Perturbing the Costa Surface

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The Costa Surface

The Costa surface was the first constructed example of a complete embedded minimal surface in $\mathbb{R}^3$ with non-trivial finite topology.

It is homeomorphic to a torus with 3 points removed.

It has 2 catenoidal ends and 1 planar end.

Numerology of the Costa Surface

The Costa Surface can be perturbed along a smooth 3-dimensional family of complete **embedded** minimal surfaces.

It can also be perturbed along a smooth 9-dimensional family of complete **immersed** minimal surfaces.

If the planar end is held fixed, then the perturbation family becomes 6-dimensional.

If we impose in addition symmetry in the $x-z$-plane, then the family now becomes 4-dimensional.

Where do these numbers come from?

The Jacobi Operator

The Jacobi operator measures the infinitesimal variation of mean curvature resulting from an infinitesimal normal perturbation of the surface.

$$Jf = \mathrm{Tr}(A^2)f + \Delta f,$$

where A here denotes the shape operator of the surface.

There exists a constant B such that, for all $x \in C$,

$$\|A(x)\| \leq \frac{B}{1 + \|x\|^2}.$$

In particular,

$$Jf = \Delta f + V_K f + V_S f,$$

where V_K is compactly supported and V_S is as small as we wish.

Fredholm Theory

J maps $H^2(C)$ into $L^2(C)$.

J has the same Fredholm properties as Δ.

However, Δ is **not** Fredholm!

Indeed, by Stokes Theorem, if $\phi \in C_0^\infty(C) \subseteq L^2(C)$ satisfies

$$\int_C \phi \mathrm{dVol} = 0,$$

then ϕ is not in the image of Δ.

On the other hand, $\mathrm{Ker}(\Delta) = 0$.

Weighted Sobolev Spaces

We consider the cylinder $S^1 \times \mathbb{R}$.

For $m \in \mathbb{Z}$, for $\delta \in \mathbb{R}$, and for $f \in C_0^\infty(S^1 \times \mathbb{R})$, we define the δ-weighted m'th order Sobolev norm of f by

$$\|f\|_{m,\delta} = \|e^{-\delta t} f\|_m,$$

where $\|\cdot\|_m$ is the normal m'th order Sobolev norm and t is the $\mathbb{R}$-coordinate.

$H_\delta^m(S^1 \times \mathbb{R})$ is the completion of $C_0^\infty(S^1 \times \mathbb{R})$ with respect to this norm.

Heuristically, elements of H_δ^m grow like $e^{\delta t}$ as t tends to $+\infty$ and decay like $e^{\delta t}$ as t tends to $-\infty$.

The Schrödinger and Heisenberg Pictures

For all δ, let M_δ be the operator of multiplication by $e^{-\delta t}$. That is,

$$M_\delta f = e^{-\delta t} f.$$

By definition of the norms, $M_\delta : H^m_\delta \to H^m$ is an isometry.

In fact, the following diagram commutes.

$$\begin{array}{ccc} H^{m+2}_\delta & \xrightarrow{M_\delta} & H^{m+2} \\ \downarrow \Delta & & \downarrow \Delta_\delta := M_\delta \Delta M_{-\delta} \\ H^m_\delta & \xrightarrow{M_\delta} & H^m \end{array}$$

So we can equally well study Δ acting on H^{m+2}_δ, or Δ_δ acting on H^{m+2}.

Dual Spaces

The natural dual space to H^m is H^{-m}.

That is, the natural pairing

$$\langle f, g\rangle := \int fg \mathrm{dVol}$$

defines a non-degenerate bounded bilinear form over $H^m \oplus H^{-m}$.

However, for $(f, g) \in H^m_\delta \oplus H^{-m}_{-\delta}$,

$$\begin{aligned} |\langle f, g\rangle| &= \left|\langle e^{-\delta t} f, e^{\delta t} g\rangle\right| \\ &\leq \|e^{-\delta t} f\|_m \|e^{\delta t} g\|_{-m} = \|f\|_{m,\delta} \|g\|_{-m,-\delta}. \end{aligned}$$

That is, $\langle \cdot, \cdot \rangle$ defines a bounded bilinear form over $H^m_\delta \oplus H^{-m}_{-\delta}$.

The natural dual space to H^m_δ is thus $H^{-m}_{-\delta}$.

The Small Miracle

Theorem A

For all $\delta \notin \mathbb{Z}$, Δ defines a linear isomorphism from H_δ^{m+2} into H_δ^m.

Proof. Δ_δ defines a linear isomorphism from H^{m+2} into H^m.

$$\Delta_\delta f = \Delta f + 2\delta\partial_t f + \delta^2 f.$$

The Fourier transform of this operator is

$$(\hat{\Delta}_\delta \hat{f})(n,\eta) = \left(-n^2 + (\delta - i\eta)^2\right) \hat{f}(n,\eta).$$

When $\delta \notin \mathbb{Z}$, this is inverted by $\hat{K}_\delta$, where

$$(\hat{K}_\delta \hat{f})(n,\eta) = \frac{1}{-n^2 + (\delta - i\eta)^2} \hat{f}(n,\eta).$$

Weighted Sobolev Spaces Again

Let S be a complete surface and suppose that

$$S = S_0 \cup S_\infty,$$

where S_0 is compact, and S_∞ is isometric to $S^1 \times [0, \infty[$.

Let $\tau : S \to \mathbb{R}$ be such that over $S_\infty = S^1 \times [0, \infty[$,

$$\tau(\theta, t) = t.$$

For all $m \in \mathbb{Z}$, for all $\delta \in \mathbb{R}$, and for $f \in C_0^\infty(S^1 \times \mathbb{R})$, we define

$$\|f\|_{m,\delta} = \|e^{-\delta\tau} f\|_m.$$

$H_\delta^m(S)$ is the completion of $C_0^\infty(S)$ with respect to this norm.

Elliptic Estimates

Let E and F be Banach spaces. Let $A : E \to F$ be a bounded linear map.

We say that A satisfies an **elliptic estimate** whenever there exists a Banach space G and a **compact** linear map $K : E \to G$ such that

$$\|e\| \leq B(\|Ke\| + \|Ae\|).$$

Theorem B

If A satisfies an elliptic esimate, then $\mathrm{Ker}(A)$ is finite-dimensional and $\mathrm{Im}(A)$ is closed.

The Fredholm Property of the Laplacian

Theorem C

If $\delta \notin \mathbb{Z}$, then Δ defines a Fredholm mapping from H^{m+2}_δ into H^m_δ.

Proof. Let $\chi \in C_0^\infty(S)$ be such that

$$\chi(x) = \begin{cases} 1 & \text{if } x \in S_0, \\ 0 & \text{if } x \in S^1 \times [1, \infty[. \end{cases}$$

Then,

$$\begin{aligned} \|\chi f\|_{m+2,\delta} &\leq B\left(\|\chi f\|_{m,\delta} + \|\Delta \chi f\|_{m,\delta}\right), \\ \|(1-\chi) f\|_{m+2,\delta} &\leq B\|\Delta(1-\chi) f\|_{m,\delta}. \end{aligned}$$

Thus

$$\|f\|_{m+2,\delta} \leq B\left(\|f|_{S_0 \cup S^1 \times [0,1]}\|_{m+1,\delta} + \|\Delta f\|_{m,\delta}\right).$$

Towards Calculating the Index - Some Definitions

We introduce the following spaces of harmonic functions.

$$\mathcal{H}_\delta(S) := \left\{ f \in H^2_\delta(S) \mid \Delta f = 0 \right\},$$
$$\mathcal{H}(S_0) := \left\{ f \in H^2(S_0) \mid \Delta f = 0 \right\},$$
$$\mathcal{H}_\delta(S_\infty) := \left\{ f \in H^2_\delta(S_\infty) \mid \Delta f = 0 \right\}.$$

The **Cauchy data operators** are given by

$$C_0 : \mathcal{H}_\delta(S) \to H^{\frac{3}{2}}(S^1); f \mapsto f|_{S^1 \times \{0\}}.$$
$$C_1 : \mathcal{H}_\delta(S) \to H^{\frac{1}{2}}(S^1); f \mapsto \partial_t f|_{S^1 \times \{0\}}.$$

The **Cauchy jump operator** is given by

$$J_\delta : \mathcal{H}(S_0) \oplus \mathcal{H}_\delta(S_\infty) \to H^{1/2}(S^1) \oplus H^{3/2}(S^1),$$
$$(f, g) \mapsto (C_1 f - C_1 g, C_0 f - C_0 g).$$

An Exact Sequence

We have the following exact sequence.

$$0 \longrightarrow \mathcal{H}_\delta(S) \longrightarrow \mathcal{H}(S_0) \oplus \mathcal{H}_\delta(S_\infty) \xrightarrow{J_\delta} H^{\frac{1}{2}}(S^1) \oplus H^{\frac{3}{2}}(S^1).$$

What is the image of J_δ?

In other words, for which pairs of functions (ϕ, ψ) in $H^{\frac{1}{2}}(S^1) \oplus H^{\frac{3}{2}}(S^1)$ does there exists a pair of harmonic functions (f, g) in $\mathcal{H}(S_0) \oplus \mathcal{H}_\delta(S_\infty)$ such that

$$J_\delta(f, g) = (\phi, \psi)?$$

This is a relative of the **Neumann problem**.

Solving the Neumann Problem

Fix $(\phi, \psi) \in H^{\frac{1}{2}}(S^1) \oplus H^{\frac{3}{2}}(S^1)$.

Let $(u, v) := E(\phi, \psi)$ be any extension of (ϕ, ψ) to $H^2(S_0) \oplus H^2_\delta(S_\infty)$ such that

$$J_\delta(u, v) = (\phi, \psi).$$

Define $f := \Phi(\phi, \psi) \in L^2_\delta(S)$ by

$$f(x) := \begin{cases} \Delta u(x) & \text{if } x \in S_0, \\ \Delta v(x) & \text{if } x \in S_\infty. \end{cases}$$

If $g \in H^2_\delta(S)$ satisfies $\Delta g = f$, then $(u - g, v - g)$ solves the Neumann Problem.

It thus suffices to determine when $f = \Phi(\phi, \psi)$ is in the image of Δ.

Solving the Cauchy Problem

Theorem D

$\Phi(\phi, \psi)$ is in the image of Δ if and only if

$$\int_{S^1} \phi\beta - \psi\alpha dl = 0,$$

for all $(\alpha, \beta) = (C_1 h, C_0 h)$, where $h \in \mathcal{H}_{-\delta}(S)$.

Proof. By Stokes' Theorem, for all $h \in \mathcal{H}_{-\delta}(S)$, denoting $(\alpha, \beta) = (C_1 h, C_0 h)$,

$$\int_{S^1} \psi\alpha - \phi\beta dl = \int_S \Phi(\phi, \psi) h \mathrm{dVol}$$

However, since Δ is self-adjoint, and since the dual space to $H^m_\delta(S)$ is $H^{-m}_{-\delta}(S)$, this is exactly the condition for being in the image of Δ.

A First Index Formula

Theorem E

J_δ is Fredholm, and

$$\mathrm{Ind}(J_\delta) = \mathrm{Ind}(\Delta; H^2_\delta(S)).$$

Proof. The kernel of J_δ is isomorphic to $\mathcal{H}_\delta = \mathrm{Ker}(\Delta; H^2_\delta(S))$.

(C_1, C_0) maps $\mathcal{H}_{-\delta}(S)$ injectively into $H^{\frac{1}{2}}(S^1) \oplus H^{\frac{3}{2}}(S^1)$.

The codimension of $\mathrm{Im}(J_\delta)$ therefore equals the dimension of $\mathcal{H}_{-\delta} = \mathrm{Ker}(\Delta; H^2_{-\delta}(S))$.

J_δ is therefore Fredholm and

$$\begin{aligned}\mathrm{Ind}(J_\delta) &= \mathrm{Dim}(\mathrm{Ker}(\Delta; H^2_\delta(S))) - \mathrm{Dim}(\mathrm{Ker}(\Delta; H^2_{-\delta}(S))) \\ &= \mathrm{Ind}(\Delta; H^2_{-\delta}(S)).\end{aligned}$$

Harmonic Functions Over The Half-Cylinder

By separation of variables,

$$\mathcal{H}_\delta = \left\{ \sum_{n<\delta} a_n e^{in\theta} e^{nt} \mid \sum n^3 a_n^2 < \infty \right\}$$

$$\oplus \left\{ \sum_{-\delta<n} b_n e^{in\theta} e^{-nt} \mid \sum n^3 b_n^2 < \infty \right\}.$$

In particular,

$$\mathcal{H}_\delta = \mathcal{H}_{-\delta} \oplus \left\{ \sum_{-\delta<n<\delta} a_n e^{in\theta} e^{nt} \right\} \oplus \left\{ \sum_{-\delta<n<\delta} b_n e^{in\theta} e^{-nt} \right\}.$$

That is,

$$\mathrm{Dim}\left(\mathcal{H}_\delta / \mathcal{H}_{-\delta}\right) = 2\left(2\lfloor\delta\rfloor + 1\right).$$

The Index Formula

Theorem F

For all non-integer δ,

$$\mathrm{Ind}\left(\Delta; H^2_\delta\right) = 2\lfloor\delta\rfloor + 1.$$

Proof. Indeed,

$$\begin{aligned}\mathrm{Ind}\left(\Delta; H^2_\delta\right) - \mathrm{Ind}\left(\Delta; H^2_{-\delta}\right) &= \mathrm{Ind}\left(J_\delta\right) - \mathrm{Ind}\left(J_{-\delta}\right) \\ &= 2\left(2\lfloor\delta\rfloor + 1\right).\end{aligned}$$

Since Δ is self-adjoint, and since the dual space to $\mathcal{H}^m_\delta$ is $H^{-m}_{-\delta}$,

$$\mathrm{Ind}\left(\Delta; H^2_\delta\right) + \mathrm{Ind}\left(\Delta; H^2_{-\delta}\right) = 0.$$

Thankyou!