The Kulkarni-Pinkall form and locally strictly convex immersions in $\mathbb{H}^3$.

Graham Andrew Smith

Universidade Federal do Rio de Janeiro

30/04/2021

Framing the problem - general definitions

$\mathbb{H}^3$ is 3-dimensional hyperbolic space.

An **immersed surface** is a pair (S, e) where S is a smooth surface and $e : S \to \mathbb{H}^3$ is a smooth immersion.

$N_e : S \to \mathbb{H}^3$ is the unit normal vector field of e.

I_e, II_e and III_e are respectively the 1st, 2nd and 3rd fundamental forms of e.

By convention

$$\mathrm{II}_e(\xi, \mu) := \langle De \cdot \xi, \nabla_\mu N_e \rangle.$$

Framing the problem - k-surfaces

(S, e) is **Infinitesimally Strictly Convex** (ISC) whenever II is positive definite.

(S, e) is **quasicomplete** whenever $\mathrm{I} + \mathrm{III}$ is complete.

$K := \mathrm{Det}(\mathrm{II}_e)/\mathrm{Det}(\mathrm{I}_e)$ is the **extrinsic** (or Gaussian) curvature of e.

For $k \in]0, 1[$, a ***k*-surface** is a quasicomplete ISC immersed surface (S, e) of constant extrinsic curvature equal to k.

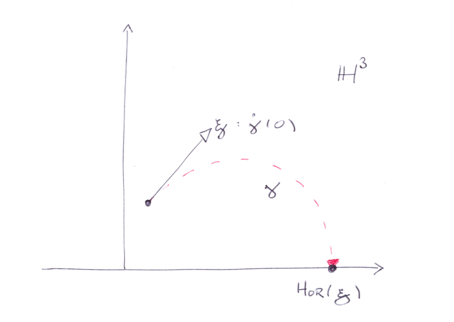
Framing the problem - the horizon map

$\mathrm{U}\mathbb{H}^3$ is the unit sphere bundle, $\partial_\infty \mathbb{H}^3$ the ideal boundary.

The **horizon map** $\mathrm{Hor} : \mathrm{U}\mathbb{H}^3 \to \partial_\infty \mathbb{H}^3$ is

$$\mathrm{Hor}(\dot\gamma(0)) := \lim_{t \to +\infty} \gamma(t),$$

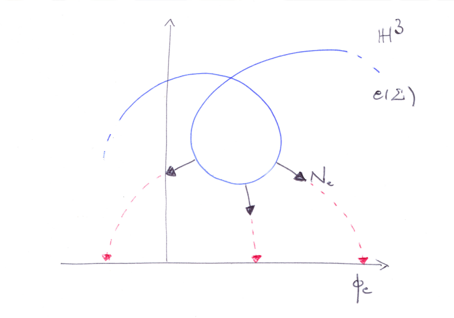
for every unit-speed geodesic $\gamma : \mathbb{R} \to \mathbb{H}^3$.



Framing the problem - the asymptotic Gauss map

The **asymptotic Gauss map** of (S, e) is

$$\phi_e := \mathrm{Hor} \circ N_e.$$



When (S, e) is ISC, ϕ_e is a local diffeomorphism.

Labourie's asymptotic Plateau problem

An **asymptotic Plateau problem** (APP) is a pair (S, ϕ) where S is surface and $\phi : S \to \partial_\infty \mathbb{H}^3$ is a local diffeomorphism.

(S, ϕ) and (S', ϕ') are **equivalent** whenever there exists a diffeomorphism $\alpha : S \to S'$ such that

$$\phi = \phi' \circ \alpha.$$

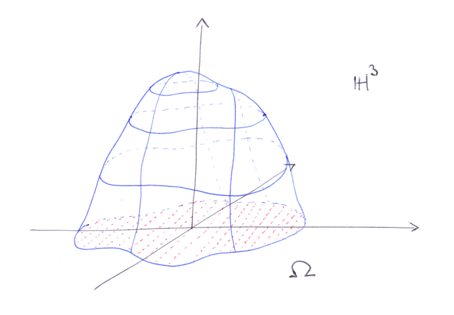
For $k > 0$, a k-surface (S, e) **solves** the APP (S, ϕ) with curvature k whenever

$$\phi_e = \phi.$$

Known results - Part I

Theorem, Rosenberg-Spruck, 1994

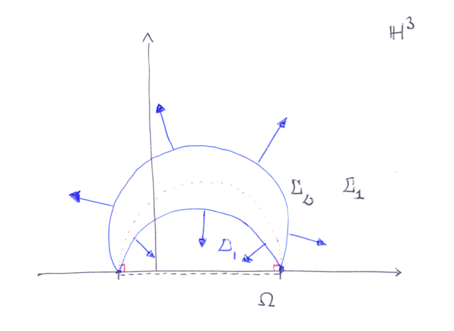
If Ω is a Jordan domain in $\partial_\infty \mathbb{H}^3$, then, for all $k \in]0,1[$, there exists a unique solution to the APP (Ω, Id) with curvature k.



Known results - Part II

Theorem, Labourie, 2000

For all $k \in]0,1[$, there exists at most 1 solution to any APP with curvature k.



Known results - Part III

Theorem, Labourie, 2000

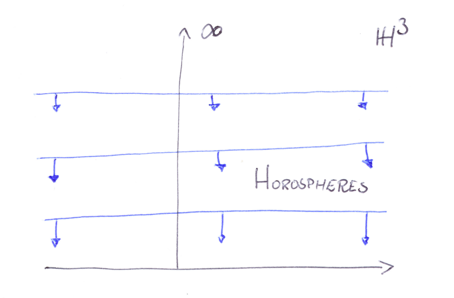
For all $k \in]0,1[$ and for every APP (S,ϕ), there exists a unique solution to the APP (Ω,ϕ) for every relatively compact open subset of S.

Theorem, Labourie, 2000

Let P be a finite subset of $\partial_\infty \mathbb{H}^3$ of cardinality at most 2. The APP $(\partial_\infty \mathbb{H}^3 \setminus P, \mathrm{Id})$ has no solution for any $k \in]0,1[$.

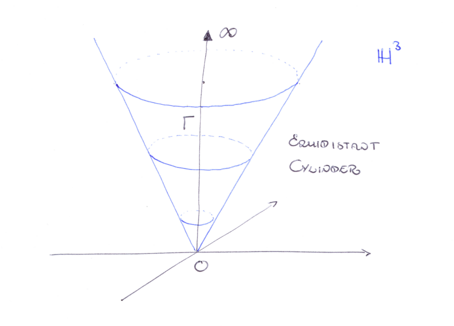
Non existence I

Non-existence is shown using folations by horospheres...



Non existence II

...and foliations by cylinders about geodesics.



The hyperbolic case

$\partial_\infty \mathbb{H}^3$ naturally identifies with the Riemann sphere $\hat{\mathbb{C}}$.

For every APP (S, ϕ), $\phi^*\hat{\mathbb{C}}$ defines a holomorphic structure over S.

It suffices to consider $S \in \{\mathbb{D}, \mathbb{C}\}$ and ϕ locally conformal.

Theorem, Smith, (2014)

Let $\phi : \mathbb{D} \to \partial_\infty \mathbb{H}^3$ be locally conformal. For all $k \in]0, 1[$, there exists a unique solution e_ϕ to the APP $(\mathbb{D}, \phi)$.

Furthermore, e_ϕ varies continuously in the C^∞_{loc} sense as ϕ varies continuously in the C^0_{loc} sense.

The parabolic case

Theorem, Smith, (2021)

Let $\phi : \mathbb{C} \to \partial_\infty \mathbb{H}^3$ *be a locally conformal function which is not a cover of* $\partial_\infty \mathbb{H}^3 \setminus P$*, where P consists of at most* 2 *distinct points.*

For all k $\in]0, 1[$*, there exists a unique solution* e_ϕ *to the APP* $(\mathbb{C}, \phi)$*.*

We do not expect e_ϕ to depend continously on ϕ.

It is an interesting problem is detemine when e_ϕ is *complete*.

The monotone convergence theorem

Theorem, Smith, (2021)

Let (S, ϕ) be an APP not equivalent to $(\partial_\infty \mathbb{H}^3 \setminus P, \mathrm{Id})$, where P consists of at most 2 distinct points.

Let $(S_m)_{m\in\mathbb{N}}$ be a nested sequence of open subsets of S exhausting S.

Suppose that, for all m, the APP (S_m, ϕ) with curvature k has a solution (S_m, e_m).

If $S = \cup_{m\in\mathbb{N}} S_m$, then $(e_m)_{m\in\mathbb{N}}$ converges in the C^∞_{loc} sense to the unique solution e of the APP (S, ϕ) with curvature k.

Existence in the parabolic case now follows from known existence results.

Labourie's compactness theorem

For an immersed surface (S, e), denote $\hat{e} := N_e$.

$(S, \hat{e})$ is an immersed surface in $\mathrm{U}\mathbb{H}^3$ called the **Gauss lift** of (S, e).

$(S, \hat{e})$ is complete if and only if (S, e) is quasicomplete.

Theorem, Labourie, ($\leqslant$1997)

Let $(S_m, e_m, x_m)_{m\in\mathbb{N}}$ be pointed k-surfaces in $\mathbb{H}^3$. If $(e_m(x_m))_{m\in\mathbb{N}}$ is contained within a compact set, then there exists a complete, pointed, immersed surface $(S_\infty, \hat{e}_\infty, x_\infty)$ towards which $(S_m, \hat{e}_m, x_m)_{m\in\mathbb{N}}$ subconverges in the Cheeger-Gromov sense.

Labourie's dichotomy

Γ is a complete geodesic in $\mathbb{H}^3$.

$\mathrm{N}\Gamma$ is the bundle of unit normal vectors over Γ.

$(S, \hat{e})$ is a **curtain surface** if and only if it covers $\mathrm{N}\Gamma$ for some Γ.

Theorem, Labourie, ($\leqslant$1997)

Let $(S, \hat{e})$ be a Cheeger-Gromov limit of a sequence of Gauss lifts of k-surface. If $(S, \hat{e})$ is the not the Gauss lift of a k-surface, then it is a curtain surface.

Flat conformal surfaces

A (developed) **Flat Conformal Surface** (FCS) is a pair (S, ϕ) where S is a Riemann surface and $\phi : S \to \hat{\mathbb{C}}$ is locally conformal.

(S, ϕ) and (S', ϕ') are **equivalent** whenever there exists a diffeomorphism $\alpha : S \to S'$ such that

$$\phi = \phi' \circ \alpha.$$

The concepts of APP and FCS are trivially synonymous.

A **conformal disk** in (S, ϕ) is a pair (D, α) where $D \subseteq \hat{\mathbb{C}}$ is a disk and $\alpha : D \to S$ is a conformal map such that

$$\phi \circ \alpha = \mathrm{Id}.$$

The Kulkarni-Pinkall metric and form

For every disk $D \subseteq \hat{\mathbb{C}}$, let $g(D)$ denote its Poincaré metric.

The **Kulkarni-Pinkall (KP) metric** g_ϕ of (S, ϕ) is

$$g_\phi(p) := \inf_{p \in \alpha(D)} (\alpha_* g(D))(p),$$

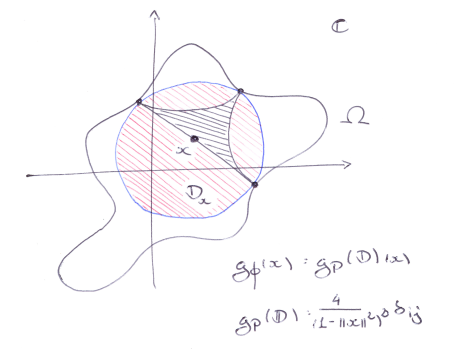
The **Kulkarni-Pinkall form** ω_ϕ of (S, ϕ) is

$$\omega_\phi := g_\phi(\cdot, J\cdot).$$

Non-degeneracy of the KP metric and form

Theorem, Kulkarni-Pinkall, (1994)

If (S, ϕ) is not equivalent to $(\hat{\mathbb{C}}, \mathrm{Id})$ or $(\hat{\mathbb{C}} \setminus \{z\}, \mathrm{Id})$, then g_ϕ is everywhere positive-definite and is complete.



Geometry of open horoballs in $\mathbb{H}^3$

Let B be an open horoball in $\mathbb{H}^3$.

The **asymptotic centre** of B is the point p of intersection of $\overline{B}$ with $\partial_\infty \mathbb{H}^3$.

Let H be an open half-space exterior tangent to B at some point q.

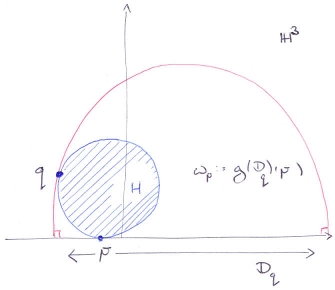
Let $D := \partial_\infty H$ denote its ideal boundary.

D contains p and $g(D)$ does not depend on q.

$\omega_p := g(D)(q)(\cdot, J\cdot)$ is the **asymptotic curvature** of B.

Parametrising open horoballs in $\mathbb{H}^3$

Open horoballs in $\mathbb{H}^3$ are uniquely determined by their asymptotic curvatures and centres.



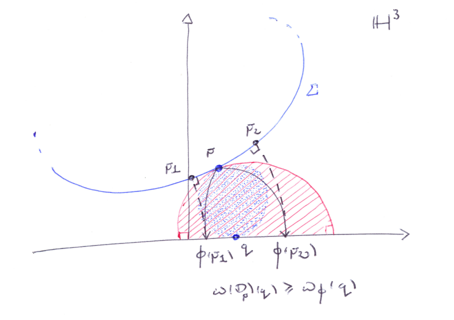
For $\omega_p \in \Lambda^2 \partial_\infty \mathbb{H}^3$, $B(\omega_p)$ is the open horoball with asymptotic centre p and asymptotic curvature ω_p.

An a priori C^0 estimate

Theorem, Smith, (2021)

Let (S, e) be a quasicomplete ISC immersed surface in $\mathbb{H}^3$, let $\phi := \phi_e$ denote its asymptotic Gauss map and let $\omega := \omega_e$ denote the Kulkarni-Pinkall form of ϕ_e.

For all $x \in S$, $e(x) \in \overline{B}(\phi_*\omega(x))$.



Uniform quasicompleteness: the horospherical metric

The proof also requires *uniform* quasicompleteness.

Lemma

Let (S, e) be an quasicomplete, ISC immersed surface, let ϕ_e denote its asymptotic Gauss map and let g_e denote the Kulkarni-Pinkall metric of ϕ_e. Then

$$g_e \leqslant \mathrm{I}_e + 2\mathrm{II}_e + \mathrm{III}_e \leqslant 2(\mathrm{I}_e + \mathrm{III}_e).$$

This is the **horospherical metric**, introduced by Schlenker and studied by Espinar-Galvez-Mira.

Schwarzian derivatives

Ω is a domain in $\mathbb{C}$.

$\phi : \Omega \to \hat{\mathbb{C}}$ is locally conformal.

The **Schwarzian derivative** of ϕ is

$$\mathrm{S}[\phi] := \left(\frac{\phi''}{\phi'}\right)' - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2.$$

Theorem, Folklore

Suppose that Ω is simply connected. For all $f : \Omega \to \mathbb{C}$, there exists $\phi : \Omega \to \hat{\mathbb{C}}$ such that

$$\mathrm{S}[\phi] = f.$$

Furthermore, ϕ is unique up to composition with a Möbius map.

The linearised parametrisation

Theorem, The hyperbolic case

For all $k \in]0,1[$ and for all holomorphic $f : \mathbb{D} \to \mathbb{C}$, there exists a k-surface $e : \mathbb{D} \to \mathbb{H}^3$ whose asymptotic Gauss map ϕ_e satisfies $\mathrm{S}[\phi_e] = f$.

Furthermore, modulo post-composition by isometries of $\mathbb{H}^3$, e is unique and varies continuously with f.

Theorem, The parabolic case

For all $k \in]0,1[$ and for all non-constant holomorphic $f : \mathbb{C} \to \mathbb{C}$, there exists a k-surface $e : \mathbb{C} \to \mathbb{H}^3$ whose asymptotic Gauss map ϕ_e satisfies $\mathrm{S}[\phi_e] = f$.

Furthermore, modulo post-composition by isometries of $\mathbb{H}^3$, e is unique.

¡Gracias!