FROBENIUS MANIFOLDS AND FORMALITY OF LIE ALGEBRAS OF POLYVECTOR FIELDS

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Abstract. We construct a generalization of the variations of Hodge structures on Calabi-Yau manifolds. It gives a Mirror partner for the theory of $genus = 0$ Gromov-Witten invariants.

Introduction

Probably the best mathematically understood part of the Mirror Symmetry program is the theory of Gromov - Witten invariants (see [KM]). In this paper we will construct a Mirror partner for the $genus = 0$ sector of this theory. It may be considered as a generalization of the theory of variations of Hodge structures on Calabi-Yau manifolds.

One of the puzzles in Mirror symmetry was to find an interpretation of the mysterious objects involved in the famous predictions of the numbers of rational curves. Such an interpretation should, in particular, give the meaning to the "extended" moduli space $H^*(M, \Lambda^* T_M)[2]$, thought as generalized deformations of complex structure. This moduli space should be equipped with the analog of the 3-tensor $C_{ijk}(t)$ ("Yukawa coupling") arising from a generalization of the variation of Hodge structure on $H^*(M)$. To find such structure is essential for the extension of the predictions of Mirror Symmetry in the dimensions $n > 3$.

0.1 Background philosophy.

The Mirror Symmetry conjecture, as it was formulated in [K1], states that the derived category of coherent sheaves on a Calabi-Yau manifold M is equivalent to the derived category constructed from (conjectured) Fukaya category associated with the dual Calabi-Yau manifold $\widetilde{M}$. The conjecture implies existence of the structure of Frobenius manifold on the moduli space of A_∞-deformations of the derived category of coherent sheaves on M. This structure coincides conjecturally with the Frobenius structure on formal neighborhood of zero in $H_*(\widetilde{M}, \mathbb{C})$ constructed via Gromov-Witten classes of the dual Calaby-Yau manifold $\widetilde{M}$.

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0.2 Formulation of the results.

Consider the differential graded Lie algebra

$$\mathbf{t} = \bigoplus_k \mathbf{t}^k,\ \mathbf{t}^k = \bigoplus_{q+p-1=k} \Gamma(M, \Lambda^q \overline{T}^*_M \otimes \Lambda^p T_M) \tag{0.1}$$

endowed with the differential $\bar{\partial}$ and the bracket coming from the cup-product on $\bar{\partial}$-forms and standard Schouten-Nijenhuys bracket on holomorphic polyvector fields. Deformation theory associates a formal (super)moduli space $\mathcal{M}_{\mathbf{t}}$ to the Lie algebra $\mathbf{t}$. It can be described informally as universal moduli space of solutions to the Maurer-Cartan equation over $\mathbb{Z}$-graded Artin algebras modulo gauge equivalence. The formal moduli space $\mathcal{M}_{\mathbf{t}}$ can be identified with a formal neighborhood of zero in graded vector space $H^*(M, \Lambda^* T_M)[2]$[1]. The tangent sheaf to $\mathcal{M}_{\mathbf{t}}$ after the shift by $[-2]$ has natural structure of graded commutative associative algebra over $\mathcal{O}_{\mathcal{M}_{\mathbf{t}}}$. In this note we show that this algebra structure gives rise to the structure of (formal) Frobenius manifold on $H^*(M, \Lambda^* T_M)[2]$. More specifically, using a universal solution to the analog of Maurer-Cartan equation in $\mathbf{t}$, we construct generalized holomorphic volume element for generalized deformations of complex structure. The integrals of this element, which can be thought of as generalized periods, produce the Frobenius manifold structure on $H^*(M, \Lambda^* T_M)[2]$.

0.3 Connection with A_∞-deformations of $\mathcal{D}^b Coh(M)$.

The Formality theorem (see [K2]) identifies the germ of the moduli space of A_∞-deformations of the derived category of coherent sheaves on M with the moduli space $\mathcal{M}_{\mathbf{t}}$. The tangent bundle of this moduli space after the shift by $[-2]$ has natural structure of the graded commutative associative algebra. The multiplication arises from the Yoneda product on Ext-groups. The identification of moduli spaces provided by the Formality theorem respects the algebra structure on the tangent bundles of the moduli spaces. This implies, in particular, that the usual predictions of numerical Mirror Symmetry can be deduced from the homological Mirror Symmetry conjecture proposed in [K1]. We hope to elaborate on this elsewhere.

1. Frobenius manifolds

Remind the definition of formal Frobenius (super) manifold as given in [D], [M], [KM]. Let $\mathbf{H}$ be a finite-dimensional $\mathbf{Z}_2$-graded vector space over $\mathbb{C}$.[2] It is convenient to choose some set of coordinates $x_{\mathbf{H}} = \{x^a\}$ which defines the basis $\{\partial_a := \partial/\partial x^a\}$ of vector fields. One of the given coordinates is distinguished and is denoted by x_0. Let $A_{ab}^c \in \mathbb{C}[[x_{\mathbf{H}}]]$ be a formal power series representing 3-tensor field, g_{ab} be a nondegenerate symmetric pairing on $\mathbf{H}$. To simplify notations in superscripts we replace $\deg(x^a)$ by $\bar{a}$.

One can use the A_{ab}^c in order to define a structure of $\mathbb{C}[[x_{\mathbf{H}}]]$-algebra on $\mathbf{H} \otimes \mathbb{C}[[x_{\mathbf{H}}]]$, the (super)space of all continuous derivations of $\mathbb{C}[[x_{\mathbf{H}}]]$, with multiplication denoted by $\circ$:

$$\partial_a \circ \partial_b := \sum_c A_{ab}^c \partial_c$$

[1] For a graded object $\mathbf{t}$ denote $\mathbf{t}[n]$ the tensor product of $\mathbf{t}$ with the trivial object concentrated in degree $-n$.

[2] One can use an arbitrary field of characteristic zero instead of $\mathbb{C}$ everywhere

One can use g_{ab} to define the symmetric $\mathbb{C}[[x_{\mathbf{H}}]]$-pairing on $\mathbf{H}\otimes\mathbb{C}[[x_{\mathbf{H}}]]$:

$$\langle\partial_a,\partial_b\rangle := g_{ab}$$

These data define the structure of formal Frobenius manifold on $\mathbf{H}$ iff the following equations hold:

(1) (Commutativity/Associativity)

$$\forall a,b,c \quad A^c_{ba} = (-1)^{\bar a\bar b}A^c_{ab} \tag{1a}$$

$$\forall a,b,c,d \quad \sum_e A^e_{ab}A^d_{ec} = (-1)^{\bar a(\bar b+\bar c)}\sum_e A^e_{bc}A^d_{ea} \tag{1b}$$

equivalently, A^c_{ab} are structure constants of a supercommutative associative $\mathbb{C}[[x_{\mathbf{H}}]]$-algebra

(2) (Invariance) Put $A_{abc}=\sum_e A^e_{ab}g_{ec}$

$$\forall a,b,c \quad A_{abc} = (-1)^{\bar a(\bar b+\bar c)}A_{bca} \quad ,$$

equivalently, the pairing g_{ab} is invariant with respect to the multiplication $\circ$ defined by A^c_{ab}.

(3) (Identity)

$$\forall a,b \quad A^b_{0a} = \delta^b_a$$

equivalently ∂_0 is an identity of the algebra $\mathbf{H}\otimes\mathbb{C}[[x_{\mathbf{H}}]]$

(4) (Potential)

$$\forall a,b,c,d \quad \partial_d A^c_{ab} = (-1)^{\bar a\bar d}\partial_a A^c_{db} \quad ,$$

which implies, assuming (1a) and (2), that the series A_{abc} are the third derivatives of a single power series $\Phi\in\mathbf{H}\otimes\mathbb{C}[[x_{\mathbf{H}}]]$

$$A_{abc} = \partial_a\partial_b\partial_c\Phi$$

2. Moduli space via deformation functor

The material presented in this section is standard (see [K2] and references therein).

Let us remind the definition of the functor $\mathrm{Def}_{\mathfrak{g}}$ associated with a differential graded Lie algebra $\mathfrak{g}$. It acts from the category of Artin algebras to the category of sets. Let $\mathfrak{A}$ be an Artin algebra with the maximal ideal denoted by $\mathfrak{m}$. Define the set

$$\mathrm{Def}_{\mathfrak{g}}(\mathfrak{A}) := \{d\gamma+\frac{[\gamma,\gamma]}{2}=0|\gamma\in(\mathfrak{g}\otimes\mathfrak{m})^1\}/\Gamma^0_{\mathfrak{A}}$$

where the quotient is taken modulo action of the group $\Gamma^0_{\mathfrak{A}}$ corresponding to the nilpotent Lie algebra $(\mathfrak{g}\otimes\mathfrak{m})^0$. The action of the group can be described via the infinitesimal action of its Lie algebra:

$$\alpha\in\mathfrak{g}\otimes\mathfrak{m}\to\dot\gamma = d\alpha+[\gamma,\alpha]$$

Sometimes functor $\mathrm{Def}_{\mathfrak{g}}$ is represented by some topological algebra $\mathcal{O}_{\mathcal{M}_{\mathfrak{g}}}$ (projective limit of Artin algebras) in the sense that the functor $\mathrm{Def}_{\mathfrak{g}}$ is equivalent to the functor $\mathrm{Hom}_{continuous}(\mathcal{O}_{\mathcal{M}_{\mathfrak{g}}}, \cdot)$. For example, $H^0(\mathfrak{g}) = 0$ is a sufficient condition for this. If $\mathrm{Def}_{\mathfrak{g}}$ is representable then one can associate formal moduli space to $\mathfrak{g}$ by defining the "algebra of functions" on the formal moduli space to be the algebra $\mathcal{O}_{\mathcal{M}_{\mathfrak{g}}}$.

We will need the $\mathbb{Z}$-graded extension of the functor $\mathrm{Def}_{\mathfrak{g}}$. The definition of $\mathrm{Def}^{\mathbb{Z}}_{\mathfrak{g}}$ is obtained from the definition of $\mathrm{Def}_{\mathfrak{g}}$ via inserting $\mathbb{Z}-$graded Artin algebras instead of the usual ones everywhere. A sufficient and probably necessary condition for the functor $\mathrm{Def}^{\mathbb{Z}}_{\mathfrak{g}}$ to be representable is that $\mathfrak{g}$ must be quasi-isomorphic to an abelian graded Lie algebra. We will see in §2.1 that this is the case for $\mathfrak{g} = \mathbf{t}$. Hence one can associate formal (graded) moduli space[3] $\mathcal{M}_{\mathbf{t}}$ to the Lie algebra $\mathfrak{g}$.

2.1 Extended moduli space of complex structure.

Let M be a connected compact complex manifold of dimension n, with vanishing 1-st Chern class $c_1(T_M) = 0 \in \mathrm{Pic}(M)$. We assume that there exists a Kähler metric on M, although we will not fix it. By Yau's theorem there exists a Calabi-Yau metric on M.

It follows from the condition $c_1(T_M) = 0$ there exists an everywhere nonvanishing holomorphic volume form $\Omega \in \Gamma(X, \Lambda^n T^*_M)$. It is defined up to a multiplication by a constant. Let us fix a choice of Ω. It induces isomorphism of cohomology groups: $H^q(M, \Lambda^p T_M) \simeq H^q(M, \Omega^{n-p})$; $\gamma \mapsto \gamma \vdash \Omega$. Define differential Δ of degree -1 on $\mathbf{t}$ by the formula :

$$(\Delta\gamma) \vdash \Omega = \partial(\gamma \vdash \Omega)$$

The operator Δ satisfies the following identity (Tian-Todorov lemma) :

$$[\gamma_1, \gamma_2] = (-1)^{\deg\gamma_1 + 1}(\Delta(\gamma_1 \wedge \gamma_2) - (\Delta\gamma_1) \wedge \gamma_2 - (-1)^{\deg\gamma_1+1}\gamma_1 \wedge \Delta\gamma_2) \qquad (2.1)$$

where $\deg\gamma = p + q - 1$ for $\gamma \in \Gamma(M, \Lambda^q \bar{T}^*_M \otimes \Lambda^q T_M)$.

Denote by $\mathbf{H}$ the total homology space of Δ acting on $\mathbf{t}[1]$. Let $\{\Delta_a\}$ denote a graded basis in the vector space $\oplus_{p,q} H^q(M, \Lambda^p T_M)$, $\Delta_0 = 1 \in H^0(M, \Lambda^0 T_M)$. Let us redefine the degree of Δ_a as follows

$$|\Delta_a| := p + q - 2 \quad \text{for} \quad \Delta_a \in H^q(M, \Lambda^p T_M)$$

Then $\{\Delta_a\}$ form a graded basis in $\mathbf{H}$. Denote by $\{t^a\}, t^a \in \mathbf{H}^*$, $\deg t^a = -|\Delta_a|$ the basis dual to $\{\Delta_a\}$. Denote by $\mathbb{C}[[t_{\mathbf{H}}]]$ the algebra of formal power series on $\mathbb{Z}$-graded vector space $\mathbf{H}$.

Lemma 2.1. *The functor $\mathrm{Def}^{\mathbb{Z}}_{\mathbf{t}}$ associated with $\mathbf{t}$ is canonically equivalent to the functor represented by the algebra $\mathbb{C}[[t_{\mathbf{H}}]]$.*

Proof. It follows from (2.1) that the maps

$$(\mathbf{t}, \bar\partial) \leftarrow (\mathrm{Ker}\,\Delta, \bar\partial) \rightarrow (\mathbf{H}[-1], d = 0) \qquad (2.2)$$

are morphisms of differential graded Lie algebras. Then the $\partial\bar\partial$-lemma, which says that

$$\mathrm{Ker}\,\bar\partial \cap \mathrm{Ker}\,\Delta \cap (\mathrm{Im}\,\Delta \oplus \mathrm{Im}\,\bar\partial) = \mathrm{Im}\,\Delta \circ \bar\partial, \qquad (2.3)$$

shows that these morphisms are quasi-isomorphisms (this argument is standard in the theory of minimal models, see [DGMS]). Hence (see e.g. theorem in §4.4 of [K2]) the deformation functors associated with the three differential graded Lie algebras are canonically equivalent. The deformation functor associated with trivial algebra $(\mathbf{H}[-1], d = 0)$ is represented by the algebra $\mathbb{C}[[t_{\mathbf{H}}]]$. $\square$

[3]We will omit the superscript $\mathbb{Z}$ where it does not seem to lead to a confusion.

Corollary 2.2. *The moduli space $\mathcal{M}_{\mathbf{t}}$ associated to* $\mathbf{t}$ *is smooth. The dimension of* $\mathcal{M}_{\mathbf{t}}$ *is equal to* $\sum_{p,q} \dim H^q(M, \Lambda^p T_M)$ *of the dimension of the space of first order deformations associated with* $\mathbf{t}$.

Remark. The Formality theorem proven in [K2] implies that the differential graded Lie algebra controlling the A_∞-deformations of $\mathcal{D}^b Coh(M)$ is quasi-isomorphic to $\mathbf{t}$. Here we have proved that $\mathbf{t}$ is quasi-isomorphic to an abelian graded Lie algebra. Therefore, the two differential graded Lie algebras are formal, i.e. quasi-isomorphic to their cohomology Lie algebras endowed with zero differential.

Corollary 2.3. *There exists a solution to the Maurer-Cartan equation*

$$\bar{\partial}\hat{\gamma}(t) + \frac{[\hat{\gamma}(t), \hat{\gamma}(t)]}{2} = 0 \tag{2.4}$$

in formal power series with values in $\mathbf{t}$

$$\hat{\gamma}(t) = \sum_a \hat{\gamma}_a t^a + \frac{1}{2!} \sum_{a_1, a_2} \hat{\gamma}_{a_1 a_2} t^{a_1} t^{a_2} + \ldots \in (\mathbf{t} \otimes \mathbf{C}[[t_{\mathbf{H}}]])^1$$

such that the cohomology classes $[\hat{\gamma}_a]$ *form a basis of cohomology of the complex* $(\mathbf{t}, \bar{\partial})$

Remark. The deformations of the complex structure are controlled by the differential graded Lie algebra

$$\mathbf{t}_{(0)} := \bigoplus_k \mathbf{t}^k_{(0)}, \ \mathbf{t}^k_{(0)} = \Gamma(M, \Lambda^k \overline{T}^*_M \otimes T_M)$$

The meaning of this is that the completion of the algebra of functions on the moduli space of complex structures on M represents $\mathrm{Def}_{\mathbf{t}_{(0)}}$ (or $\mathrm{Def}^{\mathbb{Z}}_{\mathbf{t}_{(0)}}$ restricted to the category of Artin algebras concentrated in degree 0). The natural embeddings $\mathbf{t}_{(0)} \hookrightarrow \mathbf{t}$ induces embedding of the corresponding moduli spaces. In terms of the solutions to Maurer-Cartan equation the deformations of complex structure are singled out by the condition $\gamma(t) \in \Gamma(M, \Lambda^1 \overline{T}^*_M \otimes \Lambda^1 T_M)$.

Remark. Similar thickening of the moduli space of complex structures were considered by Z. Ran in [R].

3. Algebra structure on the tangent sheaf of the moduli space

Let R denotes a $\mathbb{Z}$-graded Artin algebra over $\mathbb{C}$, $\gamma^R \in (\mathbf{t} \otimes R)^1$ denotes a solution to the Maurer-Cartan equation (2.4).

The linear extension of the wedge product gives a structure of graded commutative algebra on $\mathbf{t} \otimes R[-1]$. Let γ^R be a solution to the Maurer-Cartan equation in $(\mathbf{t} \otimes R)^1$. It defines a differential $\bar{\partial}_{\gamma^R} = \bar{\partial} + \{\gamma^R, \cdot\}$ on $\mathbf{t} \otimes R[1]$. Denote the cohomology of $\bar{\partial}_{\gamma^R}$ by T_{γ^R}. The space of first order variations of γ^R modulo gauge equivalence is identified with T_{γ^R}. Geometrically one can think of γ^R as a morphism from the formal variety corresponding to algebra R to the formal moduli space. An element of T_{γ^R} corresponds to a section of the preimage of the tangent sheaf. Note that $\bar{\partial}_{\gamma^R}$ acts as a differentiation of the (super)commutative $R-$algebra

$t \otimes R[-1]$. Therefore $T_{\gamma^R}[-2]$ inherits the structure of (super)commutative associative algebra over R. This structure is functorial with respect to the morphisms $\phi_* : T_{\gamma^{R_1}} \to T_{\gamma^{R_2}}$ induced by homomorphisms $\phi : R_1 \to R_2$.

Let $\hat{\gamma}(t) \in (\mathbf{t}\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]])^1$ be a solution to the Maurer-Cartan satisfying the condition of corollary 2.3. It follows from this condition that the $\mathbb{C}[[t_{\mathbf{H}}]]$-module $T_{\hat{\gamma}(t)}$ is freely generated by the classes of partial derivatives $[\partial_a \hat{\gamma}(t)]$. Therefore we have

Proposition 3.1. *There exists formal power series $A_{ab}^c(t) \in \mathbf{t}\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]]$ satisfying*

$$\partial_a \hat{\gamma} \wedge \partial_b \hat{\gamma} = \sum_c A_{ab}^c \partial_c \hat{\gamma} \ mod\, \bar{\partial}_{\hat{\gamma}(t)}$$

The series $A_{ab}^c(t)$ are the structure constants of the commutative associative $\mathbb{C}[[t_{\mathbf{H}}]]$-algebra structure on $\mathbf{H} \otimes \mathbb{C}[[t_{\mathbf{H}}]][-2]$.

□

Remark. Note that on the tangent space at zero this algebra structure is given by the obvious multiplication on $\oplus_{p,q} H^q(M, \Lambda^p T_M)$. This is "Mirror dual" to the ordinary multiplication on $\oplus_{p,q} H^q(M, \Omega^p_{\widetilde{M}})$.

4. Integral

Introduce linear functional on $\mathbf{t}$

$$\int \gamma := \int_M (\gamma \vdash \Omega) \wedge \Omega$$

Claim 4.1. *It satisfies the following identyties:*

$$\begin{aligned} \int \bar{\partial}\gamma_1 \wedge \gamma_2 &= (-1)^{\deg \gamma_1} \int \gamma_1 \wedge \bar{\partial}\gamma_2 \\ \int \Delta\gamma_1 \wedge \gamma_2 &= (-1)^{\deg \gamma_1 + 1} \int \gamma_1 \wedge \Delta\gamma_2 \end{aligned} \tag{4.1}$$

for $\gamma_i \in \Gamma(M, \Lambda^{q_i} \overline{T}^*_M \otimes \Lambda^{p_i} T_M)$, $i = 1, 2$ where $deg\, \gamma_i = p_i + q_i - 1$.

□

5. Metric on $\mathcal{T}_{\mathcal{M}}$

There exists a natural metric (i.e. a nondegenerate (super)symmetric $\mathcal{O}_{M_{\mathbf{t}}}$-linear pairing) on the sheaf $\mathcal{T}_{\mathcal{M}_{\mathbf{t}}}$. In terms of a solution to the Maurer-Cartan equation $\gamma^R \in (\mathbf{t} \otimes R)^1$ it means that there exists an R-linear graded symmetric pairing on T_{γ^R}, which is functorial with respect to R. Here T_{γ^R} denotes the cohomology of $\bar{\partial}_{\gamma^R}$ defined in §3. The pairing is defined by the formula

$$\langle h_1, h_2 \rangle := \int h_1 \wedge h_2 \ \text{ for } h_1, h_2 \in T_{\gamma^R}$$

where we assumed for simplicity that $\gamma^R \in \operatorname{Ker} \Delta \otimes R$. It follows from (2.2) (see also lemma 6.1) that such a choice of γ^R in the given class of gauge equivalence is always possible.

Claim 5.1. *The pairing is compatible with the algebra structure.*

□

6. Flat coordinates on moduli space.

Another ingredient in the definition of Frobenius structure is the choice of affine structure on the moduli space associated with $\mathbf{t}$. The lemma 2.1 identifies $\mathcal{M}_{\mathbf{t}}$ with the moduli space associated with trivial algebra $(\mathbf{H}[-1], d = 0)$. The latter moduli space is the affine space $\mathbf{H}$. The affine coordinates $\{t_a\}$ on $\mathbf{H}$ give coordinates on $\mathcal{M}_{\mathbf{t}}$. This choice of coordinates on the moduli space corresponds to a specific choice of a universal solution to the Maurer-Cartan equation over $\mathbb{C}[[t_{\mathbf{H}}]]$.

Lemma 6.1. *There exists a solution to the Maurer-Cartan equation in formal power series with values in* $\mathbf{t}$

$$\bar{\partial}\hat{\gamma}(t) + \frac{[\hat{\gamma}(t), \hat{\gamma}(t)]}{2} = 0,\ \hat{\gamma}(t) = \sum_a \hat{\gamma}_a t^a + \frac{1}{2!}\sum_{a_1,a_2} \hat{\gamma}_{a_1 a_2} t^{a_1} t^{a_2} + \ldots \in (\mathbf{t}\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]])^1$$

such that

1. *(Universality) the cohomology classes* $[\hat{\gamma}_a]$ *form a basis of cohomology of the complex* $(\mathbf{t}, \bar{\partial})$
2. *(Flat coordinates)* $\hat{\gamma}_a \in \operatorname{Ker}\Delta,\ \hat{\gamma}_{a_1\ldots a_k} \in \operatorname{Im}\Delta$ *for* $k \geq 2$
3. *(Flat identity)* $\partial_0\hat{\gamma}(t) = \mathbf{1}$, *where* ∂_0 *is the coordinate vector field corresponding to* $[\mathbf{1}] \in \mathbf{H}[-1]$

Proof. The theorem of §4.4 in [K2] shows that there exists L_∞ morphism f homotopy inverse to the natural morphism $(\operatorname{Ker}\Delta, \bar{\partial}) \to \mathbf{H}[-1]$ (for the definition of L_∞-morphism see §4.3 in [K2]). Put $\Delta(t) = \sum_a (\Delta_a[-1])t^a$, where $\Delta_a[-1]$ denotes the element Δ_a having degree shifted by one. Then

$$\gamma(t) = \sum_n \frac{1}{n!} f_n(\Delta(t) \wedge \cdots \wedge \Delta(t))$$

satisfies the conditions $(1) - (2)$. To fulfill the condition (3) $\gamma(t)$ must be improved slightly. Define the differential graded Lie algebra $\widetilde{\operatorname{Ker}}$ as follows

1. $\widetilde{\operatorname{Ker}}_i = \operatorname{Ker}\Delta \subset \mathbf{t}_i$ for $i \geq 0$
2. $\widetilde{\operatorname{Ker}}_{-1} = \operatorname{Im}\Delta \subset \mathbf{t}_{-1}$

Note that the algebra $\operatorname{Ker}\Delta$ is the sum of the algebra $\widetilde{\operatorname{Ker}}$ and trivial algebra of constants $\mathbb{C}\otimes\mathbf{1}[-1]$. Let $\tilde{f}$ be a homotopy inverse to the natural quasi-isomorphism $\widetilde{\operatorname{Ker}} \to \mathbf{H}[-1]_{\geq 0}$. Put $\tilde{\Delta}(t) = \sum_{a\neq 0}(\Delta_a[-1])t^a$. Then

$$\hat{\gamma}(t) = \mathbf{1}t_0 + \sum_n \frac{1}{n!}\tilde{f}_n(\tilde{\Delta}(t) \wedge \cdots \wedge \tilde{\Delta}(t))$$

satisfies all the conditions. □

Remark. Any two formal power series satisfying conditions of lemma 6.1 are equivalent under the natural action of the group associated with the Lie algebra $(\widetilde{\operatorname{Ker}}\Delta\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]])^0$.

Remark. It is possible to write down an explicit formula for the components of the morphism f in terms of Green functions of the Laplace operator acting on differential forms on M.

Remark. After the identification of the moduli space $\mathcal{M}_{\mathbf{t}}$ with $\mathbf{H}$, provided by lemma 2.1, the complex structure moduli space corresponds to the subspace $\mathbf{H}^1(M, \Lambda^1 T_M)$. In the case of classical moduli space of the complex structures on M the analogous lemma was proved in [T]. The coordinates arising on the classical moduli space of complex structures coincide with so called "special" coordinates of [BCOV].

Notation. Denote

$$\hat{\gamma}(t) = \sum_a \hat{\gamma}_a t^a + \frac{1}{2!} \sum_{a_1, a_2} \hat{\gamma}_{a_1 a_2} t^{a_1} t^{a_2} + \ldots \in (\mathbf{t} \otimes \mathbf{C}[[t_{\mathbf{H}}]])^1$$

a solution to the Maurer-Cartan equation satisfying conditions of lemma 6.1.

The parameters of a miniversal solution to the Maurer-Cartan equation over $\mathbb{C}[[t_{\mathbf{H}}]]$ serve as coordinates on the moduli space. The specific choice of coordinates corresponding to the solution to the Maurer-Cartan equation satisfying conditions (1)-(2) of lemma 6.1 corresponds to choice of coordinates on moduli space that are flat with respect to the natural (holomorphic) metric g_{ab}.

Claim 6.2. *The power series $\langle \partial_a \hat{\gamma}(t), \partial_b \hat{\gamma}(t) \rangle \in \mathbb{C}[[t_{\mathbf{H}}]]$ has only constant term in the power series expansion at $t = 0$.*

Proof. $\langle x, y \rangle = 0$ for $x \in \operatorname{Ker} \Delta, y \in \operatorname{Im} \Delta$. □

Notation. Denote $g_{ab} := \langle \partial_a \hat{\gamma}(t), \partial_b \hat{\gamma}(t) \rangle$.

Thus we have constructed all the ingredients of the Frobenius structure on $\mathcal{M}_{\mathbf{t}}$: the tensors $A^c_{ab}(t)$, g_{ab} and the coordinates $\{t_a\}$ that are flat with respect to g_{ab}.

Remark. The 3-tensor $A^c_{ab}(t)$ on $\mathcal{M}_{\mathbf{t}}$ does not depend on the choice of Ω. The 2-tensor g_{ab} is multiplied by λ^2 when Ω is replaced by $\lambda\Omega$

Claim 6.3. *The structure constants satisfy $A^b_{0a} = \delta^b_a$.*

Proof. It follows from the condition (3) imposed on $\hat{\gamma}(t)$ □

We have checked that the tensors $A^c_{ab}(t)$, g_{ab} have the properties (1)-(3) from the definition of the Frobenius structure. It remains to us to check the property (4).

7. Flat connection and periods

Let $\hat{\gamma}(t) \in (\mathbf{t} \otimes \mathbf{C}[[t_{\mathbf{H}}]])^1$ be a solution to the Maurer-Cartan equation satisfying conditions (1)-(2) of lemma 6.1. Then the formula

$$\Omega(t) := e^{\hat{\gamma}(t)} \vdash \Omega$$

defines a closed form of mixed degree depending formally on $t \in \mathbf{H}$. For $t \in \mathbf{H}^{-1,1}$ $\hat{\gamma}(t) \in \Gamma(M, \overline{T} \otimes T^*)$ represents a deformation of complex structure. Then $\Omega(t)$ is a holomorphic n-form in the complex structure corresponding to $t \in \mathbf{H}^{-1,1}$, where $n = \dim_{\mathbb{C}} M$.

Let $\{p^a\}$ denote the set of sections of $\mathcal{T}^*_{\mathbf{H}}$ that form a framing dual to $\{\partial_a\}$. Define a (formal) connection on $\mathcal{T}^*_{\mathbf{H}}$ by the covariant derivatives:

$$\nabla_{\partial_a}(p^c) := \sum_b A^c_{ab} p^b \tag{7.1}$$

Strictly speaking this covariant derivatives are formal power series sections of $\mathcal{T}^*_{\mathbf{H}}$.

Let us put

$$\Pi_{ai} = \frac{\partial}{\partial t^a} \int_{\Gamma_i} \Omega(t) \tag{7.2}$$

where $\{\Gamma_i\}$ form a basis in $\mathbf{H}_*(M, \mathbb{C})$. In particular

$$\Pi_{0i} = \int_{\Gamma_i} \Omega(t)$$

if $\hat{\gamma}(t)$ satisfies the condition (3) of lemma 6.1.

Lemma 7.1. *The periods* $\Pi_i = \sum_a \Pi_{ai} p^a$ *are flat sections of* ∇

Proof. Let $\partial_\tau = \sum_a \tau^a \partial_a$ be an even constant vector field, i.e. τ^a are even constants for even ∂_a and odd for odd ∂_a. It is enough to prove that

$$\partial_\tau \partial_\tau \int_{\Gamma_i} \Omega(t) = \sum_c A^c_{\tau\tau} \partial_c \int_{\Gamma_i} \Omega(t)$$

where $A^c_{\tau\tau}$ are the algebra structure constants defined via $\sum_a \tau^a \partial_a \circ \sum_a \tau^a \partial_a = \sum_c A^c_{\tau\tau} \partial_c$ (see §1). Note that the operators Δ and $\bar{\partial}_{\hat{\gamma}(t)}$ acting on $\mathbf{t}\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]]$ satisfy a version of $\partial\bar{\partial}$-lemma :

$$\operatorname{Ker} \Delta \cap \operatorname{Ker} \bar{\partial}_{\hat{\gamma}(t)} \cap (\operatorname{Im} \Delta \oplus \operatorname{Im} \bar{\partial}_{\hat{\gamma}(t)}) = \operatorname{Im} \Delta \circ \bar{\partial}_{\hat{\gamma}(t)} \tag{7.3}$$

Equivalently, there exists decomposition of $\mathbf{t}\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]] = X_0 \oplus X_1 \oplus X_2 \oplus X_3 \oplus Y$ into direct sum of graded vector spaces, such that the only nonzero components of $\bar{\partial}_{\hat{\gamma}(t)}$, Δ are isomorphisms $\bar{\partial}_{\hat{\gamma}(t)} : X_0 \mapsto X_1,\ \ X_2 \mapsto X_3;\ \ \ \Delta : X_0 \mapsto X_2,\ \ X_1 \mapsto X_3$. Differentiating twice the Maurer-Cartan equation (2.4) with respect to ∂_τ and using (2.1) one obtaines

$$\Delta(\partial_\tau \hat{\gamma}(t) \wedge \partial_\tau \hat{\gamma}(t) - \sum_c A^c_{\tau\tau} \partial_c \hat{\gamma}(t)) = -\bar{\partial}_{\hat{\gamma}(t)} \partial_\tau \partial_\tau \hat{\gamma}(t) \tag{7.4}$$

It follows from $\partial\bar{\partial}$-lemma for Δ, $\bar{\partial}_{\hat{\gamma}(t)}$ and the equation (7.4) that there exist formal power series $\alpha_\tau(t) \in \mathbf{t}\widehat{\otimes}\mathbb{C}[[t_{\mathbf{H}}]]$ such that

$$\begin{aligned} &\partial_\tau \hat{\gamma}(t) \wedge \partial_\tau \hat{\gamma}(t) - \sum_c A^c_{\tau\tau} \partial_c \hat{\gamma}(t) = \bar{\partial}_{\hat{\gamma}(t)} \alpha_\tau(t), \\ &\partial_\tau \partial_\tau \hat{\gamma}(t) = \Delta(\alpha_\tau(t)) \end{aligned}$$

Therefore

$$\partial_\tau \partial_\tau \Omega(t) = \sum_c A^c_{\tau\tau} \partial_c \Omega(t) + d(\alpha_\tau e^{\hat{\gamma}(t)} \vdash \Omega)$$

□

It follows from the condition (1) imposed on $\hat{\gamma}(t)$ that Π_i form a (formal) framing of $\mathcal{T}^*_{\mathbf{H}}$.

Corollary 7.2. *The connection* ∇ *is flat*

Claim 7.3. *The structure constants A_{ab}^c satisfy the potentiality condition (4) in flat coordinates.*

Proof. If one puts symbolically $\nabla = \nabla_0 + A$ then the flatness of ∇ implies that

$$\nabla_0 A + \frac{1}{2}[A, A] = 0$$

Notice that associativity and commutativity of the algebra defined by A_{ab}^c imply that

$$[A, A] = 0.$$

Therefore

$$\nabla_0(A) = 0.$$

□

We have completed the proof of the fact that $A_{ab}^c(t)$ and g_{ab} define the Frobenius structure on $\mathcal{M}_{\mathbf{t}}$ in the flat coordinates $\{t_a\}$.

Remark. In fact one can write an explicit formula for the potential of the Frobenius structure. Let us put $\hat{\gamma}(t) = \sum_a \hat{\gamma}_a t^a + \Delta\alpha(t), \alpha(t) \in (\mathbf{t}\widehat{\otimes} t_{\mathbf{H}}^2 \mathbb{C}[[t_{\mathbf{H}}]])^0$. Put

$$\Phi = \int -\frac{1}{2}\bar{\partial}\alpha \wedge \Delta\alpha + \frac{1}{6}\hat{\gamma} \wedge \hat{\gamma} \wedge \hat{\gamma}$$

Then one checks easily that $A_{abc} = \partial_a \partial_b \partial_c \Phi$ (see Appendix). In the case $\dim_{\mathbb{C}} M = 3$, $\hat{\gamma} \in \mathbf{t}^{-1,1} = \Gamma(M, \overline{T}_M^* \otimes T_M)$ this formula gives the critical value of so called Kodaira-Spencer Lagrangian of [BCOV].

Remark. Define differential Batalin-Vilkovisky algebra as $\mathbb{Z}_2$-graded commutative associative algebra A equipped with odd differentiation $\bar{\partial}$, $\bar{\partial}^2 = 0$ and odd differential operator Δ of order ≤ 2 such that $\Delta^2 = 0$, $\Delta\bar{\partial} + \bar{\partial}\Delta = 0$, $\Delta(1) = 0$. One can use the formula (2.1) to define the structure of $\mathbb{Z}_2$-graded Lie algebra on ΠA. Assume that the operators $\bar{\partial}, \Delta$ satisfy $\partial\bar{\partial}$-lemma (2.3). Assume in addition that A is equipped with a linear functional $\int : A \to \mathbb{C}$ satisfying (4.1) such that the metric defined as in §5 is nodegenerate. Then the same construction as above produces the Frobenius structure on the $\mathbb{Z}_2$-graded moduli space $\mathcal{M}_{\Pi A}$. One can define the tensor product of two such Batalin-Vilkovisky algebras. Operator Δ on $A_1 \otimes A_2$ is given by $\Delta_1 \otimes 1 + 1 \otimes \Delta_2$. Also, $\bar{\partial}$ on $A_1 \otimes A_2$ is $\bar{\partial}_1 \otimes 1 + 1 \otimes \bar{\partial}_2$. It is naturally to expect that the Frobenius manifold corresponding to $A_1 \otimes A_2$ is equal to the tensor product of Frobenius manifolds corresponding to A_1, A_2, defined in [KM] in terms of the corresponding algebras over operad $\{H_*(\overline{M}_{0,n+1})\}$.

8. Scaling transformations

The vector field $E = \sum_a -\frac{1}{2}|\Delta_a| t^a \partial_a$ generates the scaling symmetry on $\mathbf{H}$.

Proposition 8.1. *$\mathcal{L}ie_E A_{abc} = (\frac{1}{2}(|\Delta_a| + |\Delta_b| + |\Delta_c|) + 3 - dim_{\mathbb{C}} M) A_{abc}$*

Proof. $A_{abc} = \int_M \partial_a \hat{\gamma} \wedge \partial_b \hat{\gamma} \wedge \partial_c \hat{\gamma}$. Note that $\int \gamma \neq 0$ implies that $\gamma \in \mathbf{t}_{2n-1}$. The proposition follows from the grading condition on $\hat{\gamma}(t)$. □

Corollary 8.2. $\mathcal{L}ie_E A^c_{ab} = (\frac{1}{2}(|\Delta_a| + |\Delta_b| - |\Delta_c|) + 1)A^c_{ab}$

Note that the vector field E is conformal with respect to the metric g_{ab}. Therefore the proposition 8. 1 shows that E is the Euler vector field of the Frobenius structure on $\mathbf{H}$ (see [M]). Such a vector field is defined uniquely up to a multiplication by a constant. The simplest invariant of Frobenius manifolds is the spectrum of the operator $[E, \cdot]$ acting on infinitesimal generators of translations and the weight of the tensor A^c_{ab} under the $\mathcal{L}ie_E$-action. Usually the normalization of E is chosen so that $[E, \partial_0] = 1$.

In our case the spectrum of $[E, \cdot]$ is equal to

$$\bigcup_d \{1 - d/2\} \text{ with multiplicity} \sum_{q-p=d-n} \mathrm{dim} H^q(M, \Omega^p)$$

Note that this spectrum and the weight of A^c_{ab} coincide identically with the corresponding quantities of the Frobenius structure arising from the Gromov-Witten invariants of the dual Calabi-Yau manifold $\widetilde{M}$.

9.Further developments.

Conjecturally the constructed Frobenius manifold is related to the Gromov-Witten invariants of $\widetilde{M}$ in the following way. One can rephrase the present construction in purely algebraic terms using Čech instead of Dolbeault realization of the simplicial graded Lie algebra $\Lambda^* \mathcal{T}_M$. The only additional "antiholomorphic" ingredient that is used is the choice of a filtration on $H^*(M, \mathbb{C})$ complementary to the Hodge filtration. The Frobenius structure, arising from the limiting weight filtration corresponding to a point with maximal unipotent monodromy on moduli space of complex structures on M, coincides conjecturally with Frobenius structure on $H^*(\widetilde{M}, \mathbb{C})$, obtained from the Gromov-Witten invariants.

Our construction of Frobenius manifold is a particular case of a more general construction. Other cases of this construction include the Frobenius manifold structure on the moduli space of singularities of analytic functions found by K. Saito, the Frobenius manifold structure on the moduli space of "exponents of algebraic functions". The latter case is a Mirror Symmetry partner to the structure arising from Gromov-Witten invariants on Fano varieties. All these cases seem to be related with yet undiscovered generalization of theory of Hodge structures. We hope to return to this in the next paper.

Appendix

Let $\hat{\gamma}(t) = \sum_a \hat{\gamma}_a t^a + \Delta\alpha(t), \alpha(t) \in (\mathbf{t} \otimes t^2_{\mathbf{H}} \mathbf{C}[[t_{\mathbf{H}}]])^1$ be a solution to Maurer-Cartan equation satisfying conditions (1)-(2) of lemma 6.1. Put

$$\Phi = \int -\frac{1}{2}\bar{\partial}\alpha \wedge \Delta\alpha + \frac{1}{6}\hat{\gamma} \wedge \hat{\gamma} \wedge \hat{\gamma}$$

Proposition. $A_{abc} = \partial_a \partial_b \partial_c \Phi$

Proof. . Let $\partial_\tau = \sum_a \tau_a \partial_a$ be an even constant vector field in $\mathbf{H}$. It is enough to prove that $\int(\partial_\tau \hat{\gamma} \wedge \partial_\tau \hat{\gamma} \wedge \partial_\tau \hat{\gamma}) = \partial^3_{\tau\tau\tau}\Phi$. Let us differentiate the terms in Φ

$$\partial^3_{\tau\tau\tau}(\hat{\gamma} \wedge \hat{\gamma} \wedge \hat{\gamma}) = 18\partial^2_{\tau\tau}\hat{\gamma} \wedge \partial_\tau\hat{\gamma} \wedge \hat{\gamma} + 3\partial^3_{\tau\tau\tau}\hat{\gamma} \wedge \hat{\gamma} \wedge \hat{\gamma} + 6\partial_\tau\hat{\gamma} \wedge \partial_\tau\hat{\gamma} \wedge \partial_\tau\hat{\gamma}$$

$$\partial^3_{\tau\tau\tau}(\bar\partial\alpha\wedge\Delta\alpha)=\bar\partial\alpha\wedge(\partial^3_{\tau\tau\tau}\Delta\alpha)+(\partial^3_{\tau\tau\tau}\bar\partial\alpha)\wedge\Delta\alpha+3(\partial_\tau\bar\partial\alpha)\wedge(\partial^2_{\tau\tau}\Delta\alpha)+ \\ +3(\partial^2_{\tau\tau}\bar\partial\alpha)\wedge(\partial_\tau\Delta\alpha) \qquad (*)$$

Notice that

$$\int(\partial^3_{\tau\tau\tau}\bar\partial\alpha)\wedge\Delta\alpha=(-1)^{\deg\bar\partial\alpha+1}\int(\partial^3_{\tau\tau\tau}\Delta\bar\partial\alpha)\wedge\alpha=\int(\partial^3_{\tau\tau\tau}\Delta\bar\partial\alpha)\wedge\alpha=$$
$$=-\int(\partial^3_{\tau\tau\tau}\bar\partial\Delta\alpha)\wedge\alpha=-(-1)^{\deg\Delta\alpha}\int(\partial^3_{\tau\tau\tau}\Delta\alpha)\wedge\bar\partial\alpha=\int(\partial^3_{\tau\tau\tau}\Delta\alpha)\wedge\bar\partial\alpha=$$
$$=(-1)^{(\deg\Delta\alpha+1)(\deg\bar\partial\alpha+1)}\int\bar\partial\alpha\wedge(\partial^3_{\tau\tau\tau}\Delta\alpha)=\int\bar\partial\alpha\wedge(\partial^3_{\tau\tau\tau}\Delta\alpha)$$

Hence, the first two terms in (*) give the same contribution.
We have

$$\int\bar\partial\alpha\wedge(\partial^3_{\tau\tau\tau}\Delta\alpha)=(-1)^{\deg\bar\partial\alpha+1}\int\Delta\bar\partial\alpha\wedge(\partial^3_{\tau\tau\tau}\alpha)=\int\Delta\bar\partial\alpha\wedge(\partial^3_{\tau\tau\tau}\alpha)=$$
$$=-\int\bar\partial\Delta\alpha\wedge(\partial^3_{\tau\tau\tau}\alpha)=-\int\bar\partial\hat\gamma\wedge(\partial^3_{\tau\tau\tau}\alpha)=\frac{1}{2}\int[\hat\gamma,\hat\gamma]\wedge(\partial^3_{\tau\tau\tau}\alpha)=$$
$$=\frac{1}{2}\int\Delta(\hat\gamma\wedge\hat\gamma)\wedge(\partial^3_{\tau\tau\tau}\alpha)=(-1)^{(\deg(\hat\gamma\wedge\hat\gamma)+1)}\frac{1}{2}\int(\hat\gamma\wedge\hat\gamma)\wedge(\partial^3_{\tau\tau\tau}\Delta\alpha)=$$
$$=\frac{1}{2}\int(\hat\gamma\wedge\hat\gamma)\wedge\partial^3_{\tau\tau\tau}\hat\gamma$$

Similarly,

$$\int(\partial_\tau\bar\partial\alpha)\wedge(\partial^2_{\tau\tau}\Delta\alpha)=\frac{1}{2}\int\partial_\tau(\hat\gamma\wedge\hat\gamma)\wedge\partial^2_{\tau\tau}\hat\gamma$$
$$\int(\partial^2_{\tau\tau}\bar\partial\alpha)\wedge(\partial_\tau\Delta\alpha)=\frac{1}{2}\int(\partial^2_{\tau\tau}\hat\gamma)\wedge\partial_\tau(\hat\gamma\wedge\hat\gamma)$$

□

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