Titel: A_∞-algebras in Mirror Symmetry
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Mirror Symmetry gives a correspondence between symplectic and complex manifolds. We propose a way to "explain" MS by identifying both moduli spaces of complex and symplectic manifolds with $c_1 = 0$ with the moduli space of A_∞-algebras. Our guess leads to the following prediction: there should be a kind of twistor correspondence from Lagrangian subvarieties on one side to complexes of holomorphic vector bundles on the other side.

In Sections 1-3 we are trying to give more or less precise formulation of the Mirror Conjecture. This part can be considered as a complement to the talk of Yu.Manin. In Sections 4-6 we show that A_∞-algebras arise naturally from complicated structures of MS. We hope that our point of view will lead to the proof of the Mirror Conjecture in some future.

We want to mention that only few results here are proved rigorously.

1. Basic example. (After P.Candelas et al., see [Y])

Let V be a generic quintic in $\mathbf{C}P^4$. Denote by n_d the "number" of rational curves of degree d on V (see Sect.2 and talks of Yu.Manin and Y.Ruan for the meaning of quote-marks). Consider the following generating function in variable t):

$$F(t) = \frac{5}{6}t^3 + \sum_{d\geq 1} n_d \mathrm{Li}_3(e^{td}), \qquad \text{where} \quad \mathrm{Li}_3(x) = \sum_{k\geq 1}\frac{x^k}{k^3}, \quad \mathrm{Re}(t) < t_0 = -7.590\ldots$$

Here t can be considered as a local coordinate on the moduli space of symplectic manifolds. We endow V by 2-form $t\times$(the pullback of the Fubini-Studi symplectic form on $\mathbf{C}P^4$).

On the other hand, let $W = W(\lambda)$ be 1-parameter family of Calabi-Yau 3-folds obtained by the resolution of singularities of

$$\{(x_1 : x_2 : x_3 : x_4 : x_5) \in P^4 | \sum x_i^5 - \lambda x_1x_2x_3x_4x_5 = 0\}/(\mathbf{Z}/5\mathbf{Z})^3$$

where $(\mathbf{Z}/5\mathbf{Z})^3$ is the group of transformations $x_i \mapsto \xi_i x_i$, $\xi_i^5 = 1$, $\prod \xi_i = 1$ preserving the volume element on the quintic. We have the variation of 4-dimensional Hodge structures $\mathrm{H}^3(W(\lambda);\mathbf{C})$ over one parameter λ. For example, one of the periods is $\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\lambda^{-5n}$.

The conjectured mirror relation between V and W is the following. Let us consider the trivial 4-dimensional complex vector bundle with 4 linearly independent sections $e_1,\ldots,e_4$ over 1-dimensional base $\{t\in\mathbf{C}|\mathrm{Re}(t) < t_0,\ 0<\mathrm{Im}(t)<2\pi\}$. Introduce connection ∇ by

$$\dot e_1 = e_2,\ \dot e_2 = F'''(t)e_3,\ \dot e_3 = e_4,\ \dot e_4 = 0, \qquad \dot e_i := \frac{\nabla(e_i)}{dt}$$

and the Hodge filtration $0 \subset \langle e_1\rangle \subset \langle e_1e_2\rangle \subset \langle e_1e_2e_3\rangle \subset \langle e_1e_2e_3e_4\rangle$. This gives a complex variation of Hodge structures which is conjecturally eqivalent (in some coordinates $\lambda = \lambda(t)$) to the variation $\mathrm{H}^3(W(\lambda);\mathbf{C})$. One can show that t as a function of λ is equal to the ratio of two periods.

2. Gromov-Witten invariants.

Let (V,ω) be a compact *semi-positive* symplectic manifold. Semipositivity means that there exists an almost complex structure on V compatible with ω such that the canonical 2-form representing $c_1(V)$ is non-negative. In this case we expect that invariants which we call Gromov-Witten invariants are defined. These invariants depend on homology class $\beta \in \mathrm{H}_2(V;\mathbf{Z})$ and a pair (g,n) of non-negative integers satisfying inequality $2-2g-n<0$,

$$I_{g,n;\beta} \in \mathrm{H}_{\mathrm{even}}\left(\overline{\mathcal{M}}_{g,n}\times V^n;\mathbf{Z}\right).$$

Here $\overline{\mathcal{M}}_{g,n}$ denotes the coarse Deligne-Mumford compactification of the moduli space of n-punctured complex curves of genus g.

Denote by $\mathcal{X}_{g,n;\beta}$ the space of equivalence classes of $(C, x_1, \ldots, x_n, \phi)$ where C is a smooth complex curve of genus g, x_i are distinct points on C and $\phi : C \to V$ is a pseudo-holomorphic map (=the solution of a generic perturbation of the Cauchy-Riemann equation). There is a natural map from $\mathcal{X}_{g,n;\beta}$ to $\mathcal{M}_{g,n} \times V^n$ by associating with (C, x_*, ϕ) the equivalence class of (C, x_*) and the sequence of points $(\phi(x_1), \ldots, \phi(x_n))$.

It seems that using results of M.Gromov, D.McDuff and Y.Ruan one can construct a natural compactification $\overline{\mathcal{X}}_{g,n;\beta}$ which a) maps to $\overline{\mathcal{M}}_{g,n} \times V^n$ and b) has a finite Whitney stratification by even-dimensional oriented strata. We define $I_{g,n;\beta}$ to be the image of the fundamental class of $\overline{\mathcal{X}}_{g,n;\beta}$.

3. Potential and related differential-geometric structures.

Here we consider semi-positive (V, ω) as above. Denote by $\mathrm{H} := \oplus \mathrm{H}^k(V; \mathbf{C})$ the total cohomology space of V considered as a super-vector space and also as a complex flat super-manifold. Following E.Witten [W] we introduce a function Φ on H by the formula

$$\Phi(\gamma) = \sum_{\beta \in \mathrm{H}_2(V;\mathbf{Z})} e^{-\int_\beta \omega} \sum_{n=3}^{\infty} \frac{1}{n!} \int_{I_{g,n;\beta}} 1_{\overline{\mathcal{M}}_{0,n}} \otimes \gamma \otimes \cdots \otimes \gamma\,.$$

Here γ denotes a non-homogeneous cohomology class on V, $1_{\overline{\mathcal{M}}_{0,n}}$ is a zero-dimensional cohomology class 1 on $\overline{\mathcal{M}}_{0,n}$.

We expect that this series is absolutely convergent in some open domain in H if the cohomology class $[\omega] \in h^2(V, \mathbf{R})$ is sufficiently large.

Function Φ must satisfy a remarkable system of non-linear differential equations of the third order. Let us choose a base x_α of the space H. Denote by $g = (g_{\alpha\beta})$, $g_{\alpha\beta} = \int_V \alpha \wedge \beta$ the matrix of the Poincaré pairing, $(g^{\alpha\beta}) = g^{-1}$ will be the inverse matrix. For all $\alpha, \beta, \gamma, \delta$

$$\boxed{\sum_{\epsilon,\epsilon'} \frac{\partial^3 \Phi}{\partial x_\alpha \partial x_\beta \partial x_\epsilon} g^{\epsilon\epsilon'} \frac{\partial^3 \Phi}{\partial x_\gamma \partial x_\delta \partial x_{\epsilon'}} = \sum_{\epsilon,\epsilon'} \frac{\partial^3 \Phi}{\partial x_\alpha \partial x_\gamma \partial x_\epsilon} g^{\epsilon\epsilon'} \frac{\partial^3 \Phi}{\partial x_\beta \partial x_\delta \partial x_{\epsilon'}}}$$

We show now why this equation should be satisfied at point $0 \in \mathrm{H}$ (for simplicity). Third derivatives $\partial_{\alpha\beta\gamma}\Phi$ at zero point count number of rational pseudo-holomorphic curves in V passing trough 3 cycles in V with the weights $exp(-\text{area})$. Let us fix now 4 points $x_1, \ldots, x_4$ on $\mathbf{C}P^1$ and 4 cycles $C_1, \ldots C_4$ on V. We can count also the number of maps ϕ such that $\phi(x_i) \in C_i$. This number will not change under the permutations of indices and will not depend on the cross-ratio of (x_i). As the cross-ratio tends to infinity the curves must degenerate to 2 copies of $\mathbf{C}P^1$ glued at some point. At the limit we obtain the problem of counting pairs of maps $\phi_1, \phi_2 : \mathbf{C}P^1 \to V$ with restr[illegible]ions

$$\phi_1(0) \in C_1,\ \phi_1(1) \in C_2,\ \phi_2(0) \in C_3, \phi_2(1) \in C_4,\ \phi_1(\infty) = \phi_2(\infty)\,.$$

It is the same as maps $\phi_1 \times \phi_2$ from $\mathbf{C}P^1$ to $V \times V$ with certain restrictions on the values at $0, 1, \infty$. Using Künneth decomposition of the homology class of diagonal we reduce the question back to V. Hence we obtain the equation at zero.

The equation above was studied by B.Dubrovin [D]. He [illegible]iscovered that it is a completely integrable system, and it is equivalent to the following classical problem: find curvilinear coordinates in the standard flat Euclidean space $\mathbf{R}^n$ in which the metric is be diagonal.

This equation can be reformulated as the condition of associativity of the algebra given by structure constants $A^\gamma_{\alpha\beta} := \sum_{\gamma'} g^{\gamma\gamma'} \partial_{\alpha\beta\gamma'}\Phi$. In invariant terms it means that there exists a new commutative associative multiplication on H (Quantum Ring) depending on the point of H.

Let us introduce a connection ∇ on the tangent bundle to H by the formula $\nabla = \nabla_0 + A$ where ∇_0 is the standard trivial connection. One can deduce from the main equation that A is flat.

Now we are ready to formulate the general form of the Mirror Conjecture. Suppose that $c_1(V) = 0$ and V carries at least one integrable complex structure compatible with ω. For each such complex structure we have a Hodge decomposition $\mathrm{H} = \oplus \mathrm{H}^{p,q}$. We expect that all cycles $I_{g,n;\beta}$ are Hodge cycles (algebraic cycles for the algebraic case $[\omega] \in \mathrm{H}^2(V;\mathbf{Z})$) of dimensions independent on β. It follows that the restriction of ∇ to the submanifold $\mathrm{H}^{1,1}$ of H maps $\mathrm{H}^{p,q}$ to $\mathrm{H}^{p+1,q+1} \otimes \Omega^1(\mathrm{H}^{1,1})$.

We introduce filtrations $\oplus_{p \leq p_0} \mathrm{H}^{p,q}$ on the trivial bundles $\oplus_{p-q \text{ is fixed}} \mathrm{H}^{p,q}$ on $\mathrm{H}^{1,1}$. Hence we have flat connections and filtrations. One can prove using formal arguments with Hodge-Tate groups that the equivalence classes of this variations of Hodge structures are not changing under deformations of the complex structure on V used for the Hodge decomposition.

Mirror Conjecture. *These variations of Hodge structures are algebro-geometric. Sometimes they are Hodge structures on all cohomology groups of mirror manifolds.*

We wrote *sometimes* because there are examples of rigid Calabi-Yau manifolds which can't be dual to projective manifolds.

In the case of quintic V in $\mathbf{C}P^4$ the function Φ is the sum of 2 terms: the contribution of the maps to points of V and the contribution of rational curves (and their multiple covers). We introduce coordinates t_i, $i = 0, \ldots 3$ in one-dimensional spaces $\mathrm{H}^{i,i}(V)$ and odd coordinates ξ_j, η_j, $j = 1, 102$ in $\mathrm{H}^3(V)$. In these coordinates we have (up to adding a polynomial of degree 2)

$$\Phi(t_i, \xi_j, \eta_j) = \frac{5}{6} \sum_{i+j+k=3} t_i t_j t_k + t_0 \sum_j \xi_j \eta_j + \sum_{d \geq 1} n_d \mathrm{Li}_3(e^{t_1 d}) \ .$$

One can deduce from this formula the Candelas example.

4. Extended moduli spaces.

When we restrict the flat bundle to the subspace $\mathrm{H}^{1,1}$ a lot of information will be lost. It seems very reasonable to extend the moduli space of symplectic structures to the whole domain in H in which the potential Φ is defined. Hence the the tangent space to the extended moduli space is equal to $\mathrm{H} = \oplus \mathrm{H}^k$.

Now we want to construct some extended moduli space $\mathcal{M}$ for complex Calabi-Yau W containing the ordinary moduli space $Mod(W)$. The natural candidate to the tangent bundle to $\mathcal{M}$ at classical points $Mod(W)$ should be equal to the direct sum $\oplus \mathrm{H}^p(W, \wedge^q T)$. This problem was already dicussed by E.Witten (see his paper in [Y]).

Our guess is that $\oplus \mathrm{H}^p(W, \wedge^q T)$ can be interpreted as total Hochschild cohomology of the sheaf $\mathcal{O}_W$ of holomorphic functions on W.

The usual definition of Hochschild cohomology needs an associative algebra A and bimodule over it. The second Hochschild cohomology of A with coefficients in A classifies infinitesimal deformations of A. One can define HH also for graded differential algebras as well. We can replace now the sheaf of algebras $\mathcal{O}_W$ by the sheaf of differential algebras $(\Omega^{0,*}, \overline{\partial})$. This sheaf has cohomology only in degree 0. Let us pass to the differential algebra of sections, i.e. global $\overline{\partial}$-forms.

Theorem. *There exists a natural isomorphism* $HH(\Omega^{0,*}(W), \overline{\partial}) \cong \oplus H^p(W, \wedge^q T)$

Another approach to the definition of Hochschild cohomology of the sheaf $\mathcal{O}$ of holomorphic functions on algebraic manifolds was developed by M.Gerstenhaber and D.Shack.

The main question now is to give some interpretation in terms of deformations of *all* Hochschild cohomology groups of differentail graded associative algebras.

5. A_∞-algebras and their deformations.

A_∞-algebras were introduced by J.Stasheff in 1964 in [S]. We will give two equivalent definitions.

First definition. Let (x_i, ξ_j) be a set of indeterminates divided into two parts (even and odd variables). Denote by $\mathcal{A} := \overline{\mathbf{C}\langle x_\bullet, \xi_\bullet \rangle}$ the completed free associative algebra generated by $x_\bullet, \xi_\bullet$. $\mathcal{A}$ consists of all (infinite) formal linear combinations of monomials. We consider $\mathcal{A}$ as $\mathbf{Z}/2\mathbf{Z}$-graded algebra.

By definition A_∞-algebra is continuous map $D : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying conditions:

(1) $\deg(Df) = \deg f + 1 \bmod 2$,
(2) $D(fg) = D(f)g + (-1)^{\deg f} f D(g)$,
(3) $D^2 = 0$.

It is enough to define D on generators because we have super-Leibniz rule 2). For example let A be an associative algebra and c_{ij}^k be the structure constants of A in some base. We define $\mathcal{A}$ to be generated by pure odd indeterminates ξ_i and define D by $D\xi_k = \sum c_{ij}^k \xi_j \xi_i$. The associativity of A is equivalent to the condition 3).

We will say that two A_∞-algebras $(\mathcal{A}, D)$ and $(\mathcal{A}', D')$ are equivalent if there exists a continuous isomorphism between $\mathbf{Z}/2\mathbf{Z}$-graded algebras $\mathcal{A}$ and $\mathcal{A}'$ identifying D and D'.

Second definition. Let A be $\mathbf{Z}/2\mathbf{Z}$-graded vector space. The structure of A_∞-algebra on A is the infinite sequence of linear maps $m_k : A^{\otimes k} \to A$ satisfying (higher) associativity conditions:

(1) $m_1^2 = 0$, (we can consider m_1 as a differential and (A, m_1) as a complex),
(2) $m_1(m_2(a \otimes b) = m_2(m_1(a) \otimes b) \pm m_2(a \otimes m_1(b))$, ($m_2$ is a morphism of complexes),
(3) $m_3(m_1(a) \otimes b \otimes c) \pm m_3(a \otimes m_1(b) \otimes c) \pm m_3(a \otimes b \otimes m_1(c)) \pm m_1(m_3(a \otimes b \otimes c) =$

$$= m_2(m_2(a \otimes b) \otimes c) - m_2(a \otimes m_2(b \otimes c)),\ (m_2 \text{ is associative up to homotopy}),$$

(4) and so on...

One can pass from the second definition to the first one by defining the space of generators to be A^* with the reversed parity and D on generators to be equal $\sum m_k^*$.

It is very easy to see that infinitesimal deformations of a differntial graded algebra considered as an A_∞-algebra is equal to total Hochschild cohomology of A.

We expect that the formal moduli space of A_∞-algebras near Calabi-Yau manifolds is smooth.

There are several reasons to beleive that the moduli $\mathcal{M}$ of A_∞-algebras are relevant for the situation of Mirror Symmetry:

(1) on the tangent space to $\mathcal{M}$ at each point there exists a natural associative commutative multiplication (cup-product on Hochschild cohomology),
(2) there exists a natural bundle with flat connection on $\mathcal{M}$ with the fiber equal to the periodic cyclic homology of A_∞-algebras (see [G]),
(3) any finite-dimensional A_∞-algebra with some additional data (a scalar product compatible with all higher multiplications) gives cohomology classes of $\mathcal{M}_{g,n}$ (see [K]).

Now we describe natural A_∞-structure arising in the framework of symplectic geometry.

6. Fukaya's A_∞-category.

At the end of last year Kenji Fukaya introduced an A_∞-category for semi-positive symplectic manifolds for the purposes of Donaldson invariants of 4-manifolds. First of all, A_∞-category is not a category in the strict sense. It consists of objects, the space of morphisms between any two objects and a ladder of (higher) compositions of morphisms. The axioms are such that the set of morphisms of any object into itself will be A_∞-algebra (second definition).

Let (V, ω) will be semi-positive symplectic manifold with sufficiently large $[\omega]$. Objects of A_∞-category $F(V, \omega)$ are Lagrangian submanifolds of V. Morphisms are defined only when Lagrangian submanifolds are in general position, $F(\mathcal{L}, \mathcal{L}') := \mathbf{C}^{\mathcal{L} \cap \mathcal{L}'}$ with $\mathbf{Z}/2\mathbf{Z}$-grading arising from the Maslov index.

The differential on $F(\mathcal{L}, \mathcal{L}')$ will be a modified Floer differential. The matrix coefficient of differential associated with two intersection points $p_1, p_2 \in \mathcal{L} \cap \mathcal{L}'$ is the number of pseudo-holomorphic discs ϕ : $D \to V$ with

$$\phi(-1) = p_1,\ \phi(1) = p_2,\ \phi(\text{upper boundary}) \subset \mathcal{L},\ \phi(\text{lower boundary}) \subset \mathcal{L}'$$

counted with the weight $\exp(-\text{area of } D)$.

Higher multiplications are defined analogously using maps from polygons to V.

One can show (not rigorously) that the cap-product on the Hochschild cohomology of this category coinsides with the quantum multiplication on the ordinary cohomology $\mathrm{H}(V, \mathbf{C})$.

We don't know at the moment any mathematical definition of an A_∞-algebra $A(V, \omega)$ such that Fukaya's category is equivalent to the A_∞-category of modules over it. Also we expect that one has to extend in some way $F(V, \omega)$ (fro example, consider local systems over Lagrangian submanifolds) and obtain an equivalence between (extended) $F(V, \omega)$ and the derived category of coherent sheaves on the dual complex manifold W.

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