Complete embedded minimal surfaces of Costa-Hoffman-Meeks type in 3-dimensional hyperbolic space.

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Motivations

In arxiv:1501.04149 finite genus translating solitons of the mean curvature flow were constructed by singular perturbation.

The components of this construction were

(1) Costa-Hoffman-Meeks surfaces; and

(2) rotationally symmetric solitons.

A finer analysis of Costa-Hoffman-Meeks surfaces than had hitherto been carried out was required.

The analysis of the rotationally symmetric components rested on their having strictly negative intrinsic curvature near infinity.

This property is shared by rotationally symmetric minimal surfaces in hyperbolic space.

“Because it’s there.”

The construction had not been carried out before, so, heck!

George Mallory, 1886-1924

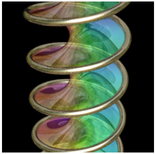
Died 8th June 1924, aged 37, attempting the final ascent towards the summit of Mount Everest.

Background - Euclidean space

Complete, properly embedded minimal surfaces in $\mathbb{R}^3$ have quite rigid structures. Heuristically, there is little room to move about.

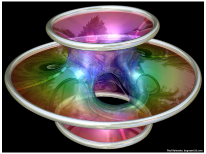
Do Carmo & Peng and Fischer-Colbrie & Schoen show that the only complete, stable, embedded minimal surfaces in $\mathbb{R}^3$ are planes.

For a long time, the only known complete, properly embedded minimal surfaces were the plane, the catenoid, and the helicoid.

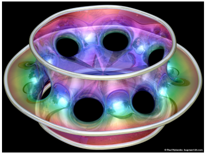


Background - “Bringing coals to Newcastle.”

This changed with the discovery in 1984 by Costa of the surface with bears his name.



The construction was later generalised to higher genus by Hoffman & Meeks.



Background - Ossermann's classification

In 1986, Ossermann described all complete, properly embedded minimal surfaces in $\mathbb{R}^3$ of finite topology.

The ends are all graphs over the same plane, asymptotic to

$$z = a + b\log(r),$$

for suitable constants a and b, where

$$r^2 = x^2 + y^2.$$

Background - Hyperbolic space I

The situation in $\mathbb{H}^3$ is different: heuristically $\mathbb{H}^3$ is much bigger.

We do not expect a classification as straightforward as Ossermann's.

In 1982 Anderson showed that every Jordan curve in $\partial\mathbb{H}^3$ bounds a properly embedded, stable minimal disk in $\mathbb{H}^3$.

In 1998, Oliveira & Soret constructed complete, properly embedded, stable minimal surfaces of arbitrary finite topology.

In 2014, Martín & White constructed *different* complete, properly embedded, stable minimal surfaces of arbitrary (finite or infinite) topology.

Oliveira & Soret's construction I

Oliveira & Soret construct a complete, properly embedded surface Σ and a mean convex subset $K \subseteq \mathbb{H}^3$ such that

(1) $\Sigma \subseteq K$,

(2) $\partial_\infty \Sigma = \partial_\infty K$, and

(3) the canonical homomorphism $\pi_1(\Sigma) \to \pi_1(K)$ is injective (i.e. Σ is incompressible in K).

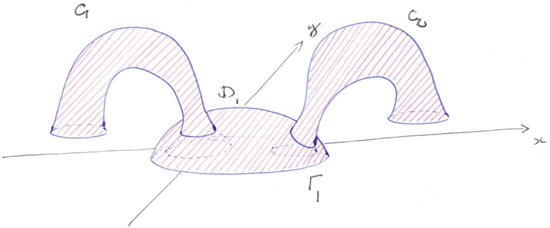
A variational (i.e. area minimizing) argument yields a complete, properly embedded, stable minimal surface $\tilde{\Sigma}$ homeomorphic to Σ.

Oliveira & Soret's construction II

Σ and K are constructed by successively removing from $\mathbb{H}^3$ the interiors of

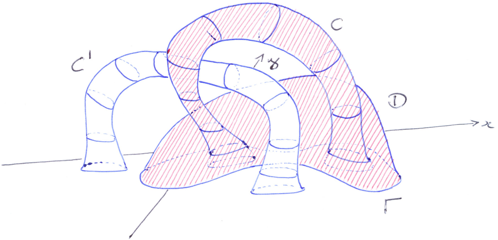
(1) Anderson's minimal disks; and

(2) rotationally symmetric minimal cylinders.



Oliveira & Soret's construction III

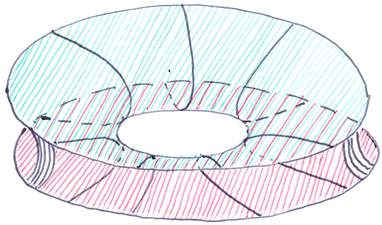
Handles are constructed by removing interiors of rotationally symmetric minimal cylinders from K.



Incompressibility is guaranteed by removing two cylinders at a time.

Martín & White's construction

Complete, stable, area minimizing surfaces are constructed by successively adding "bridges" at infinity: simple curves which meet boundary components orthogonally.



Given Σ and a "bridge" Γ, a new surface Σ' is obtained by taking the area minimiser with boundary $\partial\Sigma$ modified by fattening of Γ.

Σ' can be chosen arbitrarily close to Σ over any compact region.

Surfaces of prescribed geometry

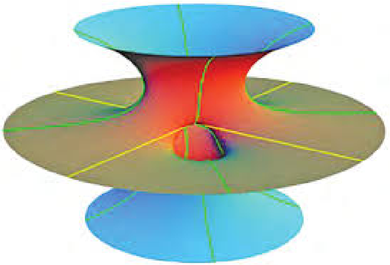
Oliveira & Soret provide little information on the *geometries* of the surfaces constructed.

Martín & White provide precise information on the *geometries* of the surfaces constructed.

However, since topology is created by surgery at infinity, not all geometries are obtainable.

Symmetries of Costa-Hoffman-Meeks surfaces

The symmetry group of the Costa-Hoffman-Meeks surface of genus g is the dihedral group generated by

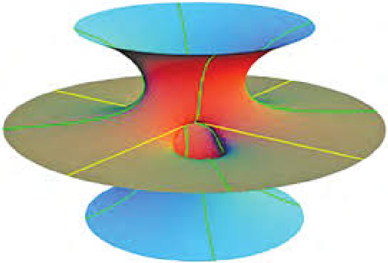


(1) Reflection in the $x - z$ plane; and

(2) Rotation about the z-axis by an angle of $k\pi/(g+1)$ followed by reflection in the $x - y$ plane.

Horizontal symmetries of Costa-Hoffman-Meeks surfaces

Horizontal symmetries preserve every point of the z-axis.



They are reflections in the plane containing the z-axis making an angle of $\pi/(g+1)$ with the x-axis.

Surfaces of Costa-Hoffman-Meeks type

Theorem A

Fix $g \in \mathbb{N}$ and $\eta \ll 1$. For all $\Lambda \gg 1$, and for all $\epsilon > 0$ and $R > 0$ satisfying

$$\epsilon R^{5-2\eta} \leq \frac{1}{\Lambda} \text{ and } \epsilon R^{5-\eta} \geq \Lambda,$$

there exists a complete, embedded minimal surface Σ in $\mathbb{H}^3$ of genus g with 3 ends.

(1) Σ is preserved by the horizontal symmetries of Σ_g;

(2) $\Sigma \setminus B(\epsilon R)$ consists of three disjoint ends each of which converges towards the same horizontal, totally geodesic plane as Λ tends to infinity;

(3) Upon rescaling by a factor of $1/\epsilon$, $\Sigma \cap B(2\epsilon R)$ converges towards the Costa-Hoffman-Meeks surface Σ_g as Λ tends to infinity.

Fermi coordinates of hyperbolic space

The Fermi metric of hyperbolic space is

$$g = dr^2 + \sinh^2(r)d\theta^2 + \cosh^2(r)dz^2.$$

(1) Every horizontal radial line is a unit speed geodesic;

(2) The z-axis is unit speed geodesic;

(3) Every horizontal plane is totally geodesic and orthogonal to the z-axis; and

(4) Every vertical plane containing the z-axis is totally geodesic.

Rotationally symmetric minimal surfaces

Given a smooth function $u : [R_0, \infty[\to \mathbb{R}$, let Σ_u be its surface of revolution about the z-axis.

Σ_u is minimal if and only if

$$u_r = \frac{(\mathcal{F}/r)}{\cosh(r)\sqrt{\sinh^2(2r) - (\mathcal{F}/\pi)^2}}$$

for some constant $\mathcal{F}$ called the **flux**.

The flux is related to the **neck radius** R_0 by

$$\mathcal{F} = \pi \sinh(2R_0).$$

The modified normal vector field

Over some regions it is better to work with perturbations in the vertical direction.

Over other regions it is better to work with perturbation in the normal direction.

We thus define the modified normal

$$\hat{N} := \chi_1 \partial_t + (1 - \chi_1) N,$$

where

(1) N is the upward-pointing unit normal vector field over Σ_u; and

(2) χ_1 is a cut-off function equal to 1 over $B_1(0)$ and 0 over $A(2, \infty)$.

The modified Jacobi operator

For $f \in C_0^\infty(\Sigma_u)$, define

$$\mathcal{E}_t(x) := x + tf(x)\hat{N}(x).$$

Define $\mathcal{H}(t,x)$ to be the mean curvature of $\mathcal{E}_t$ at x.

The **modified Jacobi operator** is

$$\hat{J}f := \frac{1}{\phi}\frac{\partial}{\partial t}\mathcal{H}(t,x)\bigg|_{t=0},$$

where

$$\phi := g\big(\hat{N}, N\big) = \chi_1 g(\partial_t, N) + (1-\chi_1).$$

Since Σ_u is minimal, this simplifies to

$$\hat{J}f = \frac{1}{\phi}J(\phi f).$$

The hybrid norm

Let $H^m(\mathbb{R}^2)$ be the L^2 Sobolov space wrt. the hyperbolic metric.

Let $C^{m,\alpha}(\mathbb{R}^2)$ be the Hölder space wrt. the hyperbolic metric.

Given $\epsilon, R > 0$, the **hybrid norm** is defined by

$$\|f\|_m := \|f\|_{C^{m,\alpha}} + \frac{1}{\epsilon R}\|f\|_{H^m}.$$

The hybrid norm refines the Hölder norm by providing stronger control of the *first* derivative.

Indeed, for sufficiently small α,

$$\|f\|_{C^{1,\alpha}} \lesssim (\epsilon R)^{1-2\alpha}\|f\|_2.$$

Towards uniform invertibility

For $\epsilon, R > 0$ and $c \in [-C, C]$, choose $\mathcal{F} = 2\pi\epsilon c$ and consider the restriction of u to the annulus $A(\epsilon R, \infty)$.

Over the annulus $A(\epsilon R, \epsilon R^4)$,

$$\hat{J}f = \Delta f - \frac{\epsilon^2 c^2}{r^2} f_{rr} + 2\left(r + \frac{\epsilon^2 c^2}{r^3}\right) f_r + \mathcal{R}f,$$

where the remainder $\mathcal{R}$ is small.

We require uniform invertibility of $\hat{J}$ for large Λ where

$$\epsilon R^{5-2\eta} \leq \frac{1}{\Lambda} \text{ and } \epsilon R^{5-\eta} \geq \Lambda.$$

Canonical extensions

$\hat{J}$ is not defined over the whole of $\mathbb{R}^2$.

For a smooth function $a : A(\epsilon R, \infty) \to \mathbb{R}$, its **canonical extension** $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ is defined such that

(1) $\tilde{f}(0)$ equals the average value of f over $C(\epsilon R)$;

(2) $\tilde{f}$ is equal to f over $A(\epsilon R, \infty)$; and

(3) $\tilde{f}$ is linear over every radial line from 0 to $C(\epsilon R)$.

For a smooth operator L over $A(\epsilon R, \infty)$, its **canonical extension** $\tilde{L}$ is that operator whose coefficients are the canonical extensions of the coefficients of L.

We identify all operators with their canonical extensions.

Uniform invertibility

Theorem

For all sufficiently small $\alpha \in]0,1[$ and for all sufficiently large Λ, the modified Jacobi operator $\hat{J}$ defines a linear isomorphism with respect to the hybrid norm.

Furthermore, the norms of $\hat{J}$ and its inverse are uniformly bounded as Λ tends to infinity.

The pertubation argument

When $u = 0$, over $A(\epsilon R, \epsilon R^4)$, the modified Jacobi operator is

$$\hat{J}_0 f = \Delta f + 2\tanh(r) f.$$

The only problematic term in $\hat{J} - \hat{J}_0$ is

$$\frac{\epsilon^2 c^2}{r^3} f_r,$$

This term grows rapidly as Λ tends to infinity...

...but it is first order in f.

The stronger estimates on Df from the hybrid norm compensate this singular behaviour.

Thankyou!