

GROMOV-THURSTON MANIFOLDS, PSEUDO-HYPERBOLIC GEOMETRY AND HIGHER HIGHER TEICHMÜLLER SPACES

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ABSTRACT. In this master's thesis we give an account of the results arrived at by Monclair-Schlenker-Tholozan in their recent paper, generalise their results to the pseudo-hyperbolic space of signature (p, q) , and explore how these give us new exotic examples of higher higher Teichmüller spaces.

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1. INTRODUCTION

In this master's thesis we: (1) give an account of the results arrived at by Monclair, Schlenker and Tholozan in their beautiful paper [32], (2) give a generalisation of their construction for the pseudo-hyperbolic space $\hat{\mathbb{H}}^{p,q}$ for $q \geq 1$, and (3) interpret these results in the context of *higher dimensional higher rank Teichmüller theory*.

(1) More precisely, we will show how they geometrised *Gromov-Thurston manifolds* M_n^k (Section 2) by realising them as the future boundary of the convex core of an anti-de Sitter manifold N of one dimension higher (Section 3).

The strategy for doing this is to construct a *kind* of (G, X) structure: a *spacelike AdS structure* (Definition 3.15 for $q = 1$), where our model manifold X is anti-de Sitter space $\hat{\text{AdS}}^{p+1}$ (p is the dimension of our Gromov-Thurston manifold) and the group of isometries is $G = \text{SO}(p, 2)$, but where the charts are what will be called *spacelike immersions* (Definition 3.11). By carefully setting up the situation, they arrive at a notion which has an equivalent of *developing map* (Definition 3.19) and *holonomy representation* (Definition 3.22). This in turn allows them to get a correspondence between such spacelike AdS structures and very nice (*globally hyperbolic maximal Cauchy compact* Definition 3.31) anti-de Sitter manifolds N (Lorentzian manifolds of constant negative curvature) containing the Gromov-Thurston manifold M_n^k as the *future boundary of the convex core* (Lemma 3.41).

Initially, it seems they have traded a difficult problem for one which is just as difficult. Namely, constructing these anti-de Sitter manifolds N to constructing these mysterious spacelike AdS structures on M_n^k . Part of the brilliance of the paper is to show that, in fact, for a particular subclass of such structures, we have a correspondence with *spacelike de Sitter polygons* (Definition 4.6) (roughly polygons on de Sitter 2-space which have spacelike edges). And so, that the anti-de Sitter manifolds N which realise M_n^k as the future boundary of the convex core exist, by existence of said polygons. Needless to say, dealing with these polygons is easier than dealing with the spacelike AdS structures.

Therefore, the first goal of this thesis is to prove the main results of the Monclair-Schlenker-Tholozan paper (see next subsection for the precise statements) by developing the machinery therein.

(2) The Monclair-Schlenker-Tholozan paper deals chiefly with anti-de Sitter geometry, namely with the geometry of the space AdS^{p+1} which is the pseudo-hyperbolic space $\hat{\mathbb{H}}^{p,1}$. However, excluding certain *convexity* considerations, their construction generalises to the pseudo-hyperbolic space $\hat{\mathbb{H}}^{p,q}$ where $q \geq 1$. Indeed, the second goal of this thesis is to generalise a portion of the Monclair-Schlenker-Tholozan construction. More precisely to develop a generalised notion of *spacelike* $\hat{\mathbb{H}}^{p,q}$ *structure* (Definition 3.15) and show that, as in the case when $q = 1$, for a special subclass of these, they are in correspondence with *spacelike* $\hat{\mathbb{S}}^{1,q}$ *polygons* (Definition 4.2).

(3) It turns out that the holonomy representations associated to spacelike AdS structures are $\hat{\mathbb{H}}^{p,1}$ -*convex cocompact* (Definition 6.4), and so single out new exotic examples of *higher higher Teichmüller spaces* (Definition 6.2). Furthermore, the correspondence ‘spacelike de Sitter polygons to spacelike AdS structures’ gives a correspondence ‘spacelike de Sitter polygons to holonomy representations associated to spacelike AdS structures’. And so, we have a hold of these $\hat{\mathbb{H}}^{p,1}$ -convex cocompact representations within the *character variety* (Definition 6.2) $\text{Hom}(\pi_1(M_n^k), \text{SO}(p, 1))$ (where this space is topologised by the compact-open topology and where we quotient by the conjugation action of G on the target) via these spacelike de Sitter polygons. Our construction for $q \geq 1$ lends itself to generalisation in this direction too (and it was the motivation for attempting to generalise the case when $q = 1$). In other words, we get that within a connected component of the set of spacelike $\hat{\mathbb{S}}^{1,q}$ polygons there are $\hat{\mathbb{H}}^{p,q}$ -*convex cocompact representations* (Definition 6.4), which once again give new exotic examples of higher higher Teichmüller spaces.

After this brief outline we proceed to introduce the main results to be proven and to give a more detailed roadmap of how we are to arrive at them.

1.1. Main results. (1) The main results of the Monclair-Schlenker-Tholozan paper are the following (see Subsection 5.1 for the proofs).

Theorem 1.1. Let M_n^k be a k -ramified Gromov-Thurston manifold of dimension $p > 3$ such that $k > n$. Then we can isometrically embed M_n^k (with its cone hyperbolic metric see Section 2) in a GHMC AdS manifold N of dimension $p + 1$, as the future boundary of the convex core of N .

Theorem 1.2. Let M_n^k be a k -ramified Gromov-Thurston manifold of dimension $p > 3$ such that $k > n$. Then there is a $2k - 3$ parameter family of GHMC AdS manifolds N for which M_n^k is isometric (with its cone hyperbolic metric) to the future boundary of the convex core.

(2) To arrive at the above results, Monclair-Schlenker-Tholozan prove the $q = 1$ case of the following (see Subsection 5.2 and Terminology 5.13 for what we mean by the natural action).

Theorem 1.3. There is a one-to-one correspondence between pairs (dev, ρ) of developing maps and associated holonomy representations giving spacelike $\hat{\mathbb{H}}^{p,q}$ -structures on the Gromov-Thurston manifold M_n^k up to the natural action of $SO_o(p, q+1)$, and equilateral spacelike $\hat{\mathbb{S}}^{1,q}$ $2k$ -polygons with sidelength π/n up to isometry. In other words,

$$\begin{array}{c} \left\{ \begin{array}{l} \text{Pairs } (\text{dev}, \rho) \text{ associated to} \\ \text{spacelike } \hat{\mathbb{H}}^{p,q} \text{ structures on } M_n^k \end{array} \right\} / \text{natural action} \\ \updownarrow \\ \left\{ \begin{array}{l} \text{Equilateral spacelike } \hat{\mathbb{S}}^{1,q} \\ 2k\text{-polygons with sidelength } \pi/n \end{array} \right\} / \text{isometry.} \end{array}$$

(3) From Theorem 1.3 we get the following theorems. The first bears witness to the fact that Monclair, Schlenker and Tholozan have constructed new exotic examples of higher higher Teichmüller spaces within the character variety

$$\text{Hom}(\pi_1(M_n^k), SO(p, 2))/SO(p, 2),$$

while the second shows that their construction can be carried out in general to get higher higher Teichmüller spaces within the character variety

$$\text{Hom}(\pi_1(M_n^k), SO(p, q+1))/SO(p, q+1)$$

for $q \geq 1$. Recall we denote higher higher Teichmüller spaces by HHTS for short (see Terminology 6.3).

Theorem 1.4. There are HHTS $\mathcal{T}(M_n^k, SO(p, 2))$ associated to the holonomy representations of folded convex ruled spacelike AdS structures on M_n^k , with $k > n$. In other words, for every holonomy representation of a folded convex ruled spacelike AdS structure, the connected component in the character variety containing it is a HHTS.

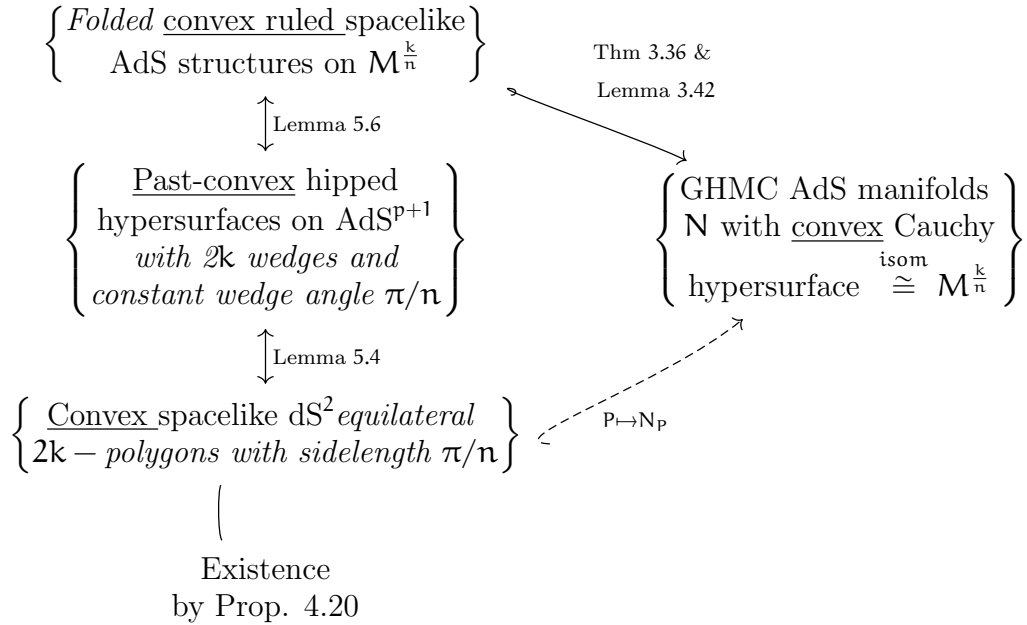
Theorem 1.5. There are HHTS $\mathcal{T}(M_n^k, SO(p, q+1))$ associated to certain folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures, which contain representations with Zariski-dense image.

1.2. Organisation. This master's thesis is born out of the study of Monclair, Schlenker, and Tholozan's paper [32] in which they geometrize Gromov-Thurston manifolds by realising them as the future boundary of the convex core of some anti-de Sitter manifold of one dimension higher. This is the cause of our (at times clumsy) pursuit of two paths in this work. The first of them is what has been discussed in the Monclair-Schlenker-Tholozan paper proper, with some adjustments, additions and, generally

more verbose explanations to suit the short-sightedness characteristic of a student (like the author)¹. The second is what I expect to be a generalisation of the main constructions of [32]: we give a way of carrying out the construction of the *holonomy representations*, associated to the *spacelike* $\hat{\mathbb{H}}^{p,q}$ *structures*, for $q \geq 1$ (the result of [32] in the $q = 1$ case). The reason for not merging (all of) the results given for the case $q = 1$ with the proofs for $q \geq 1$ is that by doing so we would preclude us from seeing first-hand the intuition which the $q = 1$ case gives: it is here where all the geometry insights can be traced to. Moreover, we do not generalise the punch-line of [32]: there is no mention of convexity for the $q > 1$ case and so there is no geometrisation of Gromov-Thurston manifolds into pseudo-hyperbolic manifolds.

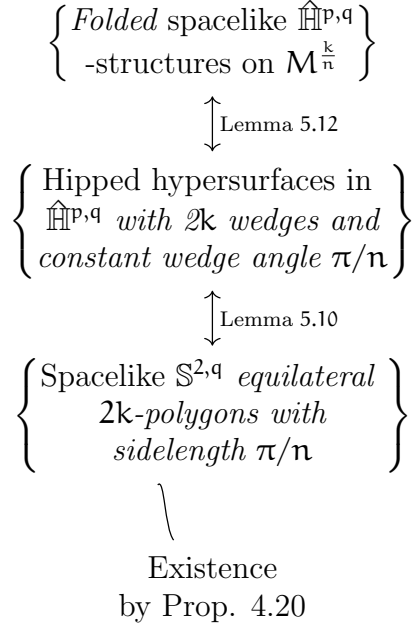
Having said that, we proceed to give two diagrammatic renderings of these two *paths*, which should henceforth be understood to be the case when $q = 1$, and the case for general $q \geq 1$. These diagrams will be alluded to throughout so as to ease understanding of *what we are doing*, and *why we are doing it*.

Diagram 1.6 (The case $q = 1$).



¹The goal of writing this master's thesis (certainly achieved after much effort) is to diminish this short-sightedness.

Diagram 1.7 (The case $q \geq 1$).



1.2.1. *Outline.* In Section 2 we introduce Gromov-Thurston manifolds and the main properties proved by Gromov-Thurston in [19]. Sections 3,4 are the technical heart of the thesis. In Section 3 we begin with some reminders on anti-de Sitter and pseudo hyperbolic geometry, we use this material to introduce spacelike $\hat{\mathbb{H}}^{p,q}$ -structures and then develop some of their properties. In Section 4, we study spacelike de Sitter polygons and their generalisations: spacelike $\hat{\mathbb{S}}^{1,q}$ polygons. In Section 5, we use this to prove Theorems 1.1 and 1.2, and the generalised statement Theorem 1.3. Lastly, in Section 6, we explore how the results above fit-in with the theory of higher higher Teichmüller spaces, proving Theorem 1.4 and its generalised version Theorem 1.5. At the end of this work there is an appendix of pictures which should really be considered to be an appendix of *doodles*: the things the reader might want to have done for himself in order to understand what is being explained².

1.3. **Acknowledgements.** I would like to warmly thank Prof. Fanny Kassel, for her patience, kindness and generosity in helping me understand all these wonderful things (and for introducing me to them!) Prof. Kassel's clarity of thought and goal-oriented mind have served as good examples in my writing of this thesis for whenever I was stuck or hopelessly confused. For this and the many other things she has taught me, together with her unwavering willingness to indulge me in conversation for hours, she deserves all of my gratitude. I would also like to thank Profs. Nicolas Tholozan and

²The doodles are not really necessary but they greatly aid in understanding, hence why I have added them despite them being hand-drawn.

Daniel Monclair for their openness in discussing their paper, and them together with Prof. Jean-Marc Schlenker, for having written such interesting mathematics.

2. GROMOV-THURSTON MANIFOLDS

The manifolds which we will be interested in are *Gromov-Thurston manifolds*, which are ramified coverings of *dihedral hyperbolic manifolds*. I like to think of them naively as follows: take your favourite piece of the hyperbolic space with you, start gluing copies of this piece one next to the other. Then it should be intuitively clear that if you do this with a lot of copies you start getting something more and more *curved*. Indeed if you think of doing this with cloth, what you will get is something that has a lot of *area*, i.e. has very negative curvature. This is (very roughly) what we will do below. We begin by making precise what we mean by *dihedral hyperbolic manifold*. Then we introduce the notion of Gromov-Thurston manifold and lastly we give an example of a 2-dimensional dihedral hyperbolic manifold and what the associated Gromov-Thurston manifold would look like. (Notice, Gromov-Thurston manifolds should exist only for dimension ≥ 4 but the construction below can be carried out in any dimension and it is nice to have an example one can visualise. This is the reason for not restricting our attention to manifolds with 4 or more dimensions).

Definition 2.1 (Dihedral hyperbolic manifold). A **n-dihedral hyperbolic manifold** M is a closed orientable hyperbolic p -manifold, with $p \geq 2$, endowed with two isometric involutions $\sigma_1, \sigma_2 : M \rightarrow M$ with the following properties:

- (1) The fixed point sets $\text{Fix}(\sigma_i)$ are connected embedded totally geodesic hypersurfaces.
- (2) The intersection $S = \text{Fix}(\sigma_1) \cap \text{Fix}(\sigma_2)$ is a connected (unless $p = 2$).
- (3) The sets $\text{Fix}(\sigma_1), \text{Fix}(\sigma_2)$ intersect at an angle of π/n .
- (4) The sets $\text{Fix}(\sigma_1), \text{Fix}(\sigma_2)$ are homologically trivial.

What we mean by the fixed point sets being *homologically trivial* is explained by the intersection theory interpretation of homology. In other words, recall that,

$$H_{n-1}(M, \mathbb{Z}/2\mathbb{Z}) \cong H^1(M, \mathbb{Z}/2\mathbb{Z}) \quad (1)$$

$$\cong \text{Hom}_{\text{Grp}}(H_1(M, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \quad (2)$$

$$\cong \text{Hom}_{\text{Grp}}(\pi_1(M), \mathbb{Z}/2\mathbb{Z}) \quad (3)$$

where the first isomorphism is given by Poincaré duality, the second is given by the Universal Coefficients Theorem and the third one is given by the fact that $H_1(M, \mathbb{Z})$ is the abelianisation of $\pi_1(M)$ (and that $\mathbb{Z}/2\mathbb{Z}$ is abelian). Then intersection theory (see [22, Chapter 3]) tells us that any element of $H_{n-1}(M, \mathbb{Z}/2\mathbb{Z})$, through the isomorphism above, can be seen as a map $\phi_H : \pi_1(M) \rightarrow \mathbb{Z}/2\mathbb{Z}$ giving the intersection number modulo 2 with the hypersurface H . Then under this interpretation a hypersurface H is *homologically trivial* if ϕ_H is the zero map (in other words the homology class in $H_{n-1}(M, \mathbb{Z}/2\mathbb{Z})$ associated to ϕ_H is zero).

Now we can define Gromov-Thurston manifolds as certain ramified coverings of a subspace of a dihedral hyperbolic manifold.

Construction 2.2. Let M be the p -dimensional n -dihedral hyperbolic manifold and let $\mathcal{H}_i = \text{Fix}(\sigma_i)$ for $i = 1, 2$. Then we denote by H_1^M, H_2^M the closure of connected components of $\mathcal{H}_1 \setminus S, \mathcal{H}_2 \setminus S$ in M such that the oriented angle between H_1^M and H_2^M is π/n . Observe that by construction the group $D = \langle \sigma_1, \sigma_2 \rangle$ is an n -dihedral group. Which means that $\sigma_1\sigma_2$ is (a ‘rotation by $2\pi/n$ ’) an element of order n . Then we can define $H_{2i}^M = (\sigma_1\sigma_2)^{i-1}(H_2^M)$ and $H_{2i+1}^M = (\sigma_1\sigma_2)^i(H_1^M)$ for $i = 1, \dots, n-1$. Notice that together they divide M into $2n$ pieces $V_1, \dots, V_{2n}$. In other words, V_i is bounded by H_i^M and H_{i+1}^M (we consider V_i to be the closed set bounded by these two hypersurfaces). Observe that a fundamental domain for the action of $R_n = \langle \sigma_1\sigma_2 \rangle$ on M is a union of two consecutive pieces V_i, V_{i+1} . We call **fundamental piece** to the union of $V_1 \cup V_2$ containing H_2^M and bounded by H_1^M and H_3^M . Observe that $\mathcal{H}_1 = H_1^M \cup H_{n+1}^M$ and $\mathcal{H}_2 = H_2^M \cup H_{n+2}^M$.

Given D acts isometrically on M we can define the quotient $\overline{M} = \langle \sigma_1\sigma_2 \rangle \backslash M$. Then M is an n -to-1 covering of $\overline{M}$, ramified over S (we denote the image of S under the canonical quotient map $M \rightarrow \overline{M}$ also by S). In other words, for any point not in S it is an n -to-1 covering, for points in S it is injective.

Definition 2.3 (Gromov-Thurston manifolds). Let M be the p -dimensional n -dihedral hyperbolic manifold. A **Gromov-Thurston manifold** $M^{\frac{k}{n}}$ is a ramified covering of $\overline{M}$ of degree k .

Remark 2.4. (1) Note that M is then a Gromov-Thurston manifold for $k = n$. The quotient $\overline{M}$ is also a Gromov-Thurston manifold this time with $k = 1$.
 (2) One can think of this construction as taking k copies of the fundamental piece and gluing them side-by-side, along the boundary H_i^M .
 (3) Given our ramified coverings are injective at S we will also denote by S all the lifts of this set to $M^{\frac{k}{n}}$.

Notice that the hyperbolic metric on M induces a **cone hyperbolic metric** on $M^{\frac{k}{n}}$ with cone singularity of angle $2\pi k/n$ at S . Recall that a hyperbolic cone p -manifold is one endowed with a hyperbolic metric everywhere except at a codimension 2 singular locus, which is a union of $p-2$ dimensional totally geodesic subspaces, where the metric ‘breaks down’. In other words, we can triangularise M in such a way that in the interior of every p -simplex we have a well-defined hyperbolic metric, which is undefined at the intersection of p -simplices at $p-2$ simplices. This means that M without its singular locus is a hyperbolic manifold. Moreover, the sum of the dihedral angles between the $p-1$ simplices intersecting along a $p-2$ simplex is called the **cone angle** at that $p-2$ simplex.

Denote by $\overline{H}_1, \overline{H}_2$ the image of H_1^M, H_2^M under the projection map into $\overline{M} = R_n \backslash M$ and denote their lifts into $M^{\frac{k}{n}}$ by $H_1, \dots, H_{2k}$ (which are being numbered cyclically

along S). Then we will also call V_j the connected components of $M_n^k \setminus \bigcup_{i=1}^{2k} H_i$ bounded by H_{j-1} and H_j .

In their paper Gromov and Thurston showed that their construction gave manifolds which could be endowed with a Riemannian metric of negative (variable) curvature but no hyperbolic metric. In particular they show:

Fact 2.5. Let M_n^k (with $k > n$) be a Gromov-Thurston p -manifold with $p \geq 4$. Then the fundamental group $\pi_1(M_n^k)$ is not commensurable to a lattice in any Lie group.

2.1. A (visualisable) example. Fact 2.5 does not apply for $p < 4$ (because for these dimensions we can always hyperbolise our Gromov-Thurston manifolds). And so, given this is such an interesting and particular property of Gromov-Thurston manifolds it would make more sense to add the condition that the dimension has to be at least 4, but the construction works in all dimensions and it is instructive to play with the dimension 2 case to better understand the general construction, if only by analogy.

This is the spirit in which we will undertake to obtain an example of a Gromov-Thurston manifold: When I was reading about this it was not clear to me that dihedral hyperbolic manifolds even existed. Of course, Gromov and Thurston [19] show they do, and Pansu's wonderful expository paper [35] on their work clarifies many things, but the uninitiated (like the author) may find that these two references do not give a clear roadmap of where to look for the facts they quote despite the exposition being clear otherwise. So we will content ourselves with a clear roadmap on how to get a dihedral hyperbolic manifold in arbitrary dimension, and so, a Gromov-Thurston manifold in arbitrary dimension.

Ultimately we will *see* (the doodle will appear in the Appendix) a 2-dimensional dihedral hyperbolic manifold, which is really the goal of this subsection.

Recipe 2.6. The recipe to obtain an example of a dihedral hyperbolic p -manifold goes as follows.

- (1) Consider the hyperboloid model of hyperbolic space, namely $\mathbb{H}^p = \{x \in \mathbb{R}^{p,1} \mid \langle x, x \rangle_{p,1} = -1, x_{p+1} > 0\}$. Find a cocompact lattice $\Gamma \subset SO(p, 1)$ that contains the reflection $\phi : \mathbb{R}^{p,1} \rightarrow \mathbb{R}^{p,1}$ defined by,

$$\phi : (x_1, \dots, x_{p+1}) \mapsto (-x_1, x_2, \dots, x_{p+1})$$

i.e. reflection along the $\{x_1 = 0\}$ hyperplane, and the rotation $\psi : \mathbb{R}^{p,1} \rightarrow \mathbb{R}^{p,1}$ given by,

$$\psi : (x_1, x_2, \dots, x_{p+1}) \mapsto (-x_2, x_1, \dots, x_{p+1})$$

i.e. rotation of $\pi/2$ radians on the x_1x_2 -plane. Moreover, find this Γ such that it contains a finite index, torsion free, normal subgroup Γ_0 . Then we can define the manifold $M_0 = \Gamma_0 \backslash \mathbb{H}^p$.

- (2) Let $H = \text{Fix}(\phi) \cap \mathbb{H}^p$ and let $H_0 = p_0(H)$ where $p_0 : \mathbb{H}^p \rightarrow M_0$ is the canonical projection. If H_0 is not *separating* find an index 2 subgroup $\Gamma_1 \subset \Gamma_0$ such that $M_1 = \Gamma_1 \backslash \mathbb{H}^p$ has $H_1 = p_1(H)$ separating, where $p_1 : \mathbb{H}^p \rightarrow M_1$ is the canonical projection map.
- (3) Conclude that $(M_1, \phi, \phi\psi^2)$ is a dihedral hyperbolic manifold by Pansu's [35, Lemma 2.3].

Construction 2.7. Now we clarify each step of the recipe.

- (1) To get the lattice (see [33, Definition 1.3.5]) Γ let $F = \mathbb{Q}(\sqrt{2})$ (a totally real algebraic number field $\mathbb{Q} \subset F$: for what this is see Definition 5.5.6 and Example 6.1.2(2) of [33]). And let $\mathbb{Z}[\sqrt{2}] = \mathcal{O} \subset F$ be its ring of integers (the set of all elements of the field F satisfying a monic polynomial of $\mathbb{Z}[x]$). Let $G = \text{SO}(x_1^2 + \cdots + x_p^2 - \sqrt{2}x_{p+1}^2)$. We want to show that the group

$$G_{\mathcal{O}} = G \cap \text{SL}_{p+1}(\mathcal{O})$$

is a cocompact lattice (i.e. a lattice such that $G/G_{\mathcal{O}}$ is compact). We first show that the group $G_{\mathcal{O}}$ is discrete. Let $\sigma : F \rightarrow F$ be the field automorphism given by conjugation (i.e. $\sqrt{2} \mapsto -\sqrt{2}$). Consider the map

$$\begin{aligned} \tilde{\Delta} : \text{SL}_{p+1}(\mathcal{O}) &\rightarrow \text{SL}_{p+1}(\mathbb{R}) \times \text{SL}_{p+1}(\mathbb{R}) \\ \gamma &\mapsto (\gamma, \sigma(\gamma)) \end{aligned}$$

where $\sigma(\gamma)$ means: apply the field automorphism σ to each entry of the matrix γ . It has discrete image because the map $\Delta : \mathcal{O} \rightarrow \mathbb{R}^2$ defined by $x \mapsto (x, \sigma(x))$ has discrete image. To see this latter fact, think of the Galois norm $x\sigma(x)$ and recall that, given $x \in \mathcal{O}$, it must be an integer. Therefore, given $\tilde{\Delta}$ has discrete image we know that $G_{\mathcal{O}} \subset \text{SL}_{p+1}(\mathcal{O})$ must be discrete (observe that $\tilde{\Delta}$ is the identity in the first component). Foreshadowing set $\Gamma = G_{\mathcal{O}}$, and consider $\tilde{\Delta}(\Gamma)$ which lands in $G \times K$, where $K = \sigma(G) \cong \text{SO}(p+1)$ (and therefore compact). We will show that $\tilde{\Delta}(\Gamma)$ is an arithmetic group (see Definition 5.1.19 of [33]), and hence a lattice. But this is easy: let $\mathcal{L} = \{(x, \sigma(x)) \mid x \in \mathcal{O}^{p+1}\}$, then

$$\tilde{\Delta}(\Gamma) = \{g \in G \times K \mid g\mathcal{L} = \mathcal{L}\}.$$

In other words, the group $\tilde{\Delta}(\Gamma)$ is the group that preserves the $\mathbb{Z}$ -lattice $\mathcal{L}$ (see Section 5.4 of [33]). Indeed, the equality above is not difficult to see, the inclusions follows by the definitions of Γ and $\mathcal{L}$. Therefore, we can ignore the compact factor K and conclude that Γ is an arithmetic group. Indeed, we can conclude that

$$\Gamma = \{g \in G \mid g\mathcal{O}^{p+1} = \mathcal{O}^{p+1}\}. \quad (4)$$

To show cocompactness of Γ notice that the Γ cannot contain any nontrivial unipotent elements (this is the proof using Godement's Criterion, see Example 5.5.4 of [33] for the details. One can also show this directly using Mahler's

- Criterion, see Ghazouani [16] for a proof). Apply Selberg's Lemma ([33, Theorem 4.8.2]) to conclude that there exists a finite index, torsion free, normal subgroup Γ_0 of Γ (the proof of Selberg's Lemma quoted indeed gives that the torsion free, finite index subgroup is also normal). By Equation 4 we get that ϕ and ψ are in Γ as required. Let $M_0 = \Gamma_0 \backslash \mathbb{H}^p$. Then we get that Γ acts by isometries on M_0 (the action descends by normality of Γ_0).
- (2) Recall that a hypersurface H in M_0 is separating if the complement of H has two connected components. Then H is separating if and only if $\phi_H = 0$ (see immediately below Definition 2.1). As before, let $H = \text{Fix}(\phi) \cap \mathbb{H}^p$ and $H_0 = p_0(H)$ where $p_0 : \mathbb{H}^p \rightarrow M_0$ is the canonical projection. Assume then that H_0 is not separating. Then this means that $\phi_{H_0} : \Gamma_0 \rightarrow \mathbb{Z}/2\mathbb{Z}$ must be surjective. Then $\Gamma_1 = \ker(\phi_{H_0})$ is an index 2 subgroup of Γ_0 with the property that $H_1 = p_1(H)$ is separating in $M_1 = \Gamma_1 \backslash \mathbb{H}^p$ (where $p_1 : \mathbb{H}^p \rightarrow M_1$ is the canonical projection). This is because, by construction, we have that $\phi_{H_1} = 0$.
- (3) Then we have a manifold M_1 together with two isometric involutions ϕ and $\phi\psi^2$, such that the fixed point sets $\text{Fix}(\phi)$ and $\text{Fix}(\phi\psi^2)$ intersect at an angle of $\pi/2$, and $\text{Fix}(\phi) = H_1$ and so separating. Therefore, we can conclude that $(M_1, \phi, \phi\psi^2)$ is a 2-dihedral hyperbolic p -manifold by Lemma 2.3 of Pansu [35].

Getting a Gromov-Thurston p -manifold then reduces to taking a ramified covering of this dihedral hyperbolic manifold (in fact, the dihedral hyperbolic manifold in question already gives you a Gromov-Thurston manifold $M_{\frac{k}{n}}$ with $k = n$). Indeed, notice that in the construction of Gromov-Thurston manifolds we could have just as well cut the dihedral hyperbolic manifold along H_1^M and then glued any number of copies of it along this H_1^M (this is somewhat easier to visualise). To make this clear let us see this in 2 dimensions. In this case there is no need for arithmetic groups and all the things we said above as we can simply see two isometric involutions acting on a hyperbolic surface. Indeed, we have that Σ_4 (the genus 4 surface) exactly fits the paradigm spelled out in Construction 2.7. To see a picture of this dihedral hyperbolic manifold, the fixed point set of the isometric involutions, and some associated Gromov-Thurston manifolds see the Appendix (Figures 1, 2, 3).

3. PSEUDO-HYPERBOLIC AND ANTI-DE SITTER GEOMETRY

In this section we introduce the relevant background material on the geometry of the pseudo hyperbolic space (and its double cover). The goal of this section is to use this background knowledge to develop a notion of spacelike $\hat{\mathbb{H}}^{p,q}$ -structure for manifolds.

3.1. Background. Let $\langle \cdot, \cdot \rangle_{p,q+1}$ be the standard signature $(p, q+1)$ non-degenerate bilinear form on $\mathbb{R}^{p+q+1}$. Let the latter space endowed with this form be denoted by

$\mathbb{R}^{p,q+1}$. Define pseudo-hyperbolic space $\mathbb{H}^{p,q}$ as the set

$$\mathbb{H}^{p,q} = \{[x] \in \mathbb{P}(\mathbb{R}^{p,q+1}) \mid \langle x, x \rangle_{p,q+1} < 0\}$$

and its double cover $\hat{\mathbb{H}}^{p,q}$ to be the level set

$$\hat{\mathbb{H}}^{p,q} = \{x \in \mathbb{R}^{p,q+1} \mid \langle x, x \rangle_{p,q+1} = -1\}.$$

The latter space inherits a pseudo-Riemannian structure from the ambient $\mathbb{R}^{p,q+1}$. Indeed the form $\langle \cdot, \cdot \rangle_{p,q+1}$ restricts to a non-degenerate signature (p, q) form on the tangent spaces of $\hat{\mathbb{H}}^{p,q}$, and so $\hat{\mathbb{H}}^{p,q}$ is naturally endowed with a pseudo-Riemannian metric $\hat{g}$ of signature (p, q) . Then the covering map $\hat{\mathbb{H}}^{p,q} \rightarrow \mathbb{H}^{p,q}$ endows $\mathbb{H}^{p,q}$ with a pseudo-Riemannian metric g of signature (p, q) . Observe that the non-trivial automorphism of the double cover $\hat{\mathbb{H}}^{p,q} \rightarrow \mathbb{H}^{p,q}$ is the antipodal map $x \mapsto -x$.

The unparametrised complete geodesics of $\hat{\mathbb{H}}^{p,q}$ are its intersection with linear 2-planes in $\mathbb{R}^{p,q+1}$. The unparametrised complete geodesics of $\mathbb{H}^{p,q}$ are its intersection with the projectivisation of linear 2-planes in $\mathbb{R}^{p,q+1}$. These geodesics divide into three types: spacelike, lightlike and timelike. They are of the first kind if the defining 2-plane has a restricted non-degenerate form of signature $(1, 1)$, of the second kind if it has a restricted degenerate form of signature $(0, 1)$, and of the third kind if it has a restricted non-degenerate form of signature $(0, 2)$. For a parametrised geodesic γ this is equivalent to $\|\gamma\|_{\hat{g}}, \|\gamma\|_g$ being > 0 , $= 0$, or < 0 respectively.

We say two distinct points $x, y \in \mathbb{H}^{p,q}$ are in spacelike, lightlike or timelike position (or spacelike, lightlike or timelike related) if for some (hence any) lifts $\hat{x}, \hat{y} \in \hat{\mathbb{H}}^{p,q}$ the absolute value $|\langle \hat{x}, \hat{y} \rangle_{p,q+1}|$ is > 1 , $= 1$ or < 1 respectively.

The totally geodesic subspaces of $\hat{\mathbb{H}}^{p,q}$ are its intersection with linear subspaces of $\mathbb{R}^{p,q+1}$. The totally geodesic subspaces of $\mathbb{H}^{p,q}$ are its intersection with the projectivisation of linear subspaces of $\mathbb{R}^{p,q+1}$.

When $q = 1$ we will denote $\hat{\mathbb{H}}^{p,1}$ by AdS^{p+1} because the former space is the double cover of the **anti-de Sitter space** AdS^{p+1} , which, predictably, is equal to $\mathbb{H}^{p,1}$.

3.1.1. The conformal model of $\hat{\mathbb{H}}^{p,q}$. We now introduce a useful model for $\hat{\mathbb{H}}^{p,q}$ known as the **conformal model** for the pseudo-hyperbolic space. Let $\langle \cdot, \cdot \rangle \stackrel{\text{def}}{=} \langle \cdot, \cdot \rangle_{p,q+1}$. Consider some timelike $(q+1)$ -subspace T (i.e. such that the standard $(p, q+1)$ form of $\mathbb{R}^{p,q+1}$ restricts to a negative definite non-degenerate form of signature $(0, q+1)$ on T). Then we can consider the orthogonal projection map $p_T : \mathbb{R}^{p,q+1} \rightarrow T$ and with it define a map $r_T : \mathbb{R}^{p,q+1} \rightarrow \mathbb{R}_{\geq 0}$ as follows:

$$r_T(x) = \sqrt{|\langle p_T(x), p_T(x) \rangle|}.$$

In other words, we are taking the distance of the T -component of x in the orthogonal decomposition $T \oplus T^\perp = \mathbb{R}^{p,q+1}$. Define a **standard basis** for the plane T to be an orthonormal basis $\{v_1, \dots, v_{p+q+1}\}$ of $\mathbb{R}^{p,q+1}$ where the last $q+1$ vectors form an orthonormal basis of T .

In what follows, let $\mathbb{S}^q = \{\mathbf{x} \in \mathbb{R}^{q+1} \mid x_1^2 + \dots + x_{q+1}^2 = 1\}$ with its standard round metric $\mathbf{g}_{\mathbb{S}^q}$ inherited from the ambient $\mathbb{R}^q$, and $\mathbb{B}^p = \{\mathbf{x} \in \mathbb{R}^{p+1} \mid x_1^2 + \dots + x_{p+1}^2 = 1, x_1 > 0\}$ with the round metric $\mathbf{g}_{\mathbb{B}^p}$ it inherits from $\mathbb{S}^p$ as its ‘northern hemisphere’.

Proposition 3.1. The map $\Psi_T : \hat{\mathbb{H}}^{p,q} \rightarrow \mathbb{B}^p \times \mathbb{S}^q$ defined by

$$\Psi_T(\mathbf{x}) = \Psi_T(x_1, \dots, x_{p+q+1}) = \left(\frac{1}{r_T(\mathbf{x})}(1, x_1, \dots, x_p), \frac{1}{r_T(\mathbf{x})}(x_{p+1}, \dots, x_{p+q+1}) \right)$$

(where the coordinates are given by a standard basis for T) is a diffeomorphism. Moreover, it is an isometry from $(\hat{\mathbb{H}}^{p,q}, \mathbf{g})$ with its induced metric as a submanifold of $\mathbb{R}^{p,q+1}$ and $(\mathbb{B}^p \times \mathbb{S}^q, \mathbf{y}_0^{-2}\mathbf{g}_{\mathbb{B}^p} \oplus -\mathbf{y}_0^{-2}\mathbf{g}_{\mathbb{S}^q})$.

Proof. That the above map is a diffeomorphism follows from inspection of the coordinate maps. That it is an isometry follows from the fact (refer to Figure 7) that the tangent space $T_{x_0}\hat{\mathbb{H}}^{p,q}$ at each point $x_0 = (v_1, \dots, v_{p+q+1})$ of $\hat{\mathbb{H}}^{p,q}$ can be decomposed into a direct sum $T_{x_0}\mathbf{M} \oplus T_{x_0}\mathbf{N}$ where, writing everything in the chosen standard coordinates $\mathbf{x} = (x_1, \dots, x_{p+q+1})$,

$$\begin{aligned} \mathbf{M} &= \left\{ \mathbf{x} \in \hat{\mathbb{H}}^{p,q} \mid \exists t \geq \frac{1}{r_T(x_0)}, (x_{p+1}, \dots, x_{p+q+1}) = t(v_{p+1}, \dots, v_{p+q+1}) \right\} \\ \mathbf{N} &= \{\mathbf{x} \in \hat{\mathbb{H}}^{p,q} \mid r_T(\mathbf{x}) = r_T(x_0)\}. \end{aligned}$$

We get that the pseudo-Riemannian metric $\mathbf{g}$ restricts to a positive definite form on $\mathbf{M}$ and to a negative definite form on $\mathbf{N}$. Writing $\Psi_T(x_0) = (\mathbf{u}, \mathbf{u}')$, the tangent space $T_{(\mathbf{u}, \mathbf{u}')} \mathbb{B}^p \times \mathbb{S}^q$ equals $T_{(\mathbf{u}, \mathbf{u}')} \mathbb{B}^p \times \{\mathbf{u}'\} \oplus T_{(\mathbf{u}, \mathbf{u}')} \{\mathbf{u}\} \times \mathbb{S}^q$. Then the map $d_{x_0}\Psi_T$ defines isometries $T_{x_0}\mathbf{M} \rightarrow T_{(\mathbf{u}, \mathbf{u}')} \mathbb{B}^p \times \{\mathbf{u}'\}$ and $T_{x_0}\mathbf{N} \rightarrow T_{(\mathbf{u}, \mathbf{u}')} \{\mathbf{u}\} \times \mathbb{S}^q$ where the targets have the claimed metric $\mathbf{y}_0^{-2}\mathbf{g}_{\mathbb{B}^p} \oplus -\mathbf{y}_0^{-2}\mathbf{g}_{\mathbb{S}^q}$. $\blacksquare$

Remark 3.2. The space $(\mathbb{B}^p, \mathbf{g}_{\mathbb{B}^p})$ is the upper hemisphere model of hyperbolic p -space. The image of the plane T under Ψ_T is the set $\{\mathbf{u}_0\} \times \mathbb{S}^q$ where $\mathbf{u}_0 = (1, 0, \dots, 0)$.

Lemma 3.3. Let $\mathbf{x}, \mathbf{y} \in \hat{\mathbb{H}}^{p,q}$ and $\Psi_T(\mathbf{x}) = (\mathbf{u}, \mathbf{u}'), \Psi_T(\mathbf{y}) = (\mathbf{w}, \mathbf{w}')$, where Ψ_T is as in Proposition 3.1 for some chosen timelike $(q+1)$ -subspace T . The points $\mathbf{x}, \mathbf{y} \in \hat{\mathbb{H}}^{p,q}$ are spacelike related if and only if $d_{\mathbb{B}^p}(\mathbf{u}, \mathbf{w}) > d_{\mathbb{S}^q}(\mathbf{u}', \mathbf{w}')$.

Proof. We know that two points $\mathbf{x}, \mathbf{y}$ of $\hat{\mathbb{H}}^{p,q}$ are spacelike related if and only if $\langle \mathbf{x}, \mathbf{y} \rangle < -1$. We compute,

$$\langle \mathbf{x}, \mathbf{y} \rangle = r_T(\mathbf{x})r_T(\mathbf{y})(\langle \mathbf{u}, \mathbf{w} \rangle_{p+1} - \langle \mathbf{u}', \mathbf{w}' \rangle_{q+1}) - 1$$

(this follows from the definition of Ψ_T , see Proposition 3.1), where $\langle \cdot, \cdot \rangle_{n+1}$ is the standard symmetric nondegenerate bilinear positive definite form of signature $(n+1, 0)$. Then we know that $\mathbf{x}, \mathbf{y}$ are spacelike related if and only if $\langle \mathbf{u}, \mathbf{w} \rangle_{p+1} - \langle \mathbf{u}', \mathbf{w}' \rangle_{q+1} < 0$. And so, given $\mathbb{B}^p$ and $\mathbb{S}^q$ are endowed with their spherical metrics, we get that $\mathbf{x}, \mathbf{y}$ are spacelike related if and only if,

$$\cos(d_{\mathbb{B}^p}(\mathbf{u}, \mathbf{w})) - \cos(d_{\mathbb{S}^q}(\mathbf{u}', \mathbf{w}')) < 0$$

which is equivalent to $d_{\mathbb{B}^p}(\mathbf{u}, \mathbf{w}) > d_{\mathbb{S}^q}(\mathbf{u}', \mathbf{w}')$, as required. $\blacksquare$

3.2. Spacelike $\hat{\mathbb{H}}^{p,q}$ -structures. With the background behind us, we will now introduce *spacelike $\hat{\mathbb{H}}^{p,q}$ -structures* which will roughly make our Gromov-Thurston manifold $M^{\frac{k}{n}}$ look like a p -dimensional subspace of $\hat{\mathbb{H}}^{p,q}$ which has the property that locally any two points are spacelike related. This will then setup the first *bridge* in Diagram 1.6, namely from spacelike AdS structures (spacelike $\hat{\mathbb{H}}^{p,1}$ -structures) to GHMC AdS manifolds containing $M^{\frac{k}{n}}$ as the future boundary of the convex core. Moreover, for the general situation, it will give us the necessary technical tools to define and proof (after Section 4) the *bridge* from such structures to hipped p -surfaces (see Diagram 1.7).

Definition 3.4 (Piece-wise geodesic p -surface in $\hat{\mathbb{H}}^{p,q}$). A **piece-wise geodesic p -surface** H in $\hat{\mathbb{H}}^{p,q}$ is a Lipschitz p -submanifold of $\hat{\mathbb{H}}^{p,q}$ which is a finite union $H = \bigcup_{i=1}^k X_i$ where:

- The X_i are convex subsets of totally geodesic copies of $\mathbb{H}^p$ in $\hat{\mathbb{H}}^{p,q}$.
- The intersections $Y_{i+1} \stackrel{\text{def}}{=} X_i \cap X_{i+1}$ are half-spaces of totally geodesic copies of $\mathbb{H}^{p-1}$.
- Under the relative topology in $\hat{\mathbb{H}}^{p,q}$, the boundary of X_i is the union $Y_i \cup Y_{i+1}$ for all $i \in \mathbb{Z}/k\mathbb{Z}$.
- If $Z_{i+1} \stackrel{\text{def}}{=} Y_i \cap Y_{i+1}$, then $Z_i = Z_j$ for all $i, j \in \mathbb{Z}/k\mathbb{Z}$. Denote this intersection by Z , then we have that it is a totally geodesic copy of $\mathbb{H}^{p-2}$ called the **stem** of H . (In other words, $Z = \bigcap_{i=1}^k Y_i$ is also equal to $Y_i \cap Y_{i+1}$ for all $i \in \mathbb{Z}/k\mathbb{Z}$.)

The sets X_i are called the **wedges** of H .

Notation 3.5. In the following we will denote the relative interior of a set X by $\overset{\circ}{X}$.

Definition 3.6 (Spacelike (p, q) -chart). Let $\mathcal{U}$ be an open contractible set of $\mathbb{R}^p$. A map $\phi : \mathcal{U} \rightarrow \hat{\mathbb{H}}^{p,q}$, is a **spacelike (p, q) -chart** if,

- (1) it is Lipschitz continuous and a homeomorphism onto its target,
- (2) any two points $\mathbf{x}, \mathbf{y} \in \phi(\mathcal{U})$ are in spacelike position,
- (3) the image $\phi(\mathcal{U})$ lies in a ball $B_r(\mathbf{x})$, for $\mathbf{x} \in \phi(\mathcal{U})$, such that r is strictly less than the injectivity radius of the exponential map of $\hat{\mathbb{H}}^{p,q}$ at $\mathbf{x}$,
- (4) the image $\phi(\mathcal{U})$ lands in a piece-wise geodesic p -surface $H = \bigcup_{i=1}^k X_i$, and
- (5) for all $\mathbf{x} \in \phi(\mathcal{U}) \cap \overset{\circ}{X}_i$, (notice that at all these points $\phi(\mathcal{U})$ has a tangent space), we have a *coherent choice* of a tuple of $q-1$ orthonormal timelike vectors living in $T_{\mathbf{x}}\phi(\mathcal{U})^\perp \subset T_{\mathbf{x}}\hat{\mathbb{H}}^{p,q}$, where coherence is given by parallel translation. More precisely, let $\mathbf{q}_x^\phi = (v_1, \dots, v_{q-1})$ be the tuple of $q-1$ orthonormal timelike vectors associated to $\mathbf{x}$ and consider any two points $\mathbf{x}, \mathbf{y} \in \phi(\mathcal{U}) \cap \overset{\circ}{X}_i$ with associated $\mathbf{q}_x^\phi, \mathbf{q}_y^\phi$ and let γ be the unique spacelike geodesic between $\mathbf{x}, \mathbf{y}$ (it exists because of the second item and it is unique because of the third item).

Denote by $\tau_\gamma|_{t_0}^{t_1}$ the parallel translation between times t_0, t_1 along γ , and choose these times such that $\gamma(t_0) = x$ and $\gamma(t_1) = y$. Then, we require

$$q_y^\phi = \tau_\gamma|_{t_0}^{t_1}(q_x^\phi)$$

where $\tau_\gamma|_{t_0}^{t_1}(q_x^\phi)$ is the image of the (ordered) set q_x^ϕ under $\tau_\gamma|_{t_0}^{t_1}$.

Remark 3.7. Note that the notation q_x^ϕ includes a mention of the chart ϕ : in practice we will suppress it when there is no possibility of confusion.

Remark 3.8. (1) Notice that the coherence condition is independent between different X_i defining the piece-wise geodesic p -surface. In other words, the coherent choice of $q-1$ timelike orthonormal vectors chosen for $x \in \phi(U) \cap \mathring{X}_i$ has nothing to do with the tuples chosen for $y \in \phi(U) \cap \mathring{X}_j$ for $i \neq j$.

- (2) If $q = 1$ then the fifth point of Definition 3.6 becomes vacuous. By Rademacher's Theorem points at which $\phi(U)$ is not differentiable form a set of measure zero.
- (3) For any point $x \in \phi(U)$ in which the notion of tangent space makes sense, we have that $T_x\phi(U) = T_xX_i$ for some i (because $\phi(U) \cap X_i$ is an open set of X_i by condition (1)). Therefore, given X_i is totally geodesic and any two points of $\phi(U)$ are spacelike related we get that the tangent space $T_x\phi(U)$ is a spacelike p -plane, i.e. such that the standard signature (p, q) form on $T_x\hat{\mathbb{H}}^{p,q}$ restricts to a non-degenerate signature $(p, 0)$ form on $T_x\phi(U)$.

We will need a characterisation of the image of a spacelike (p, q) -chart, in terms of it being the graph of a function in the product structure of $\hat{\mathbb{H}}^{p,q}$ given by Proposition 3.1. This is what the following proposition achieves.

Proposition 3.9. Let M be a p -dimensional topological submanifold of $\hat{\mathbb{H}}^{p,q}$ without boundary such that any two points of M are in spacelike position. Then under the product structure $\hat{\mathbb{H}}^{p,q} \cong \mathbb{B}^p \times \mathbb{S}^q$ given by a choice of timelike $(q+1)$ -plane T , (see Proposition 3.1), M is the graph of a **strictly** 1-Lipschitz map $f : U \rightarrow \mathbb{S}^q$ where U is an open subset of $\mathbb{B}^p$. Moreover, if M is closed, $U = \mathbb{B}^p$.

Proof. Let x, y be any two points of M and consider $\Psi_T(x) = (u, u')$ and $\Psi_T(y) = (w, w')$. The points x, y are in spacelike position, so $\langle x, y \rangle < -1$. Then, by Lemma 3.3, we know that $d_{\mathbb{B}^p}(u, w) > d_{\mathbb{S}^q}(u', w')$ which means that the 1st factor projection from M to $\mathbb{B}^p$ is injective (because failure of injectivity would imply that the condition on the distances would fail). And so there exists a map $f : U \rightarrow \mathbb{S}^q$ which is strictly 1-Lipschitz from some subset U of $\mathbb{B}^p$ to $\mathbb{S}^q$ whose image is $\Psi_T(M)$. What we miss to show is that U is open. But notice that given f is continuous and $\Psi_T(M)$ is its graph, we get that U is homeomorphic to M via the 1st factor projection. The latter is a topological p -manifold and so contains an open neighbourhood around each of its points, therefore U must be open. If M is closed then by the same argument we get that U must also be closed, and so, given $\mathbb{B}^p$ is connected, we get $U = \mathbb{B}^p$. ■

Remark 3.10. (1) The converse is also true ([BK25]) but we will not use it.
 (2) The conclusion of Proposition 3.9 when M is not closed exactly characterises the image of our spacelike (p, q) charts.

Definition 3.11 (Spacelike (p, q) -immersion, Spacelike p -surface). Let M be a Lipschitz p -manifold. The map $\phi : M \rightarrow \hat{\mathbb{H}}^{p,q}$ is a **spacelike (p, q) -immersion** if it is locally a spacelike (p, q) -chart, i.e. if for all $x \in M$ there is an open neighbourhood U containing x such that $\phi|_U$ is a spacelike (p, q) -chart (after precomposition with a bi-Lipschitz chart given by the Lipschitz manifold structure of M). A Lipschitz submanifold $M \subset \hat{\mathbb{H}}^{p,q}$ is called a **spacelike p -surface** if the natural inclusion $i : M \hookrightarrow \hat{\mathbb{H}}^{p,q}$ is a spacelike (p, q) immersion.

Remark 3.12. Notice that a spacelike (p, q) immersion is a locally Lipschitz map which is also a local homeomorphism. By Proposition 3.9 and Remark 3.10, a spacelike p -surface is locally the graph of a strictly 1-Lipschitz map $f : U \rightarrow \mathbb{S}^q$ for some open $U \subset \mathbb{B}^p$.

As is usually the case in Riemannian geometry, to be able to globalise results, we need some version of completeness, (geodesically complete, completeness of the induced distance function, etc). Our situation is no different: we want completeness of a certain natural distance function on our spacelike p -surfaces to be able to globalise key properties (see Proposition 3.24). We proceed to make this precise.

Construction 3.13. Let $\phi : \mathcal{H} \rightarrow \hat{\mathbb{H}}^{p,q}$ be a spacelike (p, q) immersion and let $\gamma : [0, 1] \rightarrow \mathcal{H}$ be a Lipschitz path. Then the composite $\phi \circ \gamma$ is also Lipschitz and so it is differentiable almost everywhere (by Rademacher's Theorem). This means we can define a notion of length for our paths by using the metric of $(\hat{\mathbb{H}}^{p,q}, g)$, namely,

$$L_{\mathcal{H}}^{\phi}(\gamma) = \int_{[0,1]} \sqrt{g((\phi \circ \gamma)'(t), (\phi \circ \gamma)'(t))} dt.$$

From this we can define a distance function on $\mathcal{H}$ by,

$$d_{\mathcal{H}}^{\phi}(x, y) = \inf_{\gamma(0)=x, \gamma(1)=y} L_{\mathcal{H}}^{\phi}(\gamma)$$

where the infimum is taken along all Lipschitz paths γ from x to y . Notice that the distance function makes sense because of the spacelike nature of the image of ϕ locally.

In what follows we will use this notion when ϕ is the natural inclusion into $\hat{\mathbb{H}}^{p,q}$, i.e. when $\mathcal{H}$ is a spacelike p -surface.

Definition 3.14 (Complete spacelike (p, q) -immersion, complete spacelike p -surface). A spacelike (p, q) immersion ϕ is said to be **complete** if the associated distance function $d_{\mathcal{H}}^{\phi}$ is complete. A spacelike p -surface $\mathcal{H}$ is said to be **complete** if the natural inclusion $i : \mathcal{H} \hookrightarrow \hat{\mathbb{H}}^{p,q}$ is a complete spacelike (p, q) immersion.

Definition 3.15 (Spacelike $\hat{\mathbb{H}}^{p,q}$ -structure). A **spacelike $\hat{\mathbb{H}}^{p,q}$ -structure** on a p -dimensional Lipschitz manifold M comprises the following data:

- (1) An open cover $\{\mathcal{U}_i\}$ of M .
- (2) For each $\mathcal{U}_i$ in the open cover, a spacelike (p, q) -chart $\phi_i : \mathcal{U}_i \rightarrow \hat{\mathbb{H}}^{p,q}$.
- (3) For each $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, a transition isometry $g \in \text{SO}(p, q+1)$ such that

$$g \circ \phi_i = \phi_j,$$

and such that $g(q_x^{\phi_i}) = q_{gx}^{\phi_j}$ for all $x \in \phi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ in which $\phi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ is differentiable, and where x and gx land in the same wedge of a piece-wise geodesic p -surface ($q_x^{\phi_i}$ is as in Definition 3.6).

A set of spacelike (p, q) charts $\{(\mathcal{U}_i, \phi_i)\}_i$ defining a spacelike $\hat{\mathbb{H}}^{p,q}$ structure is called an **atlas**. Two spacelike $\hat{\mathbb{H}}^{p,q}$ structures are **equivalent** if the atlases they define are subsets of the same maximal atlas.

The purpose of the above definitions (Definitions 3.6 and 3.11) is to have uniqueness of the transition maps defining a spacelike $\hat{\mathbb{H}}^{p,q}$ -structure, as we now show.

Lemma 3.16. Let M be a p -dimensional manifold and $\mathcal{U}$ an open subset such that $\phi : \mathcal{U} \rightarrow \hat{\mathbb{H}}^{p,q}$ is a spacelike (p, q) -chart. If $g \in \text{SO}_o(p, q+1)$ is such that $g \circ \phi = \phi$ then $g = \text{id}$.

Proof. Let $x \in \phi(\mathcal{U})$ be a point at which $\phi(\mathcal{U})$ is differentiable (existence given by Rademacher's Theorem or condition (1) of Definition 3.6). Then, by definition of spacelike (p, q) -chart and Remark 3.8(3), g must be such that it fixes an orthonormal basis of $T_x\phi(\mathcal{U})$ and the set q_x . These form an orthonormal basis of a copy of $\mathbb{R}^{p,q-1}$ in $T_x\hat{\mathbb{H}}^{p,q}$. Therefore, $g = \text{id}$. ■

With the uniqueness of the transition maps of our spacelike $\hat{\mathbb{H}}^{p,q}$ -structures we can obtain, just as in the classical (G, X) -structure situation, a *developing map* and a *holonomy representation*. We first define the developing map.

Definition 3.17 (Analytic continuation). Let M be a Lipschitz p -dimensional manifold endowed with a spacelike $\hat{\mathbb{H}}^{p,q}$ -structure and denote by $\widetilde{\pi} : \widetilde{M} \rightarrow M$ its universal cover. Let $x_0 \in M$ be a basepoint, $\tilde{x}_0$ a preferred lift in $\widetilde{M}$, and $\phi_0 : \mathcal{U}_0 \rightarrow \hat{\mathbb{H}}^{p,q}$ a spacelike (p, q) -chart around x_0 (i.e. $x_0 \in \mathcal{U}_0$). The **analytic continuation** $\phi_0^{[\alpha]} : \widetilde{M} \rightarrow \hat{\mathbb{H}}^{p,q}$ of ϕ_0 along the (homotopy class of the) path $\alpha : [0, 1] \rightarrow M$ based at x_0 is the map defined as follows:

- (1) Pick a partition $0 = t_0 < \dots < t_n = 1$ of the interval $[0, 1]$.
- (2) For $0 \leq i \leq n$ pick spacelike (p, q) -charts $\phi_i : \mathcal{U}_i \rightarrow \hat{\mathbb{H}}^{p,q}$ such that $\mathcal{U}_i$ contains $\alpha([t_i, t_{i+1}])$ (these exist up to making our partition finer, i.e. n larger).
- (3) Lift these charts (uniquely because of our choice of basepoints) to $\widetilde{M}$ and consider the transition isometries g_i between consecutive charts ϕ_{i-1}, ϕ_i .

- (4) Define $\phi_0^{[\alpha]} : \widetilde{M} \rightarrow \widehat{H}^{p,q}$ to be equal to $\phi_0 \circ \pi$ on the lift of U_0 and equal to $(g_1 \cdots g_i)\phi_i \circ \pi$ on the lift of U_i .

Proposition 3.18. Analytic continuation is well-defined.

Proof. We have to show two things: analytic continuation does not depend on the choice of charts and that it does not depend on the representative within the chosen homotopy class. (1) That it does not depend on the charts can be seen as follows. Consider the first charts $\phi_1 : U_1 \rightarrow \widehat{H}^{p,q}$, $\phi'_1 : U'_1 \rightarrow \widehat{H}^{p,q}$ after ϕ_0 associated to two different coverings of the path α . Assume moreover, that the transition isometries of the definition of analytic continuation are g_1, g'_1 (i.e. $\phi_0 = g_1\phi_1$ on $U_0 \cap U_1$, etc). Then we have a transition isometry g between ϕ_1, ϕ'_1 , namely such that $g\phi'_1 = \phi_1$ on $U_1 \cap U'_1$. By Lemma 3.16 we have that $g'_1 = g_1g$, and so $g_1\phi_1 = g'_1\phi'_1$. The general statement now follows by induction. (2) To see that analytic continuation does not depend on the representative of the homotopy class, observe that we can homotope any two paths by ‘small’ homotopies (i.e. remaining within some chosen charts). Then by the invariance of analytic continuation regardless of the choice of charts, we get the result. $\blacksquare$

Definition 3.19 (Developing map of spacelike $\widehat{H}^{p,q}$ -structure). The **developing map** $\text{dev} : \widetilde{M} \rightarrow \widehat{H}^{p,q}$ associated to a spacelike $\widehat{H}^{p,q}$ -structure on a Lipschitz p -dimensional manifold M (after a choice of basepoint x_0 and lift, and preferred spacelike (p, q) -chart ϕ_0 around x_0) is the map which coincides with the analytic continuation of ϕ_0 along each path: in symbols,

$$\text{dev}(\tilde{\alpha}(1)) = \phi_0^{[\alpha]}(\tilde{\alpha}(1))$$

where $\tilde{\alpha}$ is the unique lift of the path α based at x_0 .

Remark 3.20. By Proposition 3.18 the developing map is well-defined. Notice that by definition it is a spacelike (p, q) -immersion.

With this we have the following theorem that will give us the notion of holonomy representation and give us up-to-what is the developing map unique.

Theorem 3.21. Let M be a p -dimensional manifold endowed with a spacelike $\widehat{H}^{p,q}$ -structure. Let $\widetilde{M}$ be the universal cover of M . Then there is a representation $\rho : \pi_1(M) \rightarrow SO_o(p, q+1)$ such that,

$$\rho([\alpha]) \circ \text{dev} = \text{dev} \circ [\alpha]$$

where on the right-hand side we see $[\alpha]$ as the Deck transformation associated to it as an element of $\pi_1(M)$. Moreover, if there is another such pair (dev', ρ') then there exists a unique $g \in SO_o(p, q+1)$ such that $\text{dev}' = g \circ \text{dev}$ and $\rho' = g \circ \rho \circ g^{-1}$.

Proof. The proof of the same statement in the classical (G, X) -structure case follows through. Here we give the details. Consider a loop $[\alpha] \in \pi_1(M)$ and the analytic

continuation $\phi_0^{[\alpha]}$ associated to this loop (where ϕ_0 is the preferred chart used to define the developing map $\text{dev} : \widetilde{M} \rightarrow \widehat{\mathbb{H}}^{p,q}$). Then in the last chart (which is ϕ_0 because α is a loop) $\phi_0^{[\alpha]} = g_\alpha \phi_0$. And so we can define a map $\rho : \pi_1(M) \rightarrow \text{SO}_o(p, q+1)$ by $\rho([\alpha]) = g_\alpha$. Then by the well-definedness of analytic continuation ρ is well-defined. It is a homomorphism because if we denote the concatenation of loops by $\alpha\gamma$ to mean “first α and then γ ”, then we get that $g_{\alpha\gamma} = g_\alpha g_\gamma$. Let γ be a path in M based at x_0 and $\tilde{\gamma}$ its unique lift then,

$$\begin{aligned} g_\alpha \circ \text{dev}(\tilde{\gamma}(1)) &= g_\alpha \phi_0^{[\tilde{\gamma}]}(\tilde{\gamma}(1)) \\ &= g_\alpha g_\gamma \phi_0(\gamma(1)) \\ &= g_{\alpha\gamma} \phi_0(\gamma(1)) \\ &= \phi_0^{[\alpha\gamma]}(\tilde{\alpha}\tilde{\gamma}(1)) \\ &= \text{dev} \circ [\alpha](\tilde{\gamma}(1)). \end{aligned}$$

To see uniqueness consider another pair (dev', ρ') satisfying the equivariance condition of the theorem. Then we have that around any point $x \in M$ there is an isometry g_x such that $\text{dev}' = g_x \circ \text{dev}$. Then by Lemma 3.16 we get that the map $x \mapsto g_x$ is locally constant and hence constant. The equivariance condition forces $\rho' = g \circ \rho \circ g^{-1}$. ■

Definition 3.22 (Holonomy representation of spacelike $\widehat{\mathbb{H}}^{p,q}$ structure). The representation

$$\rho : \pi_1(M) \rightarrow \text{SO}_o(p, q+1)$$

associated to a developing map $\text{dev} : \widetilde{M} \rightarrow \widehat{\mathbb{H}}^{p,q}$ as in Theorem 3.21 is called the **holonomy representation** associated to the spacelike $\widehat{\mathbb{H}}^{p,q}$ -structure giving the developing map.

Corollary 3.23. Let M be a p -dimensional manifold. Then a representation

$$\rho : \pi_1(M) \rightarrow \text{SO}_o(p, q+1)$$

and a ρ -equivariant spacelike (p, q) immersion

$$\phi : \widetilde{M} \rightarrow \widehat{\mathbb{H}}^{p,q}$$

define a spacelike $\widehat{\mathbb{H}}^{p,q}$ structure on M .

Proof. Choose an open cover $\{U_i\}_i$ of M such that the covering map $\pi : \widetilde{M} \rightarrow M$ is invertible on the U_i and such that ϕ restricted to the U_i is a spacelike (p, q) chart. Then the charts given the spacelike $\widehat{\mathbb{H}}^{p,q}$ structure are the maps $\phi \circ \pi^{-1}|_{U_i}$. ■

3.3. The case $q = 1$. In this section we specialise to the case when $q = 1$. We will denote by spacelike AdS structures our spacelike $\widehat{\mathbb{H}}^{p,1}$ -structures because of the equality $\widehat{\text{AdS}}^{p+1} = \widehat{\mathbb{H}}^{p,1}$. Here we build the promised *bridge* between spacelike AdS structures and GHMC AdS manifolds containing $M^{\frac{k}{n}}$ as the future boundary of the

convex core. We begin with some properties of spacelike AdS immersions, the first of which actually applies for general q .

Proposition 3.24. Let $i : \mathcal{H} \rightarrow \hat{\mathbb{H}}^{p,q}$ be a complete spacelike immersion. Then $\mathcal{H}$ is homeomorphic to $\mathbb{B}^p$. If $q = 1$ then

- (1) any two points of $\mathcal{H}$ are spacelike related,
- (2) the manifold $\mathcal{H}$ is **globally** the graph of a strictly 1-Lipschitz map $f : \mathbb{B}^p \rightarrow \mathbb{S}^q$ (after having chosen some timelike $(q+1)$ -plane, see Proposition 3.1), and
- (3) the map i is an embedding (homeomorphism onto image).

Proof. Consider a decomposition of $\hat{\mathbb{H}}^{p,q} \cong \mathbb{B}^p \times \mathbb{S}^q$ given by a choice of timelike $(q+1)$ -plane T as in Proposition 3.1. Then we can see i as $i(y) = \Psi_T(x(y), f(y))$, where $x : \mathcal{H} \rightarrow \mathbb{B}^p$ is the composite $p_{\mathbb{B}^p} \circ i$, ($p_{\mathbb{B}^p} : \mathbb{B}^p \times \mathbb{S}^q \rightarrow \mathbb{B}^p$ is first projection). We can use the map x to define another distance function on $\mathcal{H}$ as follows. Let $\gamma : [0, 1] \rightarrow \mathcal{H}$ be a Lipschitz path, then, given x is the composite of a Lipschitz map (the map i) with a smooth map (the map $p_{\mathbb{B}^p}$), we have that the composite $x \circ \gamma$ is differentiable almost everywhere. As before, we can define a length associated to γ by

$$L_{\mathcal{H}}^x(\gamma) = \int_{[0,1]} \sqrt{g_{\mathbb{B}^p}((x \circ \gamma)'(t), (x \circ \gamma)'(t))} dt$$

and from this we can get a distance function,

$$d_{\mathcal{H}}^x(u, w) = \inf_{\gamma(0)=u, \gamma(1)=w} L_{\mathcal{H}}^x(\gamma)$$

where the infimum is again taken over all Lipschitz paths from u to w . By definition we get $L_{\mathcal{H}}^i(\gamma) \leq L_{\mathcal{H}}^x(\gamma)$ (because recall that the metric on the product $\mathbb{B}^p \times \mathbb{S}^q$ is $y_0^{-2}g_{\mathbb{B}^p} \oplus -y_0^{-2}g_{\mathbb{S}^q}$). And so, $d_{\mathcal{H}}^i \leq d_{\mathcal{H}}^x$. By completeness of i we get that closed balls are compact for $d_{\mathcal{H}}^i$ (by the Hopf-Rinow Theorem for length spaces [8, Proposition I.3.7]). Then, by the inequality between distance functions, we get that $d_{\mathcal{H}}^x$ is also complete (again by Hopf-Rinow because closed balls are compact). This means that x is a local isometry between a complete metric space $(\mathcal{H}, d_{\mathcal{H}}^x)$ and $\mathbb{B}^p$ with its distance function induced by the spherical metric $g_{\mathbb{B}^p}$. Therefore, it is a covering map ([8, Proposition I.3.28]), and given the target is simply connected we conclude x has to be a homeomorphism. This shows that $\mathcal{H}$ is homeomorphic to $\mathbb{B}^p$.

If $q = 1$ then $\mathcal{H}$ is ‘weakly spacelike’ in the sense of Bonsante-Schlenker ([7, Section 3.2]). And so, given $\mathcal{H}$ is locally spacelike we get that any two points must be spacelike related by [7, Proposition 3.5]. Then, by Proposition 3.9, we know that $\mathcal{H}$ must be globally the graph of a strictly 1-Lipschitz function $f : \mathbb{B}^p \rightarrow \mathbb{S}^q$ (completeness gives us closedness of $\mathcal{H}$). Lastly, given $\mathcal{H}$ is the graph of the map f and x is a homeomorphism, we get $i(y) = \Psi_T(x(y), f(x(y)))$ which immediately gives us that i is an embedding. ■

For the remainder of this section we will work exclusively with the case $q = 1$, so when we say *spacelike p -surface* we mean spacelike p -surface in $\hat{\mathbb{H}}^{p,1} = \text{AdS}^{p+1}$ or simply a *spacelike hypersurface*.

3.3.1. Convexity in AdS^{p+1} . We will need a notion of convexity of a spacelike p -surface in AdS^{p+1} . Choose a time-orientation of AdS^{p+1} .

Definition 3.25 (Spacelike hyperplane). Let Π be a $(p+1)$ -plane in $\mathbb{R}^{p,2}$ such that the restriction of the $(p, 2)$ form of the ambient $\mathbb{R}^{p,2}$ on Π is a nondegenerate signature $(p, 1)$ -form. A **spacelike hyperplane** W of AdS^{p+1} is a connected component of the intersection $\Pi \cap \text{AdS}^{p+1}$. We say that a point $w \in \text{AdS}^{p+1}$ lies **in the past of W** if there is a future-oriented timelike geodesic γ going through w and intersecting W without intersecting the other connected component of $\Pi \cap \text{AdS}^{p+1}$.

Remark 3.26. Indeed the intersection $\Pi \cap \text{AdS}^{p+1}$ is a two-sheeted p -dimensional hyperboloid. Each sheet is a totally geodesic copy of $\mathbb{H}^p$ in AdS^{p+1} . Under the isometry of Proposition 3.1 for $q = 1$, each $\mathbb{B}^p \times \{\theta\}$ is a spacelike hyperplane for all $\theta \in \mathbb{S}^1$.

Definition 3.27 (Past-convexity). We say that a spacelike p -surface $\mathcal{H} \subset \text{AdS}^{p+1}$ is **past-convex** if for all $x \in \mathcal{H}$ there is a spacelike hyperplane W containing x and such that (every point of) $\mathcal{H}$ lies in the past of W . We say $\mathcal{H}$ is **locally past-convex** if for all $x \in \mathcal{H}$ there is a neighbourhood U of x in $\mathcal{H}$ such that for all $w \in U$ there is a spacelike hyperplane W containing w and such that (every point of) U lies in the past of W , in other words, such that $\mathcal{H} \cap U$ is past-convex.

Proposition 3.28. Let $i : \mathcal{H} \hookrightarrow \text{AdS}^{p+1}$ be a complete spacelike p -surface. Then $\mathcal{H}$ is locally past-convex if and only if it is past-convex.

Proof. Assume $\mathcal{H}$ is locally past-convex. By completeness, we get that $\mathcal{H}$ is globally the graph of a strictly 1-Lipschitz map $\theta : \mathbb{B}^p \rightarrow \mathbb{S}^1$ (see Proposition 3.24). Given θ is contracting we know it cannot be surjective (no point lies at a distance of π from the image of a certain point). And so we can construe θ as a real-valued map. Let $(x, \theta(x)) \in \mathcal{H}$ and let W be a spacelike hyperplane witnessing local convexity at this point. Up to conjugation by elements of $\text{SO}_o(p, 2)$ we may assume that $W = \mathbb{B}^p \times \{0\}$ (see Figures 7,7). Seeking a contradiction, assume that $\mathcal{H}$ is not past-convex. Then we get that there is an element $y \in \mathbb{B}^p$ such that $\theta(y) > 0$ (because of our assumption on W). Let γ be the unique spherical geodesic from x to y in $\mathbb{B}^p$. Then there must be a local minimum z of $\theta \circ \gamma$ between x and y . By local past-convexity there must be a spacelike hyperplane W' that witnesses local past-convexity at $(z, \theta(z))$. Notice that W' is given by the graph of a function $\alpha : \mathbb{B}^p \rightarrow \mathbb{S}^1$. Then we get that close to $z = \gamma(t_0)$, $\alpha \circ \gamma(t) \geq \theta \circ \gamma(t) \geq \theta(z)$. In other words, the graph of $\theta \circ \gamma$ close to z is sandwiched between two spacelike hyperplanes W' and $\mathbb{B}^p \times \{\theta(z)\}$ (Figure 7). But

these two spacelike hyperplanes are tangential and hence are one and the same (the planes determining them have to coincide). Therefore, $\theta \circ \gamma$ must be constant in a neighbourhood of z , and hence must be locally constant around every local minimum. But this latter fact would imply that θ would have to be constant between x and y which is a contradiction. $\blacksquare$

3.3.2. Globally hyperbolic spacetimes. To my understanding, *global hyperbolicity* is the right (and strongest) notion of what it means to have an *arrow of time*. In other words, globally hyperbolic spacetimes are those anti-de Sitter manifolds to which you would like to give a time-orientation.

Definition 3.29 (Globally hyperbolic spacetime. Cauchy hypersurface). A **globally hyperbolic spacetime** is a Lorentzian manifold N which admits a **Cauchy hypersurface** $\mathfrak{C}$. The latter being a *spacelike* (metric restricts to a positive definite form) topological submanifold of one dimension lower with the property that every *inextendible* (defined for all time) *causal* (nowhere spacelike) geodesic in N intersects $\mathfrak{C}$ exactly once.

Remark 3.30. If N admits a smooth Cauchy hypersurface then it is diffeomorphic to $\mathfrak{C} \times \mathbb{R}$. In general we only get a homeomorphism.

Definition 3.31 (GHMC AdS manifold). A **GHMC AdS manifold** is a Lorentzian manifold which is,

- (1) An *AdS-manifold*: it has constant negative sectional curvature $= -1$,
- (2) *Globally hyperbolic*, (see Definition 3.29),
- (3) *Cauchy compact*: all admitted Cauchy hypersurfaces are compact,
- (4) *Maximal*: it is not isometric to a proper subset of another globally hyperbolic AdS manifold.

There is another model for anti-de Sitter space which we now introduce because it will let us define the *boundary* of anti-de Sitter space.

The anti-de Sitter space $\hat{\text{AdS}}^{p+1}$ can be identified with an open set of the $(p+1)$ -sphere, seen as the double cover of $\mathbb{P}(\mathbb{R}^{p,2})$. This is achieved via the map,

$$\begin{aligned} \hat{\text{AdS}}^{p+1} &\rightarrow \mathbb{S}^{p+1} = (\mathbb{R}^{p,2} \setminus \{0\}) / \mathbb{R}_{\geq 0} \\ x &\mapsto \mathbb{R}_{\geq 0} x. \end{aligned}$$

Then its *boundary*, which we call the **Einstein space**, is defined as follows.

Definition 3.32. The **Einstein space** Ein^p is

$$\text{Ein}^p = \partial_{\infty} \hat{\text{AdS}}^{p+1} = \{x \in \mathbb{R}^{p,2} \mid \langle x, x \rangle_{p,2} = 0\} / \mathbb{R}_{>0} \text{id}.$$

Remark 3.33. Imagine $\hat{\text{AdS}}^2$ as $\mathbb{B}^1 \times \mathbb{S}^1$ given by Proposition 3.1 (choose the standard timelike $(q+1)$ -plane, namely $\mathbb{R}^{0,q+1}$). Then this can be visualised as the *top half*

of the torus $\mathbb{S}^1 \times \mathbb{S}^1$, by which is meant: slice the torus by a plane which intersects its equatorial circle (which is the same as cutting your (hollow) donut in half in the obvious way), then the top half of the resulting space is your model for anti de Sitter 2-space. In this scenario, the Einstein 1-space is the (topological) boundary of $\mathbb{B}^1 \times \mathbb{S}^1$ as seen in $\mathbb{R}^3$, namely, a disjoint union of two circles.

Definition 3.34 (Invisible domain). Let Λ be a closed set subset of Ein^p . The **invisible domain** of Λ in AdS^{p+1} is the open set

$$\Omega(\Lambda) = \left\{ x \in \text{AdS}^{p+1} \mid \langle x, y \rangle_{p+2} < 0, \text{ for all } [y] \in \Lambda \right\}.$$

The following characterisation of GHMC AdS manifolds is due to work of Mess [31] and Barbot [1].

Theorem 3.35. Let Γ be a subgroup of $\text{SO}_o(p, 2)$ acting freely, properly discontinuously and cocompactly on a spacelike hypersurface $\mathcal{H}$ and let $\partial_\infty \mathcal{H}$ be the boundary of $\mathcal{H}$ in Ein^p (i.e. $= \partial \mathcal{H} \cap \text{Ein}^p$ where $\partial \mathcal{H}$ is the topological boundary of $\mathcal{H}$). Then Γ acts properly discontinuously on $\Omega(\partial_\infty \mathcal{H})$, the quotient $N = \Gamma \backslash \Omega(\partial_\infty \mathcal{H})$ is a GHMC AdS manifold and $\Gamma \backslash \mathcal{H}$ is a Cauchy hypersurface. Conversely, let N be a GHMC AdS manifold of dimension $p+1$. Then there exists a discrete subgroup Γ of $\text{SO}_o(p, 2)$ and a Γ -invariant complete spacelike hypersurface $\mathcal{H} \subset \text{AdS}^{p+1}$ such that Γ acts properly discontinuously and cocompactly on $\mathcal{H}$ and N is isometric to $\Gamma \backslash \Omega(\partial_\infty \mathcal{H})$.

Theorem 3.36. Let M be a closed p -manifold, $\rho : \pi_1(M) \rightarrow \text{SO}_o(p, 2)$ a representation and $\phi : \widetilde{M} \rightarrow \text{AdS}^{p+1}$ a ρ -equivariant spacelike immersion. Then there exists a unique open set $\Omega_\rho \subset \text{AdS}^{p+1}$ such that:

- (1) the group $\pi_1(M)$ acts properly discontinuously on Ω_ρ via ρ ,
- (2) the quotient $N_\rho = \rho(\pi_1(M)) \backslash \Omega_\rho$ is GHMC AdS manifold,
- (3) the map ϕ factors into an embedding (homeomorphism onto image) of M into N_ρ as a Cauchy hypersurface.

Proof. By compactness of M we get that the distance function d_M^ϕ (as defined in Construction 3.13) is complete and so the spacelike immersion ϕ is also complete. By Proposition 3.24(3), the map ϕ is a homeomorphism onto its image. This implies that $\rho(\pi_1(M))$ acts properly discontinuously on $\phi(\widetilde{M})$. Moreover, notice that ρ must be discrete and injective. To see injectivity, assume there is an element $[\alpha] \in \pi_1(M)$ such that $g_\alpha \stackrel{\text{def}}{=} \rho([\alpha])$ is equal to the identity. Then (after having chosen basepoints) the unique lift $\tilde{\alpha}$ of α must also be a loop (because we can transport the situation in the image to the universal cover given ϕ is a homeomorphism). And so, given $\widetilde{M}$ is simply connected, we get that α must be homotopic to the trivial loop. To see discreteness, notice that otherwise, we would have that the fibers of the universal covering map would not be discrete which is a contradiction. Therefore, by Theorem 3.35 we get

that ρ is the holonomy of a GHMC AdS manifold N having $\rho(\pi_1(M)) \backslash \phi(\widetilde{M})$ as Cauchy hypersurface. $\blacksquare$

3.3.3. Convex ruled spacelike $\widehat{\text{AdS}}^{p+1}$ structures.

Definition 3.37 (Convex ruled spacelike AdS structure). Let N be a GHMC AdS manifold. Then a Lipschitz p -submanifold M of N

- (1) is **spacelike** if its lift $\widetilde{M}$ to the universal cover of M (which is an open set of $\widehat{\mathbb{H}}^{p,1} = \widehat{\text{AdS}}^{p+1}$ by Theorem 3.35) is a spacelike p -surface,
- (2) is **past-convex** if $\widetilde{M}$ is past-convex (Definition 3.27),
- (3) is **ruled** if each $x \in M$ is contained in the relative interior of a geodesic segment in N .

A spacelike AdS structure (spacelike $\widehat{\mathbb{H}}^{p,1}$ structure) on M is **convex ruled** if for all spacelike $(p, 1)$ charts $\phi_i : \mathcal{U}_i \rightarrow \widehat{\mathbb{H}}^{p,1}$, $\phi_i(\mathcal{U}_i)$ lands in a past-convex ruled spacelike Lipschitz p -manifold.

The following theorem characterises when the Cauchy hypersurface of a GHMC AdS manifold is past-convex. It is due to work by Barbot-Merigot [3], Barbot [2], and Danciger-Gu  rita  ud-Kassel [13].

Theorem 3.38. Let $N = \Gamma \backslash \Omega$ be a GHMC AdS manifold. The following are equivalent:

- (1) The group Γ is Gromov-hyperbolic,
- (2) The manifold N contains a past-convex Cauchy hypersurface,
- (3) The group Γ acts convex cocompactly on Ω .

Definition 3.39 (Limit set. Convex core). Let $N = \Gamma \backslash \Omega$ be a GHMC AdS manifold. The **limit set** Λ of N is the set

$$\Lambda = \partial\Omega \cap \partial_\infty \widehat{\text{AdS}}^{p+1},$$

where $\partial\Omega$ denotes the topological boundary and $\partial_\infty \widehat{\text{AdS}}^{p+1}$ is as in Definition 3.32. The **convex core** $C(N)$ of N is the set defined by

$$C(N) = \Gamma \backslash (\text{Conv}(\Lambda) \backslash \Lambda),$$

where $\text{Conv}(\Lambda)$ is the convex hull of Λ (i.e. the union of all geodesics joining points of Λ).

Remark 3.40. In the proof of the following proposition whenever the word *convex* is used, we mean to say that any geodesic joining two points of the set in question is completely contained in the set. This is not to be confused with our definition of *past-convex*.

Proposition 3.41. Let N be a GHMC AdS manifold admitting a past-convex Cauchy hypersurface. Then N admits a **unique** past-convex ruled Cauchy hypersurface.

Proof. Let $N = \Gamma \backslash \Omega$, let $\Lambda = \partial\Omega \cap \partial_\infty \widehat{\text{AdS}}^{p+1}$ its limit set and $C(N) = \Gamma \backslash (\text{Conv}(\Lambda) \backslash \Lambda)$ its convex core. Consider the boundary $\partial C(N)$. It has two connected components. By [3, Lemma 4.9] we know that these connected components are Cauchy hypersurfaces and by [3, Lemma 3.16] that they are spacelike hypersurfaces. Pick the future component $\partial^+ C(N)$ of the boundary of the convex core. Then by definition and the above, we have that it is past-convex. That it is ruled follows from the fact we are taking convex hulls in the construction of the convex core. More precisely, let $x \in \partial^+ C(N)$ and let $\tilde{x} \in \partial \text{Conv}(\Lambda)$ be a lift. Given $x \in \partial^+ C(N)$ we know that $\tilde{x}$ cannot be in Λ , and in particular, it cannot be an extreme point. Therefore, it is included in the relative interior of a geodesic contained in $\text{Conv}(\Lambda)$. But this geodesic must be fully contained in $\partial \text{Conv}(\Lambda)$ because the interior of $\text{Conv}(\Lambda)$ is convex.

To see uniqueness, let $S \subset N$ be a past-convex ruled Cauchy hypersurface, and let $\tilde{S}$ be the lift of S to the universal cover of N . By [7, Corollary 3.8] we have that $\partial_\infty \tilde{S} = \partial \tilde{S} \cap \partial_\infty \widehat{\text{AdS}}^{p+1} = \Lambda$ because they have the same invisible domains. Consider the closed convex hull $\mathcal{C}$ of $\tilde{S}$ in $\widehat{\text{AdS}}^{p+1}$. Then it is compact and convex, and so, by the Krein-Milman Theorem, $\mathcal{C}$ is the convex hull of the set of its extreme points E . Given S is ruled then so is $\tilde{S}$, which means that $\tilde{S} \cap E = \emptyset$. But then $E \subset \partial_\infty \tilde{S} = \Lambda$ because $E \subset \tilde{S} \cup \partial_\infty \tilde{S}$. In other words, $\tilde{S} \subset \text{Conv}(\Lambda)$. But $\tilde{S}$ is past-convex by hypothesis. Therefore, given the convex hull is the smallest convex set containing the extreme points (in the compact and convex case), we must have that $S \subset \partial^+ C(N)$, and so they must coincide as they are both Cauchy hypersurfaces. ■

Now we easily get the bridge from spacelike AdS structures on M to GHMC AdS manifolds containing M as the future boundary of the convex core (see Diagram 1.6). This is encapsulated in the following lemma.

Lemma 3.42. Let M be a Lipschitz p -manifold. Then there is a one-to-one correspondence between convex ruled spacelike AdS structures **up to equivalence** and GHMC AdS manifolds **N up to isometry** containing M homeomorphically as the future boundary of the convex core.

Remark 3.43. Notice that the condition on M being homeomorphically contained as the future boundary of the convex core of N is the same as saying that N is homeomorphic to $M \times \mathbb{R}$.

Proof of Lemma 3.42. This follows from Proposition 3.41, Theorem 3.36 and Theorem 3.38 item (2). ■

4. POLYGONS

In this section we will study *spacelike $\hat{\mathbb{S}}^{1,q}$ polygons* (roughly polygons on the surface of $\hat{\mathbb{S}}^{1,q}$ with spacelike edges). These will give us the tools to prove the main theorems of this paper. Simply put, these polygons parametrise the GHMC AdS manifolds

in which we can embed a Gromov-Thurston manifold isometrically when $q = 1$ (in which case we call them *spacelike de Sitter polygons*), and more generally, they parametrise the folded spacelike $\hat{\mathbb{H}}^{p,q}$ structures on $M^{\frac{k}{n}}$. See Diagrams 1.6 and 1.7.

4.1. The case $q \geq 1$. Let $\hat{\mathbb{S}}^{1,q}$ be the (double cover of) the pseudo-Riemannian sphere in $\mathbb{R}^{2,q}$. Namely,

$$\hat{\mathbb{S}}^{1,q} = \{x \in \mathbb{R}^{2,q} \mid \langle x, x \rangle_{2,q} = 1\}.$$

The geodesics of $\hat{\mathbb{S}}^{1,q}$ are branches of hyperbolas if the geodesic is timelike, straight lines if the geodesic is lightlike, and ellipses if the geodesic is spacelike. These are obtained by intersecting linear 2-planes with $\hat{\mathbb{S}}^{1,q}$ in exactly the same way as with $\hat{\mathbb{H}}^{p,q}$. The difference in causal character comes from the anti-isometry between $\hat{\mathbb{H}}^{p,q}$ and $\hat{\mathbb{S}}^{q,p}$. Notice that there is a natural action of $SO(2, q)$ by isometries on $\hat{\mathbb{S}}^{1,q}$, the latter being a level set of the bilinear form preserved by the former group.

4.1.1. Spacelike $\hat{\mathbb{S}}^{1,q}$ polygons.

Definition 4.1 (Positive vector). Let $v \in \hat{\mathbb{S}}^{1,q}$ and let $u \in T_v \hat{\mathbb{S}}^{1,q}$ be spacelike. The vector u is said to be **positive** if there is a q -tuple $(w_1, \dots, w_q)$ of unit timelike orthonormal vectors such that the matrix $[v, u, w_1, \dots, w_q]$ (i.e. the matrix with columns $v, u, w_1, \dots, w_q$) is in $SO_o(2, q)$.

Definition 4.2 (Spacelike $\hat{\mathbb{S}}^{1,q}$ k -polygons). A **spacelike $\hat{\mathbb{S}}^{1,q}$ k -polygon**, for $k \geq 3$, is a k -tuple $(v_1, \dots, v_k)$, where $v_i \in \hat{\mathbb{S}}^{1,q} \subset \mathbb{R}^{2,q}$, together with a k -tuple of $(q-1)$ timelike frames (sets of $q-1$ timelike orthonormal vectors) $(q_1, \dots, q_k)$ with the following properties:

- (1) The vectors v_i, v_{i+1} are in spacelike position, with the length of $[v_i, v_{i+1}]$ between $(0, \pi)$. (Notice this implies uniqueness of the spacelike geodesic $[v_i, v_{i+1}]$).
- (2) The sets $[v_i, v_{i+1}]$, (v_{i-1}, v_i) are disjoint.
- (3) The frame q_i lives in the orthogonal complement of $\text{Span}(v_i, v_{i+1})$ in $\mathbb{R}^{2,q}$, i.e. $q_i \subset \text{Span}(v_i, v_{i+1})^\perp \subset \mathbb{R}^{2,q}$.
- (4) The vector directing the geodesic $[v_i, v_{i+1}]$ is positive.

We will denote such a polygon by the decorated tuple $(v_1, q_1; \dots, v_k, q_k)$. The frames q_i should be thought of being decorations on the *edges* $[v_i, v_{i+1}]$.

Construction 4.3. Observe that the elements of a frame q_i live in $\hat{\mathbb{H}}^{2,q-1}$. And so the set of frames can be seen as an open subset of the manifold $(\hat{\mathbb{H}}^{2,q-1})^{q-1}$. Accordingly, the set of all spacelike $\hat{\mathbb{S}}^{1,q}$ k -polygons, denote it by $\mathcal{U}_k(\hat{\mathbb{S}}^{1,q})$, can be seen as an open subset of the manifold $(\hat{\mathbb{S}}^{1,q} \times (\hat{\mathbb{H}}^{2,q-1})^{q-1})^k$. The reason for it being open is that all conditions in Definition 4.2 are open conditions. In other words, $\mathcal{U}_k(\hat{\mathbb{S}}^{1,q})$ is a

$q(q+1)k$ -dimensional manifold. Moreover, note that $SO_o(2, q)$ acts on $\mathcal{U}_k(\hat{\mathbb{S}}^{1,q})$ via its natural linear action: $(v_1, q_1; \dots; v_k, q_k) \in \mathcal{U}_k(\hat{\mathbb{S}}^{1,q})$ and $g \in SO_o(2, q)$, then

$$g \cdot (v_1, q_1; \dots; v_k, q_k) = (gv_1, g(q_1); \dots; gv_k, g(q_k))$$

where $g(q_i)$ denotes the image of the set q_i under the map g . This action is free because the element g is uniquely determined by what it does to v_1, q_1, v_2 . The action is proper by the following argument: Observe that the action of $SO_o(2, q)$ is completely determined by its action on a hyperplane of $\mathbb{R}^{2,q}$, where the standard $(2, q)$ form restricts to a $(2, q-1)$ form. This means that the properness of the action of $SO_o(2, q)$ reduces to the properness of the action of $SO_o(2, q-1)$. And so, we conclude the action is proper by induction and the fact we know the action of $SO(2)$ is proper on $\mathbb{R}^2$.

Therefore, we can define the **moduli space of spacelike $\hat{\mathbb{S}}^{1,q}$ k -polygons** to be the quotient manifold

$$\mathcal{P}_k(\hat{\mathbb{S}}^{1,q}) = SO_o(2, q) \backslash \mathcal{U}_k(\hat{\mathbb{S}}^{1,q}).$$

Which means that the manifold $\mathcal{P}_k(\hat{\mathbb{S}}^{1,q})$ is of dimension $(q+1)(qk - \frac{q+2}{2})$.

4.1.2. Equilateral polygons. We define an equilateral k -polygon to be one which has all sidelengths equal. More precisely, we can define functions $\ell_i : \mathcal{U}_k(\hat{\mathbb{S}}^{1,q}) \rightarrow \mathbb{R}$ by letting $\ell_i(P)$ to be equal to the length of the geodesic segment $[v_i, v_{i+1}]$. Notice that these functions descend to $\mathcal{P}_k(\hat{\mathbb{S}}^{1,q})$ because they are measure quantities which are isometric invariants.

Definition 4.4 (Equilateral spacelike $\hat{\mathbb{S}}^{1,q}$ k -polygon). An **equilateral spacelike $\hat{\mathbb{S}}^{1,q}$ k -polygon** is a spacelike $\hat{\mathbb{S}}^{1,q}$ k -polygon $P \in \mathcal{P}_k(\hat{\mathbb{S}}^{1,q})$ where

$$\ell_1(P) = \dots = \ell_k(P).$$

4.2. The case $q = 1$. Now we specialise what we said above for the case when $q = 1$. We repeat definitions so as to make it clear what they become in this special case.

4.2.1. Spacelike de Sitter polygons. Let $\hat{\mathbb{S}}^{1,1} = dS^2$ be the double cover of de Sitter 2-space. More precisely let,

$$dS^2 \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^{2,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle_{2,1} = x_1^2 + x_2^2 - x_3^2 = 1\}.$$

Then dS^2 is the (double cover of the) *pseudo-Riemannian 2-sphere*. In other words, dS^2 has constant curvature $+1$ under the induced pseudo-Riemannian metric it inherits from Minkowski 3-space $\mathbb{R}^{2,1}$. Note that a vector of $\mathbb{R}^{2,1}$ is **future pointing** if its third coordinate is positive.

Definition 4.5 (Positive vector). Let $\mathbf{v} \in dS^2$ and $\mathbf{u} \in T_{\mathbf{v}}dS^2$ be a unit spacelike vector. Let $\mathbf{w}$ be the future pointing unit timelike vector in $T_{\mathbf{v}}dS^2$ orthogonal to $\mathbf{u}$. We say the vector $\mathbf{u}$ is **positive** if $\mathbf{w} = \mathbf{v} \times \mathbf{u}$ (the Lorentzian cross product).

Notice that the definition of positive vector retrieves the notion of spacelike orientation by declaring dS^2 to have the spacelike orientation given by the positive vectors $\mathbf{u}$ (more precisely by the bases $\mathbf{v}, \mathbf{u}, \mathbf{w}$.)

We have that, being a level set of the standard bilinear symmetric form of signature $(2, 1)$, it is preserved by $SO(2, 1)$ via its linear action on Minkowski 3-space. The group $SO_o(2, 1)$ preserves both time and space orientations.

The geodesics of dS^2 are its intersection with linear planes. Spacelike geodesics are ellipses, lightlike geodesics straight lines and timelike geodesics are branches of hyperbolas.

Definition 4.6 (Spacelike de Sitter polygon). A spacelike de Sitter k -polygon is a k -tuple $(\mathbf{v}_1, \dots, \mathbf{v}_k) \in (dS^2)^k$ such that

- (1) The geodesics $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ are spacelike of length in $(0, \pi)$.
- (2) The vector directing the geodesic $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ is positive.
- (3) The segments $(\mathbf{v}_{i-1}, \mathbf{v}_i)$ and $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ do not intersect.

Remark 4.7. The first condition ensures that there is a unique geodesic between $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$. The second condition ensures that our polygon is a positively oriented circle. And the third condition ensures that the polygon does not intersect itself.

Construction 4.8. Denote by $\mathcal{U}_k(dS)$ the set of all spacelike de Sitter polygons. Then (given all three conditions in the definition are open conditions) we have that $\mathcal{U}_k(dS)$ is an open set of $(dS^2)^k$ and so a $2k$ -manifold. Moreover, observe that $SO_o(2, 1)$ acts on dS^2 freely and smoothly by isometries. To see it acts properly, observe that an element of $SO_o(2, 1)$ is completely determined by its action on a non-isotropic (i.e. not lightlike) plane, like the plane spanned by two consecutive vertices of a spacelike polygon. And so the properness of the action reduces to the properness of the action of $SO(2)$ on the plane.

Therefore, we can take the quotient,

$$\mathcal{P}_k(dS) \stackrel{\text{def}}{=} SO_o(2, 1) \backslash \mathcal{U}_k(dS)$$

and get that it is a manifold of dimension $2k - 3$. We will call the manifold $\mathcal{P}_k(dS)$ the **moduli space of spacelike de Sitter polygons**.

We will need the following auxiliary vectors:

- The unit vector $\mathbf{u}_i^+ \in T_{\mathbf{v}_i} dS^2$ directs the geodesic $[\mathbf{v}_i, \mathbf{v}_{i+1}]$.
- The unit vector $\mathbf{u}_i^- \in T_{\mathbf{v}_i} dS^2$ directs the geodesic $[\mathbf{v}_i, \mathbf{v}_{i-1}]$.
- The (timelike) unit vector $\mathbf{w}_i \in T_{\mathbf{v}_i} dS^2$ is the Lorentzian cross product of $\mathbf{v}_i$ and $\mathbf{u}_i^+$.

We define the map $\ell_i : \mathcal{U}_k(dS) \rightarrow (0, \pi)$ which takes in a polygon and gives the length of the geodesic $[\mathbf{v}_i, \mathbf{v}_{i+1}]$. From the auxiliary vectors we can define as well the map $\theta_i : \mathcal{U}_k(dS) \rightarrow \mathbb{R}$ defined to be the hyperbolic (Lorentzian) angle between $\mathbf{u}_i^+$ and

$-\mathbf{u}_i^-$. In other words, such that

$$-\mathbf{u}_i^- = \cosh(\theta_i)\mathbf{u}_i^+ + \sinh(\theta_i)\mathbf{w}_i.$$

These maps are smooth and descend to the quotient (because they measure things which are isometrically invariant).

4.2.2. Lie group actions. In what follows we will need a technical statement (Theorem 4.15). For it we review some of the theorems we will need from the general theory of Lie group actions on manifolds. See [30] for an account.

Let G a Lie group, M a manifold with a G -action. Then we have the following notion of Lie algebra action.

Definition 4.9 (Lie algebra actions). Let $\mathfrak{g}$ be the Lie algebra of G . Then a **Lie algebra action** on M is a smooth map,

$$\begin{aligned} \mathfrak{g} \times M &\rightarrow TM \\ (X, m) &\mapsto X.m \in T_m M \end{aligned}$$

such that the resulting map $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism. The set $\mathfrak{g}_m = \{X \in \mathfrak{g} \mid X.m = 0\}$ is called the **stabiliser algebra** of m . A Lie algebra action is said to be **free** if all stabiliser algebras are trivial.

Fact 4.10. [30, Theorem 1.6] Let $\mathfrak{g}$ be the Lie algebra of G . For $g \in G$, $m \in M$, let $(g, m) \mapsto g.m$ be a G -action on M , then there is a corresponding Lie algebra action is given by,

$$X.m = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX).m)$$

where $\exp(tX).m$ is the G -action on M .

Fact 4.11. [30, Proposition 1.20] The orbits $\mathcal{O}$ associated to this G -action are diffeomorphic to G/G_m for any $m \in \mathcal{O}$. Moreover, if the G -action on M is proper then,

$$T_m \mathcal{O} = \{X.m \mid X \in \mathfrak{g}\}$$

(where $\mathfrak{g}$ is the Lie algebra of G and the Lie algebra action in braces is the one of Proposition 4.10).

Fact 4.12. [30, Proposition 1.17] If the G -action on M is free then the corresponding Lie algebra action is free.

Remark 4.13. By Propositions 4.11 and 4.12 we get that if M is endowed with a free G -action then $T_m \mathcal{O} \cong \mathfrak{g}$, where the explicit isomorphism is given by $X.m \mapsto X$.

Fact 4.14. [30, Theorem 1.21] If M is endowed with a proper and free (left) G -action then the quotient $G \backslash M$ admits a unique smooth structure such that the quotient map $\pi : M \rightarrow G \backslash M$ is a submersion making M into a **G -principal bundle over $G \backslash M$** . (In other words, for all $x \in G \backslash M$ there exists a neighbourhood U of x in $G \backslash M$ such that $\pi^{-1}(U)$ is diffeomorphic to $U \times G$.)

With this we can finally prove the technical result alluded to above.

Theorem 4.15. The map

$$\begin{aligned}\Phi : \mathcal{P}_k(\text{dS}) &\rightarrow (0, \pi)^k \times \mathbb{R}^k \\ \bar{P} &\mapsto (\ell_1(\bar{P}), \dots, \ell_k(\bar{P}), \theta_1(\bar{P}), \dots, \theta_k(\bar{P}))\end{aligned}$$

is an embedding. Moreover, the image of its differential at any point can be described explicitly as follows:

$$\text{im}(T_{\bar{P}}\Phi) = \left\{ (\dot{\ell}_1, \dots, \dot{\ell}_k, \dot{\theta}_1, \dots, \dot{\theta}_k) \mid \sum_{i=1}^k \dot{\theta}_i \mathbf{v}_i + \dot{\ell}_i \mathbf{w}_i = 0 \right\} \quad (5)$$

for $\bar{P} \in \mathcal{P}_k(\text{dS})$.

Proof. A polygon is uniquely determined by the length of its sides and the angles between them so we get that Φ is a homeomorphism. To show it is an immersion we will have to work harder.

Recall that $\mathcal{P}_k(\text{dS}) = \text{SO}_o(2, 1) \backslash \mathcal{U}_k(\text{dS})$. Let $\bar{P} \in \mathcal{P}_k(\text{dS})$ be the image of $P \in \mathcal{U}_k(\text{dS})$ under the quotient map where $P = (\mathbf{v}_1, \dots, \mathbf{v}_k)$. By Fact 4.14, $\mathcal{U}_k(\text{dS})$ is a $\text{SO}_o(2, 1)$ -principal bundle over $\mathcal{P}_k(\text{dS})$, so we get that,

$$T_P \mathcal{U}_k(\text{dS}) = V \oplus T_P(\text{SO}_o(2, 1).P), \quad (6)$$

where $V \cong T_{\bar{P}} \mathcal{P}_k(\text{dS})$ and $\text{SO}_o(2, 1).P$ is the orbit containing P .

Let $\dot{\mathbf{v}} = (\dot{\mathbf{v}}_1, \dots, \dot{\mathbf{v}}_k) \in T_{\bar{P}} \mathcal{P}_k(\text{dS})$ (think of a tangent vector $\dot{\mathbf{v}}_i$ at each vertex $\mathbf{v}_i$ of P). Injectivity of the differential $T_{\bar{P}}\Phi$ is equivalent to the simultaneous injectivity of the maps $T_{\bar{P}}\theta_i$ and $T_{\bar{P}}\ell_i$. By the identification given by equation (6), we can construe the map $T_{\bar{P}}\theta_i$ as $T_P\theta_i|_V$, (where here we are thinking about the map that descends to the quotient and the map upstairs respectively), and likewise for $T_{\bar{P}}\ell_i$. And so, $T_P\theta_i(\dot{\mathbf{v}}) = T_P\ell_i(\dot{\mathbf{v}}) = 0$ if and only if $\dot{\mathbf{v}} \in T_P(\text{SO}_o(2, 1).P)$, implies $T_{\bar{P}}\theta_i(\dot{\mathbf{v}}) = T_{\bar{P}}\ell_i(\dot{\mathbf{v}}) = 0$ if and only if $\dot{\mathbf{v}} = 0$ (where we are once again using the identification in equation (6) to see $\dot{\mathbf{v}}$ in $T_P \mathcal{U}_k(\text{dS})$ and in $T_{\bar{P}} \mathcal{P}_k(\text{dS})$). By Remark 4.13 we get that $T_P \text{SO}_o(2, 1).P \cong \mathfrak{so}(2, 1)$, where the isomorphism is given by $X.P \mapsto X$. Therefore, what we want is to prove that for each $\dot{\mathbf{v}}$ such that $T_P\theta_i(\dot{\mathbf{v}}) = T_P\ell_i(\dot{\mathbf{v}}) = 0$ there is an element $X \in \mathfrak{so}(2, 1)$ such that $X.P = X.(\mathbf{v}_1, \dots, \mathbf{v}_k) = \dot{\mathbf{v}}$. More concretely,

$$\dot{\mathbf{v}}_i = X.\mathbf{v}_i = \left. \frac{d}{dt} \right|_{t=0} (\exp(tX).\mathbf{v}_i) = \left. \frac{d}{dt} \right|_{t=0} \sum_{k=0}^{\infty} \frac{t^k X^k}{k!} \mathbf{v}_i = X\mathbf{v}_i, \text{ for all } i,$$

where the last expression is ordinary matrix multiplication.

To show this let $\gamma : (-\epsilon, \epsilon) \rightarrow \mathcal{U}_k(\text{dS})$ be a geodesic such that $\gamma(0) = P$ and $\gamma'(0) = \dot{\mathbf{v}}$, and consider the vectors $\mathbf{u}_i^+$, $\mathbf{u}_i^-$ and $\mathbf{w}_i$ as the polygon P varies along γ . More precisely, consider how these vectors vary as the vertices $\mathbf{v}_i$ of P vary along the tangent vectors $\dot{\mathbf{v}}_i$. (refer to Figure 7). Let α_i to be the unique spacelike geodesic between $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ with unit speed vector at $\mathbf{v}_i$ equal to $\mathbf{u}_i^+$ and γ_i be the unique

(inside a sufficiently small neighbourhood of $\mathbf{v}_i$) geodesic going through $\mathbf{v}_i$ with unit speed vector $\dot{\mathbf{v}}_i$. Then $\alpha_i(0) = \mathbf{v}_i$, $\alpha_i(\phi_i) = \mathbf{v}_{i+1}$ where $\cos(\phi_i) = \langle \mathbf{v}_i, \mathbf{v}_{i+1} \rangle$ (this holds because $\mathbf{v}_i, \mathbf{v}_{i+1}$ are in spacelike position), and $\alpha'_i(0) = \mathbf{u}_i^+$. Define

$$\mathbf{v}_i^t \stackrel{\text{def}}{=} \gamma_i(t) = (x_i(t), y_i(t), z_i(t)), \quad (7)$$

and let α_i^t be the unique spacelike geodesic between $\mathbf{v}_i^t$ and $\mathbf{v}_{i+1}^t$ (such a geodesic exists for sufficiently small t because being in spacelike position is an open condition). Note that $\alpha_i^0 = \alpha_i$. Then, recalling that spacelike geodesics of dS^2 are arcs of ellipses, and so circles in some plane under appropriate base change, ([34, Proposition 28, p.112]) we get,

$$\alpha_i^t(\eta) = (\cos(\eta) - \sin(\eta)\tan(\phi_i))\mathbf{v}_i^t + \frac{\sin(\eta)}{\sin(\phi_i)}\mathbf{v}_{i+1}^t. \quad (8)$$

From this we may compute the vectors $\mathbf{u}_{i,t}^+$ and $\mathbf{w}_i^t$ which are the analogues for $\mathbf{v}_i^t$ of $\mathbf{u}_i^+$ and $\mathbf{w}_i$. In other words, $(\alpha_i^t)'(0) = \mathbf{u}_{i,t}^+$ and $\mathbf{w}_i^t = \mathbf{v}_i^t \times \mathbf{u}_{i,t}^+$. We get,

$$\mathbf{u}_{i,t}^+ = -\tan(\phi_i)\mathbf{v}_i^t + \frac{1}{\sin(\phi_i)}\mathbf{v}_{i+1}^t$$

and from this, and equation (7),

$$\begin{aligned} \sin(\phi_i)\mathbf{w}_i^t &= (y_i(t)z_{i+1}(t) - z_i(t)y_{i+1}(t))\mathbf{e}_1 \\ &\quad + (z_i(t)x_{i+1}(t) - x_i(t)z_{i+1}(t))\mathbf{e}_2 \\ &\quad + (y_i(t)x_{i+1}(t) - x_i(t)y_{i+1}(t))\mathbf{e}_3. \end{aligned}$$

(Note that $\mathbf{u}_{i,0}^+ = \mathbf{u}_i^+$ and $\mathbf{w}_i^0 = \mathbf{w}_i$). We now compute

$$\dot{\mathbf{w}}_i = \left. \frac{d}{dt} \right|_{t=0} \mathbf{w}_i^t = \frac{1}{\sin(\phi_i)}(W_1\mathbf{e}_1 + W_2\mathbf{e}_2 + W_3\mathbf{e}_3)$$

where

$$\begin{aligned} W_1 &= y'_i(0)z_{i+1}(0) + y_i(0)z'_{i+1}(0) - z'_i(0)y_{i+1}(0) - z_i(0)y'_{i+1}(0) \\ W_2 &= x'_i(0)z_{i+1}(0) + x_i(0)z'_{i+1}(0) - z'_i(0)x_{i+1}(0) - z_i(0)x'_{i+1}(0) \\ W_3 &= -x'_i(0)y_{i+1}(0) - x_i(0)y'_{i+1}(0) + y'_i(0)x_{i+1}(0) + y_i(0)x'_{i+1}(0). \end{aligned}$$

By carrying out the routine (but tedious) computations we get

$$\langle \dot{\mathbf{w}}_i, \mathbf{w}_i \rangle = -\langle \mathbf{v}_i, \mathbf{v}_{i+1} \rangle (\langle \dot{\mathbf{v}}_i, \mathbf{v}_{i+1} \rangle + \langle \mathbf{v}_i, \dot{\mathbf{v}}_{i+1} \rangle) \quad (9)$$

$$\langle \dot{\mathbf{w}}_i, \mathbf{v}_i \rangle = -\langle \mathbf{w}_i, \dot{\mathbf{v}}_i \rangle \quad (10)$$

$$\langle \dot{\mathbf{w}}_i, \mathbf{v}_{i+1} \rangle = -\langle \mathbf{w}_i, \dot{\mathbf{v}}_{i+1} \rangle \quad (11)$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{2,1}$ — the standard symmetric bilinear form of signature $(2, 1)$.

Claim 4.16. There exists a unique $X_i \in \mathfrak{so}(2, 1)$ such that $X_i \mathbf{v}_i = \dot{\mathbf{v}}_i$ and $X_i \mathbf{v}_{i+1} = \dot{\mathbf{v}}_{i+1}$.

Proof of claim. Pick an $X_i \in \mathfrak{so}(2, 1)$ such that $X_i v_i = \dot{v}_i$. Then we have that the curves $[t \mapsto \exp(tX_i).v_i]$ and $[t \mapsto \exp(tX_i).v_{i+1}]$ have speed vectors at 0 equal to $\dot{v}_i = X_i v_i$ and $X_i v_{i+1}$ respectively (see Fact 4.10). Observe that given the variation along $\dot{v}$ leaves lengths and angles infinitesimally invariant ($T_P \theta_i(\dot{v}) = T_P \ell_i(\dot{v}) = 0$) we can choose X_i such that the speed vector of the curve $[t \mapsto \exp(tX_i).v_{i+1}]$ is $\dot{v}_{i+1}$. This is because the fact that the variation along $\dot{v}$ acts as an infinitesimal isometry exactly says that $\dot{v}_{i+1}$ is one of the possible velocities (directional vector) which an isometry can take at v_{i+1} having already taken the velocity $\dot{v}_i$ at v_i (the isometry in question being the exponential of (some multiple of) X_i). Finally observe that, v_i, v_{i+1}, w_i are a basis of $\mathbb{R}^{2,1}$, so the equations $X_i v_i = \dot{v}_i$ and $X_i v_{i+1} = \dot{v}_{i+1}$ uniquely determine an element of $\mathfrak{so}(2, 1)$ (the first two columns determine an element of $\mathfrak{so}(2, 1)$). ■

Due to the claim and equation (9), we conclude that $\langle \dot{w}_i, w_i \rangle = 0$. Define a linear endomorphism $\alpha_i \in \text{End}(\mathbb{R}^3)$ by

$$\begin{aligned} \alpha_i v_i &= \dot{v}_i, \\ \alpha_i v_{i+1} &= \dot{v}_{i+1}, \\ \alpha_i w_i &= \dot{w}_i. \end{aligned}$$

Then the relation $\langle \alpha_i v, w \rangle = -\langle v, \alpha_i w \rangle$ is satisfied on the basis (v_i, v_{i+1}, w_i) and so $\alpha_i \in \mathfrak{so}(2, 1)$ (notice $\alpha_i = X_i$ as defined in the claim). We want to show that $\alpha_i = \alpha_{i-1}$. We have

$$w_{i-1} = \cosh(\theta_i) w_i + \sinh(\theta_i) u_i^+.$$

To check this is the case recall that $w_{i-1} = v_{i-1} \times u_{i-1}^+$: the relevant orthogonality relations can be checked by computing $v_{i-1} = \sin(\phi_{i-1}) u_i^- + \cos(\phi_{i-1}) v_i$ and $u_{i-1}^+ = -\cos(\phi_{i-1}) u_i^- + \sin(\phi_{i-1}) v_i$ and recalling that $\langle w_i, -u_i^- \rangle = -\sinh(\theta_i)$. Then, given $T_P(\theta_i)(\dot{v}) = 0$ we get,

$$\begin{aligned} \alpha_{i-1} w_{i-1} &= \dot{w}_{i-1} \\ &= \cosh(\theta_i) \dot{w}_i + \sinh(\theta_i) \dot{u}_i^+ \\ &= \cosh(\theta_i) \alpha_i w_i + \sinh(\theta_i) \alpha_i u_i^+ \\ &= \alpha_i w_{i-1}. \end{aligned}$$

Therefore, the map $\alpha_i - \alpha_{i-1} \in \mathfrak{o}(2, 1)$ has kernel of dimension at least 2 (because we also have $\alpha_{i-1} v_i = \dot{v}_i$ by definition) and so it must be the 0-map. In other words, $\alpha_i = \alpha_{i-1}$. Thus, by induction, we get that all the α_i so constructed are the same and that this common value α is such that $\alpha.v = \dot{v}$, which implies that Φ is an immersion and so an embedding.

To characterise the image of the differential we will proceed as follows. Note that the set of tuples $(\ell_1, \dots, \ell_k, \dot{\theta}_1, \dots, \dot{\theta}_k)$ satisfying equation (5) is a $2k - 3$ dimensional space because the vectors v_{i-1}, v_i, w_i form a basis of $\mathbb{R}^{2,1}$. So, because of injectivity of the differential we get that it suffices that the condition of equation (5) is satisfied

in the image of $T_{\bar{P}}\Phi$. Moreover, by linearity, it suffices to show that this condition holds when a single vertex is varying, i.e. for $\mathbf{v}_2$, say, varying and all other vertices remaining fixed. In other words, it suffices to show that the condition holds for speed vectors $\dot{\mathbf{v}} = (\dot{\mathbf{v}}_1, \dots, \dot{\mathbf{v}}_k)$ where a single $\dot{\mathbf{v}}_i$ is nonzero.

Let $\bar{P}$ be a k -gon with no three vertices collinear. Then the set $\mathbf{u}_i^-, \mathbf{u}_i^+$ forms a basis of $T_{\mathbf{v}_i}dS^2$ for all i , and so it suffices to show the condition holds for speed vectors $\dot{\mathbf{v}}$ with $\dot{\mathbf{v}}_2 = \mathbf{u}_2^+$ and all other $\dot{\mathbf{v}}_i = 0$ (the case $\mathbf{u}_2^-$ is exactly the same argument). By a density argument, doing this will conclude the proof.

We compute the first order variations of $\dot{\ell}_1, \dot{\ell}_2, \dot{\theta}_1, \dot{\theta}_2$ all others being $= 0$.

We get from our expressions for $\mathbf{v}_i^t$ and the associated vectors $\mathbf{u}_{i,t}^-, \mathbf{u}_{i,t}^+$ that (see Figure 7)

$$\begin{aligned}\cosh(\theta_1^t) &= \langle -\mathbf{u}_1^-, \mathbf{u}_{1,t}^+ \rangle \\ \cosh(\theta_2^t) &= \langle -\mathbf{u}_{2,t}^-, \mathbf{u}_{2,t}^+ \rangle \\ \ell_1^t &= \arccos(\langle \mathbf{v}_1, \mathbf{v}_2^t \rangle) \\ \ell_2^t &= \arccos(\langle \mathbf{v}_2, \mathbf{v}_3 \rangle) - t.\end{aligned}$$

From which we get,

$$\begin{aligned}\dot{\theta}_1 &= \frac{\sinh(\theta_2)}{\sin(\ell_1)} \\ \dot{\theta}_2 &= -\sinh(\theta_2)\cotan(\ell_1) \\ \dot{\ell}_1 &= \cosh(\theta_2) \\ \dot{\ell}_2 &= -1 \\ \dot{\ell}_i &= \dot{\theta}_i = 0, \quad \forall i \neq 1, 2.\end{aligned}$$

In the orthonormal frame $(\mathbf{v}_2, \mathbf{u}_2^+, \mathbf{w}_1)$ we have,

$$\begin{aligned}\mathbf{v}_1 &= (\cos(\ell_1), -\sin(\ell_1), 0) \\ \mathbf{v}_2 &= (1, 0, 0) \\ \mathbf{w}_1 &= (0, 0, 1) \\ \mathbf{w}_2 &= (0, -\sinh(\theta_2), \cosh(\theta_2)).\end{aligned}$$

Finally, plugging in we get,

$$\sum_{i=1}^k \dot{\theta}_i \mathbf{v}_i + \dot{\ell}_i \mathbf{w}_i = \dot{\theta}_1 \mathbf{v}_1 + \dot{\theta}_2 \mathbf{v}_2 + \dot{\ell}_1 \mathbf{w}_1 + \dot{\ell}_2 \mathbf{w}_2 = 0$$

as required. ■

4.2.3. Convex polygons.

Definition 4.17 (Convex polygon). A spacelike de Sitter polygon P is said to be **convex** if for all i we have

$$\theta_i(P) \geq 0.$$

By definition, a spacelike de Sitter polygon is a topological circle and so separates dS^2 into two connected components. Another way of saying such a polygon is convex is to say that the connected component in its past is past-convex.

4.2.4. Equilateral de Sitter polygons.

Definition 4.18. The set of all spacelike equilateral de Sitter (i.e. $S^{1,1}$) polygons will be denoted by $\mathcal{P}_k^{\text{eq}}(dS) \subset \mathcal{P}_k(dS)$ and the set of all such equilateral polygons with sidelength ℓ will be denoted by $\mathcal{P}_k^\ell(dS) \subset \mathcal{P}_k^{\text{eq}}(dS)$.

Proposition 4.19. The space $\mathcal{P}_k^{\text{eq}}(dS)$ is a submanifold of dimension $k - 2$. For all $\ell > 2\pi/k$, the space $\mathcal{P}_k^\ell(dS)$ is a submanifold of dimension $k - 3$.

Proof. Consider the maps

$$\begin{aligned} f : \mathcal{P}_k(dS) &\rightarrow \mathbb{R}^k \\ P &\mapsto (\ell_1(P), \dots, \ell_k(P)), \end{aligned}$$

and

$$\begin{aligned} g : \mathcal{P}_k(dS) &\rightarrow \mathbb{R}^{k-1} \\ P &\mapsto (\ell_2(P) - \ell_1(P), \dots, \ell_k(P) - \ell_{k-1}(P)). \end{aligned}$$

Then, $\mathcal{P}_k^{\text{eq}}(dS) = g^{-1}(0, \dots, 0)$ and $\mathcal{P}_k^\ell(dS) = f^{-1}(\ell, \dots, \ell)$. In other words, we want to show that the points $(0, \dots, 0)$ and $(\ell, \dots, \ell)$ are regular values of g, f respectively. For f , by Theorem 4.15 the value $(\dot{\ell}_1, \dots, \dot{\ell}_k)$ lies in the image of $d_P f$ if and only if there exists $(\dot{\theta}_i)_i$ such that,

$$\sum_{i=1}^k \dot{\theta}_i v_i = - \sum_{i=1}^k \dot{\ell}_i w_i.$$

And this happens if the v_i span $\mathbb{R}^3$. Else, if the v_i do not span an $\mathbb{R}^3$, the polygon P lies in an equator and so has sidelength $\ell = 2\pi/k$. This proves the second claim. For g , note that the same argument as for f shows that it is a submersion for all points in the inverse image of $(0, \dots, 0)$ whenever the polygon P does not lie in an equator. Assume then that it does. Then the vectors w_i are all equal to some vector w orthogonal to the plane defining the equator. Let $(\delta_i)_{1 \leq i \leq k}$ be any vector. Define $s = \frac{1}{k} \sum_{i=1}^k \sum_{j=1}^{i-1} \delta_i$ and $\dot{\ell}_i = -s + \sum_{j=1}^{i-1} \delta_i$. Then we have that

$$\sum_{i=1}^k \dot{\theta}_i v_i + \dot{\ell}_i w_i = \sum_{i=1}^k \dot{\ell}_i w = 0.$$

We conclude by observing that $\dot{\ell}_{i+1} - \dot{\ell}_i = \delta_i$. In other words, $\mathbf{d}_p \mathbf{g}$ contains $(\delta_i)_i$ in its image, and so it is a submersion if $\mathbf{P}$ lies inside of an equator as well. ■

We will need the following easy proposition in Subsection 6.2.

Proposition 4.20. The tuple $(\mathbf{v}_j)_{j=1}^{2k}$ given by

$$\mathbf{v}_j = \sqrt{\alpha_{k,n}} \left(\cos \left(\frac{\pi j}{k} \right) \mathbf{e}_1 + \sin \left(\frac{\pi j}{k} \right) \mathbf{e}_2 \right) + \sqrt{\alpha_{k,n} - 1} \mathbf{e}_3, \quad (12)$$

where $\alpha_{k,n} = \frac{1 - \cos(\pi/n)}{1 - \cos(\pi/k)}$, is an equilateral spacelike de Sitter $2k$ -polygon with sidelength π/n and the property that $\text{Span}(\mathbf{v}_j, \mathbf{v}_{j+2})$ is a spacelike plane.

Proof. Let $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{2,1}$. To prove it is a spacelike de Sitter polygon we have to show that $\text{Span}(\mathbf{v}_j, \mathbf{v}_{j+1})$ is a spacelike plane. This is equivalent to having $\mathbf{v}_j, \mathbf{v}_{j+1}$ in spacelike position, i.e. $\langle \mathbf{v}_j, \mathbf{v}_{j+1} \rangle < 1$. We compute and find that $\langle \mathbf{v}_j, \mathbf{v}_{j+1} \rangle = \cos(\pi/n)$ (which notice also proves that the polygon is of sidelength π/n). Likewise, we compute $\langle \mathbf{v}_j, \mathbf{v}_{j+2} \rangle = 1 - 2(1 - \cos(\pi/n))(\cos(\pi/k) + 1)$ which is < 1 . ■

Remark 4.21. Notice that the existence of equilateral polygons in the general case of $q \geq 1$ is given by Proposition 4.20 because of the natural inclusion $\hat{\mathbb{S}}^{1,1} \subset \hat{\mathbb{S}}^{1,q}$ given by

$$\begin{aligned} \mathbb{R}^{2,1} &\hookrightarrow \mathbb{R}^{2,q} \\ (x_1, x_2, x_3) &\mapsto (x_1, x_2, x_3, 0, \dots, 0). \end{aligned}$$

5. PROOF OF MAIN RESULTS

In this section we prove some of the main theorems of this thesis, namely Theorems 1.1, 1.2 and 1.3. The first two are the main results of [32] and the third is the generalisation of much of the technical machinery developed therein. Theorem 1.3 will ultimately let us prove Theorem 1.5 which constructs exotic examples of higher higher Teichmüller spaces.

We begin with the case when $q = 1$ (Subsection 5.1), which is the case dealt with in [32]. Then we go on to generalise for $q > 1$ in Subsection 5.2. The reason for taking this as organisational principle instead of the slicker, ‘first show the generalised case and then get the $q = 1$ case as corollary plus some minor additions to deal with convexity’, is that by studying the $q = 1$ case we will develop the intuition needed to make the proofs in the general case readily understandable. Indeed, the proofs in both cases are nearly the same, so having worked in the $q = 1$ case lets us sit back and enjoy the fruit of our labour once we reach Subsection 5.2.

5.1. Proof when $q = 1$. We introduce the concept of *hipped hypersurface* and the *bridges* from these surfaces to polygons and to GHMCs (see Diagram 1.6).

We begin with the definition of hipped hypersurface.

Definition 5.1 (Hipped hypersurface in AdS^{p+1}). A **hipped hypersurface** H in AdS^{p+1} is a Lipschitz submanifold which is a piece-wise geodesic p -surface in $\hat{\text{AdS}}^{p+1}$ (Definition 3.4) which is also a spacelike p -surface, i.e. where the inclusion is a spacelike $(p, 1)$ immersion (Definition 3.11), see Figure 7.

It will be useful to have a notion of angles between the Y_i and between the X_i . We call the angles between X_i and X_{i+1} *dihedral* and those between Y_i and Y_{i+1} *wedge* angles.

Definition 5.2 (Dihedral and wedge angles). Let $H = \bigcup_{i=1}^k X_i$ be a hipped hypersurface in AdS^{p+1} . The **dihedral angle** between X_i and X_{i+1} is the *oriented* angle between the future pointing timelike normal vector to X_i at any point of Y_{i+1} and the future pointing timelike normal vector to X_{i+1} at the same point.

The **wedge angle** between Y_i and Y_{i+1} is determined as follows: Pick a point $z \in Z$ (in the *stem* of H), and consider the tangent spaces $T_z Y_i$, $T_z Y_{i+1}$. By construction, these subspaces of $T_z X_i \cong \mathbb{R}^p$ intersect at $T_z Z \cong \mathbb{R}^{p-2}$. We define the wedge angle between Y_i and Y_{i+1} to be the angle of the rotation necessary to move $T_z Y_i$ into $T_z Y_{i+1}$ while fixing $T_z Z$. In other words, we have that, up to conjugation by an element of $\text{SO}(p, 2)$, there is an element $g \in \text{SO}(p, 2)$ which acts on the tangent space of AdS^{p+1} at z as an element of the form

$$\begin{bmatrix} \text{id}_{p-2} & & & \\ & \cos(\theta) & -\sin(\theta) & \\ & \sin(\theta) & \cos(\theta) & \\ & & & \text{id}_1 \end{bmatrix}$$

where θ is the **wedge angle** between Y_i and Y_{i+1} .

At first it seems like the wedge angle between Y_i and Y_{i+1} depends on the point z of the stem we happen to have chosen. But notice, the X_i , the Y_i , and Z are totally geodesic and convex, so we can parallel transport *isometrically* (using the pseudo-Riemannian structure of the ambient AdS^{p+1}) the information from one point $z \in Z$ to another such point, say w , along a geodesic of Z connecting these two points. And so, the wedge angle does not depend on the point of the stem chosen.

Lemma 5.3. A hipped hypersurface H in AdS^{p+1} is past-convex if and only if all dihedral angles are nonnegative.

Proof. ($\Rightarrow$) Without loss of generality, assume that the dihedral angle between X_1 and X_2 is negative. Pick a point x in the interior of Y_2 , and consider the future of x . By assumption there must be a geodesic which lies in the future of x and intersects X_1 and X_2 transversally. If H were past-convex this would mean that the intersection between the past of the spacelike hyperplane W witnessing past-convexity at x and the above mentioned geodesic would have to be disconnected. But this is absurd, as this would mean that the intersection of some halfspace and some line would have to

be disconnected

($\Leftarrow$) If all angles are nonnegative then H lies in the past of the X_i . These witness past-convexity at every point of H (more precisely the spacelike hyperplanes containing them witness past-convexity). $\blacksquare$

Hipped hypersurfaces should be thought of as the appropriate landing ground for our spacelike $(p, 1)$ charts. And so, it should be suggestive that they are somehow related to spacelike AdS structures on our Gromov-Thurston manifold $M^{\frac{k}{n}}$. Lemma 5.6 makes this relation rigorous. But this in no way gives existence of these hipped hypersurfaces. The following lemma reduces their existence to that of spacelike de Sitter polygons.

Lemma 5.4. Let $\alpha_1, \dots, \alpha_k \in (0, \pi)$ and $\theta_1, \dots, \theta_k \in \mathbb{R}$, then there is a one-to-one correspondence,

$$\left\{ \begin{array}{l} \text{Hipped hypersurfaces } H = \bigcup_{i=1}^k X_i \text{ in} \\ \text{AdS}^{p+1} \text{ with wedge angles } \alpha_1, \dots, \alpha_k \\ \text{and dihedral angles } \theta_1, \dots, \theta_k. \end{array} \right\} \Big/ \text{isometry}$$

$$\updownarrow$$

$$\left\{ \begin{array}{l} \text{Spacelike polygons in } dS^2 \\ \text{with sidelengths } \alpha_1, \dots, \alpha_k \\ \text{and angles } \theta_1, \dots, \theta_k. \end{array} \right\} \Big/ \text{isometry}$$

that sends past-convex hypersurfaces to convex polygons.

Proof. ($\mapsto$) Let $H = \bigcup_{i=1}^k X_i$ be a hipped hypersurface in AdS^{p+1} . The idea is to see dS^2 in the orthogonal complement of $T_z Z$ inside $T_z \text{AdS}^{p+1}$ and intersect it with the tangent spaces of X_i and Y_i at z to get an associated polygon. Fix an identification of the ambient space $\mathbb{R}^{p+1}$ with $T_z \text{AdS}^{p+1}$ and where $T_z Z^\perp \cong \{0\} \times \mathbb{R}^3$. Through the identification we may see dS^2 in $T_z Z^\perp$. (Indeed, seeing AdS^{p+1} in the chart $x_{p+2} = 1$, and intersecting it with $T_z Z^\perp$ via the identification, we get a copy of AdS^3 whose boundary is exactly dS^2 in $T_z Z^\perp$.) Next, intersect the tangent spaces $T_z Y_i$ with this copy of dS^2 . The intersections $T_z Y_i \cap T_z Z^\perp$ are one dimensional because $T_z Y_i \cap T_z Z$ are $(p-2)$ dimensional by construction. And so, letting v_i be the unique point in $T_z Y_i \cap T_z Z^\perp \cap dS^2$ (where this dS^2 is the copy in $T_z Z^\perp$) such that $\exp_z(v_i) \in Y_i$, gives us the vertices of the associated de Sitter polygon. Likewise, intersect $T_z X_i$ with this copy of dS^2 to get the edges $[v_i, v_{i+1}]$ by prescribing that the image via the exponential map of any point of the intersection must land in X_i . (Note that $T_z X_i \cap T_z Z^\perp$ is 2 dimensional.)

($\mapsto$) Conversely, take a spacelike de Sitter polygon P in dS^2 , and consider the latter space in the orthogonal of the tangent space of some point of AdS^{p+1} , call it z . Let

Z be any totally geodesic copy of $\mathbb{H}^{p-2}$ going through z . Then we can construct the Y_i and X_i as follows:

$$Y_i = \{\exp_z(u + tv_i) \in \text{AdS}^{p+1} \mid u \in T_z Z, t \in \mathbb{R}_{\geq 0}\},$$

$$X_i = \{\exp_z(u + tw) \in \text{AdS}^{p+1} \mid u \in T_z Z, t \in \mathbb{R}_{\geq 0}, w \in [v_i, v_{i+1}]\}.$$

These two constructions are inverses of each other. To conclude, notice that the dihedral angle θ_i is equal to (angle at v_i of P) and so past-convex hipped hypersurfaces get sent to convex spacelike polygons by Lemma 5.3. $\blacksquare$

With these lemmas we are almost done: the map advertised in Diagram 1.6 uses these two to bridge between spacelike de Sitter polygons and spacelike AdS structures, which then are related to GHMCs via the theorems proved in Section 3. But there is still one hurdle to overcome, namely, that spacelike AdS structures are not well-suited to the construction of our Gromov-Thurston manifolds $M^{\frac{k}{n}}$. In some sense, we want something that resembles our glue and paste construction for $M^{\frac{k}{n}}$. This is what the following definition achieves.

Definition 5.5 (Folded spacelike AdS structures on $M^{\frac{k}{n}}$). Let $\widetilde{M}$ be the universal cover of $M^{\frac{k}{n}}$ (or equivalently of the dihedral hyperbolic manifold $M = M^1$ we used to construct $M^{\frac{k}{n}}$). Let $\{\widetilde{H}_i\}_i$ be *all* of the lifts of $H_1, \dots, H_{2k}$ along $\pi: \widetilde{M} \rightarrow M^{\frac{k}{n}}$ and consider $X \stackrel{\text{def}}{=} \widetilde{M} \setminus \bigcup_i \widetilde{H}_i$. Then, a **spacelike AdS structure** on $M^{\frac{k}{n}}$ is **folded** if every connected component of X is included in a totally geodesic copy of $\mathbb{H}^p$ (via the developing map associated to the spacelike AdS structure on $M^{\frac{k}{n}}$, $\text{dev}: \widetilde{M} \rightarrow \text{AdS}^{p+1}$), *and* the induced metric on $M^{\frac{k}{n}} \setminus S$ is the original hyperbolic metric. (Recall that S is the lift to $M^{\frac{k}{n}}$ of the intersection of the fix point sets of the involutions defining the dihedral hyperbolic manifold M^1).

Lemma 5.6. Let $M^{\frac{k}{n}}$ be a Gromov-Thurston manifold with $k > n$ and of dimension p . Then there is a one-to-one correspondence,

$$\begin{array}{c} \left\{ \begin{array}{l} \text{Folded spacelike AdS} \\ \text{structures on } M^{\frac{k}{n}} \end{array} \right\} / \text{equivalence} \\ \updownarrow \\ \left\{ \begin{array}{l} \text{Hipped hypersurfaces in} \\ \text{AdS}^{p+1} \text{ with } 2k \text{ wedges} \\ \text{and wedge angles } \pi/n \end{array} \right\} / \text{isometry,} \end{array}$$

which sends convex spacelike AdS structures to past-convex hipped hypersurfaces.

Proof. ($\mapsto$) The idea is that a folded spacelike AdS structure makes $M^{\frac{k}{n}}$ look like a hipped hypersurface close to S , and given hipped hypersurfaces are fully determined

by small neighbourhoods around points of the stem, we are done (note the last statement follows from the fact that the constituent parts of our hipped hypersurface are convex and totally geodesic).

Let $\text{dev} : \widetilde{M} \rightarrow \text{AdS}^{p+1}$ be the developing map associated to a *folded* spacelike AdS structure on $M^{\frac{k}{n}}$. Consider a point $x \in S$ and let $\widetilde{U}$ be a lift along the universal covering map $\pi : \widetilde{M} \rightarrow M^{\frac{k}{n}}$ of an open set U containing x . Then, after potentially shrinking U , we get that $\text{dev}(\widetilde{U})$ defines a neighbourhood of a point in the stem of a hipped hypersurface in AdS^{p+1} . Indeed, by the folded structure of $M^{\frac{k}{n}}$ we have that away from S the metric of $M^{\frac{k}{n}}$ is the original hyperbolic metric and that excluding the H_i we have the totally geodesic copies of $\mathbb{H}^p$ defining the X_i in a hipped hypersurface. Moreover, the fact that the H_i are totally geodesic hypersurfaces in a Gromov-Thurston manifold, gives us that the boundaries of the X_i are our desired totally geodesic copies of half spaces of $\mathbb{H}^{p-1}$, which we termed Y_i above. We have defined a neighbourhood of a point of the stem of a hipped hypersurface in AdS^{p+1} , and so, we have the complete hipped hypersurface because we can use the exponential map as in the proof of Lemma 5.4 to retrieve it whole.

($\Leftarrow$) The idea for this direction is to use a hipped hypersurface as a landing ground for the charts of our folded spacelike AdS structure.

Let $\Sigma = \bigcup_{i=1}^{2k} X_i$ be a hipped hypersurface in AdS^{p+1} with wedge angles π/n . Then we can define a folded spacelike AdS structure on $M^{\frac{k}{n}}$ as follows:

- Let $x \in M^{\frac{k}{n}} \setminus \bigcup_{i=1}^{2k} H_i$. Then we can define a chart $\phi : U_x \rightarrow \text{AdS}^{p+1}$ mapping the neighbourhood U_x of x isometrically to one of the X_i . (This is possible because away from S , $M^{\frac{k}{n}}$ has a hyperbolic metric.)
- Let $x \in H_i \setminus S$. Then we can define a continuous chart $\phi : U_x \rightarrow \text{AdS}^{p+1}$ mapping $U_x \cap V_{i-1}$ isometrically to X_{i-1} , $U_x \cap H_i$ to Y_i and $U_x \cap V_i$ to X_i .
- Let $x \in S$ then we can define a continuous chart $\phi : U_x \rightarrow \text{AdS}^{p+1}$ mapping $U_x \cap V_i$ to X_i and $U_x \cap H_i$ to Y_i .

Observe that by construction these two maps are inverses to one another.

To conclude, note that a past-convex hipped hypersurface gets sent to a locally past-convex folded AdS structure on $M^{\frac{k}{n}}$, (locally they look the same). Moreover, by compactness of $M^{\frac{k}{n}}$, we have completeness of the AdS structure (i.e. via the developing map) and so by Proposition 3.28 we get global past-convexity. $\blacksquare$

We are finally ready to prove Theorems 1.1, 1.2 which we restate here for clarity.

Theorem 1.1. Let $M^{\frac{k}{n}}$ be a k -ramified Gromov-Thurston manifold of dimension $p > 3$ such that $k > n$. Then we can isometrically embed $M^{\frac{k}{n}}$ (with its cone hyperbolic metric see Section 2) in a GHMC AdS manifold N of dimension $p + 1$, as the future boundary of the convex core of N .

Proof. By Theorem 3.36 and Lemma 3.42, it suffices to construct a folded convex spacelike AdS structure on $M^{\frac{k}{n}}$. By Lemma 5.6 this is equivalent to finding some hipped hypersurface in AdS^{p+1} with $2k$ pieces and wedge angles π/n . By Lemma 5.4 this is equivalent to finding some equilateral spacelike de Sitter $2k$ -polygon with sidelength π/n . By Proposition 4.20 such polygons exist. $\blacksquare$

Theorem 1.2. Let $M^{\frac{k}{n}}$ be a k -ramified Gromov-Thurston manifold of dimension $p > 3$ such that $k > n$. Then there is a $2k - 3$ parameter family of GHMC AdS manifolds N for which $M^{\frac{k}{n}}$ is isometric (with its cone hyperbolic metric) to the future boundary of the convex core.

Proof. In the proof of Theorem 1.1 we constructed a bijective map $P \mapsto N_P$, from equilateral de Sitter $2k$ -polygons of sidelength π/n to the set of GHMC AdS manifolds N containing $M^{\frac{k}{n}}$ isometrically as the future boundary of the convex core of N (see Diagram 1.6). By Proposition 4.19 the set of such polygons is a $2k - 3$ dimensional manifold. $\blacksquare$

5.2. Proof when $q \geq 1$. In this subsection we give the general proof. The situation has been rigged (see Section 3) so as to have the proofs be almost exactly the same as in the $q = 1$ case. We begin with some needed definitions.

Definition 5.7 (Hipped p -surface in $\hat{\mathbb{H}}^{p,q}$). A **hipped p -surface** $\mathcal{H}$ in $\hat{\mathbb{H}}^{p,q}$ is a piece-wise geodesic p -surface (Definition 3.4) where the natural inclusion $i : \mathcal{H} \hookrightarrow \hat{\mathbb{H}}^{p,q}$ is a spacelike (p, q) immersion (Definition 3.11).

As in the anti-de Sitter case we define wedge angles. Notice, as is shown in Diagram 1.7, we no longer need the definition of dihedral angle as there is no mention of convexity in the generalised case.

Definition 5.8 (Wedge angle). Let $\mathcal{H} = \bigcup_{i=1}^k X_i$ be a hipped p -surface in $\hat{\mathbb{H}}^{p,q}$. Then the **wedge angle** between Y_i and Y_{i+1} (see Definition 3.4 for what these are) is defined as follows. Pick an element $z \in Z$ and consider $T_z Y_i$ and $T_z Y_{i+1}$. Define the wedge angle between Y_i and Y_{i+1} to be the angle given by the rotation, up to conjugation, necessary to move $T_z Y_i$ to $T_z Y_{i+1}$, as subspaces of $T_z \hat{\mathbb{H}}^{p,q} \cong \mathbb{R}^{p,q}$, while fixing $T_z Z$ and the subspace $(T_z X_i)^\perp$. In other words, up to conjugation, there is an element $g \in \text{SO}(p, q + 1)$ which acts on the tangent space of $\hat{\mathbb{H}}^{p,q}$ at z as an element,

$$\begin{bmatrix} \text{id}_{p-2} & & & \\ & \cos(\theta) & -\sin(\theta) & \\ & \sin(\theta) & \cos(\theta) & \\ & & & \text{id}_q \end{bmatrix}$$

which sends $T_z Y_i$ to $T_z Y_{i+1}$. Then, the **wedge angle** between Y_i and Y_{i+1} is θ .

Remark 5.9. Notice that the reason the matrix above has that form, is because we are carrying out the rotation to get from $T_z Y_i$ to $T_z Y_{i+1}$ within $T_z X_i$. In other words,

everything outside of $T_z X_i \cong \mathbb{R}^p$ remains fixed together with $p-2$ dimensions of $T_z X_i$ which correspond to the space $T_z Z \cong \mathbb{R}^{p-2}$. And so, given our convention of putting the timelike vectors in an orthonormal basis last, we get the result.

Now we can give an equivalent of Lemma 5.4.

Lemma 5.10. Let $\alpha_1, \dots, \alpha_k \in (0, \pi)$, then there is a one-to-one correspondence,

$$\left\{ \begin{array}{l} \text{Hipped } p\text{-surfaces } H = \bigcup_{i=1}^k X_i \text{ in } \hat{\mathbb{H}}^{p,q} \\ \text{with wedge angles } \alpha_1, \dots, \alpha_k. \end{array} \right\} \Big/ \text{isometry}$$

$$\updownarrow$$

$$\left\{ \begin{array}{l} \text{Spacelike } k\text{-polygons in } \hat{\mathbb{S}}^{1,q} \\ \text{with sidelengths } \alpha_1, \dots, \alpha_k. \end{array} \right\} \Big/ \text{isometry}.$$

Proof. ($\mapsto$) Let $H = \bigcup_{i=1}^k X_i$ be a hipped p -surface in $\hat{\mathbb{H}}^{p,q}$. Pick a point $z \in Z$ (a point in the *stem* of H). The idea is to see $\hat{\mathbb{S}}^{1,q}$ in the orthogonal complement of $T_z Z$ inside $T_z \hat{\mathbb{H}}^{p,q}$ and intersect it with the tangent spaces of X_i and Y_i at z to get an associated polygon. Fix an identification of $\mathbb{R}^{p,q}$ with $T_z \hat{\mathbb{H}}^{p,q}$ and where $T_z Z^\perp \cong \{0\} \times \mathbb{R}^{2,q}$. Through the identification we may see $\hat{\mathbb{S}}^{1,q}$ in $T_z Z^\perp$. (Indeed, seeing $\mathbb{H}^{p,q}$ in the chart $x_{p+q+1} = 1$, and intersecting it with $T_z Z^\perp$ via the identification, we get a copy of $\mathbb{H}^{2,q}$, seen in the chart $x_{p+q+1} = 1$, whose boundary is exactly $\hat{\mathbb{S}}^{1,q}$ in $T_z Z^\perp$.) Next, intersect the tangent spaces $T_z Y_i$ with this copy of $\hat{\mathbb{S}}^{1,q}$. The intersections $T_z Y_i \cap T_z Z^\perp$ are one dimensional because $T_z Y_i \cap T_z Z$ are $(p-2)$ dimensional by construction and Y_i is a half space of a totally geodesic copy of $\mathbb{H}^{p-1}$. And so, letting v_i be the unique point in $T_z Y_i \cap T_z Z^\perp \cap \hat{\mathbb{S}}^{1,q}$ (where this $\hat{\mathbb{S}}^{1,q}$ is the copy in $T_z Z^\perp$) such that $\exp_z(v_i) \in Y_i$, gives us the vertices of the associated spacelike $\hat{\mathbb{S}}^{1,q}$ polygon. Likewise, intersect $T_z X_i$ with this copy of $\hat{\mathbb{S}}^{1,q}$ to get the edges $[v_i, v_{i+1}]$ by prescribing that the image via the exponential map of any point of the intersection must land in X_i . (Note that $T_z X_i \cap T_z Z^\perp$ is 2 dimensional). This gives us the tuple $(v_1, \dots, v_k)$ of the definition of spacelike $\hat{\mathbb{S}}^{1,q}$ k -polygon, we are missing the k -tuple of $q-1$ frames. To get this consider X_i and pick a point in it, call it x . Take its associated q_x (given by any spacelike (p, q) chart containing the point x). Then we can parallel transport q_x to $T_z \hat{\mathbb{H}}^{p,q}$ within the totally geodesic copy of $\mathbb{H}^p$ of which X_i is a subset (see Definition 3.4). Call the resultant frame q_i . Then by construction $q_i \subset (T_z X_i)^\perp \subset (T_z Y_i)^\perp$ and likewise for $(T_z Y_{i+1})^\perp$. And so q_i lands in the orthogonal complement of $\text{Span}(v_i, v_{i+1})$. This construction is unique up to isometry and so gives us our desired tuple of $q-1$ frames.

($\leftarrow$) Conversely, take a spacelike $\hat{\mathbb{S}}^{1,q}$ polygon $P = (v_1, q_1; \dots; v_k, q_k)$, and a point $z \in \hat{\mathbb{H}}^{p,q}$. Let Z be any spacelike totally geodesic copy of $\mathbb{H}^{p-2}$ going through z . Consider $\hat{\mathbb{S}}^{1,q}$ in $(T_z Z)^\perp \cong \mathbb{R}^{2,q}$. Then we can construct the Y_i and X_i of our desired

hipped $\mathbf{p}$ -surface as follows:

$$\begin{aligned} Y_i &= \{\exp_z(\mathbf{u} + t\mathbf{v}_i) \in \hat{\mathbb{H}}^{p,q} \mid \mathbf{u} \in T_z Z, t \in \mathbb{R}_{\geq 0}\}, \\ X_i &= \{\exp_z(\mathbf{u} + t\mathbf{w}) \in \hat{\mathbb{H}}^{p,q} \mid \mathbf{u} \in T_z Z, t \in \mathbb{R}_{\geq 0}, \mathbf{w} \in [\mathbf{v}_i, \mathbf{v}_{i+1}]\}. \end{aligned}$$

To get the frames we need to have the inclusion (of the spacelike piece-wise geodesic $\mathbf{p}$ -surface we have just constructed) be a spacelike $(\mathbf{p}, \mathbf{q})$ immersion, we do the following. By construction we know that $\mathbf{q}_i$ is a subset of $\text{Span}(\mathbf{v}_i, \mathbf{v}_{i+1})^\perp$, where the orthogonal complement is being taken within $(T_z Z)^\perp \cong \mathbb{R}^{2,q}$, seen in $T_z \hat{\mathbb{H}}^{p,q} \cong \mathbb{R}^{p,q}$. In fact, we have that, under appropriate coordinates, $T_z X_i \cong \mathbb{R}^{p,0}$, $T_z Z \cong \mathbb{R}^{p-2,0}$ and $\text{Span}(\mathbf{v}_i, \mathbf{v}_{i+1}) \cong \mathbb{R}^{2,0}$ where $T_z X_i = T_z Z \oplus \text{Span}(\mathbf{v}_i, \mathbf{v}_{i+1})$. And so, $\text{Span}(\mathbf{v}_i, \mathbf{v}_{i+1})^\perp = \mathbb{R}^{0,q} = (T_z X_i)^\perp$, where the first orthogonal complement is being taken in $(T_z Z)^\perp \cong \mathbb{R}^{2,q}$ and the last is being taken in $T_z \hat{\mathbb{H}}^{p,q} \cong \mathbb{R}^{p,q}$. In other words, we get that $\mathbf{q}_i \subset (T_z X_i)^\perp$ which gives us the possibility of parallel transporting $\mathbf{q}_i$ to all points in the interior of X_i . This gives us the structure of a spacelike $(\mathbf{p}, \mathbf{q})$ immersion for the inclusion. Lastly, notice that these two constructions are inverses of each other. ■

Definition 5.11 (Folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures on $M^{\frac{k}{n}}$). Let $\widetilde{M}$ be the universal cover of $M^{\frac{k}{n}}$ (or equivalently of the dihedral hyperbolic manifold $M = M^1$ we used to construct $M^{\frac{k}{n}}$). Let $\{\widetilde{H}_i\}_i$ be *all* of the lifts of $H_1, \dots, H_{2k}$ along $\pi : \widetilde{M} \rightarrow M^{\frac{k}{n}}$ and consider $X \stackrel{\text{def}}{=} \widetilde{M} \setminus \bigcup_i \widetilde{H}_i$. Then, a spacelike $\hat{\mathbb{H}}^{p,q}$ structure on $M^{\frac{k}{n}}$ is **folded** if every connected component of X is included in a totally geodesic copy of $\mathbb{H}^p$ (via the developping map associated to the spacelike $\hat{\mathbb{H}}^{p,q}$ -structure on $M^{\frac{k}{n}}$, $\text{dev} : \widetilde{M} \rightarrow \hat{\mathbb{H}}^{p,q}$), *and* the induced metric on $M^{\frac{k}{n}} \setminus S$ is the original hyperbolic metric. (Recall that S is the lift to $M^{\frac{k}{n}}$ of the intersection of the fix point sets of the involutions defining the dihedral hyperbolic manifold M^1).

Lemma 5.12. Let $M^{\frac{k}{n}}$ be a Gromov-Thruston manifold with $k > n$ and of dimension p . Then there is a one-to-one correspondence,

$$\begin{array}{c} \left\{ \begin{array}{l} \text{Folded spacelike } \hat{\mathbb{H}}^{p,q} \\ \text{-structures on } M^{\frac{k}{n}} \end{array} \right\} \Big/ \text{equivalence} \\ \updownarrow \\ \left\{ \begin{array}{l} \text{Hipped } \mathbf{p}\text{-surfaces in} \\ \hat{\mathbb{H}}^{p,q} \text{ with } 2k \text{ wedges} \\ \text{and wedge angles } \pi/n \end{array} \right\} \Big/ \text{isometry.} \end{array}$$

Proof. ($\mapsto$) The idea is that a folded spacelike $\hat{\mathbb{H}}^{p,q}$ structure makes $M^{\frac{k}{n}}$ look like a hipped $\mathbf{p}$ -surface close to S , and given hipped $\mathbf{p}$ -surfaces are fully determined by small neighbourhoods around points of the stem, we are done (note the last statement follows from the fact that the constituent parts of our hipped $\mathbf{p}$ -surface are convex

and totally geodesic).

Let $\text{dev} : \widetilde{M} \rightarrow \widehat{\mathbb{H}}^{p,q}$ be the developing map associated to a *folded* spacelike $\widehat{\mathbb{H}}^{p,q}$ structure on $M^{\frac{k}{n}}$. Consider a point $x \in S$ and let $\widetilde{U}$ be a lift along the universal covering map $\pi : \widetilde{M} \rightarrow M^{\frac{k}{n}}$ of an open set U containing x . Then, after potentially shrinking U , we get that $\text{dev}(\widetilde{U})$ defines a neighbourhood of a point in the stem of a hipped p -surface in $\widehat{\mathbb{H}}^{p,q}$. Indeed, by the folded structure of $M^{\frac{k}{n}}$ we have that, away from S , the metric of $M^{\frac{k}{n}}$ is the original hyperbolic metric. This means that, excluding the H_i from $\text{dev}(\widetilde{U})$, we obtain the sets $X_i \cap \text{dev}(\widetilde{U})$ in the totally geodesic copies of $\mathbb{H}^p$ defining a hipped p -surface. Moreover, the fact that the H_i are totally geodesic p -surfaces in a Gromov-Thurston manifold, gives us that the boundaries of the $X_i \cap \text{dev}(\widetilde{U})$ are our desired totally geodesic copies of half spaces of $\mathbb{H}^{p-1}$, which we termed Y_i above (see Definition 3.4). We have defined a neighbourhood of a point of the stem in a hipped p -surface in $\widehat{\mathbb{H}}^{p,q}$, and so, we have the complete hipped p -surface because we can use the exponential map as in the proof of Lemma 5.10 to retrieve it whole. To get the frames associated to the X_i of our hipped p -surface, we can use the frames associated to our spacelike $\widehat{\mathbb{H}}^{p,q}$ structure. Namely, as constructed in this proof, the X_i carry a coherent choice of frame q_x for each point $x \in X_i$ given by the spacelike $\widehat{\mathbb{H}}^{p,q}$ structure. Then we can parallel translate this frame q_x to get the needed condition on a hipped p -surface. Notice that we do not get conflicting choices of frame because we can always shrink U so as to make $\text{dev}(\widetilde{U})$ land in a single spacelike (p, q) chart.

($\leftarrow$) The idea for this direction is to use a hipped p -surface as a landing ground for the charts of our folded spacelike $\widehat{\mathbb{H}}^{p,q}$ structure.

Let $\Sigma = \bigcup_{i=1}^{2k} X_i$ be a hipped p -surface in $\widehat{\mathbb{H}}^{p,q}$ with wedge angles π/n . Then we can define a folded spacelike $\widehat{\mathbb{H}}^{p,q}$ -structure on $M^{\frac{k}{n}}$ as follows:

- Let $x \in M^{\frac{k}{n}} \setminus \bigcup_{i=1}^{2k} H_i$. Then we can define a chart $\phi : U_x \rightarrow \widehat{\mathbb{H}}^{p,q}$ mapping the neighbourhood U_x of x isometrically to one of the X_i . (This is possible because away from S , $M^{\frac{k}{n}}$ has a hyperbolic metric.)
- Let $x \in H_i \setminus S$. Then we can define a continuous chart $\phi : U_x \rightarrow \widehat{\mathbb{H}}^{p,q}$ mapping $U_x \cap V_{i-1}$ isometrically to X_{i-1} , $U_x \cap H_i$ to Y_i and $U_x \cap V_i$ to X_i .
- Let $x \in S$ then we can define a continuous chart $\phi : U_x \rightarrow \text{AdS}^{p+1}$ mapping $U_x \cap V_i$ to X_i and $U_x \cap H_i$ to Y_i .

The coherent choice of frames is induced by that of the hipped p -surface. In other words, in each of the three bullets above, everytime we land somewhere we take with us the coherent choice of frame given by the hipped p -surface at one of the X_i . Observe that by construction these two maps are inverses to one another. ■

This gives us the tools to prove the main result of this section (and one of the main results of this thesis).

Terminology 5.13. We will term by the **natural action of $SO_o(p, q + 1)$** on the set of pairs (dev, ρ) of developing maps and associated holonomy representations giving spacelike $\hat{\mathbb{H}}^{p,q}$ -structures, the action naturally given by Theorem 3.21. Namely, $g \in SO_o(p, q + 1)$ acts on (dev, ρ) by

$$g \cdot (\text{dev}, \rho) = (g \circ \text{dev}, g \circ \rho \circ g^{-1}).$$

Recall that the main goal of the generalisation was to prove the following theorem.

Theorem 1.3. There is a one-to-one correspondence between pairs (dev, ρ) of developing maps and associated holonomy representations giving spacelike $\hat{\mathbb{H}}^{p,q}$ -structures on the Gromov-Thurston manifold $M^{\frac{k}{n}}$ up to the natural action of $SO_o(p, q + 1)$, and equilateral spacelike $\hat{\mathbb{S}}^{1,q}$ $2k$ -polygons with sidelength π/n up to isometry. In other words,

$$\begin{array}{c} \left\{ \begin{array}{l} \text{Pairs } (\text{dev}, \rho) \text{ associated to} \\ \text{spacelike } \hat{\mathbb{H}}^{p,q} \text{ structures on } M^{\frac{k}{n}} \end{array} \right\} / \text{natural action} \\ \updownarrow \\ \left\{ \begin{array}{l} \text{Equilateral spacelike } \hat{\mathbb{S}}^{1,q} \\ 2k\text{-polygons with sidelength } \pi/n \end{array} \right\} / \text{isometry.} \end{array}$$

Proof. This follows immediately from Theorem 3.21, Corollary 3.23, Lemma 5.12 and Lemma 5.10. ■

6. HIGHER (DIMENSIONAL) HIGHER (RANK) TEICHMÜLLER SPACES (HHTS)

As stated in the Introduction, the results of this paper (in particular, Monclair, Schlenker and Tholozan's proof of Theorem 1.3 for $q = 1$ and our generalisation for $q > 1$) give us a way of constructing new exotic examples of *higher dimensional higher rank Teichmüller spaces* (as they are usually called). To explain what this means, we first introduce what these higher higher Teichmüller spaces are and how they came to be studied (Subsection 6.1). Then, in the same subsection, we give the construction of the higher higher Teichmüller space associated with the Monclair-Schlenker-Tholozan construction. In Subsection 6.2 we follow Beyrer-Kassel [5], and *deform*, within the character variety $\text{Hom}(\pi_1(M^{\frac{k}{n}}), SO(p, q + 1))/SO(p, q + 1)$, the representations given to us by the holonomy representation of a spacelike AdS structure. We do this so as to get representations with *Zariski-dense image*. In Subsection 6.3, we carry out the construction of the higher higher Teichmüller space associated with the generalised Monclair-Schlenker-Tholozan construction (i.e. for $q > 1$) and show that the representations of Subsection 6.3 are included in our examples.

6.1. HHTS-perspective of main results $q = 1$. Classical Teichmüller theory studies the space of all marked complex structures on some oriented closed surface S .

This is known as the *Teichmüller space* $\mathcal{T}(S)$ associated to S , whose study has been shown to be a wellspring of rich connections between many areas of mathematics — as evidenced in the works of Thurston and others working in the former’s Geometrisation programme (see [24], [25] for details).

Via the Uniformisation Theorem one can construe the Teichmüller space $\mathcal{T}(S)$ as the space of all marked *hyperbolic* structures on S , for genus ≥ 2 . From this perspective we can obtain $\mathcal{T}(S)$ as a connected component of the **G-character variety** associated to $\pi_1(S)$ where $G = \mathrm{PSL}(2, \mathbf{R})$. More precisely, the holonomy representation of $\pi_1(S)$ (associated to a hyperbolic structure) naturally realises $\mathcal{T}(S)$ as a connected component of $\mathrm{Hom}(\pi_1(S), G)$ (modulo conjugation by G) consisting entirely of discrete and injective representations — where $\mathrm{Hom}(\pi_1(S), G)$ denotes the set of representations of $\pi_1(S)$ to G with the compact-open topology.

An interesting (and perhaps surprising) phenomenon, that has been the cause of a wealth of research in the past twenty years, is that this fact holds within the character variety of higher rank simple Lie groups, like $\mathrm{PSL}(\mathfrak{n}, \mathbf{R})$ for $\mathfrak{n} \geq 3$, (recall that the rank of a semisimple Lie group is the dimension of its maximal torus). In other words, that the character variety has connected components consisting entirely of discrete and injective representations.

Definition 6.1 (Higher(-rank) Teichmüller space. Character variety). Let S be a closed, oriented, topological surface and G a simple Lie group of higher rank (≥ 2). A **higher(-rank) Teichmüller space** $\mathcal{T}(S, G)$ is a connected component of $\mathrm{Hom}(\pi_1(S), G)/G$ consisting entirely of discrete and injective representations and which is not a singleton. The space $\mathrm{Hom}(\pi_1(S), G)/G$ is the **G-character variety** of all representations of $\pi_1(S)$ into G modulo conjugation by G on the target. In other words, $\mathrm{Hom}(\pi_1(S), G)/G$ is the set of all representations $\rho : \pi_1(S) \rightarrow G$ topologised by the compact-open topology, where we quotient by conjugation by G on the target of said representations. Henceforth we will denote the character variety by $\mathfrak{C}(S, G)$, i.e.

$$\mathfrak{C}(S, G) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\pi_1(S), G)/G.$$

Historically, it was Hitchin [23] who first discovered higher rank Teichmüller spaces via Higgs bundles, examples which are now known as *Hitchin components* (although that these spaces were higher Teichmüller spaces as in Definition 6.1 was unknown to Hitchin at the time). It was later proved by Labourie [28] and Fock-Goncharov [15], that these Hitchin components are indeed higher Teichmüller spaces in the sense of Definition 6.1: connected components of the character variety consisting entirely of discrete and injective representations. Independently, a second family of higher Teichmüller spaces known as *maximal components* — defined as level sets of the Toledo number (an invariant on the character variety generalising the Euler number) — was being studied (see [9], [10] for details). It was later realised that these two families share many properties despite having been approached from very different areas and

explored through very different methods. Indeed, they have a common characterisation in terms of positive structures of flag manifolds [21] which conjecturally classifies all such higher rank Teichmüller spaces (see [38] for an expository account).

The study of these higher Teichmüller spaces has been shown to be as rich a theory as its classical incarnation: higher Teichmüller spaces parametrise (in analogy with the classical Teichmüller space) connected components of the deformation space of certain geometric structures on manifolds [20], they admit Fenchel-Nielsen type coordinates [18], [37], together with a symplectic structure [17], Hitchin components are homeomorphic to vector spaces [23], etc (see [38] for more). So it is natural to ask whether or not there is a similarly rich theory for *higher dimensional* analogues of higher rank Teichmüller spaces (a question posed in Wienhard's ICM survey [38]). More precisely, we can ask: What do we get when we let the rank of the simple Lie group G and the dimension of S as a manifold vary to larger values? Are there connected components of the character variety $\text{Hom}(\pi_1(M), G)/G$, where G is a real simple Lie group of rank ≥ 2 and M is a closed oriented manifold of dimension ≥ 3 , consisting entirely of injective and discrete representations? The answer is *yes* and what we get is higher *dimensional* higher rank Teichmüller spaces (which we call higher higher Teichmüller spaces for short).

Definition 6.2 (Higher(-dimensional) higher(-rank) Teichmüller space.

Character variety). Let M be a closed, oriented, topological manifold of dimension ≥ 3 , and G a simple Lie group of higher rank (≥ 2). A **higher(-dimensional) higher(-rank) Teichmüller space** $\mathcal{T}(M, G)$ is a connected component of the character variety $\text{Hom}(\pi_1(M), G)/G$ consisting entirely of discrete and injective representations and which is not a singleton. The space $\text{Hom}(\pi_1(M), G)/G$ is the **G -character variety** of all representations of $\pi_1(M)$ into G , modulo conjugation by G on the target. In other words, $\text{Hom}(\pi_1(M), G)/G$ is the set of all representations $\rho : \pi_1(M) \rightarrow G$ topologised by the compact-open topology, where we quotient by conjugation by G on the target of said representations. As before we will denote the character variety by $\mathfrak{C}(M, G)$.

Terminology 6.3. Henceforth we will refer to higher higher Teichmüller spaces as **HHTS**.

The first family of higher higher Teichmüller spaces was found by Benoist in [4] with Choi-Goldman [11] having preceded the general construction in the higher rank case. These authors focused on representations of fundamental groups of closed manifolds M , with $\dim(M) = n - 1$, into $\text{PSL}(n, \mathbf{R})$ that preserve a properly convex open set in projective space $\mathbb{P}(\mathbb{R}^n)$. They showed that the set of such representations in the character variety is a union of connected components consisting entirely of injective and discrete representations. Interesting examples within this family came in the works of Kapovich [27] when $\pi_1(M)$ is Gromov hyperbolic but not commensurable to

a discrete subgroup of $SO(\mathfrak{n} - 1, 1)$, and when $\pi_1(M)$ is not Gromov hyperbolic by Benoist and in Choi-Lee-Marquis [12] for $4 \leq \mathfrak{n} \leq 7$, and Blayac-Viaggi [6] for $\mathfrak{n} \geq 4$.

The second known family of higher higher Teichmüller spaces was found by Barbot [2] in the anti-de Sitter setting and more generally by Beyrer-Kassel in [5]. The description of this second family needs crucially the concept of $\mathbb{H}^{p,q}$ -convex cocompactness.

Definition 6.4 ($\mathbb{H}^{p,q}$ -convex cocompact representation). A representation $\rho : \pi_1(M) \rightarrow SO(p, q + 1)$ is said to be $\mathbb{H}^{p,q}$ -convex cocompact if it is injective and its image is a discrete group which acts properly discontinuously and cocompactly on some properly convex set Ω of the pseudo-Riemannian hyperbolic space $\mathbb{H}^{p,q}$, and where the ideal boundary $\partial_\infty \Omega = \partial \Omega \cap \partial_\infty \mathbb{H}^{p,q}$ does not contain any projective segments.

What these authors showed (Barbot for $q = 1$ and Beyrer-Kassel for general q) was that the set of $\mathbb{H}^{p,q}$ -convex cocompact representations in $\text{Hom}(\pi_1(M), SO(p, q + 1))$ modulo conjugation by $SO(p, q + 1)$ is a union of connected components.

Indeed, we have the following fact.

Fact 6.5. [5, Theorem 1.3] Let $p \geq 2$ and $q \geq 1$ and let Γ be a discrete Gromov hyperbolic subgroup of $G = PO(p, q + 1)$ with $\text{vcd}(\Gamma) = p$. Then the set of all $\mathbb{H}^{p,q}$ -convex cocompact representations is a union of connected components of $\text{Hom}(\Gamma, G)$.

Remark 6.6. Note that this means that if a connected component contains an $\mathbb{H}^{p,q}$ -convex cocompact representation then it consists entirely of such representations. This then easily gives HHTS as we will see below.

A natural example of such higher higher Teichmüller spaces is given by the fundamental group of a closed *hyperbolic* manifold M : The holonomy representation of $\pi_1(M)$ gives us a map $\rho : \pi_1(M) \rightarrow SO(p, 1)$ which we can then compose with the canonical injection of $SO(p, 1)$ into $SO(p, q + 1)$ and then *deform* (see Subsection 6.2). These can be thought of as the *classical* examples of HHTS in this context: the group $\pi_1(M)$ is a lattice in $SO(p, 1)$. Which means that finding HHTS when the group $\pi_1(M)$ is not (commensurable to) a lattice would give us *exotic* examples. Such examples, with $\pi_1(M)$ Gromov hyperbolic but not commensurable to a lattice in $SO(p, 1)$, were found by Lee-Marquis for $q = 1$ and $4 \leq p \leq 7$ in [29] using Coxeter groups, and for $q = 1$ and $p \geq 4$ by Monclair, Schlenker and Tholozan [32] via Gromov-Thurston manifolds. Indeed, we saw in Section 2 that the fundamental group of a Gromov-Thurston manifold is not commensurable to a lattice in any Lie group (Fact 2.5). We proceed to construct the Monclair-Schlenker-Tholozan higher higher Teichmüller space.

Theorem 6.7. The holonomy representations associated to folded convex ruled space-like AdS structures on $M_{\mathfrak{n}}^k$, with $k > \mathfrak{n}$, are $\mathbb{H}^{p,1}$ -convex cocompact.

Proof. This follows easily from the work we have done above. Indeed, by Theorem 3.21 for $q = 1$, let $\rho : \pi_1(M_n^k) \rightarrow \mathrm{SO}(p, 2)$ be the holonomy representation associated with the given spacelike AdS structure. Then by Theorem 3.36(1) we know that there is a GHMC AdS manifold $N = \rho(\pi_1(M_n^k)) \backslash \Omega$ where $\rho(\pi_1(M_n^k))$ acts on Ω . But then by Theorem 3.38(2), (3) and Lemma 3.42 (here is where we use that the spacelike AdS structure is convex ruled) we get that the action on Ω must be convex cocompact. ■

Terminology 6.8. We say a HHTS is **associated** to our folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures (folded spacelike AdS structures when $q = 1$) if the HHTS contains a connected component of the space of folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures.

From the previous theorem we easily get HHTS associated to folded convex ruled spacelike AdS structures.

Theorem 1.4. There are HHTS $\mathcal{T}(M_n^k, \mathrm{SO}(p, 2))$ associated to the holonomy representations of folded convex ruled spacelike AdS structures on M_n^k , with $k > n$. In other words, for every holonomy representation of a folded convex ruled spacelike AdS structure, the connected component in the character variety containing it is a HHTS.

Proof. Let $\rho : \pi_1(M_n^k) \rightarrow \mathrm{SO}(p, 2)$ be the holonomy representation of a folded convex ruled spacelike AdS structure of M_n^k . Then by Theorem 6.7 we know that ρ is $\mathbb{H}^{p,1}$ -convex cocompact. By Fact 6.5 we know that the connected component of $\mathrm{Hom}(\pi_1(M_n^k), \mathrm{SO}(p, 2))/\mathrm{SO}(p, 2)$ containing ρ consists entirely of $\mathbb{H}^{p,1}$ -convex cocompact representations. To prove that this connected component is a HHTS we need to show it is not a singleton. But notice that the parametrisation of the holonomies of folded spacelike AdS structures on M_n^k given by Theorem 1.3 for $q = 1$ is *up to the natural action* (see Terminology 5.13). And so, given the set of equilateral spacelike de Sitter polygons of sidelength π/n is a $(k - 3)$ -dimensional manifold (Proposition 4.19) we get that the connected component of the character variety cannot be a singleton. ■

Remark 6.9. (1) Notice that Theorem 1.4 is unclear in one direction: we do not know if we get a *single* HHTS or several. Indeed, this is a product of not knowing (yet!) whether $\mathcal{P}_k^{\pi/n}(\mathrm{dS})$ (the space of equilateral spacelike de Sitter polygons of sidelength π/n) is connected. If so, we indeed get a unique HHTS associated to folded convex ruled spacelike AdS structures (i.e. a HHTS containing all the holonomies of such structures). If $\mathcal{P}_k^{\pi/n}(\mathrm{dS})$ is *not* connected then we would get a HHTS per connected component of $\mathcal{P}_k^{\pi/n}(\mathrm{dS})$.
 (2) It is unclear how much of the topology of the HHTS so constructed we get by studying the set of all holonomies of folded convex ruled spacelike AdS

structures. Indeed, we do not even know whether the connected component containing the set of these representations contains it properly or not.

Having proven the existence of these exotic HHTS associated to the holonomy representations of folded convex ruled spacelike AdS structures, we proceed to do something similar for the case when $q > 1$. In other words, for the holonomy representations of folded spacelike $\hat{\mathbb{H}}^{p,q}$ structures for general q . Before doing this (Subsection 6.3) we give an account of Beyrer and Kassel's [5] deformation procedure by which they obtain, from the representations constructed by Monclair-Schlenker-Tholozan, $\mathbb{H}^{p,q}$ -convex cocompact representations in the character variety $\text{Hom}(\pi_1(M^{\frac{k}{n}}), \text{SO}(p, q+1))/\text{SO}(p, q+1)$ with Zariski-dense image.

6.2. Representations with Zariski-dense image. We have gotten HHTS

$\mathcal{T}(M^{\frac{k}{n}}, G)$ where $G = \text{SO}(p, 2)$ associated to the holonomy representations of folded convex ruled spacelike AdS structures. We did this by proving that such holonomy representations are $\mathbb{H}^{p,1}$ -convex cocompact (Theorem 6.7). We would like to massage our construction to get HHTS $\mathcal{T}(M^{\frac{k}{n}}, G')$ where $G' = \text{SO}(p, q+1)$ for $q > 1$. One way of doing this is to *deform*, in the sense of Johnson-Millson [26], our $\mathbb{H}^{p,1}$ -convex cocompact representations within the character variety $\mathfrak{C}(M^{\frac{k}{n}}, G')$, to obtain $\mathbb{H}^{p,q}$ -convex cocompact representations with Zariski-dense image. In this subsection we will explain how to do exactly this. We first justify wanting to find representations with Zariski-dense image.

One way in which HHTS $\mathcal{T}(M, G)$ are considered to be interesting (nontrivial) is if they contain representations with Zariski dense image. In some sense these representations properly land in the Lie group G (see [33, Theorem A.4.9]). For instance, if our HHTS were to consist of representations $\rho : \pi_1(M) \rightarrow \text{SO}(p, q+1)$ with image landing entirely in the copy of $\text{SO}(p, 1)$ given by the natural inclusion, then we would not want to say that this is a HHTS associated to the Lie group $\text{SO}(p, q+1)$, but more so that it is associated to $\text{SO}(p, 1)$ in disguise (and this we dislike because $\text{SO}(p, 1)$ is not a Lie group of higher rank!)

In this section we show how the Monclair-Schlenker-Tholozan construction can be *deformed* to give examples of HHTS associated to $\text{SO}(p, q+1)$ (for q in some range) that contain representations with Zariski-dense image, but first we introduce the method by which these *deformations* will be carried through. It was introduced by Johnson and Millson in [26]. En route, we will prove that we can obtain HHTS from the holonomy representations of hyperbolic structures on manifolds. More precisely, that the holonomy representation associated to a hyperbolic structure can be so deformed to give HHTS associated to $\text{SO}(p, q+1)$ for $q > 1$ containing representations with Zariski-dense image. We have the following proposition. Set $G = \text{SO}(p, q+1)$.

Proposition 6.10. Let M be a closed orientable hyperbolic real p -manifold with q two-sided, totally geodesic, connected, pairwise disjoint hypersurfaces. Let

$\rho_0 : \pi_1(M) \rightarrow G$ be the composite of the holonomy representation of M with the map $PO_o(p, 1) \cong O_o(p, 1) \hookrightarrow SO(p, q+1)$ (where the inclusion is the natural one). Then any neighbourhood of ρ_0 in $\text{Hom}(\pi_1(M), G)$ contains representations with Zariski dense image.

To show this we will need the following lemma.

Lemma 6.11. Let $p \geq 1$ and $q \geq q' \geq 0$.

- (1) If $\mathfrak{h} \subset \mathfrak{o}(p, q+1)$ is a Lie subalgebra such that $\mathfrak{o}(p, q) \subsetneq \mathfrak{h}$ then $\mathfrak{h} = \mathfrak{o}(p, q+1)$.
- (2) There exists an element $X \in \mathfrak{o}(p, q+1)$ centralising $\mathfrak{o}(p-1, 1)$ such that the Lie subalgebra of $\mathfrak{o}(p, q+1)$ generated by $\mathfrak{o}(p, q')$ and $\text{ad}(X)(\mathfrak{o}(p, 1))$ is $\mathfrak{o}(p, q'+1)$. Moreover, this X will also give that the Lie subalgebra of $\mathfrak{o}(p, q+1)$ generated by $\mathfrak{o}(p, q')$ and $\text{ad}(X)(\mathfrak{o}(p, q'))$ is $\mathfrak{o}(p, q'+1)$.

Proof. (1) We can take the Lie algebra $\mathfrak{o}(p, q+1)$ to be

$$\left\{ \begin{bmatrix} A & B \\ B^t & C \end{bmatrix} \in \mathbb{M}_{p+q+1}(\mathbb{R}) \mid A \in \mathfrak{o}(p), C \in \mathfrak{o}(q+1), B \in \mathbb{M}_{p \times (q+1)}(\mathbb{R}) \right\}.$$

(This is by choosing the standard form for the signature $(p, q+1)$ bilinear form, namely the diagonal matrix J with 1 repeated p times and -1 repeated $q+1$ times). Then $\mathfrak{o}(p, q)$ can be seen as the set of submatrices with coordinates $1 \leq i, j \leq p+q$. We get that $\mathfrak{o}(p, q+1) = \mathfrak{o}(p, q) \oplus \mathbb{R}^{p,q}$, where the adjoint action of $O(p, q)$ on $\mathfrak{o}(p, q+1)$ becomes the adjoint action in $\mathfrak{o}(p, q)$ and the linear action on $\mathbb{R}^{p,q}$. For sufficiently large p, q we have that the group $O(p, q)$ is simple which implies that the Lie algebra $\mathfrak{o}(p, q)$ is simple, which exactly says that the ad-action is irreducible. Therefore the Ad-action is also irreducible, and so $\mathfrak{o}(p, q)$ is an irreducible $O(p, q)$ -module. In other words, we have a decomposition of $\mathfrak{o}(p, q+1)$ into irreducible $O(p, q)$ -modules (the linear action on $\mathbb{R}^{p,q}$ is irreducible because of transitivity.) By hypothesis we have that $\exp(\mathfrak{h}) \supset O(p, q)$ and so $\mathfrak{h}$ is an $O(p, q)$ -module strictly containing $\mathfrak{o}(p, q)$ and so it must contain $\mathbb{R}^{p,q}$ as well.

(2) By using the matrix representation in (1) we can see $\mathfrak{o}(p-1, 1)$ as submatrices in the coordinates $2 \leq i, j \leq p+1$. Let $X = E_{1, p+q'+1} + E_{p+q'+1, 1}$ then X centralises $\mathfrak{o}(p-1, 1)$. Moreover, $\text{ad}(X)(\mathfrak{o}(p, 1))$ lies in $\mathfrak{o}(p, q'+1)$ but not in $\mathfrak{o}(p, q')$ (because the left/right action of X exchanges the first and last columns/rows in a matrix in $\mathfrak{o}(p, q'+1)$.) Therefore, by part (1) the Lie subalgebra of $\mathfrak{o}(p, q+1)$ generated by $\mathfrak{o}(p, q')$ and $\text{ad}(X)(\mathfrak{o}(p, 1))$ is $\mathfrak{o}(p, q'+1)$. $\blacksquare$

Proof of Proposition 6.10. The idea is to cut our manifold M along the surfaces assumed to exist in the theorem-statement, construct a graph of groups out of this decomposition of M , and then to build a deformation of the representation ρ_0 with Zariski dense image by inducting along our graph of groups.

Let $\mathcal{H}_1, \dots, \mathcal{H}_q$ be the q two-sided, connected, totally geodesic, pairwise disjoint hypersurfaces in M . Denote by $M_1, \dots, M_r$ the connected components of $M \setminus \bigcup_{i=1}^q \mathcal{H}_i$.

Then we can define an oriented graph as follows (this is the graph of groups construction).

Let Y be the oriented graph with vertex set $v_1, \dots, v_r$, one corresponding for each connected component M_i ; and with edge set defined by: there is an edge e_i (with arbitrary orientation) joining v_{j_i} to $v_{j'_i}$ if there is a hypersurface $\mathcal{H}_i$ separating M_{j_i} and $M_{j'_i}$. Associate to each edge e_i the group $G_{e_i} = \pi_1(\mathcal{H}_i)$ and to each vertex v_{j_i} the group $G_{v_{j_i}} = \pi_1(M_{j_i})$. Then considering the maps $\phi_{0,e_i} : G_{e_i} \rightarrow G_{v_{j_i}}$, $\phi_{1,e_i} : G_{e_i} \rightarrow G_{v_{j'_i}}$ induced on fundamental groups by the inclusion of $\mathcal{H}_i$ in M_{j_i} and $M_{j'_i}$ respectively, we get the structure of a graph of groups on Y . By Bass-Serre theory [36] we know that the fundamental group Γ of this graph of groups is $\pi_1(M)$.

Let T be a maximal tree in Y . Consider the group Γ_T generated by the vertex groups $G_{v_1}, \dots, G_{v_r}$ with the relations $\phi_{0,e}(g) = \phi_{1,e}(g)$ for all edges e in T and all $g \in G_e$. Bass-Serre theory tells us that Γ is generated by Γ_T and an element γ_e for each edge e not in T , with relations $\gamma_e \phi_{0,e_i}(g) \gamma_e^{-1} = \phi_{1,e}(g)$ for all $g \in G_e$. After a possible renumbering, we may assume that the edges in T are $e_1, \dots, e_r$ and such that e_i joins v_{j_i} , which lies in the graph generated by $e_1, \dots, e_{i-1}$ in T , to $v_{j'_i}$ which does not lie in the subgraph of T generated by $e_1, \dots, e_{i-1}$. Define $\Gamma_0 \stackrel{\text{def}}{=} \pi_1(M_{j_1})$ and inductively, for all $1 \leq i \leq r$, $\Gamma_i = \Gamma_{i-1} *_{\pi_1(\mathcal{H}_i)} \pi_1(M_{j'_i})$. With this, $\Gamma_r = \Gamma_T$. We can now define a representation $\tilde{\rho} : \Gamma_T \rightarrow G$ as follows (here is where the deformation construction kicks in: we will be twisting along each gluing of $\mathcal{H}_i$ in M_{j_i} and $M_{j'_i}$):

- $\tilde{\rho}|_{\Gamma_0} = \rho_0|_{\Gamma_0}$.
- Assume inductively that $\tilde{\rho}|_{\Gamma_{i-1}}$ has been defined. Let X_i be an element of the Lie algebra of the centraliser of $\rho_0(\pi_1(\mathcal{H}_i))$ and let the geodesic path between v_{j_1} and $v_{j'_i}$ in T be $e_{k_1}, \dots, e_{k_{m_i}}, e_i$. Then

$$\tilde{\rho}|_{\pi_1(M_{j'_i})} = e^{X_i} e^{X_{k_{m_i}}} \dots e^{X_{k_1}} \circ \rho_0|_{\pi_1(M_{j'_i})} \circ (e^{X_i} e^{X_{k_{m_i}}} \dots e^{X_{k_1}})^{-1}$$

where the composition above means: compose with the map which is conjugation by $e^{X_i} e^{X_{k_{m_i}}} \dots e^{X_{k_1}}$. This defines $\tilde{\rho}$ on Γ_i .

This makes sense because the fact that the X_i are elements of the Lie algebra of the centraliser implies $\tilde{\rho}|_{\Gamma_{i-1}}$ and $\tilde{\rho}|_{\pi_1(M_{j'_i})}$ coincide in $\pi_1(\mathcal{H}_i)$.

The claim now is: by choosing the X_i appropriately, this twisting that defines $\tilde{\rho}$ is such that the Zariski closure of the image of $\tilde{\rho}$ in $SO(p, q+1)$ is $SO(p, r+1)$. To see this, first notice that $\rho_0(\pi_1(\mathcal{M}_{j_1}))$ is a lattice in (a conjugate of) $SO(p, 1)$ in $SO(p, q+1)$. Its Zariski closure in $SO(p, 1)$ is then $SO(p, 1)$ by ([26, Lemma 5.9]). So we have that the Zariski closure of $\tilde{\rho}(\Gamma_1)$ is the same as the Zariski closure of the group generated by $SO(p, 1)$ and $e^{X_1} SO(p, 1) (e^{X_1})^{-1}$. The latter group has its Lie algebra generated by $\mathfrak{o}(p, 1)$ and $\text{ad}(X_1)(\mathfrak{o}(p, 1))$ where X_1 is in the Lie algebra of the centraliser of $\rho_0(\pi_1(\mathcal{H}_1))$. Moreover, notice that $\rho_0(\pi_1(\mathcal{H}_1))$ is a lattice in $SO(p-1, 1)$ and so its Zariski closure has Lie algebra $\mathfrak{o}(p-1, 1)$. Then by Lemma 6.11(2), for

$q' = 1$, we may choose X_1 such that the Lie algebra of the Zariski closure of the group generated by $SO(p, 1)$ and $e^{X_1}SO(p, 1)(e^{X_1})^{-1}$ is $\mathfrak{o}(p, 2)$ (this follows from [33, Theorem A.4.9] and the fact that $\mathfrak{o}(p, 2)$ is simple). And so the Zariski closure of this group (and hence of $\tilde{\rho}(\Gamma_1)$) in $SO(p, q+1)$ is $SO(p, 2)$. Now we can proceed inductively. Consider $\tilde{\rho}(\Gamma_{i-1})$, which has Zariski closure $SO(p, i)$ in $SO(p, q+1)$. Then $\tilde{\rho}(\Gamma_i)$ has the same Zariski closure as the group generated by $SO(p, i)$ and $e^{X_i}SO(p, i)(e^{X_i})^{-1}$. Again, this group has Lie algebra generated by $\mathfrak{o}(p, i)$ and $\text{ad}(X_i)(\mathfrak{o}(p, i))$ where X_i is an element of the centraliser of $\tilde{\rho}(\mathcal{H}_i)$. By Lemma 6.11(2) for $q' = i$, we have that the Zariski closure of this group has Lie algebra $\mathfrak{o}(p, i+1)$ and so the Zariski closure of $\tilde{\rho}(\Gamma_i)$ in $SO(p, q+1)$ is $SO(p, i+1)$. Therefore, for $i = r$ we get that the Zariski closure of $\tilde{\rho}(\Gamma_r)$ is $SO(p, r+1)$.

So far, in this process of twisting our representation ρ_0 inductively along our graph of groups, we have only use those edges which correspond to separating hypersurfaces. Now it is time to use the remaining edges (i.e. corresponding to those edges associated to a non-separating hypersurface, or equivalently corresponding an edge not in the maximal tree T). To that effect define, for $r+1 \leq i \leq q$, $\Gamma_i = \Gamma_{i-1} *_{\pi_1(\mathcal{H}_i)}$, where, as before, $\Gamma_r = \Gamma_T$, and $\Gamma_q = \Gamma$. This means that Γ_i is generated by Γ_{i-1} and by γ_{e_i} subject to the relations $\gamma_{e_i} \phi_{0,e_i}(g) \gamma_{e_i}^{-1} = \phi_{1,e_i}(g)$, where recall e_i is not in the maximal tree T and $g \in G_{e_i} = \pi_1(\mathcal{H}_i)$. Define $\rho : \Gamma \rightarrow G$ as follows:

- $\rho|_{\Gamma_r} = \tilde{\rho}|_{\Gamma_r}$.
- Assume inductively that $\rho|_{\Gamma_{i-1}}$ has been defined. Let e_i be the edge joining the vertices v_{j_i} and $v_{j'_i}$ with an arbitrary orientation and let the geodesic paths in T starting at v_{j_1} and ending at v_{j_i} and $v_{j'_i}$ be $e_{k_1}, \dots, e_{k_{m_i}}$ and $e_{\ell_1}, \dots, e_{\ell_{n_i}}$ respectively. Moreover, let X_i be an element of the Lie algebra of the centraliser of $\rho_0(\pi_1(\mathcal{H}_i))$. Then $\rho|_{\Gamma_i}$ is defined as $\rho|_{\Gamma_{i-1}}$ and

$$\rho(\gamma_{e_i}) = e^{X_{\ell_{n_i}}} \dots e^{X_{\ell_1}} \rho_0(\gamma_{e_i}) e^{X_i} (e^{X_{k_{m_i}}} \dots e^{X_{k_1}})^{-1}.$$

This last bullet makes sense because X_i is in the Lie algebra of the centraliser of $\rho_0(\pi_1(\mathcal{H}_i))$. Indeed, $\rho|_{\pi_1(M_{j_i})} = e^{X_{k_{m_i}}} \dots e^{X_{k_1}} \circ \rho_0|_{\pi_1(M_{j_i})} \circ (e^{X_{k_{m_i}}} \dots e^{X_{k_1}})^{-1}$ and $\rho|_{\pi_1(M_{j'_i})} = e^{X_{\ell_{n_i}}} \dots e^{X_{\ell_1}} \circ \rho_0|_{\pi_1(M_{j'_i})} \circ (e^{X_{\ell_{n_i}}} \dots e^{X_{\ell_1}})^{-1}$ so we have that for $g \in G_{e_i} = \pi_1(\mathcal{H}_i)$ the relation $\rho(\gamma_{e_i}) \rho(\phi_{0,e_i}(g)) \rho(\gamma_{e_i})^{-1} = \rho(\phi_{1,e_i}(g))$ is satisfied.

Once again, the claim is that if we choose the X_i appropriately then the image of ρ in $SO(p, q+1)$ will be Zariski dense. The group $\rho(\Gamma_{r+1})$ is generated by $\rho(\Gamma_r)$ and by $\rho(\gamma_{e_i})$. The group $\rho(\Gamma_r)$ has Zariski closure $SO(p, r+1)$ in $SO(p, q+1)$, and the element $\rho(\gamma_{e_{r+1}})$ is $e^{X_{r+1}}$ multiplied on the left and on the right by elements in $SO(p, r+1)$ ($\rho_0(\gamma_{e_{r+1}})$ is in a copy of $SO(p, 1)$ and the elements e^X are all in Γ_T , because they are associated to edges in the maximal tree T). Therefore, the Zariski closure of $\rho(\Gamma_{r+1})$ contains the Zariski closure of the group generated by $SO(p, r+1)$ and $e^{X_{r+1}}SO(p, r+1)(e^{X_{r+1}})^{-1}$. Then by Lemma 6.11(2), for $q' = r+1$, we may choose X_{r+1} such that the Lie algebra generated by the $\mathfrak{o}(p, r+1)$ and $\text{ad}(X_{r+1})(\mathfrak{o}(p, r+1))$

is $\mathfrak{o}(\mathfrak{p}, \mathfrak{r} + 2)$. And so, the Zariski closure of the group generated by $\mathrm{SO}(\mathfrak{p}, \mathfrak{r} + 1)$ and $e^{X_{\mathfrak{r}+1}}\mathrm{SO}(\mathfrak{p}, \mathfrak{r} + 1)(e^{X_{\mathfrak{r}+1}})^{-1}$ must be $\mathrm{SO}(\mathfrak{p}, \mathfrak{r} + 2)$. In other words, $\rho(\Gamma_{\mathfrak{r}+1})$ has $\mathrm{SO}(\mathfrak{p}, \mathfrak{r} + 2)$ as Zariski closure. Likewise, proceeding inductively and using Lemma 6.11(2) for $\mathfrak{q}' = \mathfrak{i}$, we may choose $X_{\mathfrak{i}}$ such that the Zariski closure of $\rho(\Gamma_{\mathfrak{i}})$ in $\mathrm{SO}(\mathfrak{p}, \mathfrak{q} + 1)$ is $\mathrm{SO}(\mathfrak{p}, \mathfrak{i} + 1)$. Which means that $\rho(\Gamma_{\mathfrak{q}})$ is Zariski dense in $\mathrm{SO}(\mathfrak{p}, \mathfrak{q} + 1)$.

Lastly, if $\mathcal{U}$ is a neighbourhood of ρ_0 in $\mathrm{Hom}(\Gamma, \mathbf{G})$, then for a sufficiently small choice of ϵ_i , the deformation given by replacing the X_i by the $\epsilon_i X_i$ will be contained in $\mathcal{U}$. This is because the conjugation maps defined by the $e^{\epsilon_i X_i}$ will not distance themselves much from the identity and so under the compact-open topology of $\mathrm{Hom}(\Gamma, \mathbf{G})$ the resultant ρ will lie in a small neighbourhood of ρ_0 as desired. $\blacksquare$

Therefore, by [5, Theorem 1.1] we get a HHTS $\mathcal{T}(\mathbf{M}, \mathrm{SO}(\mathfrak{p}, \mathfrak{q} + 1))$, where $\mathbf{M}$ is hyperbolic.

Having seen how to *bend* representations following Johnson-Millson, we can construct the HHTS gotten from the Monclair-Schlenker-Tholozan representations. Note that this is fundamentally different from the preceding example because, before, we had the fundamental group of our manifold being a lattice in $\mathrm{SO}(\mathfrak{p}, 1)$ while we know (see Section 2) that the fundamental group of a Gromov-Thurston manifold with dimension ≥ 4 is not commensurable to a lattice in any Lie group. We have the following proposition.

Proposition 6.12. Let $\mathfrak{p} \geq 4$ and $\mathfrak{k} > \mathfrak{n} \geq 2$. Consider a $\mathfrak{p}$ -dimensional $\mathfrak{k}$ -ramified Gromov-Thurston manifold $\mathbf{M}^{\frac{\mathfrak{k}}{\mathfrak{n}}}$ over a $\mathfrak{p}$ -dimensional $\mathfrak{n}$ -dihedral hyperbolic manifold $\mathbf{M}$. Then the $\mathbb{H}^{\mathfrak{p}, 1}$ -convex cocompact representations $\rho : \pi_1(\mathbf{M}^{\frac{\mathfrak{k}}{\mathfrak{n}}}) \rightarrow \mathrm{SO}(\mathfrak{p}, 2)$ constructed in the Monclair-Schlenker-Tholozan paper [32] can be deformed continuously in $\mathfrak{C}(\mathbf{M}^{\frac{\mathfrak{k}}{\mathfrak{n}}}, \mathrm{SO}(\mathfrak{p}, \mathfrak{q} + 1))$ to $\mathbb{H}^{\mathfrak{p}, \mathfrak{q}}$ -convex cocompact representations with Zariski dense image as long as $1 \leq \mathfrak{q} \leq 2\mathfrak{k} - 2$.

Remark 6.13. By Fact 6.5 we get that Proposition 6.12 gives examples of HHTS of the form $\mathcal{T}(\mathbf{M}^{\frac{\mathfrak{k}}{\mathfrak{n}}}, \mathrm{SO}(\mathfrak{p}, \mathfrak{q} + 1))$ for $\mathfrak{q} > 1$.

For the proof of Proposition 6.12 we will need the following lemma.

Lemma 6.14. Consider the pseudo-Riemannian Euclidean space $\mathbb{R}^{2, \mathfrak{q}}$ of signature $(2, \mathfrak{q})$. Let $\mathbf{v}_0$ and $\mathbf{v}_2$ be two non-collinear vectors such that the standard bilinear form of signature $(2, \mathfrak{q})$ restricts to a positive definite form. Then for all $\alpha \in \mathbb{R}$ the group $\mathrm{SO}(2, \mathfrak{q})$ acts transitively on the set of vectors

$$\mathcal{V}_{\mathbf{v}_0, \mathbf{v}_2, \alpha} = \{\mathbf{v} \in \mathbb{R}^{2, \mathfrak{q}} \mid \langle \mathbf{v}, \mathbf{v} \rangle_{2, \mathfrak{q}} = 1, \langle \mathbf{v}, \mathbf{v}_0 \rangle_{2, \mathfrak{q}} = \langle \mathbf{v}, \mathbf{v}_2 \rangle_{2, \mathfrak{q}} = \alpha\}.$$

Proof. Up to conjugation (base change) we can let $\mathbf{v}_0, \mathbf{v}_2$ be the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ (see Figure 7). Consider the unique vector $\mathbf{v}'_1 \in \mathrm{Span}\{\mathbf{v}_0, \mathbf{v}_2\}$ such that $\langle \mathbf{v}'_1, \mathbf{v}_0 \rangle_{2, \mathfrak{q}} = \alpha$ and $\langle \mathbf{v}'_1, \mathbf{v}_2 \rangle_{2, \mathfrak{q}} = \alpha$. In other words, such that $\mathbf{v}'_1 = (\alpha, \alpha, 0)$. Then the set $\mathcal{V}_{\mathbf{v}_0, \mathbf{v}_2, \alpha}$ is the $(\mathfrak{q} - 1)$ -sphere of radius $2\alpha^2 - 1$ in the affine subspace $\mathbf{v}'_1 + \mathrm{Span}\{\mathbf{v}_0, \mathbf{v}_2\}^\perp$ (or is empty). This is because $\mathcal{V}_{\mathbf{v}_0, \mathbf{v}_2, \alpha}$ will be the set of all vectors $\mathbf{v} = \mathbf{v}'_1 + \mathbf{v}_1$ such

that $\mathbf{v}_1 \in \text{Span}\{\mathbf{v}_0, \mathbf{v}_2\}^\perp$ and $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle_{2,q} = 1 - 2\alpha^2$, the latter equation being exactly the $(q-1)$ -sphere of radius $2\alpha^2 - 1$ in $\text{Span}\{\mathbf{v}_0, \mathbf{v}_2\}^\perp$. Depending on the value of α this latter set is empty or an honest $(q-1)$ -sphere. In either case, the subgroup which fixes $\text{Span}\{\mathbf{v}_0, \mathbf{v}_2\}$ acts transitively on $\mathcal{V}_{\mathbf{v}_0, \mathbf{v}_2, \alpha}$ because it is a conjugate of $\text{SO}(q)$ in $\text{SO}(2, q)$. $\blacksquare$

Proof of Proposition 6.12. Let $M^{\frac{k}{n}}$ be a p -dimensional k -ramified Gromov-Thurston manifold as in the proposition-statement. By what was done in Section 5 we know that for each equilateral spacelike de Sitter $2k$ -polygon of sidelength π/n there is associated a spacelike AdS atlas of $M^{\frac{k}{n}}$ (see Lemmas 5.4 and 5.6). By what was done in Section 3 we know that these spacelike AdS atlases give, in analogy with the classical theory of (G, X) -structures, a holonomy representation $\rho : \pi_1(M^{\frac{k}{n}}) \rightarrow \text{SO}(p, 2)$ and a developping map $\text{dev} : \widetilde{M} \rightarrow \text{AdS}^{p+1}$ with image a spacelike hypersurface (Theorem 3.21). In Subsection 6.1 we saw that this holonomy representation is actually an $\mathbb{H}^{p,1}$ -convex cocompact representation. Here we prove that this representation can be deformed, in $\text{Hom}(\pi_1(M^{\frac{k}{n}}), \text{SO}(p, q+1))$, in the sense of Johnson-Millson to an $\mathbb{H}^{p,q}$ -convex cocompact representation with Zariski dense image.

The idea will be to see $\text{AdS}^{p+1} = \mathbb{H}^{p,1}$ inside $\mathbb{H}^{p,q}$ and to carry out the Monclair-Schlenker-Tholozan spacelike de Sitter polygon construction in $\text{AdS}^{p+1} \subset \mathbb{H}^{p,q}$ for a specific polygon on which we will apply Lemma 6.14 inductively. Then we will see that the representation for this polygon can be deformed in the sense of Johnson-Millson into a representation with Zariski dense image. A path-connectedness argument will conclude the proof.

Let M be the p -dimensional n -dihedral hyperbolic manifold covered by $M^{\frac{k}{n}}$, and let $\mathcal{H}_i = \text{Fix}(\sigma_i)$ for $i = 1, 2$. Then we denote (as in Section 2) by H_1^M, H_2^M the closure of connected components of $\mathcal{H}_1 \setminus S, \mathcal{H}_2 \setminus S$ in M such that the oriented angle between H_1^M and H_2^M is π/n . Denote by $\overline{H}_1, \overline{H}_2$ the image of H_1^M, H_2^M under the projection map into $\overline{M} = R_n \setminus M$ (see Section 2) and denote their lifts into $M^{\frac{k}{n}}$ by $H_1, \dots, H_{2k}$. Then we will call V_j the connected components of $M^{\frac{k}{n}} \setminus \bigcup_{i=1}^{2k} H_i$ bounded by H_{j-1} and H_j .

Observe that the groups $\rho(\pi_1(H_i)), \rho(\pi_1(V_i))$ are lattices in a copy of $\text{SO}(p-1, 1)$ and $\text{SO}(p, 1)$ respectively. Moreover, we have the inclusions $\rho(\pi_1(H_i)) \hookrightarrow \rho(\pi_1(V_i)) \hookrightarrow \rho(\pi_1(M^{\frac{k}{n}}))$. Which means that the Lie algebra of the Zariski closure of $\rho(\pi_1(M^{\frac{k}{n}}))$ contains $\mathfrak{o}(p, 1)$. This inclusion must be strict because $M^{\frac{k}{n}}$ is a Gromov-Thurston manifold and so its fundamental group is not commensurable to a lattice in $\text{SO}(p, 1)$ (for $p \geq 4$). Therefore, by Lemma 6.11 the Lie algebra of the Zariski closure of $\rho(\pi_1(M^{\frac{k}{n}}))$ is $\mathfrak{o}(p, 2)$ and so the Zariski closure of $\rho(\pi_1(M^{\frac{k}{n}}))$ is $\text{SO}(p, 2)$.

Let P be the $2k$ -polygon given by $(\mathbf{v}_j)_{j=1}^{2k}$

$$\mathbf{v}_j = \sqrt{\alpha_{k,n}} \left(\cos \left(\frac{\pi j}{k} \right) \mathbf{e}_1 + \sin \left(\frac{\pi j}{k} \right) \mathbf{e}_2 \right) + \sqrt{\alpha_{k,n} - 1} \mathbf{e}_3 \quad (13)$$

as in Proposition 4.20 i.e. where $\alpha_{k,n} = \frac{1-\cos(\pi/n)}{1-\cos(\pi/k)}$ (the e_i are an orthonormal basis in which the standard bilinear form of signature $(2, 1)$ has matrix $\text{diag}(1, 1, -1)$). Then we know that the planes $\text{Span}(v_j, v_{j+2})$ are spacelike (we will need this to be able to apply Lemma 6.14 inductively). Let $\rho : \pi_1(M^{\frac{k}{n}}) \rightarrow \text{SO}(p, 2)$ be the holonomy representation of the spacelike AdS structure associated to P . Then ρ is $\mathbb{H}^{p,1}$ -convex cocompact and so its composition with the natural inclusion $\text{SO}(p, 2) \hookrightarrow \text{SO}(p, q+1)$ is $\mathbb{H}^{p,q}$ -convex cocompact (see [13] and [14]). Call this composite $\rho : \pi_1(M^{\frac{k}{n}}) \rightarrow \text{SO}(p, q+1)$ as well. We will continuously deform ρ in the sense of Johnson-Millson to obtain a representation with Zariski dense image. Notice that by Fact 6.5 the resulting representation is still $\mathbb{H}^{p,q}$ -convex cocompact.

Just as in Lemma 5.4 this polygon lives in the tangent space to a point $z \in S \subset M^{\frac{k}{n}}$, where S is the lift to $M^{\frac{k}{n}}$ of the intersection $\mathcal{H}_1 \cap \mathcal{H}_2$ in M . Indeed, by Lemma 5.4 we have a folded hipped hypersurface associated to our polygon P such that $z \in Z$ where Z is a totally geodesic copy of $\mathbb{H}^{p-2}$ in AdS^{p+1} . Thus we can see the polygon P as living in $T_z \text{AdS}^{p+1} \subset T_z \mathbb{H}^{p,q}$, where we are seeing AdS^{p+1} naturally included in $\mathbb{H}^{p,q}$. More precisely, P lives in an $\mathbb{R}^{2,1}$ in the orthogonal complement $T_z Z^\perp \cong \mathbb{R}^{2,q}$ inside $T_z \mathbb{H}^{p,q}$.

Let $\{e_1, \dots, e_{p+q}\}$ be an orthonormal basis of $T_z \mathbb{H}^{p,q}$ such that the standard signature (p, q) symmetric bilinear form has matrix $\text{diag}(1, \dots, 1, -1, \dots, -1)$ with 1 repeated p times and -1 repeated q times. By Lemma 6.14 (with $\alpha = \cos(\pi/n)$) there exist elements $g_t^{(1)}$ in $\text{SO}(p, q+1)$, with $g_0^{(1)} = \text{id}$, acting on the tangent space $T_z \mathbb{H}^{p,q} \cong \mathbb{R}^{p,q}$ fixing $\text{Span}(e_1, \dots, e_{p-2})$ and preserving the complement, which fix v_0 and v_2 (because the $\text{Span}(v_0, v_2)$ is a spacelike plane), and such that $v_{1,t} \stackrel{\text{def}}{=} g_t^{(1)} \cdot v_1$ satisfy $\langle v_{1,t}, v_{1,t} \rangle = 1$, $\langle v_{1,t}, v_0 \rangle = \cos(\pi/n)$, $\langle v_{1,t}, v_2 \rangle = \cos(\pi/n)$, lies in $\text{Span}(e_1, e_2, e_3, e_4) \setminus \text{Span}(e_1, e_2, e_3)$ and $\text{Span}(v_{1,t}, v_3)$ remains a spacelike plane. The last two claims can be seen as follows. That the elements can be chosen so that $v_{1,t}$ lies in $\text{Span}(e_1, e_2, e_3, e_4) \setminus \text{Span}(e_1, e_2, e_3)$ follows from dimensionality considerations. Given $q \geq 2$ the $(q-1)$ -sphere of freedom we have in moving $v_{1,t}$ within $\text{Span}(e_1, e_2, e_3, e_4)$ is at worst a circle, and so we have enough space to move solely in the first 4-coordinates. For the last statement note that, by construction,

$$\text{Span}(v_{j-1}, v_j, v_{j+1}) = \text{Span}(e_1, e_2, e_3)$$

(because our polygon lives wholly in an $\mathbb{R}^{2,1}$ and no three consecutive vertices are co-linear). Thus, we can move entirely above v_1 (in the orthogonal direction to $\text{Span}(v_0, v_2)$ at v_1), which means $v_{1,t}$ has the same first three coordinates as v_1 and so $\langle v_1, v_3 \rangle = \langle v_{1,t}, v_3 \rangle$.

Moreover, these elements act trivially on $T_z Z$ because (up to conjugation) they look like

$$\begin{bmatrix} \text{id}_{p-2} & \\ & \text{SO}(2, q) \end{bmatrix}$$

for they are elements in the natural copy of $SO(2, q)$ in $SO(p, q + 1)$ (note that this means the $g_t^{(1)}$ fix Z).

Consider

$$M^{\frac{k}{n}} = (M^{\frac{k}{n}} \setminus \overline{V}_j \cup \overline{V}_{j+1}) \cup (H_j \cup H_{j+1}) \cup (V_j \cup V_{j+1} \cup (H_j \setminus S)).$$

Then given $H_j \cup H_{j+2}$ is path-connected and the complement spaces are also, we get by Van Kampen's theorem,

$$\pi_1(M^{\frac{k}{n}}) = \pi_1(M^{\frac{k}{n}} \setminus \overline{V}_j \cup \overline{V}_{j+1}) *_{\pi_1(H_j \cup H_{j+2})} \pi_1(V_j \cup V_{j+1} \cup (H_j \setminus S)). \quad (14)$$

Then, letting $j = 0$, we can define a representation $\rho_t^{(1)}$ by prescribing it be equal to ρ on $\pi_1(M^{\frac{k}{n}} \setminus \overline{V}_0 \cup \overline{V}_1)$ and to $g_t^{(1)} \rho(\cdot) (g_t^{(1)})^{-1}$ on $\pi_1(V_0 \cup V_1 \cup (H_1 \setminus S))$. Indeed, the $g_t^{(1)}$ centralise $\rho(\pi_1(H_0))$ because they act trivially on the tangent space $T_z H_0$. This is because $T_z H_0 = \{u + tv_0 \mid u \in T_z Z, t \in \mathbb{R}\}$ (see the proof of Lemma 5.4) and the $g_t^{(1)}$ act trivially on $T_z Z$ and fix v_0 . Likewise, the $g_t^{(1)}$ centralise $\rho(\pi_1(H_2))$ and so they centralise $\rho(\pi_1(H_0 \cup H_2))$ (by van Kampen's theorem $\pi_1(H_0 \cup H_2) = \pi_1(H_0) *_{\pi_1(S)} \pi_1(H_2)$). Therefore, we get a continuous family $\rho_t^{(1)}$ of representations within $\text{Hom}(\pi_1(M^{\frac{k}{n}}), SO(p, q + 1))$. By construction $v_{2k-2}, v_{2k-1}, v_{2k} = v_0$ and the $v_{1,t}$ span an $\mathbb{R}^{2,2}$. This means that the image of $\rho_t^{(1)}$, for $t \neq 0$, cannot lie entirely within a copy of $SO(p, 2)$ in $SO(p, 3)$. And so the Zariski closure of the image of $\rho_t^{(1)}$ is $SO(p, 3)$ by Lemma 6.11(1) (notice that the image must properly contain an $SO(p, 1)$ because of the inclusion $\pi_1(V_j) \hookrightarrow \pi_1(M^{\frac{k}{n}})$). Set $\rho^{(1)} = \rho_1^{(1)}$.

Then, proceeding by induction, for all j, q such that $1 \leq j \leq q-1 \leq 2k-3$, we get by Lemma 6.14 that there is a continuous family of elements $g_t^{(j)} \in SO(p, q+1)$ preserving $T_z Z^\perp = \text{Span}(e_{p-1}, \dots, e_{p+q})$ fixing $v_{j-1,1} = g_1^{(j-1)} \cdot v_{j-1}$ and v_{j+1} , such that $g_0^{(j)} = \text{id}$ and such that for all $t \neq 0$, $\langle v_{j,t}, v_{j,t} \rangle = 1$, $\langle v_{j,t}, v_{j-1} \rangle = \langle v_{j,t}, v_{j+1} \rangle = \cos(\pi/n)$ and the plane $\text{Span}(v_{j,t}, v_{j+1})$ is spacelike. So as before we can define a representation $\rho_t^{(j)}$ by prescribing it to be equal to $\rho^{(j-1)}$ on $\pi_1(M^{\frac{k}{n}} \setminus \overline{V}_{j-1} \cup \overline{V}_j)$ and equal to $g_t^{(j)} \rho^{(j-1)}(\cdot) (g_t^{(j)})^{-1}$ on $\pi_1(V_{j-1} \cup V_j \cup (H_j \setminus S))$. Then we get that $\rho_0^{(j)} = \rho^{(j-1)}$. By construction we have that $v_{2k-2}, v_{2k-1}, v_{2k} = v_0$ and the vertices $v_{1,1}, \dots, v_{j,1}$ span an $\mathbb{R}^{2,1+j}$ and so, as before, by Lemma 6.11(1) the Zariski closure of the image of $\rho_1^{(j)}$ is $SO(p, 2 + j)$. Set $\rho^{(j)} = \rho_1^{(j)}$. Lastly, letting $j = q - 1$ we get that the representation $\rho^{(q-1)}$ is a continuous deformation of ρ and has Zariski dense image. By Fact 6.5 it must be $\mathbb{H}^{p,q}$ -convex cocompact.

We have that the chosen representation ρ can be continuously deformed in the character variety $\mathfrak{C}(M^{\frac{k}{n}}, SO(p, q + 1))$ into an $\mathbb{H}^{p,q}$ -convex cocompact representation with Zariski dense image. By Proposition 4.19 the set of equilateral $2k$ -polygons with sidelength π/n is path-connected and so the set of all representations given by the Monclair-Schlenker-Tholozan construction is also path-connected within the character variety $\mathfrak{C}(M^{\frac{k}{n}}, SO(p, q + 1))$. This latter fact is due to the continuity of

the map from the set of such polygons, which is a manifold by Proposition 4.19, to the set of $\mathbb{H}^{p,1}$ -convex cocompact representations under the compact-open topology. By connectedness we get that all such representations (after composing with the natural inclusion $\mathrm{SO}(p, 2) \hookrightarrow \mathrm{SO}(p, q + 1)$) can be continuously deformed in $\mathrm{Hom}(\pi_1(M_n^k), \mathrm{SO}(p, q + 1))$ into an $\mathbb{H}^{p,q}$ -convex cocompact representation with Zariski dense image. ■

6.3. HHTS-perspective of generalised results $q \geq 1$. In this subsection we do the same as we did in Subsection 6.1 for $q > 1$. In other words, we prove the existence of HHTS associated to folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures on M_n^k (in other words containing the holonomies of such structures). Moreover, we show that the representations with Zariski-dense image constructed in Subsection 6.2 are the holonomies of folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures, hence showing that the HHTS constructed contain representations with Zariski-dense image.

Setup 6.15. For this, consider any holonomy representation $\rho_0 : \pi_1(M_n^k) \rightarrow \mathrm{SO}(p, 2)$ associated to a folded convex ruled spacelike AdS structure on M_n^k , where $k > n$. Then by Theorem 6.7 we know such a representation is $\mathbb{H}^{p,1}$ -convex cocompact. Composing with the natural inclusion $\mathrm{SO}(p, 2) \hookrightarrow \mathrm{SO}(p, q + 1)$ then yields an $\mathbb{H}^{p,q}$ -convex cocompact representation (see [13] and [14]), call it $\rho : \pi_1(M_n^k) \rightarrow \mathrm{SO}(p, 2) \hookrightarrow \mathrm{SO}(p, q + 1)$ as well. In Subsection 6.2 we saw how to continuously deform this representation into a representation with Zariski-dense image, which, by Fact 6.5, remains $\mathbb{H}^{p,q}$ -convex cocompact. Recall that the deformation procedure of Proposition 6.12 gave us a set $\rho^{(j)}$ (of which $\rho^{(0)} = \rho$) of deformed representations each with Zariski-dense image in a larger subgroup of $\mathrm{SO}(p, q + 1)$ (namely $\rho^{(j)}$ had Zariski-dense image in a conjugate of $\mathrm{SO}(p, 2 + j)$). We get the following.

Theorem 6.16. In the notation of the proof of Proposition 6.12, let

$$\rho^{(q-1)} : \pi_1(M_n^k) \rightarrow \mathrm{SO}(p, q + 1)$$

be the result of applying the deformation procedure to ρ (as in Proposition 6.12). Then ρ is the holonomy representation of a folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structure on M_n^k .

Proof. This follows by analysing the proof of Proposition 6.12. Namely, recall we constructed ρ by an inductive process, so it suffices to show that the first step in this inductive process is the holonomy representation of a folded spacelike $\hat{\mathbb{H}}^{p,q}$ structure. In other words, (see proof of Proposition 6.12) that $\rho^{(1)}$ is the holonomy representation of a spacelike $\hat{\mathbb{H}}^{p,2}$ -structure.

To be precise, recall that $\rho^{(1)} : \pi_1(M_n^k) \rightarrow \mathrm{SO}(p, q + 1)$ had Zariski-dense image in (a conjugate of) $\mathrm{SO}(p, 3)$ within $\mathrm{SO}(p, q + 1)$. We constructed $\rho^{(1)}$ by deforming the first vertex of the equilateral spacelike de Sitter $2k$ -polygon $P = (v_0, \dots, v_{2k-1})$ giving ρ (via Theorem 1.3 for $q = 1$). Indeed, we defined $\rho^{(1)}$ by letting it be equal

to ρ on $\pi_1(M_{\mathbb{H}^n}^{\mathbb{K}} \setminus (\overline{V_0} \cup \overline{V_1}))$ and equal to (in the notation of the proof of Proposition 6.12) $g_1^{(1)} \rho(\cdot) (g_1^{(1)})^{-1}$ on $\pi_1(V_0 \cup V_1 \cup (H_1 \setminus S))$. By Lemma 6.14, the element $g_1^{(1)} \in \text{SO}(\mathfrak{p}, \mathfrak{q} + 1)$ could be chosen so as to fix v_0, v_2 and act of v_1 so as to get that $v_1^{(1)} = g_1^{(1)} v_1$ is in spacelike position with respect to v_0, v_2, v_3 and such that $v_1^{(1)}$ does not live in the $\mathbb{R}^{2,1}$ which is spanned by the v_i (i.e. that $\text{Span}(v_0, v_1^{(1)}, v_2, \dots, v_{2k-1})$ is an $\mathbb{R}^{2,2}$).

With this, we would like to get a spacelike $\hat{\mathbb{H}}^{\mathfrak{p},2}$ -structure associated (as in Theorem 3.21) to the representation $\rho^{(1)}$. In other words, $\rho^{(1)}$ gives us our transition maps, now we want to find the spacelike $(\mathfrak{p}, 2)$ -charts associated to this holonomy representation. But by construction we have an obvious candidate for our spacelike $(\mathfrak{p}, 2)$ -charts, namely, the charts $\phi_i : \mathcal{U}_i \rightarrow \hat{\mathbb{H}}^{\mathfrak{p},2}$ where $\mathcal{U}_i \in M_{\mathbb{H}^n}^{\mathbb{K}} \setminus (\overline{V_0} \cup \overline{V_1})$ and $g_1^{(1)} \phi_j (g_1^{(1)})^{-1} : \mathcal{U}_j \rightarrow \hat{\mathbb{H}}^{\mathfrak{p},2}$ where $\mathcal{U}_j \in V_0 \cup V_1 \cup (H_1 \setminus S)$: the $\{\phi_i, \phi_j\}$ are the spacelike $(\mathfrak{p}, 1)$ -charts giving the spacelike AdS structure of $M_{\mathbb{H}^n}^{\mathbb{K}}$ associated to the polygon P . (In what follows it will be clear that we can ignore the technical detail of those chart whose domain $\mathcal{U}$ lands *in both* $M_{\mathbb{H}^n}^{\mathbb{K}} \setminus (\overline{V_0} \cup \overline{V_1})$ and $V_0 \cup V_1 \cup (H_1 \setminus S)$), essentially by condition (3) of Definition 3.15.)

The reader will of course notice that we are missing the 1-frames we need to get a spacelike $(\mathfrak{p}, 2)$ -structure out of the charts $\{\phi_i, g_1^{(1)} \phi_j (g_1^{(1)})^{-1}\}$. We proceed to construct such frames. Recall that our polygon P lives in the tangent space of a point of $\hat{\mathbb{H}}^{\mathfrak{p},2}$. In particular, it lives in an $\mathbb{R}^{2,1}$ within it. And this is also the case for most of the polygon $P' = (v_0, v_1^{(1)}, v_2, \dots, v_{2k-1})$ (which is, for clarity, neither a spacelike de Sitter polygon nor a spacelike $\hat{\mathbb{S}}^{1,2}$ polygon). Namely, all vertices except $v_1^{(1)}$, live in this same $\mathbb{R}^{2,1}$, but P' lives in an $\mathbb{R}^{2,2}$. Hence, pick the unit timelike vector w_1 in the orthogonal complement of $\mathbb{R}^{2,1}$ within this $\mathbb{R}^{2,2}$ with the property that the matrix $[v_{2k-2}, v_{2k-1}, v_0, w_1]$ is in $\text{SO}_o(2, 2)$ (note that $(\mathbb{R}^{2,1})^\perp = \mathbb{R}^{0,1} \subset \mathbb{R}^{2,2}$ so the preceding phrase makes sense). Next, let w_1 be the 1-frame associated to each edge of P' except the edges $[v_0, v_1^{(1)}]$ and $[v_1^{(1)}, v_2]$. For these let the associated 1-frame be $g_1^{(1)} w_1$. Then, by definition of $g_1^{(1)}$, we get that P' is an equilateral spacelike $\hat{\mathbb{S}}^{1,2}$ $2k$ -polygon. Indeed, given $g_1^{(1)}$ fixes v_0, v_2 and w_1 is in the orthogonal complement of $\text{Span}(v_0, v_1)$ and of $\text{Span}(v_1, v_2)$, we get that $g_1^{(1)} w_1$ will be in the orthogonal complement of $\text{Span}(v_0, v_1^{(1)}, v_2)$.

With this we are done, because now the charts $\{\phi_i, g_1^{(1)} \phi_j (g_1^{(1)})^{-1}\}$ are honest spacelike $(\mathfrak{p}, 2)$ charts as they are retrieved from the spacelike $\hat{\mathbb{S}}^{1,2}$ polygon P' . Indeed, one can check this directly, namely, that the charts $\{\phi_i, g_1^{(1)} \phi_j (g_1^{(1)})^{-1}\}$ give a spacelike $\hat{\mathbb{H}}^{\mathfrak{p},2}$ structure, by computing that in fact the transition maps work and give us the needed coherence condition (Definition 3.15(3)).

The argument for general $\rho^{(q-1)}$ then follows by applying an equivalent procedure to the above at each step $\rho^{(i)}$. ■

With this we can easily get HHTS containing a connected component of the space of folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures, which, moreover, contains a representation with Zariski-dense image. Recall we say these HHTS are associated to our folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures.

Theorem 1.5. There are HHTS $\mathcal{T}(\mathcal{M}_{\mathfrak{n}}^{\frac{k}{n}}, \mathrm{SO}(p, q + 1))$ associated to certain folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structures, which contain representations with Zariski-dense image.

Proof. This is an immediate consequence of Theorem 6.16. Namely, take ρ as in Theorem 6.16, then we know it is the holonomy representation of a folded spacelike $\hat{\mathbb{H}}^{p,q}$ -structure and which is moreover, $\hat{\mathbb{H}}^{p,q}$ -convex cocompact, and has Zariski-dense image. Then by Fact 6.5 we know that the connected component containing it must be a HHTS. $\blacksquare$

6.3.1. *Concluding thoughts.* The reader will have noticed that I have shoved one crucial thing under the rug in this last section: There was never any talk about the regularity of the obvious map that Theorem 1.3 gives us between the character variety $\mathfrak{C}(\mathcal{M}_{\mathfrak{n}}^{\frac{k}{n}}, \mathrm{SO}(p, q + 1))$ and the set of all equilateral spacelike $\hat{\mathbb{S}}^{1,q}$ $2k$ -polygons of sidelength π/n .

Indeed, we used implicitly that the map is continuous to justify, in Theorems 1.4 and 1.5 that the resultant connected component was not a singleton and hence was a HHTS.

This is something discussed in Monclair-Schlenker-Tholozan [32, Section 6] for the anti-de Sitter case (the map is definitely continuous, hence why the results are true for all cases $q \geq 1$) and I expect their remarks to follow through to the case when $q > 1$. In fact, more can be said of the regularity of the map (see [32, Section 6]).

Another thing that must be addressed is the connectedness of the space of equilateral spacelike $\hat{\mathbb{S}}^{1,q}$ $2k$ -polygons (Remark 6.9(1)), and how much of the topology we get from knowledge of this subset of the character variety (Remark 6.9(2)). One idea to start would be to try to generalise the theorems, for the case when $q > 1$, found in Section 4 when $q = 1$. This would give us more of a hold on the topology of the higher q case of the set of equilateral spacelike $\hat{\mathbb{S}}^{1,q}$ $2k$ -polygons (and I believe this is possible).

One last thing is that, depending on the connectedness of the set of equilateral spacelike $\hat{\mathbb{S}}^{1,q}$ $2k$ -polygons, we may actually get more of the character variety in the case when $q > 1$ than in the case when $q = 1$. Notice that in the latter we know we always land in representations that are $\mathbb{H}^{p,1}$ -convex cocompact, while in the former, because of the potential for several connected components, we may land in representations which are not $\mathbb{H}^{p,q}$ -convex cocompact. This is a result of us not having dealt with the convexity conditions of the $q > 1$ case, in other words, of not having generalised what *convex ruled* means for a spacelike AdS structure to the case when $q > 1$.

Why did we not pursue any of these things in this master's thesis? Because “a work of art is never finished, it is merely abandoned.”³

³I am unsure whether or not this qualifies as art. I surely think it was beautiful, and had plenty of fun doing it!

7. APPENDIX OF PICTURES

In this appendix we gather the pictures (doodles) the reader might want to have done for himself while thinking about the mathematics above. We gather them here for convenience.

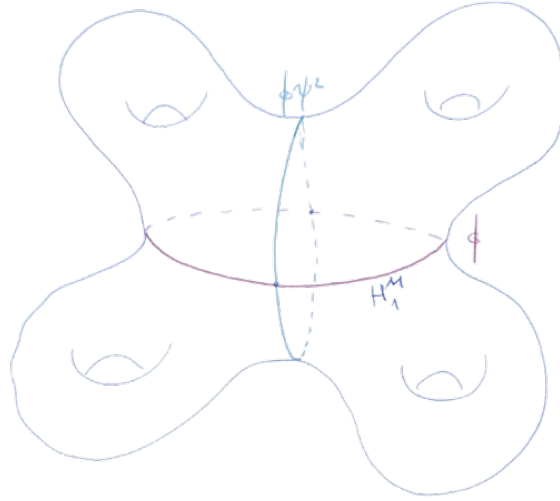


FIGURE 1. Section 2. Here we have Σ_4 with the two isometric involutions constructed. Note that this is the only dimension where S is not connected: a feature that this is the only dimension in which codimension 2 spheres are disconnected (0-spheres are disconnected).

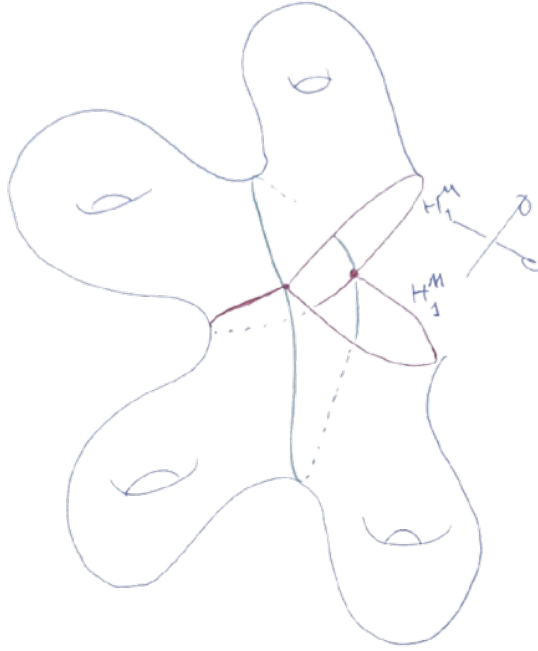


FIGURE 2. Section 2. Here we cut along H_1^M to then glue two such copies along the boundary to get a Gromov-Thurston manifold as in the next picture.

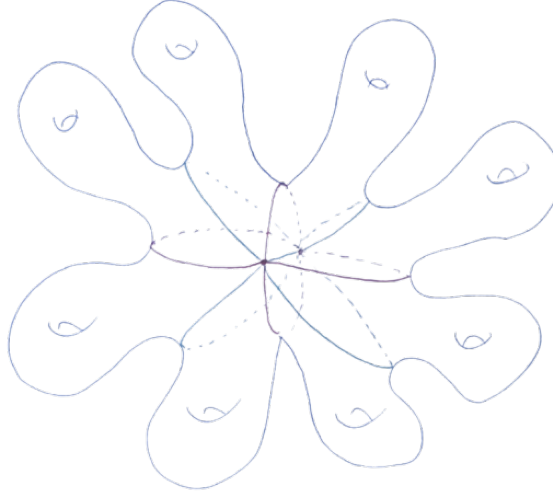


FIGURE 3. Section 2. The promised visualisable Gromov-Thurston manifold. Notice this is $M^{\frac{4}{2}}$ where $M = \Sigma_4$ as in Figure 1.

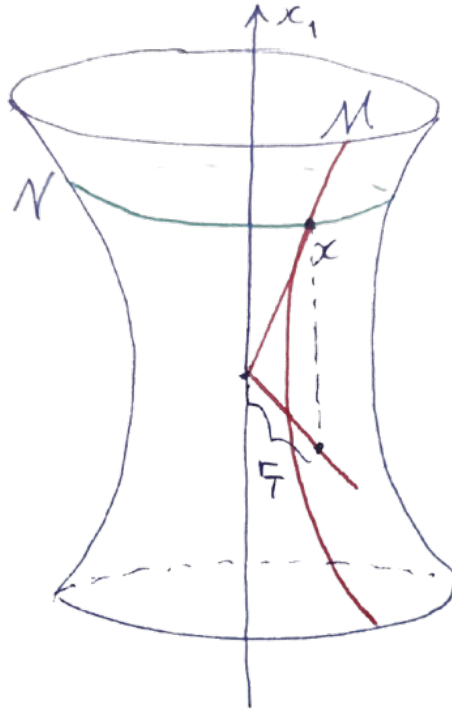


FIGURE 4. Proposition 3.24. This is the situation in $\widehat{\text{AdS}}^2$. Notice that if we wanted to be precise, N would have to be two disjoint circles, but this does not change the argument of the proof of Proposition 3.24.

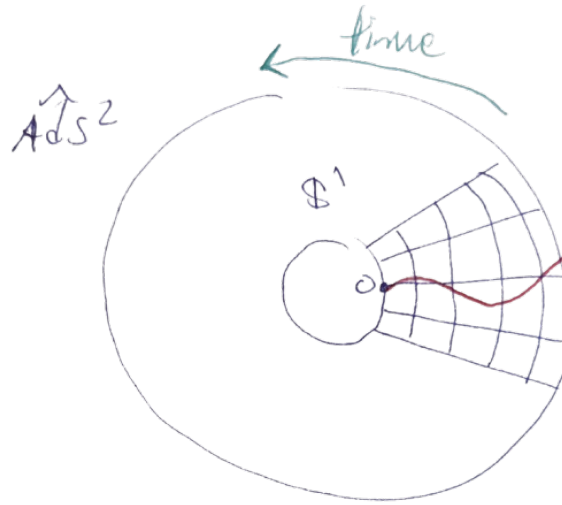


FIGURE 5. Proposition 3.28. The annulus is the view *from above* of the conformal model for $\widehat{\text{AdS}}^2 \cong \mathbb{B}^1 \times \mathbb{S}^1$. The graphs show how the situation looks like locally.

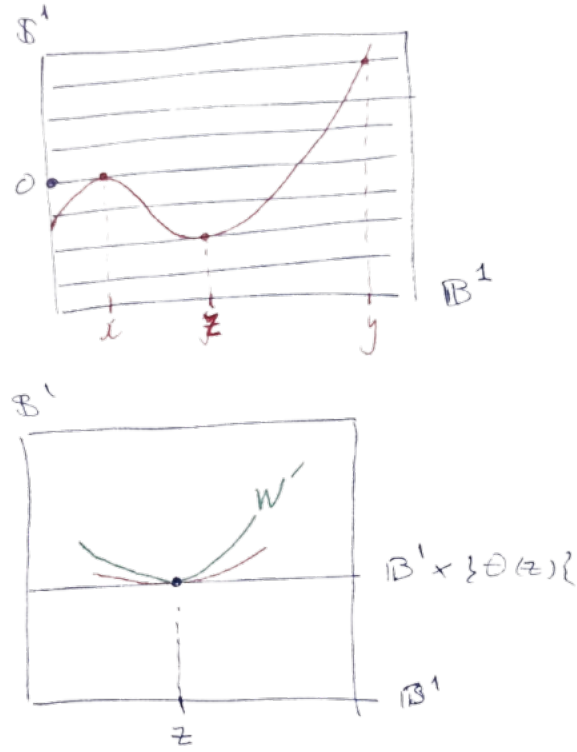


FIGURE 6. Proposition 3.28. The graphs show how the situation looks like locally.

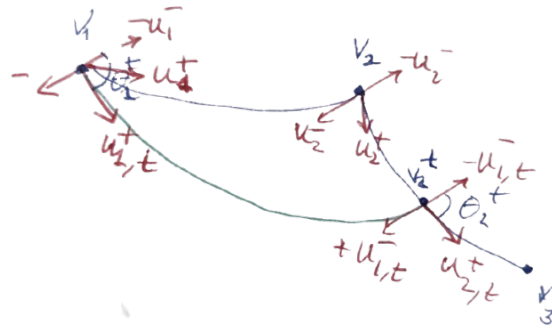
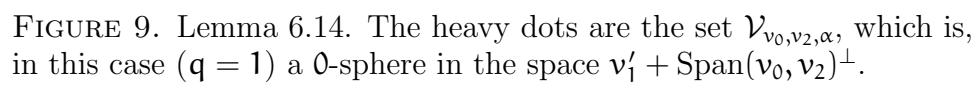
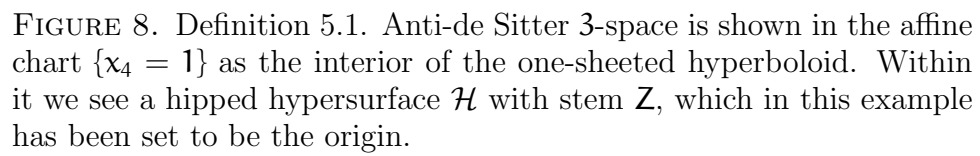


FIGURE 7. Theorem 4.15. Here we see how the polygon varies along a tangent vector $\dot{v}$ where its only non-zero coordinate is $\dot{v}_2 = u_2^+$. This means that the only vertex that varies is v_2 as shown.



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