

ON THE PRODUCT OF A BUILDING AS A POSET OF STANDARD COSETS

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ABSTRACT. This thesis explores the product in a Coxeter complex and in a building, both from an algebraic viewpoint. In the case of Coxeter complexes we consider the product when the simplicial complex is the poset of standard cosets of a Coxeter group W ordered by reverse inclusion. We find a solution method which allows us to compute this product solely from the algebraic data and with it we carry out a similar process when we consider thick buildings where there is a group G acting on them first by strong and then by Weyl transitivity.

1. INTRODUCTION

The goal of this thesis is to understand a certain operation on the faces of very nice simplicial complexes. To be precise we care to think about Coxeter complexes and of buildings in general (for definitions see sections 2.1 and 3.1). These two constructions are intimately linked. In these simplicial complexes we get that there is a binary operation on the faces which turns out to describe and illuminate many of their properties, so understanding it is crucial to the mathematician wishing to study and use these objects for his purposes.

There is a robust literature dealing with the behavior of this product in the case of a generic building. There is, however, a gap in understanding on how this product behaves when there is a group G acting on a building Δ in a sufficiently nice way. What ‘sufficiently nice’ means will be explained later (see section 3.1), but so far the reader must know that in these cases we can consider Δ as some poset of subsets of G under reverse inclusion. A natural question arises: how does the product behave with respect to these subsets? This thesis answers this question in its various avatars and explains the relationship between this representation of the building Δ as a poset of subsets of the group G and the product in it.

2. THE FIRST QUESTION: THE PRODUCT IN A COXETER COMPLEX

To understand the more general case of the product in a building we must first explore what it means to take a product of two faces inside a Coxeter complex. The reason for this is that a building is, informally, a collection of Coxeter complexes, (commonly called apartments in this context), that are stitched together in a coherent way. We now go through a quick review of some fundamental concepts necessary to understand the discussion that follows.

2.1. Background: Coxeter complexes. Before we even start discussing what a Coxeter complex is we need to recall the idea of a Weyl group.

Definition 1 (section 2.4 [Abramenko and Brown, 2008]). (Weyl group) A group W is a Weyl group if it has a set S of generators of order 2 such that

$$\langle S; (st)^{m(s,t)} = 1 \rangle$$

is a presentation of W where $m(s, t)$ is the order of st and there is one relation for each pair s, t with $m(s, t) < \infty$.

Example 1. To fix ideas the reader may want to think of W as the symmetric group S_n or the dihedral group D_{2n} which, as he can check, satisfy the conditions above. For an in-depth treatment of Weyl groups the reader is referred to [Abramenko and Brown, 2008] chapters 1 and 2.

Definition 2 (Definition 2.12 [Abramenko and Brown, 2008]). (Standard coset of a Weyl group) A standard coset of a Weyl group W is a set of the form wW_J where $w \in W$ and $W_J := \langle J \rangle$ where $J \subseteq S$.

Now that we have our basis we can jump right into Coxeter complexes.

Definition 3 (Definition 3.1 [Abramenko and Brown, 2008]). (Coxeter complex). The Coxeter complex $\Sigma(W, S)$ associated to the pair (W, S) , where S is a set of order two generators of the Weyl group W , is the set of standard cosets in W ordered by reverse inclusion. That is, if $B \geq A$ in $\Sigma(W, S)$ then this means that $B \subseteq A$ as subsets of W and vice versa. We say that B is a *face* of A if $B \leq A$.

A Coxeter complex is a simplicial complex of a very special kind (see [Abramenko and Brown, 2008] section 1.6). It is for instance a chamber complex. The next definition serves to remind the reader what some of these terms are.

Definition 4 (Chambers, panels, dimension and rank). A simplicial complex is said to be a chamber complex if the rank of the maximal simplices C are all the same, where by *rank* we mean the rank of the Boolean lattice of faces which are $\leq C$. These maximal simplices are called *chambers* and their co-dimension one faces are called *panels*. The dimension of a simplex A is $= \text{rank}(A) - 1$. [There is technically another requirement for a simplicial complex to be a chamber complex and that is that the chambers are connected by *galleries*, which are sequences of chambers in which any two consecutive ones share a panel. We will not need this second condition in what follows].

Remark 1. The concept of Coxeter complex is regularly weakened as follows: Let any simplicial complex Σ which is isomorphic to $\Sigma(W, S)$ as in **Definition 3** be called a Coxeter complex as well. This is the naming convention that we will follow in this thesis.

It will be useful to introduce the fact that a Coxeter complex $\Sigma(W, S)$ is colorable and thus that there is *type function* which we can choose to be the following:

Definition 5 (Type function). A Coxeter complex is colorable under the type function defined by

$$\begin{aligned} \tau : \Sigma &\rightarrow S \\ wW_J &\mapsto S \setminus J. \end{aligned}$$

or equivalently the standard coset wW_J has *cotype* J .

The reader is referred to [Abramenko and Brown, 2008] section 3.1 and to appendix A, in particular sections A.1 and A.2, if he wants to see the results on the type function proved and explained in detail.

Finally, we must introduce the Weyl distance function which can be thought of as ‘vector’ which points from one chamber to the next, in the sense that multiplying the former chamber by the ‘vector’ will give you the latter chamber.

Definition 6 (Weyl distance function). Let $\mathcal{C}$ be the set of chambers of $\Sigma := \Sigma(W, S)$. Recall that because the order in $\Sigma(W, S)$ is reverse inclusion we get that the maximal simplices are the standard coset of the form $w\{1\} = \{w\}$, i.e. we can identify the chambers of $\Sigma(W, S)$ with the elements of W . Then the Weyl distance function of Σ is,

$$\begin{aligned} \delta : \mathcal{C} \times \mathcal{C} &\rightarrow W \\ (u, w) &\mapsto u^{-1}w. \end{aligned}$$

Remark 2. This function can be extended to all simplices as follows. Let $A, B \in \Sigma$ be such that $A = wW_J$ and $B = uW_K$. Then define $\delta(A, B)$ to be *the* element of minimal length in the double coset $A^{-1}B = W_Jw^{-1}uW_K$. The existence of such a unique element in the double coset mentioned follows from the proposition in [Abramenko and Brown, 2008] stated below.

Proposition 1 (Proposition 2.23 in [Abramenko and Brown, 2008]). Let W_J and W_K be standard cosets (subgroups in this case) where as usual $J, K \subseteq S$. Then every double coset of the form W_JwW_K has a unique representative w_1 of minimal length.

2.2. The Product of a Coxeter Complex. With the background behind us we can phrase the following question: if we are given two simplices $A = wW_J, B = uW_K \in \Sigma$ where $\Sigma := \Sigma(W, S)$ is a Coxeter complex of type (W, S) and $J, K \subset S$, then what is the product AB equal to as a standard coset? In particular, we would like to be able to get this standard coset from the the information given to us in the cosets wW_J and uW_K .

We start with the simpler case where $B = C \in \mathcal{C}(\Sigma)$, i.e. where what we project onto A is a chamber. This means that $C = u \in W$. By the characterization of AC given in section 3.6 of [Abramenko and Brown, 2008] (Proposition 3.103), we get that AC is the unique chamber having A as a face that is at *minimal distance* from C . In the algebraic setting we can think of distances by using the Weyl distance function

$$\delta : \mathcal{C}(\Sigma) \times \mathcal{C}(\Sigma) \rightarrow W,$$

where we identify $\mathcal{C}(\Sigma)$ with W . If we denote by $\ell(w)$ the minimal length of the element w written as a word in the letters S , then we get we can define the distance d between two chambers as follows: $d(uD, wD) = \ell(\delta(u, w)) = \ell(u^{-1}w)$ where D is assumed to be the fundamental chamber. In other words, for the case $B = C \in \mathcal{C}(\Sigma)$ the question above reduces to finding the element of wW_J (i.e. the chamber among those which have A as a face) which is at a minimal distance from u , where *minimal distance* refers to those elements $w' \in wW_J$ which minimize $\ell(\delta(u, w'))$.

Before we can introduce the answer to this first case we must introduce an important result commonly known as the *gate property*.

Proposition 2 (Proposition 3.105 in [Abramenko and Brown, 2008]). For any simplex A and any chambers C, D with $D > A$,

$$d(C, D) = d(C, AC) + d(AC, D).$$

With this we can state the following modest generalization of Prop. 2.20 [Abramenko and Brown, 2008] which provides an answer to this simpler case.

Proposition 3. Let W_J be a standard parabolic subgroup (i.e. $J \subset S$). Then every coset wW_J has a unique representative $\hat{w}$ that is at a minimal distance from $u \in W$. It is characterized by the property,

$$l(u^{-1}\hat{w}s) = l(u^{-1}\hat{w}) + 1 \quad (1)$$

for all $s \in J$. Moreover,

$$l(u^{-1}\hat{w}w_J) = l(u^{-1}\hat{w}) + l(w_J) \quad (2)$$

for all $w_J \in W_J$.

Proof. The proof is essentially an application of the gate property. We prove equation (2) as (1) follows from it. Let Σ be the Coxeter complex associated to the Coxeter group W , i.e. $\Sigma = \Sigma(W, S)$, and let $A \in \Sigma$ be the simplex such that $A = wW_J$. Likewise let $E \in \mathcal{C}(\Sigma)$ the chamber such that $E = u$. The existence of an element $\hat{w} \in wW_J$ is given to us by Proposition 3.103 [Abramenko and Brown, 2008]. Let C be the fundamental chamber (i.e. $C = 1$) and $D := \hat{w}w_JC$ for some $w_J \in W_J$. Then $E = uC$, $D \geq A$ and by Proposition 3.103 [Abramenko and Brown, 2008], $AE = \hat{w}C$. The gate property gives us,

$$\begin{aligned} d(E, D) &= d(E, AE) + d(AE, D) \\ d(uC, \hat{w}w_JC) &= d(uC, \hat{w}C) + d(\hat{w}C, \hat{w}w_JC) \\ l(\delta(u, \hat{w}w_J)) &= l(\delta(u, \hat{w})) + l(\delta(\hat{w}, \hat{w}w_J)) \\ l(u^{-1}\hat{w}w_J) &= l(u^{-1}\hat{w}) + l(w_J), \end{aligned}$$

which is exactly equation (2). To get equation (1) simply let $w_J = s \in J$. To see that $\hat{w}$ is characterized by (1) observe first that (2) tells us that if $w_J \neq 1$ and we have reduced decompositions $u^{-1}\hat{w} = s_1 \dots s_d$ and $w_J = t_1 \dots t_r$ where $t_i \in J$, then $u^{-1}\hat{w}w_J = s_1 \dots s_d t_1 \dots t_r$ is a reduced decomposition. Then $l(u^{-1}\hat{w}w_J t_r) < l(u^{-1}\hat{w}w_J)$ which means that $\hat{w}$ is the unique representative of wW_J satisfying (1). ■

With this case settled we move on to the more general case of two simplices $A, B \in \Sigma$, where $A = wW_J$ and $B = uW_K$, (the previous case being the special case when $K = \emptyset$). Once again we want to find AB represented as a standard parabolic coset of W (the Weyl group of the Coxeter complex Σ) where the main ingredients we are given are

the representations of A and B as parabolic cosets. In other words, we want to find $AB = vW_L$ where dependent on $A = wW_J$ and $B = uW_K$.

Finding what L is dependent on J, K is given to us by Theorem 3.108 [Abramenko and Brown, 2008] which we state below for completeness.

Theorem 1. Let A be a simplex of cotype J , let B be a simplex of cotype K , and let $r := \delta(A, B)$. Then $\mathcal{C}_{A,B}$ is a residue of type $L := J \cap rKr^{-1}$. In other words there is a simplex AB of cotype L such that $\mathcal{C}_{A,B} = \mathcal{C}_{AB}$, i.e. the chambers that can start a minimal gallery from A to B are precisely those having AB as a face.

Finding what v is dependent on u, w is not as simple because it is unclear what may be a relevant way to characterize a representative v of vW_L . But another way of seeing the problem is to try to get a hold of the elements of vW_L (which in turn are all the possible representatives of the coset AB) by studying $A = wW_J$ and $B = uW_K$. The following proposition provides an answer in this direction.

Proposition 4. Let $A = wW_J$, $B = uW_K$, and $r = \delta(A, B)$ then

$$AB := vW_L = wW_J \cap uW_K r^{-1}.$$

Proof. If $x \in wW_J \cap uW_K r^{-1}$ then $x \in wW_J$ which means that the chamber $C := x$ has A as a face. Moreover, we have that $x \in uW_K r^{-1}$ i.e. there exists $t \in uW_K$ such that $x = tr^{-1}$ where $D := t$ has B as a face. Then given $r = \delta(A, B)$ we get

$$\delta(x, t) = \delta(A, B).$$

Which means there exists a minimal gallery from C to D such that its length is

$$= l(\delta(x, t)) = l(\delta(A, B)) = d(A, B).$$

This implies that $x \in vW_L$ [see Theorem 1]. Now assume $x \in vW_L$, then there exists $t \in uW_K$ such that $\delta(x, t) = \delta(A, B)$ because by the characterization of AB given in Theorem 1, the chamber $C = x$ can start a minimal gallery from A to B . Then we get

$$\begin{aligned} x^{-1}t &= \delta(A, B) \\ \Rightarrow x &= tr^{-1} \\ \Rightarrow x &\in uW_K r^{-1}. \end{aligned}$$

And given $x \in vW_L \subset wW_J$ we get the desired result. ■

3. THE GENERALIZED QUESTION: THE PRODUCT IN A BUILDING

Once again we introduce some reminders on the relevant background needed for this section.

3.1. Background: Buildings. All of what appears in this subsection is taken from chapters 4 and 6 [Abramenko and Brown, 2008].

Definition 7 (Building). A *building* is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ (called *apartments*) satisfying the following axioms:

- (B0) Each apartment Σ is a Coxeter complex.
- (B1) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.
- (B2) If Σ, Σ' are two apartments containing A and B , then there is an isomorphism $\Sigma \rightarrow \Sigma'$ fixing A and B pointwise (where a map fixes a simplex pointwise if it fixes every vertex of the simplex).

Remark 3 (Thickness). Note that buildings are chamber complexes and as such there is a well-defined notion of chamber and panel. Thus, we say that the building is *thick* if every panel is adjacent to at least three chambers.

As in the case of an apartment (Coxeter complex) Σ , a building Δ is also colorable as the following proposition testifies.

Proposition 5 (Proposition 4.6 in [Abramenko and Brown, 2008]). A building Δ is colorable. Moreover, the isomorphism $\Sigma \rightarrow \Sigma'$ in axiom (B2) can be taken to be type-preserving.

Remark 4. In fact, the type function which colors Δ can be chosen to have values in S where the apartments of Δ are $\cong \Sigma(W, S)$ in other words where the apartments are of type (W, S) (see the discussion after Proposition 4.6 [Abramenko and Brown, 2008]). Moreover, this type function coincides with the type function in each apartment as defined in **Definition 5** (see Corollary 4.8 in [Abramenko and Brown, 2008]).

In this thesis we are concerned with understanding the product algebraically, so to say, and what gives an excellent venue for this vague statement to be meaningful is having a certain group G acting on our building Δ . The way in which the action behaves is crucial to our understanding of the building from the algebraic viewpoint and so we explore two such types of actions: strongly transitive and Weyl transitive. We assume throughout that G acts on Δ preserving types and that Δ has a system of apartments $\mathcal{A}$ which is G -invariant.

Definition 8 (Strong transitivity). We say that a G -action is strongly transitive with respect to $\mathcal{A}$ if G acts transitively on the set of pairs (Σ, C) consisting of an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$.

To understand Weyl transitivity we need to note that the Weyl distance function defined before for Coxeter complexes can be extended from these to the whole building in a coherent way. See section 4.8 [Abramenko and Brown, 2008]. Assume, moreover, that G acts as a set of isometries on Δ , i.e.

$$\delta(gC, gD) = \delta(C, D)$$

for all chambers C, D of Δ and all $g \in G$. (Recall that δ is the Weyl distance function on the building Δ).

Definition 9 (Weyl Transitivity definition 6.10 in [Abramenko and Brown, 2008]). We say that the action of G on Δ is Weyl transitive if for each $w \in W$, the action is transitive on the set of ordered pairs (C, D) of chambers with $\delta(C, D) = w$.

Moreover, we get that a building Δ is such that we can choose a *fundamental pair* (Σ, C) of an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$.

Finally we collect some results and definitions that will be useful in what follows.

Proposition 6 (Corollary 6.12 [Abramenko and Brown, 2008]). Strong transitivity (with respect to some apartment system) implies Weyl transitivity.

Definition 10 (Standard parabolic coset/subgroup of G). We will call the subgroups of the form P_J standard parabolic subgroups, where P_J is the stabilizer in G of the cotype J face of the fundamental chamber of Δ . And we will call the left cosets gP_J standard parabolic cosets.

Proposition 7 (Corollary 6.29 in [Abramenko and Brown, 2008]). The building Δ with a Weyl transitive action of G is isomorphic, as a poset, to the set of standard parabolic cosets ordered by reverse inclusion.

Definition 11 ($\Delta(G, B)$). We denote by $\Delta(G, B)$ the poset of standard parabolic cosets ordered by reverse inclusion, where B is the stabilizer of the fundamental chamber.

3.2. The Product in a Building. With this quick review of the background and having settled the question we outlined in the beginning of this thesis, namely, to describe the product of two simplices (of two faces) of a Coxeter complex Σ in a purely algebraic fashion, we can ask the equivalent question for a building Δ . Then the question vaguely

resembles the following: If Δ is a building and A, D are two of its faces; how can we understand the product AD ‘algebraically’?

We proceed by making this question precise. The first thing we will do is to give a framework in which it is possible to understand the product of two faces *algebraically*. Recall first that if Δ is a building then the product of two faces A, D can be computed, unambiguously, by choosing an apartment in which these two faces live and then taking the product in it. In this case then we are satisfied by the description of the product because given any two faces it is, in principle, possible to deduce what their product must be. However, if Δ is a building where there is a group G acting on it in a ‘sufficiently nice’ way then the faces of the building may be represented as certain subsets of G . In which case we may ask: how do these sets interact with the product operation inside of the building?

This is exactly what we want because then we have that the question of understanding the product algebraically acquires meaning. We will consider a group G acting strongly transitively on a thick building Δ , and then we will relax the conditions on these assumptions as we settle the more restrictive cases. Observe that even though we are starting with the narrowest case, the one in which the most conditions apply, we are also dealing with the most important one: the one most widely dealt with in the literature. The structure of a thick building is very rich and when given a strongly transitive action on it, we only enrich this structure further.

With this, we can phrase a tractable question. Let Δ be a thick building of type (W, S) , where G acts strongly transitively. Then we know that $\Delta \cong \Delta(G, B)$ (because of **Proposition 6**) where B is the stabilizer of the fundamental chamber (i.e. of some chosen chamber). Then we have that each face of Δ can be thought of having a ‘representation’, so to say, in $\Delta(G, B)$. In other words, $A \in \Delta$ is ‘represented’ by the parabolic coset in $\Delta(G, B)$ to which it maps under the canonical isomorphism $\Delta \xrightarrow{\sim} \Delta(G, B)$. Then, we can ask: If $A, D \in \Delta$ are represented by gP_J, hP_K respectively, where $g, h \in G$ and $J, K \subset S$, what is the representation of the product AD ? Or more simply, what is the product of gP_J and hP_K in $\Delta(G, B)$?

3.3. The Strongly Transitive Case. Recall that in the strongly transitive case where the building is thick we get that there exists a group N which stabilizes the fundamental apartment and so that $W = N/T$ where $T := B \cap N$. By Proposition 6.52 [Abramenko and Brown, 2008] and its proof we get that we can identify the fundamental apartment Σ of $\Delta(G, B)$ with the apartment whose chambers are $\{wB \mid w \in W\}$

where the apartment system is $G\Sigma$ and where we are thinking of wB as nB for any $n \in w \in N/T$. Thus we get that there is a natural way of thinking about the product in terms of parabolic cosets in Σ as a product in $\Sigma(W, S)$. Indeed, there is a natural (poset) isomorphism between Σ and $\Sigma(W, S)$ that preserves the product, namely

$$\begin{aligned} \phi : \Sigma &\rightarrow \Sigma(W, S) \\ [wP_J &\mapsto wW_J]. \end{aligned}$$

To see that ϕ is in fact a semigroup isomorphism observe that the only thing that ϕ changes is the labelling of the elements from standard parabolic subgroups of G , like P_J , to standard parabolic subgroups of W , like W_J . In other words, the product in Σ and in $\Sigma(W, S)$ behaves in the exact same way. Therefore, ϕ is also a semigroup isomorphism, i.e.

$$\phi(AD) = \phi(wP_JvP_K) = \phi(A)\phi(D) = wW_JvW_K.$$

This provides us with the first tool we will need to explore the product in $\Delta(G, B)$.

We will also need to explore the remaining apartments in our apartment system $G\Sigma = \{g\Sigma \mid g \in G\}$ (this is an apartment system by Lemma 6.13 [Abramenko and Brown, 2008]). Define

$$\begin{aligned} m_g : \Sigma &\rightarrow g\Sigma \\ [A &\mapsto gA] \end{aligned}$$

to be the map ‘left multiplication by g ’. Then we know given our action is strongly transitive that m_g is a type preserving isomorphism from Σ to $g\Sigma$ by Proposition 6.6 [Abramenko and Brown, 2008].

Lemma 1. The map m_g is a semigroup isomorphism.

Proof. This reduces to asking whether the map m_g preserves the product. The lemma then follows by observing that it preserves the defining terms of the product. Indeed, the product in an apartment (i.e. a Coxeter complex) is defined as an operation on the sign sequences of the simplices in question (see section 3.6 [Abramenko and Brown, 2008] in particular Proposition 3.103 and Theorem 3.108). And the sign sequences are in turn determined by the walls of the complex which, as we know from the general theory of Coxeter complexes, are uniquely determined by panels. Finally, panels are uniquely determined by what chambers they are a face of and what is their cotype. Given m_g preserves the latter being a type-preserving isomorphism we get that it must preserve the product. ■

In other words, $gAD = m_g(AD) = m_g(A)m_g(D) = gAgD$, for all $A, D \in \Sigma$. This means that when thinking about products in the apartment $g\Sigma \subset \Delta(G, B)$ we can ‘take g out’. More precisely, if we are given the product $(gwP_J)(gw'P_K)$ in $g\Sigma$ then we get that

$$(gwP_J)(gw'P_K) = g(wP_J)(w'P_K)$$

where we know how to deal with the product $(wP_J)(w'P_K)$ because it happens in Σ — our fundamental apartment. Indeed, $(wP_J)(w'P_K) = (wW_J)(w'W_K)$ which is a product we know how to deal with by what we have done previously. In other words, if we know an apartment that contains both of A, D as given, then we know how to answer the question of determining the product algebraically. Which is something we should have expected by what we did before (the product inside an apartment is something we understand well).

With this we can reduce our problem to the following: if we have $A, D \in \Delta \cong \Delta(G, B)$ what is an element $g \in G$ such that $g\Sigma \ni A, D$? (In fact, we may want to think about the set of all such elements, call it $H \subseteq G$. What is $H := \{g \in G \mid g\Sigma \ni A, D\}$? Note that the set H is not empty by the building axioms so our search cannot be in vain.)

To answer this question we attack a simplified problem: Again we are in a thick building Δ where a group G acts strongly transitively, i.e. $\Delta \cong \Delta(G, B)$, where B is the stabilizer of the fundamental chamber C and N is the stabilizer of the fundamental apartment Σ . In other words, B, N form a BN -pair. We want to find the product AD where $A \leq C$ and D is an arbitrary face of Δ .

We have as before that Σ is the apartment whose chambers are wB for $w \in W$. Here we will usually confound the simplex $A \in \Delta$ with its representation in $\Delta(G, B)$, saying for instance, $A = gP_J$ when really we mean to say that gP_J is the image of A under the canonical isomorphism $\Delta \xrightarrow{\sim} \Delta(G, B)$. Thus let $A = P_J$ (because $A \leq C := B$) and $D = hP_K$ where $h \in G$ and $K, J \subseteq S$. By the strong transitivity of the action of G on Δ we get that $G = BNB$ by Proposition 6.3 [Abramenko and Brown, 2008]. Thus we have that we can write $h = bnb'$ for $b, b' \in B$ and $n \in N$. Then we get that $D = bnP_J$ and thus that $D \in b\Sigma$. This is because there exists $w \in W := N/T$ such that $w = nT$ (the coset whose representative is n), and so $bnP_J = bwP_J$ which is an element of Σ acted on by b and thus must be an element of $b\Sigma$. (That this is possible should not be a surprise in view of Proposition 6.11 [Abramenko and Brown, 2008]). Observe that $h\Sigma$ and $b\Sigma$ need not be the same apartment. Notice, moreover, that given B is the stabilizer of C , then it is also *in* the stabilizer of A (indeed the stabilizer of A is $P_J \geq B$). Therefore, we get that the element $b \in B$

stabilizes A which means that $A \in b\Sigma$. In other words, we have found an element $b \in G$ such that $A, D \in b\Sigma$, which was exactly what we set out to find. We then get that,

$$\begin{aligned} AD &= (P_J)(bnP_K) \\ &= b(P_J)(nP_K) \end{aligned} \tag{3}$$

$$= b(W_J)(wW_K) \tag{4}$$

$$= b(\hat{w}W_L), \quad L := J \cap rKr^{-1} \tag{5}$$

$$= b\hat{n}P_L.$$

Where (3) follows from the fact that m_b is a semigroup isomorphism by Lemma 1. Equation (4) is a convenience in which we are once more identifying Σ with $\Sigma(W, S)$ under ϕ . In other words, the equality means ‘consider the product in $\Sigma(W, S)$ and then translate back’. And equation (5) follows from Theorem 1, where we get the L , and Proposition 4, from where we get the $\hat{w}$. Here we also have $\hat{n}T := \hat{w}$, in other words, our solution involves a choice of representative of $\hat{w}$.

With this case settled we can move to the general case. Namely, finding an element $g^* \in G$ such that $g^*\Sigma$ contains both A, D where $A = gP_J$ and $D = hP_K$. To solve this problem we will set it up so that we can change perspectives and end up reducing it to the special case just solved.

The first thing we must notice is that $C := gB$ is a chamber having A as a face, whose stabilizer is gBg^{-1} . The idea is then to consider $B' := gBg^{-1}$ as the ‘new’ B and to think of C as the fundamental chamber, where we would then be working over $\Delta(G, B')$ instead of over $\Delta(G, B)$.

If we let C , as defined, be the fundamental chamber and B' its stabilizer, then $A = P'_J = gP_Jg^{-1}$. In other words, the representation of A in $\Delta(G, B')$ is $P'_J = gP_Jg^{-1} \geq B'$. What is D in the new perspective, i.e. what is the representation of D in $\Delta(G, B')$? Notice first that if we are considering the building $\Delta(G, B')$ then the fundamental apartment becomes $\Sigma' = g\Sigma$, which means that the stabilizer of the fundamental apartment is $N' = gNg^{-1}$.

Lemma 2. If $D \in \Delta$ is so that its representation in $\Delta(G, B)$ is hP_K then its representation in $\Delta(G, B')$, where $B' = gBg^{-1}$, is $hg^{-1}P'_K$ where $P'_K = gP_Kg^{-1}$.

Proof. Let $D = hP_K$ then there is chamber $E \in \Delta$ whose representation in $\Delta(G, B)$ is hB . In other words, D is a face of E . Now observe that given $C = gB$ then $E = hg^{-1}C$. Therefore, given D is the unique cotype- K face of E we get that $D = hg^{-1}P'_K$. ■

Now the stage is set. We have $\Delta \cong \Delta(G, B')$ where $C = B'$ is the fundamental chamber, $P'_J = A \leq C$ and $D = hg^{-1}P'_K$ and we want to find AD . Let $hg^{-1} = bnb' \in B'N'B' = G$, then this is the case we have already solved so we can apply the same method:

$$\begin{aligned}
 AD &= (P'_J)(bnP'_K) \\
 &= b(P'_J)(nP'_K) \\
 &= b(W_J)(wW_K) \\
 &= b(\hat{w}W_L), \quad L := J \cap rKr^{-1} \\
 &= b\hat{n}P'_L.
 \end{aligned}$$

Here we get once more that L is given by Theorem 1 and $\hat{w}$ is given by Proposition 4. Moreover, $\hat{n}T = \hat{w}$, i.e. $\hat{n}$ is a representative of $\hat{w}$. Now what we miss is to translate back to our original perspective, in other words, to figure out what $b\hat{n}P'_L$ is in $\Delta(G, B)$. This poses no difficulties. Observe that $AD = b\hat{n}P'_L$ is a face of a chamber $F \in \Delta$ represented by $b\hat{n}B'$ in $\Delta(G, B')$, i.e. $F = b\hat{n}B'$. This means that $F = b\hat{n}C$ because $C = B'$, thus in $\Delta(G, B)$ we get that $F = b\hat{n}gB$ (because $C = gB$ in $\Delta(G, B)$) which then means that $AD = b\hat{n}gP_L$ because AD is the unique cotype- L face of F . This settles the general case.

There is one final thing we need to settle before we can move on, and that is that we never justified that B', N' form a BN -pair, and so we do not know that $\Delta \cong \Delta(G, B')$, in fact we do not even know that the claim makes any sense at all. This is not an issue as, morally, we could have chosen any chamber of Δ to be our fundamental chamber and any apartment containing this chamber as the fundamental apartment. Then the stabilizers of these two in G would have to give us a BN -pair and this is exactly what we have done with B', N' . Indeed, B' is the stabilizer of chamber C which is represented by gB in $\Delta(G, B)$ and N' is the stabilizer of $g\Sigma \in G\Sigma$. Therefore, by Theorem 6.56 (2) [Abramenko and Brown, 2008] we get that B', N' is a BN -pair. We have thus proven the following.

Proposition 8. Let B, N be a BN -pair, and $g \in G$. Then $B' := gBg^{-1}$ and $N' := gNg^{-1}$ also form a BN -pair. ■

3.4. The Weyl Transitive Case. In the Weyl transitive case we are not so lucky. The main difficulty is that in this case there is no way in which to get a canonical representation of the fundamental apartment out of the algebra. In other words, we do not have something like we did in the strongly transitive case where the fundamental apartment Σ was specified by having chambers of the form $\{wB \mid w \in W\}$ where

B is the stabilizer of the fundamental chamber and W is the Weyl group specifying the type of the building. Yet another way of putting it is that we do not know how the fundamental apartment looks like in $\Delta(G, B)$. This means that in the Weyl transitive case the proof we gave for the strongly transitive case breaks down because we have no way of reducing the problem to the product inside of an apartment.

However, if we are given an isomorphism $\phi : \Sigma \rightarrow \Sigma(W, S)$ from the fundamental apartment Σ to $\Sigma(W, S)$ (the Coxeter complex of the Weyl group W) then the previous proof goes through. This should not come as a surprise because we then have a way of multiplying faces within the fundamental apartment and given $G\Sigma$ is an apartment system then we can deduce how the product behaves in all apartments. The remainder of this section is devoted to stating this argument precisely, but before we undertake this, note first that having an isomorphism ϕ , although an exogenous element, does not represent an unrealistic scenario. If we are studying a building it is likely we know how one of the apartments of the building looks like and if this is the case then we know how the product works within this apartment. This is what the isomorphism ϕ represents, namely, knowing how to multiply in the fundamental apartment (in other words, in any apartment).

Having said this we undertake to give a method of finding the representation of a product AD in $\Delta(G, B)$ where we know the representations of A, D and where the building Δ we are dealing with is thick, of type (W, S) , and has an action of a group G on it which is type preserving and Weyl transitive. Moreover, we have that C is the fundamental chamber and Σ is the fundamental apartment where we have an isomorphism $\phi : \Sigma \rightarrow \Sigma(W, S)$.

Recall that a building Δ with an action by a group G which is Weyl transitive can be represented by (is canonically isomorphic to) $\Delta(G, B)$ where B is the stabilizer of C . Another thing to note is that if Σ is the fundamental apartment then $G\Sigma$ is an apartment system by Lemma 6.13 [Abramenko and Brown, 2008]. Let $\psi : \Delta \xrightarrow{\sim} \Delta(G, B)$ be the canonical isomorphism discussed above. We get,

$$\xi := \phi \circ \psi^{-1}|_{\psi(\Sigma)} : \psi(\Sigma) \rightarrow \Sigma(W, S)$$

is an isomorphism between the representation (the image) of the fundamental apartment Σ in $\Delta(G, B)$, in other words of $\psi(\Sigma)$, and $\Sigma(W, S)$. This means that we know how the product behaves in $\psi(\Sigma)$. We will once again be careless and will write $\Sigma = \psi(\Sigma)$ saying that we are considering the representation of Σ in $\Delta(G, B)$.

Observe next that Lemma 1 holds for the Weyl transitive case because the only things we use in the proof is that the action of G is type

preserving and that the system of apartments is $G\Sigma$, two facts we also have in this case. Therefore, for all $g \in G$ we get that $m_g : \Sigma \rightarrow g\Sigma$ is a semigroup isomorphism besides being a type preserving poset isomorphism. In other words, we get as before that when taking the product of two faces of Δ ‘we can take g out’, i.e. that if $A = gP_J$ and $D = gP_K$ then $AD = (gP_J)(gP_K) = g(P_J P_K)$ where the product occurs in $g\Sigma$.

This once again gets us to the place (as in the proof of the strongly transitive case) where our goal is to find an element $g^* \in G$ such that given $A = gP_J$ and $D = hP_K$ we get that $A, D \in g^*\Sigma$. We turn now to answering this question.

Let $A = gP_J \in g\Sigma$ and let $D = hP_K$. Then we want to once more change perspectives and say that A is a face of the fundamental chamber, call it C . By this we mean to say that we are considering $B' := gBg^{-1}$, which is the stabilizer of the chamber $C := gB \geq A$, as our new B . In other words, We are considering a new way of seeing the building Δ not as $\Delta(G, B)$ but as $\Delta(G, B')$. Then we get as before that $A = P'_J := gP_Jg^{-1}$, i.e. that A is represented by P'_J in $\Delta(G, B')$. An old question now arises again: what is the representation of D in $\Delta(G, B')$? But we already know the answer because Lemma 2 also holds in this case. To see this simply glance at the proof given and realize that the only thing we used was that Δ could be represented both as $\Delta(G, B)$ and as $\Delta(G, B')$. Therefore, we know that $D = hg^{-1}P'_K \in \Delta(G, B')$. Observe now that by Proposition 6.11 [Abramenko and Brown, 2008] we know that there exists an element $b' \in B'$ such that $D \in b'\Sigma'$ and thus, given B' is the stabilizer of C (the fundamental chamber in $\Delta(G, B')$) we get that $A \in b'\Sigma'$ as well. So we know that there exists an element g^* such that $A, D \in g^*\Sigma'$ that is also $\in B'$.

Here, however, we no longer have that this element depends directly on the ways we may represent h as a product of elements of important subgroups of G : what in the strongly transitive case were elements of $B'N'B'$. In other words, we have no immediate way to know where to look for these elements b' which will give us the apartment we need in which to perform the product. Nonetheless, there is a method we might want to apply to pose the problem in such a way that may ease finding these elements. It goes as follows. We know that $A \leq C$ and if $E := hg^{-1}C = hg^{-1}B'$ then $D \leq E$. Let $\delta(C, E) = \delta(C, hg^{-1}C) = w$. And define C_w as the unique chamber in Σ' which is at a distance w from C , i.e. such that $\delta(C, C_w) = w$. Such a chamber exists and is unique because if we identify Σ' with $\Sigma(W, S)$ then C_w is the chamber $wC \in \Sigma(W, S)$. Then given we have our isomorphism ξ above we know what its representation is going to be in Σ' , namely we can find an

element $t \in G$ such that $C_w = tC \in \Sigma'$. Indeed,

$$\xi \circ m_{g^{-1}} : \Sigma' \rightarrow \Sigma(W, S)$$

is the isomorphism that would let us do this. Then by Weyl transitivity we know that there exists $b' \in B'$ such that $b'tC = hg^{-1}C$ and given $hg^{-1}C = E \geq D$ then we know that $D \in b'\Sigma'$ and thus that $A \in b'\Sigma$. In other words, we may pose the problem of finding an element $b' \in B'$ such that $A, D \in b'\Sigma$ as finding an element $b' \in B'$ such that $b'tC = hg^{-1}C$. The reasoning being that given we have a good grasp of how the action of G on $\Delta(G, B')$ and given we know what tC and $hg^{-1}C$ are then it should be feasible to find a $b' \in B'$ such that $b'tC = hg^{-1}C$.

With this then we get that $D = hg^{-1}P'_K = b'tP'_K$ and so thus

$$\begin{aligned} AD &= (P'_J)(b'tP'_K) \\ &= b'(P'_J t P'_K) \end{aligned}$$

where $P'_J, tP'_K \in \Sigma'$. But this is a product we know how to deal with because we have the isomorphism $\xi \circ m_{g^{-1}}$, i.e. with it we can think of this product in $\Sigma(W, S)$ compute it there and then pullback with the inverse of $\xi \circ m_{g^{-1}}$ to Σ' . Finally, we would have to revert to our initial perspective, in other words to the world where we are considering the representation of Δ by $\Delta(G, B)$ instead of by $\Delta(G, B')$. This is because the procedure above has computed things in $\Delta(G, B')$. But the argument we gave at the end of our consideration of the strongly transitive case goes through without change, namely, find a chamber of the final product and consider how that chamber is represented in $\Delta(G, B)$ and $\Delta(G, B')$. These representations will then allow us to revert perspectives.

Here we see that the need of using this exogenous ϕ made the solution of the problem less concrete as we do not know what this black box will look like, contrary to the strongly transitive case where the equivalent of isomorphism ϕ was given to us canonically. It is worth noting, however, that in specific cases when we do know what this map ϕ is, the solution is no less powerful.

REFERENCES

- [Abramenko and Brown, 2008] Abramenko, P. and Brown, K. S. (2008). *Buildings*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York. Theory and applications.