Noncommutative motives

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Abstract

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Plan:

- Noncommutative algebraic geometry
- Examples of saturated spaces
- Hodge and de Rham cohomology
- NC pure Hodge structures pure and mixed motives over $\mathbb{C}$
- $\mathbb{Z}_p$-case; Frobenius isomorphism, Euler factors, L-functions

1 Basic "derived" noncommutative algebraic geometry

Definition. A **noncommutative space** X is a small triangulated category $\mathcal{C}_X$, which is Karoubi closed (= every projector splits) and appropriately enriched either

- by **spectra**: $\mathrm{Hom}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F}[i]) = \pi_{-i}\, \mathbf{Hom}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F})$, or
- by **complexes** of $\mathbf{k}$-vector spaces: $\mathrm{Hom}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F}[i]) = H^i(\mathbf{Hom}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F}))$. Here X is $\mathbf{k}$-linear, where $\mathbf{k}$ is a field, so we write $X/\mathbf{k}$

Remark. One can define X/R for every commutative ring R. In that case, we rather enrich over complexes of R-modules which are *cofibrant*.

Definition. $X/\mathbf{k}$ is **algebraic** if for every dg-algebra $A/\mathbf{k}$ such that $\mathcal{C}_X$ is equivalent (in enriched sense) to the category $\mathrm{Perf}(A - mod)$. By definition, $\mathrm{Perf}(A - mod)$ is the closure of one-object full subcategory $\{A\}$ by shifts, cones and direct summands in appropriate triangulated category $A - mod$.

Theorem. (Bondal-van den Bergh) If $X/\mathbf{k}$ is a scheme of finite type, then X is algebraic in noncommutative sense. Here, by definition, X is replaced by $\mathcal{C}_X := \mathrm{Perf}(X)$, the category of perfect complexes of quasicoherent sheaves. $\mathcal{C}_X$ has a split-generator $\mathcal{E}$, and $A = \mathrm{RHom}(\mathcal{E}, \mathcal{E})^{\mathrm{op}}$.

Example. (A. Beilinson) $X = \mathbb{P}^n_{\mathbf{k}}$
$A = \mathrm{End}(\mathcal{O}(0) \oplus \ldots \oplus \mathcal{O}(n))^{\mathrm{op}}$
$D^b(\mathrm{Coh}X) = \mathrm{Perf}(X) = D^b(\text{fin.gen. } A - mod) = \mathrm{Perf}(A - mod)$

Definition. *Algebraic noncommutative space $X/\mathbf{k}$ is*

- **proper** *if* $\sum_{i\in\mathbb{Z}} \mathrm{rk}\, H^i(A, d) < +\infty$
- **smooth** *if* $A \in \mathrm{Perf}(A \otimes A^{\mathrm{op}} - mod)$

Theorem. *The notions of properness and smoothness of noncommutative spaces do not depend on the choice of generator A, and they coincide with the usual properness and smoothness for schemes of finite type.*

Examples of algebras A (in degree 0) such that $''\,\mathrm{Spec}\, A''$, where $\mathcal{C}_{''\,\mathrm{Spec}\, A''} = \mathrm{Perf}(A - mod)$, is smooth:

- $\mathcal{O}(X)$ where X is smooth affine scheme over bfk
- $T(V) = \oplus_{n\geq 0} V^{\otimes n}$, $\mathrm{rk}\, V < \infty$ (free finitely generated algebra)
- $\mathfrak{U}_q\mathfrak{g}$ Drinfeld-Jimbo quantized enveloping algebra

Finiteness for sheaves:

If $X/\mathbf{k}$ is **proper** then $\forall \mathcal{E}, \mathcal{F} \in \mathcal{C}_X$, such that $\sum_{i\in\mathbb{Z}} \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i]) < +\infty$, there is a bilinear form $\chi_{\mathrm{RHom}} : K_0(\mathcal{C}_X) \otimes K_0(\mathcal{C}_X) \to \mathbb{Z}$ (which is neither symmetric nor skew-symmetric).

– a noncommutative version of Riemann-Roch theorem

We have a correspondence

$$\{\text{Objects in } \mathcal{C}_X/\text{iso}\} \leftrightarrow \begin{array}{c}\mathbf{k}\text{-points in a}\\ \coprod_{\text{countable}}(\mathbf{k}\text{-schemes of finite type})/\sim\end{array}$$

where $\sim$ is an equivalence relation of a similar nature.

Finiteness for spaces

$$\{\text{smooth proper } X/\mathbf{k} \text{ /equiv. of cats.} \mathcal{C}_X \sim \mathcal{C}_{X'}\} \leftrightarrow \mathbf{k}\text{-points in...}$$

Definition. A noncommutative space is said to be **saturated** if it is **smooth** and **proper**. (The name comes from *saturated categories* of Bondal and Kapranov)

Manipulations with saturated spaces

$X \mapsto X^{\text{op}}$ is given by $\mathcal{C}_{X^{\text{op}}} := \mathcal{C}_X^{\text{op}}$, $A_{X^{\text{op}}} := A_X^{\text{op}}$
$X, Y \mapsto X \otimes Y$ is given by $A_{X\otimes Y} = A_X \otimes_{\mathbf{k}} A_Y$
$X, Y \mapsto \mathbf{Maps}(X, Y) := X^{\text{op}} \otimes Y$ where $\mathcal{C}_{\mathbf{Maps}(X,Y)} := \text{Funct}(\mathcal{C}_X \to \mathcal{C}_Y)$

Glueing $f : X \to Y \mapsto (X \xrightarrow{f} Y)$ where f is given by a bimodule $M \in A_Y - mod - A_X$ and $A_{(X \xrightarrow{f} Y)} = \begin{pmatrix} A_X & 0 \\ M & A_Y \end{pmatrix}$. Glueing is analogous to cones in triangulated categories. $\mathbb{P}^n_{\mathbf{k}}$ is glued iteratively from $(n + 1)$ points. Braid group acts on { iterated glueings }.

Duality theory

For every saturated X there is a canonical Serre functor $S_X \in \mathbf{Maps}(X, X)$
$\boxed{\text{Hom}(\mathcal{E}, \mathcal{F})^* = \text{Hom}(\mathcal{F}, S_X(\mathcal{E}))}$ (in schematic case $S_X := K_X[\dim X]\otimes$).

2 Examples of saturated spaces

2.1

- smooth proper schemes
- smooth proper algebraic spaces
- smooth proper Deligne-Mumford stacks – particularly those which are locally crossed products $\mathbf{k}[\Gamma]\#\mathcal{O}_X$, where Γ is a finite group acting on X and $\text{char}\,\mathbf{k} = 0$
- (X, α) where $X/\mathbf{k}$ is a smooth proper scheme, and $\alpha \in \text{Br}(X)$ is a class of Azumaya algebra $\mathcal{A}/X$. In that case, $C_{(X,\alpha)} := \text{Perf}(\mathcal{A} - mod)$
- deformation quantization of smooth projective variety $X/\mathbf{k}$, $\text{char}\,\mathbf{k} = 0$. Here the following data: ample line bundle $\mathcal{L} = \mathcal{O}(\infty) \to \mathcal{X}$; homogeneous Poisson structure $\gamma \in \Gamma(\mathcal{L} - X, \Lambda^2 T_{\mathcal{L}})^{\mathbb{G}_m}$ – under the assumption $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ – give rise to a quantized space $X_q/\mathbf{k}((\hbar))$ with the star-product $f \star g = fg + \hbar\langle\gamma, df \otimes dg\rangle + \ldots$.

  Subexamples are Feigin-Odesski "elliptic" projective spaces, quantized del Pezzo surfaces etc.
- Artin-Zhang noncommutative projective spaces

2.2 Landau-Ginzburg models

(name comes from topological B-strings)

Definition. A $\mathbb{Z}/2\mathbb{Z}$-graded noncommutative space X is $\mathcal{C}_X$ together with an isomorphism $[0] \sim [2]$. The notions of algebraic, smooth and proper noncommutative space extend to the $\mathbb{Z}/2\mathbb{Z}$-graded case.

Suppose we are given a smooth scheme X over $\mathbf{k}$ and $f : X \to \mathbb{A}^1$ (or view as $f \in \mathcal{O}(X)$). This datum gives rise to a $\mathbb{Z}/2\mathbb{Z}$-graded space (X, f).

Locally, $\mathcal{C}_{(X,f)}$ is a category of supervector bundles $\mathcal{E} = \mathcal{E}^0 \oplus \mathcal{E}^1$ with a "differential" $d_{\mathcal{E}} \in \mathrm{End}(\mathcal{E})^{\mathrm{odd}}$, $d_{\mathcal{E}}^2 = f \cdot \mathrm{Id}_{\mathcal{E}}$. The inner homs are given as follows:

$$\mathbf{Hom}((\mathcal{E}, d_{\mathcal{E}}), (\mathcal{F}, d_{\mathcal{F}})) := \begin{cases} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}), \text{ with differential } d = d_{(\mathcal{E},\mathcal{F})} \\ d\phi = \phi \circ d_{\mathcal{E}} - d_{\mathcal{F}} \circ \phi, d^2 = 0 \end{cases}$$

Globally (D. Orlov) Assume $f \not\equiv 0$ at each component of X. Then $\mathcal{C}_{(X,f)} := D^b(\mathrm{Coh}Z)/\mathrm{Perf}Z$, where $Z = f^{-1}(0)$.

We expect that $\mathcal{C}_{(X,f)}$ is saturated whenever $X_0 := \mathrm{Crit}(f) \cap f^{-1}(0)$ is proper. Moreover, X can be a formal smooth thickening of X_0.

Example. f – a germ of an analytic function in $\mathbb{C}^n$, with $f(0) = 0$ and isolated singularity.

3 (Co)homology theories

In this section A is a unital associative algebra over a field $\mathbf{k}$,

Definition. Homological Hochschild complex

$$\begin{array}{ccccccc} & & -2 & & -1 & & 0 \\ C_\bullet(A, A) = \ldots \to & & A \otimes A \otimes A & \xrightarrow{\partial} & A \otimes A & \xrightarrow{\partial} & A \end{array}$$

(the top row shows the degrees) where $\partial(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots a_i a_{i+1} \otimes \ldots a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}$

Analogously one defines the reduced Hochschild complex $C_\bullet^{\mathrm{red}}(A, A)$. This is certain quotient complex of the Hochschild complex, which is actually quasiisomorphic to $C_\bullet(A, A)$.

$$\begin{array}{ccccccc} & & -2 & & -1 & & 0 \\ C_\bullet^{\mathrm{red}}(A, A) = \ldots \to & & A \otimes A/1 \otimes A/1 & \xrightarrow{\partial} & A \otimes A/1 & \xrightarrow{\partial} & A \end{array}$$

Theorem. (Hochschild-Kostant-Rosenberg) For $A = \mathcal{O}(X)$, where X is a *smooth* affine variety over $\mathbf{k}$

$$\boxed{H^{-i}(C_\bullet(A,A)) = \Omega^i(X/\mathbf{k})}$$

Theorem. (Charles Weibel [10], in other formulation) For a smooth scheme X over $\mathbf{k}$ where $\operatorname{char}\mathbf{k} = 0$ or $\operatorname{char}\mathbf{k} > \dim X$,

$$\boxed{H^n(C_\bullet(A,A)) = \oplus_{i-j=n} H^i(X, \Omega^j)}$$

Definition. For every *algebraic* noncommutative space X Hodge cohomology $H^\bullet_{\mathrm{Hodge}}(X)$ is simply the Hochschild homology $H_\bullet(A,A)$.

There is also an intrinsic definition in terms of $\mathcal{C}_X$. For saturated X

$$\boxed{H^\bullet_{\mathrm{Hodge}} = \mathrm{RHom}_{\mathbf{Maps}(X,X)}(\mathrm{Id}_X, S_X)}$$

Definition. Connes' operator B acting on $C^{\mathrm{red}}_\bullet(A,A)$, of degree -1, is given by

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^{n_i} 1_A \otimes a_i \otimes a_{i+1} \otimes \ldots \otimes a_{i-1}$$

(here $a_i \otimes a_{i+1} \otimes \ldots \otimes a_{i-1}$ is obtained by a cyclic permutation of tensor factors in $a_1 \otimes \ldots \otimes a_n$). The following holds: $B^2 = 0$, $B\partial + \partial B = 0$.

In the case $A = \mathcal{O}(X)$, where X is a smooth affine scheme over $\mathbf{k}$, B induces the *de Rham differential* on $\Omega^{-\bullet} = H^\bullet_{\mathrm{Hodge}}(X)$.

Everything generalizes to dg-algebras and to the $\mathbb{Z}/2\mathbb{Z}$-graded case. For example, we have $C_\bullet(A,A) = \oplus_{n\geq 0} A \otimes A[1]^{\otimes n}$ etc.

Definition. Periodic cyclic cohomology for algebraic noncommutative space $X/\mathbf{k}$ is given by

$$HP^\bullet(X) := HP_\bullet(A) := H^\bullet(C^{\mathrm{red}}_\bullet(A,A)((u)), \partial + uB)$$

Here $\partial + uB$ is the differential, $C^{\mathrm{red}}_\bullet(A,A)((u))$ and $HP^\bullet(X)$ are $\mathbf{k}((u))$-modules, where u is an even variable, with $\deg u = +2$ in $\mathbb{Z}$-graded case. Furthermore, in $\mathbb{Z}$-graded case, $HP^i(X) = HP^{i+2}(X)$ for all i, hence it gives rise to a $\mathbb{Z}/2\mathbb{Z}$-graded space over $\mathbf{k}$. Now $HP(X) = HP^{\mathrm{even}}(X) \oplus HP^{\mathrm{odd}}(X)$; if $X/\mathbf{k}$ is smooth and if either $\operatorname{char}\mathbf{k} = 0$ or $\operatorname{char}\mathbf{k} > \dim X$, then

$$HP^{\mathrm{even}}(X) = \oplus H^{2i}_{\mathrm{dR}}(X), \qquad HP^{\mathrm{odd}}(X) = \oplus H^{2i+1}_{\mathrm{dR}}(X).$$

HP is a noncommutative analog of de Rham cohomology.

Finiteness: for saturated $\mathbf{Z}$-graded or $\mathbf{Z}/2\mathbf{Z}$-graded X, we have

$$+\infty > \mathrm{rk}_{\mathrm{total}}\, H^\bullet_{\mathrm{Hodge}}(X)/\mathbf{k} \geq \mathrm{rk}\, HP^\bullet(X)/\mathbf{k}((u)) \geq 0$$

Definition. For algebraic noncommutative space $X/\mathbf{k}$ the spectral sequence **Hodge $\Rightarrow$ de Rham collapses** at E_1 if $\forall n \geq 1$ $n < +\infty$

$$H^\bullet(C^{\mathrm{red}}_\bullet(A,A)[u]/(u^n), \partial + uB)$$

is free (= flat) $\mathbf{k}[u]/(u^n)$-module,

For *saturated* X this is equivalent to the statement

$$\mathrm{rk}_{\mathrm{total}}\, H^\bullet_{\mathrm{Hodge}}(X)/\mathbf{k} = \mathrm{rk}\, HP^\bullet(X)/\mathbf{k}((u))$$

Conjecture. *For saturated X over a field $\mathbf{k}$, with* $\mathrm{char}\,\mathbf{k} = 0$ *Hodge $\Rightarrow$ de Rham collapses.*

This is true for schemes, quantum deformations, stacks, Azumaya algebras, (X, f) Landau-Ginzburg models.

In the commutative case there are 2 types of proofs: those using Kähler geometry and those which are finite characteristics (Deligne-Illusie) or p-adic (Faltings). There is a good chance in noncommutative case (discussed below).

Assume conjecture

Then we have a super-vector bundle H_u over $\mathbf{k}[[u]]$ with Sections $= H^\bullet(C^{\mathrm{red}}_\bullet(A, A)[[u]], \partial + uB)$.

Furthermore, there is a canonical connection ∇ on H_u, $u \neq 0$:

- in $\mathbb{Z}$-graded case: it comes from $\mathbb{G}_m$-equivariance $\lambda \in \mathbf{k}^\times$, $u \mapsto \lambda^2 u$; the monodromy, which is equal id on HP^{even} and id on HP^{odd} has 1st order pole at $u = 0$. This is equivalent to a filtration by $\frac{1}{2}\mathbf{Z}$ on $H^\bullet_{\mathrm{dR}}(X)$. In the case of schemes, $\boxed{F_q H^n_{\mathrm{dR}}(X) = \oplus_{n/2-p=a} F^p H^n_{\mathrm{dR}}(X)}$

- in $\mathbb{Z}/2\mathbb{Z}$-graded case: it comes from Gauβ-Manin connection on $HP^\bullet(X_\lambda)$ where $(X_\lambda)_{\lambda \in \mathbb{G}_m}$ is the orbit of RG (renormalization group) flow acting on $\{\mathbb{Z}/2\mathbb{Z} - \text{graded spaces }\}$.

  In $\mathbb{Z}/2\mathbb{Z}$-graded case the connection ∇ has a second order pole at $u = 0$ (this follows from an explicit formula), still with regular (?) singularity and with quasi-unipotent (?) monodromy.

  $A' = A_\lambda = A$ as a space over $\mathbf{k}$; $a \cdot' b = ab$, $d'(a) = \lambda da$.

We obtain here

$$(,) : H_u \otimes H_{-u} \to \mathbf{k} \tag{1}$$

– a non-degenerate ∇-covariant pairing (neither symmetric nor antisymmetric).

Example. When (X, f) is a Landau-Ginzburg model, with $\mathrm{Crit}(f) \cap f^{-1}(0)$ proper,

$$\boxed{\Gamma(\mathbf{k}[[u]], H_u) = \mathbb{H}^\bullet(X_{\mathrm{Zar}}, \Omega^\bullet_{X/\mathbf{k}}[[u]],\ \text{differential } u \cdot d_{\mathrm{dR}} + df\wedge)}$$

$$= \mathbb{H}(X_{\mathrm{Zar}}, e^{f/u}\Omega^\bullet_{X/\mathbf{k}}[[u]], ud_{\mathrm{dR}}).$$

In this case, for the degeneration of Hodge $\Rightarrow$ de Rham spectral sequence, there are 3 proofs

1) S. Barannikov and M. K. using harmonic theory for $e^{f/u}$
2) C. Sabbah, using M. Saito's Hodge modules
3) V. Vologodsky, A. Ogus, proof *a la* Deligne-Illusie

An application of collapse Hodge $\Rightarrow$ de Rham

Construction:
- algebraic B-model
- generalization of Deligne's conjecture on cohomological operations

INPUT: Saturated $\mathbb{Z}/2\mathbb{Z}$-graded NC space X such that

1) Hodge $\Rightarrow$ de Rham s.s. collapses

2) X is even or odd Calabi-Yau; there is an isomorphism $S_X \sim \mathrm{Id}_X$ or $S_X \sim \prod \mathrm{Id}_X$

\+ some choices:

1') trivialization of bundle H_u compatible with the pairing $(,)$ from (1),

2') choice of isomorphism $S_X \sim \mathrm{Id}_X$ or $S_X \sim \prod \mathrm{Id}_X$ with "higher homotopies"; this is equivalent to some purely cohomological data $\in \Gamma(\mathbf{k}[[u]], H_u)$ satisfying some non-degeneracy.

OUTPUT: Cohomological 2dTQFT in the sense M.K.-Yu.Manin.

$H := H^\bullet_{\mathrm{Hodge}}(X)$ $\forall g, n \geq 0$ $2-2g-n<0$ $H^{\otimes n} \to H^\bullet_{\mathrm{Betti}}(\overline{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbf{k})$.

4 Noncommutative pure Hodge strutcures, $\mathbf{k} = \mathbb{C}$

Pre-Definition. (putative) A noncommutative pure Hodge struture is given by

- (H_u) holomorphic super vector bundle over $\{u \in \mathbb{C} \,|\, |u| << 1\}$
- ∇ – flat connection on $u \neq 0$ with the second order pole at $u = 0$ and with regular singularity
- K_u^{top} – a local system over $u \neq 0$ of finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups, together with a ∇-flat isomorphism of super spaces over $\mathbb{C}$: $K_u^{\mathrm{top}} \otimes \mathbb{C} \simeq H_u$.

MAIN PROBLEM: How should we define lattice K_u^{top} ? The answer is clear in (almost)-commutative examples, e.g. in LG model.

Vague idea (in general $\mathbb{Z}$-graded case): $\exists$ (?) another "algebraic" noncommutative space, with nuclear (?) algebra A' together with a map $\phi : X' \to X$ such that

1) ϕ induces an isomorphism $HP^\bullet(X) \xrightarrow{\sim} HP^\bullet_{\text{cont}}(X)$

2) K-theory of X' has Bott perioditicty $K_i(X') \simeq K_{i+2}(X')$

3) $\forall i \geq 0$ Chern character ch : $K_i(X_i) \to HP^i_{\text{cont}}(X')$ induces an isomorphism $K_i(X') \otimes \mathbb{C} \simeq HP^i_{\text{cont}}(X')$

E.g. for a smooth proper scheme $X/\mathbb{C}$ we can take $A' := C^\infty_{\mathbb{C}}(X(\mathbb{C}))$.

Fact: for any C^∞-manifold X

$HP^\bullet_{\text{cont}}(X) \simeq H^\bullet_{\text{dR}}(X)$

image of $K_0(X)$ in $HP^{\text{even}}_{\text{cont}}(X)$ is (up to finite torsion) $\oplus_{n\in\mathbb{Z}}(2\pi\sqrt{-1})^n \cdot H^{2n}(X, \mathbb{Z})$

image of $K_1(X)$ in $HP^{\text{odd}}_{\text{cont}}(X)$ divided by $\sqrt{2\pi\sqrt{-1}}$ is (up to finite torsion) $\oplus_{n\in\frac{1}{2}+\mathbb{Z}}(2\pi\sqrt{-1})^n H^{2n}(X, \mathbb{Z})$

Hodge conjecture: For saturated $\mathbb{Z}/2\mathbb{Z}$-graded noncommutative space

$\mathbb{Q}\otimes$ (image of $K_0(\mathcal{C}_X)$ in $\Gamma(\mathbb{C}[[u]], H_u)$ by Chern character)

$= \mathbb{Q} \otimes \text{Hom}_{\text{NC pure Hodge str.}}(\mathbf{1}, H^\bullet(X))$;

here $H^\bullet(X)$ is equipped with the structure coming from the formal bundle H_u *canonically* extended to $|u| << 1$, because of regular singularity + putative lattice K^{top}_u.

Theorem (L. Katzarkov, M. K.) *For LG model (X, f) this "Hodge conjecture" follows from the usual Hodge conjecture.*

Polarizations

Definition. A poolarization of a noncommutative Hodge structure $H = (H_u, \nabla, K^{\text{top}}_u)$ at radius $r.0$, $r \in \mathbb{R}$ is an isomorphism $\Psi : H \xrightarrow{\sim} (H^{\text{op}})^\vee$ of noncommutative Hodge structures satisfying certain symmetry and positivity condition.

The operation $(H^{\text{op}}_u)^\vee = H^\vee_{-u}$ corresponds to $X \mapsto X^{\text{op}}$.

Suppose we are given the following data:

holomorphic vector bundle $\begin{array}{c}\mathcal{H}\\ \downarrow\\ \mathbb{C}P^1\end{array}$

holomorphic pairing $\psi_{\mathcal{H}} : \mathcal{H}_u \otimes \mathcal{H}_{\sigma(u)} \to \mathbb{C}$, where $\sigma(u) = -\frac{r^2}{\bar{u}}$ such that

1) $\mathcal{H}$ is holomorphically trivial $\mathcal{H} \cong \oplus\mathcal{O}$

2) $\psi_{\mathcal{H}}$ induces positive Hermitean form on $\Gamma(\mathbb{C}P^1, \mathcal{H})$

Such $\mathcal{H}$ can be constructed from H and ψ

$$\mathcal{H}|_{|u|\leq r} \simeq H|_{|u|\leq r}$$

pairing $\psi_{\mathcal{H}}|_{S^1:|u|=r}$ is given by ψ composed with complex conjugation $u \neq 0 : H_u \to \bar{H}_u$, associated with $\mathbb{R}$-structure $K_u^{\rm top} \otimes \mathbb{R} \subset H_u$.

Recent results of C. Sabbah imply that in LG model there is a polarization for all sufficiently small r.

Conjecture. *For saturated $X/\mathbb{C}$ polarizations on $H^\bullet(X)$ exist.*

They should come from certain endofunctors $F : X \to X$ (as $\mathrm{ch}(F)$); presumably F is something like $\otimes\mathcal{O}(n)$, $n >> 1$.

If there exist a polarization on $H^\bullet(X)$ then the image of $K_0(X)$ in $H^\bullet(X)$ equals $K_0(X)$ modulo the numerical equivalence; in this setup this is defined to be the kernel of the canonical pairing $\chi_{\mathrm{RHom}} : K_0(X) \otimes K_0(X) \to \mathbb{Z}$. Here, using S_X, one observes that it is irrelevant wheather we take the kernel in left or right factor.

Definition. *For every field* **k***, the category of pure motives over* **k** *is the Karoubi envelope of "effective motives".*

Objects: saturated $X/\mathbf{k}$

$$\mathbf{Hom}_{EM}(X,Y) = \mathbb{Q} \otimes K_0(\mathbf{Maps}(X,Y))/\text{numerical equivalence}$$

Previous conjectures $\Rightarrow$ noncommutative pure motives over a field of $\mathrm{char}\,\mathbf{k} = 0$ is a *semisimple rigid tensor category.*

Corollary. *(By Tannakian reconstruction) we get a pro-reductive group over $\mathbb{Q}$.*

The noncommutative motivic Galois group such that there is a surjective map $G_{\rm mot}^{\rm NC} \xrightarrow{\neq} \mathrm{Ker}(G_{\rm mot} \to GL(1))$ where $G_{\rm mot}$ is the usual (pure) motivic Galois group and the map in brackets is the Tate motive representation.

There are interesting things in the kernel, e.g. the G. Anderson's "t-motives" ([1])

$H = \oplus H^{pq}_{p+q \in \mathbb{Z}, p,q \in \mathbb{Q}}$ for $\Gamma(t)$, $t \in \mathbb{Q}, \ldots$

Definition. *Triangulated category of noncommutative mixed motives over* **k** *:= triangulated + Karoubi envelope of category enriched over spectra*

Objects = saturated $X/\mathbf{k}$

Morphism spaces

$$\boldsymbol{Hom}(X,Y) = K\text{-theory spectrum of category } \mathcal{C}_{\boldsymbol{Maps}(X,Y)}$$

5 Frobenius isomorphism

Conjecture: *For every saturated noncommutative space $X/\mathbb{Z}_p$ $\exists$ canonical Frobenius isomorphism*

$$\boxed{H^\bullet(C^{\rm red}_\bullet((u)), \partial + uB) \sim H^\bullet(C^{\rm red}_\bullet((u)), \partial \pm puB)}$$

of $\mathbb{Z}_p((u))$*-modules, preserving the connection* ∇*.*

Using holonomy of ∇ (it is not enturely canonical if the monodromy $\neq$ id) from u to pu we get operator Fr_p with coefficients in $\mathbb{Q}_p$.

Weil conjecture: *Let* (λ_a) *be the eigenvalues of* Fr_p *are algebraic* $\subset \bar{Q} \subset \bar{Q}_p$*, and* $\forall l \neq p$ $|\lambda_a|_l = 1$*,* $\lambda_a|_{\mathbb{C}} = 1$*.*

Example. $(X, f) = (\mathbb{A}^1, x^2)$. Frobenius comes from the intertwining operator $\cdot \exp(f + \frac{f^p}{p})$; $\dim H(X, f) = 1$, $\mathrm{Fr}_p = \lambda \in \mathbb{Q}_p^\times$. In fact, $\lambda \in \mathbb{Z}_p^\times$

?? $\lambda = \left(\frac{p-1}{2}\right)! \bmod p\mathbb{Z}_p$, $\lambda^4 = 1$

Why I am optimistic

Observation. (D. Kaledin, Spring 2005) For every associative algebra $A/\mathbb{Z}/p\mathbb{Z}$ $\exists$ a $\mathbf{k}$-linear endomorphism of $H_0(A, A)$ given by $[a] \mapsto [a^p]$, where $H_0(A, A) = A/[A, A]$. Moreover, it lifts to a map $H_0(A, A) \to HP_0(A)$. Here $[a] \mapsto$ finite sum which is of the form

$$a^p + \sum_{i_1 + \ldots + i_n = p} c_{i_1, \ldots, i_n} a^{i_0} \otimes \ldots a^{i_n} \cdot u^{\frac{n-1}{2}}, \quad p > 2,$$

$$a^2 + 1 \otimes a \otimes a \cdot u, \quad p = 2.$$

The last term, when $p \geq 3$, is $\left(\frac{p-1}{2}\right) a \otimes \ldots \otimes a \cdot u^{\frac{p-1}{2}} \neq 0$.

Conjecture. *For saturated* $X/\mathbb{Z}/p\mathbb{Z}$*,* $H^\bullet(C^{\rm red}_\bullet[u], \partial + uB)$ *is a* coherent $\mathbb{Z}/p\mathbb{Z}[u]$*-module.*

This is completely opposite to the case $\mathrm{char}\,\mathbf{k} = 0$:

$H^\bullet(C^{\rm red}_\bullet[u, u^{-1}, \partial + uB) = 0$,

$H^\bullet(C^{\rm red}_\bullet[u], \partial + uB)$ is ∞ torsion module over $u = 0$.

Conjecture. *Let* $A/\mathbb{Z}$ (no finiteness condition!) *dg-algebra which is flat over p; denote* $A_0 := A \otimes \mathbb{Z}/p\mathbb{Z}$ *then there is a canonical isomorphism*

$$H^\bullet(C^{\rm red}_\bullet(A_0, A_0)[u, u^{-1}], \partial + uB) \cong H^\bullet(C^{\rm red}_\bullet(A_0, A_0)[u, u^{-1}], \partial)$$

of $\mathbb{Z}/p\mathbb{Z}[u, u^{-1}]$*-modules.* The two above conjectures together imply the degeneration Hodge $\Rightarrow$ de Rham s.s.

D. Kaledin anounced the proof of the degeneration. Now paper [5].

Reason in favour of the second conjecture: use increasing filtration on $C^{\rm red}_\bullet(A, A)$:

$$\mathrm{Fil}_{\leq n} := A \otimes (A/1)^{\otimes \leq (n-1)} \oplus a \otimes (A/1)^{\otimes n}$$

On associated graded module gr for this filtration we get

$$\partial + B \ : \qquad V^{\otimes n} \underset{1+\sigma+\ldots+\sigma^{n-1}}{\overset{1-\sigma}{\rightleftarrows}} V^{\otimes n}$$

$$\partial \ : \qquad V^{\otimes n} \xrightarrow{1-\sigma} V^{\otimes n}$$

where σ is the generator of $\mathbb{Z}/n\mathbb{Z}$ and $V = H^\bullet(A/1)$

$(\mathrm{gr}, \partial + B)$ is acyclic complex if $(n,p) = 1$

if $n = kp$, *canonically* we have a quasiisomorphism

(gr, ∂) in degree $k = \frac{n}{p} : V^{\otimes k} \to V^{\otimes k}$

Works $\forall$ free $\mathbb{Z}/p^l\mathbb{Z}$-module V also.

L-functions

If monodromy equals $(-1)^{\text{parity}}$ (e.g. $\mathbb{Z}$-graded case) then the L-factors for HP^{odd}, HP^{even} are the usual $L_p(s)$ normalized to have the eigenvalues of Frobenius Fr_p in $U(1)$.

Shift $s \mapsto s - \frac{\text{weight}}{2}$

Beilinson conjectures: multiplicity of zero and leading term

$K_0^{(0)} := \mathrm{Ker}(\text{ numerical } \sim)$

$L^{\text{even}}(s)$ (picture)

$L^{\text{odd}}(s)$ (picture)

It is quite possible that all noncommutative motives come from commutative schemes X (in $\mathbb{Z}$-graded case) and LG models (X, f) (in $\mathbb{Z}/2\mathbb{Z}$-graded case). Still they have a potential use in Langlands correspondence

? $H^i(GL(n,\mathbb{Z}))$ related to natural NC spaces ??

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