A short proof of an assertion of Thurston concerning convex hulls.

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Abstract: Let X be a closed subset of the ideal boundary $\partial_\infty\mathbb{H}^3$ of 3-dimensional hyperbolic space $\mathbb{H}^3$ and let K be its convex hull in $\mathbb{H}^3$. We provide a short proof of the fact that the set-theoretic boundary ∂K of K is intrinsically hyperbolic.

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1 - Introduction. In this paper, we study the intrinsic geometries of the boundaries of convex hulls in space-forms. For m a positive integer and $\kappa \in \{-1, 0, 1\}$, let M^m_κ denote the m-dimensional space-form of constant sectional curvature equal to κ. We will say that a closed subset K of M^m_κ is *convex* whenever any length-minimising geodesic arc whose two extremities lie in K is also wholly contained in K. Observe that, in the positive-curvature case, with convexity defined in this manner, a convex subset of M^m_1 is either contained in an open hemisphere or is equal to the whole of M^m_1. Finally, given a closed subset X of M^m_κ, we define its *convex hull*, denoted by $\mathrm{Conv}(X)$, to be the intersection of all closed, convex subsets of M^m_κ containing X.

Let K now be a convex subset with non-trivial interior of some space-form M^m_κ. Let ∂K denote its set-theoretic boundary. The intrinsic metric (distance function) of ∂K is defined by

$$d(x, y) := \operatorname*{Inf}_{\gamma} \mathrm{Length}(\gamma),$$

where γ varies over all rectifiable curves $\gamma : [0, 1] \to \partial K$ with $\gamma(0) = x$ and $\gamma(1) = y$. Since ∂K is everywhere locally a Lipschitz graph (see, for example, Theorem 4.12 of [7]), the topology generated over this subset by d coincides with the topology that it inherits from M^m_κ.

Suppose now that K is the convex hull of some closed subset X. In this case, the set $\partial K \setminus X$ is known to satisfy at the every point x the *local geodesic property* (c.f. Section 4.5 of [7] and Chapter 8 of [8]), namely, that there exists an *open* geodesic segment $\gamma :]-\epsilon, \epsilon[\to \partial K \setminus X$ such that $\gamma(0) = x$. Furthermore, this property characterises convex hulls (see Theorem 4.18 of [7]).

Having established these preliminaries, we now consider the case where the ambient space is 3-dimensional, so that ∂K is 2-dimensional. If $\partial K \setminus X$ were smooth, then the local geodesic property would make this surface extrinsically flat, and thus instrinsically everywhere locally isometric to M^2_κ. In Chapter 8 of [8], Thurston argues heuristically to show that this property continues to hold even in the non-smooth case. He then explains, furthermore, that the canonical embedding of $\partial K \setminus X$ into M^3_κ is totally geodesic except over a singular set given by the union of disjoint geodesics. These observations play a key role in Thurston's approach to Teichmüller theory by providing a bridge between hyperbolic geometry, on the one hand, and the theory of measured geodesic laminations,

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on the other. In this paper, we provide a new proof of Thurston's result which is both shorter and more direct than those currently available in the literature (see [3] and [6]).

We show

Theorem 1.1

Let X be a closed subset of M^3_κ. If Conv(X) has non-trivial interior, then $\Sigma := \partial\text{Conv}(X)\setminus X$ is everywhere locally isometric to M^2_κ. Furthermore, the canonical embedding of Σ into M^3_κ is totally geodesic except over a closed set which is a union of disjoint geodesic segments.

When $\kappa = -1$, M^3_κ is 3-dimensional hyperbolic space $\mathbb{H}^3$. Recall that the ideal boundary $\partial_\infty\mathbb{H}^3$ of $\mathbb{H}^3$ is defined to be the space of equivalence classes of oriented geodesic rays (see [1]). A more useful description of the ideal boundary for our current applications is given by the Kleinian parametrisation, which maps $\mathbb{H}^3$ onto the open unit ball $B_1(0)$ in $\mathbb{R}^3$ in such a manner as to send hyperbolic geodesics to straight lines. With this parametrisation, the ideal boundary of $\mathbb{H}^3$ identifies with the unit sphere $S_1(0)$ in $\mathbb{R}^3$, and the convex hull in $\mathbb{H}^3$ of any given subset X of $S_1(0)$ likewise identifies with its convex hull in $B_1(0)$. Theorem 1.1 now yields

Theorem 1.2

Let X be a closed subset of $\partial_\infty\mathbb{H}^3$. Let Conv$(X)$ denote its convex hull in $\mathbb{H}^3$. If Conv(X) has non-trivial interior, then $\Sigma := \partial\text{Conv}(X)\setminus X$ is everywhere locally isometric to $\mathbb{H}^2$. Furthermore, the canonical embedding of Σ into $\mathbb{H}^3$ is totally geodesic except over a closed set which is a union of complete, non-intersecting geodesics.

2 - Convex subsets viewed extrinsically. Consider first an arbitrary metric space (Y,δ). In what follows, for any subset X of Y and for any $r>0$, we will denote by $B_r(X)$ the open neighbourhood of radius r about X and, in the case where $X=\{x\}$ consists of a single point, we will write $B_r(x)$ instead of $B_r(X)$. Let CB(Y) denote the set of closed, bounded subsets of Y. Recall (see Section 45 of [5]) that the *Hausdorff metric* is defined over this set by

$$d_{\text{H}}(X_1,X_2) = \text{Inf}\{r>0 \mid X_1\subseteq B_r(X_2)\ \&\ X_2\subseteq B_r(X_1)\}\,.$$

Recall also that the metric space $(\text{CB}(Y), d_{\text{H}})$ is compact (resp. complete) if and only if Y is compact (resp. complete).

Suppose now that $Y=\mathbb{R}^m$ is m-dimensional euclidean space. Consider the set CC(Y) of compact, convex subsets of Y. Observe that CC(Y) is a closed subset of CB(Y) and that, furthermore, the operator Conv defines a projection from CB(Y) onto this subset. In this section, we study the topological properties of these objects. Our results will also extend to convex subsets of arbitrary space-forms via affine charts.

Recall now that a closed *half-space* in $\mathbb{R}^m$ is a subset of the form

$$H_{\alpha,\lambda} := \{y \mid \alpha(y)\leq\lambda\}\,,$$

where $\alpha:\mathbb{R}^m\to\mathbb{R}$ is a linear form and $\lambda>0$ is a real number. It is straightforward to show (see Theorem 5.2 of [7]) that the convex hull of X is also the intersection of all closed half-spaces containing X.

Lemma 2.1

Let (X_n) be a sequence of compact subsets of $\mathbb{R}^m$. If this sequence converges in the Hausdorff sense to the compact subset X_∞, then the sequence $(\mathrm{Conv}(X_n))$ of convex hulls also converges in the Hausdorff sense to $\mathrm{Conv}(X_\infty)$. In other words, Conv maps $CB(\mathbb{R}^m)$ continuously onto $CC(\mathbb{R}^m)$.

Proof: For all $n \in \mathbb{N} \cup \{\infty\}$, denote $K_n := \mathrm{Conv}(X_n)$. First observe that there exists $R > 0$ such that, for all n, $X_n \subseteq B_R(0)$ so that, in particular, $K_n \subseteq B_R(0)$. Since the set of compact, convex subsets of the closed ball $\overline{B}_R(0)$ is compact in the Hausdorff topology, it suffices to show that K_∞ is the only concentration point of the sequence (K_n) in this topology. Suppose therefore that another such concentration point K'_∞ exists. In particular, K'_∞ is compact and convex and $X_\infty \subset K'_\infty$ so that, by definition of the convex hull, $K_\infty \subseteq K'_\infty$. Let x be a point of $K'_\infty \setminus K_\infty$. Let $H_{\alpha,\lambda}$ be a half-space which contains K_∞ but which does not contain x. Let $\epsilon > 0$ be such that $\alpha(x) = \lambda + 2\epsilon$. Since X_∞ is contained in $H_{\alpha,\lambda}$, for all sufficiently large n, X_n is contained in $H_{\alpha,\lambda+\epsilon}$. In particular, for all such n, K_n is also contained in $H_{\alpha,\lambda+\epsilon}$, so that $K_n \cap B_\epsilon(x) = \emptyset$. This is absurd, since some subsequence of (K_n) converges to K_∞ in the Hausdorff sense, and the result follows. □

Consider now a point $x \in \mathbb{R}^m$. For $\epsilon, r > 0$, we say that the subset X of $\mathbb{R}^m$ is *ϵ-dense* in $B_r(x)$ whenever every point of $B_r(x)$ lies at a distance of less than ϵ from X.

Lemma 2.2

If X is a compact, convex subset of $\mathbb{R}^m$ which is r-dense in $B_r(x)$, then x lies in X.

Proof: Indeed, if H is a half-space that contains X, then H must also contain x, for otherwise there would be a point of $B_r(x)$ lying at a distance of greater than r from H, and therefore also from X, which is absurd. The result follows. □

For $x \in \mathbb{R}^m$ and $\lambda > 0$, let D^λ_x denote the affine transformation which dilates by a factor of λ about the point x, that is,

$$D^\lambda_x y = x + \lambda(y - x).$$

Lemma 2.3

Let (K_n) be a sequence of compact, convex subsets of $\mathbb{R}^m$ converging in the Hausdorff sense to the compact, convex subset K_∞. If x is an interior point of K_∞ then, for all $\lambda > 1$, and for all sufficiently large n,

$$D^{\frac{1}{\lambda}}_x K_\infty \subseteq K_n \subseteq D^\lambda_x K_\infty.$$

Proof: Suppose that $B_{2r}(x)$ is contained in K_∞. We first show that $B_r(x)$ is also contained in K_n for sufficiently large n. Indeed, since (K_n) converges to K_∞ in the Hausdorff sense, for sufficiently large n, the set K_n is r-dense in $B_r(y)$ for all $y \in B_r(x)$. It follows by Lemma 2.2 that, for all such n, $B_r(x)$ is also contained in K_n, as asserted.

Observe now that, for any convex subset K, if $B_r(x) \subseteq K$, then for all $\lambda > 1$,

$$B_{(\lambda-1)r}(K) \subseteq D_x^\lambda K.$$

Thus, since (K_n) converges to K_∞ in the Hausdorff sense, for sufficiently large n,

$$K_n \subseteq B_{(\lambda-1)r}(K_\infty) \subseteq D_x^\lambda K_\infty,$$

and the second inclusion follows. Likewise, for sufficiently large n,

$$K_\infty \subseteq B_{(\lambda-1)r}(K_n) \subseteq D_x^\lambda K_n,$$

so that

$$D_x^{\frac{1}{\lambda}} K_\infty \subseteq K_n,$$

and the first inclusion follows. This completes the proof. $\square$

3 - Convex subsets viewed intrinsically. First recall that, given two compact metric spaces (X_1, d_1) and (X_2, d_2), their *Gromov-Hausdorff distance* (see [4]) is defined by

$$d_{\mathrm{GH}}((X_1, d_1), (X_2, d_2)) := \operatorname*{Inf}_{\phi_1, \phi_2, (Y, \delta)} d_{\mathrm{H}}(\phi_1(X_1), \phi_2(Y_2)),$$

where the infimum is taken over all metric spaces (Y, δ) and functions $\phi_1 : X_1 \to Y$ and $\phi_2 : X_2 \to Y$ which are isometries onto their images. The following technical result will prove useful.

Lemma 3.1

Let X_1 and X_2 be compact metric spaces with metrics d_1 and d_2 respectively. For $\epsilon \in]0, 1]$, suppose that there exist surjective maps $\Phi : X_1 \to X_2$ and $\Psi : X_2 \to X_1$ such that

$$d_2(\Phi(x), \Phi(y)) \leq (1 + \epsilon) d_1(x, y),$$
$$d_1(\Psi(x), \Psi(y)) \leq (1 + \epsilon) d_2(x, y),$$

and

$$d_1(x, \Psi\Phi(x)) \leq \epsilon.$$

Then the Gromov-Hausdorff distance between X_1 and X_2 satisfies

$$d_{\mathrm{GH}}(X_1, X_2) \leq \epsilon(2 + Max(Diam(X_1), Diam(X_2))).$$

Proof: Indeed, consider first a compact metric space (X, d). Observe that the map $D : X \to L^\infty(X)$ given by $D(x)(y) := d(x, y)$ is an isometry onto its image. Furthermore, given another compact metric space (X', d') and a surjective map $\Phi : X \to X'$, the composition operator $\Phi^* : L^\infty(X') \to L^\infty(X)$ also defines an isometry onto its image and, in particular, restricts to an isometry from $D'(X')$ onto a subset of $L^\infty(X)$. For each i, define $D_i : X_i \to L^\infty(X_i)$ in this manner. From the above relations, we deduce that, for all $x, y \in X_1$,

$$\begin{aligned} \Phi^* D_2(\Phi(x))(y) - D_1(x)(y) &\leq \epsilon \mathrm{Diam}(X_1), \\ (\Psi\Phi)^* D_1(\Psi\Phi(x))(y) - \Phi^* D_2(\Phi(x))(y) &\leq \epsilon \mathrm{Diam}(X_2), \text{ and} \\ D_1(x)(y) - (\Psi\Phi)^* D_1(\Psi\Phi(x))(y) &\leq 2\epsilon. \end{aligned}$$

Together these relations yield

$$\|\Phi^* D_2(\Phi(x)) - D_1(x)\|_{L^\infty} \leq \epsilon(2 + \mathrm{Max}(\mathrm{Diam}(X_1), \mathrm{Diam}(X_2))),$$

from which the result follows. □

Now let Ω be an affine chart of M^m_κ in $\mathbb{R}^m$. That is, when $\kappa = -1$, and $M^m_\kappa = \mathbb{H}^m$ is hyperbolic space, Ω is the open unit ball in $\mathbb{R}^m$ which identifies with $\mathbb{H}^m$ via the Kleinian parametrisation; when $\kappa = 0$ and $M^m_\kappa = \mathbb{R}^m$ is Euclidean space, Ω is simply the whole of $\mathbb{R}^m$; and when $\kappa = 1$ and $M^m_\kappa = \mathbb{S}^m$ is the unit sphere, Ω is also the whole of $\mathbb{R}^m$ which now identifies with an open hemisphere also via the Kleinian parametrisation. Let $\overline{g}$ denote the riemannian metric of this affine chart and let $\overline{d}$ denote the topological metric (distance function) that it defines. Throughout the rest of this section, for any subset X of Ω, and for all $r > 0$, $B_{r,\overline{d}}(X)$ will denote the open neighbourhood of radius r about X with respect to $\overline{d}$.

Consider now a compact, convex subset K of Ω, and let $\Pi : \Omega \to K$ be the closest point projection.

Lemma 3.2

If $B_{r,\overline{d}}(K) \subseteq \Omega$, then for all $x, y \in B_{r,\overline{d}}(K)$,

$$\overline{d}(\Pi(x), \Pi(y)) \leq \frac{1}{cos(r)} \overline{d}(x, y).$$

Remark: In fact, when $\kappa \in \{-1, 0\}$, the closest point projection is a contraction (see [1]).

Proof: It suffices to consider the case where x and y are elements of $B_{r,\overline{d}}(K) \setminus K$, as the remaining cases are similar and simpler. Consider the geodesic quadrilateral determined by the ordered sequence of points $(x, y, \Pi(y), \Pi(x))$. By convexity, the geodesic segment $\Pi(x)\Pi(y)$ is contained in K. In particular, since $\Pi(x)$ and $\Pi(y)$ are the closest points in K to x and y respectively, the angles $x\Pi(x)\Pi(y)$ and $y\Pi(y)\Pi(x)$, taken with respect to the metric $\overline{g}$, are both at least $\pi/2$, and the result now follows by standard comparison theory (see [2]). □

Lemma 3.3

Let (K_n) be a sequence of compact, convex subsets of Ω with non-trivial interiors, let K_∞ be another compact, convex subset of Ω with non-trivial interior, and for all $n \in \mathbb{N} \cup \{\infty\}$, let d_n denote the intrinsic metric of ∂K_n with respect to $\overline{g}$. If (K_n) converges to K_∞ in the Hausdorff sense, then $(\partial K_n, d_n)$ converges to $(\partial K_\infty, d_\infty)$ in the Gromov-Hausdorff sense.

Proof: Let x be an interior point of K_∞. Choose $\lambda > 1$. By Lemma 2.3, for sufficiently large n,

$$D_x^{\frac{1}{\lambda}} K_\infty \subseteq K_n \subseteq D_x^\lambda K_\infty.$$

Now denote $\Phi := D_x^\lambda \Pi_\infty$ and $\Psi := \Pi_n D_x^\lambda$, where Π_∞ and Π_n are respectively the closest point projections onto $D_x^{1/\lambda} K_\infty$ and K_n with respect to the metric $\overline{g}$. Since Φ and Ψ are continuous with unit degree, they are surjective. Thus, by Lemma 3.2 and the smoothness of $\overline{g}$, for any given $\epsilon > 0$, Φ and Ψ satisfy the hypotheses of Lemma 3.1 provided that λ is chosen sufficiently close to 1. The result follows. □

Proof of Theorem 1.1: Let (X_n) be a sequence of finite subsets of Ω converging to X in the Hausdorff sense. For all n, $\text{Conv}(X_n)$ is a convex polyhedron with vertices in X_n. In particular, for all n, the intrinsic metric of $\Sigma_n := \partial\text{Conv}(X_n) \setminus X_n$ is locally isometric to M^2_κ. Since, by Lemma 2.1, the sequence $(\text{Conv}(X_n))$ of convex hulls converges in the Hausdorff sense to the convex hull $\text{Conv}(X_\infty)$, the first assertion now follows by Lemma 3.3. To prove the second assertion, consider a totally geodesic supporting plane P to $\text{Conv}(X_\infty)$ at some point of $\Sigma := \text{Conv}(X_\infty) \setminus X$. Since $\text{Conv}(X_\infty)$ is a convex hull, the intersection of P with $\text{Conv}(X)$ is either a geodesic segment with end-points in X, or a convex polygon with geodesic edges and vertices in X. From this the second assertion readily follows, and this completes the proof. □

Proof of Theorem 1.2: Fix a point $x \in \mathbb{H}^3$, and for all $r > 0$, let $\overline{B}_r(x)$ denote the closed ball of radius r about x in $\mathbb{H}^3$. For all r, $\text{Conv}(X) \cap \overline{B}_r(x)$ is the convex hull of the compact set $\text{Conv}(X) \cap \partial B_r(x)$, and the result follows by Theorem 1.1. □

4 - Bibliography.

[1] Ballmann W., Gromov M., Schroeder V., *Manifolds of Nonpositive Curvature*, Birkhäuser Verlag, 1985

[2] Cheeger J., Ebin D. G., *Comparison Theorems in Riemannian Geometry*, American Mathematical Society, 2008

[3] Epstein D. B. A., Marden A., Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, in *Fundamentals of Hyperbolic Geometry: Selected Expositions*, Cambridge University Press, 2006, 117–266

[4] Gromov M., *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser Verlag, 2007

[5] Munkres J. R., *Topology*, Prentice-Hall, 1975

[6] Rourke C., Convex ruled surfaces, in *Analytical and geometric aspects of hyperbolic space*, Cambridge University Press, 1987, 255–272

A short proof of an assertion of Thurston concerning convex hulls.

[7] Smith G. A. C., *Global singularity theory for the Gauss curvature equation*, Ensaios Mathematics, 2015

[8] Thurston W. P., *The geometry and topology of three-manifolds*, Mathematical Sciences Research Institute, 2002