Formal (non)-commutative symplectic geometry

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Some time ago B. Feigin, V. Retakh and I had tried to understand a remark of J. Stasheff [15] on open string theory and higher associative algebras [16]. Then I found a strange construction of cohomology classes of mapping class groups using as initial data any differential graded algebra with finite-dimensional cohomology and a kind of Poincaré duality.

Later generalizations to the commutative and Lie cases appeared. In attempts to formulate all this I have developed a kind of (non)-commutative calculus. The commutative version has fruitful applications in topology of smooth manifolds in dimensions ≥ 3. The beginnings of applications are perturbative Chern-Simons theory (S. Axelrod and I.M. Singer [1] and myself), V. Vassiliev's theory of knot invariants and discriminants (see [19], new results in [2]) and V. Drinfeld's works on quasi-Hopf algebras (see [6]), also containing elements of Lie calculus.

Here I present the formal aspects of the story. Theorem 1.1 is the main motivation for my interest in non-commutative symplectic geometry. Towards the end the exposition becomes a bit more vague and informal. Nevertheless, I hope that I will convince the reader that non-commutative calculus has every right to exist.

I have benefited very much from conversations with B. Feigin, V. Retakh, J. Stasheff, R. Bott, D. Kazhdan, G. Segal, I.M. Gelfand, I. Zakharevich, J. Cuntz, Yu. Manin, V. Ginzburg, M. Kapranov and many others.

1 Three infinite-dimensional Lie algebras

Let us define three Lie algebras. The first one, denoted by ℓ_n, is a certain Lie subalgebra of derivations of the free Lie algebra generated by $2n$ elements $p_1, \ldots, p_n, q_1, \ldots, q_n$.

By definition, ℓ_n consists of the derivations acting trivially on the element $\Sigma[p_i, q_i]$.

The second Lie algebra a_n is defined in the same way for the free associative algebra without unit generated by $p_1, \ldots, p_n, q_1, \ldots, q_n$.

The third Lie algebra c_n is the Lie algebra of polynomials

$$F \in \mathbf{Q}[p_1, \ldots, p_n, q_1, \ldots, q_n]$$

such that $F(0) = F'(0) = 0$, with respect to the usual Poisson bracket

$$\{F, G\} = \sum \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right) .$$

One can define c_n also as the Lie algebra of derivations of a free polynomial algebra $\mathbf{Q}[p_*, q_*]$ preserving the form $\Sigma\, dp_i \wedge dq_i$ and the codimension one ideal $(p_1, \ldots, p_n, q_1, \ldots, q_n)$.

In Section 4 we shall give an interpretation of the algebras ℓ_n, a_n as Poisson algebras in some versions of non-commutative geometry.

Our aim is a computation of the stable homology (with trivial coefficients) of these Lie algebras. The spirit of the (quite simple) computations is somewhere between Gelfand-Fuks computations (see [8] and [7]) and cyclic homology.

It is well known that all classical series of locally-transitive infinite-dimensional Lie algebras (formal vector fields, hamiltonian fields, contact fields, ...) have trivial or uninteresting stable (co)homology (see [11]). Our algebra h_n is a subalgebra of the algebra of hamiltonian vector fields, consisting of the vector fields preserving a point. Applying the Shapiro lemma one can relate its cohomology with the cohomology of the algebra of all polynomial (or formal) hamiltonian vector fields with coefficients in the adjoint representation. We want to mention here the recent work of I.M. Gelfand and O. Mathieu (see [9]) where some nonstable classes for the Lie algebra of all formal hamiltonian vector fields were constructed using cylic homology and non-commutative deformations.

If we denote by h_n one of these three series of algebras, then we have a sequence of natural embeddings $h_1 \subset h_2 \subset \cdots \subset h_\infty$ where the last algebra corresponds to the case of a countable infinite number of generators. Of course, $H_*(h_\infty) = \varinjlim H_*(h_n)$.

Let us denote by $\hat{h}_n$ the completion of h_n with respect to the natural grading on it. Then the continuous cohomology of $\hat{h}_n$ is in a sense dual to $H_*(h_n)$. More precisely, the grading on h_n induces a grading on its homology, $H_k(h_n) = \bigoplus_i H_k^{(i)}(h_n)$.

$$H^k_{\mathrm{cont}}(\hat{h}_n) = \bigoplus_i (H_k^{(i)}(h_n))^* .$$

For the limit algebras h_∞ we have a structure of Hopf algebra on its homology (as is usual in K-theory). The multiplication comes from the homomorphism $h_\infty \oplus h_\infty \to h_\infty$ and the comultiplication is dual to the multiplication in cohomology.

This Hopf algebra is commutative and cocommutative. Thus $H_*(h_\infty)$ is a free polynomial algebra (in the $\mathbf{Z}/2\mathbf{Z}$-graded sense) generated by the subspace $PH_*(h_\infty)$ of primitive elements.

In all three cases we have an evident subalgebra $sp(2n) \subset h_n$ consisting of linear derivations. The primitive homology of $sp(2\infty)$ is well-known:

$$PH_k(sp(2\infty), \mathbf{Q}) = \left\{ \begin{matrix} \mathbf{Q}, & k = 3 (\operatorname{mod} 4) \\ 0, & k \neq 3 (\operatorname{mod} 4) \end{matrix} \right\}$$

Now we can state our main result:

Theorem 1.1. *$PH_k(h_\infty)$ is equal to the direct sum of $PH_k(sp(\infty))$ for all three cases and*

(1) *(for the case ℓ_∞)*

$$\bigoplus_{n\geq 2} H^{2n-2-k}(\mathit{OutFree}(n), \mathbf{Q}),$$

where $\mathit{OutFree}(n)$ denotes the group of outer automorphisms of a free group with n generators,

(2) *(for the case a_∞)*

$$\bigoplus_{m>0, 2-2g-m<0} H^{4g-4+2m-k}(\mathcal{M}_{g,m}/\Sigma_m, \mathbf{Q}),$$

where $\mathcal{M}_{g,m}/\Sigma_m$ denotes the (coarse) moduli space of smooth complex algebraic curves of genus g with m punctures, (the quotient space modulo the action of the symmetric group is equal to the moduli space of curves with unlabeled punctures),

(3) *(for the case c_∞)*

$$\bigoplus_{n\geq 2} (\mathit{Graph\ homology})_k^{(n)}$$

(see the definition below).

The grading on the homology groups arising from the natural grading on h_∞ is equal to $(2n-2)$, $(4g-4+2m)$ and $(2n-2)$, respectively.

2 Hamiltonian vector fields in the ordinary sense

Before starting the proof of the third case of Theorem 1.1, we define the graph complex. By a *graph* we mean a finite 1-dimensional CW-complex. Let us call an *orientation* of the graph Γ a choice of orientation of the real vector space $\mathbf{R}^{\{\text{edges of }\Gamma\}} \oplus H^1(\Gamma, \mathbf{R})$. For $n \geq 2$, $k \geq 1$ denote by $G_k^{(n)}$ the vector space over $\mathbf{Q}$ generated by the equivalence classes of pairs (Γ, or) where Γ is a connected nonempty graph with Euler characteristic $1-n$ and k vertices, such that degrees of all vertices are greater than or equal to 3 and (or) is an orientation of Γ. We impose the relation $(\Gamma, -\text{or}) = -(\Gamma, \text{or})$.

It follows that $(\Gamma, \text{or}) = 0$ for every graph Γ containing a simple loop (i.e., an edge attached by both ends to one vertex). The reason is that such graphs have automorphisms reversing orientation in our sense.

It is easy to see that $G_k^{(n)}$ is finite-dimensional for all k, n.

Define a differential on the vector space $G_*^{(n)} = \oplus_k G_k^{(n)}$ by the formula (for Γ without simple loops):

$$d(\Gamma, \mathrm{or}) = \sum_{e \in \{\text{edges of } \Gamma\}} (\Gamma/e, \text{induced orientation})\,.$$

Here Γ/e denotes the result of the contraction of the edge e, the "induced orientation" is the product of the natural orientation on the codimension-1 co-oriented subspace $\mathbf{R}^{\{\text{edges of } \Gamma\}/e} \subset \mathbf{R}^{\{\text{edges of } \Gamma\}}$ and the orientation on $H^1(\Gamma/e, \mathbf{R}) \simeq H^1(\Gamma, \mathbf{R})$.

One can easily check that $d^2 = 0$. Hence we have an infinite sequence of finite-dimensional complexes $G_*^{(n)}$. Define the graph complex without specification as the direct sum of the complexes $G_*^{(n)}$.

The homology groups of graph complexes have important topological applications. In a sense they are universal characteristic classes for diffeomorphism groups of manifolds in odd dimensions ≥ 3. The idea of this relation cornes from perturbative Chern-Simons theory. We shall describe this somewhere later.

Proof of the third case of Theorem 1.1. Recall that our Lie algebras c_n are $\mathbf{Z}_{\geq 0}$-graded. Thus the standard chain complex $\bigwedge^*(c_n)$ is graded. We consider the case when n is much larger than the grading degree.

It is well known that every Lie algebra acts (through the adjoint representation) trivially on its homology. The algebra $sp(2n) \subset c_n$ acts reductively on $\bigwedge^*(c_n)$. Hence the chain complex is canonically quasi-isomorphic to the subcomplex of $sp(2n)$-invariants.

The underlying vector space of the Lie algebra c_n as a representation of $sp(2n)$ is equal to

$$\bigoplus_{j \geq 2} S^j(V)\,,$$

where $V = \mathbf{Q}\langle p_1, \ldots, p_n, q_1, \ldots, q_n \rangle$ is the defining $2n$-dimensional representation of $sp(2n)$.

Thus our chain complex as a representation of $sp(2n)$ is equal to the sum

$$\bigoplus_{k_2 \geq 0, k_3 \geq 0, \ldots} (\wedge^{k_2}(S^2(V) \otimes \wedge^{k_3}(S^3(V) \otimes \ldots)\,.$$

Every summand is a space of tensors on V satisfying some symmetry conditions.

We can construct $(N-1)!! = 1 \cdot 3 \cdot \ldots \cdot (N-1)$ invariant elements in $V^{\otimes N}$ for every even N. Namely, each decomposition of the finite set $\{1, \ldots, N\}$ into pairs $(i_1, j_1), \ldots, (i_{N/2}, j_{N/2})$ where $i_1 < j_1, \ldots, i_{N/2} < j_{N/2}$; $i_1 < \cdots < i_{N/2}$ gives the tensor $\omega_{i_1 j_1} \ldots \omega_{i_{N/2} j_{N/2}}$, where ω_{ij} denotes the tensor of the standard skew-symmetric product on V^*.

By the Main Theorem of Invariant Theory these tensors will form a base of the space $(V^{\otimes N})^{sp(2n)}$ and there are no nonzero invariants for odd N if $2n = \dim(V)$ is sufficiently large.

Let us consider the space $\wedge^{k_2}(S^2(V)\otimes\wedge^{k_3}(S^3(V)\otimes\ldots$ as a quotient space of $(V^{\otimes 2})^{\otimes k_2}\otimes(V^{\otimes 3})^{\otimes k_3}\otimes\ldots$ Every pairing on the set $\{1,2,\ldots,2k_2+3k_3+\ldots\}$ gives a graph with labeled vertices and edges in the following way: we consider $\{1,2,\ldots,k_2+k_3+\ldots\}$ as a set of vertices, and the set of pairs as a set of edges. One can also provide in a canonical way these graphs with orientations. The passing to the quotient spaces modulo the action of symmetric groups corresponds to the consideration of graphs *without* labelings.

It is easy to see that we obtain a vector space analogous to our graph complex with two differences: 1) we consider now graphs not necessarily empty or connected, 2) vertices have degrees greater than or equal to 2. The differential in the new complex can be described in the same way as for the graph complex. The homology of the new complex is equal to the homology of c_∞.

One can check that the multiplication in the stable homology can be identified with the operation of disjoint union of graphs. Thus the primitive part arises from the subcomplex corresponding to the nonempty connected graphs.

There is a direct summand subcomplex of the last complex, consisting of "polygons", i.e. connected graphs with degrees of all vertices equal 2. It is easy to see that for $k\neq 3(\mathrm{mod}\,4)$ there exists an automorphism of k-gon reversing the orientation. Hence we obtain the trivial $sp(2\infty)$-part of primitive stable homology.

Let us consider now the subcomplex consisting of connected nonempty graphs containing at least one vertex of degree ≥ 3. We can associate with such a graph a new graph with degrees of all vertices greater than or equal to 3. The new graph is just the old graph with removed vertices of degree 2.

One can introduce a partial order on the set of equivalence classes of graphs by the possibility of obtaining one graph from another by a sequence of edge contractions.

In such a way we define a certain filtration on the bigger complex by the ordered set of graphs with degrees ≥ 3. For any such graph Γ the corresponding graded subquotient complex is the quotient complex of tensor products over the set of edges of Γ of some standard complexes modulo the action of the finite group $\mathrm{Aut}\,(\Gamma)$.

The standard complex for an edge has dimension 1 in each degree $k\geq 0$, because there exists unique up to an isomorphism way to put k points to the interior of the standard interval (edge). The differential in this standard complex kills all classes in positive degrees. Hence it has only one nontrivial homology in degree 0.

We see that the spectral sequence associated with the filtration by graphs with degrees ≥ 3 collapses at the first term to the graph complex. It is clear that Euler characteristic of a graph is preserved under any edge contraction. Thus the graph complex is the direct sum of its subcomplexes over all possible Euler characteristics. The degree in the sense of the natural grading on $c_{2\infty}$ of a cycle associated with a graph Γ is equal to $-2\chi(\Gamma)$. $\square$

There are a lot of nontrivial classes in the graph complex. For example, any finite-dimensional Lie algebra $\mathfrak{g}$ with fixed nondegenerate invariant scalar product on it defines a sequence of classes of graph homology in positive even degrees.

One can choose an orthogonal base in $\mathfrak{g}$ with respect to the scalar product. The tensor of structure constants will be skew-symmetric 3-tensor. Each 3-valent graph defines up to a sign a way to contract indices in some tensor power of the tensor of structure constants. It is easy to see that we obtain a function $\Phi_{\mathfrak{g}}(\Gamma, \text{or})$ on the set of equivalence classes of 3-valent graphs with orientation.

The immediate consequence of the Jacobi identity is the fact that

$$\sum_{\text{equiv. classes of } (\Gamma,\text{or})} \frac{\Phi_{\mathfrak{g}}(\Gamma, \text{or})}{\# \operatorname{Aut}(\Gamma)} (\Gamma, \text{or})$$

gives closed chains in all $G_*^{(n)}$. Thus one can construct some classes for every simple Lie algebra using the Killing scalar product.

3 Moduli spaces of graphs

It will be useful for us to describe graph homology as a kind of homology of topological spaces.

Denote by $\mathcal{G}^{(n)}$ for $n \geq 2$ the set of equivalence classes of pairs (Γ, metric) where Γ is a nonempty connected graph with Euler characteristic equalling $(1-n)$ and degrees of all vertices greater than or equal to 3, (metric) is a map from the set of edges to the set of positive real numbers $\mathbf{R}_{>0}$. One can introduce a topology on $\mathcal{G}^{(n)}$ using Hausdorff distance between metrized spaces associated in the evident way with pairs (Γ, metric). It is better to consider $\mathcal{G}^{(n)}$ not as an ordinary space, but as an orbispace (i.e. don't forget automorphism groups). Mention here that $\mathcal{G}^{(n)}$ is a non-compact and non-smooth locally polyhedral space. It has a finite stratification by combinatorial types of graphs with strata equal to some qoutient spaces of Euclidean spaces modulo actions of finite groups.

A fundamenatal fact on the topology of $\mathcal{G}^{(n)}$ is the following theorem of M. Culler, K. Vogtmann (see [4]):

Theorem 3.1. *$\mathcal{G}^{(n)}$ is a classifying space of the group OutFree(n) of outer automorphisms of a free group with n generators.*

The virtual cohomological dimension of $\mathcal{G}^{(n)}$ is equal to $2n-3$ and the actual dimension is equal to $3n-3$.

Any representation of the group OutFree(n) gives a local system on $\mathcal{G}^{(n)}$. We can define (co)homology, also homology with closed support and cohomology with compact support of $\mathcal{G}^{(n)}$ with coefficients in any local system.

Let us denote by ϵ the 1-dimensional local system with the fiber over (Γ, metric) equal to $\wedge^n(H^1(\Gamma, \mathbf{Q}))$. A simple check shows that the chain complex computing $H_*^{\text{closed}}(\mathcal{G}^{(n)}, \epsilon)$ arising from the stratification above coincides with the shifted graph complex $G_{*+n-1}^{(n)}$.

Now we have a geometric realization of homology arising in the first and the third cases of Theorem 1.1.

Define a *ribbon graph* (or a *fatgraph* in other terms) as a graph with fixed cyclic orders on the sets of half-edges attached to each vertex. One can associate an oriented surface with boundary to each ribbon graph by replacing edges by thin oriented rectangles (ribbons) and glueing them together at all vertices according to the chosen cyclic order.

Denote by $\mathcal{R}^{(g,m)}$ the moduli space of connected ribbon graphs with metric, such that degrees of all vertices greater than or equal to 3 and the corresponding surface has genus g and m boundary components.

Theorem 3.2. *$\mathcal{R}^{(g,m)}$ is canonically isomorphic as an orbispace to $\mathcal{M}_{g,m} \times \mathbf{R}^m/\Sigma_m$, (and, hence is a classifying space of some mapping class group).*

This theorem follows from results of K. Strebel and/or R. Penner (see [17], [13] or an exposition in [12]).

The space $\mathcal{R}^{(g,m)}$ is a non-compact but smooth orbispace (orbifold), so there is a rational Poincaré duality. We want to mention here that due to the factor $\mathbf{R}^m$ and to the action of the symmetric group the orbifold $\mathcal{R}^{(g,m)}$ is *not* oriented for $m > 1$.

The virtual cohomological dimension of $\mathcal{R}^{(g,m)}$ is equal to $4g-4+m$ for $g \geq 1$ and to $m-3$ for $g=0$ (see [12]), the actual dimension is equal to $6g-6+3m$.

Thus vector spaces arising in Theorem 1.1 could be written as

$$\bigoplus_{n\geq 2} H^{2n-2-k}(\mathcal{G}^{(n)}, \mathbf{Q}), \quad \bigoplus_{m>0, 2-2g-m<0} H^{4g-4+2m-k}(\mathcal{R}^{(g,m)}, \mathbf{Q}), \bigoplus_{n} \geq 2$$

respectively.

The evident forgetful map $\mathcal{R}^{(g,m)} \to \mathcal{G}^{(2g+m-1)}$ is proper. The orientation sheaf on $\mathcal{R}^{(g,m)}$ coincides with the pullback of ϵ under this map. Hence we obtain a sequence of linear maps

$$H^{2n-2-k}(\mathcal{G}^{(n)}, \mathbf{Q}) \to \bigoplus_{g,m:2g+m-1=n} H^{4g-4+2m-k}(\mathcal{R}^{(g,m)}, \mathbf{Q}) \simeq$$

$$\simeq \bigoplus_{g,m:2g+m-1=n} H^{\text{closed}}_{2g+m-1-k}(\mathcal{R}^{(g,m)}, \epsilon) \to H^{\text{closed}}_{k+n-1}(\mathcal{G}^{(n)}, \epsilon)\,.$$

We shall see in Section 5 that the composition map is zero.

4 (Non)-commutative symplectic geometry

We shall describe here a (non)-commutative formalism surprisingly parallel to the usual calculus of differential forms and Poisson brackets. Almost everything will work literally at the same way in three possible worlds: Lie algebras, associative algebras and commutative algebras. Our formalism could be extended to the case of "Koszul dual pairs of quadratic operads" (see [10]) including Poisson algebras and, probably, operator algebras etc.

Let us fix a wor(l)d $\mathcal{A} \in \{\text{Lie, associative, commutative}\}$.

Definitions. *A formal $\mathcal{A}$-supermanifold is a complete free finitely generated $\mathbf{Z}/2\mathbf{Z}$-graded $\mathcal{A}$-algebra (nonunital in the case $\mathcal{A} \in \{Lie, associative\}$).*

A local coordinate system on a manifold is a choice of generators of the corresponding algebra.

A (formal) diffeomorphism between two manifolds is a continuous isomorphism between graded algebras.

A vector field is a continuous derivation.

Submanifold is a free quotient algebra.

The tangent space at zero is dual to the space of generators of algebra (= the quotient space of algebra by the maximal proper ideal).

All definitions above are just general categorical nonsense.

In ordinary calculus we can consider differential forms as functions on the odd tangent bundle to the manifold. We can define this object without difficulties in our situation:

Definition. *For supermanifold X, "the total space of the odd tangent bundle" ΠTX is the free differential envelope of X.*

For example, if X has coordinates $x_1, \ldots, x_n$ then ΠTX has coordinates $x_1, \ldots, x_n, dx_1, \ldots, dx_n$. The algebra corresponding to ΠTX is $\mathbf{Z}_{\geq 0}$-graded by the number of differentials. In other words, there is a canonical action of the multiplicative group scheme $\mathbf{G}_m$ on ΠTX. The presence of differential on the free differential envelope means that there is a canonical action of the odd affine group scheme $\mathbf{G}_a^{0|1}$ on ΠTX.

Now we are coming to the delicate point: what is the notion of function? We propose the following strange definition (where $\cdot$ denotes the operation in A):

Definition. *For $\mathcal{A}$-algebra A the space of 0-forms $F(A)$ is the quotient space*

$$A \otimes A/(\textit{subspace generated by } a \otimes b - b \otimes a \textit{ and } a \otimes (b \cdot c) - (a \cdot b) \otimes c)\,.$$

Of course, in the supercase one has to make appropriate sign corrections.

Functor $F(A)$ coincides with A^2 in the commutative case, with $A^2/[A, A]$ in the associative case (for unital A, $F(A)^*$ is equal to the space of traces on A), and with the functor considered by Drinfeld (see [6]) for the Lie case.

By functoriality we obtain the action of $\mathbf{G}_m$ and $\mathbf{G}_a^{0|1}$ on $F(\Pi TX)$ for any X. In other words, $F(\Pi TX)$ is a $\mathbf{Z}_{\geq 0}$-graded complex. We shall call it the de Rham complex of the manifold X.

Notations. *$F^i(X)$ for $i \geq 0$ is the i-th homogeneous comportent of $F(\Pi TX)$, d is the differential $F^i(X) \to F^{i+1}(X)$.*

It is clear that $F^0(X)$ coincides with the functor F applied to the algebra corresponding to the manifold.

One can define for a vector field ξ on a manifold two vector fields L_ξ, i_ξ on ΠTX by formulas

$$L_\xi(a) = \xi(a)\,,\ L_\xi(da) = d(\xi(a))\,,\ i_\xi(a) = 0\,,\ i_\xi(da) = \xi(a)$$

for every $a \in A$. The following commutator relations hold:

$$L_\xi = i_\xi d + d i_\xi\,,\ i_\xi i_\eta + i_\eta i_\xi = 0\,,\ [L_\xi, i_\eta] = i_{[\xi,\eta]}\,,\ [L_\xi, L_\eta] = L_{[\xi,\eta]}\,.$$

By functoriality we have analogous operations on the de Rham complex.

Using these formulas we can prove easily that for local manifolds the de Rham complex is exact. It follows from the fact that $L_e = [i_e, d]$ is an invertible operator on $F^*(X)$ where e denotes the Euler vector field $x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$ on a manifold with coordinates $x_1, \ldots, x_n$.

The mini-theory developed above works well for many other functors (algebras) $\to$ (vector spaces) instead of $F(A)$. The advantage of our definition is the existence of symplectic theory.

It follows easily from the definitions that any 2-form on a manifold defined a skew-symmetric bilinear form on the tangent space at 0 (through the first coefficient in its Taylor expansion).

Definition. *Symplectic supermanifold is a pair (X, ω) where ω is closed even 2-form on X with nondegenerate restriction to T_0X.*

One can check that for any nondegenerate 2-form ω the operator $\xi \to i_\xi \omega$ is an isomorphism between the space of vector fields and the space of 1-forms. Thus by usual arguments vector fields preserving symplectic structure are in one-to-one correspondence with 0-forms. In fact, the Lie algebra of hamiltonian vector fields depends (up to inner automorphism) only on the dimension of the symplectic manifold.

Theorem 4.1. (Darboux theorem) *A symplectic manifold is isomorphic to the flat manifold, i.e. with*

$$\omega = \sum c_{\alpha\beta} dt_\alpha \otimes dt_\beta\,.$$

We shall not use here this theorem, so the proof will be omitted.

For the case of associative or Lie manifolds there exists a simple description of closed 2-forms:

Theorem 4.2. *For (associative or Lie) free algebra A there exists a canonical isomorphism $F^2_{\text{closed}}(A) \simeq [A, A]$.*

Proof. First of all, we define a map $t : F^1(A) \to [A, A]$ by the formula $t(a \otimes db) = [a, b]$. It is clear that this map is onto and it vanishes on $dF^0(A)$. Thus for the associative case we obtain the short sequence

$$A \to A^2/[A, A] \xrightarrow{d} F^1(A) \xrightarrow{t} [A, A] \to 0$$

exact everywhere, but the middle term. If we choose coordinates then we obtain a grading on all terms of this sequence. Simple dimension count shows that Euler characteristics of all graded components are zero (we know generating function of $F^1(A)$ because there exists an isomorphism $F^1(A) \simeq \{\text{derivations of } A\}$). Thus the sequence above is exact and coincides with the exact sequence

$$0 \to F^0(A) \xrightarrow{d} F^1(A) \xrightarrow{d} F^2_{\text{closed}}(A) \to 0\,.$$

Analogous but more lengthy arguments work for Lie algebras too. (For another approach see [6]). □

5 Sketch of the proof of Theorem 1.1 for the associative and the Lie cases

Now we can combine all facts together.

It follows from Theorem 4.2 that algebras ℓ_n, a_n are algebras of hamiltonian vector fields on flat symplectic manifolds in non-commutative geometries. Thus they are canonically equivalent as vector spaces to 0-forms.

In the associative case the vector space of 0-forms on a flat manifold with the cotangent space V at zero as $GL(V)$-module is equal to

$$\bigoplus_{n\geq 2} (V^{\otimes n})^{\mathbf{Z}/n\mathbf{Z}}$$

where cyclic group $\mathbf{Z}/n\mathbf{Z}$ acts by permutations of factors in $V^{\otimes n}$. The same arguments as in the commutative case lead to the ribbon version of the graph complex. By Theorem 3.2 we obtain at the end *all* cohomology groups of all moduli spaces of complex curves with unlabeled punctures.

In the case of Lie algebras the situation is a bit more complicated. We say (without proof) that the space of 0-forms is now equal to

$$\bigoplus_{n\geq 2} (V^{\otimes n} \otimes L_n)^{\Sigma_n}$$

where L_n is a certain $(n-2)!$-dimensional representation of the symmetric group Σ_n. Again using the same strategy as in Section 2 we obtain the Lie version of the graph complex. As the vector space it will be the direct sum over equivalence classes of graphs of some vector spaces. The vector space associated with graph Γ will be the subspace of $\mathrm{Aut}(\Gamma)$-invariants in the tensor product of natural (degree of vertex -2)!-dimensional vector spaces over all vertices twisted with the 1-dimensional local system ϵ.

On the other hand, we can construct a finite cell-complex homotopy equivalent to $B\mathrm{OutFree}(n)$ passing from the natural stratification of $\mathcal{G}^{(n)}$ to its baricentric subdivision. The corresponding cochain complex carries some filtration by graphs (by the minimal graph corresponding to strata attached to the cell).

Computations show that the spectral sequence associated with this filtration collapses at the second term to the Lie version of graph complex. □

Two evident functors

$$\{\text{commutative algebras}\} \to \{\text{associative algebras}\} \to \{\text{Lie algebras}\}$$

lift to correspondences between 3 types of calculus, in particular, to homomorphisms of Lie algebras

$$c_\infty \to a_\infty \to \ell_\infty \,.$$

The composite map goes through sub- and quotient algebra $sp(2\infty)$, so it is zero on the nontrivial part of primitive homology. One can identify arising maps with geometrie maps from Section 3.

The entire story above has an odd analogue. One has to consider superalgebras and odd symplectic structures. Then the stable homology will be described in the same way but with the twisted by ϵ coefficients.

The odd version of (commutative) graph homology plays the same role for smooth even-dimensional manifolds ($\dim \geq 4$) as the even version for odd dimensions.

6 Poisson brackets: Formulas and interpretations

In all 3 worlds the space of 1-forms on the flat manifold with coordinates $x_1, \ldots, x_n$ can be identified with the direct sum of n copies of the corresponding free algebra A:

$$(a_1, \ldots, a_n) \leftrightarrow \sum a_i \otimes dx_i \,.$$

Thus we can define linear operators $\frac{\partial}{\partial x_i} : F(A) \to A$ by formula $dH = \sum \frac{\partial H}{\partial x_i} \otimes dx_i$.

In the associative case 0-forms are linear combinations of cyclic words (of length ≥ 2) in alphabet $x_1, \ldots, x_n$. For example,

$$\frac{\partial(xxyxz)}{\partial x} = xyxz + yxzx + zxxy \,, \ \frac{\partial(xxyxz)}{\partial y} = xzxx \,,$$

where $xxyxz$ is considered as a cyclic word.

The following basic identity holds in all 3 cases:

$$\sum \left[x_i, \frac{\partial H}{\partial x_i} \right] = 0 \,.$$

It is just nothing in the commutative world. In the associative world one can prove this identity immediately using the description of F^0 above. The Lie case follows from the associative case by embeddings of free Lie algebras into free associative algebras.

There are universal formulas for Poisson brackets:

$$\{G,H\} = \sum \left(\frac{\partial G}{\partial p_i} \otimes \frac{\partial H}{\partial q_i} - \frac{\partial G}{\partial q_i} \otimes \frac{\partial H}{\partial p_i} \right) ,$$

hamiltonian vector field, corresponding to H is $\dot{p}_i = \frac{\partial H}{\partial q_i}$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$. The invariance of $\Sigma[p_i, q_i]$ is equivalent to the identity above.

V. Drinfeld in [6] used another Poisson bracket on $F(A)$ (for the Lie case):

$$\{G,H\} = \sum_{i=1}^{n} x_i \otimes \left[\frac{\partial G}{\partial x_i}, \frac{\partial H}{\partial x_i} \right] ,$$

$$H \mapsto \text{vector field } \dot{x}_i = \left[x_i, \frac{\partial H}{\partial x_i} \right] .$$

Later we shall give an interpretation of this bracket as a Kirillov bracket on the dual space to the "Lie" algebra $\mathbf{C} \oplus \cdots \oplus \mathbf{C}$ (n summands). In the Lie world "Lie" algebras are commutative algebras. As an abstract Lie algebra $F(A)$ with this bracket is a trivial central extension by $\langle x_1 \otimes x_1, \ldots, x_n \otimes x_n \rangle$ of the Lie algebra of derivations D of the free Lie algebra generated by x_i such that

$$\forall\, i\ \exists\, y_i \text{ such that } D(x_i) = [x_i, y_i]\,, \quad D\left(\sum x_i\right) = 0\,.$$

Recall, that

(1) Teichmüller group $T_{g,1}$ is the group of automorphisms of the free group generated by $\{p_1, \ldots, p_g, q_1, \ldots, q_g\}$ preserving the element $\Pi p_i q_i p_i^{-1} q_i^{-1}$,

(2) pure braid group with n strings is the group of automorphisms of the free group generated by $\{x_1, \ldots, x_n\}$ preserving conjugacy classes of x_i and the element $x_1 x_2 \ldots x_n$.

Thus we see that in the Lie world Poisson algebra is an analogue of the Teichmüller group for flat symplectic manifolds, and an analogue of the pure braid group for Kirillov brackets.

Also, if K is a subfield of $\mathbf{C}$ containing all roots of unity and C is a smooth algebraic curve defined over K of genus g with one puncture or of genus 0 with $n+1$ punctures then the Galois group $\mathrm{Gal}\,(\bar{K}/K)$ acts on the ℓ-adic completion of the fundamental group of $C(\mathbf{C})$ through the ℓ-adic pro-nilpotent group with the corresponding Poisson algebra as the Lie algebra (ℓ is an arbitrary prime).

7 How big are stable homologies?

We collect here some attempts to understand the "size" of stable homologies of Poisson algebras. The situation is not clear because different approachs give contradictory hints.

7.1 Explicit constructions of stable classes

There is a generalization of the construction mentioned at the end of Section 2 for the commutative case.

Let us start from some general remark. If $\mathfrak{G}$ is a Lie superalgebra and $D \in \mathfrak{G}^1$ is an odd element such that $[D, D] = 0$ then one can associate with D for any $k \geq 0$ some homology class of $\mathfrak{G}$ in degree k. The reason is that D produces a homomorphism from $0|1$-dimensional commutative algebra $\mathbf{A}^{0|1}$ to $\mathfrak{G}$ and $\dim H_k(\mathbf{A}^{0|1}) = (1|0)$ for k even and $(0|1)$ for k odd.

We can construct superanalogs of algebras ℓ_n, a_n, c_n starting from flat symplectic supermanifolds in the sense of the previous section. One can see that the stable homology are the same as in the pure even case because Main Theorem of Invariant Theory works with appropriate corrections also in the supercase.

Thus any odd Hamiltonian with vanishing Poisson bracket with itself produces stable classes in even degrees. For example, a finite-dimensional Lie algebra $\mathfrak{g}$ with a nondegenerate scalar product on it gives: 1) a pure odd symplectic (in ordinary super-commutative sense) manifold $X = \Pi\mathfrak{g}$, 2) an odd cubic polynomial H on X arising from the structure constants. Jacobi identity implies $[H, H] = 0$.

Formally the same construction works in all three cases. Define duality between types of algebras as

$$\text{Lie} \leftrightarrow \text{commutative}\,, \ \text{associative} \leftrightarrow \text{associative}\,.$$

For any finite-dimensional $\mathcal{A}$-algebra V with nondegenerate invariant scalar product on it, the odd vector space ΠV considered as a manifold of the dual type carries a symplectic structure and an odd hamiltonian vector field with square equal to 0.

If we restrict ourselves to finite-dimensional simple algebras over the field of complex numbers, then we obtain many examples (Dynkin diagrams) for the case of Lie algebras, essentially one example (matrix algebra) for the associative case, and *no* nontrivial examples in the commutative case (one-dimensional commutative algebra gives zero classes in nontrivial part of $PH_*(\ell_\infty)$).

We have tried to deform these examples. It turns out that there are many new classes in the case c_∞, some classes for a_∞ and no classes for ℓ_∞.

The basic example for the associative case is the following: symplectic manifold X is $0|1$-dimensional with the coordinate x and symplectic structure $\omega = dx \otimes dx$, odd hamiltonian H is arbitrary linear combination of $x \otimes x^{2k}$, $k \geq 0$. One can prove that the linear span of all stable classes via isomorphism of Theorem 1.1 is equal to the space of all polynomials in Morita-Miller-Mumford classes on moduli spaces of curves.

Thus the conclusion from this approach is that the nontrivial primitive part of stable homology of Poisson algebras looks big for the commutative case, moderate for the associative case and small or zero for the Lie case.

7.2 Euler characteristics of (generalized) graph complexes

The absolute value of Euler characteristic of a finite complex of vector spaces gives an estimate from below for the total dimension of its homology. It is much easier to compute Euler characteristics for our generalized graph complexes in an "orbifold" sense (see [3]). The last adjective means that we count each graph Γ with the weight equal to $1/\#\,\mathrm{Aut}\,(\Gamma)$. It is reasonable to expect that the "most" part of graphs has no nontrivial automorphisms.

The generating function

$$\sum_{k\geq 1} t^k \times (\text{orbifold Euler characteristic of the subcomplex of graphs with } \chi = -k)$$

in all cases is an asymptotic expansion for $t \to 0$ of

$$\log\left(\frac{\int_{\mathrm{near}\,0} \exp(-F(x)/t)dx}{\sqrt{2\pi t}}\right)$$

where $F(x)$ is a series in x equal to

(1) $\sum_{n\geq 2} \frac{x^n}{n(n-1)}$ for the Lie case,

(2) $\sum_{n\geq 2} \frac{x^n}{n}$ for the associative case,

(3) $\sum_{n\geq 2} \frac{x^n}{n!}$ for the commutative case.

These formulas follows from Feynman rules. The second and the third integral coincide! (It is a simple exercise in calculus.)

Thus we obtain quite big but the same numbers (Bernoulli numbers) for the second and the third cases and some bigger numbers for the first (Lie) case. It is absolutely different from the previous picture.

7.3 Conjecture

Computations show that $\dim(H_2(c_\infty)) = 1$. The unique up to a factor class corresponds to the ordinary "quantization", i.e. deformation of the Lie algebra structure on ordinary hamiltonian vector fields. (Recall that by the Shapiro lemma $H^*(c_n)$ is more or less equal to the deformation cohomology of hamiltonian fields). The graph representing this class has 2 vertices and 3 edges connecting both vertices.

We conjecture that for all 3 cases (or 6, if one takes into account also odd versions) stable homology of Poisson algebras are finite-dimensional.

This conjecture has a non-trivial consequence that the difference between the virtual and the actual rational Euler characteristic for moduli spaces of open curves tends to $+\infty$ when genus tends to $+\infty$.

7.4 Poisson world

As we mentioned before, the whole story can be told for some more general classes of algebras with a set of basic binary operations and quadratic relations between these operations (like Jacobi identity or associativity). One example of such situation is the case of Poisson algebras, i.e. vector spaces V with structures of commutative and Lie algebra on it satisfying condition

$$[a, bc] = b[a, c] + c[a, b]\,.$$

As in Section 4 one can define "Poisson algebras in the Poisson world".

Poisson world is a degenerated relative of the associative world. For example, like in the associative case, there are $n!$ linearly independent polylinear monomials in n indeterminates $x_1, \ldots, x_n$ for any $n \geq 1$.

On the other hand, generalized graph complex for the Poisson world contains as a direct summands graph complexes for commutative and Lie cases. We expect that better understanding of the underlying geometry of Poisson graph complexes gives more clear picture in three classical cases.

8 Duality

Here we shall be very concise.

First of all, any differential graded $\mathcal{A}$-algebra V defines a manifold ΠV^* in the dual world with the action of $\mathbf{A}^{0|1}$ (ΠV^* could be infinite-dimensional). It is just the usual (co)bar construction. Applying the bar construction twice we obtain a differential graded algebra quasi-isomorphic to the initial one. Hence suitably defined homotopy categories for dual types of algebras are dual (see [14]).

Define *strong homotopy* $\mathcal{A}$-algebra as a manifold in the dual world with the action of $\mathbf{A}^{0|1}$. Homotopy theories of differential graded algebras and strong homotopy algebras coincide. The advantage of strong homotopy algebras is that their homotopy types are in one-to-one correspondence with equivalence types of so called *minimal* strong homotopy algebras, i.e. manifolds with odd vector fields with square equal zero and with the vanishing first Taylor coefficient at zero point (see [18]).

Another quite different aspect of duality is a kind of Lie theory. On the tensor product $V \otimes U$ of $\mathcal{A}$-algebra V and $\mathcal{A}^{\mathrm{dual}}$-algebra U there is a canonical structure of Lie algebra. Category of $\mathcal{A}^{\mathrm{dual}}$-algebras is (more or less) equivalent to the category of functors

$$\{\mathcal{A}\text{-algebras}\} \to \{\text{Lie algebras}\}$$

preserving limits. At the moment we don't understand why the homotopy theory gives the same duality as the Lie theory.

In general, we expect 4 constructions. If V is an $\mathcal{A}^{\mathrm{dual}}$-algebra, then as formal flat $\mathcal{A}$-manifolds

(1) V is a group-like object in the category of $\mathcal{A}$-manifolds (Lie theory),

(2) there is an odd vector field of homogeneity degree 1 with the square equal to 0 on ΠV (bar construction),

(3) there is an even Poisson bracket of homogeneity degree 1 on V^* (Kirillov bracket),

(4) there is an odd Poisson bracket of homogeneity degree 1 on ΠV^* (odd Kirillov bracket).

In the last three cases Taylor coefficients of corresponding structures are structure constants of algebra V. To construct group law we use a) the structure of Lie algebra on $V \otimes U$ for arbitrary $\mathcal{A}$-algebra U and b) Campbell-Dynkin-Hausdorff formula.

9 Towards a global geometry

J. Cuntz and D. Quillen ([5]), following A. Grothendieck, define *smooth* non-commutative associative algebra (with or without unit) as an algebra having the lifting property with respect to the nilpotent extensions. They proved that this property is equivalent to the existence of "connection with zero torsion on the tangent bundle". The last notion means the following: starting from an algebra A one can construct a new algebra TA adding *even* symbols da, $a \in A$ satisfying the Leibniz rule $d(a \cdot b) = a \cdot db + da \cdot b$. There is a $\mathbf{Z}_{\geq 0}$-grading on TA by the number of differentials. Connection with zero torsion on the tangent bundle is a derivation (= vector field) D of TA of homogeneity degree 1 such that $Da = da$ for $a \in A$.

It seems that both definitions of smoothness are equivalent in other cases.

It follows from results of J. Cuntz and D. Quillen that for smooth algebras co-homology of the de Rham complex (which is Karoubi-de Rham complex in the associative unital case) gives the "right" cohomology, i.e. cyclic homology of algebras.

We propose the following picture:

Let V be a finite-dimensional $\mathcal{A}$-algebra. Then V defines a functor

$$\{\text{finitely generated } \mathcal{A}\text{-algebras}\} \to \{\text{affine schemes over } \mathbf{C}\}$$

by associating with $\mathcal{A}$-algebra A the scheme $A(V)$ of its homomorphisms to V. We expect that if A is smooth then $A(V)$ is smooth, vector fields on A go to vector fields on $A(V)$, and if V carries a nondegenerate invariant scalar product then differential forms for A go to differential forms on $A(V)$ and symplectic structures go to symplectic structures.

Examples.

$\mathbf{G}_m = \langle x, y : xy = 1 \rangle$ is smooth unital associative algebra. Its de Rham cohomologies are 1-dimensional in degrees $1, 3, 5, \ldots$ and zero in even degrees (compare with the cohomology of the representation space $\mathbf{G}_m(\mathrm{Mat}_N(\mathbf{C})) = GL(N, \mathbf{C})$).

$P = \langle p : p^2 = p \rangle$ is 1-dimensional smooth nonunital associative algebra. And what is more, this non-commutative manifold is symplectic. De Rham cohomologies are 1-dimensional in degrees $0, 2, 4, \ldots$ and zero in odd degrees. Representation spaces are symplectic manifolds homotopy equivalent to the disjoint union of complex Grassmanians.

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