

COBORDISM

1. INTRODUCTION

The purpose of this essay is to explain the main theorems in the theory of unoriented cobordism, (with a small hike into oriented cobordism at the end). Much of mathematics, to my understanding, is about trying to classify objects up to some relation that approximates equality up to some flexibility. For instance, we would like to say that two topological spaces are ‘the same’ if they are homeomorphic. Why? Because being homeomorphic means that for considerations of the underlying topology the spaces can be construed as one and the same. Similarly, homotopy equivalence, in algebraic topology and being diffeomorphic, in differential topology, play the rôle of this equality up to some flexibility. The problem with some of these relations is that they are so fine that they can become impossible to study, like being homeomorphic. Somehow, they do not distance themselves enough from equality for there to be a genuine study of the relation. One outlier among these, is the relation of *cobordism*, which relates two manifolds M, N if their disjoint union forms the boundary of a manifold-with-boundary of one dimension higher. Cobordism is coarse enough so as to be of interest, and it has turned out to be a wonderful motivating force in algebraic topology since it was first conceived. It is this relation that will form the focus of the present essay. In other words, we will study manifolds up to cobordism.

Definition 1. Two compact, smooth n -manifolds M, N are said to be *cobordant* if there is a compact $n + 1$ dimensional manifold-with-boundary X such that ∂X is diffeomorphic to $M \amalg N$. The manifold X is said to be a *cobordism* between M and N .

The relation of being cobordant defines an equivalence relation (for the verification of this see [Stong, Chapter I]) in the class of n -manifolds. The set of all such cobordism classes is denoted by $\mathfrak{N}_n$ and the class represented by the manifold M by $[M]$. We will call $\mathfrak{N}_n$ the n th cobordism group, where the additive operation is disjoint union of topological spaces (i.e. $[M] + [N] := [M \amalg N]$), and the identity is, as a consequence, the cobordism class containing the empty set $[\emptyset]$. (Checking that this is indeed a group and in fact an abelian group is easy and so it is omitted). There are several things to notice,

- If M is the boundary of a manifold-with-boundary X , then $[M] = [\emptyset]$. Henceforth, we will say that M is a *boundary* or that M *bounds*.
- Every element in $\mathfrak{N}_n$ has order 2, i.e. $[M] + [M] = [\emptyset]$. This is because $M \amalg M$ is the boundary of $M \times [0, 1]$.

- We want the condition of compactness for our cobordisms for otherwise we would get a trivial theory, namely all manifolds would be boundaries. To see this simply note that a manifold M is the boundary of $M \times [0, \infty)$.

Moreover, the cartesian product of manifolds defines a bilinear map $\mathfrak{N}_n \times \mathfrak{N}_k \rightarrow \mathfrak{N}_{n+k}$, sending $([M], [N])$ to $[M \times N]$. Then we can define a commutative graded ring $\mathfrak{N}_* := \bigoplus_{k \geq 0} \mathfrak{N}_k$, which we will call the *unoriented cobordism ring*.

Then, it is clear that if we manage to compute these cobordism groups and we get to figure out the ring structure of $\mathfrak{N}_*$ we would have disposed of much of the cobordism problem. So it should be no surprise that doing exactly this will be the main focus of this essay. To be precise the following structure theorems will be our goals, where the parenthesis show where these results appear in the body of the essay.

Theorem (Theorem 14). For every dimension k , the k th cobordism group $\mathfrak{N}_k$ is isomorphic to the direct sum of $d(k)$ copies of $\mathbf{Z}/2$.

Theorem (Theorem 18). The unoriented cobordism ring $\mathfrak{N}_* = \bigoplus_k \mathfrak{N}_k$ is isomorphic to a free algebra over $\mathbf{Z}/2$ with a generator for every dimension k which is not of the form $2^m - 1$.

These theorems were proven by R. Thom in [Thom, 1954]. Our references for this essay are [Thom, 1954], [Milnor and Stasheff, 1974], [Stong, 1968], [Hatcher, 2017], [Hatcher, 2002], [Miller, 1994], [Milnor, 1997], [Weston, 1997], [Whitehead, 1978], [Adams, 1969] and [Spanier, 1995].

This essay is divided into six sections: Section 2 gathers some facts we will need for the remainder of the essay. Section 3 introduces the tool of Stiefel-Whitney classes which will be crucial for our study of cobordism. Section 4 forms the technical heart of the essay because in it we prove the Pontrjagin-Thom construction which translates the topology problem of cobordism into a homotopy problem. Section 5 has the proofs of the main theorems and Section 6 gives a glimpse into the theory of oriented cobordism.

2. BACKGROUND

Theorem 1 (Tubular Neighborhoods). Let M^n be a submanifold of A^{n+k} . Then there is a tubular neighborhood of M in A diffeomorphic to the total space of the normal bundle of M as a submanifold of A sending points $x \in M$ to the zero in the fiber covering x .

If one thinks of simple examples, like some curve in the surface of the sphere, one sees easily that the theorem holds. Why? Because if we look close enough (a small enough neighborhood) then all the complication our curve may have globally will disappear (because it is a manifold) and the tubular neighborhood will then be easily identifiable with the normal bundle (small lines perpendicular to our curve one next to the other). The proof of the theorem boils down to showing that the exponential

map $\text{Exp} : E(\epsilon) \rightarrow A$ is a diffeomorphism onto a tubular neighborhood of M for small enough ϵ , where $E(\epsilon)$ is the set of all vectors of the total space of the normal bundle of M in A of length $\leq \epsilon$. That this is possible should be at least half believable because locally (by the Inverse Function Theorem) the exponential map is already a diffeomorphism.

It will be convenient to set some notation for the rest of the essay. Let ξ be a general rank n -vector bundle, then I will usually denote the total space by E , the base space by B , and the fiber by F . I also set E_0 to be the complement of the zero-section and F_0 to be the complement of 0 . Recall that unless otherwise stated, our coefficients are always taken to be $\mathbf{Z}/2$.

Theorem 2 (Thom Isomorphism). For a vector bundle ξ as above, the group $H^i(E, E_0)$ is trivial for $i < n$ and for $i = n$ there is a unique class $u \in H^n(E, E_0)$ such that for all fibers F ,

$$u|_{(F, F_0)} \in H^n(F, F_0) \cong \mathbf{Z}/2$$

is the unique nontrivial class. Moreover, the map

$$H^i(E) \xrightarrow{\sim u} H^{n+i}(E, E_0)$$

is an isomorphism for all i . (We call u the Thom class of the bundle).

I will omit this proof as we saw this theorem in Ivan's class. However, recall that what we did was first to show the statement held for a trivial bundle. Then the idea is that given all bundles look like trivial bundles locally, we could somehow use this statement over the trivialising opens and with luck get the statement for the whole bundle. In other words, we proved the theorem 'locally' and then used this to prove it 'globally'.

Some intuition as to why the first part of this theorem holds goes as follows: recall that the closed n -cell is a deformation retract of $\mathbf{R}^n$ and the $n - 1$ sphere is a deformation retract of $\mathbf{R}^n \setminus \{0\}$. Then if we assume B (the base space of our bundle ξ) to be paracompact, we can choose a Euclidean metric for ξ . It is clear then that the set of all vectors x in E of length ≤ 1 (i.e. $x \cdot x \leq 1$) is a deformation retract of E and likewise the set of all vectors such that $x \cdot x = 1$ is a deformation retract of E_0 . Call these spaces E', E'' respectively. Then we have that $H^*(E', E'') \cong H^*(E, E_0)$. If we assume further that B is a cell-complex with a fine enough cell structure so that each cell c^i lies in a trivialising open, then the preimage of c^i is a cell of dimension $n + i$. This means that we can cook up E' out of E'' by attaching cells of dimensions $\geq n$ to the latter. It follows, $H^i(E', E'') = 0$ for all $i < n$ as we have a good pair where E'/E'' has cells of dimensions $> n$.

If we let $\pi : E(\xi) \rightarrow B$ be the projection map of the bundle ξ then given E is homotopy equivalent to its zero section B , we get that the induced map π^* is an isomorphism on cohomology.

Definition 2. The map $\phi : H^k(B) \rightarrow H^{k+n}(E, E_0)$ defined as the composite,

$$H^k(B) \xrightarrow{\pi^*} H^k(E) \xrightarrow{\smile u} H^{k+n}(E, E_0)$$

is an isomorphism for all k . We say ϕ is the Thom isomorphism.

We will also need the cohomology ring of the real projective space P^n . For the proof of the following theorem, reminisce to Ivan's class (it is a good example of how to use the cup product and the Künneth theorem to study the ring structure of a cohomology ring), or see Spanier p. 264.

Theorem 3 (Cohomology of P^n). The group $H^i(P^n)$ is cyclic of order 2 for all $0 \leq i \leq n$ and is zero for $i > n$. Moreover, if a denotes the nonzero element of $H^1(P^n)$ then each $H^i(P^n)$ is generated by the i th-fold cup product a^i .

So we get that $H^*(P^n) \cong (\mathbf{Z}/2)[a]/a^{n+1}$, and in the limit $H^*(P^\infty) \cong (\mathbf{Z}/2)[a]$.

In this essay we will see a phenomenon that occurs repeatedly in algebraic topology, namely, we will have some questions about some objects (vector bundles) and we will see that most of them can be answered if we first answer the same question for some universal object. In our case, the universal object is the universal bundle over the Grassmann manifold of $\mathbf{R}^\infty$. (Another example of such a phenomenon is the theorem that relates homotopy classes of maps to an Eilenberg-MacLane space to the cohomology of the domain of such maps.)

Definition 3. Let the canonical rank n -vector bundle $\gamma^n(\mathbf{R}^{n+k}) = \gamma_k^n$ over the Grassmann manifold $G_n(\mathbf{R}^{n+k})$ be defined by

$$\gamma_k^n = \{(\ell, v) \in G_n(\mathbf{R}^{n+k}) \times \mathbf{R}^{n+k} \mid v \in \ell\}.$$

In particular, we have in the limit the *universal bundle* γ^n over the Grassmann manifold $G_n := G_n(\mathbf{R}^\infty)$.

Recall that we saw this bundle in Ivan's class: we called it 'the tautological bundle over the Grassmannian'. For a more detailed treatment see [Milnor-Stasheff, §5].

The following theorem justifies calling γ^n *universal*, see [Hatcher VBKT, Theorem 1.16, p.29]

Theorem 4. For paracompact B the map

$$\begin{aligned} [B, G_n] &\rightarrow \text{Vect}^n(B) \\ [f] &\mapsto f^*(\gamma^n) \end{aligned}$$

from the set of homotopy classes of maps $B \rightarrow G_n$ to the set of isomorphism classes of n -plane vector bundles over B , is a bijection.

In other words, we have that if $\eta : E \rightarrow B$ is an n -plane vector bundle, then there is a map $f : B \rightarrow G_n$ such that $f^*(\gamma^n) \cong E(\eta)$ as bundles.

Lastly, we will need that the Grassmann manifolds can be endowed with the structure of a CW complex (it pains me to have to omit the proof and the discussion of this theorem, but alas, not everything can survive!). For a sharper statement and a more detailed discussion see [Milnor-Stasheff, §6, Theorem 6.4 and Corollary 6.7].

Theorem 5. The Grassman manifold $G_n(\mathbf{R}^m)$ has a cell structure such that the number of cells in the r -skeleton is equal to the number of partitions of r into at most n integers each of which is $\leq m - n$.

In particular, if both n and $m - n$ are $\geq r$, then the number of r -cells in $G_n(\mathbf{R}^m)$ is equal to the number of partitions of r . The theorem also holds if we let m tend to infinity.

3. STIEFEL-WHITNEY CLASSES

Stiefel-Whitney classes are certain elements of the cohomology ring of the base space of a vector bundle $\xi \rightarrow B$. In some sense, these classes give us information about how far a certain bundle is from being the trivial bundle. In what follows they will be crucial to determine the main theorems of the theory of cobordism in its unoriented flavour. The reason being that with them we will be able to define cobordism invariants through what will later be called the *Stiefel-Whitney numbers*.

To define these classes we will first need some background on the Steenrod squaring operations Sq^i , which can be characterised by the following four properties:

1. For each pair $Y \subset X$ of spaces and each pair of nonnegative integers n, i there is a well-defined homomorphism of abelian groups,

$$\text{Sq}^i : H^n(X, Y) \rightarrow H^{n+i}(X, Y).$$

2. If $f : (X, Y) \rightarrow (X', Y')$ is a map of pairs then

$$f^* \circ \text{Sq}^i = \text{Sq}^i \circ f^*.$$

3. If $a \in H^n(X, Y)$ then $\text{Sq}^0(a) = 1$, $\text{Sq}^n(a) = a \smile a$ and $\text{Sq}^i a = 0$ for all $i > n$. (Intuitively this is why we would like to think of Sq^n as a squaring operation).
4. This property is called Cartan's formula and holds whenever the cup-product $a \smile b$ makes sense:

$$\text{Sq}^i(a \smile b) = \sum_{j=0}^i \text{Sq}^j(a) \smile \text{Sq}^{i-j}(b).$$

It will be useful in what follows to define the *total* Steenrod square $\text{Sq}a$ for $a \in H^n(X, Y)$,

$$\text{Sq}(a) := \text{Sq}^0(a) + \text{Sq}^1(a) + \cdots + \text{Sq}^n(a).$$

Note this is a formal sum. With this definition we have that property (4) above can be written as

$$\text{Sq}(a \smile b) = \text{Sq}(a) \smile \text{Sq}(b)$$

and likewise the Steenrod square of a cross product can be expressed by

$$\text{Sq}(a \times b) = \text{Sq}(a) \times \text{Sq}(b).$$

(Expand and chase definitions).

Before we introduce Stiefel-Whitney classes, it will be useful in what follows to have the definition of the *Steenrod algebra* $\mathfrak{A}_2$. For this we need to introduce the Adém relations which are the only relations that exist among the Steenrod squares under composition,

$$\text{Sq}^a \text{Sq}^b = \sum_j \binom{b-j-1}{a-2j} \text{Sq}^{a+b-j} \text{Sq}^j, \text{ if } a < 2b.$$

Definition 4. The Steenrod algebra $\mathfrak{A}_2$ is defined to be the $\mathbf{Z}/2$ -algebra that is the quotient of the algebra generated by the noncommuting variables Sq^i for $i > 0$ over the ideal generated by the Adém relations.

In other words, if $\text{Sq}^{i_1} \text{Sq}^{i_2} \cdots \text{Sq}^{i_r}$ is an iterated Steenrod square such that $i_{m-1} \geq 2i_m$ then we cannot hit it with the Adém relations. Indeed, if $I = (i_1, \dots, i_r)$ is a partition of k and $\text{Sq}^I := \text{Sq}^{i_1} \text{Sq}^{i_2} \cdots \text{Sq}^{i_r}$ then it is clear that the elements Sq^I , where I runs over all partitions such that $i_{m-1} \geq 2i_m$, form a basis of the degree k part of $\mathfrak{A}_2$.

Definition 5. Let $I = (i_1, \dots, i_r)$ be a partition of k some nonnegative integer. We say I is

- Admissible, if $i_{m-1} \geq 2i_m$ for all $m < r$,
- dyadic, if all i_m are of the form $2^n - 1$, and
- nondyadic, if *none* of the i_m are of the form $2^n - 1$.

The number of admissible partitions is the same as the number of dyadic partitions (see [Weston, Lemma 8.1]). The number of partitions of k will be denoted by $p(k)$, the number of nondyadic partitions by $d(k)$, and the number of dyadic partitions by $c(k)$.

With this we conclude that $\dim_{\mathbf{Z}/2} \mathfrak{A}_2^m = c(m)$.

Now we can introduce Stiefel-Whitney classes.

Definition 6. Let ξ be a rank k vector bundle with total space E and base space B . Then the i th Stiefel-Whitney class associated to ξ , denoted by $W_i(\xi)$ is defined as

$$W_i(\xi) = \phi^{-1} \text{Sq}^i \phi(1) = \phi^{-1} \text{Sq}^i(u).$$

where ϕ is the Thom isomorphism defined in Definition 2 and u the Thom class in $H^k(E, E_0)$. In other words, we push the identity of the cohomology ring of B along the following composite, and where we land we define as $W_i(\xi)$,

$$H^0(B) \xrightarrow{\phi} H^k(E, E_0) \xrightarrow{\text{Sq}^i} H^{k+i}(E, E_0) \xrightarrow{\phi^{-1}} H^i(B) .$$

Similarly define the total Stiefel-Whitney class as a formal sum of the Stiefel-Whitney classes, i.e.

$$W(\xi) = 1 + W_1(\xi) + \cdots + W_k(\xi).$$

for a k -vector bundle. Then we have that

$$W(\xi) = \phi^{-1}\text{Sq}(\phi(1)) = \phi^{-1}\text{Sq}(u).$$

It's here that we need to introduce the Steenrod algebra and some facts about it.

Theorem 6. The Stiefel-Whitney classes $W_i(\xi)$ associated to a rank k bundle $\xi \rightarrow B$ have the following properties,

- (1) For each $i = 0, 1, 2, \dots$ there corresponds a Stiefel-Whitney class $W_i(\xi) \in H^i(B)$ where $W_0(\xi)$ is equal to the unit element $1 \in H^0(B)$ and $W_i(\xi) = 0$ for all $i > k$.
- (2) If $f : B(\xi) \rightarrow B(\xi')$ is covered by a bundle map from ξ to ξ' , then

$$f^*(W_i(\xi')) = W_i(\xi).$$

- (3) If ξ and ξ' are bundles over the same base space, then

$$W_i(\xi \oplus \xi') = \sum_{j=0}^i W_j(\xi) \smile W_{i-j}(\xi').$$

- (4) For the line bundle γ_1^1 over the real projective line $\mathbf{RP}^1$, the Stiefel-Whitney class $W_1(\gamma_1^1)$ is nonzero.

Proof. (1) That $W_i(\xi) \in H^i(B)$ and $W_0(\xi) = 1$ follows by definition (see properties (1) and (3) of Steenrod squares). Moreover by property (3) we get that $W_i(\xi) = 0$ for $i > k$.

(2) Given bundle maps send fibers to fibers isomorphically, we have that a bundle map $f : \xi \rightarrow \xi'$ induces a map on pairs

$$g : (E, E_0) \rightarrow (E', E'_0).$$

If we let $u' \in H^k(E', E'_0)$ be the associated Thom class, then by definition of Thom class we get $g^*(u') = u \in H^k(E, E_0)$. Given this, and after a fairly easy chase of definitions, we get that the Thom isomorphism ϕ and ϕ' satisfy the following relation,

$$g^* \circ \phi' = \phi \circ \overline{f}^*$$

where $\bar{f}$ is the map on base spaces covered by f . Finally, by property (3) of the Steenrod squares we get

$$\begin{aligned}\phi \bar{f}^*(W_i(\xi')) &= g^* \phi'(W_i(\xi')) \\ &= g^* \text{Sq}^i(\phi'(1)) \\ &= \text{Sq}^i g^*(u') \\ &= \text{Sq}^i u.\end{aligned}$$

Therefore, we have

$$\bar{f}^*(W_i(\xi')) = \phi^{-1} \text{Sq}^i \phi(1) = W_i(\xi).$$

- (3) To prove this we compute the Stiefel-Whitney classes of a product bundle and then we pullback along the diagonal map. To be precise, let ξ, ξ' be bundles of rank m, n over B, B' respectively. Consider $\xi \times \xi'$ with projection map $\pi \times \pi' : E \times E' \rightarrow B \times B'$. If we let u, u' be the respective Thom classes, I claim that the Thom class of the product bundle is $u \times u'$ where we are taking the cross product in the latter. In other words, given $E_0 \times E'$ and $E \times E'_0$ are open in the product space $E \times E'$, and because $E_0 \times E' \cup E \times E'_0$ is exactly the complement of the zero section in $E \times E'$ we get that

$$u \times u' \in H^{m+n}(E \times E', E_0 \times E' \cup E \times E'_0)$$

makes sense. To show $u \times u'$ is the Thom class of the product bundle it suffices to show that this class restricts to the unique nontrivial class in $H^{m+n}(F'', F''_0)$ where $F'' := F \times F'$, F, F' are the fibers in the respective bundles and the 0 subscript means the complement of 0. But this follows from the fact that the restriction of $u \times u'$ is the cross product of the restrictions of u, u' , which cannot be zero.

Then we have that the Thom isomorphism ϕ'' for the product bundle behaves as follows,

$$\phi''(a \times b) = \phi(a) \times \phi'(b).$$

This relation follows in part from the following: if $\bar{a} = \pi^*(a) \in H^*(E)$ and $\bar{b} = \pi'^*(a) \in H^*(E')$ then,

$$(\bar{a} \times \bar{b}) \smile (u \times u') = (\bar{a} \smile u) \times (\bar{b} \smile u')$$

(a chase of definitions) where there is no sign because we are working over $\mathbf{Z}/2$ coefficients. So we can compute the total Stiefel-Whitney class of the product bundle as follows,

$$\phi''(W(\xi \times \xi')) = \text{Sq}(u \times u') = \text{Sq}(u) \times \text{Sq}(u').$$

From which we get,

$$\phi''(W(\xi \times \xi')) = \phi(W(\xi)) \times \phi'(W(\xi')) = \phi''(W(\xi)) \times (W(\xi')).$$

Applying ϕ''^{-1} we get $W(\xi) \times W(\xi') = W(\xi \times \xi')$. Finally, if we let ξ, ξ' be bundles over the same base space B , we can apply the pullback under the diagonal map $d : B \rightarrow B \times B$ to $W(\xi) \times W(\xi') = W(\xi \times \xi')$ and we immediately get Whitney's formula (by definition of Whitney sum).

- (4) Let γ_1^1 be the tautological line bundle over P^1 , then we know that the set of all vectors in the fibers of length at most one is homeomorphic to the Möbius band M , where we denote its boundary circle ∂M . Therefore

$$H^*(M, \partial M) \cong H^*(E, E_0)$$

where $E = E(\gamma_1^1)$ is the total space of the bundle. Moreover, given the real projective plane P^2 is homeomorphic to a 2-cell attached to the boundary circle of the Möbius band we get that the closure of $P^2 - D^2$ (where D^2 is an embedded 2-cell) is homeomorphic to M . Thus we get the following sequence of isomorphisms by excision and the long exact sequence of the pair,

$$H^i(E, E_0) \rightarrow H^i(M, \partial M) \leftarrow H^i(P^2, D^2) \rightarrow H^i(P^2)$$

for all $i \neq 0$. Note that the Thom class $u \in H^1(E, E_0)$ cannot be 0 and so under the isomorphism it must map to $a \in H^1(P^2)$. Therefore, $Sq^1 u = u \smile u$ must correspond to $Sq^1 a = a \smile a$ which cannot be zero by Theorem 3. It follows then that

$$W_1(\gamma_1^1) = \phi^{-1} Sq^1(u)$$

cannot be zero either as required. ■

Now we can show the following result which will be important in the sequel.

Theorem 7. The cohomology ring $H^*(G_n)$ of $G_n = G_n(\mathbf{R}^\infty)$ is isomorphic, as a $\mathbf{Z}/2$ algebra, to $\mathbf{Z}/2[W_1(\gamma^n), \dots, W_n(\gamma^n)]$, where γ^n is the universal bundle over G_n .

To prove this theorem we first gather some facts.

Lemma 1. The total Stiefel-Whitney class of the canonical line bundle γ_n^1 over P^n is given by

$$W(\gamma_n^1) = 1 + a$$

where $a \in H^1(P^n)$ is as in Theorem 3.

Proof. Note that the canonical inclusion $j : P^1 \rightarrow P^n$ is covered by a bundle map $\gamma_1^1 \rightarrow \gamma_n^1$. Thus by property (2) of Stiefel-Whitney classes we get

$$j^*(W_1(\gamma_n^1)) = W_1(\gamma_1^1).$$

But the right hand side is nonzero by property (4), so $W_1(\gamma_n^1)$ cannot be zero either. Therefore, by Theorem 3, it must equal a . Given the remaining Stiefel-Whitney classes are 0 by property (1) we are done. ■

Lemma 2. The classes $W_i(\gamma^n)$ are algebraically independent over $\mathbf{Z}/2$ in $H^*(G_n)$, i.e. there is no nonzero polynomial p in n variables with coefficients in $\mathbf{Z}/2$ such that $p(W_1(\gamma^n), \dots, W_n(\gamma^n)) = 0$.

Proof. Assume otherwise, i.e. that there is such a polynomial p . Then for any n -bundle $\xi \rightarrow B$ over a paracompact base space we have that there is a classifying map $g : B \rightarrow G_n$ such that by property (2) of Stiefel-Whitney classes,

$$g^*(W_i(\gamma^n)) = W_i(\xi).$$

So we get that the $W_i(\xi)$ satisfy

$$p(W_1(\xi), \dots, W_n(\xi)) = g^*(p(W_1(\gamma^n), \dots, W_n(\gamma^n))) = 0.$$

Thus, to get a contradiction, it suffices to find a bundle ξ as above, so that its Stiefel-Whitney classes satisfy no polynomial relation. Consider then γ^1 the canonical line bundle over P^∞ , the infinite real projective space. Let $X = P^\infty \times \dots \times P^\infty$ and consider the n plane bundle

$$\xi = \gamma^1 \times \dots \times \gamma^1$$

over X . As before, let $H^*(P^\infty) = \mathbf{Z}/2[a]$ where a is of cohomological dimension 1. Moreover, let $p_i : X \rightarrow P^\infty$ be the i th canonical projection, then we define $a_i = p_i^*(a)$. By the Künneth theorem we have,

$$H^*(X) \cong \mathbf{Z}/2[a_1, \dots, a_n].$$

By the proof of property (4) of Theorem 6 we know,

$$W(\xi) = W(\gamma^1) \times \dots \times W(\gamma^1) = p_1^*(W(\gamma^1)) \smile \dots \smile p_n^*(W(\gamma^1)).$$

(where the second equation we get by the definition of cross product). But we know that $W(\gamma^1) = 1 + a$ (by Lemma 1) so the above becomes,

$$W(\xi) = (1 + a) \cdots (1 + a) = (1 + a_1) \cdots (1 + a_n)$$

which gives,

$$\begin{aligned} W_1(\xi) &= a_1 + \dots + a_n, \\ W_2(\xi) &= a_1 a_2 + a_1 a_3 + \dots + a_1 a_n + \dots + a_{n-1} a_n, \\ &\dots \\ W_n(\xi) &= a_1 \cdots a_n. \end{aligned}$$

In other words, we get that the $W_i(\xi)$ are the symmetric polynomials in $H^*(X)$, but we know that the subalgebra generated by the symmetric polynomials in $\mathbf{Z}/2[a_1, \dots, a_n]$ is a free algebra and so we get that there is no polynomial relation among the $W_i(\xi)$. ■

Proof of Theorem 7. By the previous lemma we know that $H^*(G_n)$ contains a polynomial algebra in the $W_i(\gamma^n)$. We will show these two must coincide by computing dimensions as $\mathbf{Z}/2$ vector spaces of the graded parts of $H^*(G_n)$. By the cell structure of G_n given Theorem 4, we know G_n has at most m cells of dimension r where m is the total number of partitions of r into n parts. Therefore we get that $H^r(G_n)$ has rank at most m .

It is clear that a basis of the r th part of $\mathbf{Z}/2[W_1(\gamma^n), \dots, W_n(\gamma^n)]$ in $H^*(G_n)$ is $W_1(\gamma^n)^{r_1} \dots W_n(\gamma^n)^{r_n}$ where $r_1 + 2r_2 + \dots + nr_n = r$. But notice that the size of this basis is m (the number of partitions of r into at most n parts). This is because to any sequence $(r_1, \dots, r_n)$ we can associate the partition,

$$r_n, r_n + r_{n-1}, \dots, r_n + r_{n-1} + \dots + r_1$$

ignoring any zeros that might appear. Therefore, the dimension of $H^r(G_n)$ is m by Lemma 2 and we are done as $\mathbf{Z}/2[W_1(\gamma^n), \dots, W_n(\gamma^n)]$ must be the whole of $H^*(G_n)$. ■

In what follows we will also make use of the following well-known theorem, see [Thom, p.37] and [Milnor-Stasheff, §8, Problem 8-A].

Theorem 8 (Wu's Formula). If ξ is a vector bundle and $W_i = W_i(\xi)$ its associated Stiefel-Whitney classes then

$$\text{Sq}^r(W_i) = \sum_t \binom{r-i+t-1}{t} W_{r-t} \cdot W_{i+t}.$$

Definition 7. Let M be an n -manifold, μ_M its fundamental homology class, and τ_M its tangent bundle. Then the numbers mod 2

$$\langle W_1(\tau_M)^{r_1} \dots W_n(\tau_M)^{r_n}, \mu_M \rangle$$

are its *Stiefel-Whitney numbers*. Note that we will usually drop the τ_M from our notation.

Now our study of cobordism begins in earnest! The following is the first theorem on cobordism theory of this essay. It says that if a manifold is a boundary then its Stiefel-Whitney numbers are all 0.

Theorem 9 (Pontrjagin's Theorem). If B is a smooth compact $n+1$ manifold with boundary M , then M has all its Stiefel-Whitney numbers equal to zero.

Proof. Denote the fundamental homology class of the pair

$$\mu_B \in H_{n+1}(B, M)$$

and the natural boundary homomorphism in the long exact sequence of the pair by

$$d_* : H_{n+1}(B, M) \rightarrow H_n(M).$$

Then it is clear that d_* sends μ_B to the fundamental homology class of M , call it μ_M . In other words, if $d^* : H^n(M) \rightarrow H^{n+1}(B, M)$ is the natural boundary map of the long exact sequence of the pair in cohomology, we have

$$\langle a, d_*\mu_B \rangle = \langle d^*a, \mu_B \rangle$$

for $a \in H^n(M)$ (no signs because our coefficients are $\mathbf{Z}/2$).

Consider the tangent bundle $\tau_B|_M$ of B restricted to M and consider the subbundle τ_M . Then choosing a Euclidean metric on τ_B we can find a unique outwards normal vector field (remember the definition to manifold with boundary) to M spanning ϵ^1 (the trivial bundle of rank 1). In other words,

$$\tau_B|_M \cong \tau_M \oplus \epsilon^1.$$

Which means that the Stiefel-Whitney classes of B , restricted to M , are the same as the Stiefel-Whitney classes of M (this is because trivial bundles have null Stiefel-Whitney classes in every degree except 0). Using the exact sequence,

$$H^n(B) \xrightarrow{i^*} H^n(M) \xrightarrow{d^*} H^{n+1}(B, M)$$

we get

$$d^*(W_1^{r_1} \cdots W_n^{r_n}) = 0$$

where the W_i are the Stiefel Whitney classes associate to τ_M . And so,

$$\begin{aligned} \langle W_1^{r_1} \cdots W_n^{r_n}, \mu_M \rangle &= \langle W_1^{r_1} \cdots W_n^{r_n}, d_*\mu_B \rangle \\ &= \langle d^*W_1^{r_1} \cdots W_n^{r_n}, \mu_B \rangle \\ &= 0 \end{aligned}$$

Therefore, all Stiefel-Whitney numbers of M are zero. ■

Pontrjagin's theorem motivates the question: is the converse true? Namely, is it true that if a manifold has all its Stiefel-Whitney numbers equal to 0 then it has to be a boundary? The answer is *yes*, and proving this is our next big goal!

4. THE PONTRJAGIN-THOM CONSTRUCTION

In this section we prove the Pontrjagin-Thom construction (Theorem 11), which lets us translate the cobordism problem into a homotopy one. For a reminder about transversality and regular values see [Milnor-Stasheff, pp.208-210].

We first introduce some lemmas that we will need for the proof of the Pontrjagin-Thom construction.

Let $f : M \rightarrow N$ be a smooth map and $Y \in N$ a submanifold of co-dimension k such that f is transverse to Y .

Lemma 3. If ν^k is the normal bundle of $Y \in N$, then the normal bundle of the manifold $f^{-1}(Y)$ can be naturally identified with $f^*\nu^k$, the pullback under map $f : M \rightarrow N$.

Proof. Let ν' be the normal bundle of $f^{-1}(Y)$. Chasing definitions we get that,

$$\begin{aligned} f^*\nu^k &= \{(x, f(x), v) \mid x \in f^{-1}(Y), v \in \nu^k|_{f(x)}\} \\ \nu' &= \{(x, w) \mid x \in f^{-1}(Y), w \in Df^{-1}(Y)_x^\perp\}. \end{aligned}$$

Then there is a clear bundle map given by $(x, w) \mapsto (x, f(x), Df_x(w))$ that, as is easily checked, sends the fibers over a point isomorphically to the corresponding fiber. ■

Let $W \subset \mathbf{R}^m$ be an open subset, K a compact subset of W , $X \subset W$ a relatively closed subset, and let $f : W \rightarrow \mathbf{R}^k$ be a smooth map such that 0 is a regular value of f throughout X . The next lemma says that we can ‘enlarge’ the set of points throughout which 0 is a regular value at the small cost of disturbing f just a little bit.

Lemma 4. Let $f : W \rightarrow \mathbf{R}^k$ be as above. Then we can find a smooth map $g : W \rightarrow \mathbf{R}^k$ that coincides with f outside some compact set and that has 0 as regular value throughout $X \cup K$. Moreover, we may choose g uniformly close to f , that is, for an $\epsilon > 0$ we may choose g such that $|f(x) - g(x)| < \epsilon$.

Proof. Given W is open and K compact it is not difficult to cook up a smooth map $\lambda : W \rightarrow [0, 1]$, using a partition of unity, that is 1 on a neighborhood of K and vanishes outside some larger compact set K' . Assume y is a regular value of f such that $|y| < \epsilon$. Then if we define,

$$g(x) = f(x) - \lambda(x)y$$

we get that

- (a): the map g has 0 as regular value throughout K ,
- (b): g coincides with f outside of K' , and
- (c): $|f(x) - g(x)| < \epsilon$.

Items (b) and (c) are clear. To see (a) note simply that in K the map $g(x)$ equals $f(x) - y$ and so 0 is regular value of g throughout K because y is a regular value of f (we have just shifted f by y to obtain g essentially).

Observe that by Brown’s Theorem (reference) we have that we may choose y arbitrarily close to 0, and so we can make g uniformly close to f . This means that the partial derivatives $\delta f_j / \delta x_i$ are uniformly close to the partial derivatives $\delta g_j / \delta x_i$. And so, given 0 is a regular value of f throughout $K' \cap X$ (because it is a regular value throughout X), we get that 0 is a regular value of g throughout $K' \cap X$. Finally, given g has 0 as regular value throughout K , by (a), we get that 0 is a regular value of g throughout $K \cup X$. (Why? Because $f = g$ outside K' , by (b), and so g has 0 as regular value throughout $X \setminus K'$.) ■

Definition 8. Let ξ be a bundle over a paracompact base space B . Then (see [Milnor-Stasheff, §§5, 6]) we can attach a Euclidean metric on E . Let A be the set of all vectors in E of length ≥ 1 , then we set $T := T(\xi)$ to be the quotient E/A . We

call t_0 the point in the quotient corresponding to A/A . Note that the complement $T - t_0$ corresponds to all vectors in E of length < 1 .

In the particular case when $\xi = \gamma^k$, we denote $T\gamma^k$ by $MO(k)$ (for why it is called this way, see [Thom, Chapter II]).

From the Thom Isomorphism Theorem (Theorem 2) and the above definition it is easy to deduce the following.

Lemma 5. If ξ is a k -plane vector bundle over B , then

$$H^{k+i}(T\xi, t_0) \cong H^i(B).$$

Recall that we want to reduce our problem from a geometric one to an algebraic one, more precisely, to find a way of thinking about the cobordism groups through the homotopy groups of $MO(k)$. To do this we will want to approximate maps $f : S^m \rightarrow T(\xi)$ (from spheres to the Thom space of vector bundles) via ‘nice’ maps. By ‘nice’, we would like to say ‘smooth’, but we must be careful for even if we assume that the bundle ξ is smooth, we may not have that $T(\xi)$ is smooth. However, if we let $\xi \rightarrow B$ be a smooth vector bundle of rank k , by using techniques found in [Steenrod, §6.7], we can find a map $f_0 : S^m \rightarrow T(\xi)$, homotopic to f , such that it coincides with f on $f^{-1}(t_0) = f_0^{-1}(t_0)$ and that is smooth throughout $f_0^{-1}(T - t_0)$ where $T = T(\xi)$.

With this we can prove the following theorem.

Theorem 10. For any map $f : S^m \rightarrow T(\xi)$ there is a homotopic map $g : S^m \rightarrow T(\xi)$ which is smooth on $g^{-1}(T - t_0)$ and transverse to B (construed in $T(\xi)$ via the zero-section). Moreover, the cobordism class of the $m - k$ dimensional manifold $g^{-1}(B)$ depends only on the homotopy type of the map g . Therefore, there is a well-defined homomorphism

$$[g] \mapsto [g^{-1}(B)]$$

from $\pi_m(T, t_0)$ to the cobordism group $\mathfrak{N}_{m-k}$.

Proof. As explained above, we can replace f by f_0 , where the latter is homotopic to f , smooth throughout $f_0^{-1}(T - t_0)$ and coincides with f on $f^{-1}(t_0)$. Given $f_0^{-1}(B)$ is compact (it is closed inside of a compact space), we can choose a cover $O_1, \dots, O_r$ of opens in $f_0^{-1}(T - t_0)$ such that $f_0(O_i)$ (which is $\subset T - t_0 \subset E(\xi)$) is small enough to be contained in

$$\pi^{-1}(U_i) \cong U_i \times \mathbf{R}^k$$

where $\pi : \xi \rightarrow B$ is the bundle map and $U_i \subset B$ is small enough so as to be contained in a trivialising open. Let $K_i \subset O_i$ be a compact subset such that $f_0^{-1}(B) \subset K_1 \cup \dots \cup K_r$. With this setup we will try to construct maps $f_1, \dots, f_r$ in succession, by focusing on what is happening in each O_i , such that each will have the following three properties,

- (a): f_i is smooth throughout $f_0^{-1}(T - t_0)$, and coincides with f_{i-1} outside of a compact subset of O_i ,
- (b): f_i is transverse to B throughout $K_1 \cup \dots \cup K_i$, and

(c): $\pi(f_i(x)) = \pi(f_0(x))$ for all $x \in f_0^{-1}(T - t_0)$.

For $i = 0$ the three conditions are satisfied, (a) and (c) by definition of f_0 and (b) vacuously. So assume, inductively, that we have constructed f_{i-1} . By (a), we want to construct a map $x \mapsto f_i(x)$ in O_i . By (c), we have that f_i must map O_i to the product coordinate patch $\pi^{-1}(U_i) \cong U_i \times \mathbf{R}^k$ ‘in the same way’ as f_0 does. In other words, the first coordinate of f_i is determined by condition (c). So we need only figure out what is the second coordinate. If we let $\rho_i : U_i \times \mathbf{R}^k \rightarrow \mathbf{R}^k$ be the canonical projection onto the second factor, then what we want is to determine the map $x \mapsto \rho_i(f_i(x))$.

Note that, by (b), the map $\rho_i \circ f_{i-1}$ from O_i to $\mathbf{R}^k$ has zero as regular value throughout the relatively closed set $(K_1 \cup \dots \cup K_{i-1}) \cap O_i$. Then by Lemma 4 we can enlarge this set, i.e. we can find a smooth map which

- (i): coincides with $\rho_i \circ f_{i-1}$ outside of a compact set in O_i and
- (ii): has the origin as regular value throughout $(K_1 \cup \dots \cup K_{i-1} \cup K_i) \cap O_i$.

(Note that by the lemma we also get to choose this map uniformly close to $\rho_i \circ f_{i-1}$, we will use this fact a bit later). Then we may define $\rho_i \circ f_i$ as the resulting smooth map. By (a) and (c) of f_{i-1} , we have also defined f_i for all other x . Moreover by (ii) we get that condition (b) is also satisfied by f_i .

Therefore, we may proceed inductively defining $f_1, \dots, f_r$. Take $g = f_r$, then we have that g is transverse to B throughout $K_1 \cup \dots \cup K_r$. So the only thing we would be missing to prove to show is that $g^{-1}(B)$ is a subset of $K_1 \cup \dots \cup K_r$. To do this we recur to the only conclusion of Lemma 4 that we did not use, namely, that we could choose the $\rho_i \circ f_i$ uniformly close to the previous iteration.

Let $t \in T - t_0 \cong E - A$. Then taking the Euclidean norm on the fibers we may say that $0 < |t| < 1$. Observe then that, given $f_0^{-1}(B) \subset K_1 \cup \dots \cup K_r$ and B is construed as the zero-section in T , we get that there must exist a positive constant c such that

$$|f_0(x)| > c$$

for all $x \notin K_1 \cup \dots \cup K_r$. Thus, if we choose each f_i such that

$$|f_{i-1}(x) - f_i(x)| < \frac{c}{r}$$

then we have that

$$|g(x) - f_0(x)| < c.$$

And so $|g(x)| > 0$ for $x \notin K_1 \cup \dots \cup K_r$ which means that $g^{-1}(B) \subset K_1 \cup \dots \cup K_r$. Therefore, g is transverse to B and $g^{-1}(B)$ is a smooth, compact, $m - k$ dimensional manifold. This proves the first claim of the theorem.

Next, consider homotopic $g, g' : S^m \rightarrow T$ smooth along the preimage of $T - t_0$, and transverse along B . Then by the same techniques quoted in the preamble to the theorem statement (see [Steenrod §6.7]) we get that we can construct a smooth homotopy $h_0 : S^m \times [0, 3] \rightarrow T$ between g, g' , such that h_0 is smooth throughout

$h_0^{-1}(T - t_0)$ and such that,

$$\begin{aligned} h_0(x, t) &= g(x), \quad \forall x \in [0, 1] \\ h_0(x, t) &= g'(x), \quad \forall x \in [2, 3]. \end{aligned}$$

As in the first part of this proof, (think of h_0 as f_0) we can construct an $h : S^m \times [0, 3] \rightarrow T$ that it coincides with h_0 outside of a compact set in $S^m \times (0, 3)$ and is transverse to B . The construction of h is inductive making sure that transversality throughout $S^m \times [0, 1] \cup S^m \times [2, 3]$ is preserved. Therefore, we get that $h^{-1}(B)$ gives us a cobordism between $g^{-1}(B)$ and $g'^{-1}(B)$. Thus we get that the function

$$[g] \mapsto [g^{-1}(B)]$$

is well-defined. To see this is a homomorphism of abelian groups, observe that the composition operation in $\pi_m(T, t_0)$ naturally corresponds to the disjoint union operation in $\mathfrak{N}_{m-k}$ by construing the composition operation as follows,

$$\begin{array}{ccccc} & & S^m & & \\ & & \searrow g & & \\ S^m & \longrightarrow & \vee & \longrightarrow & T \\ & & \nearrow g' & & \\ & & S^m & & \end{array}$$

where the horizontal map is ‘crush the equator’ and the diagonal maps represent the wedge product of maps. Therefore, we have a well-defined homomorphism as claimed. $\blacksquare$

Now we can state the main theorem of this section, namely, the Pontrjagin-Thom construction.

Theorem 11 (Pontrjagin-Thom Construction). For $k > n + 1$ the homotopy group $\pi_{n+k}(MO(k))$ is canonically isomorphic to the cobordism group $\mathfrak{N}_n$.

For the proof we will need the following lemma.

Lemma 6. If $k \geq n$ and $p \geq n$ then the homomorphism of abelian groups

$$\begin{aligned} \pi_{n+k}(T\gamma_p^k) &\rightarrow \mathfrak{N}_n \\ [g] &\mapsto [g^{-1}(G_k(\mathbf{R}^{k+p}))] \end{aligned}$$

of Theorem 10 is surjective.

Proof. Let M^n be a smooth, compact, n -dimensional manifold. By Whitney’s Embedding Theorem we may embed M^n into $\mathbf{R}^{n+k}$. Consider the normal bundle of M^n

as a submanifold of $\mathbf{R}^{n+k}$, call it ν^k , and let U be a tubular neighborhood of M^n . We will work with the following composite

$$U \cong E(\nu^k) \xrightarrow{g} E(\gamma_n^k) \subset E(\gamma_p^k) \twoheadrightarrow T\gamma_p^k$$

where $U \cong E(\nu^k)$ is given by Tubular Neighborhoods, the inclusion $E(\gamma_n^k) \subset E(\gamma_p^k)$ is induced by $\mathbf{R}^{k+n} \hookrightarrow \mathbf{R}^{k+p}$ (including the former in the first $k+n$ variables of the latter), the rightmost map is the canonical projection, and g is given by

$$\begin{aligned} g : E(\nu^k) &\rightarrow E(\gamma_n^k) \\ (x, v) &\mapsto (DM_x^\perp, v). \end{aligned}$$

I claim that g is transverse to $B := G_k(\mathbf{R}^{k+n})$ construed as the zero section in $E(\gamma_n^k)$. The first thing to note is that g is a bundle map covering $\bar{g} : M^n \rightarrow B$. The map g is transverse to B if

$$DE(\nu^k)_{(x,0)} \xrightarrow{Dg_{(x,0)}} DE(\gamma_n^k)_{(x,DM_x^\perp)} \quad (1)$$

surjects onto the complement of $DB_{DM_x^\perp}$ in the codomain. But given bundles are locally product spaces (1) looks like the following commutative diagram,

$$\begin{array}{ccc} DM_x \times DM_x^\perp & \xrightarrow{Dg_{(x,0)}} & DB_{DM_x^\perp} \times DM_x^\perp \\ & \searrow & \swarrow \\ & DM_x^\perp & \end{array}.$$

We get commutativity from the fact that g can be construed as a projection that “kills n dimensions” by comparing with the commutative diagram we get from the pullback of $\bar{g}$,

$$\begin{array}{ccccc} E(\nu^k) & \cong & \bar{g}^* \gamma_n^k & \xrightarrow{p_2} & E(\gamma_n^k) \\ & \searrow & \downarrow p_1 & & \downarrow \\ & & M^n & \xrightarrow{\bar{g}} & B \end{array}$$

where p_i is the projection into the i th factor and recall $\bar{g}^* \gamma_n^k = \{(x, DM_x^\perp, v) \in M^n \times E(\gamma_n^k)\}$. Therefore we get that g is transverse to B , and so we get that the composite above, (call it g as well), $g : U \rightarrow T\gamma_n^k$ is transverse to B . Which means that $g^{-1}(B)$ is a manifold (equal to M^n by construction). Finally extend g to the one point compactification $\mathbf{R}^{k+n} \cup \infty \cong S^{k+n}$ by sending $S^{k+n} - U$ to the point t_0 . Then clearly, the resultant g maps to M^n under the homomorphism of Theorem 10. $\blacksquare$

Proof of Theorem 11. Lemma 6 is not quite what we want, because we would like the surjectivity of a map from $\pi_{n+k}(MO(k))$ to $\mathfrak{N}_n$ as claimed in Theorem 11. To remedy this we can compose the map $g : S^{n+k} \rightarrow T\gamma_p^k$ of Theorem 10 with the

inclusion $T\gamma_p^k \subset T\gamma^k = MO(k)$ induced by the map $\mathbf{R}^{k+p} \subset \mathbf{R}^\infty$ that includes the former space into the first $k+p$ coordinates of the latter, the rest being 0. This gives us that the composite $g : S^{n+k} \rightarrow T\gamma_p^k \subset T\gamma^k = MO(k)$ represents a homotopy class in $\pi_{n+k}(MO(k))$ and, by construction, the map of Theorem 10 induces a homomorphism

$$\begin{aligned} \pi_{n+k}(MO(k)) &\rightarrow \mathfrak{N}_n \\ [g] &\mapsto [g^{-1}(G_k)] \end{aligned}$$

that by Lemma 6 must be surjective. So to conclude the proof of Theorem 11 we must show injectivity of the above map. To do this we will show that there is a well-defined inverse function, where by Lemma 6 this reduces to showing that the map insinuated in the proof of the lemma $[M^n] \mapsto [g]$ is well-defined (here we are using the notation of the lemma). The latter boils down to showing that a boundary is always sent to a map homotopic to a constant map. Consider then, a manifold $M = \partial W$, where we are considering M embedded in the sphere S^{n+k} . By theorems of differential topology (see Hirsch, Theorems 4.3 and 6.4) we can construct the map g such that it can be extended to a map from the disc D^{n+k+1} containing W , and such that any tubular neighborhood of M in S^{n+k} is the intersection of the sphere with a tubular neighborhood of W in D^{n+k+1} . In other words, we get a commutative diagram,

$$\begin{array}{ccccc} S^{n+k} & \longrightarrow & T\nu_M^k & \longrightarrow & MO(k) \\ \downarrow & & \downarrow & & \parallel \\ D^{n+k+1} & \longrightarrow & T\nu_W^k & \longrightarrow & MO(k) \end{array}$$

where the top row is the map g (ν_M^k is the normal bundle of M in the sphere and ν_W^k is the normal bundle of W in the disc). So, given we have an extension from a map on the sphere to a map on the disc, we get g must be null-homotopic as required. This concludes the proof of Theorem 11. $\blacksquare$

5. PROOFS OF THE MAIN THEOREMS

As advertised, in this section we will prove the main structure theorems stated in the introduction, together with the converse to Pontrjagin's Theorem.

We will need the following technical statement that can be found in [Hatcher, p. 393, Theorem 4.57].

Theorem 12. There are natural bijections $T : [X, K(G, n)] \rightarrow H^n(X, G)$ for all CW complexes X and all $n > 0$, with G any abelian group. Such a T has the form $T([f]) = f^*(\iota)$ for a certain distinguished class ι in $H^n(K(G, n); G)$ which we call the *fundamental class*.

For reminders about Eilenberg-MacLane spaces see [Thom, pp. 37-38] or [Hatcher, pp. 365-366].

The proof of the first structure theorem is a consequence of the following technical statement.

Theorem 13. The space $MO(k)$ has the same $(2k-1)$ -homotopy type as the product,

$$Y = K(\mathbf{Z}/2, k) \times K(\mathbf{Z}/2, k+2) \times \cdots \\ \cdots \times (K(\mathbf{Z}/2, k+h))^{d(h)} \times \cdots \times (K(\mathbf{Z}/2, 2k))^{d(k)}.$$

By definition of Eilenberg-MacLane space we immediately get the following,

Corollary 1. The homotopy group $\pi_{k+h}(MO(k))$ for $h < k$ is isomorphic to the direct sum of $d(h)$ copies of $\mathbf{Z}/2$.

From these two statements it is easy to deduce the first structure theorem.

Theorem 14. For every dimension k , the k th cobordism group $\mathfrak{N}_k$ is isomorphic to the direct sum of $d(k)$ copies of $\mathbf{Z}/2$.

Proof. By the Pontrjagin-Thom construction we get,

$$\mathfrak{N}_k \cong \pi_{n+k}(MO(n)) \quad k < n$$

and, by the preceding corollary, we get the latter is

$$\pi_{n+k}(MO(n)) \cong (\mathbf{Z}/2)^{d(k)}.$$

■

Now we proceed with the proof of Theorem 13 which we divide in several lemmas.

Lemma 7. Every linear combination of iterated Sq^i 's, of total degree $h \leq k$, which is zero on the top Stiefel-Whitney class $W_k \in H^k(G_k; \mathbf{Z}/2)$, is identically zero.

Remark 1. We defer this proof for after the following definition and lemma.

By Wu's formula (Theorem 8) and Cartan's formula we get that all classes of the form $\text{Sq}^I(W_k)$, for I not necessarily an admissible partition, are $= W_k \cdot Q_I$ where $Q_I \in H^h(G_k)$ is a homogeneous polynomial in the classes W_i of degree h .

Definition 9. Let $\prec$ be an order relation on the set of monomials $W_k^{r_k} \cdots W_1^{r_1}$ defined as lexicographic order obtained from: $W_m \prec W_n$ if $m < n$. For instance, $W_5 \prec W_5 \cdot W_2 \prec W_5 \cdot (W_3)^2 \prec W_5 \cdot W_4 \prec (W_5)^2$.

Lemma 8. Let $\text{Sq}^I = \text{Sq}^{i_1} \text{Sq}^{i_2} \cdots \text{Sq}^{i_r}$ where $I = (i_1, \dots, i_r)$ is an admissible partition (i.e. $i_{m-1} \geq 2i_m$), and let $\text{Sq}^I W_k = W_k \cdot Q_I$. Then

$$Q_I = W_{i_1} W_{i_2} \cdots W_{i_r}$$

+ monomials strictly smaller than $W_{i_1} W_{i_2} \cdots W_{i_r}$ under $\prec$.

Proof. The proof is by induction on r . For $r = 1$ we get by Wu's formula,

$$\mathrm{Sq}^{i_1} W_k = W_k \cdot W_{i_1}$$

where $Q_{i_1} = W_{i_1}$ and so the statement holds vacuously. Now assume inductively that the statement holds for $r - 1$. We will establish it for r . Consider,

$$\mathrm{Sq}^{i_1}(\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_1} W_k) = \mathrm{Sq}^{i_1}(W_k \cdot P)$$

where by the inductive hypothesis we have that

$$\begin{aligned} P = & W_{i_2} \cdots W_{i_r} \\ & + \text{monomials strictly smaller than } W_{i_2} \cdots W_{i_r}. \end{aligned}$$

Applying Cartan's formula we get,

$$\mathrm{Sq}^{i_1}(W_k \cdot P) = \sum_{0 \leq j \leq i_1} \mathrm{Sq}^j(W_k) \mathrm{Sq}^{i_1-j}(P) = \sum_j W_k \cdot W_j \cdot \mathrm{Sq}^{i_1-j}(P).$$

Which means that $\mathrm{Sq}^I(W_k) = W_k \cdot Q_I$ where

$$Q_I = \sum_j W_j \cdot \mathrm{Sq}^{i_1-j}(P).$$

For $j = i_1$ we get the term

$$\begin{aligned} W_{i_1} \cdot P = & W_{i_1} W_{i_2} \cdots W_{i_r} \\ & + \text{strictly smaller monomials.} \end{aligned}$$

What we want to show then is that there is no monomial in the terms $W_j \cdot \mathrm{Sq}^{i_1-j} P$ for $j \neq i_1$ that is greater than or equal to $W_{i_1} W_{i_2} \cdots W_{i_r}$. And this we do by showing that there is no W_s in any monomial in $\mathrm{Sq}^{i_1-j} P$, for $j \neq i_1$, that is greater than or equal to W_{i_1} . Indeed, property (3) of the Steenrod squares and Wu's formula (Theorem 8), $\mathrm{Sq}^{i_1-j} W_s$ can have W_i 's of maximum index $2s$, and so $i \leq 2s < 2i_2 \leq i_1$. It follows that $\mathrm{Sq}^{i_1-j} P$ for $j \neq i_1$ can have to W_i of degree greater than or equal to i_1 , which proves the claim. $\blacksquare$

Proof of Lemma 7. This statement will follow once we show that the $\mathrm{Sq}^I(W_k)$ are linearly independent. From Lemma 8 we get that the $\mathrm{Sq}^I(W_k)$ where the I runs over all $c(h)$ admissible partitions of h are linearly independent in $H^{k+h}(G_k)$ construed as a $\mathbf{Z}/2$ vector space (recall that admissible partitions of h are in 1-1 correspondence with the dyadic partitions of h , and there are $c(h)$ of the latter). This is because if there were a linear relation among the $\mathrm{Sq}^I(W_k)$ then we could take the largest term for $\prec$ and conclude that we could express it as a linear combination of terms that are strictly smaller, a contradiction. $\blacksquare$

If we take the perspective of $H^*(G_k)$ as the $\mathbf{Z}/2$ algebra freely generated by the symmetric polynomials in the variables $t_1, \dots, t_k$, where W_i is identified with the i th symmetric polynomial (see Theorem 7), then Lemma 7 takes the following form.

Lemma 9. The classes $\text{Sq}^I(t_1 \dots t_k)$, where I runs along the admissible partitions of $h \leq k$, are linearly independent symmetric functions in the t_i .

We will also need the following lemma.

Lemma 10. There is a commutative diagram

$$\begin{array}{ccc} H^k(G_n) & \xrightarrow[\cong]{\phi^*} & H^{k+n}(MO(n)) \\ id \downarrow & & \downarrow \\ H^k(G_n) & \xrightarrow{\smile_{\omega_n}} & \langle \omega_n \rangle \cap H^{k+n}(G_n) \end{array}$$

where the dashed arrow is an isomorphism given by

$$\pi^*y \smile u \mapsto y \smile \omega_n$$

for $y \in H^k(G_n)$, and $\pi : \gamma^n \rightarrow G_n$.

Proof. By definition of Thom isomorphism, $\phi^*(y) = \pi^*y \smile u$, so the diagram commutes by construction. the bijectivity of the dashed arrow follows from its surjectivity and the dimensions, as $\mathbf{Z}/2$ vector spaces, of $H^{k+n}(MO(n))$ and $\langle \omega_n \rangle \cap H^{k+n}(G_n)$. Namely, they are both of dimension $p(k)$, the number of partitions of k . ■

Carrying on this identification of $H^*(G_k)$ with the $\mathbf{Z}/2$ algebra on the symmetric polynomials in the variables t_i , we get that $H^h(G_k)$ has a basis, as a $\mathbf{Z}/2$ vector space, given by

$$\sum t_1^{a_1} \dots t_r^{a_r} \quad (2)$$

where $\omega = (a_1, \dots, a_r)$ is a partition of h and where the sum is taken over all permutations of the $t_1, \dots, t_k$ that do not leave the monomial $t_1^{a_1} \dots t_r^{a_r}$ invariant. We will say that such a sum (over such essential permutations) is a symmetrisation sum, or that $\sum$ is a **symmetrisation sigma**. In other words, (2) is the ‘smallest’ (the one with the least number of terms) symmetric polynomial in the t_i containing $t_1^{a_1} \dots t_r^{a_r}$ as a term. A smallest set of permutations of the $t_1, \dots, t_k$ needed to produce this polynomial out of $t_1^{a_1} \dots t_r^{a_r}$ is called an **essential** set of permutations (I say ‘a smallest’ because there may be several such sets with the same cardinality). To be precise then, an inessential permutation is one which leaves the monomial $t_1^{a_1} \dots t_r^{a_r}$ invariant. With the above basis we can find a basis for $\langle W_k \rangle \cap H^{k+h}(G_k)$, namely, all monomials (2) cup W_k , the top Stiefel-Whitney class. This gives,

$$\sum t_1^{a_1+1} \dots t_r^{a_r+1} t_{r+1} \dots t_k \quad (3)$$

is a basis of $\langle W_k \rangle \cap H^{k+h}(G_k)$. To see this note that any essential permutation of (2) is an essential permutation of (3) and vice versa. Lastly, recall from Lemma 10 that $H^*(MO(k))$ can be identified with the ideal $\langle W_k \rangle$ in $H^*(G_k)$, so we have found a basis of $H^{k+h}(MO(k))$ as seen inside $H^{k+h}(G_k)$.

Definition 10. Let P be a polynomial in the variables t_i , we say that t_n is a **dyadic variable** of P if for all monomials in P the variable t_n appears with either a null exponent or an exponent of the form 2^m .

Lemma 11. If t_n is a dyadic variable P (polynomial in the t_i) then t_n is a dyadic variable of $\text{Sq}^i(P)$ where P is being construed as an element of $H^*(G_k)$.

Proof. It is not difficult to check that

$$\text{Sq}^i(t_n^m) = \binom{m}{i} t_n^{m+i}$$

(see Hatcher p. 490). However, if m is a power of 2, then the binomial coefficient is even, and so $\equiv 0 \pmod{2}$, for all values except when $i = 0$ or $i = m$, and in these cases $m + i$ is again a power of 2. $\blacksquare$

Definition 11. Name **nondyadic factor** of a monomial $t_1^{a_1} \dots t_r^{a_r}$ the monomial created by all the variables that are not dyadic, together with their exponents in $t_1^{a_1} \dots t_r^{a_r}$. Let u be the number of such variables and v the degree of the nondyadic factor. We introduce a partial order $\prec^1$ in the monomials on the variables t_i : the monomial Y is less than X , written $Y \prec X$, if $u(Y) < u(X)$, and if $u(Y) = u(X)$ then $Y \prec X$ if $v(X) < v(Y)$.

For $h \leq k$, consider the classes,

$$X_\omega^h := \sum t_1^{a_1+1} \dots t_r^{a_r+1} t_{r+1} \dots t_k \in H^{k+h}(G_k) \quad (4)$$

where $\omega = (a_1, \dots, a_r)$ runs along the nondyadic partitions of h and where the $\sum$ is symmetrisation sigma. The number of such partitions will be denoted by $d(h)$. For every dimension m such that $h \leq m \leq k$ consider the classes,

$$\{\text{Sq}^{I_h}(X_{\omega_h}^h)\}_{h \leq m}. \quad (5)$$

where I_h runs along all admissible partitions of $m - h$ and ω_h over all $d(h)$ nondyadic partitions of h . Note then that all the elements in (5) are elements of $H^{k+m}(G_k)$.

Lemma 12. The classes defined in (5) are linearly independent.

Proof. To start, I claim that in each of the expansions of $\text{Sq}^I(X_\omega^h)$, the monomials that are largest under the order $\prec$ defined in Definition 11 form the expansion of the following expression

$$\sum t_1^{a_1+1} \dots t_r^{a_r+1} \text{Sq}^I(t_{r+1} \dots t_k) \quad (6)$$

where the sum is taken over all essential permutations for the monomial (4) associated to the partition $\omega = (a_1, \dots, a_r)$. To show this recall,

$$X_\omega^h := \sum t_1^{a_1+1} \dots t_r^{a_r+1} t_{r+1} \dots t_k$$

¹We will not use the previous ordering relation defined in Definition 9 for the remainder of the essay, so reusing the same symbol should not cause confusion.

where the $\sum$ is symmetrisation sigma. So we consider

$$\mathrm{Sq}^I(t_1^{a_1+1} \cdots t_r^{a_r+1} t_{r+1} \cdots t_k) \quad (7)$$

where we observe that, by Lemma 11, this must be such that $u = r$ (because all variables $t_{r+1}, \dots, t_k$ are dyadic). So we can focus on the case $u = r$ as we are concerned with the largest monomials under $\prec$. For $u = r$ we have two cases: (i) either this monomial of (7) comes from the expansion of

$$t_1^{a_1+1} \cdots t_r^{a_r+1} \mathrm{Sq}^I(t_{r+1} \cdots t_k) \quad (8)$$

or (ii) it belongs to some term in the expansion of $t_1^{a_1+1} \cdots t_r^{a_r+1}$ by iterated Steenrod squares (this is because we are applying the only tool we know to expand (7), namely, Cartan's formula). In the first case, we see that $v = r + h = u + h$, while in the second case we get that $v > r + h = u + h$ (because of the formula quoted in the proof of Lemma 11). So we have that those monomials arising from the first case are strictly larger than those in the second case under $\prec$ (see Definition 11). And so those terms in the expansion of expressions of the form (7) are greater under $\prec$ than all other terms of the development of $\mathrm{Sq}^I X_\omega^h$, as claimed.

On the other hand, the sum (6) must be such that each (8) appears exactly once, as the essential permutations of the monomial $t_1^{a_1+1} \cdots t_r^{a_r+1} t_{r+1} \cdots t_k$ are the same as the essential permutations of $t_1^{a_1+1} \cdots t_r^{a_r+1}$ (the nondyadic factor of (8)). So we have (6) is not zero as each (8) appears exactly once.

With this we can consider a linear combination, call it L , of the $\mathrm{Sq}^I X_\omega^h$ which represents 0. Given terms cannot be represented as linear combinations of strictly smaller elements under $\prec$ (this follows from the fact that $\sum t_1^{a_1} \cdots t_r^{a_r}$ is a basis of $H^h(G_k)$ and the definition of $\prec$) we get that in L all $\mathrm{Sq}^I X_\omega^h$ must be such that all u, v are the same among the largest monomials given by (6). Notice this fact about linear combinations and $\prec$ also allows us to focus on the largest terms as given by (6). Therefore, we conclude that among the largest elements of the $\mathrm{Sq}^I X_\omega^h$ as given by (6) $v = u + h$ and so the h is the same for all. Lastly, we observe that ω must also coincide for all $\mathrm{Sq}^I X_\omega^h$ in L as the nondyadic factors of terms in (6) are all different for each ω so they would not be able to vanish in L . So we reduce the problem to the case

$$\sum_{\lambda} c_{\lambda} \mathrm{Sq}^{I_{\lambda}} X_{\omega}^h = 0.$$

Expanding and taking the largest elements once more, we have,

$$\sum_{\lambda} c_{\lambda} t_1^{a_1+1} \cdots t_r^{a_r+1} \mathrm{Sq}^{I_{\lambda}}(t_{r+1} \cdots t_k) = 0.$$

Extracting the nondyadic factor we get,

$$t_1^{a_1+1} \cdots t_r^{a_r+1} \sum_{\lambda} c_{\lambda} \mathrm{Sq}^{I_{\lambda}}(t_{r+1} \cdots t_k) = 0.$$

But by Lemma 9 we know that the $\text{Sq}^{I_\lambda}(t_{r+1} \cdots t_k)$ are linearly independent as long as $m - h \leq k - r$ which holds in this case as $m \leq k$ and $h \geq r$. Therefore, we conclude that $c_\lambda = 0$ for all λ and so the classes (5) are linearly independent. $\blacksquare$

We need a few more facts for the proof of Theorem 13, but we are almost there!

By a theorem of Serre proved in [Whitehead, Chapter V, Theorem 8.2] we have that $H^{k+m}(K(\mathbf{Z}/2, k); \mathbf{Z}/2) \cong \mathfrak{A}_2^m$. This will be crucial in the following lemma.

Lemma 13. The dimension of $H^{k+m}(Y)$ as a $\mathbf{Z}/2$ vector space is $p(m)$ (the number of partitions of m), where recall,

$$Y = K(\mathbf{Z}/2, k) \times K(\mathbf{Z}/2, k+2) \times \cdots \\ \cdots \times (K(\mathbf{Z}/2, k+h))^{d(h)} \times \cdots \times (K(\mathbf{Z}/2, 2k))^{d(k)}.$$

Proof. The proof is an application of the Künneth and Hurewicz Theorems. By Künneth we get the nasty expression

$$H^{k+m}(Y) \cong \bigoplus_{\substack{i_0, \dots, i_k \\ i_0 + \dots + i_k = k+m}} H^{i_0}(K(\mathbf{Z}/2, k)) \otimes \cdots \\ \cdots \otimes H^{i_h}(K(\mathbf{Z}/2, k+h))^{d(h)} \otimes \cdots \otimes H^{i_k}(K(\mathbf{Z}/2, 2k))^{d(k)}.$$

However, given we are dealing with Eilenberg-MacLane spaces, and by Hurewicz, we get that we need only concern ourselves with $i_h \geq k+h$ for each h . So we get that at most one i_h can be nonzero (because $k \geq m$) and if it is then $i_h = k+m$, i.e. the above expression becomes,

$$H^{k+m}(Y) \cong \bigoplus_{i=0}^k H^{k+m}(K(\mathbf{Z}/2, k+i))^{d(i)}. \quad (9)$$

Our task is now to compute $H^{k+m}(K(\mathbf{Z}/2, k+i))^{d(i)}$. By Künneth and applying similar reasoning we get that

$$H^{k+m}(K(\mathbf{Z}/2, k+i))^{d(i)} \cong \bigoplus_{j=1}^{d(i)} H^{k+m}(K(\mathbf{Z}/2, k+i)) \\ \cong \bigoplus_{j=1}^{d(i)} \mathfrak{A}_2^{(m-i)}.$$

Thus, as a $\mathbf{Z}/2$ vector space, we get $H^{k+m}(K(\mathbf{Z}/2, k+i))^{d(i)}$ has dimension $= d(i)c(m-i)$ (recall from §3 that $\mathfrak{A}_2^{m-i}$ has dimension $c(m-i)$), and so

$$\dim_{\mathbf{Z}/2}(H^{k+m}(Y)) = \sum_{i=0}^m d(i)c(m-i) = p(m).$$

$\blacksquare$

We will need the following version of Whitehead's Theorem (compare [Spanier, Theorem 9, p. 399])

Theorem 15. If $f : X \rightarrow Y$ is a map between simply connected spaces and for all primes p

$$f^* : \tilde{H}^q(Y; \mathbf{Z}/p) \rightarrow \tilde{H}^q(X; \mathbf{Z}/p)$$

is an isomorphism for all $q < n$, where $n \geq 1$, and an injective morphism for $q = n$, then

$$f_* : \pi_q(X) \rightarrow \pi_q(Y)$$

is an isomorphism for $q < n$ and a surjective morphism for $q = n$.

To be able to apply Whitehead's Theorem in the proof of Theorem 13 we will need the following fact.

Lemma 14. For the space Y defined in Theorem 13,

$$\tilde{H}^i(Y; \mathbf{Z}/p) = 0, \quad \forall i < 2k - 1$$

for all odd primes p .

In turn, to show this lemma we will have to recur to the following theorem which relates homotopy groups to homology groups 'over the class of finite abelian 2-groups', (where 2-group means that every element is of order a power of 2). Let $\mathcal{C}$ denote the class of all finite abelian 2-groups. A homomorphism of abelian groups $h : A \rightarrow B$ is said to be a $\mathcal{C}$ -isomorphism if both the kernel $h^{-1}(0)$ and the cokernel $B/h(A)$ belong to $\mathcal{C}$.

Theorem 16 (Milnor-Stasheff, Theorem 18.3, p. 207). Let X be a finite cell complex which is $(k - 1)$ -connected, $k \geq 2$. Then the Hurewicz homomorphism

$$\pi_i(X) \rightarrow \tilde{H}_i(X; \mathbf{Z})$$

is a $\mathcal{C}$ -isomorphism for $0 < i < 2k - 1$.

Proof of Lemma 14. Given the definition of Y it suffices to show that

$$\tilde{H}^i(K(\mathbf{Z}/2, h + k); \mathbf{Z}/p) = 0$$

for all $i < 2k - 1$ and all odd primes p (where recall by definition $h \leq k$). By Theorem 16 there is a $\mathcal{C}$ -isomorphism

$$\pi_i(K(\mathbf{Z}/2, k + h)) \rightarrow \tilde{H}_i(K(\mathbf{Z}/2, k + h); \mathbf{Z})$$

for $0 < i < 2k - 1$. By definition of Eilenberg-MacLane space we get $\pi_i(K(\mathbf{Z}/2, k + h)) = 0$ except for $i = k + h$ where it is $\cong \mathbf{Z}/2$. So we get that

$$\tilde{H}_i(K(\mathbf{Z}/2, k + h); \mathbf{Z}) \in \mathcal{C}.$$

By the Universal Coefficients Theorem we get,

$$0 \longrightarrow \text{Ext}_{\mathbf{Z}}(\tilde{H}_{i-1}, \mathbf{Z}/p) \longrightarrow \tilde{H}^i(K; \mathbf{Z}/p) \longrightarrow \text{Hom}_{\mathbf{Z}}(\tilde{H}_i, \mathbf{Z}/p) \longrightarrow 0$$

is an exact sequence that splits, where $\tilde{H}_i$ stands for the $\mathbf{Z}$ -homology of $K = K(\mathbf{Z}/2, k+h)$. However, given $K(\mathbf{Z}/2, k+h)$ is a finite cell complex, we know (compare Spanier p. 242)

$$\text{Ext}_{\mathbf{Z}}(\tilde{H}_{i-1}, \mathbf{Z}/p) \cong \tilde{H}_{i-1} \otimes_{\mathbf{Z}} \mathbf{Z}/p.$$

But $\tilde{H}_{i-1} \in \mathcal{C}$ i.e. it is 2-torsion, and so (given p is odd),

$$\text{Ext}_{\mathbf{Z}}(\tilde{H}_{i-1}, \mathbf{Z}/p) = 0.$$

Moreover, we get the only possible $\mathbf{Z}$ -map $\tilde{H}_i \rightarrow \mathbf{Z}/p$ is the zero map, as again every element of the domain is 2-torsion and p is odd (homomorphisms preserve the parity of order). Therefore, we get,

$$\tilde{H}^i(Y; \mathbf{Z}/p) = 0, \quad \forall i < 2k - 1$$

as required. ■

Remark 2. In the proof of Theorem 13 we also need to show that the reduced cohomology in $\mathbf{Z}/p$ coefficients of $MO(k)$ is 0 in degrees $i < 2k - 1$. This boils down to the same proof as for Y where the crucial step is showing

$$\tilde{H}^i(MO(k); \mathbf{Z}) \in \mathcal{C}.$$

This follows by applying Theorem 16 to $MO(k)$: it is $(k-1)$ -connected by considering the long exact sequence on homotopy groups for the bundle map $\gamma^k \rightarrow G_k$ and recalling that $\mathbf{R}^k$ is contractible (compare [Hatcher Theorem 4.41, p. 376]). And we know $\pi_{n+k}(MO(k))$ are 2-groups by the Pontrjagin-Thom construction (Theorem 11). In other words, we know that the cobordism groups are 2-groups.

Now we are ready to prove Theorem 13.

Proof of Theorem 13. The dimension of $H^{k+m}(MO(k))$ as a $\mathbf{Z}/2$ vector space is $p(m)$, the number of partitions of m . This is because, by Lemma 10, we can construe $H^{k+m}(MO(k))$ as $\langle W_k \rangle \cap H^{k+m}(G_k)$ the latter having the same dimension as $H^m(G_k)$ which has dimension $p(m)$. Note, moreover, that the number of elements in (5) is

$$= \sum_{h \leq m} d(h)c(m-h).$$

And that this latter sum equals $p(m)$. To see this note that for any partition of m we can associate a partition of $m-h$, for some h , all of the form $2^n - 1$ and a partition of h , all of whose elements are not of this form (simply take all elements in the given partition of m that are of the form $2^n - 1$ to make the first partition, all of

the elements that remain to make the second partition). Therefore, we have that the classes in (5) form a basis of $H^{k+m}(MO(k))$.

Now for each X_ω^h in (4) associate a map (see Theorem 12)

$$F_\omega : MO(k) \rightarrow K(\mathbf{Z}/2, k+m)$$

such that, if F_ω^* is the induced map on cohomology, we get that $F_\omega^*(\iota) = X_\omega^h$. Together the F_ω define a map $F : MO(k) \rightarrow Y$, where recall

$$Y = K(\mathbf{Z}/2, k) \times K(\mathbf{Z}/2, k+2) \times \cdots \\ \cdots \times (K(\mathbf{Z}/2, k+h))^{d(h)} \times \cdots \times (K(\mathbf{Z}/2, 2k))^{d(k)},$$

with the property that $F^* : H^{k+m}(Y) \rightarrow H^{k+m}(MO(k))$ is onto. This is because the classes in (5) form a basis of $H^{k+m}(MO(k))$. Then, by Lemma 13, F^* must be an isomorphism as both domain and codomain are of dimension $p(m)$. This was all in $\mathbf{Z}/2$ coefficients, in $\mathbf{Z}/p$ coefficients, for $p > 2$, we have that the cohomology of Y is zero in degrees $i < 2k-1$ (Lemma 14), and that of $MO(k)$ is also zero for all cohomological degrees $< 2k-1$ (Remark 2). Thus F^* is still an isomorphism for all degrees $< 2k-1$ and is injective in degree $2k-1$. Then finally we can apply Whitehead's Theorem (Theorem 15), from which we get the claim. ■

That concludes the proof of the first structure theorem (Theorem 14). Now we move on to showing the converse of Pontrjagin's theorem and the second structure theorem which determines the ring structure of $\mathfrak{N}_*$.

Definition 12. As for Stiefel-Whitney numbers we can define the *characteristic normal numbers* of an n -manifold M as

$$\langle W_1(\nu_M)^{r_1} W_2(\nu_M)^{r_2} \cdots W_k(\nu_M)^{r_k}, \mu_M \rangle$$

where ν_M is the rank k normal bundle of the manifold M as a submanifold of $n+k$ space. We will at times call these simply *normal numbers*.

Lemma 15. Let $k \leq n$. If $f : S^{n+k} \rightarrow MO(n)$ is homologically null, i.e. if

$$f : H^{n+k}(MO(n); \mathbf{Z}/2) \rightarrow H^{n+k}(S^{n+k}; \mathbf{Z}/2)$$

is the zero map, then f is homotopically null, i.e. $[f] = 0 \in \pi_{n+k}(MO(n))$.

Proof. For every nondyadic partition of ω of k , we have defined above a map,

$$F_\omega : MO(n) \rightarrow K(\mathbf{Z}/2, k+n)$$

such that $F_\omega^*(\iota) = X_\omega$, where X_ω is the class in $H^{k+n}(MO(n); \mathbf{Z}/2)$ defined by the symmetric function

$$\sum t_1^{a_1+1} t_2^{a_2+1} \cdots t_r^{a_r+1} t_{r+1} \cdots t_n$$

where we are construing $H^{k+n}(MO(n); \mathbf{Z}/2)$ as a subset of $H^{k+n}(G_n; \mathbf{Z}/2)$ via Lemma 10 and where (a_i) is the nondyadic partition ω of k (note this is also a symmetrisation

sigma). Then I claim we can choose a basis $f_{\omega'}$ of $\pi_{n+k}(MO(n))$ such that with the F_{ω} we have

$$f_{\omega'}^* F_{\omega}^*(\iota) = \begin{cases} s, & \omega' = \omega \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

where s is the nonzero class in $H^{k+n}(S^{n+k}; \mathbf{Z}/2) \cong \mathbf{Z}/2$. We first use the claim to prove the lemma and then we exhibit its proof. Let f not homotopically null, then we can write the class it represents in $\pi_{n+k}(MO(n))$ as a sum of elements in the basis $f_{\omega'}$. It is then clear that we can find a linear combination of the X_{ω} such that f^* of this linear combination is nonzero by (10).

Proof of claim. Let $[g_{\omega'}]$ in $\pi_{n+k}(K(\mathbf{Z}/2, n+k)) \cong \mathbf{Z}/2$ be the unique nonzero class (recall the definition of Eilenberg-MacLane space). Define the class $[f_{\omega'}]$ in $\pi_{n+k}(MO(k))$ by

$$[f_{\omega'}] = (F_*)^{-1}([g_{\omega'}])$$

where F is the map of Theorem 13 (constructed in the proof: the product of the maps F_{ω}) between $MO(k)$ and Y . Then to prove (10) it will suffice to compute the Kronecker index of $f_{\omega'}^* F_{\omega}^*(\iota)$ with respect to the fundamental homology class of S^{n+k} . This is because saying that $f_{\omega'}^* F_{\omega}^*(\iota)$ is the unique nonzero class of $H^{n+k}(S^{n+k})$ is the same as saying the Kronecker index evaluates to something nonzero. Consider then,

$$\langle f_{\omega'}^* F_{\omega}^*(\iota), [S^{n+k}] \rangle.$$

By definition we have that $F \circ f_{\omega'} = g_{\omega'}$ and, if we let $\pi_{\omega} : Y \rightarrow K(\mathbf{Z}/2, n+k)$ be the canonical projection onto the ω factor of Y , we have $F_{\omega} = \pi_{\omega} \circ F$. Then,

$$\begin{aligned} \langle f_{\omega'}^* F_{\omega}^*(\iota), [S^{n+k}] \rangle &= \langle \iota, F_{\omega*}(f_{\omega'})_*[S^{n+k}] \rangle \\ &= \langle \iota, (\pi_{\omega})_* F_*(f_{\omega'})_*[S^{n+k}] \rangle \\ &= \langle \iota, (\pi_{\omega})_*(g_{\omega'})_*[S^{n+k}] \rangle. \end{aligned}$$

Now consider the composite

$$S^{n+k} \xrightarrow{g_{\omega'}} K(\mathbf{Z}/2, k+n) \hookrightarrow Y \xrightarrow{\pi_{\omega}} K(\mathbf{Z}/2, k+n)$$

where the inclusion is made onto the ω' factor (note that both ω and ω' are nondyadic partitions of k , hence the apparent ambiguity as to where we are including and projecting: in $K(\mathbf{Z}/2, k+n)^{d(k)} \subset Y$ there is a factor corresponding to ω and one corresponding to ω').

Then clearly $(\pi_{\omega} \circ g_{\omega'})_*$ is only nonzero if $\omega = \omega'$. And given ι is the unique nonzero class in $H^{k+n}(K(\mathbf{Z}/2, n+k)) \cong \mathbf{Z}/2$ we get,

$$\langle f_{\omega'}^* F_{\omega}^*(\iota), [S^{n+k}] \rangle = \begin{cases} 1, & \omega = \omega' \\ 0, & \text{otherwise} \end{cases}$$

from which (10) follows at once (notice it equals 1 for $\omega = \omega'$ by the definition of $g_{\omega'}$). ■

Let $Y_\omega = \sum t_1^{a_1} \cdots t_m^{a_m}$ where $Y_\omega \in H^k(G_n; \mathbf{Z}/2)$ under our identification of $H^*(G_n)$ with the polynomial subalgebra of $\mathbf{Z}/2[t_1, \dots, t_n]$ generated by the symmetric polynomials in the t_n .

Lemma 16. If all the normal numbers of a manifold are zero, then it is a boundary.

Proof. First observe that

$$X_\omega = \phi(Y_\omega)$$

if we identify $H^{k+n}(MO(n))$ with $\langle \omega_n \rangle \cap H^{k+n}(G_n)$ via the morphism in Lemma 10 (recall the X_ω where defined in (4)). Now all of the maps in the basis $f_{\omega'}$ in (10) can be chosen tranverse to G_n in $MO(n)$, where we are construing G_n as the image of the zero section in $MO(n)$. Let $V_{\omega'} := f_{\omega'}^{-1}(G_n)$ and consider a tubular neighborhood of $V_{\omega'}$ in S^{n+k} , where $\phi^* : H^k(V_{\omega'}) \rightarrow H^{k+n}(T\nu)$ is the associated Thom isomorphism of the normal bundle ν of $V_{\omega'}$ in S^{n+k} where, by the Tubular Neighborhoods Theorem, we have have identified ν with the tubular neighborhood of $V_{\omega'}$. Let $Y'_\omega := f_{\omega'}^*(Y_\omega) \in H^k(V_{\omega'})$, where we are taking $f_{\omega'}$ restricted to $V_{\omega'}$. Then the class Y'_ω can be expressed as a polynomial in the classes $\overline{W}_i = W(\nu)$. This we get by the naturality property of Stiefel-Whitney classes (property (2) in Theorem 6) and the fact that $f_{\omega'}$ restricted to $V_{\omega'}$ is the classifying map of the normal bundle ν of $V_{\omega'}$ in S^{n+k} (the latter fact being a consequence of the Pontrjagin-Thom construction, Theorem 11 and Lemma 3).

Given $f_{\omega'}$ restricted to $V_{\omega'}$ is the classifying map of ν , we get a commutative diagram,

$$\begin{array}{ccccc} \nu & \cong & f_{\omega'}^* \gamma^n & \xrightarrow{\overline{f_{\omega'}}} & \gamma^n \\ & \searrow & \downarrow & & \downarrow \\ & & V_{\omega'} & \xrightarrow{f_{\omega'}} & G_n. \end{array}$$

And so we get in cohomology,

$$\begin{array}{ccc} H^k(\nu) & \xleftarrow{\overline{f_{\omega'}}} & H^k(\gamma^n) \\ \uparrow & & \uparrow \\ H^k(V_{\omega'}) & \xleftarrow{f_{\omega'}^*} & H^k(G_n). \end{array}$$

But note that the vertical arrows are the first step in the Thom isomorphism, so we have,

$$\begin{array}{ccccc}
H^{k+n}(T\nu) & \cong & H^{n+k}(\nu, \nu^\#) & \xleftarrow{\quad\quad\quad} & H^{n+k}(\gamma^n, \gamma^{n\#}) & \cong & H^{k+n}(MO(n)) \\
& & \uparrow & & \uparrow & & \\
& & H^k(\nu) & \xleftarrow{\quad \overline{f_{\omega'}^*} \quad} & H^k(\gamma^n) & & \\
& & \uparrow & & \uparrow & & \\
& & H^k(V_{\omega'}) & \xleftarrow{\quad f_{\omega'}^* \quad} & H^k(G_n) & &
\end{array}$$

is a commutative diagram as well, where the vertical composites are the Thom isomorphism for the respective bundles. In all, naming the top dashed arrow $\overline{f_{\omega'}^*}$ as well, we get the following commutative diagram, where Y'_ω is the image of Y_ω under $f_{\omega'}^*$,

$$\begin{array}{ccc}
H^{k+n}(T\nu) & \xleftarrow{\quad \overline{f_{\omega'}^*} \quad} & H^{k+n}(MO(n)) \\
\phi \uparrow & & \uparrow \phi \\
H^k(V_{\omega'}) & \xleftarrow{\quad f_{\omega'}^* \quad} & H^k(G_n) \\
\Downarrow & & \Downarrow \\
Y'_\omega & \xleftarrow{\quad\quad\quad} & Y_\omega.
\end{array}$$

This diagram gives us,

$$\phi(Y'_\omega) = \phi f_{\omega'}^*(Y_\omega) \quad (11)$$

$$= \overline{f_{\omega'}^*} \phi(Y_\omega) \quad (12)$$

$$= \overline{f_{\omega'}^*}(X_\omega). \quad (13)$$

But by our identification using Tubular Neighborhoods and (10) we get

$$\phi(Y'_\omega) = \overline{f_{\omega'}^*}(X_\omega) = \begin{cases} s, & \omega' = \omega \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

Now consider a nonzero cobordism class $[M] \in \mathfrak{N}_k$ such that under the Pontrjagin-Thom construction $[f]$ is the associated homotopy class in $\pi_{k+n}(MO(n))$. Then by Lemma 15 we know f is not homologically null either, which means that there is a linear combination of the X_ω such that f is nonzero when applied to it. But by (14) we get that this means that there must be normal characteristic number of $M = f^{-1}(G_n)$ that is not zero. In other words, if M is not a boundary, it has a normal characteristic number which is not zero. $\blacksquare$

You know what is coming! We can finally prove the converse of Pontrjagin's theorem!

Theorem 17. If a manifold M has all its Stiefel-Whitney numbers equal to zero, then it is a boundary.

Proof. This follows from Lemma 16 and the fact that the classes $W_i(\nu) = \overline{W}_i$ can be written as polynomials in the Stiefel-Whitney classes W_i . ■

From the preceding theorem we immediately get the following corollary.

Corollary 2. If the Stiefel-Whitney numbers of two manifolds M and N are the same, then M and N are cobordant. ■

We can now focus on proving the second structure theorem stated in the introduction. For this it will be useful to make explicit an abuse of notation that will be common in the next lemma and its proof. Let $Y_\omega = \sum t_1^{a_1} \cdots t_m^{a_m} \in H^k(G_n; \mathbf{Z}/2)$ represent a polynomial in the Stiefel-Whitney classes of the universal bundle. Then in what follows we will blur the distinction between the class Y_ω , the class Y'_ω which represents some polynomial in the Stiefel-Whitney classes of the normal bundle of a manifold (the pullback of Y_ω under the classifying map of the normal bundle of a manifold), and the normal number to which the latter evaluates. (For definiteness, the reader is advised to review the proof of Lemma 16). For instance, equation (15) below, says that the the normal number one gets by evaluating the Kronocker index of the left hand side is the same as the normal number one gets by evaluating it on the right hand side, even though as stated the equation is about an equality of polynomials in the Stiefel-Whitney classes of the universal bundle. The following definition will help clarify this issue.

Definition 13. Let V_ω^k be as in the proof of Lemma 16. In other words, $V_\omega^k := f_\omega^{-1}(G_n)$ where the f_ω , ranging over all ω nondyadic partitions of k form a basis of $\pi_{n+k}(MO(n))$ as in the proof of Lemma 15. (The superscript is, as usual, the dimension of the manifold, where we get it is equal to k by transversality). Finally, recall that by what we did in the proof of Lemma 16 we get that V_ω^k has a single nonzero normal number, namely Y_ω .

Remark 3. Following the thread started before the previous definition, we have that $f_\omega^*(Y_\omega) = Y'_\omega$.

Our strategy for proving the second structure theorem will be to study the V_ω^k showing that a subset of these manifolds actually give us a set of free generators of the cobordism ring $\mathfrak{N}_*$ as a $\mathbf{Z}/2$ algebra.

Let $k = r + s$.

Lemma 17. The normal characteristic numbers Y_ω of $V_{\omega_1}^r \times V_{\omega_2}^s$ satisfy,

$$Y_\omega = \sum_{(\omega'_1, \omega'_2) = \omega} Y_{\omega'_1} Y_{\omega'_2}, \quad (15)$$

where ω is a nondyadic partition of k and ω'_1, ω'_2 are nondyadic partitions of r, s respectively.

Proof. Recall that the normal bundle of a product manifold is the cartesian product of the normal bundles of the individual factors (there is some ambiguity here but it does not matter). In other words, if ν_1 (resp. ν_2) is the normal bundle of $V_{\omega_1}^r$ (resp. $V_{\omega_2}^s$) then the normal bundle of $V_{\omega_1}^r \times V_{\omega_2}^s$ is $\nu_1 \times \nu_2$. Let $\overline{W}_i$ be the normal numbers of the product manifold $V_{\omega_1}^r \times V_{\omega_2}^s$, U_i the normal numbers of $V_{\omega_1}^r$, and V_j the normal numbers of $V_{\omega_2}^s$. By Whitney's product formula (see the proof of property (4) in Theorem 6) we get the following formal relation

$$\sum_i \overline{W}_i t^i = \sum_i U_i t^i \times \sum_j V_j t^j \quad (16)$$

(indeed simply multiply out and match coefficients). Let u_i , resp. v_j , be the formal roots of the polynomials $\sum_i U_i t^i$, resp. $\sum_j V_j t^j$. Then we have chosen variable names in which to represent U_i , resp. V_j , as symmetric polynomials in u_i , resp. v_j (to see this recall that the coefficients of a polynomial are given as symmetric polynomials of the roots). Then what (16) says is

$$Y_\omega = \sum_{(\omega'_1, \omega'_2) = \omega} \left(\sum u_1^{a_1} \cdots u_m^{a_m} \right) \left(\sum v_1^{b_1} \cdots v_n^{b_n} \right) \quad (17)$$

where $\omega'_1 = (a_1, \dots, a_m)$ and $\omega'_2 = (b_1, \dots, b_n)$. Why? Because each $\overline{W}_\ell$, represented as some symmetric polynomial in the t_i 's, can be construed, by (16), as a sum of products of symmetric polynomials in the u_i and v_j (i.e. you have a sum of products of a symmetric polynomial in the u_i and a symmetric polynomial in the v_j). This is such that the degree of the polynomial in the u_i plus the degree of the polynomial in the v_j sum to ℓ . And so, given Y_ω is determined by ω as a polynomial in the $\overline{W}_i$, we get the above expression as a result.

Observe, moreover, that we need to exclude all pairs (ω'_1, ω'_2) where ω'_1 is a partition of something $\neq r$ because of the definition of normal number for $V_{\omega_1}^r$ (the relevant polynomials in the u_i 's must be of degree r). Thus, we get that ω'_2 is a partition of s . Lastly, the term corresponding to (ω'_1, ω'_2) appears once in the sum of (17). Otherwise, if the decomposition (ω'_1, ω'_2) of ω appeared more than once in the sum of (17) then this would mean that there is some permutation σ of the k variables t_i which in the monomial,

$$t_1^{a_1} \cdots t_m^{a_m} t_{m+1}^{b_1} \cdots t_k^{b_n}$$

never permutes a t_i into a t_j such that $i \neq j$, for it has to transform (ω'_1, ω'_2) into an isomorphic decomposition. Which means, σ is an inessential permutation, and so, is not included in the symmetrisation. We then get equation (15) from (17) (they say the exact same thing). $\blacksquare$

Corollary 3. The manifolds V_ω^k and $V_{\omega_1}^r \times V_{\omega_2}^s$ are cobordant, where $k = r + s$.

Proof. By Lemma 17 and the definition of manifolds V_ω^k , we get that these two have the same normal numbers and so they have the same Stiefel-Whitney numbers which means, by Corollary 2, that they are cobordant. (Indeed Lemma 17 says $Y_\omega = Y_{\omega_1} Y_{\omega_2}$ after having evaluated the Kronecker index.) ■

Now we are ready to prove the second structure theorem.

Theorem 18. The unoriented cobordism ring $\mathfrak{N}_* = \bigoplus_k \mathfrak{N}_k$ is isomorphic to a free algebra over $\mathbf{Z}/2$ with a generator $[V_{(k)}^k]$ for every dimension k which is not of the form $2^m - 1$.

Proof. Corollary 3 says that $[V_\omega^k] = [V_{\omega_1}^r][V_{\omega_2}^s]$, so given the only nondecomposable partitions of k are those of the form (k) (i.e. those consisting of a single term, which is k itself) then $[V_{(k)}^k]$ is the only nondecomposable cobordism class. This gives us the theorem, as the definition of the manifolds V_ω^k allows us to represent every cobordism class as a sum of products of classes of the form $[V_{(k)}^k]$ uniquely. ■

Remark 4. In symbols, Theorem 18 says,

$$\mathfrak{N}_* \cong \mathbf{Z}/2[x_2, x_4, x_5, x_6, x_8, x_9, x_{10}, \dots].$$

This concludes our efforts with regards to unoriented cobordism!

6. ORIENTED COBORDISM

(For a more detailed treatment of oriented cobordism see [Milnor-Stasheff, §§12, 15, 17, 18])

It has been a sweaty essay! Now that we have the unoriented case under our belts we can lay back and see what lies ahead. The next big milestone would be to prove similar theorems as we have above for the *oriented* cobordism ring and groups. What *oriented* means is that manifolds M, N now have an orientation and we insist that a cobordism between them is oriented in such a way that is compatible with the orientations of $M \amalg -N$ as boundary, where $-N$ is taken as the same manifold with opposite orientation. To be precise, the manifold with boundary X is an oriented cobordism between M, N if it is compact and oriented and so that ∂X is diffeomorphic with its induced orientation to $M \amalg -N$. We denote the n th cobordism group by Ω_n where the group operation is the disjoint union operation. This makes Ω_n into an abelian group and the cartesian product $M^m, N^n \mapsto M^m \times N^n$ gives rise to a bilinear operation

$$\Omega_m \times \Omega_n \rightarrow \Omega_{n+m}$$

thus making $\Omega_* = \bigoplus_{n \geq 0} \Omega_n$ into a graded ring.

To do justice to this flavour of cobordism we would need a lot more space than we have available, so I will quote some facts that will sadly remained unproven. Our goal will be to compute the oriented cobordism ring modulo torsion. The first thing to note is that as in the unoriented case, the technical heart of the theory is the

Pontrjagin-Thom construction, transforming the topology problem into a homotopy one. Indeed, with almost identical proofs Theorem 10, and Lemmas 4,6 follow through to the oriented case. In Lemma 6 we would have to change γ_p^k by $\tilde{\gamma}_p^k$ where the latter is the bundle of oriented k -planes in $(k+p)$ -space.

In fact we have an analogous construction to the classifying space G_n together with the universal bundle γ^n in the oriented case. Let $\tilde{G}_n(\mathbf{R}^{n+k})$ denote the Grassmann manifold consisting of all oriented n -planes in $\mathbf{R}^{n+k}$. Then $\tilde{\gamma}_k^n$ is a vector bundle over $\tilde{G}_n(\mathbf{R}^{n+k})$. Moreover, we have that $\tilde{G}_n(\mathbf{R}^{n+k})$ is a CW complex (and so it is paracompact), which means that the map ‘forget orientations’ from $\tilde{G}_n(\mathbf{R}^{n+k})$ to $G_n(\mathbf{R}^{n+k})$ gives us that γ^n lifts to what we will call the *oriented universal bundle*: $\tilde{\gamma}^n$. We call it *universal* because for any oriented vector bundle ξ the bundle map $\xi \rightarrow \gamma^n$ lifts uniquely to an orientation preserving map of bundles $\xi \rightarrow \tilde{\gamma}^n$.

We will need the following statements, see [Milnor-Stasheff, §§17, 18, Corollary 17.5 and Corollary 18.4].

Lemma 18. The products $P^{2i_1}(\mathbf{C}) \times \cdots \times P^{2i_r}(\mathbf{C})$ where $i_1, \dots, i_r$ ranges over all partitions of k , represent linearly independent elements of the cobordism group Ω_{4k} . Hence, Ω_{4k} has rank greater than or equal to $p(k)$ the number of partitions of k .

Lemma 19. If T is the Thom space of an oriented rank k vector bundle ξ over the finite complex B , then there is a $\mathcal{C}$ -isomorphism

$$\pi_{n+k}(T) \rightarrow H_n(B; \mathbf{Z})$$

for all dimensions $n < k - 1$.

Now we can state the main theorem of this section, see [Milnor-Stasheff, §18, Theorem 18.8 and Corollary 18.9].

Theorem 19. The oriented cobordism group Ω_n is finite for $n \not\equiv 0 \pmod{4}$, and is a finitely generated abelian group of rank equal to $p(r)$, the number of partitions of r , for $n = 4r$.

Proof. By Lemma 6, Ω_n is a quotient of $\pi_{n+k}(T\gamma_p^k)$ for k, p large enough. The latter is $\mathcal{C}$ -isomorphic to $H^n(\tilde{G}_n(\mathbf{R}^{k+p}); \mathbf{Z})$ by Lemma 19 where $\mathcal{C}$ is taken to be the class of finite abelian groups. However, it is a fact (see Theorem 15.9 and problem 15-C in [Milnor-Stasheff]) that $H^n(\tilde{G}_n(\mathbf{R}^{k+p}); \mathbf{Z})$ is finite for $n \not\equiv 0 \pmod{4}$ and it is finitely generated of rank $p(r)$ for $n = 4r$. Therefore, Ω_n is finite for $n \not\equiv 0 \pmod{4}$ and it is of rank at most $p(r)$ for $n = 4r$. Finally, by Lemma 18 we know the rank of Ω_n when $n = 4r$ must be at least $p(r)$ and so it must be exactly equal to $p(r)$ by what was said before. ■

Finally, if we tensor with $\mathbf{Q}$, killing torsion, we get that by Theorem 19 only the groups Ω_n with $n = 4r$ survive and that the products $P^{2i_1}(\mathbf{C}) \times \cdots \times P^{2i_k}(\mathbf{C})$ where $i_1, \dots, i_k$ ranges over all partitions of r form a basis of the group $\Omega_n \otimes \mathbf{Q}$. So we immediately get the following result.

Corollary 4. The tensor product $\Omega_* \otimes \mathbf{Q}$ is a polynomial algebra over $\mathbf{Q}$ freely generated by $P^2(\mathbf{C}), P^4(\mathbf{C}), P^6(\mathbf{C}), \dots$

7. CONCLUSION

Of course, the subject of cobordism does not end in the oriented case, even if treated thoroughly. There are many more flavours to explore (see [Stong, Chapter IV] for a review of some of the other types of cobordism out there).

One very interesting theorem that lies not too far ahead of where this essay has left off is Hirzebruch's Signature Theorem. In short, just as we defined Stiefel-Whitney classes above to understand unoriented cobordism, we could also define the *Pontrjagin classes* of an oriented bundle. These classes give us a grip on the oriented cobordism problem (and indeed are central in many of the proofs omitted in the previous section). What Hirzebruch's theorem says is then that the *signature* of a compact oriented manifold of dimension $4r$ is the same as evaluating a certain linear combination of the Pontrjagin classes (called the L -genus) on the fundamental class of the manifold. Here 'signature' means the classical signature of a certain real quadratic form defined on the $2r$ th cohomology group of the manifold. Thom proved that if a manifold like so is an oriented boundary then it must have its signature equal to zero.

(Oh and what lies ahead! With tools as these (that the L_2 genus is integral for any smooth compact oriented manifold), Milnor managed to show that there are exotic 7-spheres! Which are spheres topologically but with a different smooth structure than that of the Euclidean 7-sphere.)

Even in purely cobordism-theory-terms the options are varied for further pursuit: complex cobordism, spherical cobordism, special unitary cobordism, spin cobordism, cobordism of Poincaré duality spheres,... you name it, there seems to be a cobordism theory to fit everyone's fancy. *What matters is* that in this essay we have set the technical stage to tackle each and every one of them, at least conceptually. Namely, we have given the general strategy to try and compute the cobordism groups and the associated cobordism ring: set up the cobordism problem (define the equivalent of $\mathfrak{N}_k$ and $\mathfrak{N}_*$ for the appropriate cobordism relation), translate the cobordism problem to an algebra problem (prove some version of the Pontrjagin-Thom construction), and then using cobordism invariants (in our case the Stiefel-Whitney numbers) and the solution to the algebra problem to solve the cobordism problem at hand.

To conclude, in this essay we have introduced the first instance of what are called *characteristic classes* in the avatar of *Stiefel-Whitney classes*. We saw how useful these are in determining invariants for unoriented cobordism in the form of *Stiefel-Whitney numbers*. We learnt how to use much of the machinery of basic homotopy theory in our solution of the homotopy problem given to us by the clever *Pontrjagin-Thom construction*, and we saw the method to attack the basic problem in a long ladder of wonderfully interesting theories which are regularly bundled under the name of *cobordism*.

REFERENCES

- [Adams, 1969] Adams, J. F. (1969). *Stable homotopy theory*, volume No. 3 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, third edition. Lectures delivered at the University of California at Berkeley, 1961, Notes by A. T. Vasquez.
- [Hatcher, 2002] Hatcher, A. (2002). *Algebraic topology*. Cambridge University Press, Cambridge.
- [Hatcher, 2017] Hatcher, A. (2017). *Vector Bundles and K-Theory*.
- [Miller, 1994] Miller, H. (1994). Notes on cobordism.
- [Milnor, 1997] Milnor, J. W. (1997). *Topology from the differentiable viewpoint*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ. Based on notes by David W. Weaver, Revised reprint of the 1965 original.
- [Milnor and Stasheff, 1974] Milnor, J. W. and Stasheff, J. D. (1974). *Characteristic classes*, volume No. 76 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo.
- [Spanier, 1995] Spanier, E. H. ([1995?]). *Algebraic topology*. Springer-Verlag, New York. Corrected reprint of the 1966 original.
- [Stong, 1968] Stong, R. E. (1968). *Notes on cobordism theory*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo. Mathematical notes.
- [Thom, 1954] Thom, R. (1954). Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86.
- [Weston, 1997] Weston, T. (1997). An introduction to cobordism theory.
- [Whitehead, 1978] Whitehead, G. W. (1978). *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin.