslant i < j \leqslant l} L(1, \chi_i \times \chi_j^c)$ which equals to product of CM periods by Blasius's result. Therefore, the left hand side equals to product of $p(\Pi)$, the CM periods of χ_i and an archimedean factor.

We may calculate the right hand side by Ingredient A. We get that the right hand side equals to product of the arithmetic automorphic periods of Π, the CM periods of χ_i and a power of $2\pi i$.

Comparing both sides, we will see unsurprisingly that the CM periods of χ_i in two sides coincide. We will get a formula for $p(\Pi)$ in terms of arithmetic automorphic periods. Varying the Hecke characters χ_i, we get different formulas for $p(\Pi)$ in terms of arithmetic automorphic periods. We then deduce relations between arithmetic automorphic periods. The factorization property then follows.

We remark that the above procedure can only treat the case when $I(\sigma) \neq 0$ or n for all σ. The proof for general case is more tricky (see section 7.5).

Step 2: repeat step 1 with suitable $\Pi^{\#}$.

For example, if $n > n'$ and the pair (Π, Π') is in good position, we may take $\Pi^{\#}$ to be the Langlands sum of Π' and some Hecke characters $\chi_1, \chi_2, \cdots, \chi_l$ where $l = n - n' - 1$ such that $(\Pi, \Pi^{\#})$ is in good position. We have $L(m, \Pi \times \Pi^{\#}) = L(m, \Pi \times \Pi') \prod\limits_{1 \leqslant i \leqslant l} L(\Pi \otimes \chi_i)$.

Again, we calculate the left hand side by ingredient B and ingredient C. We apply step 1 to $p(\Pi)$ and $p(\Pi')$ and we will get that the left hand side equals to product of arithmetic automorphic periods for Π and Π' and CM periods of χ_i.

We then apply ingredient A to $L(\Pi \otimes \chi_i)$ and compare both sides. We will get a formula for $L(m, \Pi \times \Pi')$.

For the case where $m = 1$, we may take $\Pi^{\#}$ to be the Langlands sum of Π and Π'^c. We know that $L(1, \Pi \times \Pi')$ then appears in the calculation of $p(\Pi^{\#})$ by ingredient C.

Step 3: Simplify the archimedean factors.

Once we get a formula of $L(m, \Pi \times \Pi')$ in terms of arithmetic automorphic periods, we may replace Π and Π' by representations which are automorphic inductions of Hecke characters. Blasius's result says that $L(m, \Pi \times \Pi')$ is equivalent to the product of a power of $2\pi i$ and some CM periods. On the other hand, the arithmetic automorphic periods of Π are related to CM periods by Theorem 0.0.3. We shall deduce that the archimedean factor is equivalent to a power of $2\pi i$ if Π and Π' are induced from Hecke characters. We can finish the proof by noticing that such representations can have any infinity type.

Plan for the text:

In chapter 1 we introduce our basic notation, in particular, the split index.

In chapter 2 we introduce the base change theory for similitude unitary groups which will help us understanding the descending condition in the definition of arithmetic automorphic periods.

We summarize some results on rational structures in Chapter 3. They play an important role in the proof. In particular, the ingredients B and C are introduced in the second half of this chapter.

In chapter 4 we construct the arithmetic automorphic periods. We generalize the construction of [13] to general CM fields.

Chapter 5 contains the details for ingredient A. We remark that we have made a hypothesis here (c.f. Conjecture 5.1.1). We will prove it in a forthcoming paper.

The motivic results are contained in Chapter 6. This chapter is independent of others. We show that our main automorphic results are compatible with Deligne's conjecture for motives.

We prove the factorization of arithmetic automorphic periods in Chapter 7 (c.f. Theorem 0.0.1). This result itself is very important. It is also the crucial step to generalize our results to CM fields.

In chapter 8 we prove that the global and local arithmetic periods are functorial for automorphic induction and base change (c.f. Theorem 0.0.3). This is a direct corollary of the ingredient A in chapter 5 and the factorization property in chapter 7.

In chapter 9 we claim our main conjecture which is an automorphic analogue of Deligne's conjecture. We also claim our main theorem there, namely, Theorem 0.0.2. Moreover, in the last section of this chapter, we explain why the generalization from quadratic imaginary fields to CM fields is direct by the factorization property.

The last two chapters contain the details of the proof for Theorem 0.0.2. The calculation is not difficult but technique.

Chapter 1

Notation

1.1 Basic notation

We fix an algebraic closure $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ of $\mathbb{Q}$ and $K \hookrightarrow \overline{\mathbb{Q}}$ a quadratic imaginary field. We denote by ι the complex conjugation of the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}$.

We denote by c the complex conjugation on $\mathbb{C}$. Via the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, it can be considered as an element in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

For any number field L, let $\mathbb{A}_L$ be the adele ring of L and $\mathbb{A}_{L,f}$ be the finite part of $\mathbb{A}_L$. We denote by Σ_L the set of embeddings from L to $\overline{\mathbb{Q}}$. If L contains K, we write $\Sigma_{L;K}$ for the subset of Σ_L consisting of elements which is the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}$ when restricted to K.

Throughout the text, we fix ψ an algebraic Hecke character of K with infinity type $z^1\overline{z}^0$ such that $\psi\psi^c = ||\cdot||_{\mathbb{A}_K}$ (see Lemma 4.1.4 of [6] for its existence). It is easy to see that the restriction of $||\cdot||_{\mathbb{A}_K}^{\frac{1}{2}}\psi$ to $\mathbb{A}_{\mathbb{Q}}^{\times}$ is the quadratic character associated to the extension $K/\mathbb{Q}$ by the class field theory. Consequently our construction is compatible with that in [8].

Let F^+ be a totally real field of degree d over $\mathbb{Q}$. We define $F := F^+K$ a CM field. We take ψ_F an algebraic Hecke character of F with infinity type z^1 at each $\sigma \in \Sigma$ such that $\psi_F\psi_F^c = ||\cdot||_{\mathbb{A}_F}$.

For $z \in \mathbb{C}$, we write $\bar{z}$ for its complex conjugation. For $\sigma \in \Sigma_F$, we define $\bar{\sigma} := \sigma^c$ the complex conjugation of σ.

Let η be a Hecke character of F. We define $\check{\eta} := \eta^{-1,c}$ and $\tilde{\eta} := \eta/\eta^c$ two Hecke characters of F.

Let n be an integer greater or equal to 2.

Definition 1.1.1. *Let N be an integer and Π be an automorphic representation of $GL_n(\mathbb{A}_F)$. Let σ be an element in $\Sigma_{F;K}$. We denote the infinity type of Π at σ by $(z^{a_i(\sigma)}\overline{z}^{a_i(\sigma)'})_{1\leqslant i\leqslant n}$. We may assume that $a_1(\sigma) \geqslant a_2(\sigma) \geqslant \cdots \geqslant a_n(\sigma)$ for all $\sigma \in \Sigma_{F;K}$. The representation Π will be called:*

1. ***pure of weight*** $\omega(\Pi)$ *if* $a_i(\sigma) + a_i(\sigma)' = -\omega(\Pi)$ *for all* $1 \leqslant i \leqslant n$ *and all* σ*;*

2. ***algebraic*** *if* $a_i(\sigma), a_i(\sigma)' \in \mathbb{Z} + \frac{n-1}{2}$ *for all* $1 \leqslant i \leqslant n$ *and all* σ*;*

3. ***cohomological*** *if there exists W an irreducible algebraic finite dimensional representation of $GL_n(F \otimes_{\mathbb{Q}} \mathbb{R})$ such that $H^*(\mathfrak{g}_\infty, F_\infty; \Pi \otimes W) \neq 0$ (see section 3.3 for more details);*

4. ***regular*** *if it is pure and $a_i(\sigma) - a_{i+1}(\sigma) \geqslant 1$ for all $1 \leqslant i \leqslant n-1$ and all σ.*

5. *N-**regular** if it is pure and $a_i(\sigma) - a_{i+1}(\sigma) \geqslant N$ for all $1 \leqslant i \leqslant n-1$ and all σ.*

Finally, let $E \supset K$ be a number field. We now define the relation $\sim_{E;K}$.
Let $\{a(\sigma)\}_{\sigma \in Aut(\mathbb{C}/K)}$, $\{b(\sigma)\}_{\sigma \in Aut(\mathbb{C}/K)}$ be two families of complex numbers.

Definition 1.1.2. *We say $a \sim_{E;K} b$ if one of the following conditions is verified:*

(i) $a(\sigma) = 0$ for all $\sigma \in Aut(\mathbb{C}/K)$,

(ii) $b(\sigma) = 0$ for all $\sigma \in Aut(\mathbb{C}/K)$, or

(iii) $a(\sigma) \neq 0$, $b(\sigma) \neq 0$ for all σ and there exists $t \in E^\times$ such that $a(\sigma) = \sigma(t)b(\sigma)$ for all $\sigma \in Aut(\mathbb{C}/K)$.

Remark 1.1.1. *1. Note that this relation is symmetric but not transitive. More precisely, if $a \sim_{E;K} b$ and $a \sim_{E;K} c$, we do not know whether $b \sim_{E;K} c$ in general unless the condition $a \neq 0$ is provided.*

2. If $a \sim_{E;K} b$ with $b(\sigma) \neq 0$, we see that $\frac{a(\sigma)}{b(\sigma)}$ is contained in the Galois closure of E. In particular, it is an algebraic number.

3. If moreover, $a(\sigma) = a(\sigma')$ for $\sigma, \sigma' \in Aut(\mathbb{C})$ such that $\sigma|_E = \sigma'|_E$, we can then define $a(\sigma)$ for $\sigma \in \Sigma_{E;K}$ by taking any $\widetilde{\sigma} \in Aut(\mathbb{C})$, a lifting of σ, and define $a(\sigma) := a(\widetilde{\sigma})$. We identify $\mathbb{C}^{\Sigma_{E;K}}$ with $E \otimes_K \mathbb{C}$. We consider $A := (a(\sigma))_{\sigma \in \Sigma_{E;K}}$ as an element in $E \otimes_K \mathbb{C}$.

We assume the same condition for b and define B for b similarly. It is easy to verify that $a \sim_{E;K} b$ if and only if one of the three conditions is verified: $A = 0$, $B = 0$, or $B \in (E \otimes_K \mathbb{C})^\times$ and $AB^{-1} \in E^\times \subset (E \otimes_K \mathbb{C})^\times$.

We remark that our results will be in this case.

Lemma 1.1.1. *We assume $b(\sigma) \neq 0$ for all σ. We also assume that $a(\sigma) = a(\sigma')$ and $b(\sigma) = b(\sigma')$ if $\sigma|_E = \sigma'|_E$ for any $\sigma, \sigma' \in Aut(\mathbb{C})$. We have $a \sim_{E;K} b$ if and only if*

$$\tau\left(\frac{a(\sigma)}{b(\sigma)}\right) = \frac{a(\tau\sigma)}{b(\tau\sigma)}$$

for all $\tau \in Aut(\mathbb{C}/K)$ and $\sigma \in \Sigma_{E;K}$.

1.2 Split index and good position for automorphic pairs

Definition 1.2.1. ***(Split Index)***
Let n and n' be two positives integers.
Let Π and Π' be two regular pure representations of $GL_n(\mathbb{A}_F)$ and $GL_{n'}(\mathbb{A}_F)$ respectively. Let σ be an element of $\Sigma_{F;K}$. We denote the infinity type of Π and Π' at σ by $(z^{a_i(\sigma)}\overline{z}^{-\omega(\Pi)-a_i(\sigma)})_{1 \leqslant i \leqslant n}$, $a_1(\sigma) > a_2(\sigma) > \cdots > a_n(\sigma)$ and $(z^{b_j(\sigma)}\overline{z}^{-\omega(\Pi')-b_j(\sigma)})_{1 \leqslant j \leqslant n'}$,

$b_1(\sigma) > b_2(\sigma) > \cdots > b_{n'}(\sigma)$ respectively. We assume that $a_i(\sigma)+b_j(\sigma) \neq -\frac{\omega(\Pi)+\omega(\Pi')}{2}$ for all $1 \leqslant i \leqslant n$ all $1 \leqslant j \leqslant n'$ and all σ.

We split the sequence $(a_1(\sigma) > a_2(\sigma) > \cdots > a_n(\sigma))$ with the numbers

$$-\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_{n'}(\sigma) > -\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_{n'-1}(\sigma) > \cdots > -\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_1(\sigma).$$

This sequence is split into $n'+1$ parts. We denote the length of each part by

$$sp(0,\Pi';\Pi,\sigma), sp(1,\Pi';\Pi,\sigma), \cdots, sp(n',\Pi';\Pi,\sigma),$$

and call them the ***split indices****.*

Lemma 1.2.1. *Let n, n', Π and Π' be as in the above definition. Let σ be an element in $\Sigma_{F;K}$. Let η be an algebraic Hecke character of $\mathbb{A}_F$. Let $0 \leqslant j \leqslant n'$ be an integer. We have the following formulas:*

1. $\sum\limits_{i=0}^{n'} sp(i,\Pi';\Pi,\sigma) = n.$
2. $sp(j,\Pi';\Pi,\sigma) = sp(n'-j,\Pi'^c;\Pi^c,\sigma) = sp(n'-j,\Pi'^\vee;\Pi^\vee,\sigma).$
3. *For any $t,s \in \mathbb{R}$, $sp(j,\Pi'\otimes||\cdot||^t_{\mathbb{A}_K};\Pi,\sigma) = sp(j,\Pi';\Pi\otimes||\cdot||^s_{\mathbb{A}_K},\sigma) = sp(j,\Pi';\Pi,\sigma)$.*
4. *$sp(j,\Pi'\otimes\eta;\Pi,\sigma) = sp(j,\Pi';\Pi\otimes\eta,\sigma)$ and $sp(j,\Pi'\otimes\eta^c;\Pi,\sigma) = sp(j,\Pi'\otimes\eta^{-1};\Pi,\sigma)$. Similarly, $sp(j,\Pi';\Pi\otimes\eta^c,\sigma) = sp(j,\Pi';\Pi\otimes\eta^{-1},\sigma)$.*

The first two points of the above lemma are direct. For the remaining, we only need to notice that calculating the split index is nothing but comparing $a_i(\sigma)+b_j(\sigma)$ with $-\frac{\omega(\Pi)+\omega(\Pi')}{2}$.

Example 1.2.1. 1. *If $F^+ = \mathbb{Q}$, $n=5$, $n'=4$, $\omega(\Pi)=\omega(\Pi')=0$ and*

$$-b_4 > \boldsymbol{a}_1 > \boldsymbol{a}_2 > -b_3 > -b_2 > \boldsymbol{a}_3 > \boldsymbol{a}_4 > -b_1 > \boldsymbol{a}_5,$$

we have $sp(0,\Pi';\Pi)=0$, $sp(1,\Pi';\Pi)=2$, $sp(2,\Pi';\Pi)=0$, $sp(3,\Pi';\Pi)=2$ and $sp(4,\Pi';\Pi)=1$. We verify that $sp(0,\Pi';\Pi)+sp(1,\Pi';\Pi)+sp(2,\Pi';\Pi)+sp(3,\Pi';\Pi)+sp(4,\Pi';\Pi)=5$ as expected by the previous lemma.

2. *If $F^+=\mathbb{Q}$, $n'=n-1$, $\omega(\Pi)=\omega(\Pi')=0$ and $\boldsymbol{a}_1 > -b_{n-1} > \boldsymbol{a}_2 > -b_{n-2} > \cdots > \boldsymbol{a}_{n-1} > -b_1 > \boldsymbol{a}_n$, we have $sp(j,\Pi';\Pi)=1$ for all $0 \leqslant j \leqslant n-1$.*

   Moreover, $sp(k,\Pi;\Pi')=1$ for all $1 \leqslant k \leqslant n-1$, $sp(0,\Pi;\Pi')=0$ and $sp(n,\Pi;\Pi')=0$.

   We verify that $\sum\limits_{j=0}^{n'} sp(j,\Pi';\Pi)=n$ and $\sum\limits_{j=0}^{n} sp(j,\Pi;\Pi')=n-1$ as expected.

Definition 1.2.2. *We assume that $n > n'$. Let Π and Π' be as before. We say the pair(Π, Π') is **in good position** if for any $\sigma \in \Sigma_{F;K}$, the n' numbers*

$$-\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_{n'}(\sigma) > -\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_{n'-1}(\sigma) > \cdots > -\frac{\omega(\Pi)+\omega(\Pi')}{2} - b_{1}(\sigma).$$

lie in different gaps between $(a_1(\sigma) > a_2(\sigma) > \cdots > a_n(\sigma))$.

It is equivalent to saying that $sp(i, \Pi'; \Pi, \sigma) \neq 0$ for all $0 \leqslant i \leqslant n'$ and $\sigma \in \Sigma_{F;K}$. In particular, if $n' = n-1$, we know (Π, Π') is in good position if and only if $sp(i, \Pi'; \Pi, \sigma) = 1$ for all i and σ.

Chapter 2

Unitary groups and base change

2.1 Unitary groups

In this section, let $\mathcal{L}$ be an arbitrary number field and $\mathcal{F}/\mathcal{L}$ be a quadratic extension of number fields.

Let U_0 be the quasi-split unitary group over $\mathcal{L}$ of dimension n with respect to the extension $\mathcal{F}/\mathcal{L}$. We want to know the local behavior of inner forms of U_0. More generally, we will answer the following question:

Let G_0 be a connected reductive group over $\mathcal{L}$. If we are given $G_{(v)}$, an inner form of $G_{0,v}$ over $\mathcal{L}_v$ for each place v of $\mathcal{L}$, when does G, an inner form of G_0 over $\mathcal{L}$ such that $G_v \cong G_{(v)}$ for all v, exist?

The answer is given in section 2 of [4]. We recall some results there. We also refer to section 1.2 [18] for further details in the unitary group case.

The isomorphism classes of inner forms are classified by Galois cohomology. Let v be a place of $\mathcal{L}$. Let $L = \mathcal{L}$ or $\mathcal{L}_v$. There exists a bijection between the set of isomorphism classes of inner forms of $G_{0,L}$ and $H^1(L, G_0^{ad})$. Therefore, the global inner form exists if and only if the element in $\bigoplus_v H^1(\mathcal{L}_v, G_0^{ad})$ corresponding to the local datum is in the image of $H^1(\mathcal{L}, G_0^{ad}) \to \bigoplus_v H^1(\mathcal{L}_v, G_0^{ad})$.

We remark that if L is local then the quasi-split class corresponds to the trivial element of $H^1(L, G_0^{ad})$.

If we can calculate this Galois cohomology, then everything is done. Otherwise Kottwitz has given an alternate choice as follows.

For H a connected reductive group over L, we define $A(H) = A(H/L) :=$ the dual of $\pi_0(Z(\hat{H})^{G_L}$ where $\hat{H}$ is the neutral component of the dual group of H.

Let $A = A(G_0^{ad}/\mathcal{L})$ and $A_v = A(G_0^{ad}/\mathcal{L}_v)$.

Proposition 2.1.1. *There exists a natural map $H^1(\mathcal{L}_v, G_0^{ad}) \to A_v$. Moreover, it is an isomorphism when v is finite.*

The above proposition gives a morphism $\bigoplus_v H^1(\mathcal{L}_v, G_0^{ad}) \to \bigoplus_v A_v \to A$ where the latter is given by restriction.

Theorem 2.1.1. *The following sequence is exact:*

$$H^1(\mathcal{L}, G_0^{ad}) \to \bigoplus_v H^1(\mathcal{L}_v, G_0^{ad}) \to A.$$

In other words, the image of the first map equals the kernel of the second map.

By this theorem, our question turns to determine the kernel of the second map.

Let us now focus on unitary groups, namely when $G_0 = U_0$. Clozel has calculated A_v in the case when $\mathcal{L}$ is totally real and $\mathcal{F}$ is a quadratic imaginary extension over $\mathcal{L}$. We call it the **CM case**. This is enough for our purpose. Let us list some facts from [4]:

- If n is odd, then $A = 0$. In other words, any local datum $(U_{(v)})_v$ which is quasi-split at almost all places come from a global unitary group.
- If n is even, then
  1. $A \cong \mathbb{Z}/2$.
  2. $A_v \cong \mathbb{Z}/2$ if v is finite and inert. The non quasi-split unitary group corresponds to the non trivial element of $\mathbb{Z}/2$. The map $A_v \to A$ is identity if we identify both groups with $\mathbb{Z}/2$.
  3. $A_v \cong \mathbb{Z}/n$ if v is finite and split. The element corresponding to the unitary group of a division algebra generates A_v. The map $A_v \to A$ is the mod 2 map from $\mathbb{Z}/n$ to $\mathbb{Z}/2$.
  4. The real unitary group $U(p,q)$ has image $(p-q)/2 \mod 2$ in A.

Remark 2.1.1. *1. The idea of the proof for the last point is to consider the surjective map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G_0)$ where $T \subset G_0$ is the maximal elliptic torus over $\mathbb{R}$.*

The above calculation leads to the following theorem:

Theorem 2.1.2. *Let $F = F^+K$ Let I be as before. Let q be a finite place of $\mathbb{Q}$ inert in F^+ and split in F. There exists a Hermitian space V_I of dimension n over F with respect to F/F^+ such that the unitary group $U = U(V_I)$ over F^+ associated to V satisfies:*

- *At each $\sigma \in \Sigma$, U is of sign $(n - I(\sigma), I(\sigma))$.*
- *For $v \neq q$, a finite place of F^+, U_v is unramified;*
- *If n is even and $\sum_{\sigma\in\Sigma} \frac{n-2I(\sigma)}{2} \not\equiv 0 \mod 2$, then U_q is a division algebra. Otherwise U_q is also unramified.*

We denote by U_I the restriction of U from F^+ to $\mathbb{Q}$ and GU_I the rational similitude group associated to V_I, namely, for any $\mathbb{Q}$-algebra R,

$$GU_I(R) = \{g \in GL(V_I \otimes_{\mathbb{Q}} R) | (gv, gw) = \nu(g)(v,w), \nu(g) \in R^*\}. \tag{2.1}$$

2.2 General base change

Let G and G' be two connected quasi-split reductive algebraic groups over $\mathbb{Q}$. Let $\widehat{G}$ be the complex dual group of G. The Galois group $G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\widehat{G}$. We define the L**-group** of G by ${}^LG := \widehat{G} \rtimes G_{\mathbb{Q}}$ and we define ${}^LG'$ similarly. A group homomorphism between two L-groups ${}^LG \to {}^LG'$ is called an L**-morphism** if it is continuous, its restriction to $\widehat{G}$ is analytic and it is compatible with the projections of ${}^LG$ and ${}^LG'$ to $G_{\mathbb{Q}}$. If there exists an L-morphism ${}^LG \to {}^LG'$, the **Langlands' principal of functoriality** predicts a correspondence from automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$ to automorphic representations of $G'(\mathbb{A}_{\mathbb{Q}})$ (c.f. section 26 of [1]). More precisely, we wish to associate an L-packet of automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$ to that of $G'(\mathbb{A}_{\mathbb{Q}})$.

Locally, we can specify this correspondence for unramified representations. Let v be a finite place of $\mathbb{Q}$ such that G is unramified at v. We fix Γ_v a maximal compact hyperspecial subgroup of $G_v := G(\mathbb{Q}_v)$. By definition, for π_v an admissible representation of G_v, we say π_v is **unramified** (with respect to Γ_v) if it is irreducible and $dim\pi_v^{\Gamma_v} > 0$. One can show that $\pi_v^{\Gamma_v}$ is actually one dimensional since π_v is irreducible.

Denote $H_v := \mathcal{H}(G_v, \Gamma_v)$ the Hecke algebra consisting of compactly supported continuous functions from G_v to $\mathbb{C}$ which are Γ_v invariants on both sides. We know H_v acts on π_v and preserves $\pi_v^{\Gamma_v}$ (c.f. [24]). Since $\pi_v^{\Gamma_v}$ is one-dimensional, every element in H_v acts as a multiplication by a scalar on it. In other words, π_v thus determines a character of H_v. This gives a map from the set of unramified representations of G_v to the set of characters of H_v which is in fact a bijection (c.f. section 2.6 of [24]).

We can moreover describe the structure of H_v in a simpler way. Let T_v be a maximal torus of G_v contained in a Borel subgroup of G_v. We denote by $X_*(T_v)$ the set of cocharacters of T_v defined over $\mathbb{Q}_v$. The Satake transform identifies the Hecke algebra H_v with the polynomial ring $\mathbb{C}[X_*(T_v)]^{W_v}$ where W_v is the Weyl group of G_v (c.f. section 1.2.4 of [15]).

Let G' be a connected quasi-split reductive group which is also unramified at v. We can define Γ'_v, $H'_v := \mathcal{H}(G'_v, \Gamma'_v)$ and T'_v similarly. An L-morphism ${}^LG \to {}^LG'$ induces a morphism $\widehat{T_v} \to \widehat{T'_v}$ and hence a map $T'_v \to T_v$. The latter gives a morphism from $\mathbb{C}[X_*(T'_v)]^{W'_v}$ to $\mathbb{C}[X_*(T_v)]^{W_v}$ and thus a morphism from H'_v to H_v. Its dual hence gives a map from the set of unramified representations of G_v to that of G'_v. This is the **local Langlands's principal of functoriality** for unramified representations.

In this article, we are interested in a particular case of the Langlands' functoriality: the cyclic base change. Let $K/\mathbb{Q}$ be a cyclic extension, for example K is a quadratic imaginary field. Let G be a connected quasi-split reductive group over $\mathbb{Q}$. Let $G' = Res_{K/\mathbb{Q}}G_K$. We know $\widehat{G'}$ equals to $\widehat{G}^{[K:\mathbb{Q}]}$. The diagonal embedding is then a natural L-morphism ${}^LG \to {}^LG'$. The corresponding functoriality is called the base change.

More precisely, let v be a place of $\mathbb{Q}$ and w a place of K over v. The local Langlands's principal of functoriality gives a map from the set of unramified representations of $G(\mathbb{Q}_v)$ to that of $G(K_w)$. We call this map the base change with respect to $K_w/\mathbb{Q}_v$.

Let π be an admissible irreducible representation of $G(\mathbb{A}_{\mathbb{Q}})$. We say Π, a representation of $G(\mathbb{A}_K)$, is a **(weak) base change** of π if for almost all v, a finite place of $\mathbb{Q}$, such that π is unramified at v and all w, a place of K over v, Π_w is the base change of π_v. In this case, we say Π **descends to** π by base change.

Remark 2.2.1. *The group $Aut(K)$ acts on $G(\mathbb{A}_K)$. This induces an action of $Aut(K)$ on automorphic representations of $G(\mathbb{A}_K)$. For $\sigma \in Aut(K)$ and Π an automorphic representation of $G(\mathbb{A}_K)$, we write Π^σ to be the image of Π under the action of σ. It is easy to see that if Π is a base change of π, then Π^σ is one as well. In particular, we have $\Pi^\sigma_w \cong \Pi_w$ for almost every finite place w of K. So if we have the strong multiplicity one* *theorem for $G(\mathbb{A}_K)$, we can conclude that every representation in the image of base change is $Aut(K)$-stable.*

2.3 Base change for unitary groups and similitude unitary groups

Recall that $U_I(\mathbb{A}_K) \cong GL_n(\mathbb{A}_F)$. The following result on base change comes from Theorem 1.7 of [21]. We also refer to Corollary 2.5.9 of [25] for the quasi-split case.

Proposition 2.3.1. ***Base change for unitary group***
Let Π be a cuspidal conjugate self-dual and cohomological representation of $GL_n(\mathbb{A}_F)$. If n is odd then Π descends to a cohomological representation of $U_I(\mathbb{A}_\mathbb{Q})$ unconditionally. If n is even then it descends if Π_q descends locally.

We have an exact sequence $1 \to U_I \to GU_I \to \mathbb{G}_m \to 1$ which is split over K. Indeed, by Galois descent, it is enough to define θ_I, a Galois automorphism on $U_{I,F} \times \mathbb{G}_{m,K}$ such that the subgroup of $U_{I,K} \times \mathbb{G}_{m,K}$ fixed by θ_I is isomorphic to GU_I. We now define θ_I as follows:

For R a $\mathbb{Q}$-algebra, note that $(U_{I,K} \times \mathbb{G}_{m,K})(R) \cong GL(V_I \otimes_\mathbb{Q} R) \times (K \otimes_\mathbb{Q} R)$. We define

$$\theta_I : GL(V_I \otimes_\mathbb{Q} R) \times (K \otimes_\mathbb{Q} R) \to GL(V_I \otimes_\mathbb{Q} R) \times (K \otimes_\mathbb{Q} R)$$

by sending (g, z) to $((g^*)^{-1}\bar{z}, \bar{z})$ where g^* is the adjoint of g with respect to the Hermitian form. It is easy to verify that θ_I satisfies the condition mentioned above.

We then have that $GU_{I,K} \cong U_{I,K} \times \mathbb{G}_{m,K}$. In particular, $GU(\mathbb{A}_K) \cong GL_n(\mathbb{A}_F) \times \mathbb{A}_K^\times$. For Π a cuspidal representation of $GL_n(\mathbb{A}_F)$ and ξ a Hecke character of K, $\Pi \otimes \xi$ defines a cuspidal representation of $GU(\mathbb{A}_K)$. Conversely, by the tensor product theorem, every irreducible admissible automorphic representation of $GU(\mathbb{A}_K)$ is of the form $\Pi \otimes \xi$. The following Lemma is shown in [19] VI.2.10 and [5] Lemma 2.2.

Lemma 2.3.1. *If Π is algebraic and conjugate self-dual, then there exists ξ, an algebraic Hecke character of $\mathbb{A}_K$ such that $\Pi \times \xi$ is θ_I-stable.*

Proof It is easy to verify that $\Pi \times \xi$ is θ_I-stable if and only if $\frac{\xi(\bar{z})}{\xi(z)} = \xi_\Pi(z)$ for any $z \in \mathbb{A}_K^\times$ where ξ_Π is the central character of Π.

We define U the torus over $\mathbb{Q}$ such that $U(\mathbb{Q}) = ker\{Norm : K^\times \to \mathbb{Q}^\times\}$. We have $U(\mathbb{A}_\mathbb{Q}) = \{\frac{\bar{z}}{z} \mid z \in \mathbb{A}_K^\times\}$ by Hilbert 90.

Again by Hilbert 90, we have an exact sequence:

$$1 \to \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to K^\times \backslash \mathbb{A}_K^\times \to U(\mathbb{Q}) \backslash U(\mathbb{A}_\mathbb{Q}) \to 1$$

where the last map sends z to $\dfrac{\overline{z}}{z}$.

Therefore, such ξ exists if and only if ξ_Π is trivial on $\mathbb{A}_{\mathbb{Q}}^{\times}$.

Since Π is conjugate self-dual, we know ξ_Π is trivial on $Norm_{\mathbb{A}_K/\mathbb{A}_{\mathbb{Q}}}(\mathbb{A}_K^{\times})$. By class field theory, $\mathbb{Q}^{\times}N_{\mathbb{A}_K/\mathbb{A}_{\mathbb{Q}}}(\mathbb{A}_K^{\times})$ has index 2 in $\mathbb{A}_{\mathbb{Q}}^{\times}$. It remains to show that ξ_Π is trivial at any element t in $\mathbb{A}_{\mathbb{Q}}^{\times} - \mathbb{Q}^{\times}N_{\mathbb{A}_K/\mathbb{A}_{\mathbb{Q}}}(\mathbb{A}_K^{\times})$.

We consider $t \in \mathbb{A}_{\mathbb{Q}}^{\times}$ such that $t = 1$ at all finite places and $t = -1$ at the infinity place. It is not in $\mathbb{Q}^{\times}N_{\mathbb{A}_K/\mathbb{A}_{\mathbb{Q}}}(\mathbb{A}_K^{\times})$.

Since Π is algebraic, we know ξ_Π has infinity type $z^a\overline{z}^{-a}$ with $a \in \mathbb{Z}$. In particular, we have $\xi_\Pi(t) = 1$ as expected.

□

The following proposition follows from Proposition 2.3.1. The idea is the same with Theorem $VI.2.9$ in [19].

Proposition 2.3.2. *Let* Π *be an algebraic automorphic representation of* $GL_n(\mathbb{A}_F) \cong U_I(\mathbb{A}_K)$ *which descends to* $U_I(\mathbb{A}_F)$*. If* ξ *is an algebraic Hecke character of* K *as in Lemma 2.3.1 then* $\Pi \otimes \xi$ *descends to an automorphic representation of* $GU_I(\mathbb{A}_F)$*. Moreover, if* Π *is cohomological then its descending is also cohomological.*

In particular, let Π *be a cuspidal conjugate self-dual and cohomological representation of* $GL_n(\mathbb{A}_F)$*. We assume moreover that* Π_q *descends locally if* n *is even. Then there always exists* ξ *such that* $\Pi \otimes \xi$ *descends to an irreducible cohomological automorphic representation of* $GU_I(\mathbb{A}_F)$*.*

Chapter 3

Rational structures and Whittaker periods

In this chapter, we will recall some results on the rationality of certain algebraic automorphic representations and also the rationality of the associated Whittaker models and cohomology spaces.

In particular, we will define the Whittaker period and present a way to calculate the Whittaker period in certain cases.

3.1 Rational structures on certain automorphic representations

Let F is an arbitrary number field and n be a positive integer.
Let Π be an automorphic representation of $GL_n(\mathbb{A}_F)$.

We denote by V the representation space for Π_f. For $\sigma \in Aut(\mathbb{C})$, we define another $GL_n(\mathbb{A}_{F,f})$-representation Π_f^σ to be $V \otimes_{\mathbb{C},\sigma} \mathbb{C}$. Let $\mathbb{Q}(\Pi)$ be the subfield of $\mathbb{C}$ fixed by $\{\sigma \in Aut(\mathbb{C}) \mid \Pi_f^\sigma \cong \Pi_f\}$. We call it the **rationality field** of Π.

For E a number field, G a group and V a G-representation over $\mathbb{C}$, we say V has an **E-rational structure** if there exists an E-vector space V_E endowed with an action of G such that $V = V_E \otimes_E \mathbb{C}$ as representation of G. We call V_E an E-rational structure of V.

We denote by $\mathcal{A}lg(n)$ the set of algebraic automorphic representations of $GL_n(\mathbb{A}_F)$ which are isobaric sums of cuspidal representations as in section 1 of [3].

Theorem 3.1.1. *(Théorème 3.13 in [3])*
Let Π be a regular representation in $\mathcal{A}lg(n)$. We have that:

1. *$\mathbb{Q}(\Pi)$ is a number field.*

2. *Π_f has a $\mathbb{Q}(\Pi)$-rational structure unique up to homotheties.*

3. *For all $\sigma \in Aut(\mathbb{C})$, Π_f^σ is the finite part of a regular representation in $\mathcal{A}lg(n)$. It is unique up to isomorphism by the strong multiplicity one theorem. We denote it by Π^σ.*

Remark 3.1.1. *Let $n = n_1 + n_2 + \cdots + n_k$ be a partitian of positive integers and Π_i be regular representations in $\mathcal{A}lg(n_i)$ for $1 \leqslant i \leqslant k$ respectively.*

The above theorem implies that, for all $1 \leqslant i \leqslant k$, the rational field $\mathbb{Q}(\Pi_i)$ is a number field.

Let $\Pi = (\Pi_1 ||\cdot||_{\mathbb{A}_K}^{\frac{1-n_1}{2}} \boxplus \Pi_2 ||\cdot||_{\mathbb{A}_K}^{\frac{1-n_2}{2}} \boxplus \cdots \boxplus \Pi_k ||\cdot||_{\mathbb{A}_K}^{\frac{1-n_k}{2}}) ||\cdot||_{\mathbb{A}_K}^{\frac{n-1}{2}}$ be the normalized isobaric sum of Π_i. It is still algebraic.

We can see from definition that $\mathbb{Q}(\Pi)$ is the compositum of $\mathbb{Q}(\Pi_i)$ with $1 \leqslant i \leqslant k$. Moreover, if Π is regular, we know from the above theorem that Π has a $\mathbb{Q}(\Pi)$-rational structure.

3.2 Rational structures on the Whittaker model

Let Π be a regular representation in $\mathcal{A}lg(n)$ and then its rationality field $\mathbb{Q}(\Pi)$ is a number field.

We fix a nontrivial additive character ϕ of $\mathbb{A}_F$. Since Π is an isobaric sum of cuspidal representations, it is generic. Let $W(\Pi_f)$ be the Whittaker model associated to Π_f (with respect to ϕ_f). It consists of certain functions on $GL_n(\mathbb{A}_{F,f})$ and is isomorphic to Π_f as $GL_n(\mathbb{A}_{F,f})$-modules.

Similarly, we denote the Whittaker model of Π (with respect to) ϕ by $W(\Pi)$.

Definition 3.2.1. ***Cyclotomic character***
There exists a unique homomorphism $\xi : Aut(\mathbb{C}) \to \widehat{\mathbb{Z}}^\times$ such that for any $\sigma \in Aut(\mathbb{C})$ and any root of unity ζ, $\sigma(\zeta) = \zeta^{\xi(\sigma)}$, called the cyclotomic character.

For $\sigma \in Aut(\mathbb{C})$, we define $t_\sigma \in (\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_F)^\times = \widehat{\mathcal{O}_F}^\times$ to be the image of $\xi(\sigma)$ by the embedding $(\widehat{\mathbb{Z}})^\times \hookrightarrow (\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_F)^\times$. We define $t_{\sigma,n}$ to be the diagonal matrix

$$diag(t_\sigma^{-n+1}, t_\sigma^{-n+2}, \cdots, t_\sigma^{-1}, 1) \in GL_n(\mathbb{A}_{F,f})$$

as in section 3.2 of [26].

For $w \in W(\Pi_f)$, we define a function w^σ on $GL_n(\mathbb{A}_{F,f})$ by sending $g \in GL_n(\mathbb{A}_{F,f})$ to $\sigma(w(t_{\sigma,n}g))$. For classical cusp forms, this action is just the $Aut(\mathbb{C})$-action on Fourier coefficients.

Proposition 3.2.1. *(Lemma 3.2 of [26] or Proposition 2.7 of [8])*
The map $w \mapsto w^\sigma$ gives a σ-linear $GL_n(\mathbb{A}_{F,f})$-equivariant isomorphism from $W(\Pi_f)$ to $W(\Pi_f^\sigma)$.
For any extension E of $\mathbb{Q}(\Pi_f)$, we can define an E-rational structure on $W(\Pi_f)$ by taking the $Aut(\mathbb{C}/E)$-invariants.
Moreover, the E-rational structure is unique up to homotheties.

Proof The first part is well-known (see the references in [26]).

For the second part, the original proof in [26] works for cuspidal representations. The key point is to find a nonzero global invariant vector. It is equivalent to finding a nonzero local invariant vector for every finite place. Then Theorem 5.1(ii) of [20] is involved as in [8].

The last part follows from the one-dimensional property of the invariant vector which is the second part of Theorem 5.1(ii) of [20].

□

3.3 Rational structures on cohomology spaces and comparison of rational structures

Let Π be a regular representation in $\mathcal{A}lg(n)$. The Lie algebra cohomology of Π has a rational structure. It is described in section 3.3 of [26]. We give a brief summary here.

Let Z be the center of GL_n. Let $\mathfrak{g}_\infty$ be the Lie algebra of $GL_n(\mathbb{R}\otimes_{\mathbb{Q}} F)$. Let S_{real} be the set of real places of F, $S_{complex}$ be the set of complex places of F and $S_\infty = S_{real} \cup S_{complex}$ be the set of infinite places of F.

For $v \in S_{real}$, we define $K_v := Z(\mathbb{R})O_n(\mathbb{R}) \subset GL_n(F_v)$. For $v \in S_{complex}$, we define $K_v := Z(\mathbb{C})U_n(\mathbb{C}) \subset GL_n(F_v)$. We denote by K_∞ the product of K_v with $v \in S_\infty$, and by K_∞^0 the topological connected component of K_∞.

We fix T the maximal torus of GL_n consisting of diagonal matrices and B the Borel subgroup of G consisting of upper triangular matrices. For μ a dominant weight of $T(\mathbb{R}\otimes_{\mathbb{Q}} F)$ with respect to $B(\mathbb{R}\otimes_{\mathbb{Q}} F)$, we can define W_μ an irreducible representation of $GL_n(\mathbb{R}\otimes_{\mathbb{Q}} F)$ with highest weight μ.

From the proof of Théorème 3.13 [3], we know that there exists a dominant algebraic weight μ, such that $H^*(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu) \neq 0$.

Let b be the smallest degree such that $H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu) \neq 0$. We have an explicit formula for b in [26]. More precisely, we set r_1 and r_2 the numbers of real and complex embeddings of F respectively. We have $b = r_1 \left[\frac{n^2}{4}\right] + r_2 \frac{n(n-1)}{2}$.

We can decompose this cohomology group via the action of K_∞/K_∞^0. There exists a character ϵ of K_∞/K_∞^0 described explicitly in [26] such that:

1. The isotypic component $H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu)(\epsilon)$ is one dimensional.
2. For fixed w_∞, a generator of $H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu)(\epsilon)$, we have a $GL_n(\mathbb{A}_{F,f})$-equivariant isomorphisms:

$$\begin{aligned} W(\Pi_f) &\xrightarrow{\sim} W(\Pi_f) \otimes H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi_\infty \otimes W_\mu)(\epsilon) \\ &\xrightarrow{\sim} H^b(\mathfrak{g}_\infty, K_\infty^0; W(\Pi) \otimes W_\mu)(\epsilon) \\ &\xrightarrow{\sim} H^b(\mathfrak{g}_\infty, K_\infty^0; \Pi \otimes W_\mu)(\epsilon) \end{aligned} \tag{3.1}$$

where the first map sends w_f to $w_f \otimes w_\infty$ and the last map is given by the isomorphism $W(\Pi) \xrightarrow{\sim} \Pi$.

3. The cohomology space $H^b(\mathfrak{g}_\infty, K^0_\infty; \Pi \otimes W_\mu)(\epsilon)$ is related to the cuspidal cohomology if Π is cuspidal and to the Eisenstein cohomology if Π is not cuspidal. In both cases, it is endowed with a $\mathbb{Q}(\Pi)$-rational structure (see [26] for cuspidal case and [8] for non cuspidal case).

We denote by $\Theta_{\Pi_f,\epsilon,w_\infty}$ the $GL_n(\mathbb{A}_{F,f})$-isomorphism given in (3.1)

$$W(\Pi_f) \xrightarrow{\sim} H^b(\mathfrak{g}_\infty, K^0_\infty; \Pi \otimes W_\mu)(\epsilon).$$

Both sides have a $\mathbb{Q}(\Pi)$-rational structure. In particular, the preimage of the rational structure on the right hand side gives a rational structure on $W(\Pi_f)$. But the rational structure on $W(\Pi_f)$ is unique up to homotheties. Therefore, there exists a complex number $p(\Pi_f, \epsilon, w_\infty)$ such that the new map $\Theta^0_{\Pi_f,\epsilon,w_\infty} = p(\Pi_f, \epsilon, w_\infty)^{-1}\Theta_{\Pi_f,\epsilon,w_\infty}$ preserves the rational structure on both sides. It is easy to see that this number $p(\Pi_f, \epsilon, w_\infty)$ is unique up to multiplication by elements in $\mathbb{Q}(\Pi)^\times$.

Finally, we observe that the $Aut(\mathbb{C})$-action preserves rational structures on both the Whittaker models and cohomology spaces. We can adjust the numbers $p(\Pi^\sigma_f, \epsilon^\sigma, w^\sigma_\infty)$ for all $\sigma \in Aut(\mathbb{C})$ by elements in $\mathbb{Q}(\Pi)^\times$ such that the following diagram commutes:

$$\begin{array}{ccc} W(\Pi_f) & \xrightarrow{p(\Pi_f,\epsilon,w_\infty)^{-1}\Theta_{\Pi_f,\epsilon,w_\infty}} & H^b(\mathfrak{g}_\infty, K^0_\infty; \Pi \otimes W_\mu)(\epsilon) \\ \downarrow \sigma & & \downarrow \sigma \\ W(\Pi^\sigma_f) & \xrightarrow{p(\Pi^\sigma_f,\epsilon^\sigma,w^\sigma_\infty)^{-1}\Theta_{\Pi^\sigma_f,\epsilon^\sigma,w^\sigma_\infty}} & H^b(\mathfrak{g}_\infty, K^0_\infty; \Pi^\sigma \otimes W^\sigma_\mu)(\epsilon^\sigma) \end{array}$$

The proof is the same as the cuspidal case in [26].

In the following, we fix ϵ, w_∞ and we define the **Whittaker period** $p(\Pi) := p(\Pi_f, \epsilon, w_\infty)$. For any $\sigma \in Aut(\mathbb{C})$, we define $p(\Pi^\sigma) := p(\Pi^\sigma_f, \epsilon^\sigma, w^\sigma_\infty)$. It is easy to see that $p(\Pi^\sigma) = p(\Pi)$ for $\sigma \in Aut(\mathbb{C}/\mathbb{Q}(\Pi))$.

Moreover, the elements $(p(\Pi^\sigma))_{\sigma\in Aut(\mathbb{C})}$ are well defined up to $\mathbb{Q}(\Pi)^\times$ in the following sense: if $(p'(\Pi^\sigma))_{\sigma\in Aut(\mathbb{C})}$ is another family of complex numbers such that $p'(\Pi^\sigma)^{-1}\Theta_{\Pi^\sigma_f,\epsilon^\sigma,w^\sigma_\infty}$ preserves the rational structure and the above diagram commutes, then there exists $t \in \mathbb{Q}(\Pi)^\times$ such that $p'(\Pi^\sigma) = \sigma(t)p(\Pi^\sigma)$ for any $\sigma \in Aut(\mathbb{C})$. This also follows from the one dimensional property of the invariant vector. The argument is the same as the last part of the proof of Definition/Proposition 3.3 in [26].

3.4 Shahidi's calculation on Whittaker periods

Let us assume F is a CM field in the following sections of this chapter. In this case, K_∞ itself is connected and hence we may omit the index ϵ.

let l be a positive integer. Let n_1, n_2, $\cdots$,n_l be positive integers such that $n = n_1 + n_2 + \cdots + n_l$.

Let Π_1, Π_2, $\cdots$, Π_l be regular cohomological conjugate self-dual automorphic representations of GL_{n_1}, GL_{n_2}, $\cdots$, GL_{n_l} respectively. We assume that they are Langlands sum of cuspidal representations.

We write $P \leqslant GL_n$ for the maximal parabolic group of type $(n_1, n_2, \cdots, n_l)$ and $B \leqslant P$ be the corresponding Borel subgroup.

Let Π be the Langlands sum of $\Pi_1, \Pi_2, \cdots, \Pi_l$.

Proposition 3.4.1. *There exists a non zero complex number $\Omega_{(n_1,n_2,\cdots,n_l)}(\Pi_\infty)$ depending on Π_∞ and the parabolic type of P which is unique up to elements of $E(\Pi)^\times$ such that:*

$$p(\Pi) \sim_{E(\Pi_1)\cdots E(\Pi_l);K} \Omega_{(n_1,n_2,\cdots,n_l)}(\Pi_\infty) \prod_{1\leqslant i\leqslant l} p(\Pi_i) \prod_{1\leqslant i<j\leqslant l} L(1, \Pi_i \times \Pi_j^\vee) \tag{3.2}$$

The constructions and ideas come from [23] and [8]. We give a sketch of the proof here and will include the details in a forthcoming paper.

Sketch of the proof: For simplicity, we may assume that $l = 2$.

For $i = 1$ or 2, we denote by V_i the representation space for Π_i consisting of cusp forms.

We denote by $W_{i,f}$ the Whittaker model for $\Pi_{i,f}$. We write μ_i for the cohomological type of Π_i. We fix $\omega_{i,\infty}$, a generator of $H^{b_{n_i}}(\mathfrak{g}_{i,\infty}, K_{i,\infty}; \Pi_{i,\infty} \otimes W_{\mu_i})$ and we write Θ_i for $\Theta_{\Pi_{i,f},\omega_i}$.

We write $H^{b_i}(\Pi_i \otimes W_{\mu_i}) := H^{b_i}(\mathfrak{g}_{i,\infty}, K_{i,\infty}; \Pi_i \otimes W_{\mu_i})$ for simplicity.

We use similar notation for $\Pi = \Pi_1 \boxplus \Pi_2$.

We claim that there exists a commutative diagram as follows (c.f. (1.3) of [23]):

$$\begin{array}{ccc} W_{1,f} \otimes W_{2,f} & \xrightarrow{\Theta_1 \otimes \Theta_2} & H^{b_1}(\Pi_1 \otimes W_{\mu_1}) \otimes H^{b_2}(\Pi_2 \otimes W_{\mu_2}) \cong V_{1,f} \otimes V_{2,f} \\ \downarrow \mathcal{F}^{loc} & & \downarrow Eis \\ W(\Pi_f) & \xrightarrow{\Theta} & H^b(\Pi \otimes W_\mu) \supset V_f \end{array}$$

We now introduce the maps which appear in the diagram:

- The map $\mathcal{F}^{loc}$ is an explicit map defined locally in [23].
- The map Eis is rational and defined by the theory of Eisenstein series which sends $V_{1,f} \otimes V_{2,f}$ to V_f. (c.f. Section 1.1 in [23]).
- The isomorphism $V_{i,f} \cong H^{b_i}(\Pi_i \otimes W_{\mu_i})$ is rational. The composition of this isomorphism and Θ_i is just the isomorphism between the cuspidal forms and the corresponding Whittaker functions.
- The theory of Eisenstein cohomology (c.f. [9]) gives a rational embedding of V_f in $H^b(\Pi \otimes W_\mu)$.

  More precisely, we write S_n for $GL_n(F)\backslash GL_n(\mathbb{A}_F)/K_\infty$. We denote by $\overline{S}_n$ the Borel-Serre compactification of S_n. We write $\partial\overline{S}_n$ for its boundary and $\partial_B\overline{S}_n$ for the face corresponding to the Borel subgroup B.

  We know $H^b(\Pi \otimes W_\mu)$ embeds rationally in $H^b(\overline{S}_n, \mathcal{E}_\mu)$ (c.f. Section 3 of [8] and Section 1.2 of [9]) where $\mathcal{E}_\mu$ is a sheaf on S_n defined by μ. We restrict the latter to the face $\partial_B\overline{S}_n$ and get a rational map $H^b(\Pi \otimes W_\mu) \to H^b(\partial_B\overline{S}_n, \mathcal{E}_\mu)$ which admits a rational section (c.f. Proposition 5.2 of [8]).

We may decompose $H^b(\partial_B\overline{S}_n, \mathcal{E}_\mu)$ as in Theorem 4.2 of [23] and we see that $V_f = Ind_P(V_{1,f} \otimes V_{2,f})$ is a rational direct summand of $H^b(\Pi \otimes W_\mu) \to H^b(\partial_B\overline{S}_n, \mathcal{E}_\mu)$ as $GL_n(\mathbb{A}_{F,f})$-module. Here we should take w to be the longest element in the Weyl group and s to be trivial in the *loc.cit.*

We take $g_1 \otimes g_2 \in W_{1,f} \otimes W_{2,f}$ to be a rational element. We write $f_1 \otimes f_2$ for the image of $g_1 \otimes g_2$ under the map $\Theta_1 \otimes \Theta_2$. We denote by $F := Eis(f_1 \otimes f_2)$. We write $W = W(F)$ to be the corresponding Whittaker function (with respect to a fixed additive Hecke character).

From the diagram it is easy to see that $p_1^{-1}p_2^{-1}pW$ is a rational element and therefore:

$$p(\Pi) \sim_{E(\Pi_1)E(\Pi_2);K} p(\Pi_1)p(\Pi_2)W(Id)^{-1}. \tag{3.3}$$

Shahidi's calculation (c.f. Theorem 7.1.2 of [27] and Corollary 5.7 of [8]) implies that:

$$W(Id)^{-1} \sim_{E(\Pi);K} W_\infty(Id_\infty)^{-1} \prod_{w \text{ ramified places}} W_w(id_w)^{-1} \prod_{1\leqslant i<j\leqslant l} L(1, \Pi_i \times \Pi_j^\vee).$$

By the arguments in Corollary 5.7 of [8]) we may choose g_1, g_2 such that $W_w(id_w)$ is rational for each ramified place. At last, we conclude the proof by setting $\Omega_{(n_1,n_2)}(\Pi_\infty) := W_\infty(Id_\infty)^{-1}$. We can read from the construction that it depend onlys on Π_∞ and the parabolic data.

□

For any partition $n_1 + \cdots + n_l = n$, we may take $\Pi_1, \cdots, \Pi_l$ as above such that $\Pi_\infty = (\Pi_1 \boxplus \cdots \boxplus \Pi_l)_\infty$. Hence $\Omega_{(n_1,\cdots,n_l)}(\Pi_\infty)$ is well defined. In particular, $\Omega_{(1,1,\cdots,1)}(\Pi_\infty)$ is well-defined. We denote it by $\Omega(\Pi_\infty)$. We remark that this is the same archimedean factor appeared in Corollary 5.7 of [8].

3.5 First discussions on archimedean factors

We will discuss the archimedean factors $\Omega_{(n_1,n_2,\cdots,n_l)}(\Pi_\infty)$ defined in the last section.

One first observation is that

$$\Omega_{(n)}(\Pi_\infty) \sim_{E(\Pi);K} 1. \tag{3.4}$$

This can be read directly from Equation (3.2).

We observe that if Π_1 is also a Langlands sum of automorphic representations, we can furthermore decompose $p(\Pi_1)$. We will get relations between the archimedean factors. In fact, we have:

Lemma 3.5.1. *If* Π *is Langlands sum of* Π_1, Π_2, $\cdots$, Π_l *then we have:*

$$\Omega_{(n_1,n_2,\cdots,n_l)}(\Pi_\infty) \sim_{E(\Pi);K} \frac{\Omega(\Pi_\infty)}{\prod\limits_{1\leqslant i\leqslant n} \Omega(\Pi_{i,\infty})} \tag{3.5}$$

Proof We endow $P(n)$, the set of partitions of n, with the dictionary order. More precisely, if $(n_1, \cdots, n_l)$ and $(n'_1, \cdots, n'_{l'})$ are two partitions of n, we say $(n_1, \cdots, n_l) < n'_1, \cdots, n'_{l'})$ if there exists an integer $s \leqslant min\{l, l'\}$ such that $n_i = n'_i$ for $i < s$ and $n_s < n'_s$. The set $P(n)$ then becomes a totally ordered set.

We shall prove the lemma by induction on n. For each level n, we shall prove by induction on $P(n)$.

(1) Basis: When $n = 1$, we know both sides are equivalent to 1 by equation (3.4).

(2) Inductive step: We assume that the lemma is true for $n_1 + \cdots + n_l = n-1$ with $n \geqslant 2$. We shall prove it for $n_1 + \cdots + n_l = n$ by induction on $P(n)$.

(2.1) Basis: The smallest element in $P(n)$ is $(1, 1, \cdots, 1)$. In this case, we have $\Omega_{(1,1,\cdots,1)}(\Pi_\infty) \sim_{E(\Pi);K} \Omega(\Pi_\infty)$ by definition. Moreover, $\Omega(\Pi_{i,\infty}) \sim_{E(\Pi);K} \Omega_{(1)}(\Pi_{i,\infty}) \sim_{E(\Pi);K}$ 1 by equation (3.4) for all i. The lemma then follows.

(2.2) Inductive step: Let $(n_1, \cdots, n_l) \neq (1, 1, \cdots, 1) \in P(n)$. We assume that the lemma holds for all elements in $P(n)$ smaller than $(n_1, \cdots, n_l)$. We now prove the lemma for $(n_1, \cdots, n_l)$.

Since $(n_1, \cdots, n_l) \neq (1, 1, \cdots, 1)$, there exists an integer i such that $n_i \geqslant 2$. We take the smallest i with this property and denote it by t.

We take positive integers n'_t, n^*_t such that $n'_t + n^*_t = n_t$. For example, we may take $n'_t = n_t - 1$ and $n^*_t = 1$. We take Π'_t and Π^*_t to be cohomological conjugate self-dual regular representations of $GL_{n'_t}(\mathbb{A}_F)$ and $GL_{n_t}(\mathbb{A}_F)$ respectively such that $\Pi_{t,\infty}$ is the same with the infinity type of the Langlands sum of Π'_t and Π^*_t.

Let $\Pi^\#$ be the Langlands sum of $\Pi_1, \cdots, \Pi_{t-1}, \Pi'_t, \Pi^*_t, \Pi_{t+1}, \cdots, \Pi_l$. We apply Proposition 3.4.1 to $(\Pi_1, \cdots, \Pi_{t-1}, \Pi'_t, \Pi^*_t, \Pi_{t+1}, \cdots, \Pi_l)$ and get:

$$p(\Pi^\#) \sim_{E(\Pi);K} \Omega_{(n_1,\cdots,n_{t-1},n'_t,n^*_t,n_{t+1}\cdots,n_l)}(\Pi_\infty)[\prod_{i\neq t} p(\Pi_i)]p(\Pi'_t)p(\Pi^*_t) \prod_{i<j, i\neq t, j\neq t} L(1, \Pi_i \times \Pi_j^\vee)$$
$$\prod_{i<t}(L(1, \Pi_i \times \Pi'^{\vee}_t)L(1, \Pi_i \times \Pi^{*,\vee}_t)) \prod_{j>t}(L(1, \Pi'_t \times \Pi_j^\vee)L(1, \Pi^*_t \times \Pi_j^\vee))L(\Pi'_t \times \Pi^{*,\vee}_t)$$

Similarly, we apply Proposition 3.4.1 to $(\Pi_1, \Pi_{t-1}, \Pi'_t \boxplus \Pi^*_t, \Pi_{t+1}, \cdots, \Pi_l)$ and then to (Π'_t, Π^*_t). We will get:

$$\begin{aligned}
p(\Pi^\#) \quad & \sim_{E(\Pi);K} \quad \Omega_{(n_1,\cdots,n_{t-1},n'_t+n^*_t,n_{t+1})\cdots,n_l)}(\Pi_\infty)[\prod_{i\neq t} p(\Pi_i)]p(\Pi'_t \boxplus \Pi^*_t) \times \\
& \prod_{i<j, i\neq t, j\neq t} L(1, \Pi_i \times \Pi_j^\vee) \prod_{i<t} L(1, \Pi_i \times (\Pi'_t \boxplus \Pi^*_t)^\vee) \prod_{j>t} L(1, (\Pi'_t \boxplus \Pi^*_t) \times \Pi_j^\vee) \\
& \sim_{E(\Pi);K} \quad \Omega_{(n_1,\cdots,n_{t-1},n'_t+n^*_t,n_{t+1},\cdots,n_l)}(\Pi_\infty)\Omega_{(n'_t,n^*_t)}((\Pi'_t \boxplus \Pi^*_t)_\infty)[\prod_{i\neq t} p(\Pi_i)]p(\Pi'_t)p(\Pi^*_t) \\
& \times \prod_{i<j, i\neq t, j\neq t} L(1, \Pi_i \times \Pi_j^c) \prod_{i<t}(L(1, \Pi_i \times \Pi'^{\vee}_t)L(1, \Pi_i \times \Pi^{*,\vee}_t)) \\
& \times \prod_{j>t}(L(1, \Pi'_t \times \Pi_j^\vee)L(1, \Pi^*_t \times \Pi_j^\vee))L(\Pi'_t \times \Pi^{*,\vee}_t).
\end{aligned}$$

Recall that $n_t'+n_t^*=n_t$, we obtain that:

$$\Omega_{(n_1,\cdots,n_{t-1},n_t',n_t^*,n_{t+1}\cdots,n_l)}(\Pi_\infty)\sim_{E(\Pi;K)}\Omega_{(n_1,\cdots,n_{t-1},n_t,n_{t+1},\cdots,n_l)}(\Pi_\infty)\Omega_{(n_t',n_t^*)}((\Pi_t'\boxplus\Pi_t^*)_\infty) \tag{3.6}$$

Since $(n_1,\cdots,n_{t-1},n_t',n_t^*,n_{t+1}\cdots,n_l)<(n_1,n_2,\cdots,n_l)$, we may apply the hypothesis of the induction step (2.2) and get:

$$\Omega_{(n_1,\cdots,n_{t-1},n_t',n_t^*,n_{t+1}\cdots,n_l)}(\Pi_\infty)\sim_{E(\Pi;K)}\frac{\Omega(\Pi_\infty)}{\prod\limits_{i\neq t}\Omega(\Pi_{i,\infty})\Omega(\Pi_{t,\infty}')\Omega(\Pi_{t,\infty}^*)}. \tag{3.7}$$

If $n_t=n$ then $l=1$ and both sides of the equation of the lemma are equivalent to 1 by equation (3.4). Hence we may assume that $n_t<n$. Therefore, the hypothesis of the induction step (2) is satisfied by (Π_t',Π_t^*). We get:

$$\Omega_{(n_t',n_t^*)}((\Pi_t'\boxplus\Pi_t^*)_\infty)\sim_{E(\Pi;K)}\frac{\Omega(\Pi_{t,\infty})}{\Omega(\Pi_{t,\infty}')\Omega(\Pi_{t,\infty}^*)}. \tag{3.8}$$

Comparing the above three equations, we finally deduce that the lemma is true for $(\Pi_1,\cdots,\Pi_n)$ and complete the proof.

□

Corollary 3.5.1. *If* Π *is Langlands sum of* $\Pi_1,\ \Pi_2,\ \cdots,\ \Pi_l$ *then we have:*

$$p(\Pi)\sim_{E(\Pi);K}\frac{\Omega(\Pi_\infty)}{\prod\limits_{1\leqslant i\leqslant n}\Omega(\Pi_{i,\infty})}\prod_{1\leqslant i\leqslant l}p(\Pi_i)\prod_{1\leqslant i<j\leqslant l}L(1,\Pi_i\times\Pi_j^\vee) \tag{3.9}$$

3.6 Special values of tensor products in terms of Whittaker periods, after Grobner-Harris

Let Π be a regular cuspidal cohomological representation of $GL_n(\mathbb{A}_F)$. Let $\Pi^\#$ be a regular automorphic cohomological representation of $GL_{n-1}(\mathbb{A}_F)$ which is the Langlands sum of cuspidal representations. Equivalently, it is a regular element in $\mathcal{A}lg(n-1)$.

The arguments in section of [8] go over word for word and give the following result:

Proposition 3.6.1. *We assume that* $(\Pi,\Pi^\#)$ *is in good position.*

There exists a complex number $p(m,\Pi_\infty,\Pi_\infty^\#)$ *which depends on* m,Π_∞ *and* $\Pi_\infty^\#$ *well defined up to* $(E(\Pi)E(\Pi^\#))^\times$ *such that for* $m\in\mathbb{N}$ *with* $m+\frac{1}{2}$ *critical for* $\Pi\times\Pi^\#$*, we have*

$$L(\frac{1}{2}+m,\Pi\times\Pi^\#)\sim_{E(\Pi)E(\Pi^\#);K}p(m,\Pi_\infty,\Pi_\infty^\#)p(\Pi)p(\Pi^\#) \tag{3.10}$$

where $p(\Pi)$ *and* $p(\Pi^\#)$ *are the Whittaker periods of* Π *and* $\Pi^\#$ *respectively.*

We remark that we don't need $\Pi^\#$ to be cuspidal here. The above conditions are sufficient to guarantee that a certain Eisenstein series is holomorphic.

Moreover, we remark that the good position condition is necessary so that a certain intertwining operator exists.

We shall give a proof of this proposition in a separate article.

Chapter 4

CM periods and arithmetic automorphic periods

4.1 CM periods

Let (T,h) be a Shimura datum where T is a torus defined over $\mathbb{Q}$ and $h: Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{R}}$ a homomorphism satisfying the axioms defining a Shimura variety. Such pair is called a **special** Shimura datum. Let $Sh(T,h)$ be the associated Shimura variety and $E(T,h)$ be its reflex field.

Let (γ, V_{γ}) be a one-dimensional algebraic representation of T (the representation γ is denoted by χ in [17]). We denote by $E(\gamma)$ a definition field for γ. We may assume that $E(\gamma)$ contains $E(T,h)$. Suppose that γ is motivic (see *loc.cit* for the notion). We know that γ gives an automorphic line bundle $[V_{\gamma}]$ over $Sh(T,h)$ defined over $E(\gamma)$. Therefore, the complex vector space $H^0(Sh(T,h),[V_{\gamma}])$ has an $E(\gamma)$-rational structure, denoted by $M_{DR}(\gamma)$ and called the De Rham rational structure.

On the other hand, the canonical local system $V_{\gamma}^{\nabla} \subset [V_{\gamma}]$ gives another $E(\gamma)$-rational structure $M_B(\gamma)$ on $H^0(Sh(T,h),[V_{\gamma}])$, called the Betti rational structure.

We now consider χ an algebraic Hecke character of $T(\mathbb{A}_{\mathbb{Q}})$ with infinity type γ^{-1} (our character χ corresponds to the character ω^{-1} in *loc.cit*). Let $E(\chi)$ be the number field generated by the values of χ on $T(\mathbb{A}_{\mathbb{Q},f})$ over $E(\gamma)$. We know χ generates a one-dimensional complex subspace of $H^0(Sh(T,h),[V_{\gamma}])$ which inherits two $E(\chi)$-rational structures, one from $M_{DR}(\gamma)$, the other from $M_B(\gamma)$. Put $p(\chi,(T,h))$ the ratio of these two rational structures which is well defined modulo $E(\chi)^{\times}$.

Remark 4.1.1. *If we identify $H^0(Sh(T,h),[V_{\gamma}])$ with the set $\{f \in \mathbb{C}^{\infty}(T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathbb{Q}}),\mathbb{C} \mid f(tt_{\infty})) = \gamma^{-1}(t_{\infty})f(t), t_{\infty} \in T(\mathbb{R}), t \in T(\mathbb{A}_{\mathbb{Q}})\}$, then χ itself is in the rational structure inherits from $M_B(\gamma)$. See discussion from $A.4$ to $A.5$ in [17].*

Suppose that we have two tori T and T' both endowed with a Shimura datum (T,h) and (T',h'). Let $u:(T',h') \to (T,h)$ be a map between the Shimura data. Let χ be an algebraic Hecke character of $T(\mathbb{A}_{\mathbb{Q}})$. We put $\chi' := \chi \circ u$ an algebraic Hecke character of $T'(\mathbb{A}_{\mathbb{Q}})$. Since both the Betti structure and the De Rham structure commute with the pullback map on cohomology, we have the following proposition:

Proposition 4.1.1. *Let χ, (T,h) and χ', (T',h') be as above. We have:*

$$p(\chi,(T,h)) \sim_{E(\chi)} p(\chi',(T',h'))$$

Remark 4.1.2. *In Proposition 1.4 of [11], the relation is up to $E(\chi);E(T,h)$ where $E(T,h)$ is a number field associated to (T,h). Here we consider the action of $G_{\mathbb{Q}}$ and can thus obtain a relation up to $E(\chi)$ (see the paragraph after Proposition 1.8.1 of loc.cit).*

For F a CM field and Ψ a subset of Σ_F such that $\Psi\cap\iota\Psi=\varnothing$, we can define a Shimura datum (T_F,h_Ψ) where $T_F := Res_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ is a torus and $h_\Psi : Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to T_{F,\mathbb{R}}$ is a homomorphism such that over $\sigma\in\Sigma_F$, the Hodge structure induced on F by h_Ψ is of type $(-1,0)$ if $\sigma\in\Psi$, of type $(0,-1)$ if $\sigma\in\iota\Psi$, and of type $(0,0)$ otherwise.

Let χ be a motivic critical character of a CM field F. By definition, $p_F(\chi,\Psi) = p(\chi,(T_F,h_\Psi))$ and we call it a **CM period**. Sometimes we write $p(\chi,\Psi)$ instead of $p_F(\chi,\Psi)$ if there is no ambiguity concerning the base field F.

Example 4.1.1. *We have $p(||\cdot||_{\mathbb{A}_K},1) \sim_{\mathbb{Q}} (2\pi i)^{-1}$. See (1.10.9) on page 100 of [13].*

Let $\theta\in Gal(F/\mathbb{Q})$. We know θ induces an action on Σ_F by composition with θ. Moreover, θ acts on $\mathbb{A}_F^\times$ and hence acts on the set of Hecke characters of F.

The CM periods have many good properties. We list below some of them which will be useful in the future.

Proposition 4.1.2. *Let F be a CM field. Let $F_0\subset F$ be a sub CM field.*

Let η be a motivic critical Hecke character of F_0, χ, χ_1, χ_2 be motivic critical Hecke characters of F.

Let $\tau\in\Sigma_F$ be an embedding of F into $\bar{\mathbb{Q}}$ and Ψ be a subset of Σ_F such that $\Psi\cap\Psi^c=\varnothing$. We take $\Psi=\Psi_1\sqcup\Psi_2$ a partition of Ψ.

Let θ be an element in $Gal(F/\mathbb{Q})$. We then have:

$$p(\chi_1\chi_2),\Psi) \sim_{E(\chi_1)E(\chi_2)} p(\chi_1,\Psi)p(\chi_2^\sigma,\Psi). \tag{4.1}$$

$$p(\chi,\Psi_1\sqcup\Psi_2) \sim_{E(\chi)} p(\chi,\Psi_1)p(\chi,\Psi_2). \tag{4.2}$$

$$p(\chi^\theta,\Psi^\theta) \sim_{E(\chi)} p(\chi,\Psi). \tag{4.3}$$

$$p_F(\eta\circ N_{\mathbb{A}_F/\mathbb{A}_{F_0}},\tau) \sim_{E(\eta)} p_{F_0}(\eta,\tau|_{F_0}). \tag{4.4}$$

In particular, if we take $\theta=c$ the complex conjugation, we have:

$$p(\chi,\Psi^c) \sim_{E(\chi)} p(\chi^c,\Psi). \tag{4.5}$$

Remark 4.1.3. *The first three formulas come from Proposition 1.4, Corollary 1.5 and Lemma 1.6 in [11]. The last formula is a variation of the Lemma 1.8.3 in loc.cit. The idea was explained in the proof of Proposition 1.4 in loc.cit. We sketch the proof here.*

Proof. All the equations in Proposition 4.1.2 come from Proposition 4.1.1 by certain maps between Shimura data as follows:

1. The diagonal map $(T_F,h_\Psi)\to(T_F\times T_F,h_\Psi\times h_\Psi)$ pulls (χ_1,χ_2) back to $\chi_1\chi_2$.

2. The multiplication map $T_F\times T_F\to T_F$ sends h_{Ψ_1}, h_{Ψ_2} to $h_{\Psi_1\sqcup\Psi_2}$.

3. The Galois action $\theta : H_F \to H_F$ sends h_Ψ to h_{Ψ^θ}.

4. The norm map $(T_F, h_{\{\tau\}}) \to (T_{F_0}, h_{\{\tau|_{F_0}\}})$ pulls η back to $\eta \circ N_{\mathbb{A}_F/\mathbb{A}_{F,0}}$.

□

The special values of an L-function for a Hecke character over a CM field can be interpreted in terms of CM periods. The following theorem is proved by Blasius. We state it as in Proposition 1.8.1 in [11] where ω should be replaced by $\breve{\omega} := \omega^{-1,c}$ (for this erratum, see the notation and conventions part on page 82 in the introduction of [13]),

Theorem 4.1.1. *Let F be a CM field and F^+ be its maximal totally real subfield. Put n the degree of F^+ over $\mathbb{Q}$.*

Let χ be a motivic critical algebraic Hecke character of F and Φ_χ be the unique CM type of F which is compatible with χ.

Let D_{F^+} be the absolute discriminant of F^+. We assume that $D_{F^+}^{1/2} \in E(\chi)$ for simplicity.

For m a critical value of χ in the sense of Deligne (c.f. Lemma 6.1.1), we have

$$L(\chi, m) \sim_{E(\chi)} (2\pi i)^{mn} p(\check{\chi}, \Phi_\chi)$$

equivariant under action of $G_\mathbb{Q}$

Remark 4.1.4.

1. *Let $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ be any CM type of F. Let $(\sigma_i^{a_i}\overline{\sigma}_i^{-w-a_i})_{1\leqslant i\leqslant n}$ denote the infinity type of χ with $w = w(\chi)$. We may assume $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. We define $a_0 := +\infty$ and $a_{n+1} := -\infty$ and define $k := max\{0 \leqslant i \leqslant n \mid a_i > -\frac{w}{2}\}$. An integer m is critical for χ if and only if*

$$max(-a_k + 1, w + 1 + a_{k+1}) \leqslant m \leqslant min(w + a_k, -a_{k+1}) \tag{4.6}$$

   (c.f. Lemma 6.1.1).

2. *$D_{F^+}^{1/2}$ is well defined up to multiplication by ± 1. More generally, if $\{z_1, z_2, \cdots, z_n\}$ is any $\mathbb{Q}$-base of L, then $det(\sigma_i(z_j))_{1\leqslant i,j\leqslant n} \sim_\mathbb{Q} D_{F^+}^{1/2}$.*

4.2 Construction of cohomology spaces

Let $\Sigma = \Sigma_{F;K}$ in the current and the following chapters. Fix an index I as before. Write $s_\sigma := I(\sigma)$ and $r_\sigma := n - I(\sigma)$ for all $\sigma \in \Sigma$.

Denote $\mathbb{S} := Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Recall that $GU_I(\mathbb{R})$ is isomorphic to a subgroup of $\prod_{\sigma\in\Sigma} GU(r_\sigma, s_\sigma)$ defined by the same *similitude*. We can define a homomorphism $h_I : \mathbb{S}(\mathbb{R}) \to GU_I(\mathbb{R})$ by sending $z \in \mathbb{C}$ to $\left(\begin{pmatrix} zI_{r_\sigma} & 0 \\ 0 & \bar{z}I_{s_\sigma}\end{pmatrix}\right)_{\sigma\in\Sigma}$.

Let X_I be the $GU_I(\mathbb{R})$-conjugation class of h_I. We know (GU_I, X_I) is a Shimura datum with reflex field E_I and dimension $2\sum_{\sigma\in\Sigma} r_\sigma s_\sigma$. The Shimura variety associated to (GU_I, X_I) is denoted by Sh_I.

Let $K_{I,\infty}$ be the centralizer of h_I in $GU_I(\mathbb{R})$. Via the inclusion

$$GU_I(\mathbb{R}) \hookrightarrow \prod_{\sigma\in\Sigma} GU(r_\sigma, s_\sigma) \subset \mathbb{R}^{+,\times} \prod_{\sigma\in\Sigma} U(n,\mathbb{C}),$$

we may identify $K_{I,\infty}$ with

$$\{(\mu, \begin{pmatrix} u_{r_\sigma} & 0 \\ 0 & v_{s_\sigma} \end{pmatrix}_{\sigma\in\Sigma}) \mid u_{r_\sigma} \in U(r_\sigma,\mathbb{C}), v_{s_\sigma} \in U(s_\sigma,\mathbb{C}), \mu \in \mathbb{R}^{+,\times}\}$$

where $U(r,\mathbb{C})$ is the standard unitary group of degree r over $\mathbb{C}$. Let H_I be the subgroup of $K_{I,\infty}$ consisting of the diagonal matrices in $K_{I,\infty}$. Then it is a maximal torus of $GU_I(\mathbb{R})$. Denote its Lie algebra by $\mathfrak{h}_I$.

We observe that $H_I(\mathbb{R}) \cong \mathbb{R}^{+,\times} \times \prod\limits_{\sigma\in\Sigma} U(1,\mathbb{C})^n$. Its algebraic characters are of the form

$$(w, (z_i(\sigma))_{\sigma\in\Sigma, 1\leqslant i\leqslant n}) \mapsto w^{\lambda_0} \prod_{\sigma\in\Sigma}\prod_{i=1}^{n} z_i(\sigma)^{\lambda_i(\sigma)}$$

where $(\lambda_0, (\lambda_i(\sigma))_{\sigma\in\Sigma,1\leqslant i\leqslant n})$ is a $(nd+1)$-tuple of integers with $\lambda_0 \equiv \sum\limits_{\sigma\in\Sigma}\sum\limits_{i=1}^{n} \lambda_i(\sigma) \pmod 2$.

Recall that $GU_I(\mathbb{C}) \cong \mathbb{C}^\times \prod_{\sigma\in\Sigma} GL_n(\mathbb{C})$. We fix B_I the Borel subgroup of $GU_{I,\mathbb{C}}$ consisting of upper triangular matrices. The highest weights of finite-dimensional irreducible representations of $K_{I,\infty}$ are tuples $\Lambda = (\Lambda_0, (\Lambda_i(\sigma))_{\sigma\in\Sigma,1\leqslant i\leqslant n})$ such that $\Lambda_1(\sigma) \geqslant \Lambda_2(\sigma) \geqslant \cdots \geqslant \Lambda_{r_\sigma}(\sigma)$, $\Lambda_{r_\sigma+1}(\sigma) \geqslant \cdots \geqslant \Lambda_n(\sigma)$ for all σ and $\Lambda_0 \equiv \sum\limits_{\sigma\in\Sigma}\sum\limits_{i=1}^{n} \Lambda_i(\sigma) \pmod 2$.

We denote the set of such tuples by $\Lambda(K_{I,\infty})$. Similarly, we write $\Lambda(GU_I)$ for the set of the highest weights of finite-dimensional irreducible representations of GU_I. It consists of tuples $\lambda = (\lambda_0, (\lambda_i(\sigma))_{\sigma\in\Sigma,1\leqslant i\leqslant n})$ such that $\lambda_1(\sigma) \geqslant \lambda_2(\sigma) \geqslant \cdots \lambda_n(\sigma)$ for all σ and $\lambda_0 \equiv \sum\limits_{\sigma\in\Sigma}\sum\limits_{i=1}^{n} \lambda_i(\sigma) \pmod 2$.

We take $\lambda \in \Lambda(GU_I)$ and $\Lambda \in \Lambda(K_{I,\infty})$.

Let V_λ and V_Λ be the corresponding representations. We define a local system over Sh_I:

$$W_\lambda^\triangledown := \varprojlim_K GU_I(\mathbb{Q})\backslash V_\lambda \times X \times GU_I(\mathbb{A}_{\mathbb{Q},f})/K$$

and an automorphic vector bundle over Sh_I

$$E_\Lambda := \varprojlim_K GU_I(\mathbb{Q})\backslash V_\Lambda \times GU_I(\mathbb{R}) \times GU_I(\mathbb{A}_{\mathbb{Q},f})/KK_{I,\infty}$$

where K runs over open compact subgroup of $GU_I(\mathbb{A}_{\mathbb{Q},f})$.

The automorphic vector bundles E_Λ are defined over the reflex field E.

The local systems $W_\lambda^\triangledown$ are defined over K. The Hodge structure of the cohomology space $H^q(Sh_I, W_\lambda^\triangledown)$ is not pure in general. But the image of $H_c^q(Sh_I, W_\lambda^\triangledown)$ in $H^q(Sh_I, W_\lambda^\triangledown)$ is pure of weight $q-c$. We denote this image by $\bar{H}^q(Sh_I, W_\lambda^\triangledown)$.

Note that all cohomology spaces have coefficients in $\mathbb{C}$ unless we specify its rational structure over a number field.

4.3 The Hodge structures

The results in section 2.2 of [12] give a description of the Hodge components of $\bar{H}^q(Sh_I, W_\lambda^\nabla)$.

Denote by R^+ the set of positive roots of $H_{I,\mathbb{C}}$ in $GU_I(\mathbb{C})$ and by R_c^+ the set of positive compact roots. Define $\alpha_{j,k} = (0, \cdots, 0, 1, 0, \cdots, 0, -1, 0, \cdots, 0)$ for any $1 \leqslant j < k \leqslant n$. We know $R^+ = \{(\alpha_{j_\sigma,k_\sigma})_{\sigma\in\Sigma} \mid 1 \leqslant j_\sigma < k_\sigma \leqslant n\}$ and $R_c^+ = \{(\alpha_{j_\sigma,k_\sigma})_{\sigma\in\Sigma} \mid j_\sigma < k_\sigma \leqslant r_\sigma \text{ or } r_\sigma + 1 \leqslant j_\sigma < k_\sigma\}$.

Let $\rho = \frac{1}{2}\sum\limits_{\alpha\in R^+} \alpha = \left(\left(\frac{n-1}{2}, \frac{n-3}{2}, \cdots, -\frac{n-1}{2}\right)\right)_\sigma$.

Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ be Lie algebras of $GU_I(\mathbb{R})$, $K_{I,\infty}$ and $H(\mathbb{R})$. Write W for the Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and W_c for the Weyl group $W(\mathfrak{k}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. We can identify W with $\prod\limits_{\sigma\in\Sigma} \mathfrak{S}_n$ and W_c with $\prod\limits_{\sigma\in\Sigma} \mathfrak{S}_{r_\sigma} \times \mathfrak{S}_{s_\sigma}$ where $\mathfrak{S}$ refers to the standard permutation group. For $w \in W$, we write the length of w by $l(w)$.

Let $W^1 := \{w \in W | w(R^+) \supset R_c^+\}$ be a subset of W. By the above identification, $(w_\sigma)_\sigma \in W^1$ if and only if $w_\sigma(1) < w_\sigma(2) < \cdots < w_\sigma(r_\sigma)$ and $w_\sigma(r_\sigma + 1) < \cdots < w_\sigma(n)$ One can show that W^1 is a set of coset representatives of shortest length for $W_c\backslash W$.

Moreover, for λ a highest weight of a representation of GU_I, one can show easily that $w * \lambda := w(\lambda + \rho) - \rho$ is the highest weight of a representation of $K_{I,\infty}$. More precisely, if $\lambda = (\lambda_0, (\lambda_i(\sigma))_{\sigma\in\Sigma,1\leqslant i\leqslant n})$, then $w * \lambda = (\lambda_0, ((w * \lambda)_i(\sigma))_{\sigma\in\Sigma,1\leqslant i\leqslant n})$ with $(w * \lambda)_i(\sigma) = \lambda_{w_\sigma(i)}(\sigma) + \frac{n+1}{2} - w_\sigma(i) - (\frac{n+1}{2} - i) = \lambda_{w_\sigma(i)}(\sigma) - w_\sigma(i) + i$.

Remark 4.3.1. *The results of [12] tell us that there exists*

$$\bar{H}^q(Sh_I, W_\lambda^\nabla) \cong \bigoplus_{w\in W^1} \bar{H}^{q;w}(Sh_I, W_\lambda^\nabla) \tag{4.7}$$

a decomposition as subspaces of pure Hodge type $(p(w, \lambda), q - c - p(w, \lambda))$. We now determine the Hodge number $p(w, \lambda)$.

We know that $w * \lambda$ is the highest weight of a representation of $K_{I,\infty}$. We denote this representation by $(\rho_{w*\lambda}, W_{w*\lambda})$. We know that $\rho_{w*\lambda} \circ h_I|_{\mathbb{S}(\mathbb{R})} : \mathbb{S}(\mathbb{R}) \to K_{I,\infty} \to GL(W_{w*\lambda})$ is of the form $z \mapsto z^{-p(w,\lambda)}\bar{z}^{-r(w,\lambda)} I_{W_{w*\lambda}}$ with $p(w, \lambda)$, $r(w, \lambda) \in \mathbb{Z}$. The first index $p(w, \lambda)$ is the Hodge type mentioned above.

Recall that the map

$$\begin{aligned} h_I|_{\mathbb{S}(\mathbb{R})} : \mathbb{S}(\mathbb{R}) &\to K_{I,\infty} \subset \mathbb{R}^{+,\times} \times U(n, \mathbb{C})^\Sigma \\ z &\mapsto \left(|z|, \begin{pmatrix} \frac{z}{|z|} I_{r_\sigma} & 0 \\ 0 & \frac{\bar{z}}{|z|} I_{s_\sigma} \end{pmatrix}_{\sigma\in\Sigma}\right) \end{aligned} \tag{4.8}$$

and the map

$$\begin{aligned} \rho_{w*\lambda} : K_{I,\infty} &\to GL(W_{w*\lambda}) \\ (w, diag(z_i(\sigma))_{\sigma\in\Sigma,1\leqslant i\leqslant n}) &\mapsto w^{\lambda_0} \prod_{\sigma\in\Sigma} \prod_{i=1}^{n} z_i(\sigma)^{(w*\lambda)_i(\sigma)} \end{aligned}$$

where $diag(z_1, z_2, \cdots, z_n)$ means the diagonal matrix of coefficients $z_1, z_2, \cdots, z_n$.

Therefore we have:

$$\begin{aligned} z^{-p(w,\lambda)}\overline{z}^{-r(w,\lambda)} &= |z|^{\lambda_0}\prod_{\sigma\in\Sigma}(\prod_{1\leqslant i\leqslant r_\sigma}(\frac{z}{|z|})^{(w*\lambda)_i(\sigma)}\prod_{r_\sigma+1\leqslant i\leqslant n}(\frac{\overline{z}}{|z|})^{(w*\lambda)_i(\sigma)} \\ &= (z^{\frac{1}{2}}\overline{z}^{\frac{1}{2}})^{\lambda_0-\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}(w*\lambda)_i(\sigma)} z^{\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant r_\sigma}(w*\lambda)_i(\sigma)}\overline{z}^{\sum\limits_{\sigma\in\Sigma}\sum\limits_{r_\sigma+1\leqslant i\leqslant n}(w*\lambda)_i(\sigma)} \end{aligned}$$

Since $(w*\lambda)_i(\sigma) = \lambda_{w_\sigma(i)}(\sigma) - w_\sigma(i) + i$ and then $\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}(w*\lambda)_i(\sigma) = \sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}\lambda_i(\sigma)$, we obtain that:

$$\begin{aligned} p(w,\lambda) &= \frac{\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}\lambda_i(\sigma)-\lambda_0}{2} - \sum_{\sigma\in\Sigma}\sum_{1\leqslant i\leqslant r_\sigma}(w*\lambda)_i(\sigma) \\ &= \frac{\sum\limits_{\sigma\in\Sigma}\sum\limits_{1\leqslant i\leqslant n}\lambda_i(\sigma)-\lambda_0}{2} - \sum_{\sigma\in\Sigma}\sum_{1\leqslant i\leqslant r_\sigma}(\lambda_{w_\sigma(i)}(\sigma)-w_\sigma(i)+i) \qquad (4.9) \end{aligned}$$

The method of toroidal compactification gives us more information on $\bar{H}^{q;w}(Sh_I, W_\lambda^\nabla)$. We take $j : Sh_I \hookrightarrow \widetilde{Sh}_I$ to be a smooth toroidal compactification. Proposition 2.2.2 of [12] tells us that the following results do not depend on the choice of the toroidal compactification.

The automorphic vector bundle E_Λ can be extended to $\widetilde{Sh}_I$ in two ways: the canonical extension E_Λ^{can} and the sub canonical extension E_Λ^{sub} as explained in [12]. Define:

$$\bar{H}^q(Sh_I, E_\Lambda) = Im(H^q(\widetilde{Sh}_I, E_\Lambda^{sub}) \to H^q(\widetilde{Sh}_I, E_\Lambda^{can})).$$

Proposition 4.3.1. *There is a canonical isomorphism*

$$\bar{H}^{q;w}(Sh_I, W_\lambda^\nabla) \cong \bar{H}^{q-l(w)}(Sh_I, E_{w*\lambda})$$

Let $D = 2\sum\limits_{\sigma\in\Sigma} r_\sigma s_\sigma$ be the dimension of the Shimura variety. We are interested in the cohomology space of degree $D/2$. Proposition 2.2.7 of [13] also works here:

Proposition 4.3.2. *The space $\bar{H}^{D/2}(Sh_I, W_\lambda^\nabla)$ is naturally endowed with a K-rational structure, called the de Rham rational structure and noted by $\bar{H}_{DR}^{D/2}(Sh_I, W_\lambda^\nabla)$. This rational structure is endowed with a K-Hodge filtration $F^\cdot\bar{H}_{DR}^{D/2}(Sh_I, W_\lambda^\nabla)$ pure of weight $D/2 - c$ such that*

$$F^p\bar{H}_{DR}^{D/2}(Sh_I, W_\lambda^\nabla)/F^{p+1}\bar{H}_{DR}^{D/2}(Sh_I, W_\lambda^\nabla)\otimes_K \mathbb{C} \cong \bigoplus_{w\in W^1, p(w,\lambda)=p}\bar{H}^{D/2;w}(Sh_I, W_\lambda^\nabla).$$

Moreover, the composition of the above isomorphism and the canonical isomorphism

$$\bar{H}^{D/2;w}(Sh_I, W_\lambda^\nabla) \cong \bar{H}^{D/2-l(w)}(Sh_I, E_{w*\lambda})$$

is rational over K.

Holomorphic part: Let $w_0 \in W^1$ defined by $w_0(\sigma)(1, 2, \cdots, r_\sigma; r_{\sigma+1}, \cdots, n)_{\sigma\in\Sigma} = (s_{\sigma+1}, \cdots, n; 1, 2, \cdots, s_\sigma)$ for all $\sigma \in \Sigma$. It is the only longest element in W^1. Its length is $D/2$.

We have a K-rational isomorphism

$$\bar{H}^{D/2;w_0}(Sh_I, W_\lambda^\nabla) \cong \bar{H}^0(Sh_I, E_{w_0*\lambda}). \tag{4.10}$$

We can calculate the Hodge type of $\bar{H}^{D/2;w_0}(Sh_I, W_\lambda^\nabla)$ as in Remark 4.3.1.

By definition we have

$$w_0 * \lambda = (\lambda_0, (\lambda_{s_\sigma+1}(\sigma) - s_\sigma, \cdots, \lambda_n(\sigma) - s_\sigma; \lambda_1(\sigma) + r_\sigma, \cdots, \lambda_{s_\sigma}(\sigma) + r_\sigma)_{\sigma\in\Sigma}). \tag{4.11}$$

By the discussion in Remark 4.3.1, the Hodge number

$$p(w_0, \lambda) = \frac{\sum\limits_{\sigma\in\Sigma} \sum\limits_{1\leqslant i\leqslant n} \lambda_i(\sigma) - \lambda_0 + D}{2} - \sum_{\sigma\in\Sigma} (\lambda_{s_\sigma+1}(\sigma) + \cdots + \lambda_n(\sigma)).$$

From equation (4.9), it is easy to deduce that $p(w_0, \lambda)$ is the only largest number among $\{p(w, \lambda) \mid w \in W^1\}$. Therefore

$$F^{p(w_0,\lambda)}(Sh_I, W_\lambda^\nabla) \otimes_K \mathbb{C} \cong \bar{H}^0(Sh_I, E_{w_0*\lambda}). \tag{4.12}$$

Moreover, as mentioned in the above proposition, we know that the above isomorphism is K-rational.

We call $\bar{H}^{D/2;w_0}(Sh_I, W_\lambda^\nabla) \cong \bar{H}^0(Sh_I, E_{w_0*\lambda})$ the **holomorphic part** of the Hodge decomposition of $\bar{H}^{D/2}(Sh_I, W_\lambda^\nabla)$. It is isomorphic to the space of holomorphic cusp forms of type $(w_0 * \lambda)^\vee$.

Anti-holomorphic part: The only shortest element in W^1 is the identity with the smallest Hodge number

$$p(id, \lambda) = \frac{\sum\limits_{\sigma\in\Sigma} \sum\limits_{1\leqslant i\leqslant n} \lambda_i(\sigma) - \lambda_0}{2} - \sum_{\sigma\in\Sigma} (\lambda_1(\sigma) + \cdots + \lambda_{r_\sigma}(\sigma)).$$

We call $\bar{H}^{D/2;id}(Sh_I, W_\lambda^\nabla) \cong \bar{H}^{D/2}(Sh_I, E_\lambda)$ the **anti-holomorphic part** of the Hodge decomposition of $\bar{H}^{D/2}(Sh_I, W_\lambda^\nabla)$.

4.4 Complex conjugation

We specify some notation first.

Let $\lambda = (\lambda_0, (\lambda_1(\sigma) \geqslant \lambda_2(\sigma) \geqslant \cdots \geqslant \lambda_n(\sigma))_{\sigma\in\Sigma}) \in \Lambda(GU_I)$ as before. We define $\lambda^c := (\lambda_0, (-\lambda_n(\sigma) \geqslant -\lambda_{n-1}(\sigma) \geqslant \cdots \geqslant -\lambda_1(\sigma))_{\sigma\in\Sigma})$ and $\lambda^\vee := (-\lambda_0, (-\lambda_n(\sigma) \geqslant -\lambda_{n-1}(\sigma) \geqslant \cdots \geqslant -\lambda_1(\sigma))_{\sigma\in\Sigma})$. They are elements in $\Lambda(GU_I)$. Moreover, the representation V_{λ^c} is the complex conjugation of V_λ and the representation $V_{\lambda^\vee}$ is the dual of V_λ as GU_I-representation.

Similarly, for $\Lambda = (\Lambda_0, (\Lambda_1(\sigma) \geqslant \cdots \geqslant \Lambda_{r_\sigma}(\sigma), \Lambda_{r_\sigma+1}(\sigma) \geqslant \cdots \geqslant \Lambda_n(\sigma))_{\sigma\in\Sigma}) \in \Lambda(K_{I,\infty})$, we define $\Lambda^* := (-\Lambda_0, (-\Lambda_{r_\sigma}(\sigma) \geqslant \cdots \geqslant -\Lambda_1(\sigma), -\Lambda_n \geqslant \cdots \geqslant -\Lambda_{r_\sigma+1})_{\sigma\in\Sigma})$.

We know V_{Λ^*} is the dual of V_Λ as K_I-representation. We sometimes write the latter as $\widetilde{V_\Lambda}$.

We define I^c by $I^c(\sigma) = n - I(\sigma)$ for all $\sigma \in \Sigma$. We know $V_{I^c} = -V_I$ and $GU_{I^c} \cong GU_I$. The complex conjugation gives an anti-holomorphic isomorphism $X_I \xrightarrow{\sim} X_{I^c}$. This induces a K-antilinear isomorphism

$$\bar{H}^{D/2}(Sh_I, W_\lambda^\nabla) \xrightarrow{\sim} \bar{H}^{D/2}(Sh_{I^c}, W_{\lambda^c}^\nabla). \tag{4.13}$$

In particular, it sends holomorphic (resp. anti-holomorphic) elements with respect to (I, λ) to those respect to (I^c, λ^c). If we we denote by w_0^c the longest element related to I^c then we have K-antilinear rational isomorphisms

$$c_{DR}: \quad \bar{H}^0(Sh_I, E_{w_0*\lambda}) \xrightarrow{\sim} \bar{H}^0(Sh_{I^c}, E_{w_0^c*\lambda^c}) \tag{4.14}$$

$$\bar{H}^{D/2}(Sh_I, E_\lambda) \xrightarrow{\sim} \bar{H}^{D/2}(Sh_{I^c}, E_{\lambda^c}) \quad . \tag{4.15}$$

The Shimura datum (GU_I, h) induces a Hodge structure of wights concentrated in $\{(-1,1), (0,0), (1,-1)\}$ which corresponds to the Harish-Chandra decomposition induced by h on the Lie algebra: $\mathfrak{g} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

Let $\mathfrak{P} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^-$. Let $\mathcal{A}$ (resp. $\mathcal{A}_0$, $\mathcal{A}_{(2)}$) be the space of automorphic forms (resp. cusp forms, square-integrable forms) on $GU_I(\mathbb{Q})\backslash GU_I(\mathbb{A}_\mathbb{Q})$.

We have inclusions for all q:

$$H^q(\mathfrak{g}, K_{I,\infty}; \mathcal{A}_0 \otimes V_\lambda) \subset \bar{H}^q(Sh_I, V_\lambda^\nabla) \subset H^q(\mathfrak{g}, K_{I,\infty}; \mathcal{A}_{(2)} \otimes V_\lambda)$$
$$H^q(\mathfrak{P}, K_{I,\infty}; \mathcal{A}_0 \otimes V_\Lambda) \subset \bar{H}^q(Sh_I, E_\Lambda) \subset H^q(\mathfrak{P}, K_{I,\infty}; \mathcal{A}_{(2)} \otimes V_\Lambda).$$

The complex conjugation on the automorphic forms induces a K-antilinear isomorphism:

$$c_B: \bar{H}^0(Sh_I, E_{w_0*\lambda}) \xrightarrow{\sim} \bar{H}^{D/2}(Sh_I, E_{\lambda^\vee}) \tag{4.16}$$

More precisely, we summarize the construction in [13] as follows.

Automorphic vector bundles:

We recall some facts on automorphic vector bundles first. We refer to page 113 of [13] and [10] for notation and further details.

Let (G, X) be a Shimura datum such that its special points are all CM points. Let $\check{X}$ be the compact dual symmetric space of X. There is a surjective functor from the category of G-homogeneous vector bundles on $\check{X}$ to the category of automorphic vector bundles on $Sh(G, X)$. This functor is compatible with inclusions of Shimura data as explained in the second part of Theorem 4.8 of [10]. It is also rational over the reflex field $E(G, X)$.

Let $\mathcal{E}$ be an automorphic vector bundle on $Sh(G, X)$ corresponding to $\mathcal{E}_0$, a G-homogeneous vector bundle on $\check{X}$. Let (T, x) be a special pair of (G, X), i.e. (T, x) is a sub-Shimura datum of (G, X) with T a maximal torus defined over $\mathbb{Q}$. Since the functor mentioned above is compatible with inclusions of Shimura datum, we know that the restriction of $\mathcal{E}$ to $Sh(T, x)$ corresponds to the restriction of $\mathcal{E}_0$ to $\check{x} \in \check{X}$ by the previous functor. Moreover, by the construction, the fiber of $\mathcal{E}\mid_{Sh(T,x)}$ at any point of $Sh(T, x)$ is identified with the fiber of $\mathcal{E}_0$ at $\check{x}$. The $E(\mathcal{E}) \cdot E(T, x)$-rational structure on the fiber of $\mathcal{E}_0$ at $\check{x}$ then defines a rational structure of $\mathcal{E}\mid_{Sh(T,x)}$ and called the **canonical trivialization** of $\mathcal{E}$ associated to (T, x).

Complex conjugation on automorphic vector bundles:

Let $\mathcal{E}$ be as in page 112 of [13] and $\overline{\mathcal{E}}$ be its complex conjugation. The key step of the construction is to identify $\overline{\mathcal{E}}$ with the dual of $\mathcal{E}$ in a rational way.

More precisely, we recall Proposition 2.5.8 of the *loc.cit* that there exists a non-degenerate $G(\mathbb{A}_{\mathbb{Q},f})$-equivariant paring of real-analytic vector bundle $\mathcal{E} \otimes \overline{\mathcal{E}} \to \mathcal{E}_\nu$ such that its pullback to any CM point is rational with respect to the canonical trivializations.

We now explain the notion $\mathcal{E}_\nu$. Let $h \in X$ and K_h be the stabilizer of h in $G(\mathbb{R})$. We know $\mathcal{E}$ is associated to an irreducible complex representation of K_h, denoted by τ in the *loc.cit.* The complex conjugation of τ can be extended as an algrebraic representation of K_h, denoted by τ'. We know τ' is isomorphic to the dual of τ and then there exists ν, a one-dimensional representation K_h, such that a K_h-equivariant rational paring $V_\tau \otimes V_{\tau'} \to V_\nu$ exists. We denote by $\mathcal{E}_\nu$ the automorphic vector bundle associated to V_ν.

In our case, we have $(G, X) = (GU_I, X_I)$, $h = h_I$ and $K_h = K_{I,\infty}$. Let $\tau = \Lambda = w_0 * \lambda$ and $\mathcal{E} = E_\Lambda$. As explained in the last second paragraph before Corollary 2.5.9 in the *loc.cit*, we may identify the holomorphic sections of V_Λ with holomorphic sections of the dual of $\overline{V_\Lambda}$. The complex conjugation then sends the latter to the anti-holomorphic sections of $\widehat{V_\Lambda} = V_{\Lambda^*}$. The latter can be identified with harmonic (0,d)-forms with values in $\mathbb{K} \otimes E_\Lambda$ where $\mathbb{K} = \Omega^{D/2}_{Sh_I}$ is the canonical line bundle of Sh_I.

By 2.2.9 of [13] we have $\mathbb{K} = E_{(0,(-s_\sigma,\cdots,-s_\sigma,r_\sigma,\cdots,r_\sigma)_{\sigma\in\Sigma})}$ where the number of $-s_\sigma$ in the last term is r_σ. Therefore, complex conjugation gives an isomorphism:

$$c_B : \bar{H}^0(Sh_I, E_\Lambda) \xrightarrow{\sim} \bar{H}^{D/2}(Sh_I, E_{\Lambda^*+0,(-s_\sigma,\cdots,-s_\sigma,r_\sigma,\cdots,r_\sigma)_{\sigma\in\Sigma}}). \tag{4.17}$$

Recall equation (4.11) that

$$\Lambda = w_0 * \lambda = (\lambda_0, (\lambda_{s_\sigma+1}(\sigma) - s_\sigma, \cdots, \lambda_n(\sigma) - s_\sigma; \lambda_1(\sigma) + r_\sigma, \cdots, \lambda_{s_\sigma}(\sigma) + r_\sigma)_{\sigma\in\Sigma}).$$

We have

$$\Lambda^* = (-\lambda_0, (-\lambda_n(\sigma)+s_\sigma, \cdots, -\lambda_{s_\sigma+1}(\sigma)+s_\sigma; -\lambda_{s_\sigma}(\sigma)-r_\sigma, \cdots, -\lambda_1(\sigma)+r_\sigma)_{\sigma\in\Sigma}). \tag{4.18}$$

Therefore, $\Lambda^* + (0, (-s_\sigma, \cdots, -s_\sigma, r_\sigma, \cdots, r_\sigma)_{\sigma\in\Sigma}) = \lambda^\vee$. We finally get equation (4.16).

Similarly, if we start from the anti-holomorphic part, we will get a K-antilinear isomorphism which is still denoted by c_B:

$$c_B : \bar{H}^{D/2}(Sh_I, E_\lambda) \xrightarrow{\sim} \bar{H}^0(Sh_I, E_{w_0*\lambda^\vee}) \tag{4.19}$$

which sends anti-holomorphic elements with respect to λ to holomorphic elements for $\lambda^\vee$.

4.5 The rational paring

Let $\Lambda \in \Lambda(K_{I,\infty})$. We write $V = V_\Lambda$ in this section for simplicity. As in section 2.6.11 of [13], we denote by $\mathbb{C}_\Lambda$ the corresponding highest weight space. We know $\Lambda^* := \Lambda^\# - (2\Lambda_0, (0))$ is the tuple associated to $\check{V}$, the dual of this K_I representation. We denote by $\mathbb{C}_{-\Lambda}$ the lowest weight of $\check{V}$.

The restriction from V to $\mathbb{C}_\Lambda$ gives an isomorphism

$$Hom_{K_{I,\infty}}(V, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \xrightarrow{\sim} Hom_H(\mathbb{C}_\Lambda, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_V) \qquad (4.20)$$

where $\mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_V$ is the V-isotypic subspace of $\mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))$.

Similarly, we have

$$Hom_{K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \xrightarrow{\sim} Hom_H(\mathbb{C}_{-\Lambda}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_{\check{V}}) \qquad (4.21)$$

Proposition 2.6.12 of [13] says that up to a rational factor the perfect paring

$$Hom_H(\mathbb{C}_\Lambda, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_V) \times Hom_H(\mathbb{C}_{-\Lambda}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))_{\check{V}}) \qquad (4.22)$$

given by integration over the diagonal equals to restriction of the canonical paring (c.f. (2.6.11.4) of [13])

$$\begin{aligned} & Hom_{K_{I,\infty}}(V, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \times Hom_{K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \\ \to\ & Hom_{K_{I,\infty}}(V \otimes \check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \\ \to\ & Hom_{K_{I,\infty}}(\mathbb{C}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F))) \\ \to\ & \mathbb{C}. \end{aligned} \qquad (4.23)$$

We identify $\Gamma^\infty(Sh_I, E_\Lambda)$ with $Hom_{GU_I K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F)))$ and regard the latter as subspace of $Hom_{K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F)))$.

The above construction gives a K-rational perfect paring between holomorphic sections of E_Λ and anti-holomorphic sections of E_{Λ^*}.

If $\Lambda = w_0 * \lambda$, as we have seen in Section 4.4 that the anti-holomorphic sections of E_{Λ^*} can be identified with harmonic $(0,d)$-forms with values in $E_{\lambda^\vee}$.

We therefore obtain a K-rational perfect paring

$$\Phi = \Phi^{I,\lambda} : \bar{H}^0(Sh_I, E_{w_0 * \lambda}) \times \bar{H}^{D/2}(Sh_I, E_{\lambda^\vee}) \to \mathbb{C}. \qquad (4.24)$$

In other words, there is a rational paring between the holomorphic elements for (I, λ) and anti-holomorphic elements for $(I, \lambda^\vee)$.

It is easy to see that the isomorphism $Sh_I \xrightarrow{\sim} Sh_{I^c}$ commutes with the above paring and hence:

Lemma 4.5.1. *For any $f \in \bar{H}^0(Sh_I, E_{w_0*\lambda})$ and $g \in \bar{H}^{D/2}(Sh_I, E_{\lambda^\vee})$, we have*

$$\Phi^{I,\lambda}(f, g) = \Phi^{I^c,\lambda^c}(c_{DR} f, c_{DR} g).$$

The next lemma follows from Corollary 2.5.9 and Lemma 2.8.8 of [13].

Lemma 4.5.2. *Let* $0 \neq f \in \bar{H}^0(Sh_I, E_{w_0 * \lambda})$. *We have* $\Phi(f, c_B f) \neq 0$.

More precisely, if we consider f *as an element in* $Hom_{K_{I,\infty}}(\check{V}, \mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F)))$ *then by (4.21) and the fixed trivialization of* $\mathbb{C}_{-w_0*\lambda}$, *we may consider* f *as an element in* $\mathcal{C}^\infty(GU_I(\mathbb{F})\backslash GU_I(\mathbb{A}_F)))$. *We have:*

$$\Phi(f, c_B f) = \pm i^{\lambda_0} \int_{GU_I(\mathbb{Q}) Z_{GU_I}(\mathbb{A}_\mathbb{Q})\backslash GU_I(\mathbb{A}_\mathbb{Q})} f(g)\overline{f}(g)||\nu(g)||^c dg. \tag{4.25}$$

Recall that $\nu(\cdot)$ *is the similitude defined in (2.1).*

Similarly, if we start from anti-holomorphic elements, we get a paring:

$$\Phi^- = \Phi^{I,\lambda,-} : \bar{H}^{D/2}(Sh_I, E_\lambda) \times \bar{H}^0(Sh_I, E_{w_0 * \lambda^\vee}) \to \mathbb{C}. \tag{4.26}$$

We use the script $^-$ to indicate that is anti-holomorphic. It is still c_{DR} stable. For $0 \neq f^- \in \bar{H}^{D/2}(Sh_I, E_\lambda)$, we also know that $\Phi^-(f^-, c_B f^-) \neq 0$.

4.6 Arithmetic automorphic periods

Let π be an irreducible cuspidal representation of $GU_I(\mathbb{A}_\mathbb{Q})$ defined over a number field $E(\pi)$. We may assume that $E(\pi)$ contains the quadratic imaginary field K.

We assume that π is cohomological with type λ, i.e. $H^*(\mathfrak{g}, K_{I,\infty}; \pi \otimes W_\lambda) \neq 0$.

For M a $GU_I(\mathbb{A}_{\mathbb{Q},f})$-module, define the K-rational π_f-isotypic components of M by

$$M^\pi := Hom_{GU_I(\mathbb{A}_{F,f})}(Res_{E(\pi)/K}(\pi_f), M) = \bigoplus_{\tau \in \Sigma_{E(\pi);K}} Hom(\pi_f^\tau, M).$$

Therefore, if M has a K-rational structure then M^π also has a K-rational structure.

As in section 4.4, we have inclusions:

$$H^q(\mathfrak{P}, K_{I,\infty}; \mathcal{A}_0^\pi \otimes V_\Lambda) \subset \bar{H}^q(Sh_I, E_\Lambda)^\pi \subset H^q(\mathfrak{P}, K_{I,\infty}; \mathcal{A}_{(2)}^\pi \otimes V_\Lambda).$$

Under these inclusions, c_B sends $\bar{H}^0(Sh_I, E_{w_0*\lambda})^\pi$ to $\bar{H}^{D/2}(Sh_I, E_{\lambda^\vee})^{\pi^\vee}$.

These inclusions are compatible with those K-rational structures and then induce K-rational parings

$$\Phi^\pi : \bar{H}^0(Sh_I, E_{w_0*\lambda})^\pi \times \bar{H}^{D/2}(Sh_I, E_{\lambda^\vee})^{\pi^\vee} \to \mathbb{C} \tag{4.27}$$

$$\text{and } \Phi^{-,\pi} : \bar{H}^{D/2}(Sh_I, E_\lambda)^\pi \times \bar{H}^0(Sh_I, E_{w_0*\lambda^\vee})^{\pi^\vee} \to \mathbb{C}. \tag{4.28}$$

Definition 4.6.1. *Let* β *be a non zero* K*-rational element of* $\bar{H}^0(Sh_I, E_{w_0*\lambda})^\pi$. *We define the* ***holomorphic arithmetic automorphic period associated to*** β *by* $P^{(I)}(\beta, \pi) := (\Phi^\pi(\beta^\tau, c_B\beta^\tau))_{\tau \in \Sigma_{E(\pi);K}}$. *It is an element in* $(E(\pi) \otimes_K \mathbb{C})^\times$.

Let γ *be a non zero* K*-rational element of* $\bar{H}^{D/2}(Sh_I, E_\lambda)^\pi$. *We define the* ***anti-holomorphic arithmetic automorphic period associated to*** γ *by* $P^{(I),-}(\gamma, \pi) := (\Phi^{-,\pi}(\gamma^\tau, c_B\gamma^\tau))_{\tau \in \Sigma_{E(\pi);K}}$. *It is an element in* $(E(\pi) \otimes_K \mathbb{C})^\times$.

Definition-Lemma 4.6.1. *Let us assume now π is tempered and π_∞ is discrete series representation. In this case, $\bar{H}^0(Sh_I, E_{w_0*\lambda})^\pi$ is a rank one $E(\pi)\otimes_K\mathbb{C}$-module (c.f. [21]).*

*We define the **holomorphic arithmetic automorphic period** of π by $P^{(I)}(\pi) := P^{(I)}(\beta,\pi)$ by taking β any non zero rational element in $\bar{H}^0(Sh_I, E_{w_0*\lambda})^\pi$. It is an element in $(E(\pi)\otimes_K\mathbb{C})^\times$ well defined up to $E(\pi)^\times$-multiplication.*

*We define $P^{(I),-}(\pi)$ the **anti-holomorphic arithmetic automorphic period** of π similarly.*

Lemma 4.6.1. *We assume that π is tempered and π_∞ is discrete series representation. Let β be a non zero rational element in $\bar{H}^0(Sh_I, E_{w_0*\lambda})^\pi$ and $\beta^\vee$ be a non zero rational element in $\bar{H}^0(Sh_I, E_\lambda^\vee)^{\pi^\vee}$.*

We have $c_B(\beta)\sim_{E(\pi)} P^{(I)}(\pi)\beta^\vee$.

Proof It is enough to notice that $\Phi^\pi(\beta,\beta^\vee)\in E(\pi)^\times$.

□

Lemma 4.6.2. *If π is tempered and π_∞ is discrete series representation then we have:*

1. *$P^{(I^c)}(\pi^c)\sim_{E(\pi);K} P^{(I)}(\pi)$.*
2. *$P^{(I)}(\pi^\vee) * P^{(I),-}(\pi)\sim_{E(\pi);K} 1$.*

Proof The first part comes from Lemma 4.5.1 and the fact that c_{DR} preserves rational structures.

For the second part, recall that the following two parings are actually the same:

$$\Phi^{\pi^\vee}: \bar{H}^0(Sh_I, E_{w_0*\lambda^\vee})^{\pi^\vee}\times \bar{H}^{D/2}(Sh_I, E_\lambda)^\pi\to\mathbb{C} \tag{4.29}$$
$$\text{and } \Phi^{-,\pi}: \bar{H}^{D/2}(Sh_I, E_\lambda)^\pi\times \bar{H}^0(Sh_I, E_{w_0*\lambda^\vee})^{\pi^\vee}\to\mathbb{C}. \tag{4.30}$$

We take β a rational element in $\bar{H}^0(Sh_I, E_{w_0*\lambda^\vee})^{\pi^\vee}$ and γ a rational element in $\bar{H}^{D/2}(Sh_I, E_\lambda)^\pi$. We may assume that $\Phi^{\pi^\vee}(\beta^\tau,\gamma^\tau)=\Phi^{-,\pi}(\gamma^\tau,\beta^\tau)=1$ for all $\tau\in\Sigma_{E(\pi);K}$.

By definition $p^{(I)}(\pi^\vee)=(\Phi^{\pi^\vee}(\beta^\tau, c_B\beta^\tau))_{\tau\in\Sigma_{E(\pi);K}}$. Since $\bar{H}^{D/2}(Sh_I, E_\lambda)^\pi$ is a rank one $E(\pi)\otimes\mathbb{C}$-module, there exists $C\in(E(\pi)\otimes_K\mathbb{C})^\times$ such that $(c_B\beta^\tau)_{\tau\in\Sigma_{E(\pi);K}} = C(\gamma^\tau)_{\tau\in\Sigma_{E(\pi);K}}$. Therefore $p^I(\pi^\vee)=C(\Phi^{\pi^\vee}(\beta^\tau,\gamma^\tau))_{\tau\in\Sigma_{E(\pi);K}}=C$.

On the other hand, since $c_B^2=Id$, we have $(c_B\gamma^\tau)_{\tau\in\Sigma_{E(\pi);K}}=C^{-1}(\beta^\tau)_{\tau\in\Sigma_{E(\pi);K}}$. We can deduce that $p^{(I),-}(\pi)=C^{-1}$ as expected.

□

Definition 4.6.2. *We say I is **compact** if $U_I(\mathbb{C})$ is. In other words, I is compact if and only if $I(\sigma)=0$ or n for all $\sigma\in\Sigma$.*

Corollary 4.6.1. *If I is compact then $P^{(I)}(\pi)\sim_{E(\pi);K} P^{(I),-}(\pi)$. We have $P^{(I)}(\pi^\vee) * P^{(I)}(\pi)\sim_{E(\pi);K} 1$.*

Proof If I is compact, then $w_0 = Id$. The anti-holomorphic part and holomorphic part are the same. We then have $P^{(I)}(\pi) \sim_{E(\pi);K} P^{(I),-}(\pi)$. The last assertion comes from Lemma 4.6.2.

□

Chapter 5

Special values of automorphic L-functions: the start point

5.1 Special values of automorphic L-functions for similitude unitary group

The method in [13] should work for a general CM field. We state the predicted formula in this section.

Let π be a tempered representation of $GU_I(\mathbb{A}_{\mathbb{Q}})$ such that π_∞ is discrete series representation. In particular, the holomorphic arithmetic automorphic periods $P^{(I)}(\pi)$ is well defined.

We assume that π is cohomological with type $\lambda = (\lambda_0, (\lambda_1(\sigma) \geqslant \lambda_2(\sigma) \cdots \geqslant \lambda_n(\sigma))_{\sigma\in\Sigma})$.

We say λ or π is 2-regular if $\lambda_i(\sigma) - \lambda_{i+1}(\sigma) \geqslant 1$ for all $1 \leqslant i \leqslant n-1$ and all $\sigma \in \Sigma$.

Let χ be an algebraic Hecke character of $\mathbb{A}_F^\times$ with infinity type $(z^{-k(\sigma)})_{\sigma\in\Sigma}$. Let α be an algebraic Hecke character of $\mathbb{A}_F^\times$ with infinity type $(z^\kappa)_{\sigma\in\Sigma}$.

Definition 5.1.1. *We set $\lambda_0(\sigma) = +\infty$ and $\lambda_{n+1}(\sigma) = -\infty$. We say $m \in \mathbb{Z}$ is critical for $M(\pi, \chi, \alpha)$ (c.f. [13]) if for all $\sigma \in \Sigma$,*

$$\begin{aligned}\lambda_{r_\sigma+1}(\sigma) + k(\sigma) + s_\sigma - \kappa \leqslant m \leqslant -\lambda_{r_\sigma+1}(\sigma) - k(\sigma) + r_\sigma \\ \textit{and } -\lambda_{r_\sigma}(\sigma) - k(\sigma) + r_\sigma \leqslant m \leqslant \lambda_{r_\sigma}(\sigma) + k(\sigma) + s_\sigma - \kappa.\end{aligned}$$

This definition generalize the condition in Lemma 3.3.7 of the *loc.cit.* In the *loc.cit*, it is assumed that μ is self-dual. In general, the index Λ in that Lemma should be $\Lambda(\mu^c; r, s)$.

We assume the following conjecture throughout the text.

Conjecture 5.1.1. *Let π be a tempered representation of $GU_I(\mathbb{A}_{\mathbb{Q}})$ such that π_∞ is discrete series representation and cohomological with type λ. In particular, the holomorphic arithmetic automorphic periods $P^{(I)}(\pi)$ is well defined. If an integer $m \geqslant \frac{n-\kappa}{2}$ is critical* *for $M(\pi,\chi,\alpha)$ then a certain rational differential operator exists and we have*

$$L^{mot,S}(m,\pi\otimes\chi,St,\alpha)\sim_{E(\pi)E(\chi)E(\alpha);K} (2\pi i)^{(m-\frac{n-1}{2})nd}(2\pi)^{-\lambda_0}P^{(I)}(\pi)\prod_{\sigma\in\Sigma}(p(\widetilde{\chi}\alpha,\sigma)^{-I(\sigma)}p(\widetilde{\chi}\alpha,\overline{\sigma})^{-n+I(\sigma)}) \quad . \tag{5.1}$$

Recall that $\widetilde{\chi}=\dfrac{\chi}{\chi^c}$ is a Hecke character of $\mathbb{A}_F$.

Remark 5.1.1. *1. If $m\geqslant\dfrac{n-\kappa}{2}$ is critical for $M(\pi,\chi,\alpha)$, we have $2\lambda_{r_\sigma+1}-2(r_\sigma+1)\leqslant\kappa-2k(\sigma)-n-2$ and $2\lambda_{r_\sigma}-2r_\sigma\geqslant\kappa-2k(\sigma)-n$. We see that $r_\sigma=\max\{r\mid 2\lambda_{n'}-2r\geqslant\kappa-2k(\sigma)-n\}$.*

2. We didn't state the CM periods in the above conjecture as in Theorem 3.5.13 of [13]. Instead, the current form appears in middle steps of the proof for Theorem 3.5.13. We refer to equation (2.9.12) or the third line in page 138 of the loc.cit.

Let us examine the condition in the above conjecture. After simple calculation, we see that such m always exists. Moreover, if $\lambda_{r_\sigma}>\lambda_{r_\sigma+1}$, we may have $m\geqslant\dfrac{n-\kappa+1}{2}$. In this case, we know $L^{mot,S}(m,\pi\otimes\chi,St,\alpha)$ does not vanish.

Let GU and GU' be two rational similitude group associated to two unitary groups over F with respect to F/F^+ of dimension n. We know GU' is an inner form of GU and thus they are isomorphic to each other at almost all primes.

Let π and π' be automorphic representations of $GU(\mathbb{A}_\mathbb{Q})$ and $GU'(\mathbb{A}_\mathbb{Q})$ respectively. We say π is **nearly equivalent** to π' if they are isomorphic to each other at almost all primes. In particular, they have the same value of partial L-functions.

We then deduce that:

Corollary 5.1.1. *The arithmetic automorphic periods $P^{(I)}(\pi)$ and $P^{(I),-}(\pi)$ depends only on the nearly equivalence class of π if π is 2-regular.*

5.2 Special values of automorphic L-functions for $GL_n\times GL_1$

Let Π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ which is regular conjugate self-dual, cohomological. We assume moreover that Π_q descends locally if n is even. By Lemma 2.3.1, there exists an Hecke character of K, denoted by ξ, such that $\Pi^\vee\otimes\xi$ is θ_I-stable. By Proposition 2.3.2, we know $\Pi^\vee\otimes\xi$ descends to π, an automorphic cohomological representation of $GU_I(\mathbb{A}_\mathbb{Q})$ with is tempered and discrete series at the infinity place. In particular, the arithmetic automorphic period of π can be defined.

Definition 5.2.1. *We fix one Hecke character ξ as above. We denote its infinity type by $z^u\overline{z}^v$. We define the* ***arithmetic automorphic period*** *for Π by $P^{(I)}(\Pi,\xi):=(2\pi)^{u+v}P^{(I)}(\pi)$.*

If Π is 2-regular then $P^{(I)}(\Pi,\xi)$ does not depend on the choice of ξ up to elements in $E(\Pi)$. This is a corollary of Theorem 5.2.1. Therefore we may define $P^{(I)}(\Pi):=P^{(I)}(\Pi,\xi)$ for any fixed ξ in this case.

Lemma 5.2.1. *Let* Π *be as in the above theorem.*

1. *For all* I *we have* $P^{(I)}(\Pi) \sim_{E(\Pi);K} P^{(I^c)}(\Pi^c)$.
2. *If* I *is compact then* $P^{(I)}(\Pi^\vee) * P^{(I)}(\Pi) \sim_{E(\Pi);K} 1$. *In particular, since* Π *is conjugate self-dual, we have* $P^{(I^c)}(\Pi) * P^{(I)}(\Pi) \sim_{E(\Pi);K} 1$ *in this case.*

Proof The first part comes from the first part of Lemma 4.6.2. It is also a direct corollary from Theorem 5.2.1 below.

The second one follows from Corollary 4.6.1. Let ξ be the auxiliary Hecke character with infinity type $z^u \overline{z}^v$ as above and π be the representation of $GU_I(\mathbb{A}_{\mathbb{Q}})$ with base change $\Pi^\vee \otimes \xi$. We know $\pi^\vee$ is a representation of $GU_I(\mathbb{A}_{\mathbb{Q}})$ with base change $\Pi \otimes \xi^{-1}$.

By definition

$$\begin{aligned} & P^{(I)}(\Pi) \sim_{E(\Pi);K} (2\pi)^{u+v} P^{(I)}(\pi) \\ \text{and} \quad & P^{(I)}(\Pi^\vee) \sim_{E(\Pi);K} (2\pi)^{-u-v} P^{(I)}(\pi^\vee). \end{aligned} \tag{5.2}$$

Thus $P^{(I)}(\Pi^\vee) * P^{(I)}(\Pi) \sim_{E(\Pi);K} P^{(I)}(\pi^\vee) * P^{(I)}(\pi) \sim_{E(\Pi);K} 1$.

□

Critical points: Let n, n' be two integers. Let Π and Π' be algebraic automorphic representations of $GL_n(\mathbb{A}_F)$ and $GL_{n'}(\mathbb{A}_F)$ with pure infinity type $(z^{a_i(\sigma)}\overline{z}^{-\omega(\Pi)-a_i(\sigma)})_{1\leqslant i\leqslant n}$ and $(z^{a'_j(\sigma)}\overline{z}^{-\omega(\Pi)-a'_j(\sigma)})_{1\leqslant j\leqslant n'}$ respectively.

We assume the existence of motives $M(\Pi)$ and $M(\Pi')$ associated to Π and Π'. Let $m \in \mathbb{Z}+\dfrac{n+n'}{2}$. We say m is **critical** for $\Pi\otimes\Pi'$ if $m+\dfrac{n+n'-2}{2}$ is critical for $M(\Pi)\otimes M(\Pi')$ in the sense of Deligne (c.f. [7] or Chapter 6).

If $a_i(\sigma) + a'_j(\sigma) \neq \dfrac{-\omega(\Pi) - \omega(\Pi')}{2}$ for all i, j and σ then critical points always exist. In this case, we have an explicit description for them (c.f. (1.3.1) of [7]). More precisely, m is critical if and only if for all i, j, σ, if $-a_i(\sigma) - a'_j(\sigma) > \dfrac{\omega(\Pi)+\omega(\Pi')}{2}$ then $\omega(\Pi)+\omega(\Pi')+1+a_i(\sigma)+a'_j(\sigma) \leqslant m \leqslant -a_i(\sigma)-a'_j(\sigma)$, if $-a_i(\sigma)-a'_j(\sigma) < \dfrac{\omega(\Pi)+\omega(\Pi')}{2}$ then $1-a_i(\sigma)-a'_j(\sigma) \leqslant m \leqslant \omega(\Pi)+\omega(\Pi')+a_i(\sigma)+a'_j(\sigma)$. Roughly speaking, m should be closer to the central point than any of the $a_i(\sigma)+a'_j(\sigma)$.

If $a_i(\sigma) + a'_j(\sigma) = \dfrac{-\omega(\Pi) - \omega(\Pi')}{2}$ for some i, j and σ then there is no critical points (c.f. Lemma 1.7.1 of [13]).

The following theorem follows directly from Conjecture 5.1.1.

Theorem 5.2.1. *Let us assume that Conjecture 5.1.1 is true. Let* Π *be a regular, conjugate self-dual, cohomological, cuspidal automorphic representation of* $GL_n(\mathbb{A}_F)$ *which descends to* $U_I(\mathbb{A}_{F^+})$ *for any* I *(c.f. Proposition 2.3.1). We denote the infinity type of* Π

at $\sigma \in \Sigma$ by $(z^{a_i(\sigma)}\overline{z}^{-a_i(\sigma)})_{1\leqslant i\leqslant n}$.

Let η be an algebraic Hecke character of F with infinity type $z^{a(\sigma)}\overline{z}^{b(\sigma)}$ at $\sigma \in \Sigma$. We know that $a(\sigma)+b(\sigma)$ is a constant independent of σ, denoted by $-\omega(\eta)$.

We suppose that $a(\sigma)-b(\sigma)+2a_i(\sigma)\neq 0$ for all $1\leqslant i\leqslant n$ and $\sigma\in\Sigma$. We define $I := I(\Pi,\eta)$ to be the map on Σ which sends $\sigma\in\Sigma$ to $I(\sigma) := \#\{i : a(\sigma)-b(\sigma)+2a_i(\sigma)<0\}$.

Let $m\in\mathbb{Z}+\dfrac{n-1}{2}$. If $m\geqslant\dfrac{1+\omega(\eta)}{2}$ is critical for $\Pi\otimes\eta$, we have:

$$L(m,\Pi\otimes\eta)\sim_{E(\Pi)E(\eta);K}(2\pi i)^{mnd}P^{(I(\Pi,\eta))}(\Pi)\prod_{\sigma\in\Sigma}p(\breve{\eta},\sigma)^{I(\sigma)}p(\breve{\eta},\overline{\sigma})^{n-I(\sigma)}. \tag{5.3}$$

Lemma 5.2.2. *Let χ, α be as in Conjecture 5.1.1. Let π be a representation of GU_I with base change $\Pi^c\times\xi$ for certain auxiliary ξ. We set $\eta^c=\widetilde{\chi}\alpha$. Let $m\in\mathbb{Z}+\dfrac{n+1}{2}$. Then m is critical for $\Pi\otimes\eta$ if and only if $m+\dfrac{n-1}{2}$ is critical for $M(\pi,\chi,\alpha)$.*

Proof Since $\eta^c=\widetilde{\chi}\alpha$, we have $b(\sigma)=-k(\sigma)+\kappa$ and $a(\sigma)=k(\sigma)$. Note that $-\omega(\eta)=a(\sigma)+b(\sigma)=\kappa$.

We write the cohomology type of π by $(\lambda_0,(\lambda_1(\sigma)\geqslant\cdots\lambda_n(\sigma)))$. We order $a_i(\sigma)$ in decreasing order. The cohomology type of π is then $(-u-v;(a_1(\sigma)-\dfrac{n-1}{2}\geqslant a_2(\sigma)-\dfrac{n-3}{2}\geqslant\cdots\geqslant a_n(\sigma)+\dfrac{n-1}{2})_{\sigma\in\Sigma})$. This gives $\lambda_0=-u-v$ and $\lambda_i(\sigma)=a_i(\sigma)+i-\dfrac{n+1}{2}$.

Let $m\in\mathbb{Z}+\dfrac{n-1}{2}$. By the above discussion, m is critical for $\Pi\otimes\eta$ if and only if $a_i(\sigma)-b(\sigma)+1\leqslant m\leqslant -a_i(\sigma)-a(\sigma)$ if $a(\sigma)-b(\sigma)+2a_i(\sigma)<0$, $1-a_i(\sigma)-a(\sigma)\leqslant m\leqslant a_i(\sigma)-b(\sigma)$ if $a(\sigma)-b(\sigma)+2a_i(\sigma)>0$. Since $r_\sigma := \max\{i : a(\sigma)-b(\sigma)+2a_i(\sigma)>0\}$, we deduce that m is critical for $\Pi\otimes\eta$ if and only if $a_{r_\sigma+1}(\sigma)-b(\sigma)+1\leqslant m\leqslant -a_{r_\sigma+1}(\sigma)-a(\sigma)$ and $1-a_{r_\sigma}(\sigma)-a(\sigma)\leqslant m\leqslant a_{r_\sigma}(\sigma)-b(\sigma)$. It is easy to see that these two equations are the same with those in Definition 5.1.1.

□

Proof of Theorem 5.2.1: We can always choose χ and α as in Conjecture 5.1.1 such that $\eta^c=\widetilde{\chi}\alpha$.

As $m\geqslant\dfrac{1+\omega(\eta)}{2}$ we have $m+\dfrac{n-1}{2}\geqslant\dfrac{n+\omega(\eta)}{2}$. Moreover, the above lemma implies that $m+\dfrac{n-1}{2}$ is critical for $M(\pi,\chi,\alpha)$ and then Conjecture 5.1.1 applies, namely:

$$\begin{aligned}&L(m+\frac{n-1}{2},\Pi^\vee\otimes\widetilde{\chi}\alpha)\sim_{E(\pi)E(\chi)E(\alpha);K}\\&(2\pi i)^{mnd}(2\pi)^{-\lambda_0}P^{(I(\Pi,\eta))}(\pi)\prod_{\sigma\in\Sigma}(p(\widetilde{\chi}\alpha,\sigma)^{-s_\sigma}p(\widetilde{\chi}\alpha,\overline{\sigma})^{-n+s_\sigma})\quad.\end{aligned}\tag{5.4}$$

Recall by definition that $P^{(I(\Pi,\eta))}(\Pi,\xi)=(2\pi)^{-\lambda_0}P^{(I(\Pi,\eta))}(\pi)$. Moreover, $\eta^c=\widetilde{\chi}\alpha$ and then $p(\widetilde{\chi}\alpha,\sigma)\sim p(\eta^c,\sigma)\sim p(\check{\eta},\sigma)^{-1}$ and $p(\widetilde{\chi}\alpha,\overline{\sigma})\sim p(\check{\eta},\overline{\sigma})^{-1}$. We deduce that the right hand side of (5.4) is equivalent to $(2\pi i)^{mnd}(P^{(I(\Pi,\eta))}(\Pi,\xi)\prod\limits_{\sigma\in\Sigma}(p(\check{\eta},\sigma)^{s_\sigma}p(\check{\eta},\overline{\sigma})^{n-s_\sigma})$.

We end our proof by the fact that $L(m,\Pi\otimes\eta)=L(m,\Pi^c\otimes\eta^c)=L(m,\Pi^\vee\otimes\widetilde{\chi}\alpha)=L^{mot}(m+\dfrac{n-1}{2},\pi\otimes\chi,St,\alpha)$.

□

5.3 Arithmetic automorphic periods for conjugate self-dual representations

If we consider $\Pi^*:=\Pi\otimes\eta$, it is not conjugate self-dual but $\Pi^*\otimes\eta^{-1}$ is. We call such representations **conjugate self-dual**. We want to generalize the definition for arithmetic automorphic period to such representations.

We firstly generalize the definition for $I(\Pi,\eta)$ in Theorem 5.2.1.

Definition-Lemma 5.3.1. *Let Π^* be an algebraic regular representation of $GL_n(F)$ with infinity type $(z^{a_i(\sigma)}\overline{z}^{b_i(\sigma)})_{1\leqslant i\leqslant n}$ at $\sigma\in\Sigma$. Let η be am algebraic Hecke character of F with infinity type $z^{a(\sigma)}\overline{z}^{b(\sigma)}$ at $\sigma\in\Sigma$. We assume that $a(\sigma)-b(\sigma)+a_i(\sigma)-b_i(\sigma)\neq 0$ for all σ and i.*

We define $I(\Pi^*,\eta)$ to be the map sending $\sigma\in\Sigma$ to $\#\{i:a(\sigma)-b(\sigma)+a_i(\sigma)-b_i(\sigma)<0\}$.

It is easy to see that $I(\Pi^*,\eta_1\eta_2)=I(\Pi^*\otimes\eta_1,\eta_2)$ for any η_1, η_2.

We can now define the arithmetic automorphic periods.

Definition-Lemma 5.3.2.

We say a 3-regular cohomological cuspidal automorphic representation Π^* of $GL_n(\mathbb{A}_F)$ has ***definable arithmetic automorphic periods*** *if there exists an algebraic Hecke character η of F such that $\Pi^*\otimes\eta^{-1}$ descends to unitary groups of any sign. In particular, $\Pi^*\otimes\eta^{-1}$ is conjugate self-dual.*

In this case, for any sign I, i.e. a map from Σ to $\{0,1,\cdots,n\}$, we define the ***arithmetic automorphic period*** *for Π^* by $P^{(I)}(\Pi^*):=P^{(I)}(\Pi^*\otimes\eta^{-1})\prod\limits_{\sigma\in\Sigma}p(\check{\eta},\sigma)^{I(\sigma)}p(\check{\eta},\overline{\sigma})^{n-I(\sigma)}$.*

This definition does not depend on the choice of η and hence is compatible with Definition 5.2.1 if Π^* itself is conjugate self-dual.

Proof The last part comes from Theorem 5.2.1. In fact, for any I, let χ be an algebraic Hecke character such that $I(\Pi^*,\chi)=I$. Since Π^* is 3-regular, we may choose χ such that there exists $m\geqslant 1+\dfrac{\omega(\eta)+\omega(\chi)}{2}$ critical for $\Pi^*\otimes\chi$.

Let η be an algebraic Hecke character such that $\Pi:=\Pi^*\otimes\eta^{-1}$ is conjugate self-dual, we have $\Pi^*\otimes\chi=\Pi\otimes(\eta\chi)$.

Since $I(\Pi,\eta\chi)=I(\Pi^*,\chi)=I$, Theorem 5.2.1 gives that:

$$\begin{aligned} & L(m,\Pi^*\otimes\chi)=L(m,\Pi\otimes(\eta\chi)) \\ \sim_{E(\Pi)E(\eta)E(\chi);K}\ & (2\pi i)^{mnd}P^{(I)}(\Pi)\prod_{\sigma\in\Sigma}p(\widecheck{\eta\chi},\sigma)^{I(\sigma)}p(\widecheck{\eta\chi},\overline{\sigma})^{n-I(\sigma)} \\ \sim_{E(\Pi)E(\eta)E(\chi);K}\ & (2\pi i)^{mnd}P^{(I)}(\Pi)\prod_{\sigma\in\Sigma}p(\check{\eta},\sigma)^{I(\sigma)}p(\check{\eta},\overline{\sigma})^{n-I(\sigma)}\prod_{\sigma\in\Sigma}p(\check{\chi},\sigma)^{I(\sigma)}p(\check{\chi},\overline{\sigma})^{n-I(\sigma)} \end{aligned}$$

with both sides non zero.

In particular, $P^{(I)}(\Pi^*\otimes\eta^{-1})\prod\limits_{\sigma\in\Sigma}p(\check{\eta},\sigma)^{I(\sigma)}p(\check{\eta},\overline{\sigma})^{n-I(\sigma)}$

$$\sim_{E(\Pi^*);K}(2\pi i)^{-mnd}\Big(\prod_{\sigma\in\Sigma}p(\check{\chi},\sigma)^{I(\sigma)}p(\check{\chi},\overline{\sigma})^{n-I(\sigma)}\Big)^{-1}L(m,\Pi^*\otimes\chi)$$

which does not depend on the choice of η.

□

Remark 5.3.1. *Let Π be a 3-regular cohomological cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ which is conjugate self-dual after tensoring an algebraic Hecke character η. Let q be a prime number inert in F^+ and split in F. If $(\Pi\otimes\eta)_q$ descends locally then Π has definable arithmetic automorphic periods. In particular, this holds true if Π_q is in discrete series.*

We read from the above proof that Theorem 5.2.1 can be rewritten as follows:

Theorem 5.3.1. *Let Π be an algebraic automorphic representation of $GL_n(\mathbb{A}_F)$ which has definable arithmetic automorphic periods. Let η be an algebraic Hecke character as in Definition 5.3.1. We write $I:=I(\Pi,\eta)$.*

Let $m\in\mathbb{Z}+\frac{n-1}{2}$ be critical for $\Pi\otimes\eta$. If $m\geqslant\frac{1+\omega(\Pi)+\omega(\eta)}{2}$ then

$$L(m,\Pi\otimes\eta)\sim_{E(\Pi)E(\eta);K}(2\pi i)^{mnd}P^{(I(\Pi,\eta))}(\Pi)\prod_{\sigma\in\Sigma}p(\check{\eta},\sigma)^{I(\sigma)}p(\check{\eta},\overline{\sigma})^{n-I(\sigma)}. \tag{5.5}$$

Moreover, there always exists such m with both sides non zero.

Remark 5.3.2. *The last part comes from the fact that Π is 3-regular. The 3-regular condition is not needed to define the arithmetic automorphic periods in general. We assume it here to guarantee that Definition 5.3.2 does not depend on the choice of η. One can replace this condition by a weaker one on the non vanishing property for certain L-functions.*

Chapter 6

Motives and Deligne's conjecture

6.1 Motives over $\mathbb{Q}$

In this article, a **motive** simply means a pure motive for absolute Hodge cycles in the sense of Deligne [7].

More precisely, a motive over $\mathbb{Q}$ with coefficients in a number field E is given by its Betti realization M_B, its de Rham realization M_{DR} and its l-adic realization M_l for all prime numbers l where M_B and M_{DR} are finite dimensional vector space over E, M_l is a finite dimensional vector space over $E_l := E \otimes_{\mathbb{Q}} \mathbb{Q}_l$ endowed with:

- $I_\infty : M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes \mathbb{C}$ as $E \otimes_{\mathbb{Q}} \mathbb{C}$-module;
- $I_l : M_B \otimes \mathbb{Q}_l \xrightarrow{\sim} M_l$ as $E \otimes_{\mathbb{Q}} \mathbb{Q}_l$-module.

From the isomorphisms above, we see that $dim_E M_B = dim_E M_{DR} = dim_{E_l} M_l$ and this is called the **rank** of M. We need moreover:

1. An E-linear involution (infinite Frobenius) F_∞ on M_B and a Hodge decomposition $M_B \otimes \mathbb{C} = \bigoplus\limits_{p,q \in \mathbb{Z}} M^{p,q}$ as $E \otimes \mathbb{C}$-module such that F_∞ sends $M^{p,q}$ to $M^{q,p}$.

   For w an integer, we say M is **pure of weight** w if $M^{p,q} = 0$ for $p + q \neq w$. Throughout this paper, all the motives are assumed to be pure. We assume also that F_∞ acts on $M^{p,p}$ as a scalar for all $p \in \mathbb{Z}$.

   We say M is **regular** if $dim M^{p,q} \leqslant 1$ for all $p, q \in \mathbb{Z}$.

2. An E-rational Hodge filtration on M_{DR}: $\cdots \supset M^i \supset M^{i+1} \supset \cdots$ which is compatible with the Hodge structure on M_B via I_∞, i.e.,
$$I_\infty(\bigoplus_{p \geqslant i} M^{p,q}) = M^i \otimes \mathbb{C}.$$

3. A Galois action of $G_{\mathbb{Q}}$ on each M_l such that $(M_l)_l$ forms a compatible system of l-adic representations $\rho_l : G_{\mathbb{Q}} \longrightarrow GL(M_l)$. More precisely, for each prime number p, let I_p be the inertia subgroup of a decomposition group at p and F_p the geometric Frobenius of this decomposition group. We have that for all $l \neq p$, the polynomial $det(1 - F_p | M_l^{I_p})$ has coefficients in E and is independent of the choice of l. We can then define $L_p(s, M) := det(1 - p^{-s} F_p | M_l^{I_p})^{-1} \in E(p^{-s})$ for whatever $l \neq p$.

For any fixed embedding $\sigma : E \hookrightarrow \mathbb{C}$, we may consider $L_p(s, M, \sigma)$ as a complex valued function. We define $L(s, M, \sigma) = \prod_p L_p(s, M, \sigma)$. It converges for $Re(s)$ sufficiently large. It is conjectured that the L-function has analytic continuation and functional equation on the whole complex plane.

We can also define $L_\infty(s, M)$, the infinite part of the L-function, as in chapter 5 of [7].

Deligne has defined the critical values for M as follows:

Definition 6.1.1. *We say an integer m is **critical** for M if neither $L_\infty(M, s)$ nor $L_\infty(\check{M}, 1 - s)$ has a pole at $s = m$ where $\check{M}$ is the dual of M. We call m a **critical value** of M.*

Remark 6.1.1. *The notion $L_\infty(s, M)$ implicitly indicates that the infinity type of the L-function does not depend on the choice of $\sigma : E \hookrightarrow \mathbb{C}$. More precisely, for every $\sigma : E \hookrightarrow \mathbb{C}$, put $M_{B,\sigma} := M_B \otimes_{E,\sigma} \mathbb{C}$.*

We then have $M_B \otimes \mathbb{C} = \bigoplus_{\sigma:E\hookrightarrow\mathbb{C}} M_{B,\sigma}$. Since $M^{p,q}$ is stable by E, each $M_{B,\sigma}$ inherits a Hodge decomposition $M_{B,\sigma} = \bigoplus M^{p,q}_{B,\sigma}$. We may define $L_\infty(s, M, \sigma)$ with help of the Hodge decomposition of $M_B \otimes_{E,\sigma} \mathbb{C}$. It is a product of Γ factors which depend only on $\dim M^{p,q}_{B,\sigma}$ and the action of F_∞ on $M^{p,p}_{B,\sigma}$. The latter is independent of σ since we have assumed that F_∞ acts on $M^{p,p}$ by a scalar.

It remains to show that $\dim M^{p,q}_{B,\sigma}$ is also independent of σ. In fact, since M is pure, $M^{p,q}$ can be reconstructed from the Hodge filtration M^i. Hence $M^{p,q} = \bigoplus M^{p,q}_{B,\sigma}$ is a free $E \otimes \mathbb{C}$-module. One can show $M^{p,q}_{B,\sigma} = M^{p,q} \otimes_{E,\sigma} \mathbb{C}$ and hence $dim M^{p,q}_{B,\sigma}$ is independent of σ.

If F_∞ acts as a scalar at $M^{p,p}$ for every p then Deligne's period can be defined. We will only treat the case when M has no (p, p) class. Therefore, Deligne's period can always be defined.

Definition 6.1.2. *Let M be a motive over $\mathbb{Q}$ of weight ω which has no $(\omega/2, \omega/2)$ class. We denote by M_B^+ the subspace of M_B fixed by F_∞. We denote by $F^+(M) := F^{\omega/2}(M)$ a subspace of M_{DR}. It is easy to see that $I_\infty^{-1}(F^+(M) \otimes \mathbb{C})$ equals to $\bigoplus_{p>q} M^{p,q}$.*

The comparison isomorphism then induces an isomorphism:

$$M_B^+ \otimes \mathbb{C} \hookrightarrow M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes \mathbb{C} \to (M_{DR}/F^+(M)) \otimes \mathbb{C}. \tag{6.1}$$

Deligne's period $c^+(M)$ is defined to be the determinant of the above isomorphism with respect to fixed $E(M)$-bases of M_B^+ and $M_{DR}/F^+(M)$. It is well defined up to $E(M)^\times$,

Deligne has conjectured in [7] that:

Conjecture 6.1.1. *If 0 is critical for M (see the loc.cit for the definition of critical), then $L(0, M, \sigma) \sim_{E(M)} c^+(M)$.*

More generally, tensoring M by the Tate motive $\mathbb{Q}(m)$ (c.f. [7] chapter 1), we obtained a new motive $M(m)$. We remark that $L(s, M(m), \sigma) = L(s+m, M, \sigma)$. The following conjecture is a corollary of the previous conjecture:

Conjecture 6.1.2. *If m is critical for M, then*

$$L(m, M, \sigma) \sim_{E(M)} (2\pi i)^{d^+ m} c^+(M) \tag{6.2}$$

where $d^+ = dim_{E(M)}(M_B^+)$.

Deligne has given a criteria to determine whether 0 is critical for M (see (1.3.1) of [7]). We observe that n is critical for M if and only if 0 is critical for $M(n)$. Thus we can rewrite the criteria of Deligne for arbitrary n. In the case where $M^{p,p} = 0$ for all p, this criteria becomes rather simple.

We first define the **Hodge type** of M by the set $T = T(M)$ consisting of pairs (p, q) such that $M^{p,q} \neq 0$. Since M is pure, there exists an integer w such that $p + q = w$ for all $(p, q) \in T(M)$. We remark that if (p, q) is an element of $T(M)$, then (q, p) is also contained $T(M)$.

Lemma 6.1.1. *Let M be a pure motive of weight w. We assume that for all $(p, q) \in T(M)$, $p \neq q$ which is equivalent to that $p \neq \frac{w}{2}$.*

Let $p_1 < p_2 < \cdots < p_n$ be some integers such that

$$T(M) = \{(p_1, q_1), (p_2, q_2), \cdots, (p_n, q_n)\} \cup \{(q_1, p_1), (q_2, p_2), \cdots, (q_n, p_n)\}$$

where $q_i = w - p_i$ for all $1 \leqslant i \leqslant n$.

We set $p_0 = -\infty$ and $p_{n+1} = +\infty$. Denote by $k := max\{0 \leqslant i \leqslant n \mid p_i < \frac{w}{2}\}$. We have that m is critical for M if and only if

$$\max(p_k + 1, w + 1 - p_{k+1}) \leqslant m \leqslant \min(w - p_k, p_{k+1}).$$

In particular, critical value always exists in the case where $p_i \neq q_i$ for all i.

Proof The Hodge type of $M(m)$ is $\{(p_i - m, w - p_i - m) \mid 1 \leqslant i \leqslant n\} \cup \{(w - p_i - m, p_i - m) \mid 1 \leqslant i \leqslant n\}$. By Deligne's criteria, 0 is critical for M if and only if for all i, either $p_i - m \leqslant -1$ and $w - p_i - m \geqslant 0$, or $p_i - m \geqslant 0$ and $w - p_i - m \leqslant -1$. Hence the set of critical values for M are $\bigcap_{1 \leqslant i \leqslant n} ([w + 1 - p_i, p_i] \cup [p_i + 1, w - p_i])$.

For $i \leqslant k$, $p_i < \frac{w}{2}$ and then $p_i < w + 1 - p_i$. Therefore $\bigcap_{1 \leqslant i \leqslant k} ([w + 1 - p_i, p_i] \cup [p_i + 1, w - p_i]) = \bigcap_{1 \leqslant i \leqslant k} [p_i + 1, w - p_i] = [p_k + 1, w - p_k]$. Similarly we have $\bigcap_{k < i \leqslant n} ([w + 1 - p_i, p_i] \cup [p_i + 1, w - p_i]) = \bigcap_{k < i \leqslant n} [w + 1 - p_i, p_i] = [w + 1 - p_{k+1}, p_{k+1}]$.

We deduce, at last, that the set of critical values for M is $[\max(p_k + 1, w + 1 - p_{k+1}), \min(w - p_k, p_{k+1})]$. It is easy to verify that the latter set is non empty.

□

6.2 Motivic periods over quadratic imaginary fields

Recall that K is a quadratic imaginary field with fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}$. Let E be a number field.

Let M be a regular motive over K (with respect to the fixed embedding) with coefficients in E of dimension n pure of weight $\omega(M)$.

Recall that M_B, the Betti realization of M, is a finite dimensional E-vector space. The infinite Frobenius gives an E-linear isomorphism $F_\infty : M_B \xrightarrow{\sim} M_B^c$.

Since M is regular of dimension n, we can write its Hodge type by $(p_i, q_i = \omega(M) - p_i)_{1\leqslant i\leqslant n}$ with $p_1 > p_2 > \cdots > p_n$. The Betti realization M_B has a Hodge decomposition $M_B \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus\limits_{i=1}^{n} M^{p_i,q_i}$ as $E \otimes_{\mathbb{Q}} \mathbb{C}$-modules.

We write the Hodge type of M^c by $(p_i^c, q_i^c = \omega(M) - p_i^c)_{1\leqslant i\leqslant n}$ with $p_i^c = q_{n+1-i} = \omega(M) - p_{n+1-i}$. Note that the Hodge numbers p_i^c are still in decreasing order. We know $M_B^c \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus\limits_{i=1}^{n} (M^c)^{p_i^c,q_i^c}$ and F_∞ induces E-linear isomorphisms: $M^{p_i,q_i} \xrightarrow{\sim} (M^c)^{p_{n+1-i}^c, q_{n+1-i}^c}$.

The De Rham realization M_{DR} is also a finite dimensional E-linear space endowed with a Hodge filtration $M_{DR} = M^{p_n} \supset M^{p_{n-1}} \supset \cdots \supset M^{p_1}$. The comparison isomorphism:

$$I_\infty : M_B \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes_{\mathbb{Q}} \mathbb{C} \tag{6.3}$$

induces compatibility isomorphisms on the Hodge decomposition of M_B and the Hodge filtration on M_{DR}.

More precisely, for each $1 \leqslant i \leqslant n$, I_∞ induces an isomorphism:

$$I_\infty : \bigoplus_{p_j \geqslant p_i} M^{p_j,q_j} = \bigoplus_{j\leqslant i} M^{p_j,q_j} \xrightarrow{\sim} M^{p_i} \otimes_{\mathbb{Q}} \mathbb{C} \tag{6.4}$$

Definition 6.2.1. *For any fixed E-bases of M_B and M_{DR}, we can extend them to $E\otimes_{\mathbb{Q}}\mathbb{C}$ bases of $M_B \otimes \mathbb{C}$ and $M_{DR} \otimes \mathbb{C}$ respectively. We define $\delta^{Del}(M)$ to be the determinant of I_∞ with respect to the fixed E-rational bases, called the* ***determinant period****. It is an element in $(E \otimes \mathbb{C})^\times$ well defined up to multiplication by elements in $E^\times \subset (E \otimes \mathbb{C})^\times$.*

This is an analogue of Deligne's period δ defined in (1.7.3) of [7].

Let us now fix some bases. We take $(e_i)_{1\leqslant i\leqslant n}$ an E-base of M_B. Since F_∞ is E-linear on M_B, we know $(e_i^c := F_\infty e_i)_{1\leqslant i\leqslant n}$ forms an E-base of M_B^c.

From (6.4) we see that I_∞ induces an isomorphism $M^{p_i,q_i} \xrightarrow{\sim} (M^{p_i})\otimes_{\mathbb{Q}}\mathbb{C}/(M^{p_{i-1}})\otimes_{\mathbb{Q}}\mathbb{C}$ for any $1 \leqslant i \leqslant n$. Here we set $M^{p_0} = \{0\}$. Let ω_i be a non zero element in M^{p_i,q_i} such that the image of ω_i by the above isomorphism is in $M^{p_i}(\text{mod } (M^{p_{i-1}}) \otimes_{\mathbb{Q}} \mathbb{C})$. In other words, $I_\infty(\omega_i)$ is equivalent to an element in M^{p_i} modulo $(M^{p_{i-1}}) \otimes_{\mathbb{Q}} \mathbb{C}$.

Since $(\omega_i)_{1\leqslant i\leqslant n}$ forms an $E \otimes \mathbb{C}$-base of $M_B \otimes \mathbb{C}$, we know $(I_\infty(\omega_i))_{1\leqslant i\leqslant n}$ forms an $E \otimes \mathbb{C}$-base of $M_{DR} \otimes \mathbb{C}$. This base is not rational, i.e. is not contained in M^{DR}. But by the above construction, it can pass to a rational base of $M_{DR} \otimes_{\mathbb{Q}} \mathbb{C}$ with a unipotent matrix for change of basis. Since the determinant of a unipotent matrix is always one, we can use this base for calculating $\delta^{Del}(M)$.

We define $\omega_i^c \in (M^c)^{p_i^c, q_i^c}$ similarly. We will use $(I_\infty(\omega_i^c))_{1 \leqslant i \leqslant n}$ as an $E \otimes_Q \mathbb{C}$ base of $M_{DR} \otimes \mathbb{C}$ to calculate motivic period henceforth.

Definition 6.2.2. *Since $M^{p_i^c, q_i^c}$ is a rank one free $E \otimes \mathbb{C}$-module, we know there exists numbers $Q_i(M) \in (E \otimes \mathbb{C})^\times$, $1 \leqslant i \leqslant n$ such that $F_\infty \omega_i = Q_i(M) \omega_{n+1-i}^c$ for all i. These numbers in $(E \otimes \mathbb{C})^\times$ are called **motivic period** and well defined up to $E^\times$.*

Since $F_\infty^2 = Id$, we have $F_\infty \omega_{n+1-i}^c = Q_i(M)^{-1} \omega_i$. We deduce that:

Lemma 6.2.1. *For all $1 \leqslant i \leqslant n$, $Q_i(M^c) \sim_{E(M);K} Q_{n+1-i}(M)^{-1}$.*

We write $\omega_a = \sum\limits_{i=1}^n A_{ia} e_i$, $\omega_t^c = \sum\limits_{i=1}^n A_{it}^c e_i^c$ for all $1 \leqslant a, t \leqslant n$.

We know $\delta^{Del}(M) = det(A_{ia})_{1 \leqslant i,a \leqslant n}$ and $\delta^{Del}(M^c) = det(A_{it}^c)_{1 \leqslant i,t \leqslant n}$. This implies that $\bigwedge_{i=1}^n \omega_i = \delta^{Del}(M) \bigwedge_{i=1}^n e_i$.

We denote by $det(M)$ the determinant motive of M as in section 1.2 of [16]. We know $I_\infty(\bigwedge_{i=1}^n \omega_i)$ is an E-base of $det(M)_{DR}$ and $\bigwedge_{i=1}^n e_i$ is an E-base of $det(M)_B$. Moreover, $F_\infty(\bigwedge_{i=1}^n \omega_i) = \prod\limits_{1 \leqslant i \leqslant n} Q_i(M) \bigwedge_{i=1}^n \omega_i^c$.

We deduce that:

Lemma 6.2.2.

$$\delta^{Del}(M) \sim_{E(M);K} \delta^{Del}(det(M)) \tag{6.5}$$

$$Q_1(det(M)) \sim_{E(M);K} \prod_{i=1}^n Q_i(M) \tag{6.6}$$

Remark 6.2.1. *The determinant period $\delta^{Del}(M)$ is inverse of the period δ defined in [16]. In fact, the period $\delta(M)$ is defined by equation (1.2.4) of [16], namely, $\bigwedge_{i=1}^n e_i = \delta(M) \bigwedge_{i=1}^n \omega_i$. Therefore $\delta(M) \sim_{E(M);K} \delta^{Del}(det(M))^{-1} \sim_{E(M);K} \delta^{Del}(M)^{-1}$.*

Lemma 6.2.3. *For all motive M as above, we have:*

$$\delta^{Del}(M^c) \sim_{E(M);K} (\prod_{1 \leqslant i \leqslant n} Q_i^{-1}) \delta^{Del}(M)$$

Proof This follows directly from equation (6.10).

One can also prove this with help of Lemma 6.2.2. In fact, by Lemma 6.2.2, we may assume that $n = 1$. We take $\omega \in M_{DR}$, $\omega^c \in M_{DR}^c$ and $e \in M_B$. Then $\omega = \delta^{Del}(M)e$ and $\omega^c = \delta^{Del}(M^c)e^c$ where $e^c = F_\infty e$.

By definition of motivic period, we have $F_\infty\omega = Q_1(M)\omega^c$ and then $\omega^c = Q_1(M)^{-1}F_\infty\omega = Q_1(M)^{-1}F_\infty(\delta^{Del}(M)e) = Q_1(M)^{-1}\delta^{Del}(M)e^c$. It follows that

$$\delta^{Del}(M^c) \sim_{E(M);K} Q_1(M)^{-1}\delta^{Del}(M)$$

as expected.

□

Example 6.2.1. ***Tate motive***
Let $\mathbb{Z}(1)_K$ be the extension of $\mathbb{Z}(1)$ from $\mathbb{Q}$ to K. It is a motive with coefficients in K. As in section 3.1 of [7], $\mathbb{Z}(1)_{K,B} = H_1(\mathbb{G}_{m,K}) \cong K$ and $\mathbb{Z}(1)_{K,DR}$ is the dual of $H^1_{DR}(\mathbb{G}_{m,K})$ with generator $\frac{dz}{z}$. Therefore the comparison isomorphism $\mathbb{Z}(1)_{K,B}\otimes\mathbb{C} \cong K\otimes\mathbb{C} \to \mathbb{Z}(1)_{K,DR}\otimes\mathbb{C} \cong K\otimes\mathbb{C}$ sends K to $\oint \frac{dz}{z}K = (2\pi i)K$. We have $\delta^{Del}(\mathbb{Z}(1)_K) \sim_{K;K} 2\pi i$.

In general, let M be a motive over K with coefficients in E. We have

$$\delta^{Del}(M(n)) \sim_{E(M);K} (2\pi i)^n\delta^{Del}(M). \tag{6.7}$$

Remark 6.2.2. *All the determinants and the coefficients we consider here are elements in $(E \otimes_{\mathbb{Q}} \mathbb{C})^\times$.*

6.3 Deligne's conjecture for tensor product of motives

Let E and E' be two number fields.

Let M be a regular motive over K (with respect to the fixed embedding) with coefficients in E of dimension n pure of weight $\omega(M)$. Let M' be a regular motive over K with coefficients in E' of dimension n' pure of weight $\omega(M')$.

We denote by $R(M \otimes M')$ the restriction from K to $\mathbb{Q}$ of the motive $M \otimes M'$. It is a motive of weight $\omega := \omega(M) + \omega(M')$ with Betti realization $M_B \otimes M'_B \oplus M^c_B \otimes M'^c_B$ and De Rham realization $M_{DR} \otimes M'_{DR} \oplus M^c_{DR} \otimes M'^c_{DR}$.

We denote the Hodge type of M by $(p_i, \omega(M) - p_i)_{1\leqslant i\leqslant n}$ with $p_1 > \cdots > p_n$ and the Hodge type of M' by $(r_j, \omega(M') - r_j)_{1\leqslant j\leqslant n}$ with $r_1 > r_2 > \cdots > r_{n'}$. As before, we define $p^c_i = \omega(M) - p_{n+1-i}$ and $r^c_j = \omega(M') - r_{n'+1-j}$. There are indices for Hodge type of M^c and M'^c respectively.

We assume that $R(M\otimes M')$ has no $(w/2,w/2)$ class. In other words, $p_a+r_b\neq\frac{\omega}{2}$ and then $p_t^c+r_u^c\neq\frac{\omega}{2}$ for all $1\leqslant a,t\leqslant n$, $1\leqslant b,u\leqslant n'$.

As in the above section, we take $(e_i)_{1\leqslant j\leqslant n}$ an E-base of M_B and define $(e_i^c:=F_\infty e_i)_{1\leqslant i\leqslant n}$ which is an E-base of M_B^c. Similarly, we take $(f_j)_{1\leqslant j\leqslant n'}$ an E'-base of M'_B and define $f_j^c:=F_\infty f_j$ for $1\leqslant j\leqslant n'$.

We also take $\omega_i\in M^{p_i,\omega(M)-p_i}$, $(\omega_i^c)\in(M^c)^{p_i^c,\omega(M)-p_i^c}$ for $1\leqslant i\leqslant n$ as in previous section and $\eta_j\in M^{r_j,\omega(M')-r_j}$, $\eta_j^c\in(M'^c)^{r_j^c,\omega(M')-r_j^c}$ for $1\leqslant j\leqslant n'$ similarly.

Recall the motive periods are complex numbers Q_i, $1\leqslant i\leqslant n$ and Q'_j, $1\leqslant j\leqslant n'$ such that

$$F_\infty\omega_i=Q_i\omega^c_{n+1-i},F_\infty\mu_j=Q'_j\mu^c_{n'+1-j}. \tag{P}$$

The aim of this section is to calculate the Deligne's period for $R(M\otimes M')$ in terms of motivic periods.

Remark 6.3.1. *If we define a paring $(M_B\otimes\mathbb{C})\otimes(M_B\otimes\mathbb{C})\to\mathbb{C}$ such that $<\omega_i,\omega^c_{n+1-i}>=1$ and $<\omega_i,\omega^c_{n+1-j}>=0$ for $j\neq i$ then $Q_i=<\omega_i,F_\infty\omega_i>$.*

Let $M^\#=R(M\otimes M')$. It is a motive over $\mathbb{Q}$. We are going to calculate $c^+(M^\#)$.

We define $A=\{(a,b)\mid p_a+r_b>\frac{\omega}{2}\}$ and $T=\{(t,u)\mid p_t^c+r_u^c>\frac{\omega}{2}\}=\{(t,u)\mid p_{n+1-t}+r_{n'+1-u}<\frac{\omega}{2}\}$.

Remark 6.3.2. *Keeping in mind that*

$$(t,u)\in T \text{ if and only if } (n+1-t,n'+1-u)\notin A. \tag{6.8}$$

Proposition 6.3.1. *Let M, M' be motive over K with coefficients in E and E' respectively. We assume that $M\otimes M'$ has no $(\omega/2,\omega/2)$-class. We then have*

$$\begin{aligned} &c^+(R(M\otimes M'))\\ \sim_{E(M)E(M');K}&\left(\prod_{(t,u)\notin T(M,M')}Q_{n+1-t}(M)^{-1}Q_{n'+1-u}(M')^{-1}\right)\delta^{Del}(M\otimes M')\\ \sim_{E(M)E(M');K}&\left(\prod_{(t,u)\in A(M,M')}Q_t(M)^{-1}Q_u(M')^{-1}\right)\delta^{Del}(M\otimes M') \end{aligned}\tag{6.9}$$

Proof For simplification of notation, we identify $\omega_i\in M_B\otimes\mathbb{C}$ and $I_\infty(\omega_i)\in M_{DR}\otimes\mathbb{C}$ and similarly, we identify ω^c, μ_j, μ_j^c with their image under I_∞ in the following.

We fixe bases for M_B^+ and $M_{DR}^\#/F^+(M^\#)$ now. For M_B^+, we know $(e_i\otimes f_j+e_i^c\otimes f_j^c)_{1\leqslant i\leqslant n,1\leqslant j\leqslant n'}$ forms an EE'-base. For $M_{DR}^\#/F^+(M^\#)$, as in the above section, we consider $\mathcal{B}:=(\omega_a\otimes\mu_b,\omega_t^c\otimes\mu_u^c(\text{mod }F^+(M^\#))\mid(a,b)\notin A,(t,u)\notin T)$ as an $E\otimes\mathbb{C}$ base of $(M_{DR}^\#/F^+(M^\#))\otimes\mathbb{C}$ which is not rational but can change to a rational base with a unipotent matrix for change of basis. Therefore we can use this base to calculate Deligne's period.

If $(a,b) \notin A$ then $(n+1-a, n'+1-b) \in T$ by (6.8). Along with (P), we know that $F_\infty(\omega_a \otimes \mu_b) = Q_a Q'_b \omega^c_{n+1-a} \mu^c_{n'+1-b} \in F^+(M^\#) \otimes \mathbb{C}$. Similarly, $F_\infty(\omega^c_t \otimes \mu^c_u) \in F^+(M^\#) \otimes \mathbb{C}$ for all $(t,u) \notin T$.

Note that F_∞ is an endomorphism on $M^\#(B) \otimes \mathbb{C}$ and $M^\#_{DR} \otimes \mathbb{C}$. For any $\phi \in M^\#(B) \otimes \mathbb{C}$ or $M^\#(DR) \otimes \mathbb{C}$, we write $(1+F_\infty)\phi := \phi + F_\infty(\phi)$.

Recall that $(M^\#_{DR}/F^+(M^\#)) \otimes \mathbb{C} \cong (M^\#_{DR} \otimes \mathbb{C})/(F^+(M^\#) \otimes \mathbb{C})$. Thus $\mathcal{B} = ((1+F_\infty)\omega_a \otimes \mu_b, (1+F_\infty)\omega^c_t \otimes \mu^c_u (\text{mod } F^+(M^\#) \otimes \mathbb{C}) \mid (a,b) \notin A, (t,u) \notin T)$.

We write $\omega_a = \sum\limits_{i=1}^{n} A_{ia} e_i$, $\omega^c_t = \sum\limits_{i=1}^{n} A^c_{it} e^c_i$, $\mu_b = \sum\limits_{j=1}^{n'} B_{jb} f_j$, $\mu_u = \sum\limits_{j=1}^{n'} B^c_{ju} f^c_j$ for all $1 \leqslant a, t \leqslant n$ and $1 \leqslant b, u \leqslant n'$.

We then have

$$(1+F_\infty)\omega_a \mu_b = (1+F_\infty) \sum_{i,j} A_{ia} B_{jb} e_i \otimes f_j = \sum_{i,j} A_{ia} B_{jb} (e_i \otimes f_j + e^c_i \otimes f^c_j)$$

$$\text{and} \quad (1+F_\infty)\omega^c_t \omega^c_u = (1+F_\infty) \sum_{i,j} A^c_{it} B^c_{ju} e^c_i \otimes f^c_j = \sum_{i,j} A^c_{it} B^c_{ju} (e_i \otimes f_j + e^c_i \otimes f^c_j).$$

Up to multiplication by elements in $(EE')^\times$, the Deligne's period then equals the determinant of the matrix

$$Mat_1 := \left(A_{ia} B_{jb}, A^c_{it} B^c_{ju}\right)$$

with $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n'$, $(a,b) \notin A$, $(t,u) \notin T$.

By the relation P, we have $F_\infty \omega_{n+1-t} = Q_{n+1-t} \omega^c_t$. We get

$$\sum_{i=1}^{n} A_{i,n+1-t} e^c_i = Q_{n+1-t} \omega^c_t = Q_{n+1-t} \sum_{i=1}^{n} A^c_{it} e^c_i$$

Therefore, for all i, j, we obtain,

$$A^c_{it} = (Q_{n+1-t})^{-1} A_{i,n+1-t}, B^c_{ju} = (Q'_{n'+1-u})^{-1} B_{j,n'+1-u}. \tag{6.10}$$

We then deduce that $A^c_{it} B^c_{ju} = (Q_{n+1-t})^{-1} (Q'_{n'+1-u})^{-1} A_{i,n+1-t} B_{j,n'+1-u}$.

Thus the Deligne's period:

$$c^+(R(M \otimes M')) \sim_{E(M^\#);K} det(Mat_1) = \prod_{(t,u) \notin T} ((Q_{n+1-t})^{-1} (Q'_{n'+1-u})^{-1}) \times det(Mat_2)$$

where $Mat_2 = \left(A_{ia} B_{jb}, A_{i,n+1-t,j,n'+1-u}\right)$ with $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n'$, $(a,b) \notin A$ and $(t,u) \notin T$.

Recall that $(t,u) \notin T$ if and only if $(n+1-t, n'+1-u) \in A$. Therefore the index $(n+1-t, n'+1-u)$ above runs over the pairs in A. We see that $Mat_2 = (A_{ia} B_{jb})$ with both (i,j) and (a,b) runs over all the pair in $\{1, 2, \cdots, n\} \times \{1, 2, \cdots, n'\}$. It is noting but

$(A_{ia}) \otimes (B_{jb})$.

Let us back to the definition of A_{ia}. It is the coefficients with respect to the chosen rational bases of the map $M_B \otimes \mathbb{C} \to M_{DR} \otimes \mathbb{C}$. Therefore $(A_{ia}) \otimes (B_{jb})$ is the coefficient matrix of the comparison isomorphism $(M \otimes M')_B \otimes \mathbb{C} \to (M \otimes M')_{DR} \otimes \mathbb{C}$. We then get $det((A_{ia}) \otimes (B_{jb})) = \delta^{Del}(M \otimes M')$ which terminates the proof.

□

6.4 Motivic periods for automorphic representations over quadratic imaginary fields

Hecke character case: Let η be an arbitrary algebraic Hecke character of K with infinity type $z^a \overline{z}^b$. We assume that $a \neq b$.

Let $M(\eta)$ be the motive associated to η (c.f. [7] section 8.) It is of Hodge type $(-a, -b)$.

For the motivic period for $M(\eta)$, we use η to indicate $M(\eta)$ for simplification. For example. $\delta^{Del}(\eta) := \delta^{Del}(M(\eta))$.

On one hand, by Blasius's result, $c^+(R(M(\eta))) \sim_{E(\eta);K} p(\check{\eta}, 1)$ if $a < b$;
$c^+(R(M(\eta))) \sim_{E(\eta);K} p(\check{\eta}, \iota)$ if $a > b$.

On the other hand, by Proposition 6.3.1, we have

$$c^+(R(M(\eta))) \sim_{E(\eta);K} \prod_{t \in A} Q_t(\eta)^{-1} \delta^{Del}(\eta) \tag{6.11}$$

where $A = \{1\}$ if $-a > -b$ and $A = \varnothing$ if $-a < -b$.

Let us assume $a < b$ first. We have $Q_1(\eta)^{-1}\delta^{Del}(\eta) \sim_{E(\eta):K} p(\check{\eta}, 1)$. We apply the above to η^c and get $\delta^{Del}(M(\eta^c)) \sim_{E(\eta):K} p(\check{\eta^c}, \iota) \sim_{E(\eta):K} p(\check{\eta}, 1)$.

Notice that there is there is a rational paring: $M(\eta) \times M(\eta^c) \to M(\eta_0)(a + b)$ where η_0 is a Dirichlet character over $\mathbb{A}_{\mathbb{Q}}$ such that $\eta\eta^c = (\eta_0 \circ N_{\mathbb{A}_K/AQ})|| \cdot ||_{\mathbb{A}_K}^{a+b}$. We obtain that

$$\delta^{Del}(\eta) \times \delta^{Del}(\eta^c) \sim_{E(\eta);K} \delta^{Del}(\eta_0)(2\pi i)^{a+b} \tag{6.12}$$

by equation (6.7).

We deduce by Lemma 4.1.2 that

$$\begin{aligned}
& \delta^{Del}(\eta) \times \delta^{Del}(\eta^c) \sim_{E(\eta);K} \mathcal{G}(\eta_0)(2\pi i)^{a+b} \\
\sim_{E(\eta);K} \quad & p(\eta_0 \circ N_{\mathbb{A}_K/\mathbb{A}_{\mathbb{Q}}}, 1)^{-1} p(|| \cdot ||_{\mathbb{A}_K}^{a+b}, 1)^{-1} \sim_{E(\eta);K} p((\eta_0 \circ N_{\mathbb{A}_K/AQ})|| \cdot ||_{\mathbb{A}_K}^{a+b}, 1)^{-1} \\
\sim_{E(\eta);K} \quad & p(\eta\eta^c, 1)^{-1} \sim_{E(\eta);K} p(\check{\eta}, 1) p(\check{\eta^c}, 1)
\end{aligned}$$

Therefore,

$$\delta^{Del}(\eta) \sim_{E(\eta);K} p(\check{\eta^c}, 1) \text{ and then } Q_1(\eta) \sim_{E(\eta);K} \frac{p(\check{\eta^c}, 1)}{p(\check{\eta}, 1)} \sim_{E(\eta);K} p(\frac{\eta^c}{\eta}, 1). \tag{6.13}$$

If $a > b$, we follow the above procedure and can see easily the last two formulas are still true.

Conjugate self-dual case: Let Π be a regular cuspidal cohomological conjugate self-dual representation of $GL_n(\mathbb{A}_K)$. We denote the infinity type of Π by $(z^{a_i}\overline{z}^{-a_i})_{1\leqslant i\leqslant n}$ with $a_1 > a_2 > \cdots > a_n$.

We assume that there exists a motive M over K associated to Π with coefficients in $E(M) = E(\Pi)$.

We know M is a reguler motive of Hodge type $(-a_i + \frac{n-1}{2}, a_i + \frac{n-1}{2})_{1\leqslant i\leqslant n}$.

We define $p_i := -a_{n+1-i}+\frac{n-1}{2}$, $q_i := n-1-p_i = a_{n+1-i}++\frac{n-1}{2}$, $p_i^c = a_i+\frac{n-1}{2} = q_{n+1-i}$, $q_i^c := n-1-p_i^c$ for all $1 \leqslant i \leqslant n$.

We define $\omega_i \in M^{p_i,q_i}$, $\omega_i^c \in (M^c)^{p_i^c,q_i^c}$ as in the previous sections.

If Π is conjugate self-dual, then $M(\Pi)$ is polarized. The polarization on the De Rham realization induces an $E(M)$-rational perfect parings $<,>$:

$$M(\Pi)^{p_i,q_i} \otimes (M(\Pi)^c)^{p^c_{n+1-i},q^c_{n+1-i}} \to E(M)(1-n) \cong E(M).$$

We may assume that $<\omega_i, \omega^c_{n+1-i}>= 1$ by adjusting ω^c_{n+1-i} with multiplication by elements in $E(M)^\times$.

Let $1 \leqslant i \leqslant n$. We write $Q_i(\Pi) := Q_i(M(\Pi))$ as we did for η. The motivic period $Q_i(\Pi)$ then equals $< w_i, F_\infty w_{n+1-i} >$ up to multiplication by elements in $E(\Pi)^\times$.

conjugate self-dual case: In the general case, we write $\Pi = \Pi'\otimes\eta$ with Π' conjugate self-dual and η be an algebraic Hecke character of $\mathbb{A}_K$.

We take $\omega_i^0 \in M(\Pi')^{p_i(\Pi'),q_i(\Pi')}$ as before. Let ω be a base of $M(\eta)_{DR}$ and ω^c be a base of $M(\eta^c)_{DR}$. We know $F_\infty(\omega) = Q_1(\eta)\omega^c$ up to multiplication by elements in $E(\eta)^\times$.

Then $(\omega_i := \omega_i^0 \otimes \omega)_{1\leqslant i\leqslant n} \in M^{p_i,q_i}$ which is equivalent to a rational element in $F^{p_i(M)}$ modulo $F^{p_{i-1}}(M)\otimes\mathbb{C}$. We have similar properties for $(\omega_i^c := \omega_i^{0,c}\otimes\omega^c)_{1\leqslant i\leqslant n}$.

Moreover,

$$F_\infty(\omega_i \otimes \omega) = Q_i(\Pi')Q_1(\eta)(\omega'_{n+1-i} \otimes \omega'). \tag{6.14}$$

The motivic period for Π then equals $Q_i(\Pi) := Q_i(\Pi')Q_1(\eta)$ for $1 \leqslant i \leqslant n$ up to multiplication by elements in $E(\Pi)^\times$.

6.5 Deligne's conjecture for automorphic pairs over quadratic imaginary fields

Let Π (resp. Π') be a regular cuspidal cohomological conjugate self-dual representation of $GL_n(\mathbb{A}_K)$ (resp. $GL_{n'}(\mathbb{A}_K)$). We denote the infinity type of Π (resp. Π')

by $(z^{a_i}\overline{z}^{-\omega(\Pi)-a_i})_{1\leqslant i\leqslant n}$ (resp. $(z^{b_j}\overline{z}^{-\omega(\Pi)-b_j})_{1\leqslant j\leqslant n'}$) with $a_1 > a_2 > \cdots > a_n$ (resp. $b_1 > b_2 > \cdots > b_{n'}$). We suppose $\Pi\times\Pi'$ is regular, i.e. $a_i + b_j \neq -\frac{\omega(\Pi)+\omega(\Pi')}{2}$ for all i, j.

We assume that there exists a motive M (resp. M') over K associated to Π (resp. Π') with coefficients in $E(M)$ (resp. $E(M')$).

We know M (resp. M') is reguler motive of Hodge type $(-a_i + \frac{n-1}{2}, a_i + \omega(\Pi) + \frac{n-1}{2})_{1\leqslant i\leqslant n}$ (resp. $(-b_j + \frac{n'-1}{2}, b_j + \omega(\Pi') + \frac{n-1}{2})_{1\leqslant j\leqslant n'}$).

We define $p_i := -a_{n+1-i} + \frac{n-1}{2}$ and $p_i^c = \omega(\Pi) + n - 1 - p_{n+1-i} = a_i + \omega(\Pi) + \frac{n-1}{2}$ for all $1 \leqslant i \leqslant n$. We define like this to guarantee that $p_1 > p_2 > \cdots > p_n$ and $p_1^c > p_2^c > \cdots > p_n^c$ as in the previous sections. Similarly, we define $r_j := -b_{n'+1-j} + \frac{n'-1}{2}$, $r_j^c := b_j + \omega(\Pi') + \frac{n'-1}{2}$ for all $1 \leqslant j \leqslant n'$.

Proposition 6.3.1 implies that:

Proposition 6.5.1. *The Deligne's period of* $(M \otimes M')$ *satisfies:*

$$c^+(R(M(\Pi)\otimes M(\Pi'))) \sim_{E(M)E(M');K} \left(\prod_{(t,u)\in A(M,M')} Q_t(\Pi)^{-1}Q_u(\Pi')^{-1}\right)\delta^{Del}(M\otimes M'). \tag{6.15}$$

where the set $A(M, M') = \{(t, u) \mid p_t + r_u > \frac{\omega(\Pi)+\omega(\Pi')+(n-1)+(n'-1)}{2}\}$.

Recall $p_t = -a_{n+1-t} + \frac{n-1}{2}$ and $r_u := -b_{n'+1-u} + \frac{n'-1}{2}$. We obtain that $A(M, M') = \{(t, u) \mid p_t + r_u > \frac{\omega(\Pi)+\omega(\Pi')+(n-1)+(n'-1)}{2}\} = \{(t, u) \mid a_{n+1-t} + b_{n'+1-u} < -\frac{\omega(\Pi)+\omega(\Pi')}{2}\}$.

Therefore,

$$\prod_{(t,u)\in A(M,M')} Q_t(\Pi)^{-1} = \prod_{t=1}^{n} Q_t(\Pi)^{-\#\{u\mid b_{n+1-u}<-a_{n+1-t}-\frac{\omega(\Pi)+\omega(\Pi')}{2}\}} \tag{6.16}$$

In this section, we define $sp(j) := sp(j, \Pi; \Pi')$ for $0 \leqslant j \leqslant n$ and $sp'(k) := sp(k, \Pi'; \Pi)$ for $0 \leqslant k \leqslant n'$. Recall $sp(j)$ are the lengths of different parts of $b_1 > b_2 > \cdots > b_{n'}$ separated by $-a_n - \frac{\omega(\Pi)+\omega(\Pi')}{2} > -a_{n-1} - \frac{\omega(\Pi)+\omega(\Pi')}{2} > \cdots > -a_1 - \frac{\omega(\Pi)+\omega(\Pi')}{2}$.

Therefore $\#\{u \mid b_{n+1-u} < -a_{n+1-t} - \frac{\omega(\Pi)+\omega(\Pi')}{2}\} = \#\{u \mid b_u < -a_{n+1-t} - \frac{\omega(\Pi)+\omega(\Pi')}{2}\} = sp(t) + sp(t+1) + \cdots + sp(n)$, we have $\prod_{(t,u)\in A(M,M')} Q_t(\Pi)^{-1}$

$$\begin{aligned} &= Q_1(\Pi)^{-sp(1)-sp(2)-\cdots-sp(n)}Q_2(\Pi)^{-sp(2)-sp(3)-\cdots-sp(n)}\cdots Q_n(\Pi)^{-sp(n)} \\ &= [Q_1(\Pi)^{-1}]^{sp(1)}[Q_1(\Pi)^{-1}Q_2(\Pi)^{-1}]^{sp(2)}\cdots[Q_1(\Pi)^{-1}Q_2(\Pi)^{-1}\cdots Q_n(\Pi)^{-1}]^{sp(n)} \end{aligned}$$

We define $Q_{\leqslant j}(\Pi) = Q_1(\Pi)^{-1}Q_2(\Pi)^{-1}\cdots Q_j(\Pi)^{-1}$ for $1 \leqslant j \leqslant n$ and $Q_{\leqslant 0}(\Pi) = 1$. We define $Q_{\leqslant k}(\Pi')$ similarly for $0 \leqslant k \leqslant n'$.

We have obtained that

$$\prod_{(t,u)\in A(M,M')} Q_t(\Pi)^{-1}Q_u(M')^{-1} = \prod_{j=0}^{n} Q_{\leqslant j}(\Pi)^{sp(j)} \prod_{k=0}^{n'} Q_{\leqslant k}(\Pi')^{sp'(k)}. \tag{6.17}$$

We define $\Delta(M(\Pi)) = \Delta(\Pi) := (2\pi i)^{\frac{n(n-1)}{2}}\delta^{Del}(\Pi)$. In fact, let ξ_Π be the central character of Π. Since $\Lambda^n M(\Pi) \equiv M(\xi_\Pi)(-\frac{n(n-1)}{2})$, we have $\delta^{Del}(\Pi) = \delta^{Del}(\Lambda^n M(\Pi)) = \delta^{Del}(\xi_\Pi)(2\pi i)^{-\frac{n(n-1)}{2}}$. Therefore,

$$\Delta(\Pi) \sim_{E(\Pi):K} \delta^{Del}(\xi_\Pi). \tag{6.18}$$

We define $\Delta(\Pi')$ similarly. We have

$$\delta^{Del}(M\otimes M') = \delta^{Del}(\Pi)^{n'}\delta^{Del}(\Pi')^{n} = (2\pi i)^{-nn'(n+n'-2)}\Delta(\Pi)^{n'}\Delta(\Pi')^{n}.$$

Since $\sum\limits_{i=0}^{n} sp(i) = n'$ and $\sum\limits_{i=0}^{n} sp(i) = n'$, we know:

$$\delta^{Del}(M\otimes M') = (2\pi i)^{-\frac{nn'(n+n'-2)}{2}}\prod_{j=0}^{n}\Delta(\Pi)^{sp(j)}\prod_{k=0}^{n'}\Delta(\Pi')^{sp'(k)}. \tag{6.19}$$

At last, we define for all $0 \leqslant j \leqslant n$ that

$$Q^{(j)}(\Pi) = Q_{\leqslant j}(\Pi)\times\Delta(\Pi) \sim_{E(\Pi);K} Q_1(\Pi)^{-1}\cdots Q_j(\Pi)^{-1}\delta^{Del}(\xi_\Pi) \tag{6.20}$$

We define $Q^{(k)}(\Pi')$ for $1 \leqslant k \leqslant n'$ similarly. Comparing (6.15) with (6.17) and (6.19), we have

$$c^+(R(M\otimes M')) \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{\frac{-nn'(n+n'-2)}{2}}\prod_{j=0}^{n}Q^{(j)}(\Pi)^{sp(j,\Pi;\Pi')}\prod_{k=0}^{n'}Q^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}$$

We can now state Deligne's conjecture for automorphic pairs:

Conjecture 6.5.1. *Let n and n' be two positive integers. Let Π and Π' be regular cohomological cuspidal representation of $GL_n(\mathbb{A}_K)$ and $GL_{n'}(\mathbb{A}_K)$ respectively which are conjugate self-dual. We suppose that $\Pi\otimes\Pi'$ is regular.*

We assume that there exists motives M and M' over K associated to Π and Π' respectively.

Let $m \in \mathbb{Z} + \frac{n+n'}{2}$ be critical for $\Pi\otimes\Pi'$. It is equivalent to saying that $m + \frac{n+n'-2}{2}$ is critical for $M\otimes M'$. Deligne's conjecture predicts that:

$$\begin{aligned}L(m,\Pi\times\Pi') &= L(m+\tfrac{n+n'-2}{2}, M\otimes M')\\ &\sim_{E(\Pi)E(\Pi');K} (2\pi i)^{nn'm}\prod_{j=0}^{n}Q^{(j)}(\Pi)^{sp(j,\Pi;\Pi')}\prod_{k=0}^{n'}Q^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}\end{aligned}$$

6.6 The picture for general CM fields

Let F be a CM field containing K and F^+ be the maximal totally real subfield of F.

Let M be a motive over F with coefficients in $E(M)$ of dimension n and pure of weight $\omega(M)$.

For each $\sigma \in \Sigma_F$, we may define the motivic period $\delta^{Del}(M,\sigma)$ and $Q_i(M,\sigma)$ as in Section 6.2. We write the Hodge type of M at σ by $(p_i(\sigma), q_i(\sigma))_{1\leqslant i\leqslant n}$.

We take M' another motive over F with coefficients in $E(M')$ of dimension n' and pure of weight $\omega(M')$. Similarly, we write the Hodge type of M' at σ by $(r_j(\sigma), s_j(\sigma))_{1\leqslant i\leqslant n'}$.

We assume that $p_i(\sigma) + r_j(\sigma) \neq \dfrac{\omega(M)+\omega(M')}{2}$ for all σ, i, j.

Define $A(M,M')(\sigma) = \{(a,b) \mid p_a(\sigma) + r_b(\sigma) > \frac{\omega}{2}\}$.

For any CM type Φ of F, we have $Res_{F/\mathbb{Q}}(M\otimes M') = (\bigoplus\limits_{\sigma\in\Phi} M^{\sigma}\otimes M'^{\sigma}) \oplus (\bigoplus\limits_{\sigma\in\Phi} M^{\sigma^c}\otimes M'^{\sigma^c})$.

The proof of Proposition 6.3.1 can be easily generalized to the CM field case and we get:

Proposition 6.6.1. *Let M, M' be motive over F with coefficients in E and E' respectively. We assume that $M\otimes M'$ has no $(\omega/2,\omega/2)$-class. We have:*

$$c^+(Res_{F/\mathbb{Q}}(M\otimes M')) \sim_{E(M)E(M');K} \prod_{\sigma\in\Psi}\left(\prod_{(t,u)\in A(M,M')(\sigma)} Q_t(M,\sigma)^{-1}Q_u(M',\sigma)^{-1}\right)\prod_{\sigma\in\Psi}\delta^{Del}(M\otimes M',\sigma) \quad . \tag{6.21}$$

Let us assume that M and M' are motives associated to certain representations Π and Π' respectively. We still write the motivic period $Q_i(M(\Pi))$ as $Q_i(\Pi)$ for simplicity.

We define $Q_{\leqslant j}(\Pi,\sigma) := \prod\limits_{i=1}^{j} Q_i(\Pi,\sigma)$. We have (c.f. equation (6.17))

$$\prod_{(t,u)\in A(M,M')(\sigma)} Q_t(\Pi)^{-1}Q_u(M')^{-1} = \prod_{j=0}^{n} Q_{\leqslant j}(\Pi,\sigma)^{sp(j,\Pi;\Pi',\sigma)}\prod_{k=0}^{n'} Q_{\leqslant k}(\Pi',\sigma)^{sp'(k,\Pi';\Pi,\sigma)}. \tag{6.22}$$

Recall that $\Lambda^n M(\Pi) \cong M(\xi_\Pi)(-\frac{n(n-1)}{2})$. We have

$$\delta^{Del}(\Pi,\sigma) \sim_{E(\Pi);K} (2\pi i)^{-\frac{n(n-1)}{2}}\delta^{Del}(\xi_\Pi,\sigma). \tag{6.23}$$

As before, we define

$$\begin{aligned} &\Delta(\Pi,\sigma) := (2\pi i)^{\frac{n(n-1)}{2}}\delta^{Del}(\Pi,\sigma) = \delta^{Del}(\xi_\Pi,\sigma) \\ \text{and}\quad &Q^{(j)}(\Pi,\sigma) := Q_{\leqslant j}(\Pi,\sigma)\Delta(\Pi,\sigma). \end{aligned} \tag{6.24}$$

We have:

$$c^+(Res_{F/\mathbb{Q}}(M\otimes M')) \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{\frac{-nn'd(n+n'-2)}{2}} \times \prod_{\sigma\in\Psi}\prod_{j=0}^{n} Q^{(j)}(\Pi,\sigma)^{sp(j,\Pi;\Pi',\sigma)}\prod_{k=0}^{n'} Q^{(k)}(\Pi',\sigma)^{sp(k,\Pi';\Pi,\sigma)}$$

In particular, if we take $\Psi = \Sigma_{F;K}$, Deligne's conjecture predicts that:

Conjecture 6.6.1. *Let n and n' be two positive integers. Let Π and Π' be regular cohomological cuspidal representation of $GL_n(\mathbb{A}_F)$ and $GL_{n'}(\mathbb{A}_F)$ respectively which are conjugate self-dual. We suppose that $\Pi \otimes \Pi'$ is regular.*

We assume that there exists motives M and M' over F associated to Π and Π' respectively.

If $m \in \mathbb{Z} + \frac{n+n'}{2}$ is critical for $\Pi \times \Pi'$ then

$$L(m, \Pi \times \Pi') = L(m + \tfrac{n+n'-2}{2}, M \otimes M')$$

$$\sim_{E(\Pi)E(\Pi');K} (2\pi i)^{mnn'd} \prod_{\sigma \in \Sigma_{F;K}} (\prod_{j=0}^{n} Q^{(j)}(\Pi, \sigma)^{sp(j,\Pi;\Pi',\sigma)} \prod_{k=0}^{n'} Q^{(k)}(\Pi', \sigma)^{sp(k,\Pi';\Pi,\sigma)})$$

6.7 Motivic periods for Hecke characters over CM fields

Let η, η' be two algebraic Hecke character of F. We assume that $\eta\eta'$ is critical and then is compatible with a CM type $\Psi(\eta\eta')$. Proposition 6.6.1 can be rewritten as

$$c^+(Res_{F/\mathbb{Q}}(M(\eta) \otimes M(\eta'))) \sim_{E(\eta)E(\eta');K} \tag{6.25}$$
$$\prod_{\sigma \in \Psi} \left(\prod_{\sigma \in \Psi \cap \Psi(\eta\eta')} Q_1(\eta, \sigma)^{-1} Q_1(\eta', \sigma)^{-1} \right) \prod_{\sigma \in \Psi} \delta^{Del}(M \otimes M', \sigma) \quad .$$

Fix any Hecke character η. We may take η' such that $\Psi(\eta\eta') = \Psi^c$.
Equation (6.25) implies that

$$c^+(Res_{F/\mathbb{Q}})(M(\eta) \otimes M(\eta')) \sim_{E(\eta)E(\eta');K} \prod_{\sigma \in \Psi} \delta^{Del}(\eta, \sigma) \prod_{\sigma \in \Psi} \delta^{Del}(\eta', \sigma). \tag{6.26}$$

On the other hand, by Blasius's result, we have:

$$c^+(Res_{F/\mathbb{Q}}(M(\eta) \otimes M(\eta'))) \sim_{E(\eta)E(\eta');K} p(\widetilde{\eta\eta'}, \Psi^c)$$
$$\sim_{E(\eta)E(\eta');K} \prod_{\sigma \in \Psi} p(\check{\eta}, \sigma^c) \prod_{\sigma \in \Psi} p(\check{\eta}', \sigma^c).$$

Let η' vary. We get for any CM type Ψ that:

$$\prod_{\sigma \in \Psi} \delta^{Del}(\eta', \sigma) \sim_{E(\eta);K} \prod_{\sigma \in \Psi} p(\check{\eta}, \sigma^c). \tag{6.27}$$

Let Ψ vary now. It is easy to deduce that there exists ζ_d, an d-th root of unity, such that $\delta^{Del}(\eta', \sigma) \sim_{E(\eta);K} \zeta_d p(\check{\eta}, \sigma^c)$ for all $\sigma \in \Sigma$.

For simplicity, we assume that $E(\Pi)$ contains all d-th roots of unity then we get $\delta^{Del}(\eta', \sigma) \sim_{E(\eta);K} p(\check{\eta}, \sigma^c)$ for all $\sigma \in \Sigma$.

We can now calculate $Q_1(\eta, \sigma)$. Let σ_0 be in Ψ. We take η' such that $\Psi_{\eta\eta'} = \{\sigma\} \cup (\Psi - \{\sigma_0\})^c$. Equation (6.25) implies that

$$c^+(Res_{F/\mathbb{Q}})(M(\eta) \otimes M(\eta')) \sim_{E(\eta)E(\eta');K} Q_1(\eta, \sigma)^{-1} Q_1(\eta', \sigma)^{-1} \times$$
$$\prod_{\sigma \in (\Psi - \{\sigma_0\})} \delta^{Del}(\eta, \sigma) \prod_{\sigma \in (\Psi - \{\sigma_0\})} \delta^{Del}(\eta', \sigma) \times \delta^{Delta}(\eta, \sigma_0) \delta^{Delta}(\eta', \sigma_0).$$

Blasius's result implies that

$$c^{+}(Res_{F/\mathbb{Q}})(M(\eta)\otimes M(\eta')) \sim_{E(\eta)E(\eta');K} (-1)^{\epsilon(\Psi)} \prod_{\sigma\in(\Psi-\{\sigma_0\})} p(\breve{\eta\eta'}, \sigma^{c}) \times p(\breve{\eta\eta'}, \sigma_0). \quad (6.28)$$

Along with equation (6.27), we obtain that

$$Q_1(\eta,\sigma_0) \sim_{E(\eta);K} p(\check{\eta}, \sigma_0^{c}) p(\check{\eta}, \sigma_0)^{-1} \sim_{E(\eta);K} p(\frac{\eta^{c}}{\eta}, \sigma_0). \quad (6.29)$$

Chapter 7

Factorization of arithmetic automorphic periods and a conjecture

We want to show that the arithmetic automorphic periods can be factorized as products of local periods over infinite places. We may assume that Π is conjugate self-dual in this Chapter. The essential conjugate self-dual case then follows by Definition 5.3.2 and the fact that the CM periods is factorable.

7.1 Basic lemmas

Let X, Y be two sets and Z be a multiplicative abelian group. We will apply the result of this section to $Z = \mathbb{C}^\times/E^\times$ where E is a proper number field.

Lemma 7.1.1. *Let f be a map from $X \times Y$ to Z. The following two statements are equivalent:*

1. *There exists two maps $g : X \to Z$ and $h : Y \to Z$ such that $f(x, y) = g(x)h(y)$ for all $(x, y) \in X \times Y$.*
2. *For all $x, x' \in X$ and $y, y' \in Y$, we have $f(x, y)f(x', y') = f(x, y')f(x', y)$.*

Moreover, if the above equivalent statements are satisfied, the maps g and h are unique up to scalars.

Proof The direction that 1 implies 2 is trivial. Let us prove the inverse. We fix any $y_0 \in Y$ and define $g(x) := f(x, y_0)$ for all $x \in X$. We then fix any $x_0 \in X$ and define $h(y) := \dfrac{f(x_0, y)}{g(x_0)} = \dfrac{f(x_0, y)}{f(x_0, y_0)}$.

For any $x \in X$ and $y \in Y$, Statement 2 tells us that $f(x, y)f(x_0, y_0) = f(x, y_0)f(x_0, y)$. Therefore $f(x, y) = f(x, y_0) \times \dfrac{f(x_0, y)}{f(x_0, y_0)} = g(x)h(y)$ as expected.

□

Let n be a positive integer and $X_1, \cdots, X_n$ be some sets. Let f be a map from $X_1 \times X_2 \times \cdots \times X_n$ to Z.

The following corollary can be deduced from the above Lemma by induction on n.

Corollary 7.1.1. *The following two statements are equivalent:*

1. *There exists some maps $f_k : X_k \to Z$ for $1 \leqslant k \leqslant n$ such that*

$$f(x_1, x_2, \cdots, x_n) = \prod_{1 \leqslant k \leqslant n} f_k(x_k)$$

for all $x_k \in X_k$, $1 \leqslant k \leqslant n$.

2. *Given any $x_j, x'_j \in X_j$ for each $1 \leqslant j \leqslant n$, we have*

$$\begin{aligned} & f(x_1, x_2, \cdots, x_n) \times f(x'_1, x'_2, \cdots, x'_n) \\ = \; & f(x_1, \cdots, x_{k-1}, x'_k, x_{k+1}, x_n) \times f(x'_1, \cdots x'_{k-1}, x_k, x'_{k+1}, \cdots, x'_n) \end{aligned}$$

for any $1 \leqslant k \leqslant n$.

Moreover, if the above equivalent statements are satisfied then for any $\lambda_1, \cdots, \lambda_n \in Z$ such that $\lambda_1 \cdots \lambda_n = 1$, we have another factorization $f(x_1, \cdots, x_n) = \prod\limits_{1 \leqslant i \leqslant n} (\lambda_i f_i)(x_i)$. Each factorization of f is of the above form.

We fix $a_i \in X_i$ for each i and $c_1, \cdots, c_n \in Z$ such that $f(a_1, \cdots, a_n) = c_1 \cdots c_n$. If the above equivalent statements are satisfied then there exists a unique factorization such that $f_i(a_i) = c_i$.

Remark 7.1.1. *If $\# X_k \geqslant 3$ for all k, it is enough to verify the condition in statement 2 of the above corollary in the case $x_j \neq x'_j$ for all $1 \leqslant j \leqslant n$.*

In fact, when $\# X_k \geqslant 3$ for all k, for any $1 \leqslant j \leqslant n$ and any $y_j, y'_j \in X_j$, we may take $x_j \in X_j$ such that $x_j \neq y_j$, $x_j \neq y'_j$.

We fix any $1 \leqslant k \leqslant n$. If statement 2 is verified when $x_j \neq x'_j$ for all j then for any $y_k \neq y'_k$, we have

$$\begin{aligned} & f(y_1, y_2, \cdots, y_n) f(y'_1, y'_2, \cdots, y'_n) f(x_1, x_2, \cdots, x_n) \\ = \; & f(y_1, y_2, \cdots, y_n) f(y'_1, \cdots y'_{k-1}, x_k, y'_{k+1}, \cdots, y'_n) f(x_1, \cdots, x_{k-1}, y'_k, x_{k+1}, \cdots x_n) \\ = \; & f(y_1, y_2, \cdots, y_n) f(x_1, \cdots, x_{k-1}, y'_k, x_{k+1}, \cdots x_n) f(y'_1, \cdots y'_{k-1}, x_k, y'_{k+1}, \cdots, y'_n) \\ = \; & f(y_1, \cdots, y_{k-1}, y'_k, y_{k+1}, \cdots, y_n) f(x_1, \cdots, x_{k-1}, y_k, x_{k+1}, \cdots, x_n) \times \\ & f(y'_1, \cdots y'_{k-1}, x_k, y'_{k+1}, \cdots, y'_n) \\ = \; & f(y_1, \cdots, y_{k-1}, y'_k, y_{k+1}, \cdots, y_n) f(y'_1, \cdots y'_{k-1}, y_k, y'_{k+1}, \cdots, y'_n) f(x_1, x_2, \cdots, x_n). \end{aligned}$$

We have assumed $y_k \neq y'_k$ to guarantee that each time we apply the formula in Statement 2, the coefficients satisfy $x_j \neq x'_j$ for all $1 \leqslant j \leqslant n$.

Therefore if $y_k \neq y'_k$,

$$\begin{aligned} & f(y_1, y_2, \cdots, y_n) \times f(y'_1, y'_2, \cdots, y'_n) \\ = \; & f(y_1, \cdots, y_{k-1}, y'_k, y_{k+1}, \cdots, y_n) \times f(y'_1, \cdots y'_{k-1}, y_k, y'_{k+1}, \cdots, y'_n) \end{aligned} \tag{7.1}$$

If $y_k = y'_k$, this formula is trivially true.

We conclude that we can weaken the condition in Statement 2 of the above Corollary to $x_j \neq x'_j$ for all $1 \leqslant j \leqslant n$ when $\# X_k \geqslant 3$ for all k. We will verify this weaker condition in the application to the factorization of arithmetic automorphic periods.

7.2 Formula for the Whittaker period: even dimensional

Let Π be a regular cuspidal representation of $GL_n(\mathbb{A}_F)$ as in Theorem 5.2.1 with infinity type $(z^{a_i(\sigma)}\overline{z}^{-a_i(\sigma)})_{1\leqslant i\leqslant n}$ at $\sigma\in\Sigma$. We may assume that $a_1(\sigma)>a_2(\sigma)>\cdots>a_n(\sigma)$ for all $\sigma\in\Sigma$.

Recall that we say Π is **N-regular** if $a_i(\sigma)-a_{i+1}(\sigma)\geqslant N$ for all $1\leqslant i\leqslant n-1$ and $\sigma\in\Sigma$.

For $1\leqslant u\leqslant n-1$, let χ_u be an algebraic conjugate self-dual Hecke character of F with infinity type $z^{k_u(\sigma)}\overline{z}^{-k_u(\sigma)}$ at $\sigma\in\Sigma$.

Let us first consider the case n even. In this case, $a_i(\sigma)\in\mathbb{Z}+\dfrac{1}{2}$ for all $1\leqslant i\leqslant n$ and all $\sigma\in\Sigma$. We assume the following hypothesis:

Hypothesis 7.2.1. ***Even dimensional***
For all $\sigma\in\Sigma$, the numbers $\{k_u(\sigma)\mid 1\leqslant u\leqslant n-1\}$ lie in the $n-1$ gaps between $-a_n(\sigma)>-a_{n-1}(\sigma)>\cdots>-a_1(\sigma)$.

We define $\Pi^{\#}$ to be the Langlands sum of χ_u, $1\leqslant u\leqslant n-1$. It is an algebraic regular automorphic representation of $GL_{n-1}(\mathbb{A}_F)$. It follows by the above hypothesis that $(\Pi,\Pi^{\#})$ is in good position. By Proposition 3.6.1 we have

$$L(\frac{1}{2}+m,\Pi\times\Pi^{\#})\sim_{E(\Pi)E(\Pi^{\#});K} p(\Pi)p(\Pi^{\#})p(m,\Pi_\infty,\Pi^{\#}_\infty) \tag{7.2}$$

where $p(m,\Pi_\infty,\Pi^{\#}_\infty)$ is a complex number which depends on m,Π_∞ and $\Pi^{\#}_\infty$.

Simplification of $L(\frac{1}{2}+m,\Pi\times\Pi^{\#})$:

Since $\Pi^{\#}$ is the Langlands sum of χ_u, $1\leqslant u\leqslant n-1$, we have

$$L(\frac{1}{2}+m,\Pi\times\Pi^{\#})=\prod_{1\leqslant u\leqslant n-1}L(\frac{1}{2}+m,\Pi\times\chi_u).$$

We then apply Theorem 5.3 to the right hand side and get:

$$\begin{aligned}L(\frac{1}{2}+m,\Pi\times\Pi^{\#})&=\prod_{1\leqslant u\leqslant n-1}L(\frac{1}{2}+m,\Pi\times\chi_u)\\ \sim_{E(\Pi)E(\Pi^{\#});K}&\prod_{1\leqslant u\leqslant n-1}[(2\pi i)^{d(m+\frac{1}{2})n}P^{(I(\Pi,\chi_u))}(\Pi)\prod_{\sigma\in\Sigma}p(\widetilde{\chi_u},\sigma)^{I_u(\sigma)}p(\widetilde{\chi_u},\overline{\sigma})^{n-I_u(\sigma)}]\end{aligned}$$

Here we write I_u for $I(\Pi,\chi_u)$. In particular, $I_u(\sigma)=\#\{i\mid -a_i(\sigma)>k_u(\sigma)\}$ for $\sigma\in\Sigma$.

Note that χ_u is conjugate self-dual, we have $p(\widetilde{\chi_u},\overline{\sigma})\sim_{E(\Pi^{\#});K}p(\widetilde{\chi_u^c},\sigma)\sim_{E(\Pi^{\#});K}p(\widetilde{\chi_u^{-1}},\sigma)\sim_{E(\Pi^{\#});K}p(\widetilde{\chi_u},\sigma)^{-1}$. We deduce that:

$$L(\frac{1}{2}+m,\Pi\times\Pi^{\#})\sim_{E(\Pi)E(\Pi^{\#});K}(2\pi i)^{d(m+\frac{1}{2})n(n-1)}\prod_{1\leqslant u\leqslant n-1}[P^{(I_u)}(\Pi)\prod_{\sigma\in\Sigma}p(\widetilde{\chi_u},\sigma)^{2I_u(\sigma)-n}] \tag{7.3}$$

Calculate $p(\Pi^{\#})$: By Proposition 3.5.1, there exists a constant $\Omega(\Pi^{\#}_{\infty})\in\mathbb{C}^{\times}$ well defined up to $E(\Pi^{\#})^{\times}$ such that

$$p(\Pi^{\#})\sim_{E(\Pi^{\#});K}\Omega(\Pi^{\#}_{\infty})\prod_{1\leqslant u<v\leqslant n-1}L(1,\chi_u\chi_v^{-1}). \tag{7.4}$$

By Blasius's result, we have:

$$L(1,\chi_u\chi_v^{-1})\sim_{E(\Pi^{\#});K}(2\pi i)^d\prod_{\sigma\in\Sigma}p(\widetilde{\chi_u\chi_v^{-1}},\sigma')$$

If $k_u(\sigma)<k_v(\sigma)$ we have $\sigma'=\sigma$ and $p(\widetilde{\chi_u\chi_v^{-1}},\sigma')\sim_{E(\chi_u);K}p(\widetilde{\chi_u},\sigma)p(\widetilde{\chi_v},\sigma)^{-1}$.

Otherwise we have $\sigma'=\overline{\sigma}$ and $p(\widetilde{\chi_u\chi_v^{-1}},\sigma')\sim_{E(\chi_u);K}p(\widetilde{\chi_u},\sigma)^{-1}p(\widetilde{\chi_v},\sigma)$.

Therefore, we have the Whittaker period $p(\Pi^{\#})$

$$\sim_{E(\Pi^{\#});K}(2\pi i)^{\frac{d(n-1)(n-2)}{2}}\Omega(\Pi^{\#}_{\infty})\prod_{1\leqslant u\leqslant n-1}\prod_{\sigma\in\Sigma}p(\widetilde{\chi_u},\sigma)^{\#\{v|k_v(\sigma)>k_u(\sigma)\}-\#\{v|k_v(\sigma)<k_u(\sigma)\}} \tag{7.5}$$

We know $\#\{v\mid k_v(\sigma)<k_u(\sigma)\}=n-2-\#\{v\mid k_v(\sigma)>k_u(\sigma)\}$.

Moreover, by Hypothesis 7.2.1, we have $\#\{v\mid k_v(\sigma)>k_u(\sigma)\}=\#\{i\mid -a_i(\sigma)>k_u(\sigma)\}-1=I_u(\sigma)-1$. Therefore,

$$\#\{v\mid k_v(\sigma)>k_u(\sigma)\}-\#\{v\mid k_v(\sigma)<k_u(\sigma)\}=2I_u(\sigma)-n \tag{7.6}$$

We compare equations (7.2), (7.3), (7.5) and (7.6). If $L(\frac{1}{2}+m,\Pi\times\Pi^{\#})\neq 0$, we obtain that:

$$(2\pi i)^{d(m+\frac{1}{2})n(n-1)}\prod_{1\leqslant u\leqslant n-1}P^{(I_u)}(\Pi)\sim_{E(\Pi)E(\Pi^{\#});K}(2\pi i)^{\frac{d(n-1)(n-2)}{2}}p(\Pi)\Omega(\Pi^{\#}_{\infty})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}). \tag{7.7}$$

Hence we have

$$p(\Pi)\sim_{E(\Pi)E(\Pi^{\#};K)}(2\pi i)^{d(m+\frac{1}{2})n(n-1)-\frac{d(n-1)(n-2)}{2}}\Omega(\Pi^{\#}_{\infty})^{-1}p(m,\Pi_{\infty},\Pi^{\#}_{\infty})^{-1}\prod_{1\leqslant u\leqslant n-1}P^{(I_u)}(\Pi).$$

If we take $Z(m,\Pi_{\infty},\Pi'_{\infty})=(2\pi i)^{d(m+\frac{1}{2})n(n-1)-\frac{d(n-1)(n-2)}{2}}\Omega(\Pi^{\#}_{\infty})^{-1}p(m,\Pi_{\infty},\Pi^{\#}_{\infty})^{-1}$ then $p(\Pi)\sim_{E(\Pi)E(\Pi^{\#};K)}Z(m,\Pi_{\infty},\Pi'_{\infty})\prod\limits_{1\leqslant u\leqslant n-1}P^{(I_u)}(\Pi)$. We see that $Z(m,\Pi_{\infty},\Pi'_{\infty})$ depends only on Π_{∞}.

We may define:

$$Z(\Pi_{\infty}):=Z(m,\Pi_{\infty},\Pi'_{\infty})=(2\pi i)^{d(m+\frac{1}{2})n(n-1)-\frac{d(n-1)(n-2)}{2}}\Omega(\Pi^{\#}_{\infty})^{-1}p(m,\Pi_{\infty},\Pi^{\#}_{\infty})^{-1}. \tag{7.8}$$

It is well defined up to elements in $E(\Pi)^{\times}$.

We deduce that:

$$p(\Pi)\sim_{E(\Pi)E(\Pi^{\#};K)}Z(\Pi_{\infty})\prod_{1\leqslant u\leqslant n-1}P^{(I_u)}(\Pi). \tag{7.9}$$

7.3 Formula for the Whittaker period: odd dimensional

Let n be an odd positive integer. We keep the notation in the above section We have $a_i(\sigma) \in \mathbb{Z}$ for all $1 \leqslant i \leqslant n$ and all $\sigma \in \Sigma$. We assume that:

Hypothesis 7.3.1. ***Odd dimensional*** *For all $\sigma \in \Sigma$, the numbers $\{k_u(\sigma) + \frac{1}{2} \mid 1 \leqslant u \leqslant n-1\}$ lies in the $n-1$ gaps between $-a_n(\sigma) > -a_{n-1}(\sigma) > \cdots > -a_1(\sigma)$.*

Recall that ψ_F is an algebraic Hecke character of F with infinity type z^1 at each $\sigma \in \Sigma$ such that $\psi_F\psi_F^c = ||\cdot||_{\mathbb{A}_F}$. We take $\Pi^{\#}$ to be the Langlands sum of $\chi_u\psi_F||\cdot||_{\mathbb{A}_F}^{-\frac{1}{2}}$, $1 \leqslant u \leqslant n-1$. It is an algebraic regular automorphic representation of $GL_{n-1}(\mathbb{A}_F)$. The conditions of 3.6.1 hold.

We repeat the above process for Π and $\Pi^{\#}$ and get

$$\begin{aligned} & L(\frac{1}{2}+m, \Pi \times \Pi^{\#}) \\ \sim_{E(\Pi)E(\Pi^{\#});K} \; & (2\pi i)^{dmn(n-1)} \prod_{1 \leqslant u \leqslant n-1} [P^{(I(\Pi,\chi_u\psi_F))} \prod_{\sigma\in\Sigma} p(\widetilde{\chi_u}, \sigma)^{2I_u(\sigma)-n}] \times \\ & \prod_{\sigma\in\Sigma} (p(\widecheck{\psi_F}, \sigma)^{\sum_{1\leqslant u\leqslant n-1} I_u(\sigma)} p(\widecheck{\psi_F}^c, \sigma)^{\sum_{1\leqslant u\leqslant n-1}(n-I_u(\sigma))}) \end{aligned} \tag{7.10}$$

where $I_u := I(\Pi, \chi_u\psi_F)$ with $I_u(\sigma) = \#\{i \mid -a_i(\sigma) > k_u(\sigma) + \frac{1}{2}\}$.

It is easy to verify that Hypothesis 7.2.1 or 7.3.1 is equivalent to the following hypothesis:

Hypothesis 7.3.2. *For all $\sigma \in \Sigma$, the $(n-1)$ numbers $I_u(\sigma)$, $1 \leqslant u \leqslant n-1$, run over the numbers $1, 2, \cdots, n-1$.*

We see $\prod\limits_{1\leqslant u\leqslant n-1} I_u(\sigma) = \dfrac{n(n-1)}{2}$ and $\sum_{1\leqslant u\leqslant n-1}(n - I_u(\sigma)) = \dfrac{n(n-1)}{2}$.

We then have

$$\begin{aligned} & \prod_{\sigma\in\Sigma} (p(\widecheck{\psi_F}, \sigma)^{\sum_{1\leqslant u\leqslant n-1} I_u(\sigma)} p(\widecheck{\psi_F}^c, \sigma)^{\sum_{1\leqslant u\leqslant n-1}(n-I_u(\sigma))}) \\ \sim_{E(\psi_F);K} & \prod_{\sigma\in\Sigma} p(\widecheck{\psi_F\psi_F^c}, \sigma)^{\frac{n(n-1)}{2}} \sim_{E(\psi_F);K} \prod_{\sigma\in\Sigma} p(||\cdot||_{\mathbb{A}_F}^{-1}, \sigma)^{\frac{n(n-1)}{2}} \sim_{E(\psi_F);K} (2\pi i)^{\frac{dn(n-1)}{2}}. \end{aligned}$$

We verify that the equation (7.5) and (7.6) remain unchanged. We can see that equation (7.9) still holds here.

Let us start from the numbers $I_u(\sigma)$. If we are given some numbers $I_u(\sigma)$, $\sigma \in \Sigma$, $1 \leqslant u \leqslant n-1$, such that Hypothesis 7.3.2 is satisfied, we can always choose $k_u(\sigma) \in \mathbb{Z}$ such that $I_u(\sigma) = \#\{i \mid -a_i(\sigma) > k_u(\sigma)\}$ if n is even, $I_u(\sigma) = \#\{i \mid -a_i(\sigma) > k_u(\sigma) + \frac{1}{2}\}$ if n is odd.

We may then take χ_u, $1 \leqslant u \leqslant n-1$ with infinity type $z^{k_u(\sigma)}\overline{z}^{-k_u(\sigma)}$ at $\sigma \in \Sigma$. Equation (7.9) tells us that

$$p(\Pi) \sim_{E(\Pi);K} Z(\Pi_\infty) \prod_{1\leqslant u\leqslant n-1} P^{(I_u)}(\Pi) \tag{7.11}$$

provided a non vanishing condition of the L-function, for example, if Π is 3-regular.

Theorem 7.3.1. *Let $I_u(\sigma)$, $1 \leqslant u \leqslant n-1$, $\sigma \in \Sigma$ be some integers such that Hypothesis 7.3.2 is verified. There exists a complex number $Z(\Pi_\infty)$ such that if a non vanishing condition of a global L-function is verified, in particular, if Π is 3-regular, then:*

$$p(\Pi) \sim_{E(\Pi);K} Z(\Pi_\infty) \prod_{1 \leqslant u \leqslant n-1} P^{(I_u)}(\Pi). \tag{7.12}$$

7.4 Factorization of arithmetic automorphic periods: restricted case

We consider the function $\prod_{\sigma \in \Sigma} \{0, 1, \cdots, n\} \to \mathbb{C}^\times / E(\Pi)^\times$ which sends $(I(\sigma))_{\sigma \in \Sigma}$ to $P^{(I)}(\Pi)$.

The motivic calculation predicts that:

Conjecture 7.4.1. *There exists some non zero complex numbers $P^{(s)}(\Pi, \sigma)$ for all $0 \leqslant s \leqslant n$ and $\sigma \in \Sigma$ such that $P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod_{\sigma \in \Sigma} P^{(I(\sigma))}(\Pi, \sigma)$ for all $I = (I(\sigma))_{\sigma \in \Sigma} \in \{0, 1, \cdots, n\}^\Sigma$.*

In this section, we will prove the above conjecture restricted to $\{1, 2, \cdots, n-1\}^\Sigma$. More precisely, we will prove that

Theorem 7.4.1. *If $n \geqslant 4$ and Π satisfies a global non vanishing condition, in particular, if Π is 3-regular, then there exists some non zero complex numbers $P^{(s)}(\Pi, \sigma)$ for all $1 \leqslant s \leqslant n-1$, $\sigma \in \Sigma$ such that $P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod_{\sigma \in \Sigma} P^{(I(\sigma))}(\Pi, \sigma)$ for all $I = (I(\sigma))_{\sigma \in \Sigma} \in \{1, 2, \cdots, n-1\}^\Sigma$.*

Proof For all $\sigma \in \Sigma$, let $I_1(\sigma) \neq I_2(\sigma)$ be two numbers in $\{1, 2, \cdots, n-1\}$. We consider I_1, I_2 as two elements in $\{1, 2, \cdots, n-1\}^\Sigma$.

Let σ_0 be any element in Σ. We define $I'_1, I'_2 \in \{1, 2, \cdots, n-1\}^\Sigma$ by $I'_1(\sigma) := I_1(\sigma)$, $I'_2(\sigma) := I_2(\sigma)$ if $\sigma \neq \sigma_0$ and $I'_1(\sigma_0) := I_2(\sigma_0)$, $I'_2(\sigma_0) := I_1(\sigma_0)$.

By Remark 7.1.1, it is enough to prove that

$$P^{(I_1)}(\Pi) P^{(I_2)}(\Pi) \sim_{E(\Pi);K} P^{(I'_1)}(\Pi) P^{(I'_2)}(\Pi).$$

Since $I_1(\sigma) \neq I_2(\sigma)$ for all $\sigma \in \Sigma$, we can always find $I_3, \cdots, I_{n-1} \in \{1, 2, \cdots, n-1\}^\Sigma$ such that for all $\sigma \in \Sigma$, the $(n-1)$ numbers $I_u(\sigma)$, $1 \leqslant u \leqslant n-1$ run over $1, 2, \cdots, n-1$. In other words, Hypothesis 7.3.2 is verified.

By Theorem 7.3.1, we have

$$p(\Pi) \sim_{E(\Pi);K} Z(\Pi_\infty) P^{(I_1)}(\Pi) P^{(I_2)}(\Pi) \prod_{3 \leqslant u \leqslant n-1} P^{(I_u)}(\Pi).$$

On the other hand, it is easy to see that I'_1, I'_2, $I_3, \cdots, I_{n-1}$ also satisfy Hypothesis 7.3.2. Therefore

$$p(\Pi) \sim_{E(\Pi);K} Z(\Pi_\infty) P^{(I'_1)}(\Pi) P^{(I'_2)}(\Pi) \prod_{3 \leqslant u \leqslant n-1} P^{(I_u)}(\Pi).$$

We conclude at last $P^{(I_1)}(\Pi)P^{(I_2)}(\Pi) \sim_{E(\Pi);K} P^{(I'_1)}(\Pi)P^{(I'_2)}(\Pi)$ and then the above theorem follows.

□

7.5 Factorization of arithmetic automorphic periods: complete case

In this section, we will prove Conjecture 7.4.1 when Π is regular enough. More precisely, we have

Theorem 7.5.1. *Conjecture 7.4.1 is true if Π is 2-regular and satisfies a global non vanishing condition, in particular, if Π is 6-regular.*

Corollary 7.5.1. *If Π satisfied the conditions in the above theorem then we have:*

$$p(\Pi) \sim_{E(\Pi);K} Z(\Pi_\infty) \prod_{\sigma\in\Sigma} \prod_{1\leqslant i\leqslant n-1} P^{(i)}(\Pi,\sigma) \tag{7.13}$$

If $n = 1$, Conjecture 7.4.1 is known as multiplicity of CM periods. We may assume that $n \geqslant 2$. The set $\{0, 1, \cdots, n\}$ has at least 3 elements and then Remark 7.1.1 can apply.

For all $\sigma \in \Sigma$, let $I_1(\sigma) \neq I_2(\sigma)$ be two numbers in $\{0, 1, \cdots, n\}$. We have $I_1, I_2 \in \{0, 1, 2, \cdots, n\}^\Sigma$.

Let σ_0 be any element in Σ. We define $I'_1, I'_2 \in \{0, 1, 2, \cdots, n\}^\Sigma$ as in the proof of Theorem 7.4.1.

It remains to show that

$$P^{(I_1)}(\Pi)P^{(I_2)}(\Pi) \sim_{E(\Pi);K} P^{(I'_1)}(\Pi)P^{(I'_2)}(\Pi). \tag{7.14}$$

Let us assume that n is odd at first. Since Π is 2-regular, we can find χ_u a conjugate self-dual algebraic Hecke character of F such that $I(\Pi, \chi_u) = I_u$ for $u = 1, 2$. We denote the infinity type of χ_u at $\sigma \in \Sigma$ by $z^{k_u(\sigma)}\overline{z}^{-k_u(\sigma)}$, $u = 1, 2$. We remark that $k_1(\sigma) \neq k_2(\sigma)$ for all σ since $I_1(\sigma) \neq I_2(\sigma)$.

Let $\Pi^\#$ be the Langlands sum of Π, χ_1^c and χ_2^c. We write the infinity type of $\Pi^\#$ at $\sigma \in \Sigma$ by $(z^{b_i(\sigma)}\overline{z}^{-b_i(\sigma)})_{1\leqslant i\leqslant n+2}$ with $b_1(\sigma) > b_2(\sigma) > \cdots > b_{n+2}(\sigma)$. The set $\{b_i(\sigma), 1 \leqslant i \leqslant n + 2\} = \{a_i(\sigma), 1 \leqslant i \leqslant n\} \cup \{-k_1(\sigma), -k_2(\sigma)\}$.

Let $\Pi^\diamondsuit$ be a cuspidalconjugate self-dual cohomological representation of $GL_{n+3}(\mathbb{A}_F)$ with infinity type $(z^{c_i(\sigma)}\overline{z}^{-c_i(\sigma)})_{1\leqslant i\leqslant n+3}$ such that $-c_{n+3}(\sigma) > b_1(\sigma) > -c_{n+2}(\sigma) > b_2(\sigma) > \cdots > -c_2(\sigma) > b_{n+2}(\sigma) > -c_1(\sigma)$ for all $\sigma \in \Sigma$. We may assume that $\Pi^\diamondsuit$ has definable arithmetic automorphic periods.

Proposition 3.6.1 is true for $(\Pi^\diamondsuit, \Pi^\#)$. Namely,

$$L(\frac{1}{2} + m, \Pi^\diamondsuit \times \Pi^\#) \sim_{E(\Pi^\diamondsuit)E(\Pi^\#);K} p(\Pi^\diamondsuit)p(\Pi^\#)p(m, \Pi^\diamondsuit_\infty, \Pi^\#_\infty). \tag{7.15}$$

We know

$$L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\Pi^{\#})=L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\Pi)L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\chi_1^c)L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\chi_2^c) \quad (7.16)$$

For $u=1$ or 2, by Theorem 5.3 and the fact that χ_u is conjugate self-dual, we have

$$L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\chi_u)$$
$$\sim_{E(\Pi^{\diamondsuit})E(\Pi^{\#});K}(2\pi i)^{(\frac{1}{2}+m)d(n+3)}P^{I(\Pi^{\diamondsuit},\chi_u^c)}(\Pi)\prod_{\sigma\in\Sigma}p(\widecheck{\chi_u},\sigma)^{-2I(\Pi^{\diamondsuit},\chi_u^c)(\sigma)+(n+3)}. \quad (7.17)$$

Proposition 3.5.1 implies that

$$p(\Pi^{\#})\sim_{E(\Pi^{\#});K}\Omega(\Pi^{\#}_{\infty})p(\Pi)L(1,\Pi\otimes\chi_1)L(1,\Pi\otimes\chi_2)L(1,\chi_1\chi_2^c) \quad (7.18)$$

where $\Omega(\Pi^{\#}_{\infty})$ is a non zero complex numbers depend on $\Pi^{\#}_{\infty}$.

By Theorem 5.3 again, for $u=1,2$, we have

$$L(1,\Pi\times\chi_u)\sim_{E(\Pi^{\#});K}(2\pi i)^{dn}P^{I(\Pi,\chi_u)}\prod_{\sigma\in\Sigma}p(\widecheck{\chi_u},\sigma)^{2I(\Pi,\chi_u)(\sigma)-n}. \quad (7.19)$$

Moreover, $L(1,\chi_1\chi_2^c)\sim_{E(\Pi^{\#});K}(2\pi i)^d\prod_{\sigma\in\Sigma}p(\widecheck{\chi_1},\sigma)^{t(\sigma)}p(\widecheck{\chi_2},\sigma)^{-t(\sigma)}$ where $t(\sigma)=1$ if $k_1(\sigma)<k_2(\sigma)$, $t(\sigma)=-1$ if $k_1(\sigma)>k_2(\sigma)$.

Lemma 7.5.1. *For all $\sigma\in\Sigma$,*

$$-2I(\Pi^{\diamondsuit},\chi_1^c)(\sigma)+(n+3)=2I(\Pi,\chi_1)(\sigma)-n+t(\sigma),$$
$$-2I(\Pi^{\diamondsuit},\chi_2^c)(\sigma)+(n+3)=2I(\Pi,\chi_1)(\sigma)-n-t(\sigma).$$

Proof By definition we have

$$I(\Pi^{\diamondsuit},\chi_1^c)(\sigma)=\#\{1\leqslant i\leqslant n+3\mid -c_i(\sigma)>-k_1(\sigma)\}.$$

Recall that $-c_{n+3}(\sigma)>b_1(\sigma)>-c_{n+2}(\sigma)>b_2(\sigma)>\cdots>-c_2(\sigma)>b_{n+2}(\sigma)>-c_1(\sigma)$ and $\{b_i(\sigma),1\leqslant i\leqslant n+2\}=\{a_i(\sigma),1\leqslant i\leqslant n\}\cup\{-k_1(\sigma),-k_2(\sigma)\}$.

Therefore

$$\begin{aligned} I(\Pi^{\diamondsuit},\chi_1^c)(\sigma) &= \#\{1\leqslant i\leqslant n+2\mid b_i(\sigma)>-k_1(\sigma)\}+1\\ &= \#\{1\leqslant i\leqslant n\mid a_i(\sigma)>-k_1(\sigma)\}+\mathbb{1}_{-k_2(\sigma)>-k_1(\sigma)}+1.\end{aligned}$$

By definition we have

$$I(\Pi,\chi_1)(\sigma)=\#\{1\leqslant i\leqslant n\mid -a_i(\sigma)>k_1(\sigma)\}=n-\#\{1\leqslant i\leqslant n\mid a_i(\sigma)>-k_1(\sigma)\}.$$

Therefore, $I(\Pi^{\diamondsuit},\chi_1^c)(\sigma)=n-I(\Pi,\chi_1)(\sigma)+\mathbb{1}_{-k_2(\sigma)>-k_1(\sigma)}+1$. Hence we have $-2I(\Pi^{\diamondsuit},\chi_1^c)(\sigma))+(n+3)=2I(\Pi,\chi_1)(\sigma)-n+1-2\mathbb{1}_{-k_2(\sigma)>-k_1(\sigma)}$.

It is easy to verify that $1-2\mathbb{1}_{-k_2(\sigma)>-k_1(\sigma)}=t(\sigma)$. The first statement then follows and the second is similar to the first one.

□

We deduce that if $L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\Pi^{\#})\neq 0$, then

$$L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\Pi)(2\pi i)^{(1+2m)d(n+3)}P^{I(\Pi^{\diamondsuit},\chi_1^c)}(\Pi^{\diamondsuit})P^{I(\Pi^{\diamondsuit},\chi_2^c)}(\Pi^{\diamondsuit})\sim_{E(\Pi^{\diamondsuit})E(\Pi^{\#});K}$$
$$(2\pi i)^{d(2n+1)}p(\Pi^{\diamondsuit})\Omega(\Pi^{\#}_{\infty})p(m,\Pi^{\#}_{\infty},\Pi^{\#}_{\infty})P^{I(\Pi,\chi_1)}(\Pi)P^{I(\Pi,\chi_2)}(\Pi). \quad (7.20)$$

Now let χ'_1, χ'_2 be two conjugate self-dual algebraic Hecke characters of F such that $\chi'_{1,\sigma}=\chi_{1,\sigma}$ and $\chi'_{2,\sigma}=\chi_{2,\sigma}$ for $\sigma\neq\sigma_0$, $\chi'_{1,\sigma_0}=\chi_{2,\sigma_0}$ and $\chi'_{2,\sigma_0}=\chi_{1,\sigma_0}$.

We take $\Pi^{\#\#}$ as Langlands sum of Π, ${\chi'_1}^c$ and ${\chi'_2}^c$. Since the infinity type of $\Pi^{\#\#}$ is the same with $\Pi^{\#}$, we can repeat the above process and we see that equation (7.20) is true for $(\Pi^{\diamondsuit},\Pi^{\#\#})$. Observe that most terms remain unchanged.

Comparing equation (7.20) for $(\Pi^{\diamondsuit},\Pi^{\#})$ and that for $(\Pi^{\diamondsuit},\Pi^{\#\#})$, we get

$$\frac{P^{I(\Pi^{\diamondsuit},{\chi'_1}^c)}(\Pi^{\diamondsuit})P^{I(\Pi^{\diamondsuit},{\chi'_2}^c)}(\Pi^{\diamondsuit})}{P^{I(\Pi^{\diamondsuit},\chi_1^c)}(\Pi^{\diamondsuit})P^{I(\Pi^{\diamondsuit},\chi_2^c)}(\Pi^{\diamondsuit})}\sim_{E(\Pi^{\diamondsuit})E(\Pi);K}\frac{P^{I(\Pi,\chi'_1)}(\Pi)P^{I(\Pi,\chi'_2)}(\Pi)}{P^{I(\Pi,\chi_1)}(\Pi)P^{I(\Pi,\chi_2)}(\Pi)}. \quad (7.21)$$

By construction, $I(\Pi,\chi_u)=I_u$ and $I(\Pi,\chi'_u)=I'_u$ for $u=1,2$. Hence to prove (7.14), it is enough to show the left hand side of the above equation is a number in $E(\Pi^{\diamondsuit})^{\times}$.

There are at least two ways to see this. We observe that

$$I(\Pi^{\diamondsuit},{\chi'_1}^c)(\sigma)=I(\Pi^{\diamondsuit},{\chi_1}^c)(\sigma),I(\Pi^{\diamondsuit},{\chi'_2}^c)(\sigma)=I(\Pi^{\diamondsuit},{\chi_2}^c)(\sigma)\text{ for }\sigma\neq\sigma_0$$
$$\text{and }I(\Pi^{\diamondsuit},{\chi'_1}^c)(\sigma_0)=I(\Pi^{\diamondsuit},{\chi_2}^c)(\sigma_0),I(\Pi^{\diamondsuit},{\chi'_2}^c)(\sigma_0)=I(\Pi^{\diamondsuit},{\chi_1}^c)(\sigma_0).$$

Moreover, these numbers are all in $\{1,2,\cdots,(n+3)-1\}$. Theorem 7.4.1 gives a factorization of the holomorphic arithmetic automorphic periods through each place. In particular, it implies that the left hand side of (7.21) is in $E(\Pi^{\diamondsuit})^{\times}$ as expected.

One can also show this by taking $\Pi^{\diamondsuit}$ an automorphic induction of a Hecke character. We can then calculate $L(\frac{1}{2}+m,\Pi^{\diamondsuit}\times\chi_u^c)$ in terms of CM periods. Since the factorization of CM periods is clear, we will also get the expected result.

When n is even, we consider $\Pi^{\#}$ the Langlands sum of Π, $(\chi_1\psi_F||\cdot||^{-1/2})^c$ and $(\chi_2\psi_F||\cdot||^{-1/2})^c$ where χ_1, χ_2 are two suitable algebraic Hecke characters of F. We follow the above steps and will get the factorization in this case. We leave the details to the reader and just remark that as in section 7.3, some CM periods of ψ_F appear but they will be eliminated at the end.

7.6 Specify the factorization

Let us assume that Conjecture 7.4.1 is true. We want to specify one factorization.

We denote by I_0 the map which sends each $\sigma\in\Sigma$ to 0. By the last part of Corollary 7.1.1, it is enough to choose $c(\Pi,\sigma)\in(\mathbb{C}/E(\Pi))^{\times}$ which is G_K-equivariant such that

$P^{(I_0)}(\Pi) \sim_{E(\Pi);K} \prod_{\sigma\in\Sigma} c(\Pi,\sigma)$. Then there exists a unique factorization of $P^{(\cdot)}(\Pi)$ such that $P^{(0)}(\Pi,\sigma) = c(\Pi,\sigma)$. We may then define the **local arithmetic automorphic periods** $P^{(s)}(\Pi,\sigma)$ as an element in $\mathbb{C}^\times/(E(\pi))^\times$.

In this section, we shall prove $P^{(I_0)}(\Pi) \sim_{E(\Pi);K} p(\breve{\xi_\Pi},\overline{\Sigma}) \sim_{E(\Pi);K} \prod_{\sigma\in\Sigma} p(\breve{\xi_\Pi},\overline{\sigma})$. Therefore, we may take $c(\Pi,\sigma) = p(\breve{\xi_\Pi},\overline{\sigma})$.

More generally, we will see that:

Lemma 7.6.1. *If I is compact then* $P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod_{I(\sigma)=0} p(\breve{\xi_\Pi},\overline{\sigma}) \times \prod_{I(\sigma)=n} p(\breve{\xi_\Pi},\sigma)$.

This lemma leads to the following theorem:

Theorem 7.6.1. *If Conjecture 7.4.1 is true, in particular, if conditions in Theorem 7.5.1 are satisfied, then there exists some complex numbers $P^{(s)}(\Pi,\sigma)$ unique up to multiplication by elements in $(E(\Pi))^\times$ such that the following two conditions are satisfied:*

1. $P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod_{\sigma\in\Sigma} P^{(I(\sigma))}(\Pi,\sigma)$ *for all* $I = (I(\sigma))_{\sigma\in\Sigma} \in \{0,1,\cdots,n\}^\Sigma$,
2. *and* $P^{(0)}(\Pi,\sigma) \sim_{E(\Pi);K} p(\breve{\xi_\Pi},\overline{\sigma})$

where ξ_Π is the central character of Π.

Moreover, we know

$$P^{(n)}(\Pi,\sigma) \sim_{E(\Pi);K} p(\breve{\xi_\Pi},\sigma)$$

or equivalently

$$P^{(0)}(\Pi,\sigma) \times P^{(n)}(\Pi,\sigma) \sim_{E(\Pi);K} 1.$$

Proof of Lemma 7.6.1: Recall that $D/2 = \sum_{\sigma\in\Sigma} I_\sigma(n-I_\sigma) = 0$ since I is compact.

Let T be the center of GU_I. We have $T(\mathbb{R}) \cong \{(z_\sigma) \in (\mathbb{C}^\times)^\Sigma \mid |z_\sigma| \text{ does not depend on } \sigma\}$. We define a homomorphism $h_T : \mathbb{S}(\mathbb{R}) \to T(\mathbb{R})$ by sending $z \in \mathbb{C}$ to $((z)_{I(\sigma)=0}, (\overline{z})_{I(\sigma)=n})$.

Since I is compact, we see that h_I is the composition of h_T and the embedding $T \hookrightarrow GU_I$. We get an inclusion of Shimura varieties: $Sh_T := Sh(T,h_T) \hookrightarrow Sh_I = Sh(GU_I,h_I)$.

Let ξ be a Hecke character of K such that $\Pi^\vee \otimes \xi$ descends to π, a representation of $GU_I(\mathbb{A}_\mathbb{Q})$, as before. We write $\lambda \in \Lambda(GU_I)$ the cohomology type of π. We define $\lambda^T := (\lambda_0, (\sum_{1\leqslant i\leqslant n} \lambda_i(\sigma))_{\sigma\in\Sigma})$. Since π is irreducible, it acts as scalars when restrict to T. This gives π^T, a one dimensional representation of $T(\mathbb{A}_\mathbb{Q})$ which is cohomology of type λ^T. We denote by V_{λ^T} the character of $T(\mathbb{R})$ with highest weight λ^T.

The automorphic vector bundle E_λ pulls back to the automorphic vector bundle $[V_{\lambda^T}]$ (see [17] for notation) on Sh_T.

Let β be an element in $\bar{H}^0(Sh_I, E_\lambda)^\pi$. We fix a non zero $E(\pi)$-rational element in π and then we can lift β to ϕ, an automorphic form on $GU_I(\mathbb{A}_\mathbb{Q})$.

There is an isomorphism $H^0(Sh_T, [V_{\lambda^T}]) \xrightarrow{\sim} \{f \in \mathbb{C}^\infty(T(\mathbb{Q})\backslash T(\mathbb{A}_\mathbb{Q}), \mathbb{C} \mid f(tt_\infty)) = \pi^T(t_\infty)f(t), t_\infty \in T(\mathbb{R}), t \in T(\mathbb{A}_\mathbb{Q})\}$ (c.f. [17]). We send β to the element in $H^0(Sh_T, [V_{\lambda^T}])^{\pi^T}$ associated to $\phi|_{T(\mathbb{A}_\mathbb{Q})}$.

We then obtain rational morphisms

$$\bar{H}^0(Sh_I, E_\lambda)^\pi \xrightarrow{\sim} H^0(Sh_T, [V_{\lambda^T}])^{\pi^T} \tag{7.22}$$

$$\text{and similarly} \quad \bar{H}^0(Sh_I, E_{\lambda^\vee})^{\pi^\vee} \xrightarrow{\sim} H^0(Sh_T, [V_{\lambda^{T,\vee}}])^{\pi^{T,\vee}}. \tag{7.23}$$

These morphisms are moreover isomorphisms. In fact, since both sides are one dimensional, it is enough to show the above morphisms are injective. Indeed, if ϕ, a lifting of an element in $\bar{H}^0(Sh_I, E_\lambda)^\pi$, vanishes at the center, in particular, it vanishes at the identity. Hence it vanishes at $GU_I(\mathbb{A}_{\mathbb{Q},f})$ since it is an automorphic form. We observe that $GU_I(\mathbb{A}_{\mathbb{Q},f})$ is dense in $GU_I(\mathbb{Q})\backslash GU_I(\mathbb{A}_\mathbb{Q})$. We know $\phi = 0$ as expected.

We are going to calculate the arithmetic automorphic period. Let β be rational. We take a rational element $\beta^\vee \in \bar{H}^0(Sh_I, E_{\lambda^\vee})^{\pi^\vee}$ and lift it to an automorphic form $\phi^\vee$. We have $c_B(\phi) \sim_{E(\pi);K} P^{(I)}(\pi)\phi^\vee$ by Lemma 4.6.1.

For the torus, by Remark 4.1.1, we know

$$\phi^\vee|_{T(\mathbb{A}_\mathbb{Q})} \sim_{E(\pi);K} p(Sh(T, h_T), \pi^T)^{-1}(\phi|_{T(\mathbb{A}_\mathbb{Q})})^{-1}.$$

Recall that $c_B(\phi) = \pm i^{\lambda_0}\overline{\phi}||\nu(\cdot)||^{\lambda_0}$. Therefore $(c_B(\phi))|_{T(\mathbb{A}_\mathbb{Q})} = \pm i^{\lambda_0}(\phi|_{T(\mathbb{A}_\mathbb{Q})})^{-1}$. We then get

$$i^{\lambda_0}P^{(I)}(\pi) \sim_{E(\pi);K} p(Sh(T, h_T), \pi^T). \tag{7.24}$$

We now set $T^\# := Res_{K/\mathbb{Q}}T_K$. We have $T^\# \cong Res_{K/\mathbb{Q}}\mathbb{G}_m \times Res_{F/\mathbb{Q}}\mathbb{G}_m$. In particular, $T^\#(\mathbb{R}) \cong \mathbb{C}^\times \times (\mathbb{R} \otimes_\mathbb{Q} F)^\times \cong \mathbb{C}^\times \times (\mathbb{C}^\times)^\Sigma$.

We define $h_{T^\#} : \mathbb{S}(\mathbb{R}) \to T^\#(\mathbb{R})$ to be the composition of h_T and the natural embedding $T(\mathbb{R}) \to T^\#(\mathbb{R})$. We know $h_{T^\#}$ sends $z \in \mathbb{C}^\times$ to $(z\overline{z}, (z)_{I(\sigma)=0}, (\overline{z})_{r(\sigma)=0})$. The embedding $(T, h_T) \to (T^\#, h_{T^\#})$ is a map between Shimura datum.

We observe that $\pi^{T,\#} := ||\cdot||^{-\lambda_0} \times \xi_\Pi^{-1}$ is a Hecke character on $T^\#$. Its restriction to T is just π^T. By Proposition 4.1.1, we have $p(Sh(T, h_T), \pi^T) \sim_{E(\pi);K} p(Sh(T^\#, h_{T^\#}), \pi^{T^\#})$.

By the definition of CM period and Proposition 4.1.2, we have

$$p(Sh(T^\#, h_{T^\#}), \pi^{T^\#}) \sim_{E(\pi);K} (2\pi i)^{\lambda_0} \prod_{I(\sigma)=0} p(\xi_\Pi^{-1}, \sigma) \prod_{I(\sigma)=n} p(\xi_\Pi^{-1}, \overline{\sigma}). \tag{7.25}$$

Since ξ_Π is conjugate self-dual, we have $p(\xi_\Pi^{-1}, \overline{\sigma}) \sim_{E(\Pi);K} p(\xi_\Pi, \sigma)$.
By equation (7.24), we get:

$$i^{\lambda_0}P^{(I)}(\pi) \sim_{E(\pi);K} (2\pi i)^{\lambda_0} \prod_{I(\sigma)=0} p(\xi_\Pi^{-1}, \sigma) \prod_{I(\sigma)=n} p(\xi_\Pi, \sigma). \tag{7.26}$$

Recall that by definition $P^{(I)}(\Pi) \sim_{E(\Pi);K} (2\pi)^{-\lambda_0} P^{(I)}(\pi)$, we get finally

$$\begin{aligned} P^{(I)}(\Pi) &\sim_{E(\Pi);K} \prod_{I(\sigma)=0} p(\xi_\Pi^{-1},\sigma) \times \prod_{I(\sigma)=n} p(\xi_\Pi,\sigma) \\ &\sim_{E(\Pi);K} \prod_{I(\sigma)=0} p(\breve{\xi_\Pi},\overline{\sigma}) \times \prod_{I(\sigma)=n} p(\breve{\xi_\Pi},\sigma). \end{aligned}$$

The last formula comes from the fact that ξ_Π is conjugate self-dual.

□

We recall that the arithmetic automorphic periods can be defined for essential conjugate self-dual representations. More precisely, let Π be conjugate self-dual as in Theorem 7.6.1, let η be an algebraic Hecke character. By Definition 5.3.2, we have defined $P^{(I)}(\Pi\otimes \eta)$ as $P^{(I)}(\Pi)\prod\limits_{\sigma\in\Sigma} p(\check{\eta},\sigma)^{I(\sigma)}p(\check{\eta},\overline{\sigma})^{n-I(\sigma)}$. As we showed above that $P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod\limits_{\sigma\in\Sigma} P^{(I(\sigma))}(\Pi,\sigma)$, it is natural to define:

Definition 7.6.1. *We define* ***local arithmetic automorphic periods*** *for conjugate self-dual representations by*

$$P^{(s)}(\Pi\otimes\eta,\sigma) = P^{(s)}(\Pi,\sigma)p(\check{\eta},\sigma)^{I(\sigma)}p(\check{\eta},\overline{\sigma})^{n-I(\sigma)}. \tag{7.27}$$

Remark 7.6.1. *If $s=0$, we see that*

$$\begin{aligned} P^{(0)}(\Pi\otimes\eta,\sigma) &= P^{(0)}(\Pi,\sigma)p(\check{\eta},\overline{\sigma})^n \sim_{E(\Pi;K)} p(\breve{\xi_\Pi},\overline{\sigma})p(\check{\eta},\overline{\sigma})^n \\ &\sim_{E(\Pi;K)} p(\breve{\xi_\Pi\eta^n},\overline{\sigma}) \sim_{E(\Pi;K)} p(\breve{\xi_{\Pi\otimes\eta}},\overline{\sigma}) \end{aligned} \tag{7.28}$$

Therefore, if Π *has definable arithmetic automorphic periods and regular enough, we still have*

1. $P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod\limits_{\sigma\in\Sigma} P^{(I(\sigma))}(\Pi,\sigma)$ *for all* $I=(I(\sigma))_{\sigma\in\Sigma}\in\{0,1,\cdots,n\}^\Sigma$,
2. *and* $P^{(0)}(\Pi,\sigma) \sim_{E(\Pi);K} p(\breve{\xi_\Pi},\overline{\sigma})$.

Moreover, these two properties determine the local periods.

Remark 7.6.2. *If $n=1$ and $\Pi=\eta$ is a Hecke character, we obtain that:* $P^{(0)}(\eta,\sigma) \sim_{E(\eta);K} p(\check{\eta},\overline{\sigma})$ *and similarly* $P^{(1)}(\eta,\sigma) \sim_{E(\eta);K} p(\check{\eta},\sigma)$.

Chapter 8

Functoriality of arithmetic automorphic periods

8.1 Period relations for automorphic inductions: settings

Let F be a CM field containing K as before.
Let $\mathcal{F}/F$ be a cyclic extension of CM fields of degree l.
Let $\Pi_{\mathcal{F}}$ be a cuspidal representation of $GL_n(\mathbb{A}_{\mathcal{F}})$.

By Theorem 6.2 of [2], there exists Π_F, an automorphic representation of $GL_{nl}(\mathbb{A}_F)$ which lifts $\Pi_{\mathcal{F}}$. We assume moreover that $\Pi_{\mathcal{F}} \ncong \Pi_{\mathcal{F}}^g$ for all $g \in Gal(\mathcal{F}/F)$ non trivial. We can read from the proof of Theorem 6.2 in the *loc.cit* that Π_F is then cuspidal.

We want to compare the arithmetic automorphic periods of $\Pi_{\mathcal{F}}$ and Π_F if they are defined. For this purpose, we assume that $\Pi_{\mathcal{F}}$ has definable arithmetic automorphic periods as in Definition 5.3.2. In other words, $\Pi_{\mathcal{F}}$ is 3-regular, cohomological and descends to unitary groups of any sign after tensoring by an algebraic Hecke character.

We write the infinity type of $\Pi_{\mathcal{F}}$ as $(z^{a_i(\sigma)}\overline{z}^{b_i(\sigma)})_{1\leqslant i\leqslant n}$ at $\sigma \in \Sigma_{\mathcal{F};K}$. We remark that $a_i(\sigma), b_i(\sigma) \in \mathbb{Z} + \dfrac{n-1}{2}$.

The restriction of embeddings gives a map:

$$\Psi_{\mathcal{F}/F} : \Sigma_{\mathcal{F};K} \to \Sigma_{F;K}.$$

For $\tau \in \Sigma_{F;K}$, the infinity type of Π_F at τ is $(z^{a_i(\sigma)}\overline{z}^{b_i(\sigma)})_{1\leqslant i\leqslant n, \sigma\in\Psi^{-1}_{\mathcal{F}/F}(\tau)}$. We assume in this chapter that for any τ the nl numbers $a_i(\sigma)$, $1 \leqslant i \leqslant n$ and $\sigma \in \Psi^{-1}_{\mathcal{F}/F}(\tau)$, are different. We assume moreover their differences are at least 3. Hence Π_F is also 3-regular.

If l is odd or n is even, we know Π_F is algebraic and then cohomological. We write $\Pi'_F := \Pi_F$ in this case.

If l is even and n is odd, Π_F is no longer algebraic. We define $\Pi'_F := \Pi_F||\cdot||_{\mathbb{A}_F}^{-1/2}$. It is then a cuspidal cohomological representation of $GL_{nl}(\mathbb{A}_F)$. It is conjugate self-dual after tensoring by an algebraic Hecke character. To see this, we take ψ_F an algebraic Hecke character of F with infinity type $z^1\overline{z}^0$ at each infinity place such that $\psi_F\psi_F^c = ||\cdot||_{\mathbb{A}_F}$. We remark that the Hecke character $||\cdot||_{\mathbb{A}_F}^{-1/2}\otimes\psi_F$ is conjugate self-dual.

We also assume that Π'_F descends to unitary groups of any sign after tensoring an algebraic Hecke character. Therefore, Π'_F has definable arithmetic automorphic periods.

Let I_F be a map from $\Sigma_{F;K}$ to the set $\{0,1,\cdots,nl\}$. We want to relate $P^{(I_F)}(\Pi'_F)$ to arithmetic automorphic periods of $\Pi_{\mathcal{F}}$ in the following sections.

We take η an algebraic Hecke character of F such that $I(\Pi_F,\eta)=I_F$. We take m as in the last part Theorem 5.3.1. We assume that Conjecture 5.1.1 is true and we have:

$$L(m,\Pi'_F\otimes\eta)\sim_{E(\Pi_F)E(\eta);K}(2\pi i)^{mnld}P^{(I_F)}(\Pi'_F)\prod_{\tau\in\Sigma_{F;K}}p(\check{\eta},\tau)^{I_F(\tau)}p(\check{\eta},\overline{\tau})^{nl-I_F(\tau)} \quad (8.1)$$

with both sides non zero.

8.2 Relations of global periods for automorphic inductions

The case l is odd or n is even: In this case, $\Pi'_F=\Pi_F$ is the automorphic induction of $\Pi_{\mathcal{F}}$.

We know $L(m,\Pi_F\otimes\eta)=L(m,\Pi_{\mathcal{F}}\otimes\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times})$. It is easy to see that m is also critical for $\Pi_{\mathcal{F}}\otimes\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times}$. We can apply Theorem 5.3.1 to $(\Pi_{\mathcal{F}},\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times})$.

We write $I_{\mathcal{F}}:=I(\Pi_{\mathcal{F}},\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times})$ and get:

$$\begin{aligned}L(m,\Pi_F\otimes\eta)=L(m,\Pi_{\mathcal{F}}\otimes\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times})\sim_{E(\Pi_{\mathcal{F}})E(\eta);K}\\ (2\pi i)^{mndl}P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}})\prod_{\sigma\in\Sigma_{\mathcal{F};K}}p(\widecheck{\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times}},\sigma)^{I_{\mathcal{F}}(\sigma)}p(\widecheck{\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times}},\overline{\sigma})^{n-I_{\mathcal{F}}(\sigma)}.\end{aligned} \quad (8.2)$$

We first calculate $I_{\mathcal{F}}=I(\Pi_{\mathcal{F}},\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times})$. We write the infinity type of η at $\tau\in\Sigma_{F;K}$ by $z^{a(\tau)}\overline{z}^{b(\tau)}$.

For $\sigma\in\Sigma_{\mathcal{F};K}$, the infinity type of $\eta\circ N_{\mathbb{A}_{\mathcal{F}}/\mathbb{A}_F}$ at σ is then $z^{a(\Psi_{\mathcal{F}/F}(\sigma))}\overline{z}^{b(\Psi_{\mathcal{F}/F}(\sigma))}$.

We have by definition that

$$I_{\mathcal{F}}(\sigma)=\#\{i\mid 1\leqslant i\leqslant n, a(\Psi_{\mathcal{F}/F}(\sigma))-b(\Psi_{\mathcal{F}/F}(\sigma))+a_i(\sigma)-b_i(\sigma)<0\} \quad (8.3)$$

Recall that the infinity type of Π_F at τ is $(z^{a_i(\sigma)}\overline{z}^{b_i(\sigma)})_{1\leqslant i\leqslant n,\sigma\in\Psi_{\mathcal{F}/F}^{-1}(\tau),1\leqslant i\leqslant n}$. We have:

$$I_F(\tau)=I(\Pi_F,\eta)(\tau)=\#\{(i,\sigma)\mid 1\leqslant i\leqslant n,\sigma\in\Psi_{\mathcal{F}/F}^{-1}(\tau),a(\tau)-b(\tau)+a_i(\sigma)-b_i(\sigma)<0\}. \quad (8.4)$$

We observe that $I_{\mathcal{F}}$ is uniquely determined by I_F. More precisely, it is easy to show that:

Lemma 8.2.1. *The integer $I_{\mathcal{F}}(\sigma)$ is the number of elements in $\{a_i(\sigma) \mid 1 \leqslant i \leqslant n\}$ which is one of the $I_F(\tau)$-th smallest numbers in the set $\{a_i(\sigma') \mid 1 \leqslant i \leqslant n, \sigma' \in \Psi^{-1}_{\mathcal{F}/F}(\tau)\}$ where $\tau = \Psi_{\mathcal{F}/F}(\sigma)$.*

Moreover, it is easy to see that

$$\sum_{\sigma \in \Psi^{-1}_{\mathcal{F}/F}(\tau)} I_{\mathcal{F}}(\sigma) = I_F(\tau). \tag{8.5}$$

By Proposition 4.1.2, we get

$$\begin{aligned}\prod_{\sigma \in \Sigma_{\mathcal{F};K}} p(\widetilde{\eta \circ N_{\mathbb{A}^\times_{\mathcal{F}}/\mathbb{A}^\times_F}}, \sigma)^{I_{\mathcal{F}}(\sigma)} &\sim_{E(\eta);K} \prod_{\sigma \in \Sigma_{\mathcal{F};K}} p(\check{\eta}, \Psi_{\mathcal{F}/F}(\sigma))^{I_{\mathcal{F}}(\sigma)} \\ \sim_{E(\eta);K} \prod_{\tau \in \Sigma_{F;K}} p(\check{\eta}, \tau)^{\sum\limits_{\sigma \in \Psi^{-1}_{\mathcal{F}/F}(\tau)} I_{\mathcal{F}}(\sigma)} &\sim_{E(\eta);K} \prod_{\tau \in \Sigma_{F;K}} p(\check{\eta}, \tau)^{I_F(\tau)}\end{aligned} \tag{8.6}$$

Similarly, we have

$$\prod_{\sigma \in \Sigma_{\mathcal{F};K}} p(\widetilde{\eta \circ N_{\mathbb{A}^\times_{\mathcal{F}}/\mathbb{A}^\times_F}}, \overline{\sigma})^{n - I_{\mathcal{F}}(\sigma)} \sim_{E(\eta);K} \prod_{\tau \in \Sigma_{F;K}} p(\check{\eta}, \tau)^{nl - I_F(\tau)}. \tag{8.7}$$

Comparing the above two equations with equations (8.1) and (8.2), we deduce that:

$$P^{(I_F)}(\Pi_F) \sim_{E(\Pi_{\mathcal{F}});K} P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}}). \tag{8.8}$$

The case l is even and n is odd: In this case Π_F is no longer algebraic and we consider $\Pi'_F = \Pi_F \otimes || \cdot ||^{-1/2}$.

We know $L(m, \Pi'_F \otimes \eta) = L(m - \frac{1}{2}, \Pi_F \otimes \eta) = L(m - \frac{1}{2}, \Pi_{\mathcal{F}} \otimes \eta \circ N_{\mathbb{A}^\times_{\mathcal{F}}/\mathbb{A}^\times_F})$.

As in the previous case, we get:

$$\begin{aligned} L(m - \tfrac{1}{2}, \Pi_F \otimes \eta) &= L(m - \tfrac{1}{2}, \Pi_{\mathcal{F}} \otimes \eta \circ N_{\mathbb{A}^\times_{\mathcal{F}}/\mathbb{A}^\times_F}) \\ \sim_{E(\Pi_{\mathcal{F}})E(\eta);K} (2\pi i)^{(m-\frac{1}{2})ndl} P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}}) &\prod_{\sigma \in \Sigma_{\mathcal{F};K}} p(\widetilde{\eta \circ N_{\mathbb{A}^\times_{\mathcal{F}}/\mathbb{A}^\times_F}}, \sigma)^{I_{\mathcal{F}}(\sigma)} p(\widetilde{\eta \circ N_{\mathbb{A}^\times_{\mathcal{F}}/\mathbb{A}^\times_F}}, \overline{\sigma})^{n - I_{\mathcal{F}}(\sigma)} \\ \sim_{E(\Pi_{\mathcal{F}})E(\eta);K} (2\pi i)^{(m-\frac{1}{2})ndl} P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}}) &\prod_{\tau \in \Sigma_{F;K}} p(\check{\eta}, \tau)^{I_F(\tau)} p(\check{\eta}, \overline{\tau})^{nl - I_F(\tau)} \end{aligned}$$

We conclude that:

$$P^{(I_F)}(\Pi_F \otimes || \cdot ||^{-1/2}) \sim_{E(\Pi_{\mathcal{F}});K} (2\pi i)^{-\frac{nld}{2}} P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}}). \tag{8.9}$$

8.3 Relations of local periods for automorphic inductions

Recall that the arithmetic automorphic periods admit a factorization (c.f. Theorem 7.6.1) $P^{(I)}(\Pi) \sim_{E(\Pi);K} \prod\limits_{\sigma \in \Sigma} P^{(I(\sigma))}(\Pi, \sigma)$ such that

$$P^{(0)}(\Pi, \sigma) \sim_{E(\Pi);K} p(\xi_\Pi^{-1}, \sigma) \sim_{E(\Pi);K} p(\xi_\Pi, \sigma)^{-1}. \tag{8.10}$$

We will discuss the functoriality of local periods in this section.

Let τ be an element of $\Sigma_{F;K}$.

It is easy to see from Lemma 8.2.1 or equation (8.5) that if $I_F(\tau)=0$ then $I_{\mathcal{F}}(\sigma)=0$ for all $\sigma\in\Sigma_{\mathcal{F};K}$ over τ.

Fix any $\tau_0\in\Sigma_{F;K}$ and an integer $0\leqslant s_0\leqslant n$. We define I_F such that $I_F(\tau_0)=s_0$ and $I_F(\tau)=0$ for $\tau\neq\tau_0\in\Sigma_{F;K}$.

The case l is odd or n is even: Recall $P^{(I_F)}(AI(\Pi_{\mathcal{F}}))\sim_{E(\Pi_{\mathcal{F}});K} P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}})$ in this case. We get

$$P^{(s_0)}(\Pi_F,\tau_0)\prod_{\tau\neq\tau_0\in\Sigma_{F;K}}P^{(0)}(\Pi_F,\tau)\sim_{E(\Pi_{\mathcal{F}});K}\prod_{\sigma_0|\tau_0}P^{(I_{\mathcal{F}}(\sigma_0))}(\Pi_{\mathcal{F}},\sigma_0)\prod_{\tau\neq\tau_0\in\Sigma_{F;K}}\prod_{\sigma|\tau}P^{(0)}(\Pi_{\mathcal{F}},\sigma) \tag{8.11}$$

For $\tau\neq\tau_0$, we have $P^{(0)}(\Pi_F,\tau)\sim_{E(\Pi_{\mathcal{F}});K}p(\xi_{\Pi_F},\tau)^{-1}$. Similarly, we have $P^{(0)}(\Pi_{\mathcal{F}},\sigma)\sim_{E(\Pi_{\mathcal{F}});K}$ $p(\xi_{\Pi_{\mathcal{F}}},\sigma)^{-1}$.

Let g be a generator of $Gal(\mathcal{F}/F)$. From the construction in [3] we know Π_F has base change $\Pi_{\mathcal{F}}\times\Pi_{\mathcal{F}}^{g}\times\cdots\times\Pi_{\mathcal{F}}^{g^{l-1}}$. In particular, we know $\xi_{\Pi_F}\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times}=\prod\limits_{0\leqslant i\leqslant l-1}\xi_{\Pi_{\mathcal{F}}}^{g^i}$.

We fix any $\sigma_1\in\Sigma_{\mathcal{F};K}$ over τ. We know

$$\begin{aligned}
p(\xi_{\Pi_F},\tau) &\sim_{E(\Pi_{\mathcal{F}});K} p(\xi_{\Pi_F}\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times},\sigma_1)\\
&\sim_{E(\Pi_{\mathcal{F}});K} \prod_{0\leqslant i\leqslant l-1}p(\xi_{\Pi_{\mathcal{F}}}^{g^i},\sigma_1)\\
&\sim_{E(\Pi_{\mathcal{F}});K} \prod_{0\leqslant i\leqslant l-1}p(\xi_{\Pi_{\mathcal{F}}},\sigma_1^{g^{-i}})\\
&\sim_{E(\Pi_{\mathcal{F}});K} \prod_{\sigma|\tau}p(\xi_{\Pi_{\mathcal{F}}},\sigma)
\end{aligned}$$

Equation (8.11) then gives:

$$P^{(s_0)}(\Pi_F,\tau_0)\sim_{E(\Pi_{\mathcal{F}});K}\prod_{\sigma_0|\tau_0}P^{(I_{\mathcal{F}}(\sigma_0))}(\Pi_{\mathcal{F}},\sigma_0) \tag{8.12}$$

We can read from Lemma 8.2.1 that $I_{\mathcal{F}}(\sigma_0)$ depends only on $I_F(\tau_0)=s_0$.

It is natural to define:

Definition 8.3.1. *Let $0\leqslant s\leqslant nl$ be any integer. Let $\tau\in\Sigma_{F;K}$. For any $\sigma\in\Sigma_{\mathcal{F};K}$ over τ, we define $s(\sigma)$ to be the number of elements in $\{a_i(\sigma)\mid 1\leqslant i\leqslant n\}$ which is one of the s-th smallest numbers in the set $\{a_i(\sigma')\mid 1\leqslant i\leqslant n,\sigma'\in\Psi_{\mathcal{F}/F}^{-1}(\tau)\}$.*

Equation (8.12) can be rewritten as

$$P^{(s)}(\Pi_F,\tau)\sim_{E(\Pi_{\mathcal{F}});K}\prod_{\sigma|\tau}P^{(s(\sigma))}(\Pi_{\mathcal{F}},\sigma). \tag{8.13}$$

The case l is even and n is odd: In this case, we have:

$$\xi_{\Pi'_F} \circ N_{\mathbb{A}_{\mathcal{F}}^{\times}/\mathbb{A}_F^{\times}} = \prod_{0\leqslant i\leqslant l-1} \xi_{\Pi_{\mathcal{F}}}^{g^i} \times ||\cdot||_{\mathbb{A}_F}^{-nl/2}. \tag{8.14}$$

We repeat the above procedure and get:

$$P^{(s)}(AI(\Pi_{\mathcal{F}})\otimes||\cdot||^{-1/2},\tau) \sim_{E(\Pi_{\mathcal{F}});K} (2\pi i)^{-\frac{nl}{2}} \prod_{\sigma|\tau} P^{(s(\sigma))}(\Pi_{\mathcal{F}},\sigma). \tag{8.15}$$

We conclude the functoriality of arithmetic automorphic periods for automorphic induction:

Theorem 8.3.1. *Let $F\supset K$ be a CM field of degree d over K.*

Let $\mathcal{F}/F$ be a cyclic extension of CM fields of degree l and $\Pi_{\mathcal{F}}$ be a cuspidal representation of $GL_n(\mathbb{A}_{\mathcal{F}})$ which has definable arithmetic automorphic periods.

We assume that $\Pi_{\mathcal{F}} \ncong \Pi_{\mathcal{F}}^g$ for all $g\in Gal(\mathcal{F}/F)$ non trivial. We define $AI(\Pi_{\mathcal{F}})$ to be the automorphic induction of $\Pi_{\mathcal{F}}$. It is a cuspidal representation of $GL_{nl}(\mathbb{A}_F)$.

We assume that $AI(\Pi_{\mathcal{F}})$ (resp. $AI(\Pi_{\mathcal{F}})\otimes||\cdot||^{-1/2}$) also has definable arithmetic automorphic periods if l is odd or n is even (resp. if l is even and n is odd) (c.f. Section 8.1).

Let I_F be any map from $\Sigma_{F;K}$ to $\{0,1,\cdots,nl\}$. Let $I_{\mathcal{F}}$ be the map from $\Sigma_{\mathcal{F};K}$ to $\{0,1,\cdots,n\}$ determined by I_F and $\Pi_{\mathcal{F}}$ as in Lemma 8.2.1. Or locally let $0\leqslant s\leqslant nl$ be an integer and $s(\cdot)$ be as in Definition 8.3.1.

If l is odd or n is even, we have:

$$P^{(I_F)}(AI(\Pi_{\mathcal{F}})) \sim_{E(\Pi_{\mathcal{F}});K} P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}})$$
$$\text{or locally } P^{(s)}(AI(\Pi_{\mathcal{F}}),\tau) \sim_{E(\Pi_{\mathcal{F}});K} \prod_{\sigma|\tau} P^{(s(\sigma))}(\Pi_{\mathcal{F}},\sigma).$$

Otherwise we have:

$$P^{(I_F)}(AI(\Pi_{\mathcal{F}})\otimes||\cdot||^{-1/2}) \sim_{E(\Pi_{\mathcal{F}});K} (2\pi i)^{-\frac{nld}{2}} P^{(I_{\mathcal{F}})}(\Pi_{\mathcal{F}})$$
$$\text{or locally } \quad P^{(s)}(AI(\Pi_{\mathcal{F}})\otimes||\cdot||^{-1/2},\tau) \sim_{E(\Pi_{\mathcal{F}});K} (2\pi i)^{-\frac{nl}{2}} \prod_{\sigma|\tau} P^{(s(\sigma))}(\Pi_{\mathcal{F}},\sigma).$$

8.4 Period relations under Galois action

We are going to prove period relations for base change. Before that, we first prove that arithmetic periods are equivariant under Galois actions.

More precisely, let $F\supset K$ be a CM field and Π be a cuspidal representation of $GL_n(\mathbb{A}_F)$ which has definable arithmetic automorphic periods.

We fix any $I:\Sigma_{F;K}\to\{0,1,\cdots,n\}$ and take η, an algebraic Hecke character of F such that $I(\Pi,\eta)=I$. Assuming Conjecture 5.1.1 and Theorem 5.2.1 gives:

$$L(m,\Pi\otimes\eta) \sim_{E(\Pi)E(\eta);K} P^{(I)}(\Pi) \prod_{\sigma\in\Sigma_{F;K}} p(\check{\eta},\sigma)^{I(\sigma)} p(\check{\eta},\overline{\sigma})^{n-I(\sigma)} \tag{8.16}$$

for a critical point m with both sides non zero.

Let $g \in Gal(F/K)$. We observe that $L(s, \Pi \otimes \eta) = L(s, \Pi^g \otimes \eta^g)$.

We then get:

$$\begin{aligned} & L(m, \Pi \otimes \eta) = L(s, \Pi^g \otimes \eta^g) \\ \sim_{E(\Pi)E(\eta);K} \quad & P^{(I(\Pi^g))}(\Pi^g, \eta^g) \prod_{\sigma \in \Sigma_{F;K}} p(\check{\eta^g}, \sigma)^{I(\Pi^g,\eta^g)(\sigma)} p(\check{\eta^g}, \overline{\sigma})^{n-I(\Pi^g,\eta^g)(\sigma)} \end{aligned} \tag{8.17}$$

It is easy to see that $I(\Pi^g, \eta^g)(\sigma) = I(\Pi, \eta)(\sigma^{g^{-1}})$.

Moreover, by Proposition 4.1.2 we have $p(\check{\eta^g}, \sigma) \sim_{E(\eta);K} p(\check{\eta}, \sigma^{g^{-1}})$.

We obtain:

$$\begin{aligned} \prod_{\sigma \in \Sigma_{F;K}} p(\check{\eta^g}, \sigma)^{I(\Pi^g,\eta^g)(\sigma)} & \sim_{E(\eta);K} \prod_{\sigma \in \Sigma_{F;K}} p(\check{\eta}, \sigma^{g^{-1}})^{I(\Pi,\eta)(\sigma^{g^{-1}})} \\ & \sim_{E(\eta);K} \prod_{\sigma \in \Sigma_{F;K}} p(\check{\eta}, \sigma)^{I(\Pi,\eta)(\sigma)}. \end{aligned} \tag{8.18}$$

Similarly, $\prod\limits_{\sigma \in \Sigma_{F;K}} p(\check{\eta^g}, \overline{\sigma})^{n-I(\Pi^g,\eta^g)(\sigma)} \sim_{E(\eta);K} \prod\limits_{\sigma \in \Sigma_{F;K}} p(\check{\eta}, \overline{\sigma})^{n-I(\Pi,\eta)(\sigma)}$.

We write $I^g := I(\Pi^g, \eta^g)$. Then $I^g(\sigma) = I(\sigma^{g^{-1}})$. Compare with equation (8.16) and equation (8.17), we deduce that:

$$P^{(I)}(\Pi) \sim_{E(\Pi);K} P^{(I^g)}(\Pi^g). \tag{8.19}$$

We can moreover get relations of local periods. Let us fix $\sigma_0 \in \Sigma_{F;K}$ and $0 \leqslant s \leqslant n$ an integer.

We set $I(\sigma_0) = s$ and $I(\sigma) = 0$ for $\sigma \neq \sigma_0$. Then $I^g(\sigma_0^g) = s$ and $I^g(\sigma) = 0$ for $\sigma \neq \sigma_0^g$.
By the results in Section 7.6, we have

$$P^{(I)}(\Pi) \sim_{E(\Pi);K} P^{(s)}(\Pi, \sigma_0) \prod_{\sigma \neq \sigma_0} P^{(0)}(\Pi, \sigma) \sim_{E(\Pi);K} P^{(s)}(\Pi, \sigma_0) \prod_{\sigma \neq \sigma_0} p(\xi_\Pi, \sigma)^{-1} \tag{8.20}$$

and similarly:

$$P^{(I^g)}(\Pi^g) \sim_{E(\Pi);K} P^{(s)}(\Pi^g, \sigma_0^g) \prod_{\sigma \neq \sigma_0^g} p(\xi_{\Pi^g}, \sigma)^{-1}. \tag{8.21}$$

Again by Proposition 4.1.2, we have

$$\begin{aligned} \prod_{\sigma \neq \sigma_0^g} p(\xi_{\Pi^g}, \sigma)^{-1} & \sim_{E(\Pi);K} \prod_{\sigma \neq \sigma_0^g} p(\xi_\Pi^g, \sigma)^{-1} \\ & \sim_{E(\Pi);K} \prod_{\sigma \neq \sigma_0^g} p(\xi_\Pi, \sigma^{g^{-1}})^{-1} \\ & \sim_{E(\Pi);K} \prod_{\sigma \neq \sigma_0} p(\xi_\Pi, \sigma)^{-1}. \end{aligned} \tag{8.22}$$

We conclude that:

$$P^{(s)}(\Pi, \sigma_0) \sim_{E(\Pi);K} P^{(s)}(\Pi^g, \sigma_0^g). \tag{8.23}$$

Theorem 8.4.1. *Let $F \supset K$ be a CM field and Π be a cuspidal representation of $GL_n(\mathbb{A}_F)$ which has definable arithmetic automorphic periods. We assume that Conjecture 5.1.1 is true. Let $g \in Gal(F/K)$, $\sigma \in \Sigma_{F;K}$ and $0 \leqslant s \leqslant n$ be an integer. We have:*

$$P^{(I)}(\Pi) \sim_{E(\Pi);K} P^{(I^g)}(\Pi^g) \tag{8.24}$$

$$\textit{or locally} \quad P^{(s)}(\Pi, \sigma) \sim_{E(\Pi);K} P^{(s)}(\Pi^g, \sigma^g). \tag{8.25}$$

8.5 Relations of global periods for base change

Let $\mathcal{F}/F$ a cyclic extension of CM fields of degree l as before. Let π_F be a cuspidal representation of $GL_n(\mathbb{A}_F)$. The strong base change of π_F exists. We denote it by $\pi_{\mathcal{F}}$ or $BC(\pi_F)$.

By the class field theory, we have $(F^\times N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times}\mathbb{A}_{\mathcal{F}}^\times)\backslash\mathbb{A}_F^\times \cong Gal(\mathcal{F}/F)$. Since $Gal(\mathcal{F}/F)$ is cyclic, its dual is also cyclic. We fix any generator of $Hom(Gal(\mathcal{F}/F),\mathbb{C}^\times)$ which gives $\eta_{\mathcal{F}/F}$ a Hecke character of F.

We remark that $\eta_{\mathcal{F}/F}$ is an order l Hecke character. In particular it has trivial infinity type. It is also unitary and thus conjugate self-dual.

We assume that $\pi_F\otimes\eta_{\mathcal{F}/F}^t \not\cong \pi_F$ for all $1\leqslant t\leqslant l-1$. Then $\Pi_{\mathcal{F}}$ is cuspidal (Théorèm 4.2 of [2]). We want to compare the arithmetic automorphic periods of $\Pi_{\mathcal{F}}$ to those of π_F if they are defined.

We assume that π_F has definable arithmetic automorphic periods. In other words, it is 3-regular, cohomological and descends to unitary groups of any sign after tensoring an algebraic Hecke character. We know $\Pi_{\mathcal{F}}$ is also 3-regular and cohomological. We assume that $\Pi_{\mathcal{F}}$ also descends to unitary groups of any sign after tensoring an algebraic Hecke character.

Let I_F be any map from $\Sigma_{F;K}$ to the set $\{0,1,\cdots,n\}$.

We take η an algebraic Hecke character of F with $I(\pi_F,\eta)=I_F$ such that (π_F,η) satisfies conditions in Theorem 5.3.1. Let $\eta' := \eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times}$ be the base change of η to $\mathcal{F}$.

There is a relation between the L-function of $\pi_{\mathcal{F}}$ and that of π_F, namely:

$$L(s,\pi_{\mathcal{F}}\otimes\eta') = \prod_{i=0}^{l-1} L(s,\pi_F\otimes\eta\eta_{\mathcal{F}/F}^i). \tag{8.26}$$

We write $I_{\mathcal{F}} := I(\pi_{\mathcal{F}},\eta')$. For any $\tau\in\Sigma_{F;K}$ and any $\sigma\in\Sigma_{\mathcal{F};K}$ with $\sigma\mid\tau$, it is easy to see that $I_{\mathcal{F}}(\sigma)=I_F(\tau)$. In other words, $I_{\mathcal{F}}$ is the composition of I_F and $\Psi_{\mathcal{F}/F}$.

We assume that Conjecture 5.1.1 is true. We can interpret both sides in terms of arithmetic automorphic periods and CM periods and then deduce period relations.

More precisely, for a certain critical point m we have:

$$L(m,\pi_{\mathcal{F}}\otimes\eta') \sim_{E(\pi_F)E(\eta);K} (2\pi i)^{mnld}P^{(I_{\mathcal{F}})}(\pi_{\mathcal{F}}) \prod_{\tau\in\Sigma_{F;K}}\prod_{\sigma\mid\tau} p(\check{\eta}',\sigma)^{I_{\mathcal{F}}(\sigma)}p(\check{\eta}',\overline{\sigma})^{n-I_{\mathcal{F}}(\sigma)} \tag{8.27}$$

with both sides non zero.

Since $\eta'=\eta\circ N_{\mathbb{A}_{\mathcal{F}}^\times/\mathbb{A}_F^\times}$, we have $p(\check{\eta}',\sigma)=p(\check{\eta},\sigma\mid_F)$. Moreover, $I_{\mathcal{F}}(\sigma)=I_F(\tau)$ if $\sigma\mid\tau$. Therefore,

$$L(m,\pi_{\mathcal{F}}\otimes\eta') \sim_{E(\pi_F)E(\eta);K} (2\pi i)^{mnld}P^{(I_{\mathcal{F}})}(\pi_{\mathcal{F}}) \prod_{\tau\in\Sigma_{F;K}} p(\check{\eta},\tau)^{lI_F(\tau)}p(\check{\eta},\overline{\tau})^{l(n-I_F(\tau))}. \tag{8.28}$$

On the other hand, we apply Theorem 5.3.1 to $(\pi_F, \eta\eta^i_{\mathcal{F}/F})$ and get:

$$\begin{aligned}
& L(m, \pi_F \otimes \eta\eta^i_{\mathcal{F}/F}) && (8.29)\\
\sim_{E(\pi_F)E(\eta);K}\ & (2\pi i)^{mnd} P^{(I_F)}(\pi_F) \prod_{\tau\in\Sigma_{F;K}} p(\widecheck{\eta\eta^i_{\mathcal{F}/F}}, \tau)^{I_F(\tau)} p(\widecheck{\eta\eta^i_{\mathcal{F}/F}}, \overline{\tau})^{n-I_F(\tau)}\\
\sim_{E(\pi_F)E(\eta);K}\ & (2\pi i)^{mnd} P^{(I_F)}(\pi_F) \prod_{\tau\in\Sigma_{F;K}} p(\check{\eta}, \tau)^{I_F(\tau)} p(\check{\eta}, \overline{\tau})^{n-I_F(\tau)}\\
& \times p(\widecheck{\eta_{\mathcal{F}/F}}, \tau)^{iI_F(\tau)} p(\widecheck{\eta^{-1}_{\mathcal{F}/F}}, \tau)^{i(n-I_F(\tau))}\\
\sim_{E(\pi_F)E(\eta);K}\ & (2\pi i)^{mnd} P^{(I_F)}(\pi_F) \prod_{\tau\in\Sigma_{F;K}} p(\check{\eta}, \tau)^{I_F(\tau)} p(\check{\eta}, \overline{\tau})^{n-I_F(\tau)} p(\widecheck{\eta_{\mathcal{F}/F}}, \tau)^{2iI_F(\tau)-in}.
\end{aligned}$$

Comparing equation (8.26), (8.28) and (8.29), we get:

$$\begin{aligned}
P^{(I_{\mathcal{F}})}(\pi_{\mathcal{F}}) &\sim_{E(\pi_F);K} p^{I_F}(\pi_F) \prod_{\tau\in\Sigma_{F;K}} \prod_{i=0}^{l-1} p(\widecheck{\eta_{\mathcal{F}/F}}, \tau)^{2iI_F(\tau)-in}\\
&\sim_{E(\pi_F);K} p^{I_F}(\pi_F)^l \prod_{\tau\in\Sigma_{F;K}} p(\widecheck{\eta_{\mathcal{F}/F}}, \tau)^{l(l-1)I_F(\tau)-l(l-1)n/2}. \qquad (8.30)
\end{aligned}$$

Since $p(\widecheck{\eta_{\mathcal{F}/F}}, \tau)^l \sim_{K;K} p(\widecheck{\eta^l_{\mathcal{F}/F}}, \tau) \sim_{K;K} 1$, we have:

$$P^{(I_{\mathcal{F}})}(\pi_{\mathcal{F}}) \sim_{E(\pi_F);K} p^{I_F}(\pi_F)^l \prod_{\tau\in\Sigma_{F;K}} p(\widecheck{\eta_{\mathcal{F}/F}}, \tau)^{-l(l-1)n/2}. \qquad (8.31)$$

If l is odd, we have $p(\widecheck{\eta_{\mathcal{F}/F}}, \tau)^{-l(l-1)n/2} \sim_{K;K} 1$. Otherwise we assume that $p(\widecheck{\eta_{\mathcal{F}/F}}^{l/2}, \tau) \in E(\pi_F)^\times$ for simplicity.

We conclude that:

$$P^{(I_{\mathcal{F}})}(\pi_{\mathcal{F}}) \sim_{E(\pi_F);K} p^{I_F}(\pi_F)^l. \qquad (8.32)$$

8.6 Relations of local periods for base change

There are relations between local periods of π_F and those of $\pi_{\mathcal{F}}$.

Let $0 \leqslant s_0 \leqslant n$ be an integer. We fix $\tau_0 \in \Sigma_{F;K}$. We take I_F to be the map which sends τ_0 to s_0 and $\tau \neq \tau_0$ to 0. Equation (8.32) then becomes:

$$\prod_{\sigma_0|\tau_0} P^{(s_0)}(\pi_{\mathcal{F}}, \sigma_0) \prod_{\tau\neq\tau_0} \prod_{\sigma|\tau} P^{(0)}(\pi_{\mathcal{F}}, \sigma) \sim_{E(\pi_F);K} p^{(s_0)}(\pi_F, \tau_0)^l \prod_{\tau\neq\tau_0} p^{(0)}(\pi_F, \tau)^l. \qquad (8.33)$$

Recall that for $\sigma \mid \tau$, we have:

$$\begin{aligned}
P^{(0)}(\pi_{\mathcal{F}}, \sigma) &\sim_{E(\pi_F);K} p(\xi_{\pi_{\mathcal{F}}}, \sigma)^{-1}\\
&\sim_{E(\pi_F);K} p(\xi_{\pi_F} \circ N_{\mathbb{A}^\times_{\mathcal{F}}/\mathbb{A}^\times_F}, \sigma)^{-1}\\
&\sim_{E(\pi_F);K} p(\xi_{\pi_F}, \tau)^{-1}\\
&\sim_{E(\pi_F);K} P^{(0)}(\pi_F, \tau). \qquad (8.34)
\end{aligned}$$

We deduce that:

$$\prod_{\sigma|\tau_0} P^{(s)}(\pi_{\mathcal{F}}, \sigma) \sim_{E(\pi_F);K} P^{(s)}(\pi_F, \tau_0)^l. \qquad (8.35)$$

We observe that $\pi_{\mathcal{F}}$ is $Gal(\mathcal{F}/F)$-invariant. The local periods are then $Gal(\mathcal{F}/F)$-invariant.

Indeed, for any $g \in Gal(L/K)$, we have $\pi_{\mathcal{F}}^{g} \cong \pi_{\mathcal{F}}$. Theorem 8.25 implies that

$$P^{(s)}(\pi_{\mathcal{F}}, \sigma) \sim_{E(\pi_F);K} P^{(s)}(\pi_{\mathcal{F}}^{g}, \sigma^{g}) \sim_{E(\pi_F);K} P^{(s)}(\pi_{\mathcal{F}}, \sigma^{g}). \tag{8.36}$$

Recall that $Gal(\mathcal{F}/F)$ acts faithfully and transitively on the set $\{\sigma : \sigma \mid \tau\}$. We fix any $\sigma_0 \mid \tau$ and then:

$$\prod_{\sigma|\tau} P^{(s)}(\pi_{\mathcal{F}}, \sigma) = \prod_{g \in Gal(\mathcal{F}/F)} P^{(s)}(\pi_{\mathcal{F}}, \sigma_0^{g}) \sim_{E(\pi_F);K} P^{(s)}(\pi_{\mathcal{F}}, \sigma_0)^{l}. \tag{8.37}$$

Comparing equation (8.35) and equation (8.37), we conclude that for any $\sigma \in \Sigma_{\mathcal{F};K}$:

$$P^{(s)}(\pi_{\mathcal{F}}, \sigma)^{l} \sim_{E(\pi_F);K} P^{(s)}(\pi_F, \sigma \mid_F)^{l}. \tag{8.38}$$

Consequently, there exists an algebraic number $\lambda^{(s)}(\pi_F.\sigma)$ with $\lambda(\sigma)^l \in E(\pi_F)^{\times}$ such that $P^{(s)}(\pi_{\mathcal{F}}, \sigma) \sim_{E(\pi_F)} \lambda^{(s)}(\pi_F, \sigma) P^{(s)}(\pi_F, \sigma \mid_F)$.

It is expected that $\lambda^{(s)}(\pi_F, \sigma)$ is equivariant under Galois action. But we don't know how to prove it at this moment.

We summarize the above results on period relations for base change as follows.

Theorem 8.6.1. *Let $\mathcal{F}/F$ be a cyclic extension of CM fields of degree l. Let π_F be a cuspidal representation of $GL_n(\mathbb{A}_F)$. We denote by $BC(\pi_F)$ its strong base change to $\mathcal{F}$.*

We assume that $\pi_F \otimes \eta_{\mathcal{F}/F}^{t} \not\cong \pi_F$ for all $1 \leqslant t \leqslant l-1$ and then $BC(\Pi_F)$ is cuspidal (Théorèm 4.2 of [2]). We assume that both π_F and $BC(\pi_F)$ have definable arithmetic automorphic periods.

Let I_F be any map from $\Sigma_{F;K}$ to $\{0, 1, \cdots, n\}$. We write $I_{\mathcal{F}}$ the composition of I_F and $\Psi_{\mathcal{F}/F}$.

We then have:

$$P^{(I_{\mathcal{F}})}(BC(\pi_F)) \sim_{E(\pi_F);K} p^{I_F}(\pi_F)^{l} \tag{8.39}$$

$$\text{or locally} \quad P^{(s)}(BC(\pi_F), \sigma)^{l} \sim_{E(\pi_F);K} P^{(s)}(\pi_F, \sigma \mid_F)^{l}. \tag{8.40}$$

Consequently, we know

$$P^{(s)}(BC(\pi_F), \sigma) \sim_{E(\pi_F)} \lambda^{(s)}(\pi_F, \sigma) P^{(s)}(\pi_F, \sigma \mid_F). \tag{8.41}$$

where $\lambda^{(s)}(\pi_F, \sigma)$ is an algebraic number whose l-th power is in $E(\pi_F)^{\times}$.

Chapter 9

An automorphic version of Deligne's conjecture

9.1 A conjecture

Conjecture 9.1.1. *Let n and n' be two positive integers. Let Π and Π' be cuspidal representations of $GL_n(\mathbb{A}_F)$ and $GL_{n'}(\mathbb{A}_F)$ respectively which have definable arithmetic automorphic periods.*

Let $m \in \mathbb{Z} + \dfrac{n+n'}{2}$ be critical for $\Pi \otimes \Pi'$. We predict that:

$$L(m, \Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{nn'md} \prod_{\sigma\in\Sigma_{F;K}} (\prod_{j=0}^{n} P^{(j)}(\Pi,\sigma)^{sp(j,\Pi;\Pi',\sigma)} \prod_{k=0}^{n'} P^{(k)}(\Pi',\sigma)^{sp(k,\Pi';\Pi,\sigma)}).$$

Example 9.1.1. *(Known cases for the above conjecture:)*

Let $F = K$ be the quadratic imaginary field. Then the above conjecture is already known in the following cases:

1. *$n' = 1$ and m is strictly bigger than the central value. This is the main theorem in [13]. We keep the notation as in Therem 5.3.1. Let $\Pi' = \eta$. It is easy to verify that $sp(0,\Pi;\Pi') = n - s$, $sp(1,\Pi';\Pi) = s$, $sp(i,\Pi;\Pi') = 0$ unless $i = s$ and $sp(s,\Pi;\Pi') = 1$.*

   Recall that $P^{(0)}(\eta) \sim p(\check{\eta},\iota)$ and $P^{(1)}(\eta) \sim p(\check{\eta},1)$. The formula in the above conjecture is the same with the formula in Theorem 5.3.1.

2. *$n' = n-1$, Π, Π' conjugate self-dual in good position and $m > \frac{1}{2}$ or $m = \frac{1}{2}$ along with a non vanishing condition.*

   In this case, we have $-a_n > b_1 > -a_{n-1} > b_2 > \cdots > b_{n-1} > -a_1$. Equivalently we have $sp(k,\Pi';\Pi) = 1$ for all $0 \leqslant k \leqslant n-1$; $sp(j,\Pi;\Pi') = 1$ for $1 \leqslant j \leqslant n-1$ and $= 0$ for $j = 0$ or n. Recall that $P^{(0)}(\Pi') \sim_{E(\Pi);K} P^{(n-1)}(\Pi')^{-1}$. Above conjecture is equivalent to that:

   $$L(m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{n(n-1)m} \prod_{j=1}^{n-1} P^{(j)}(\Pi) \prod_{k=2}^{n-2} P^{(k)}(\Pi')$$

   This is Theorem 6.11 of [8] and Theorem 5.1 of [22].

The shall prove the following special cases of the above conjecture in the next chapters:

Theorem 9.1.1. *We assume that* Π *and* Π' *are* 6*-regular if* $F \neq K$ *to guarantee the factorization of the arithmetic automorphic periods. We know Conjecture 9.1.1 is true for the following cases:*

1. $n > n'$, *the pair* (Π, Π') *is in good position (see Definition 1.2.2),* m *is strictly bigger than the central value, or* m *equals to the central value along with a non vanishing condition and moreover*
   (i) Π, Π' *conjugate self-dual if* $n \not\equiv n' (mod\ 2)$,
   (i) Π *conjugate self-dual,* $\Pi' \otimes \psi_F^{-1}$ *conjugate self-dual if* $n \equiv n' (mod\ 2)$.
2. *Any* n, n' *and any position for* Π, Π' , *the pair* (Π, Π') *is very regular (11.1) and moreover:*
   (i) $m = 1$, Π_1, Π_2 *conjugate self-dual if* $n \equiv n' (mod\ 2)$;
   (i) $m = \frac{1}{2}$, Π_1, $\Pi_2 \otimes \psi_F^{-1}$ *conjugate self-dual if* $n \not\equiv n' (mod\ 2)$.

9.2 Compatibility with Deligne's conjecture over quadratic imaginary fields

One see easily that Conjecture 9.1.1 is formally compatible with Conjecture 6.5.1, Deligne's conjecture for automorphic pairs. For this, it is enough to compare the arithmetic automorphic period $P^{(j)}(\Pi)$ with the motivic period $Q^{(j)}(\Pi)$ where Π is an conjugate self-dual representation.

When F is not K, this is difficult since we don't have geometric meanings for our local periods $P^{(s)}(\Pi, \sigma)$. But for the case when $F = K$, this is already discussed in Section 4 of [8]. We now give a detailed explanation here.

First, let Π be conjugate self-dual. In the construction of the arithmetic automorphic period, we have chosen ξ, an algebraic Hecke character of $\mathbb{A}_K$, such that $\Pi^\vee \otimes \xi$ descends to a similitude unitary group. It is easy to verify that $\xi_\Pi = \dfrac{\xi}{\xi^c}$ (c.f. Theorem $VI.2.1$ or $VI.2.9$ of [19]). The arithmetic automorphic period is defined to be the Petterson inner product of a rational class in the bottom stage of the Hodge filtration of a cohomology space related to $\Lambda^j M(\Pi^c) \otimes M(\xi)$. In other words, $P^{(j)}(\Pi)$ is related to $Q_{n-j+1}(\Pi^c)Q_{n-j+2}(\Pi^c)\cdots Q_n(\Pi^c) \times Q_1(\xi)$.

By Lemma 6.2.1 we have $Q_{n-i+1}(\Pi^c) \sim_{E(M)} Q_i(\Pi)^{-1}$ for all $1 \leqslant i \leqslant n$.

By equation (6.13), we see

$$Q_1(\xi) \sim_{E(\xi);K} p(\frac{\xi^c}{\xi}, 1) \sim_{E(\xi);K} p(\xi_\Pi^c, 1) \sim_{E(\xi);K} p(\widecheck{\xi_\Pi^c}, 1) \sim_{E(\xi_\Pi);K} \delta^{Del}(\xi_\Pi).$$

We deduce that:

$$Q_{n-j+1}(\Pi^c)\cdots Q_n(\Pi^c) \times Q_1(\xi^c) \sim_{E(\Pi)E(\xi);K} Q_1(\Pi)^{-1}Q_2(\Pi)^{-1}\cdots Q_j(\Pi)^{-1}\delta^{Del}(\xi_\Pi). \tag{9.1}$$

Recall equation (6.20), the right hand side of the above formula is just $Q^{(j)}(\Pi)$ as expected.

Remark 9.2.1. *We can also deduce the above result without passing to the motivic period of* Π^c*. In fact, we can also consider* $P^{(j)}(\Pi)$ *as Petterson inner product of a rational class in the bottom degree of a cohomology space related to* $\Lambda^{n-j}M(\Pi)\otimes M(\xi^c)$*. It should be related to* $Q_{j+1}(\Pi)Q_{j+2}(\Pi)\cdots Q_n(\Pi)Q_1(\xi^c)$*.*

Lemma 6.2.3 implies that

$$\delta^{Del}(\xi_\Pi^c)\sim_{E(M);K}(\prod_{1\leqslant i\leqslant n}Q_i^{-1})\delta^{Del}(\xi_\Pi).$$

Therefore,

$$\begin{aligned} Q^{(j)}(\Pi) &= Q_1(\Pi)^{-1}Q_2(\Pi)^{-1}\cdots Q_j(\Pi)^{-1}\delta^{Del}(\xi_\Pi)\\ &= Q_{j+1}(\Pi)Q_{j+2}(\Pi)\cdots Q_n(\Pi)\delta^{Del}(\xi_\Pi^c). \end{aligned}$$

We can deduce the comparison by the fact that

$$Q_1(\xi^c)\sim_{E(\xi);K}p(\frac{\check{\xi}}{\xi^c},1)\sim_{E(\xi);K}p(\widecheck{\xi_\Pi},1)\sim_{E(\xi);K}\delta^{Del}(\xi_\Pi^c).$$

For the general cases, we write $\Pi=\Pi'\otimes\eta$ with Π' conjugate self-dual. For the automorphic part, we see from Definition-Lemma (5.3.2) that

$$P^{(j)}(\Pi)\sim_{E(\Pi);K}P^{(j)}(\Pi')p(\check{\eta},1)^j p(\check{\eta},\iota)^{n-j}.$$

For the motivic part, we have $Q_i(\Pi)=Q_i(\Pi')Q_1(\eta)$ and $\Delta(\Pi)=\Delta(\Pi')\delta^{Del}(\eta)^n$. Therefore $Q^{(j)}(\Pi)=Q^{(j)}(\Pi')Q_1(\eta)^{-j}\delta^{Del}(\eta)^n$.

By (6.13) again, we see at first that $Q_1(\eta)\sim_{E(\eta);K}\dfrac{p(\check{\eta^c},1)}{p(\check{\eta},1)}$ and $\delta^{Del}(\eta)\sim_{E(\eta);K}p(\check{\eta^c},1)$. We obtain finally:

$$Q^{(j)}(\Pi)\sim_{E(\Pi);K}Q^{(j)}(\Pi')p(\check{\eta},1)^j p(\check{\eta},\iota)^{n-j}.$$

We have already related $P^{(j)}(\Pi')$ to $Q^{(j)}(\Pi')$. The relation for the general cases then comes.

Remark 9.2.2. *We believe that the above comparison also works over general CM fields. However, the local periods* $P^{(s)}(\Pi,\sigma)$ *are not defined geometrically. It is expected that their geometric meaning can be obtained by comparing special values of L-functions.*

9.3 Simplify archimedean factors

We observe that in Conjecture 9.1.1 the right hand side only concerns arithmetic automorphic periods and a power of $2\pi i$. Sometimes we will get a formula of $L(m,\Pi\otimes\Pi')$ which also involves archimedean factors as in Theorem 6.10 of [8]. We need to show that the contribution of these archimedean factors is equivalent to a power of $2\pi i$:

Proposition 9.3.1. *Let Π and Π' be as in Conjecture 9.1.1. We assume that either the critical value m is strictly bigger then the central value, either it is equal to the central value along with a nonvanishing condition on a certain L-function that we shall see in the proof.*

If there exists an archimedean factor $a(m,\Pi_\infty,\Pi'_\infty)$ depending only on m, Π_∞ and Π'_∞ such that

$$L(m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K} \tag{9.2}$$

$$a(m,\Pi_\infty,\Pi'_\infty)\prod_{\sigma\in\Sigma_{F;K}}\left(\prod_{l=0}^{n}P^{(l)}(\Pi,\sigma)^{sp(l,\Pi;\Pi',\sigma)}\prod_{k=0}^{n'}P^{(k)}(\Pi',\sigma)^{sp(k,\Pi';\Pi,\sigma)}\right) \quad ,$$

then we have $a(m,\Pi_\infty,\Pi'_\infty)\sim_{E(\Pi);K}(2\pi i)^{nn'md}$. In particular, Conjecture 9.1.1 then follows.

Sometimes it is possible to calculate the archimedean factors directly.

A simpler way is to take Π and Π' as representations induced from Hecke characters. Then we may write the left hand side of equation (9.2) in terms of a power of $2\pi i$ and products of CM periods. For the right hand side, note that we have already related the arithmetic automorphic periods of a representation induced from Hecke characters and the CM periods by Theorem 8.3.1.

We shall deduce that the archimedean factor $a(m,\Pi_\infty,\Pi'_\infty)$ is equivalent to a power of $2\pi i$ if Π and Π' are induced from Hecke characters. But such representations can have any infinity type. The only non trivial point is that if Π is conjugate self-dual then we may take a conjugate self-dual Hecke character such that its automorphic induction has the same infinity type as Π. We prove this in the lemma below. Hence the above proposition is true for any Π and Π'. This is the idea of the proof of Theorem 5.1 in [22].

Lemma 9.3.1. *Let $L\supset F$ be a cyclic extension of CM fields of degree n. We assume that n is odd. If Π is a conjugate self-dual representation of $GL_n(\mathbb{A}_F)$ then there exists χ a conjugate self-dual algebraic Hecke character of L such that $\Pi_\infty\cong AI(\chi)_\infty$.*

Proof We denote by L^+ the maximal totally real subfield of L.

We may take an algebraic Hecke character χ' of L such that $\Pi_\infty\cong AI(\chi')_\infty$ (c.f. Lemma 4.1.1 and paragraphs before Lemma 4.1.3 in [6]).

Since Π is conjugate self-dual, we see that χ'_∞ is conjugate self-dual. In particular, $\chi'|_{L^+}$ is trivial at infinity places. By Lemma 4.1.4 of [6], we may find ϕ an algebraic Hecke character of L with trivial infinity type such that $\phi\phi^c=\chi'\chi'^c$. Put $\chi=\chi'\phi^{-1}$. It is then a conjugate self-dual Hecke character with $\Pi_\infty\cong AI(\chi)_\infty$.

□

We now give the details of the proof for Proposition 9.3.1.

Proof For simplicity, we assume that both n and n' are odd. For general case, we have to twist $AI(\chi)$ or $AI(\chi')$ by $||\cdot||^{-1/2}\psi_F$ as before. The following proof goes through as well.

We take $L\supset F$ (resp. $L'\supset F$) a CM field which is a cyclic extension of F of degree n (resp. n'). We assume that L and L' are linearly independent over F. Let $\mathcal{L}:=LL'$. It is then a CM field of degree nn' over F.

We may take χ (resp. χ') an algebraic Hecke character of L such that $\Pi_\infty = AI(\chi)_\infty$ (resp. $\Pi'_\infty = AI(\chi')_\infty$) where $AI(\chi)$ (resp. $AI(\chi')$) is the automorphic induction of χ (resp. χ') from L (resp. L') to F. Moreover, we may assume that $AI(\chi)$ and $AI(\chi')$ are cuspidal and have definable arithmetic automorphic periods.

For $\sigma \in \Sigma_{F;K}$, we write $\sigma_1, \cdots, \sigma_n$ for the elements in $\Sigma_{L;K}$ above σ and $\sigma'_1, \cdots, \sigma'_{n'}$ for the elements in $\Sigma_{L';K}$ above σ. Let $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n'$. We write $\sigma_{i,j}$ for the only element in $\Sigma_{\mathcal{L};K}$ such that $\sigma_{i,j} \mid_L = \sigma_i$ and $\sigma_{i,j} \mid_{L'} = \sigma'_j$.

We write $z^{a_i(\sigma)}\overline{z}^{-\omega(\Pi)-a_i(\sigma)}$ for the infinity type of $AI(\chi)$ at σ_i and $z^{b_j(\sigma)}\overline{z}^{-\omega(\Pi')-b_j(\sigma)}$ for the infinity type of $AI(\chi)$ at σ'_j .

Then Π has infinity type $(z^{a_i(\sigma)}\overline{z}^{-\omega(\Pi)-a_i(\sigma)})_{1\leqslant i\leqslant n}$ and Π' has infinity type $(z^{b_j(\sigma)}\overline{z}^{-\omega(\Pi')-b_j(\sigma)})_{1\leqslant j\leqslant n'}$ at σ.

By equation (9.2), we have

$$L(m, AI(\chi) \times AI(\chi')) \sim_{E(\chi)E(\chi');K} \tag{9.3}$$
$$a(m, \Pi_\infty, \Pi'_\infty) \prod_{\sigma\in\Sigma_{F;K}} \left(\prod_{l=0}^{n} P^{(l)}(AI(\chi), \sigma)^{sp(l,\Pi;\Pi',\sigma)} \prod_{k=0}^{n'} P^{(k)}(AI(\chi'), \sigma)^{sp(k,\Pi';\Pi,\sigma)}\right)$$

On one hand, we have $L(m, AI(\chi) \times AI(\chi')) = L(m, (\chi \circ N_{\mathbb{A}_\mathcal{L}/\mathbb{A}_L})(\chi' \circ N_{\mathbb{A}_\mathcal{L}/\mathcal{A}_{L'}}))$.

We observe that the infinity type of $(\chi \circ N_{\mathbb{A}_\mathcal{L}/\mathbb{A}_L})(\chi' \circ N_{\mathbb{A}_\mathcal{L}/\mathcal{A}_{L'}})$ at $\sigma_{i,j} \in \Sigma_{\mathcal{L};K}$ is $z^{a_i(\sigma)+b_j(\sigma)}\overline{z}^{-\omega(\Pi)-\omega(\Pi')-a_i(\sigma)-b_j(\sigma)}$.

We denote $J_\sigma := \{(i,j) \mid a_i(\sigma) + b_j(\sigma) < -\dfrac{\omega(\Pi) + \omega(\Pi')}{2}\}$.

We write $\chi_\mathcal{L} = (\chi \circ N_{\mathbb{A}_\mathcal{L}/\mathbb{A}_L})(\chi' \circ N_{\mathbb{A}_\mathcal{L}/\mathcal{A}_{L'}})$. By Blasius's result,

$$L(m, AI(\chi) \times AI(\chi')) \sim_{E(\chi)E(\chi');K} (2\pi i)^{mnn'd} \prod_{\sigma\in\Sigma_{F;K}} \prod_{(i,j)\in J_\sigma} p(\widecheck{\chi_\mathcal{L}}, \sigma_{i,j}) \prod_{(i,j)\notin J_\sigma} p(\widecheck{\chi_\mathcal{L}}, \overline{\sigma_{i,j}}). \tag{9.4}$$

We need to assume that we may choose χ and χ' such that $L(m, AI(\chi) \times AI(\chi')) \neq 0$. When m is strictly bigger then the central value, this is always true. When m is equal to the central value, we assume this as a hypothesis.

Recall that the CM periods are multiplicative and functorial for base change. Hence

$$\begin{aligned} p(\widecheck{\chi_\mathcal{L}}, \sigma_{i,j}) &\sim_{E(\chi)E(\chi');K} p(\widecheck{\chi \circ N_{\mathbb{A}_\mathcal{L}/\mathbb{A}_L}}, \sigma_{i,j}) p(\widecheck{\chi \circ N_{\mathbb{A}_\mathcal{L}/\mathcal{A}_{\mathcal{L}'}}}, \sigma_{i,j}) \\ &\sim_{E(\chi)E(\chi');K} p(\check{\chi}, \sigma_i) p(\check{\chi}', \sigma_j). \end{aligned} \tag{9.5}$$

We have deduced that

$$L(m, AI(\chi) \times AI(\chi')) \sim_{E(\chi)E(\chi');K} \tag{9.6}$$
$$(2\pi i)^{mnn'd} \prod_{\sigma\in\Sigma_{F;K}} \prod_{1\leqslant i\leqslant n} (p(\check{\chi}, \sigma_i)^{s_i(\sigma)} p(\check{\chi}, \overline{\sigma_i})^{n'-s_i(\sigma)}) \prod_{1\leqslant j\leqslant n'} (p(\check{\chi}', \sigma_j)^{t_j(\sigma)} p(\check{\chi}', \overline{\sigma_j})^{n-t_j(\sigma)})$$

where $s_i(\sigma) = \#\{1 \leqslant j \leqslant n' \mid (i,j) \in J_\sigma\}$ and $t_j(\sigma) = \#\{1 \leqslant i \leqslant n \mid (i,j) \in J_\sigma\}$.

On the other hand, for $0 \leqslant l \leqslant n$, we have

$$P^{(l)}(AI(\chi), \sigma) \sim_{E(\Pi_\mathcal{F});K} \prod_{1\leqslant i\leqslant n} P^{(u_i(l))}(\chi, \sigma_i) \tag{9.7}$$

where $u_i(l) = 1$ if $a_i(\sigma)$ is in the l-th smallest numbers in the set $\{a_i(\sigma) \mid 1 \leqslant i \leqslant n\}$ and $u_i(l) = 0$ otherwise by Definition 8.3.1.

We order $a_i(\sigma)$ and $b_j(\sigma)$ in decreasing order. We have $u_i(l) = 1$ if and only if $i \geqslant n-l+1$. We get

$$P^{(l)}(AI(\chi),\sigma) \sim_{E(\Pi_{\mathcal{F}});K} \prod_{1\leqslant i\leqslant n-l} P^{(0)}(\chi,\sigma_i) \prod_{n-l+1\leqslant i\leqslant n} P^{(1)}(\chi,\sigma_i) \tag{9.8}$$

Recall that $P^{(0)}(\chi,\sigma_i) \sim_{E(\chi);K} p(\breve{\chi},\overline{\sigma_i})$ and $P^{(1)}(\chi,\sigma_i) \sim_{E(\chi);K} p(\breve{\chi},\sigma_i)$ by Remark 7.6.2. We obtain that:

$$P^{(l)}(AI(\chi),\sigma) \sim_{E(\Pi_{\mathcal{F}});K} \prod_{1\leqslant i\leqslant n-l} p(\breve{\chi},\overline{\sigma_i}) \prod_{n-l+1\leqslant i\leqslant n} p(\breve{\chi},\sigma_i) \tag{9.9}$$

Comparing equations (9.3), (9.6) and (9.9), we observe that it remains to show:

$$\sum_{l=n-i+1}^{n} sp(l,\Pi;\Pi',\sigma) = s_i(\sigma) \tag{9.10}$$

$$\text{and} \sum_{l=0}^{n-i} sp(l,\Pi;\Pi',\sigma) = n' - s_i(\sigma). \tag{9.11}$$

Since $\sum\limits_{l=0}^{n} sp(l,\Pi;\Pi',\sigma) = n'$ by Lemma 1.2.1. We see the above two equations are equivalent. We now prove the first one.

Recall by definition that $sp(l,\Pi;\Pi',\sigma)$ is the length of the l-th part of the sequence $b_1(\sigma) > b_2(\sigma) > \cdots > b_{n'}(\sigma)$ split by the numbers $-\dfrac{\omega(\Pi)+\omega(\Pi')}{2} - a_n > -\dfrac{\omega(\Pi)+\omega(\Pi')}{2} - a_{n-1} > \cdots > -\dfrac{\omega(\Pi)+\omega(\Pi')}{2} - a_1$.

Therefore, $\sum\limits_{l=n-i+1}^{n} sp(l,\Pi;\Pi',\sigma) = \#\{j \mid b_j < -\dfrac{\omega(\Pi)+\omega(\Pi')}{2} - a_i\}$. This is exactly $s_i(\sigma)$ as expected.

□

Remark 9.3.1. *Roughly speaking, the above proposition tells us that if we have a formula like equation (9.2) then the archimedean factor must be equivalent to a power of $2\pi i$. If one can show that the CM periods $p(\chi,\tau)$, $\tau \in \Sigma_{L;K}$ is algebraically independent, we can moreover prove that the power of arithmetic automorphic periods must be the split indices.*

More precisely, the following statement is true:

If there exists an archimedean factor $a(m,\Pi_\infty,\Pi'_\infty)$ depending only on m, Π_∞ and Π'_∞ and integers $b(j,\Pi_\infty;\Pi'_\infty,\sigma)$, $c(k,\Pi'_\infty;\Pi_\infty,\sigma)$ for $0 \leqslant j \leqslant n$, $0 \leqslant k \leqslant n'$ and $\sigma \in \Sigma_{F;K}$ depending on Π_∞, Π'_∞ such that

$$L(m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K} \tag{9.12}$$

$$a(m,\Pi_\infty,\Pi'_\infty) \prod_{\sigma\in\Sigma_{F;K}} \left(\prod_{j=0}^{n} P^{(j)}(\Pi,\sigma)^{b(j,\Pi_\infty;\Pi'_\infty,\sigma)} \prod_{k=0}^{n'} P^{(k)}(\Pi',\sigma)^{c(k,\Pi'_\infty;\Pi_\infty,\sigma)}\right) ,$$

then we have $b(j, \Pi_\infty; \Pi'_\infty, \sigma) = sp(j, \Pi; \Pi', \sigma)$ *and* $c(k, \Pi'_\infty; \Pi_\infty, \sigma) = sp(k, \Pi'; \Pi, \sigma)$ *provided that the local CM periods are algebraically independent. In particular, Conjecture 9.1.1 then follows.*

The proof of the above statement is the same as proof for Proposition 9.3.1. We remark that the indices for the arithmetic automorphic periods are determined by equation (9.10).

This statement is very powerful. Sometimes it is easy to show that there exists a formula in the form of equation (9.12) but difficult to calculate the exact indices. In fact, one can explain in several lines that there exists such formulas for the cases in Theorem 9.1.1. But we have devoted the next two whole chapters to calculate the precise indices.

Unfortunately we don't know how to prove the algebraically independency of the CM periods. So the calculation in the next two chapters are inevitable at the moment.

9.4 More discussions on the archimedean factors

As discussed in the previous section, one can leave the archimedean factors to the end of the proof and show that they contribute as a power of $2\pi i$.

In our situation, we happen to be able to calculate the product of the archimedean factors directly. Let us first recall some archimedean factors.

Let Π_∞ (resp. $\Pi^\#_\infty$) be an algebraic regular generic representation of $GL_n(F \otimes_{\mathbb{Q}} \mathbb{R}$ (resp.$GL_{n-1}(F \otimes_{\mathbb{Q}} \mathbb{R})$). We have defined:

1. $\Omega(\Pi_\infty)$ which appears in the calculation of Whittaker period (c.f. Section 3.4).
2. $p(m, \Pi_\infty, \Pi'_\infty)$ which appears in the calculation of critical values for automorphic representations of $GL_n \times GL_{n-1}$ (c.f. Proposition 3.6.1).
3. $Z(\Pi_\infty)$ defined in equation (7.8) by

$$Z(\Pi_\infty) := (2\pi i)^{d(m+\frac{1}{2})n(n-1)-\frac{d(n-1)(n-2)}{2}}\Omega(\Pi^\#_\infty)^{-1}p(m, \Pi_\infty, \Pi^\#_\infty)^{-1}.$$

Lemma 9.4.1. *The archimedean factors satisfy:*

$$Z(\Pi_\infty)\Omega(\Pi'_\infty)p(m, \Pi_\infty, \Pi'_\infty) \sim_{E(\Pi_\infty)E(\Pi'_\infty);K} (2\pi i)^{dn(n-1)(m+\frac{1}{2})-\frac{d(n-1)(n-2)}{2}}$$

for all $m \geqslant 0$.

We now take Π and $\Pi^\#$ to be cuspidal conjugate self-dual representations of $GL_n(\mathbb{A}_F)$ and $GL_{n-1}(\mathbb{A}_F)$ respectively such that $(\Pi, \Pi^\#)$ is in good position. We assume that they all have definable arithmetic automorphic periods.

By Proposition 3.6.1, we have

$$L(\frac{1}{2} + m, \Pi \times \Pi^\#) \sim_{E(\Pi)E(\Pi^\#);K} p(m, \Pi_\infty, \Pi^\#_\infty)p(\Pi)p(\Pi^\#) \tag{9.13}$$

for some critical $\frac{1}{2} + m \geqslant \frac{1}{2}$.

Recall from equation (7.5.1) that:

$$p(\Pi) \sim_{E(\Pi)E(\Pi^\#;K)} Z(\Pi_\infty) \prod_{\sigma\in\Sigma_{F;K}} \prod_{1\leqslant i\leqslant n-1} P^{(i)}(\Pi). \tag{9.14}$$

We have a similar formula for $\Pi^\#$ since $\Pi^\#$ is also cuspidal. We then deduce that:

$$\begin{aligned} & L(\frac{1}{2}+m, \Pi\times\Pi^\#) && (9.15)\\ \sim_{E(\Pi)E(\Pi^\#);K} \quad & p(m,\Pi_\infty,\Pi^\#_\infty)Z(\Pi_\infty)Z(\Pi^\#_\infty) \prod_{\sigma\in\Sigma_{F;K}} (\prod_{1\leqslant i\leqslant n-1} P^{(i)}(\Pi,\sigma) \prod_{1\leqslant j\leqslant n-2} P^{(j)}(\Pi^\#,\sigma)) \\ \sim_{E(\Pi)E(\Pi^\#);K} \quad & p(m,\Pi_\infty,\Pi^\#_\infty)Z(\Pi_\infty)Z(\Pi^\#_\infty) \prod_{\sigma\in\Sigma_{F;K}} (\prod_{1\leqslant i\leqslant n-1} P^{(i)}(\Pi,\sigma) \prod_{0\leqslant j\leqslant n-1} P^{(j)}(\Pi^\#,\sigma)). \end{aligned}$$

Here we have used the fact that $P^{(0)}(\Pi^\#,\sigma)P^{(n-1)}(\Pi^\#,\sigma) \sim_{E(\Pi^\#);K} 1$ by Theorem 7.6.1.

Proposition 9.3.1 then gives the following result on the archimedean factors:

Proposition 9.4.1. *The archimedean factors satisfy:*

$$p(m,\Pi_\infty,\Pi^\#_\infty)Z(\Pi_\infty)Z(\Pi^\#_\infty) \sim_{E(\Pi_\infty)E(\Pi'_\infty);K} (2\pi i)^{d(m+\frac{1}{2})n(n-1)}$$

provided $m \geqslant 1$ or $m = 0$ along with a non vanishing condition for the central value of a certain L-function.

This is Theorem 5.1 of [22] when $F = K$ is a quadratic imaginary field.

Comparing Lemma 9.4.1 and Proposition 9.4.1, we change the notation $\Pi^\#$ to Π and deduce that:

Corollary 9.4.1. *We write $r = n-1$. For Π_∞ an algebraic and generic representation of $GL_{n'}(F\otimes_{\mathbb{Q}}\mathbb{R})$, we have*

$$Z(\Pi_\infty)\Omega(\Pi_\infty)^{-1} \sim_{E(\Pi_\infty);K} (2\pi i)^{\frac{d(n-1)(n-2)}{2}} = (2\pi i)^{\frac{d(r-1)r}{2}}$$

provided $m \geqslant 1$ or $m = 0$ along with a non vanishing condition for the central value of a certain L-function.

In the following, we assume that $m \geqslant 1$, or $m = 0$ along with a non vanishing condition for Π.

9.5 From quadratic imaginary fields to general CM fields

We shall prove Theorem 9.1.1 in the following two chapters over quadratic imaginary field. The proof only requires little change for general CM fields. This is because the automorphic arithmetic periods and the CM periods are all factorable. We now explain the details for the first case of Theorem 9.1.1 in the current section.

Let Π and Π' be cuspidal conjugate self-dual representations of $GL_n(\mathbb{A}_F)$ and $GL_{n'}(\mathbb{A}_F)$ which has definable arithmetic automorphic periods. We assume that (Π,Π') is in good position and both Π and Π' are regular enough. For simplicity, we also assume that n is even and n' is odd.

Let $l = n - n' - 1$. We take some conjugate self-dual Hecke characters $\chi_1, \cdots, \chi_l$ such that if we write $\Pi^{\#}$ for the Langlands sum of Π' and $\chi_1, \cdots, \chi_n$ then $(\Pi, \Pi^{\#})$ is in good position. By the assumption on the parity of n and n' we know that $\Pi^{\#}$ is algebraic.

We may assume that for each $\sigma \in \Sigma_{F;K}$, the first index of the infinity type of χ_i is in decreasing order. Therefore, $I(\Pi, \chi_i)(\sigma)$ is determined by the infinity type of Π and Π' at σ.

As explained in the introduction, the proof requires three main ingredients.

Ingredient A: Theorem 5.3.1 says that for certain Hecke characters η and critical points m we have:

$$L(m, \Pi \otimes \eta) \sim_{E(\Pi)E(\eta);K} (2\pi i)^{mnd} P^{(I(\Pi,\eta))}(\Pi) \prod_{\sigma \in \Sigma} p(\check{\eta}, \sigma)^{I(\Pi,\eta)(\sigma)} p(\check{\eta}, \overline{\sigma})^{n - I(\Pi,\eta)(\sigma)}. \tag{9.16}$$

where $I(\Pi, \eta)(\sigma)$ depends only the infinity type of Π and η at σ.

By Theorem 7.6.1, we may rewrite the above equation as:

$$L(m, \Pi \otimes \eta) \sim_{E(\Pi)E(\eta);K} (2\pi i)^{mnd} \prod_{\sigma \in \Sigma} [P^{(I(\Pi,\eta)(\sigma))}(\Pi, \sigma) p(\check{\eta}, \sigma)^{I(\Pi,\eta)(\sigma)} p(\check{\eta}, \overline{\sigma})^{n - I(\Pi,\eta)(\sigma)}]. \tag{9.17}$$

Ingredient B: Proposition 3.6.1 says that if $m \geqslant 0$ and $m + \frac{1}{2}$ is critical for $\Pi \times \Pi^{\#}$ then

$$L(\frac{1}{2} + m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#});K} p(m, \Pi_\infty, \Pi^{\#}_\infty) p(\Pi) p(\Pi^{\#}) \tag{9.18}$$

where $p(\Pi)$ and $p(\Pi^{\#})$ are the Whittaker periods.

Ingredient C: Corollary 3.5.1 implies that

$$p(\Pi^{\#}) \sim_{E(\Pi);K} p(\Pi') \frac{\Omega(\Pi_\infty)}{\Omega(\Pi'_\infty)} \prod_{1 \leqslant i \leqslant l} L(1, \Pi' \otimes \chi_i^\vee) \prod_{1 \leqslant i < j \leqslant l} L(1, \chi_i \times \chi_j^\vee) \tag{9.19}$$

Moreover, by Corollary 7.5.1, we have

$$p(\Pi) \sim_{E(\Pi);K} Z(\Pi_\infty) \prod_{\sigma \in \Sigma} \prod_{1 \leqslant i \leqslant n-1} P^{(i)}(\Pi, \sigma) \tag{9.20}$$

$$p(\Pi') \sim_{E(\Pi');K} Z(\Pi'_\infty) \prod_{\sigma \in \Sigma} \prod_{1 \leqslant j \leqslant n'-1} P^{(j)}(\Pi', \sigma) \tag{9.21}$$

On one hand, note that $L(\frac{1}{2} + m, \Pi \times \Pi^{\#}) = L(\frac{1}{2} + m, \Pi \times \Pi') \prod\limits_{1 \leqslant i \leqslant l} L(\frac{1}{2} + m, \Pi \times \chi_i)$. We replace η by χ_i in equation (9.17) and will get:

$$L(\frac{1}{2} + m, \Pi \times \Pi^{\#}) = L(\frac{1}{2} + m, \Pi \times \Pi') \times \tag{9.22}$$
$$\prod_{\sigma \in \Sigma_{F;K}} [\prod_{1 \leqslant i \leqslant l} P^{(I(\Pi,\chi_i)(\sigma))}(\Pi, \sigma) p(\check{\chi_i}, \sigma)^{I(\Pi,\chi_i)(\sigma)} p(\check{\chi_i}, \overline{\sigma})^{n - I(\Pi,\chi_i)(\sigma)}]$$

On the other hand, apply equations (9.19), (9.20) and (9.21) to the right hand side of equation (9.18), we get:

$$L(\frac{1}{2}+m,\Pi\times\Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#});K} a_0(m,\Pi_\infty,\Pi'_\infty)\times \tag{9.23}$$
$$\prod_{\sigma\in\Sigma}[\prod_{1\leqslant i\leqslant n-1} P^{(i)}(\Pi,\sigma)\prod_{1\leqslant j\leqslant n'-1} P^{(j)}(\Pi',\sigma)]\prod_{1\leqslant i\leqslant l} L(1,\Pi'\otimes\chi_i^\vee)\prod_{1\leqslant i<j\leqslant l} L(1,\chi_i\times\chi_j^\vee)$$

where $a_0(m,\Pi_\infty,\Pi'_\infty)$ is a non zero complex number depending only on m and the infinity type.

We apply equation (9.17) to (Π',χ_i) and Blasius's result to $L(1,\chi_i\times\chi_j^\vee)$, we get:

$$L(\frac{1}{2}+m,\Pi\times\Pi^{\#}) \tag{9.24}$$
$$\sim_{E(\Pi)E(\Pi^{\#});K} a(m,\Pi_\infty,\Pi'_\infty)\prod_{\sigma\in\Sigma}[\prod_{1\leqslant i\leqslant n-1} P^{(i)}(\Pi,\sigma)\prod_{1\leqslant j\leqslant n'-1} P^{(j)}(\Pi',\sigma)\times$$
$$\prod_{1\leqslant i\leqslant l} P^{(I(\Pi',\chi_i)(\sigma))}(\Pi',\sigma)p(\chi_i,\sigma)^{I_1(\Pi,\Pi')(\sigma)}p(\chi_i,\overline{\sigma})^{I_2(\Pi,\Pi')(\sigma)}] \tag{9.25}$$

where $a(m,\Pi_\infty,\Pi'_\infty)$ is an archimedean factor as before, $I_1(\Pi,\Pi')(\sigma)$ and $I_2(\Pi,\Pi')(\sigma)$ are two integers which depend only on the infinity type of Π and Π' at σ.

The first thing we need to show is that $I_1(\Pi,\chi_i)(\sigma)=I(\Pi,\chi_i)(\sigma)$ and $I_2(\Pi,\chi_i)(\sigma)=n-I(\Pi,\chi_i)(\sigma)$. Since we have ordered the first index of the infinity type of χ_i at σ in decreasing order, we know that both sides only concern the infinity type of Π and Π' at the fixed place σ. So the proof is the same with the quadratic imaginary case.

We then deduce a formula in the following form:

$$L(m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K} \tag{9.26}$$
$$a(m,\Pi_\infty,\Pi'_\infty)\prod_{\sigma\in\Sigma_{F;K}}(\prod_{j=0}^{n} P^{(j)}(\Pi,\sigma)^{b(j,\Pi_\infty;\Pi'_\infty,\sigma)}\prod_{k=0}^{n'} P^{(k)}(\Pi',\sigma)^{c(k,\Pi'_\infty;\Pi_\infty,\sigma)})$$

where $b(j,\Pi_\infty;\Pi'_\infty,\sigma)$ and $c(k,\Pi_\infty;\Pi'_\infty,\sigma)$ are integers which depend only on j, k and the infinity type of Π_∞ and Π'_∞ at σ.

If we know the CM periods are algebraically independent then we can finish the proof by Remark 9.3.1. Unfortunately this is hard to prove and hence we need to calculate $b(j,\Pi_\infty;\Pi'_\infty,\sigma)$ and $c(k,\Pi_\infty;\Pi'_\infty,\sigma)$ explicitly. Again, since they only concern infinity type of Π_∞ and Π'_∞ at σ, we may repeat our calculation for the quadratic imaginary field case for the fixed place σ. We shall see that the indices are just the split indices.

Finally we may show that the archimedean factor $a(m,\Pi_\infty,\Pi'_\infty)\sim_{E(\Pi)E(\Pi');K}(2\pi i)^{mnn'd}$ by Proposition 9.3.1 and complete the proof.

Chapter 10

Special values of L-functions for automorphic pairs over quadratic imaginary fields

10.1 Settings, the simplest case

In the current and the following chapters, let Π and Π' be conjugate self-dual cuspidal representations of $GL_n(\mathbb{A}_K)$ and $GL_{n'}(\mathbb{A}_K)$ respectively which have definable arithmetic automorphic periods. We will interpret the critical values for $L(s,\Pi\otimes\Pi')$ in terms of arithmetic automorphic periods when (Π,Π') is in good position (see Definition 1.2.2).

We write $(z^{a_i}\overline{z}^{-a_i})_{1\leqslant i\leqslant n}$ for the infinity type of Π and $(z^{b_j}\overline{z}^{-b_j})_{1\leqslant j\leqslant n'}$ for the infinity type of Π. We may order a_i and b_j such that $a_1>a_2>\cdots>a_n$ and $b_1>b_2>\cdots>b_{n'}$.

We assume n is even and n' is odd at first. Then the numbers $a_i, 1\leqslant i\leqslant n$ are half integers and the numbers $b_j, 1\leqslant j\leqslant n'$ are integers.

We assume the pair (Π,Π') is **in good position**, namely, each b_i are included in one of the intervals $]-a_{j+1},-a_j[$, $1\leqslant j\leqslant n-1$ and each such interval contains at most one b_i.

Let $w(1)>w(2)>\cdots>w(n)$ be the integers such that $a_{n-w(i)}>-b_{n'+1-i}>a_{n+1-w(i)}$ for all $1\leqslant i\leqslant n'$. More precisely, we have:

$$\begin{aligned}
\mathbf{a}_1>\cdots>\mathbf{a}_{n-w(1)}>-b_{n'} &\;>\\
\mathbf{a}_{n+1-w(1)}>\cdots>\mathbf{a}_{n-w(2)}>-b_{n'-1} &\;>\\
\cdots\\
>\cdots>\mathbf{a}_{n-w(n'+1-i)}>-b_i &\;>\\
\cdots\\
\cdots>\mathbf{a}_{n-w(n')}>-b_1 &\;>\\
\mathbf{a}_{n+1-w(n')}>\cdots>\mathbf{a}_n.
\end{aligned} \tag{10.1}$$

It is easy to see:

$$\begin{aligned}
sp(0,\Pi';\Pi)=n-w(1), sp(n',\Pi';\Pi)=w(n')\\
sp(j,\Pi';\Pi)=w(j)-w(j+1) \text{ for all } 1\leqslant j\leqslant n'-1.
\end{aligned} \tag{10.2}$$

Hence we have

$$w(j) = \sum_{k=j}^{n'} sp(k,\Pi';\Pi) \text{ for all } 1 \leqslant j \leqslant n'. \tag{10.3}$$

We put $l = n - n' - 1$. Let $\chi_1, \chi_2, \cdots, \chi_l$ be conjugate self-dual algebraic Hecke characters of $\mathbb{A}_K$ of infinity type $z^{k_1}\overline{z}^{-k_1}, z^{k_2}\overline{z}^{-k_2}, \cdots z^{k_l}\overline{z}^{-k_l}$ respectively. We assume that $k_1 > k_2 > \cdots > k_l$ lie in different intervals $]-a_{j+1}, -a_j[$ which do not contain any of b_i.

More precisely, we have

$$\begin{aligned} k_1 > k_2 > \cdots > k_{w(n')-1} > \quad & b_1 \quad > \\ > k_{w(n')} > k_{w(n')+1} > \cdots > k_{w(n'-1)-2} > \quad & b_2 \quad > \\ & \cdots \\ k_{w(n'+2-i)-i+2} > k_{w(n'+2-i)-i+3} > \cdots > k_{w(n'+1-i)-i} > \quad & b_i \quad > \\ & \cdots \\ k_{w(2)-n'+2} > k_{w(2)-n'+3} > \cdots k_{w(1)-n'} > \quad & b_{n'} \quad > \\ k_{w(1)-n'+1} > k_{w(1)-n'+2} > \cdots > k_l & \end{aligned} \tag{10.4}$$

and the above $l + n' = n - 1$ numbers lie in different gaps between the n numbers $-a_n > -a_{n-1} > \cdots > -a_1$. Note that in this case, the $n-1$ numbers above are integers and the $(a_i)_{1\leqslant i\leqslant n}$ are half integers.

Let $\Pi^{\#}$ be the Langlands sum of Π' and $\chi_1, \chi_2, \cdots, \chi_l$. It is a generic cohomological conjugate self-dual automorphic representation of $GL_{n-1}(\mathbb{A}_K)$.

Let $m \geqslant 0$ be an integer. By Proposition 3.6.1, we know that if $m + \frac{1}{2}$ is critical for $\Pi \times \Pi^{\#}$, then

$$L(\frac{1}{2} + m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#});K} p(\Pi)p(\Pi^{\#})p(m, \Pi_\infty, \Pi^{\#}_\infty) \tag{10.5}$$

We shall simplify both sides of the above formula. We first calculate the left hand side.

We know $L(\frac{1}{2} + m, \Pi \times \Pi^{\#}) = L(\frac{1}{2} + m, \Pi \times \Pi') \prod\limits_{j=1}^{l} L(\frac{1}{2} + m, \Pi \otimes \chi_j)$.

For each j with $1 \leqslant j \leqslant l$, we apply Theorem 5.2.1 to $\Pi \otimes \chi_j$ and get:

$$L(\frac{1}{2} + m, \Pi \otimes \chi_j) \sim_{E(\Pi)E(\chi_j);K} (2\pi i)^{(m+\frac{1}{2})n} P^{(s_j)}(\Pi) p(\widetilde{\chi_j}, 1)^{s_j} p(\widetilde{\chi_j}, \iota)^{n-s_j}$$

$$\sim_{E(\Pi)E(\chi_j);K} (2\pi i)^{(m+\frac{1}{2})n} P^{(s_j)}(\Pi) p(\widetilde{\chi_j}, 1)^{2s_j - n}$$

where

$$s_j = \#\{1 \leqslant i \leqslant n \mid k_j < -a_i\} = j + \#\{1 \leqslant i \leqslant n' \mid b_i > k_j\}. \tag{10.6}$$

By equation (10.4), we see that

$$\begin{aligned} s_1 = 1, s_2 = 2, \cdots, s_{w(n')-1} = w(n') - 1, \\ s_{w(n')} = w(n') + 1, s_{w(n')+1} = w(n') + 2, \cdots, s_{w(n'-1)-2} = w(n'-1) - 1, \\ \cdots \\ s_{w(1)-n'+1} = w(1) + 1, s_{w(1)-n'+2} = w(1) + 2, \cdots, s_l = l + n' = n - 1. \end{aligned}$$

Shortly, $s_1 < s_2 < \cdots < s_l$ are the numbers in $\{1,2,\cdots,n-1\}\backslash\{w(n'),w(n'-1),\cdots,w(1)\}$.

We then deduce that:

$$L(\frac{1}{2}+m,\Pi\times\Pi^{\#}) \sim_{E(\Pi)E(\Pi')E;K} L(\frac{1}{2}+m,\Pi\times\Pi')(2\pi i)^{(m+\frac{1}{2})nl}\times$$

$$\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{k=1}^{n'}P^{(w(k))}(\Pi)^{-1}\prod_{j=1}^{l}p(\widecheck{\chi_j},1)^{2s_j-n}$$

where E is the compositum of $E(\chi_j)$, $1\leqslant j\leqslant l$.

10.2 Calculate the Whittaker period, the simplest case

By Corollary 3.5.1, we know that

$$p(\Pi^{\#}) \sim_{E(\Pi);K} \Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}\prod_{1\leqslant j\leqslant l}L(1,\Pi'\otimes\chi_j^c)\prod_{1\leqslant i<j\leqslant l}L(1,\chi_i\otimes\chi_j^c).$$

Recall that $\chi_j^{\vee}=\chi_j^c$ since χ_j is conjugate self-dual.

Calculate $\prod\limits_{1\leqslant j\leqslant l}L(1,\Pi'\otimes\chi_j^c)$:

For $1\leqslant j\leqslant l$, applying Theorem 5.2.1 to $\Pi'\times\chi_j^c$, we get

$$L(1,\Pi'\otimes\chi_j^c) \sim_{E(\Pi')E(\chi_j);K} (2\pi i)^{n'}P^{(t_j)}(\Pi')p(\widecheck{\chi_j^c},1)^{t_j}p(\widecheck{\chi_j^c},\iota)^{n'-t_j}$$

$$\sim_{E(\Pi')E(\chi_j);K} (2\pi i)^{n'}P^{(t_j)}(\Pi')p(\widecheck{\chi_j},1)^{n'-2t_j}$$

where $t_j=\#\{1\leqslant i\leqslant n' \mid b_i-k_j<0\}$. The last step is due to the fact that $\chi_j^c=\chi_j^{-1}$.

It is easy to verify that 1 is critical for $\Pi'\times\chi_j^c$ by considering the Hodge type and the original definition by Deligne. Recall that Π' is of infinity type $(z^{b_i}\overline{z}^{-b_i})_{1\leqslant i\leqslant n'}$ and χ_j^c is of infinity type $z^{-k_j}\overline{z}^{k_j}$.

Compare with (10.6), we see that $t_j=n'-\#\{1\leqslant i\leqslant n' \mid b_i>k_j\}=n'+j-s_j$. Then $n'-2t_j=2s_j-n'-2j$.

Therefore, we have deduced that:

$$\prod_{1\leqslant j\leqslant l}L(1,\Pi'\otimes\chi_j^c) \sim_{E(\Pi')E;K} (2\pi i)^{rl}\prod_{j=1}^{l}P^{(t_j)}(\Pi')\prod_{j=1}^{l}p(\widecheck{\chi_j},1)^{2s_j-n'-2j}$$

Calculate $\prod\limits_{1\leqslant i<j\leqslant l}L(1,\chi_i\otimes\chi_j^c)$:

For $1\leqslant i<j\leqslant l$, since $k_i>k_j$, we have

$$L(1,\chi_i\otimes\chi_j^c) \sim_{E(\chi_j);K} (2\pi i)p(\widecheck{\chi_i\chi_j^c},\iota) \sim_{E(\chi_j);K} (2\pi i)p(\widecheck{\chi}_i,1)^{-1}p(\widecheck{\chi_j},1)$$

by Blasius's result.

Therefore, we know that

$$\prod_{1\leqslant i<j\leqslant l} L(1,\chi_i\otimes\chi_j^c) \sim_{E;K} (2\pi i)^{\frac{l(l-1)}{2}} \prod_{j=1}^{l} p(\widetilde{\chi_j},1)^{2j-l-1}. \tag{10.7}$$

Since $(2s_j - n' - 2j) + (2j - l - 1) = 2s_j - n' - l - 1 = 2s_j - n$, we get finally

$$p(\Pi^{\#}) \sim_{E(\Pi)E;K} \Omega(\Pi^{\#}_\infty)p(\Pi')\Omega(\Pi'_\infty)^{-1}(2\pi i)^{rl+\frac{l(l-1)}{2}} \prod_{j=1}^{l} P^{(t_j)}(\Pi') \prod_{j=1}^{l} p(\widetilde{\chi_j},1)^{2s_j-n}.$$

10.3 Calculate the arithmetic automorphic periods and conclude, the simplest case

Since Π and Π' are cuspidal, we may apply Corollary 7.5.1 and get:

$$p(\Pi) \sim_{E(\Pi);K} Z(\Pi_\infty)\prod_{i=1}^{n-1} P^{(i)}(\Pi) \tag{10.8}$$

$$\text{and}\quad p(\Pi') \sim_{E(\Pi');K} Z(\Pi'_\infty)\prod_{k=1}^{n'-1} P^{(k)}(\Pi'). \tag{10.9}$$

Therefore, the right hand side of equation (10.5)

$$\begin{aligned} & p(\Pi)p(\Pi^{\#})p(m,\Pi_\infty,\Pi^{\#}_\infty) \\ \sim_{E(\Pi)E(\Pi')E;K}\ & Z(\Pi_\infty)\Omega(\Pi^{\#}_\infty)Z(\Pi'_\infty)\Omega(\Pi'_\infty)^{-1}p(m,\Pi_\infty,\Pi^{\#}_\infty)(2\pi i)^{rl+\frac{l(l-1)}{2}} \times \\ & \prod_{j=1}^{l} p(\widetilde{\chi_j},1)^{2s_j-n}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{k=1}^{n'-1}P^{(k)}(\Pi')\prod_{j=1}^{l}P^{(t_j)}(\Pi'). \end{aligned}$$

Archimedean factors: Recall that by lemma 9.4.1, we have

$$Z(\Pi_\infty)\Omega(\Pi^{\#}_\infty)p(m,\Pi_\infty,\Pi^{\#}_\infty) \sim_{E(\Pi)E(\Pi')E;K} (2\pi i)^{n(n-1)(m+\frac{1}{2})-\frac{(n-1)(n-2)}{2}}.$$

By corollary 9.4.1, we know

$$Z(\Pi'_\infty)\Omega(\Pi'_\infty)^{-1} \sim_{E(\Pi');K} (2\pi i)^{\frac{n'(n'-1)}{2}}.$$

Therefore $Z(\Pi_\infty)\Omega(\Pi^{\#}_\infty)Z(\Pi'_\infty)\Omega(\Pi'_\infty)^{-1}p(m,\Pi_\infty,\Pi^{\#}_\infty)(2\pi i)^{n'l+\frac{l(l-1)}{2}}$

$$\sim_{E(\Pi)E(\Pi')E;K} (2\pi i)^{n(n-1)(m+\frac{1}{2})-\frac{(n-1)(n-2)}{2}+\frac{n'(n'-1)}{2}+n'l+\frac{l(l-1)}{2}}.$$

Note that $n-1 = l+n'$ and hence $\binom{n-1}{2} = \binom{l}{2} + ln' + \binom{n'}{2}$, we obtain that

$$Z(\Pi_\infty)\Omega(\Pi^{\#}_\infty)Z(\Pi'_\infty)\Omega(\Pi'_\infty)^{-1}p(m,\Pi_\infty,\Pi^{\#}_\infty)(2\pi i)^{n'l+\frac{l(l-1)}{2}}$$

$$\sim_{E(\Pi)E(\Pi');K} (2\pi i)^{n(n-1)(m+\frac{1}{2})}.$$

Arithmetic automorphic periods:

At last, we have to determine the value of $t_j = \#\{1 \leqslant i \leqslant n' \mid b_i - k_j < 0\}$ for $1 \leqslant j \leqslant l$.

For fixed $1 \leqslant k \leqslant n'-1$, from the equation (10.4), we see that the number of $1 \leqslant j \leqslant l$ such that $t_j = k$ is $w(k) - w(k+1) - 1$, the number of $1 \leqslant j \leqslant l$ such that $t_j = r$ is $w(n') - 1$, and the number of $1 \leqslant j \leqslant l$ such that $t_j = 0$ is $n - 1 - w(1)$.

For example, we see $t_1 = t_2 = \cdots = t_{w(n'-1)-1} = n'$, $t_{w(n')} = \cdots = t_{w(n'-1)-2} = n' - 1$, $\cdots$, $t_{w(n'+2-i)-i+2} = \cdots = t_{w(n'+1-i)-1} = n' - i + 1$, $\cdots$, and $t_{w(1)-n'+1} = \cdots = t_l = 0$.

We then deduce that

$$\begin{aligned}\prod_{k=1}^{n'-1} P^{(k)}(\Pi') \prod_{j=1}^{l} P^{(t_j)}(\Pi') &= \prod_{k=1}^{n'-1} P^{(k)}(\Pi')^{w(k)-w(k+1)} P^{(0)}(\Pi')^{n-1-w(1)} P^{(n')}(\Pi')^{w(n')-1} \\ &\sim_{E(\Pi')} \prod_{k=1}^{n'-1} P^{(k)}(\Pi')^{w(k)-w(k+1)} P^{(0)}(\Pi')^{n-w(1)} P^{(n')}(\Pi')^{w(n')} \\ &= \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}\end{aligned}$$

by the fact that $P^0(\Pi') \times P^{(n')}(\Pi') \sim_{E(\Pi')} 1$ and equation (10.2).

Finally, we get $p(\Pi)p(\Pi^{\#})p(m, \Pi_\infty, \Pi_\infty^{\#})$

$$\sim_{E(\Pi)E(\Pi')E;K} (2\pi i)^{n(n-1)(m+\frac{1}{2})} \prod_{j=1}^{l} p(\widetilde{\chi_j}, 1)^{2s_j - n} \prod_{i=1}^{n-1} P^{(i)}(\Pi) \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

Final conclusion, simplest case:

When $L(\frac{1}{2} + m, \Pi \times \Pi^{\#}) \neq 0$, we have that

$$L(\frac{1}{2} + m, \Pi \times \Pi')(2\pi i)^{(m+\frac{1}{2})nl} \prod_{i=1}^{n-1} P^{(i)}(\Pi) \prod_{k=1}^{n'} P^{(w(k))}(\Pi)^{-1} \prod_{j=1}^{l} p(\widetilde{\chi_j}, 1)^{2s_j - n}$$

$$\sim_{E(\Pi)E(\Pi')E;K} (2\pi i)^{n(n-1)(m+\frac{1}{2})} \prod_{j=1}^{l} p(\widetilde{\chi_j}, 1)^{2s_j - n} \prod_{i=1}^{n-1} P^{(i)}(\Pi) \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

We deduce that

$$L(\frac{1}{2} + m, \Pi \times \Pi') \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{(m+\frac{1}{2})nn'} \prod_{k=1}^{n'} P^{(w(k))}(\Pi) \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

We can read from (10.1) that for $0 \leqslant i \leqslant n$, $sp(i, \Pi; \Pi') = 0$ unless $i \in \{w(k) \mid 1 \leqslant k \leqslant n'\}$. Moreover, if $i \in \{w(k) \mid 1 \leqslant k \leqslant n'\}$ then $sp(i, \Pi; \Pi') = 1$. We can then write the above formula in a symmetric way:

$$L(\frac{1}{2} + m, \Pi \times \Pi') \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{(m+\frac{1}{2})nn'} \prod_{i=0}^{n} P^{(i)}(\Pi)^{sp(i,\Pi;\Pi')} \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

Remark 10.3.1. *If $L(\frac{1}{2}+m,\Pi\times\Pi')=0$, then the above formula is automatically true. Otherwise the condition $L(\frac{1}{2}+m,\Pi\times\Pi^{\#})\neq 0$ is equivalent to that $L(\frac{1}{2}+m,\Pi\times\chi_j)\neq 0$ for all $1\leqslant j\leqslant l$. For $m\geqslant 1$, we can always choose k_j and χ_j such that the above is true, see Section 3 of [14]. For $m=0$, we don't know how to prove it at the moment. We will assume this is true henceforth.*

10.4 Settings, the general cases

Let $n>r$ be arbitrary integers. We still want to apply the previous strategy to get special values of L-function for $\Pi\times\Pi'$. But if we take $\Pi^{\#}$ to be Langlands sum of Π' and some algebraic Hecke characters, it may be no longer algebraic. For example, if $n-1\not\equiv n'(\text{mod } 2)$, we know the Langlands parameters of Π' are in $\mathbb{Z}+\frac{n'-1}{2}$. But the Langlands parameters of an algebraic representation of GL_{n-1} should be in $\mathbb{Z}+\frac{n-1}{2}=\mathbb{Z}+\frac{n'}{2}$. In order to fix this, we will tensor the character $||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi$, a Hecke character of infinity type $(\frac{1}{2},-\frac{1}{2})$, when necessary.

When $n-1\equiv r(\text{mod } 2)$, we write $T_1=0$ and we will expand Π' to an algebraic representation of GL_{n-1} as previously. When $n-1\not\equiv r(\text{mod } 2)$, we write $T_1=\frac{1}{2}$ and we will expand $\Pi'\times||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi$ to an algebraic representation of GL_{n-1}. In both cases, we assume the pair (Π,Π') is **in good position**, namely,

$$\begin{aligned}&\text{each } b_i+T_1 \text{ are included in one of the intervals }]-a_{j+1},-a_j[,\ 1\leqslant j\leqslant n-1\\ &\text{and each such interval contains at most one } b_i.\end{aligned}\tag{10.10}$$

Let $w(1)>w(2)>\cdots>w(n)$ be the integers such that

$$-a_{n+1-w(i)}>b_{n'+1-i}+T_1>-a_{n-w(i)}\tag{10.11}$$

for all $1\leqslant i\leqslant n'$.

Let $\chi_1,\chi_2,\cdots,\chi_l$ be conjugate self-dual algebraic Hecke characters of $\mathbb{A}_K$ of infinity type $z^{k_1}\overline{z}^{-k_1},z^{k_2}\overline{z}^{-k_2},\cdots z^{k_l}\overline{z}^{-k_l}$ respectively. These characters will help us expand Π' or $\Pi'\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi$ to an algebraic representation of GL_{n-1}. Similarly, we will tensor them by $||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi$ if $n\not\equiv 0(\text{mod } 2)$ to settle the parity issue. We write $T_2=\frac{1}{2}$ in this case and 0 otherwise.

We assume that $k_1+T_2>k_2+T_2>\cdots>k_l+T_2$ and lie in different intervals $]-a_{j+1},-a_j[$ which doesn't contain any of b_i+T_1.

More precisely, we have

$$\begin{aligned} k_1+T_2 > k_2+T_2 > \cdots > k_{w(n')-1}+T_2 > \quad & b_1+T_1 \quad > \\ > k_{w(n')}+T_2 > k_{w(n')+1}+T_2 > \cdots > k_{w(n'-1)-2}+T_2 > \quad & b_2+T_1 \quad > \\ & \cdots \\ k_{w(n'+2-i)-i+2}+T_2 > k_{w(n'+2-i)-i+3}+T_2 > \cdots > k_{w(n'+1-i)-i}+T_2 > \quad & b_i+T_1 \quad > \\ & \cdots \\ k_{w(2)-n'+2}+T_2 > k_{w(2)-n'+3}+T_2 > \cdots k_{w(1)-n'}+T_2 > \quad & b_{n'}+T_1 \quad > \\ k_{w(1)-n'+1}+T_2 > k_{w(1)-n'+2}+T_2 > \cdots > k_l+T_2 \quad & \end{aligned} \tag{10.12}$$

and the above $l+k=n-1$ numbers lie in the gaps between the n numbers $-a_n > -a_{n-1} > \cdots > -a_1$. Note the above $n-1$ numbers are in $\mathbb{Z}+\dfrac{n}{2}$ when $a_i \in \mathbb{Z}+\dfrac{n-1}{2}$ for all $1 \leqslant i \leqslant n$.

There are four cases:

(A) n is even and n' is odd, then $T_1=0$ and $T_2=0$. We set $\Pi^{\#}=\Pi' \boxplus \chi_1 \boxplus \chi_2 \boxplus \cdots \boxplus \chi_l$ as in previous sections.

(B) n is even and n' is even, then $T_1=\frac{1}{2}$ and $T_2=0$. We set $\Pi^{\#}=(\Pi' \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi) \boxplus \chi_1 \boxplus \chi_2 \boxplus \cdots \boxplus \chi_l$.

(C) n is odd and n' is even, then $T_1=0$ and $T_2=\frac{1}{2}$. We set $\Pi^{\#}=\Pi' \boxplus (\chi_1 \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi) \boxplus (\chi_2 \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi) \boxplus \cdots \boxplus (\chi_l \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)$.

(D) n is odd and n' is odd, then $T_1=\frac{1}{2}$ and $T_2=\frac{1}{2}$. We set $\Pi^{\#}=(\Pi' \boxplus \chi_1 \boxplus \chi_2 \boxplus \cdots \boxplus \chi_l) \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi$.

In all cases, $\Pi^{\#}$ is a generic cohomological conjugate self-dual automorphic representation of $GL_{n-1}(\mathbb{A}_K)$ and Proposition 3.6.1 gives us that if $m+\frac{1}{2}$ is critical for $\Pi \times \Pi^{\#}$, then

$$L(\frac{1}{2}+m, \Pi \times \Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#});K} p(\Pi)p(\Pi^{\#})p(m, \Pi_\infty, \Pi_\infty^{\#}). \tag{10.13}$$

Again, we shall simplify both sides of this equation.

10.5 Simplify the left hand side, general cases

For the left hand side of equation (10.13), we know by construction that:

(A) $L(\frac{1}{2}+m, \Pi \times \Pi^{\#}) = L(\dfrac{1}{2}+m, \Pi \times \Pi') \prod_{j=1}^{l} L(\frac{1}{2}+m, \Pi \otimes \chi_j)$

(B)
$$\begin{aligned} L(\tfrac{1}{2}+m, \Pi \times \Pi^{\#}) &= L(\frac{1}{2}+m, \Pi \times (\Pi' \otimes ||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)) \prod_{j=1}^{l} L(\tfrac{1}{2}+m, \Pi \otimes \chi_j) \\ &= L(m, \Pi \times (\Pi' \otimes \psi)) \prod_{j=1}^{l} L(\tfrac{1}{2}+m, \Pi \otimes \chi_j) \end{aligned}$$

(C)
$$\begin{aligned} L(\tfrac{1}{2}+m,\Pi\times\Pi^{\#}) &= L(\frac{1}{2}+m,\Pi\times\Pi')\prod_{j=1}^{l}L(\tfrac{1}{2}+m,\Pi\otimes(\chi_j\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)) \\ &= L(\frac{1}{2}+m,\Pi\times\Pi')\prod_{j=1}^{l}L(m,\Pi\otimes(\chi_j\otimes\psi)) \end{aligned}$$

(D) $L(\frac{1}{2}+m,\Pi\times\Pi^{\#}) = L(\frac{1}{2},\Pi\times(\Pi'\otimes\psi))\prod_{j=1}^{l}L(m,\Pi\otimes(\chi_j\otimes\psi))$

We set $s_j = \#\{1\leqslant i\leqslant n \mid k_j+T_2 < -a_i\} = j+\#\{1\leqslant i\leqslant n' \mid b_i+T_1 > k_j+T_2\}$ and $t_j = \#\{1\leqslant i\leqslant n' \mid (b_i+T_1)-(k_j+T_2)<0\}$ as before. Recall that $s_j+t_j = n'+j$ for all $1\leqslant j\leqslant l$.

If n is even (case (A) and (B)), we have for all $1\leqslant j\leqslant l$:

$$L(\frac{1}{2}+m,\Pi\otimes\chi_j)\sim_{E(\Pi)E(\chi_j);K}(2\pi i)^{(m+\frac{1}{2})n}P^{(s_j)}(\Pi)p(\widecheck{\chi_j},1)^{2s_j-n}.$$

If n is odd (case (C) and (D)), we have for all $1\leqslant j\leqslant l$:

$$L(m,\Pi\otimes(\chi_j)\otimes\psi)\sim_{E(\Pi)E(\chi_j);K}(2\pi i)^{mn}P^{(s_j)}(\Pi)p(\widecheck{\chi_j},1)^{2s_j-n}p(\check{\psi},1)^{s_j}p(\check{\psi},\iota)^{n-s_j}.$$

Therefore for cases (A) and (B), we have

$$\prod_{j=1}^{l}L(\frac{1}{2}+m,\Pi\otimes\chi_j)\sim_{E(\Pi)E;K}(2\pi i)^{(m+\frac{1}{2})nl}\prod_{k=1}^{n-1}P^{(k)}(\Pi)\prod_{k=1}^{n'}P^{(w(k))}(\Pi)^{-1}\prod_{j=1}^{l}p(\widecheck{\chi_j},1)^{2s_j-n}.$$

For cases (C) and (D), we put $s := \sum_{j=1}^{l}s_j$ and then we have:

$$\prod_{j=1}^{l}L(m,\Pi\otimes(\chi_j\otimes\psi))\sim_{E(\Pi)EE(\psi);K}$$
$$(2\pi i)^{mnl}\times\prod_{k=1}^{n-1}P^{(k)}(\Pi)\prod_{k=1}^{n'}P^{(w(k))}(\Pi)^{-1}\prod_{j=1}^{l}p(\widecheck{\chi_j},1)^{2s_j-n}p(\check{\psi},1)^{s}p(\check{\psi},\iota)^{nl-s}$$

10.6 Simplify the right hand side, general cases

Calculate $p(\Pi^{\#})$: Apply Corollary 3.5.1, we get

(A) $p(\Pi^{\#})\sim_{E(\Pi^{\#});K}\Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}\prod_{1\leqslant j\leqslant l}L(1,\Pi'\otimes\chi_j^c)\prod_{1\leqslant i<j\leqslant l}L(1,\chi_i\otimes\chi_j^c)$

(B) $p(\Pi^{\#})\sim_{E(\Pi^{\#});K}\Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}\prod_{1\leqslant j\leqslant l}L(1,(\Pi'\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)\otimes\chi_j^c)\prod_{1\leqslant i<j\leqslant l}L(1,\chi_i\otimes\chi_j^c)$

(C) $p(\Pi^{\#})\sim_{E(\Pi^{\#});K}\Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}\prod_{1\leqslant j\leqslant l}L(1,\Pi'\otimes(\chi_j\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)^c)\prod_{1\leqslant i<j\leqslant l}L(1,\chi_i\otimes\chi_j^c)$

(D) $p(\Pi^{\#})\sim_{E(\Pi^{\#});K}\Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}\prod_{1\leqslant j\leqslant l}L(1,\Pi'\otimes\chi_j^c)\prod_{1\leqslant i<j\leqslant l}L(1,\chi_i\otimes\chi_j^c)$

Here we have used that:

Lemma 10.6.1. *If η is a conjugate self-dual Hecke character then:*

$$\frac{p(\Pi'\otimes\eta)}{\Omega((\Pi'\otimes\eta)_\infty)}\sim_{E(\Pi')E(\eta);K}\frac{p(\Pi')}{\Omega(\Pi'_\infty)}.$$

Proof By Corollary 7.5.1, we have:

$$p(\Pi'\otimes\eta)\sim_{E(\Pi')E(\eta);K}Z((\Pi'\otimes\eta)_\infty)\prod_{1\leqslant i\leqslant n'-1}P^{(i)}(\Pi'\otimes\eta). \tag{10.14}$$

By the definition of arithmetic automorphic period (c.f. Definition-Lemma 5.3.2), we know $P^{(i)}(\Pi'\otimes\eta)\sim_{E(\Pi')E(\eta);K}p(\check{\eta},1)^{i}p(\check{\eta},\iota)^{n-i}$. The latter is equivalent to $p(\check{\eta},1)^{2i-n}$ since η is conjugate self-dual.

We see that:

$$\prod_{1\leqslant i\leqslant n}P^{(i)}(\Pi'\otimes\eta)\sim_{E(\Pi')E(\eta);K}\prod_{1\leqslant i\leqslant n}[P^{(i)}(\Pi')p(\check{\eta},1)^{2i-n}]\sim_{E(\Pi')E(\eta);K}\prod_{1\leqslant i\leqslant n}P^{(i)}(\Pi'). \tag{10.15}$$

By Corollary 7.5.1, This will imply that:

$$\frac{p(\Pi'\otimes\eta)}{Z((\Pi'\otimes\eta)_\infty)}\sim_{E(\Pi')E(\eta);K}\frac{p(\Pi')}{Z(\Pi'_\infty)}.$$

But we know by Corollary 9.4.1 that $Z(\Pi'_\infty)\sim_{E(\Pi'_\infty;K)}(2\pi i)^{\frac{n'(n'-1)}{2}}\Omega(\Pi'_\infty)$ and a similar formula for $(\Pi'\otimes\eta)_\infty$. The lemma then follows.

□

By Theorem 5.2.1, for all $1\leqslant j\leqslant l$, we have

$$L(1,\Pi'\otimes\chi_j^c)\sim_{E(\Pi')E(\chi_j);K}(2\pi i)^{n'}P^{(t_j)}(\Pi')p(\widetilde{\chi_j},1)^{n'-2t_j}.$$

Similarly, we have

$$\begin{aligned}&L(1,(\Pi'\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)\otimes\chi_j^c)=L(\frac{1}{2},\Pi'\otimes(\psi\chi_j^c))\\ \sim_{E(\Pi')E(\chi_j)E(\psi);K}\quad&(2\pi i)^{\frac{n'}{2}}P^{(t_j)}(\Pi')p(\widetilde{\chi_j},1)^{n'-2t_j}p(\check{\psi},1)^{t_j}p(\check{\psi},\iota)^{n'-t_j};\end{aligned}$$

$$\begin{aligned}\text{and}\qquad&L(1,\Pi'\otimes(\chi_i\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)^c)=L(\frac{1}{2},\Pi'\otimes(\chi_i\otimes\psi)^c)\\ \sim_{E(\Pi')E(\chi_j)E(\psi);K}\quad&(2\pi i)^{\frac{n'}{2}}P^{(t_j)}(\Pi')p(\widetilde{\chi_j},1)^{n'-2t_j}p(\check{\psi},1)^{n'-t_j}p(\check{\psi},\iota)^{t_j}.\end{aligned}$$

Along with equation (10.7), we get

(A) and (D): $p(\Pi^{\#})\sim_{E(\Pi')EE(\psi);K}\Omega(\Pi^{\#}_\infty)p(\Pi')\Omega(\Pi'_\infty)^{-1}(2\pi i)^{n'l+\frac{l(l-1)}{2}}\times$

$$\prod_{j=1}^{l}P^{(t_j)}(\Pi')\prod_{j=1}^{l}p(\widetilde{\chi_j},1)^{2s_j-n}$$

(B) : $p(\Pi^{\#}) \sim_{E(\Pi')EE(\psi);K} \Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}(2\pi i)^{\frac{n'l}{2}+\frac{l(l-1)}{2}} \times$

$$\prod_{j=1}^{l} P^{(t_j)}(\Pi') \prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n} p(\check{\psi},1)^{t} p(\check{\psi},\iota)^{n'l-t}$$

(C) : $p(\Pi^{\#}) \sim_{E(\Pi')EE(\psi);K} \Omega(\Pi^{\#}_{\infty})p(\Pi')\Omega(\Pi'_{\infty})^{-1}(2\pi i)^{\frac{n'l}{2}+\frac{l(l-1)}{2}} \times$

$$\prod_{j=1}^{l} P^{(t_j)}(\Pi') \prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n} p(\check{\psi},1)^{n'l-t} p(\check{\psi},\iota)^{t}$$

where $t = \sum\limits_{j=1}^{l} t_j = \sum\limits_{j=1}^{l} (n'+j-s_j) = n'l + \dfrac{l(l+1)}{2} - s.$

We then apply equations (10.8), (10.9) and Lemma 9.4.1, Corollary 9.4.1 to get:

(A) $p(\Pi)p(\Pi^{\#})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}) \sim_{E(\Pi)E(\Pi')E;K} (2\pi i)^{n(n-1)(m+\frac{1}{2})} \times$

$$\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n} \prod_{i=1}^{n-1} P^{(i)}(\Pi) \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

(B) $p(\Pi)p(\Pi^{\#})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}) \sim_{E(\Pi)E(\Pi')EE(\psi);K} (2\pi i)^{n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}} \times$

$$\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n} p(\check{\psi},1)^{t} p(\check{\psi},\iota)^{n'l-t} \prod_{i=1}^{n-1} P^{(i)}(\Pi) \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\psi;\Pi)}.$$

(C) $p(\Pi)p(\Pi^{\#})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}) \sim_{E(\Pi)E(\Pi')EE(\psi);K} (2\pi i)^{n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}} \times$

$$\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n} p(\check{\psi},1)^{n'l-t} p(\check{\psi},\iota)^{t} \prod_{i=1}^{n-1} P^{(i)}(\Pi) \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\psi;\Pi)}.$$

(D) $p(\Pi)p(\Pi^{\#})p(m,\Pi_{\infty},\Pi^{\#}_{\infty}) \sim_{E(\Pi)E(\Pi')EE(\psi);K} (2\pi i)^{n(n-1)(m+\frac{1}{2})} \times$

$$\prod_{j=1}^{l} p(\widecheck{\chi_j},1)^{2s_j-n} \prod_{i=1}^{n-1} P^{(i)}(\Pi) \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

10.7 Compare both sides, general cases

At first, observe that

$$p(\check{\psi},1)p(\check{\psi},\iota) \sim_{E(\psi);K} p(\check{\psi},1)p(\widecheck{\psi^c},1) \sim_{E(\psi);K} p(\widecheck{\psi\psi^c},1) \sim_{E(\psi);K} p(||\cdot||_{\mathbb{A}_K}^{-1},1) \sim_{E(\psi)} 2\pi i.$$

We can then conclude:

(A) $L(\dfrac{1}{2}+m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K}$

$$(2\pi i)^{(m+\frac{1}{2})nn'} \prod_{k=1}^{n'} P^{(w(k))}(\Pi) \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}$$

(B) Since $n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}-(m+\frac{1}{2})nl = (m+\frac{1}{2})n(n-1-l)-\frac{n'l}{2} = (m+\frac{1}{2})nn'-\frac{n'l}{2} = mnn' + \frac{nn'}{2} - \frac{n'l}{2}$, we have

$$L(m,\Pi\times(\Pi'\otimes\psi)) \sim_{E(\Pi)E(\Pi')E(\psi);K} (2\pi i)^{mnn'+\frac{nn'}{2}-\frac{n'l}{2}} \times \prod_{k=1}^{n'} P^{(w(k))}(\Pi)\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\psi;\Pi)}p(\check{\psi},1)^{t}p(\check{\psi},\iota)^{n'l-t} \tag{10.16}$$

Since $(2\pi i)^{\frac{nn'}{2}-\frac{n'l}{2}} \sim_{E(\psi)} p(\check{\psi},1)^{\frac{nn'}{2}-\frac{n'l}{2}}p(\check{\psi},\iota)^{\frac{nn'}{2}-\frac{n'l}{2}}$, and

$$\begin{aligned}\frac{nn'}{2}-\frac{n'l}{2}+t &= \frac{nn'}{2}-\frac{n'l}{2}+(n'l+\frac{l(l+1)}{2}-s) = \frac{nn'}{2}+\frac{n'l}{2}+\frac{l(l+1)}{2}-s\\ &= \frac{nn'}{2}+\frac{(n'+l+1)l}{2}-s = \frac{nn'}{2}+\frac{nl}{2}-s\\ &= \frac{n(n'+l)}{2}-s = \frac{n(n-1)}{2}-s;\end{aligned}$$

$$\begin{aligned}\frac{nn'}{2}-\frac{n'l}{2}+n'l-t &= \frac{nn'}{2}-\frac{n'l}{2}+n'l-(n'l+\frac{l(l+1)}{2}-s)\\ &= s+\frac{nn'}{2}-\frac{(n'+l+1)l}{2}\\ &= s+nn'-\frac{nn'}{2}-\frac{nl}{2}\\ &= s+nn'-\frac{n(n-1)}{2}\end{aligned}$$

We get $L(m,\Pi\times(\Pi'\otimes\psi)) \sim_{E(\Pi)E(\Pi')E(\psi);K} (2\pi i)^{mnn'}\times$

$$\prod_{k=1}^{n'} P^{(w(k))}(\Pi)\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}p(\check{\psi},1)^{\frac{n(n-1)}{2}-s}p(\check{\psi},\iota)^{s+nn'-\frac{n(n-1)}{2}}$$

(C) Since $n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}-mnl = n(n-1)(m+\frac{1}{2})-\frac{n'l}{2}-(m+\frac{1}{2})nl+\frac{nl}{2} = (m+\frac{1}{2})nn'+\frac{nl}{2}-\frac{n'l}{2}$, we have

$$L(\frac{1}{2}+m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi')E(\psi);K} (2\pi i)^{(m+\frac{1}{2})nn'+\frac{nl}{2}-\frac{n'l}{2}} \times \prod_{k=1}^{n'} P^{(w(k))}(\Pi)\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}p(\check{\psi},1)^{n'l-t-s}p(\check{\psi},\iota)^{t+s-nl} \tag{10.17}$$

Moreover, we know $t+s = n'l+\frac{l(l+1)}{2}$, we have $2(t+s) = 2n'l+(l+1)l = n'l+(n'+l+1)l = n'l+nl$. Thus $n'l-t-s = t+s-nl = \frac{n'l}{2}-\frac{nl}{2}$. We then get $p(\check{\psi},1)^{n'l-t-s}p(\check{\psi},\iota)^{t+s-nl} = p(\widetilde{\psi\otimes\psi^c},1)^{\frac{n'l}{2}-\frac{nl}{2}} = (2\pi i)^{\frac{n'l}{2}-\frac{nl}{2}}$.

Therefore:

$$L(\frac{1}{2}+m,\Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{(m+\frac{1}{2})nn'}\prod_{k=1}^{n'} P^{(w(k))}(\Pi)\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

(D) Similarly, since $n(n-1)(m+\frac{1}{2})-mnl = n(n-1)m+\frac{n(n-1)}{2}-mnl = mnn'+\frac{n(n-1)}{2}$, we have

$$L(m,\Pi\times(\Pi'\otimes\psi)) \sim_{E(\Pi)E(\Pi')E(\psi);K} (2\pi i)^{mnn'} \times \quad (10.18)$$
$$\prod_{k=1}^{n'} P^{(w(k))}(\Pi)\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\psi;\Pi)}p(\check{\psi},1)^{\frac{n(n-1)}{2}-s}p(\check{\psi},\iota)^{s+nn'-\frac{n(n-1)}{2}}.$$

It is easy to verify that $s-nl+\frac{n(n-1)}{2} = s-nl+n(n-1)-\frac{n(n-1)}{2} = s+nn'-\frac{n(n-1)}{2}$.

10.8 Final conclusion: general cases

Before concluding, we notice that in case (B) or (D),

$$s = \sum_{1\leqslant j\leqslant n-1} s_j = \sum_{j=1}^{n-1} j - \sum_{j=1}^{n'} w(j) = \frac{n(n-1)}{2} - \sum_{j=1}^{n'} w(j).$$

Recall that $w(j) = \sum\limits_{k=j}^{n'} sp(k,\Pi'\otimes\psi;\Pi)$ for all $1\leqslant k\leqslant n'$ by (10.3). Therefore:

$$\begin{aligned}\frac{n(n-1)}{2}-s=\sum_{j=1}^{n'}w(j) &= \sum_{j=1}^{n'}\sum_{j\leqslant k\leqslant n'} sp(j,\Pi'\otimes\psi;\Pi) = \sum_{k=1}^{n'} k*sp(k,\Pi'\otimes\psi;\Pi)\\ &= \sum_{k=0}^{n'} k*sp(k,\Pi'\otimes\psi;\Pi); \qquad (10.19)\end{aligned}$$

$$\begin{aligned}\text{and } s+nn'-\frac{n(n-1)}{2} &= nn'-\sum_{k=0}^{n'} k*sp(k,\Pi'\otimes\psi;\Pi)\\ &= r\sum_{k=0}^{n'} sp(k,\Pi'\otimes\psi;\Pi)-\sum_{k=0}^{n'} j*sp(k,\Pi'\otimes\psi;\Pi)\\ &= \sum_{k=0}^{n'}(n'-k)sp(k,\Pi'\otimes\psi;\Pi)\end{aligned}$$

by Lemma 1.2.1 which says that $\sum\limits_{k=0}^{n'} sp(k,\Pi'\otimes\psi;\Pi)=n$.

Therefore, we get

$$\begin{aligned}&\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\psi;\Pi)}p(\check{\psi},1)^{\frac{n(n-1)}{2}-s}p(\check{\psi},\iota)^{s+nn'-\frac{n(n-1)}{2}}\\ \sim_{E(\Pi')E(\psi)}\ &\prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi'\otimes\psi;\Pi)}p(\check{\psi},1)^{\sum\limits_{k=0}^{n'} k*sp(k,\Pi'\otimes\psi;\Pi)}p(\check{\psi},\iota)^{\sum\limits_{k=0}^{n'}(n'-k)sp(k,\Pi'\otimes\psi;\Pi)}\\ \sim_{E(\Pi')E(\psi)}\ &\prod_{k=0}^{n'}\left(P^{(k)}(\Pi')p(\check{\psi},1)^k p(\check{\psi},\iota)^{n'-k}\right)^{sp(k,\Pi'\otimes\psi;\Pi)}.\end{aligned}$$

Recall that $P^{(k)}(\Pi'\otimes\psi) := P^{(k)}(\Pi')p(\check{\psi},1)^k p(\check{\psi},\iota)^{n'-k}$ by definition, we obtain that:

Theorem 10.8.1. *Let $n > n'$ be two positive integers. Let K be a quadratic imaginary field. Let Π and Π' be cuspidal representations of GL_n and $GL_{n'}$ respectively which are very regular, cohomological, conjugate self-dual and supercuspidal at at least two finite split places. We assume that (Π, Π') is in good position in the sense of Definition 1.2.2.*

(i) If $n \not\equiv n' (mod\ 2)$, then for any critical value $m + \frac{1}{2}$ for $\Pi \otimes \Pi'$ such that $m \geqslant 1$, or $m \geqslant 0$ along with a non-vanishing condition, we have:

$$L(\frac{1}{2}+m, \Pi\times\Pi') \sim_{E(\Pi)E(\Pi');K} (2\pi i)^{(m+\frac{1}{2})nn'} \prod_{i=0}^{n} P^{(i)}(\Pi)^{sp(i,\Pi;\Pi')} \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

(ii) If $n \equiv n' (mod\ 2)$, then for any critical value m for $\Pi\otimes\Pi'$ such that $m \geqslant 1$, or $m \geqslant 0$ along with a non-vanishing condition, we have:

$$L(m, \Pi \times (\Pi' \otimes \psi))$$
$$\sim_{E(\Pi)E(\Pi')E(\psi);K} (2\pi i)^{mnn'} \prod_{i=0}^{n} P^{(i)}(\Pi)^{sp(i,\Pi;\Pi'\otimes\psi)} \prod_{k=0}^{n'} P^{(k)}(\Pi' \otimes \psi)^{sp(k,\Pi'\otimes\psi;\Pi)}.$$

Chapter 11

Special values at 1 of L-functions for automorphic pairs over quadratic imaginary fields

11.1 Settings

Let r_1 and r_2 be two positive integers.

Let Π_1 and Π_2 be two cuspidal representations of $GL_{r_1}(\mathbb{A}_K)$ and $GL_{r_2}(\mathbb{A}_K)$ respectively which has definable arithmetic automorphic periods. We assume they are also conjugate self-dual.

We write the infinity type of Π_1 (resp. Π_2) by $(z^{b_j}\overline{z}^{-b_j})_{1\leqslant j\leqslant r_1}$ (resp. $(z^{c_k}\overline{z}^{-c_k})_{1\leqslant k\leqslant r_2}$). We see that $b_j \in \mathbb{Z}+\frac{r_1-1}{2}$ for all $1\leqslant j\leqslant r_1$ (resp. $c_k\in\mathbb{Z}+\frac{r_2-1}{2}$ for all $1\leqslant k\leqslant r_2$).

(A) If $r_1\equiv r_2\equiv 0(\bmod 2)$, we write $\Pi^{\#}=\Pi_1\boxplus\Pi_2^c$. We define $T_3=T_4=0$.

(B) If $r_1\equiv r_2\equiv 1(\bmod 2)$, we write $\Pi^{\#}=(\Pi_1\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)\boxplus(\Pi_2^c\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)$. We define $T_3=T_4=\frac{1}{2}$.

(C) If $r_1\not\equiv r_2(\bmod 2)$, we may assume that r_1 is even and r_2 is odd. We write $\Pi^{\#}=(\Pi_1\otimes||\cdot||_{\mathbb{A}_K}^{-\frac{1}{2}}\psi)\boxplus\Pi_2^c$. We define $T_3=\frac{1}{2}$ and $T_4=0$.

It is easy to see that $\Pi^{\#}$ is an algebraic generic representation of $GL_{r_1+r_2}(\mathbb{A}_K)$ with infinity type $(z^{b_j+T_3}\overline{z}^{-b_j-T_3}, z^{-c_k+T_4}\overline{z}^{c_k-T_4})_{1\leqslant j\leqslant r_1, 1\leqslant k\leqslant r_2}$.

We assume that $\Pi^{\#}$ is regular, i.e. for any $1\leqslant j\leqslant r_1$ and any $1\leqslant k\leqslant r_2$, we have $b_j+T_3\neq -c_k+T_4$.

Set $n=r_1+r_2+1$. We see that $\{b_j+T_3\mid 1\leqslant j\leqslant r_1\}\cup\{-c_k+T_4\mid 1\leqslant k\leqslant r_2\}$ are $n-1$ different numbers in $\mathbb{Z}+\frac{n-2}{2}$. We take $a_1>a_2>\cdots>a_n\in\mathbb{Z}+\frac{n-1}{2}$ such that the $n-1$ numbers above are in different gaps between $\{a_i\mid 1\leqslant i\leqslant n\}$. Let Π be a cuspidal conjugate self-dual representation of $GL_n(\mathbb{A}_K)$ which has arithmetic automorphic periods and infinity type $(z^{a_i}\overline{z}^{-a_i})$.

Our method also requires Π to be 3-regular. To guarantee this, we assume that

$$|(b_j+T_3)-(-c_k+T_4)| \geqslant 3 \text{ for all } 1\leqslant j\leqslant r_1, 1\leqslant k\leqslant r_2. \tag{11.1}$$

In this case, we say the pair (Π_1,Π_2) is **very regular**. We can then take a_i as above such that $1+\frac{1}{2}$ is critical for $\Pi\otimes\Pi^{\#}$. Moreover, results in [14] show the existence of Π as above, such that $L(1+\frac{1}{2},\Pi\otimes\Pi^{\#})\neq 0$.

We fix such Π and $m=1$, then $m+\frac{1}{2}$ is critical for $\Pi\times\Pi^{\#}$ and moreover

$$L(\frac{1}{2}+m,\Pi\times\Pi^{\#}) \sim_{E(\Pi)E(\Pi^{\#});K} p(\Pi)p(\Pi^{\#})p(m,\Pi_\infty,\Pi^{\#}_\infty) \tag{11.2}$$

with both sides non zero.

In the end of this section, let us show some simple facts on the split index. We can read from the construction of a_i that

$$sp(j,\Pi_1\otimes\psi^{2T_3};\Pi) = sp(j,\Pi_1\otimes\psi^{2T_3};\Pi_2\otimes(\psi)^{2T_4})+1 \text{ for all } 0\leqslant j\leqslant r_1$$

and similarly,
$$\begin{aligned} & sp(j,\Pi_2^c\otimes\psi^{2T_4};\Pi) = sp(j,(\Pi_2\otimes(\psi^c)^{2T_4})^c;(\Pi_1\otimes\psi^{2T_3})^c)+1 \\ = \; & sp(r_2-j,\Pi_2\otimes(\psi^c)^{2T_4});\Pi_1\otimes\psi^{2T_3})+1 \text{ for all } 0\leqslant j\leqslant r_2 \end{aligned}$$

Here we have used Lemma 1.2.1.

Moreover, for each $1\leqslant i\leqslant n-1$, one of $sp(i,\Pi;\Pi_1\otimes(\psi^c)^{2T_3})$ and $sp(i,\Pi;\Pi_2^c\otimes\psi^{2T_4})$ is 1 and another is 0. We also know that $sp(0,\Pi;\Pi_1\otimes\psi^{2T_3}) = sp(0,\Pi;\Pi_2^c\otimes\psi^{2T_4}) = 0$ and $sp(n,\Pi;\Pi_1\otimes\psi^{2T_3}) = sp(n,\Pi;\Pi_2^c\otimes\psi^{2T_3}) = 0$.

11.2 Simplify the left hand side

We are going to simply the left hand side of equation (11.2) now.

(A) In this case we have $L(m+\frac{1}{2},\Pi\times\Pi^{\#}) = L(m+\frac{1}{2},\Pi\times\Pi_1)\times L(m+\frac{1}{2},\Pi\times\Pi_2^c)$.

By Theorem 10.8.1, we know that

$$L(\frac{1}{2}+m,\Pi\times\Pi_1) \sim_{E(\Pi)E(\Pi_1);K} (2\pi i)^{(m+\frac{1}{2})nr_1}\prod_{i=0}^{n}P^{(i)}(\Pi)^{sp(i,\Pi;\Pi_1)}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi)} \tag{11.3}$$

and similarly
$$L(\frac{1}{2}+m,\Pi\times\Pi_2^c) \sim_{E(\Pi)E(\Pi_2);K} (2\pi i)^{(m+\frac{1}{2})nr_2}\prod_{i=0}^{n}P^{(i)}(\Pi)^{sp(i,\Pi;\Pi_2^c)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi)} .$$

Therefore, since $sp(i,\Pi;\Pi_1)+sp(i,\Pi;\Pi_2^c)=1$ for all $1\leqslant i\leqslant n-1$, we obtain that

$$\begin{aligned}
& L(m+\frac{1}{2},\Pi\times\Pi^{\#}) \qquad (11.4)\\
\sim_{E(\Pi)E(\Pi)E(\Pi_2);K}\ & (2\pi i)^{(m+\frac{1}{2})n(n-1)}\prod_{i=0}^{n}P^{(i)}(\Pi)^{sp(i,\Pi;\Pi_1)+sp(i,\Pi;\Pi_2^c)}\\
& \prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2^c;\Pi)}\\
\sim_{E(\Pi)E(\Pi)E(\Pi_2);K}\ & (2\pi i)^{(m+\frac{1}{2})n(n-1)}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\\
& \prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi)}.
\end{aligned}$$

(B) In this case, we have $L(m+\frac{1}{2},\Pi\times\Pi^{\#})=L(m,\Pi\times(\Pi_1\otimes\psi))\times L(m,\Pi\times(\Pi_2^c\otimes\psi))$.

Applying the second part of Theorem 10.8.1, we have

$$\begin{aligned}
& L(m+\tfrac{1}{2},\Pi\times\Pi^{\#}) \qquad (11.5)\\
\sim_{E(\Pi)E(\Pi)E(\Pi_2);K}\ & (2\pi i)^{mn(n-1)}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\psi;\Pi)}\\
& \prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c\otimes\psi;\Pi)}p(\check{\psi},1)^{\sum_{j=0}^{r_1}j*sp(j,\Pi_1\otimes\psi;\Pi)+\sum_{k=0}^{r_2}k*sp(k,\Pi_2^c\otimes\psi;\Pi)}\\
& \times p(\check{\psi},\iota)^{\sum_{j=0}^{n'}(r_1-j)*sp(j,\Pi_1\otimes\psi;\Pi)+\sum_{k=0}^{r_2}(r_2-k)*sp(k,\Pi_2^c\otimes\psi;\Pi)}.
\end{aligned}$$

Lemma 11.2.1. *We have:*

$$\begin{aligned}
& \sum_{j=0}^{r_1} j*sp(j,\Pi_1\otimes\psi;\Pi)+\sum_{k=0}^{r_2}k*sp(k,\Pi_2^c\otimes\psi;\Pi)\\
=\ & \sum_{j=0}^{n'}(r_1-j)*sp(j,\Pi_1\otimes\psi;\Pi)+\sum_{k=0}^{r_2}(r_2-k)*sp(k,\Pi_2^c\otimes\psi;\Pi)\\
=\ & \frac{n(n-1)}{2}
\end{aligned}$$

Proof We set $w(j,\Pi_1\otimes\psi;\Pi)$, $1\leqslant j\leqslant r_1$ (resp. $w(k,\Pi_2^c\otimes\psi;\Pi)$, $1\leqslant k\leqslant r_2$) to be the index $w(j)$ for the pair $(\Pi,\Pi_1\otimes\psi)$ (resp. $(\Pi,\Pi_2^c\otimes\psi))$) as in (10.1). We see from (10.19) that $\sum\limits_{j=0}^{r_1} j*sp(j,\Pi_1\otimes\psi;\Pi)=\sum\limits_{j=1}^{r_1}w(j,\Pi_1\otimes\psi;\Pi)$ and $\sum\limits_{k=0}^{r_2}k*sp(k,\Pi'\otimes\psi;\Pi)=\sum\limits_{k=1}^{r_1}w(k,\Pi_2^c\otimes\psi;\Pi)$.

Recall that $w(j,\Pi_1\otimes\psi;\Pi)$ (resp. $w(k,\Pi_2^c\otimes\psi;\Pi)$) is the position of the infinity type of $\Pi_1\otimes\psi$ (resp. $\Pi_2^c\otimes\psi$) in the gaps of the infinity type of Π. It is easy to see that the $n-1$ numbers $w(j,\Pi_1\otimes\psi;\Pi)$, $w(k,\Pi_2^c\otimes\psi;\Pi)$ for $1\leqslant j\leqslant r_1$ and $1\leqslant k\leqslant r_2$ runs over $1,2,\cdots,n-1$. We then deduce the first formula of the lemma.

The second equation follows easily from the first one.

□

From the lemma we see that

$$(2\pi i)^{\frac{n(n-1)}{2}} \sim_{E(\psi);K} p(\breve{\psi},1)^{\sum\limits_{j=0}^{r_1} j*sp(j,\Pi_1\otimes\psi;\Pi)+\sum\limits_{k=0}^{r_2} k*sp(k,\Pi_2^c\otimes\psi;\Pi)} \times p(\breve{\psi},\iota)^{\sum\limits_{j=0}^{n'}(r_1-j)*sp(j,\Pi_1\otimes\psi;\Pi)+\sum\limits_{k=0}^{r_2}(r_2-k)*sp(k,\Pi_2^c\otimes\psi;\Pi)}. \tag{11.6}$$

We thus obtain that

$$L(m+\tfrac{1}{2},\Pi\times\Pi^{\#}) \sim_{E(\Pi)E(\Pi)E(\Pi_2);K} (2\pi i)^{(m+\frac{1}{2})n(n-1)}\prod_{i=1}^{n-1}P^{(i)}(\Pi) \prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\psi;\Pi)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c\otimes\psi;\Pi)}. \tag{11.7}$$

(C) In this case, we have $L(m+\frac{1}{2},\Pi\times\Pi^{\#})=L(m,\Pi\times(\Pi_1\otimes\psi))\times L(m+\frac{1}{2},\Pi\times\Pi_2^c)$.

Similarly, we get:

$$L(m+\tfrac{1}{2},\Pi\times\Pi^{\#}) \sim_{E(\Pi)E(\Pi)E(\Pi_2);K} (2\pi i)^{(m+\frac{1}{2})n(n-1)-\frac{nr_1}{2}}\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\psi;\Pi)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi)}\times p(\breve{\psi},1)^{\sum\limits_{j=0}^{r_1} j*sp(j,\Pi_1\otimes\psi;\Pi)} p(\breve{\psi},\iota)^{\sum\limits_{j=0}^{n'}(r_1-j)*sp(j,\Pi_1\otimes\psi;\Pi)}. \tag{11.8}$$

11.3 Simplify the right hand side

By Corollary 3.5.1 and Corollary 9.4.1, for cases (A) and (B), we have:

$$\begin{aligned} p(\Pi^{\#}) &\sim_{E(\Pi^{\#});K} \Omega(\Pi^{\#}_\infty)p(\Pi_1)\Omega(\Pi_{1,\infty})^{-1}p(\Pi_2)\Omega(\Pi_{2,\infty})^{-1}L(1,\Pi_1\times\Pi_2)\\ &\sim_{E(\Pi^{\#});K} \Omega(\Pi^{\#}_\infty)Z(\Pi_{1,\infty})\Omega(\Pi_{1,\infty})^{-1}Z(\Pi_{2,\infty})\Omega(\Pi_{2,\infty})^{-1}L(1,\Pi_1\times\Pi_2)\times \prod_{j=1}^{r_1-1}P^{(j)}(\Pi_1)\prod_{k=1}^{r_2-1}P^{(k)}(\Pi_2^c)\\ &\sim_{E(\Pi^{\#});K} (2\pi i)^{\frac{(r_1-1)r_1}{2}+\frac{(r_2-1)r_2}{2}}\Omega(\Pi^{\#}_\infty)L(1,\Pi_1\times\Pi_2)\prod_{j=1}^{r_1-1}P^{(j)}(\Pi_1)\prod_{k=1}^{r_2-1}P^{(k)}(\Pi_2^c). \end{aligned}$$

Therefore, for cases (A) and (B), we obtain that:

$$\begin{aligned}
& p(\Pi)p(\Pi^{\#})p(m,\Pi_\infty,\Pi^{\#}_\infty) \\
\sim_{E(\Pi^{\#});K}\ & (2\pi i)^{\frac{(r_1-1)r_1}{2}+\frac{(r_2-1)r_2}{2}}\Omega(\Pi^{\#}_\infty)Z(\Pi_\infty)p(m,\Pi_\infty,\Pi^{\#}_\infty)\times \\
& L(1,\Pi_1\times\Pi_2)\prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{j=1}^{r_1-1}P^{(j)}(\Pi_1)\prod_{k=1}^{r_2-1}P^{(k)}(\Pi_2^c) \\
\sim_{E(\Pi^{\#});K}\ & (2\pi i)^{n(n-1)(m+\frac{1}{2})-\frac{n(n-1)}{2}+\frac{(r_1-1)r_1}{2}+\frac{(r_2-1)r_2}{2}}L(1,\Pi_1\times\Pi_2)\times \\
& \prod_{i=1}^{n-1}P^{(i)}(\Pi)\prod_{j=1}^{r_1-1}P^{(j)}(\Pi_1)\prod_{k=1}^{r_2-1}P^{(k)}(\Pi_2^c) \\
\sim_{E(\Pi^{\#});K}\ & (2\pi i)^{n(n-1)(m+\frac{1}{2})-r_1r_2}L(1,\Pi_1\times\Pi_2)\prod_{i=1}^{n-1}P^{(i)}(\Pi)\times \\
& \prod_{j=0}^{r_1}P^{(j)}(\Pi_1)\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c) \qquad (11.9)
\end{aligned}$$

We have used Lemma 9.4.1, the fact that $\binom{n-1}{2}=\binom{r_1+r_2}{2}=\binom{r_1}{2}+\binom{r_2}{2}+r_1r_2$ and also the fact that $P^{(0)}(\Pi_1)P^{(r_1)}(\Pi_1)\sim_{E(\Pi_1)}1$, $P^{(0)}(\Pi_2^c)P^{(r_2)}(\Pi_2^c)\sim_{E(\Pi_2)}1$.

For case (C), we only need to change $L(1,\Pi_1\times\Pi_2)$ to $L(\frac{1}{2},(\Pi_1\otimes\psi)\times\Pi_2)$ in the above formula.

11.4 Final conclusion

Comparing (11.4) and (11.9), we get for case (A):

$$\begin{aligned}
L(1,\Pi_1\times\Pi_2)\ \sim_{E(\Pi_1)E(\Pi_2);K}\ & (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi)-1}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi)-1} \\
\sim_{E(\Pi_1)E(\Pi_2);K}\ & (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2^c)^{sp(k,\Pi_2^c;\Pi_1^c)} \\
\sim_{E(\Pi_1)E(\Pi_2);K}\ & (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(r_2-k)}(\Pi_2)^{sp(r_2-k,\Pi_2;\Pi_1)} \\
\sim_{E(\Pi_1)E(\Pi_2);K}\ & (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1)}.
\end{aligned}$$

Comparing (11.7) and (11.9), we get for case (B):

$$\begin{aligned}
& L(1,\Pi_1\times\Pi_2) \\
\sim_{E(\Pi_1)E(\Pi_2);K}\ & (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\psi;\Pi_2\otimes\psi^c)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2\otimes\psi^c;\Pi_1\otimes\psi)} \\
\sim_{E(\Pi_1)E(\Pi_2);K}\ & (2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1)}.
\end{aligned}$$

Here we have used that $sp(j,\Pi_1\otimes\psi;\Pi_2\otimes\psi^c)=sp(j,\Pi_1\otimes\psi;\Pi_2\otimes\psi^{-1})=sp(j,\Pi_1;\Pi_2)$ by Lemma 1.2.1.

Similarly, for case (C), comparing (11.8) and (11.9), we obtain that:

$$\begin{aligned}
& L(\frac{1}{2},(\Pi_1\otimes\psi)\times\Pi_2) \\
\sim_{E(\Pi_1)E(\Pi_2)E(\psi);K}\quad & (2\pi i)^{r_1r_2-\frac{nr_1}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\psi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\psi)}\times \\
& p(\check{\psi},1)^{\sum\limits_{j=0}^{r_1}j*(sp(j,\Pi_1\otimes\psi;\Pi_2)+1)}p(\check{\psi},\iota)^{\sum\limits_{j=0}^{r_1}(r_1-j)*(sp(j,\Pi_1\otimes\psi;\Pi_2+1))} \\
\sim_{E(\Pi_1)E(\Pi_2)E(\psi);K}\quad & (2\pi i)^{\frac{r_1r_2}{2}-\frac{r_1(r_1+1)}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\psi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\psi)}\times \\
& p(\check{\psi},1)^{\sum\limits_{j=0}^{r_1}j*sp(j,\Pi_1\otimes\psi;\Pi_2)+\frac{r_1(r_1+1)}{2}}p(\check{\psi},\iota)^{\sum\limits_{j=0}^{r_1}(r_1-j)*sp(j,\Pi_1\otimes\psi;\Pi_2)+\frac{r_1(r_1+1)}{2}} \\
\sim_{E(\Pi_1)E(\Pi_2)E(\psi);K}\quad & (2\pi i)^{\frac{r_1r_2}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1\otimes\psi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\psi)}\times \\
& p(\check{\psi},1)^{\sum\limits_{j=0}^{r_1}j*sp(j,\Pi_1\otimes\psi;\Pi_2)}p(\check{\psi},\iota)^{\sum\limits_{j=0}^{r_1}(r_1-j)*sp(j,\Pi_1\otimes\psi;\Pi_2)} \\
\sim_{E(\Pi_1)E(\Pi_2)E(\psi);K}\quad & (2\pi i)^{\frac{r_1r_2}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1\otimes\psi)^{sp(j,\Pi_1\otimes\psi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\psi)}.
\end{aligned}$$

The last step is deduced by definition of $P^{(*)}(\Pi_1\otimes\psi)$ (c.f. Definition-Lemma 5.3.2).

Theorem 11.4.1. *Let r_1 and r_2 be two positive integers. Let Π_1 and Π_2 be two cuspidal representations of $GL_{r_1}(\mathbb{A}_K)$ and $GL_{r_2}(\mathbb{A}_K)$ respectively which are very regular, cohomological, conjugate self-dual and supercuspidal at at least two finite split places. Assume that the pair Π_1,Π_2 is very regular in the sense of (11.1).*

(i) If $r_1\equiv r_2(mod\ 2)$, then $L(1,\Pi_1\times\Pi_2)\sim_{E(\Pi_1)E(\Pi_2);K}$

$$(2\pi i)^{r_1r_2}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1)^{sp(j,\Pi_1;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1)}.$$

(ii) If $r_1\not\equiv r_2(mod\ 2)$, then $L(\frac{1}{2},(\Pi_1\otimes\psi)\times\Pi_2)\sim_{E(\Pi_1)E(\Pi_2)E(\psi);K}$

$$(2\pi i)^{\frac{r_1r_2}{2}}\prod_{j=0}^{r_1}P^{(j)}(\Pi_1\otimes\psi)^{sp(j,\Pi_1\otimes\psi;\Pi_2)}\prod_{k=0}^{r_2}P^{(k)}(\Pi_2)^{sp(k,\Pi_2;\Pi_1\otimes\psi)}.$$

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