

On translating solitons for the mean curvature flow that are of finite genus.

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Prologue - Glueing I

Let $A_{ij} : \mathbb{R} \rightarrow \text{Symm}(\mathbb{R}^n)$ be smooth functions such that

(1) $A_{ij}(t)$ tends to A_i (resp. A_j) as t tends to $-\infty$ (resp. $+\infty$);

(2) $A_{ij}^{(k)}(t)$ tends to 0 as $|t|$ tends to ∞ .

Define $D : H^1(\mathbb{R}, \mathbb{R}^n) \rightarrow L^2(\mathbb{R}, \mathbb{R}^n)$ by

$$(D_{ij}f)(t) = (\partial_t f)(t) + A_{ij}(t)f(t).$$

Prologue - Glueing II

Theorem

If $A_{ij}(t) =: A$ is constant, and if A is invertible, then D_{ij} defines an invertible linear map from $H^1(\mathbb{R}, \mathbb{R}^n)$ into $L^2(\mathbb{R}, \mathbb{R}^n)$.

Theorem

If A_i and A_j are invertible, then D_{ij} is Fredholm. Furthermore

$$\text{Ind}(D_{ij}) = \text{Index}(A_i) - \text{Index}(A_j),$$

where $\text{Index}(A_i)$ and $\text{Index}(A_j)$ are the Morse Indices of A_i and A_j respectively.

Theorem

For generic A_{ij} , $\text{Coker}(D_{ij})$ is maximal.

Prologue - Glueing III

Define $\xi : \mathbb{R} \rightarrow [0, 1]$ such that

$$\xi(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ 0 & \text{if } t \geq 1. \end{cases}$$

For $R > 0$, define $D_{02,R}^R : H^1(\mathbb{R}, \mathbb{R}^n) \rightarrow L^2(\mathbb{R}, \mathbb{R}^n)$ by

$$(D_{02}^R f)(t) = (\partial_t f)(t) + (\xi(t)A_{01}(t+R) + (1 - \xi(t))A_{12}(t-R))f(t).$$

Theorem

For sufficiently large R , if $\xi(t)f(t+R) \in \text{Ker}(D_{01})^\perp$, if $(1 - \xi(t))f(t-R) \in \text{Ker}(D_{12})^\perp$, and if $f \in \text{Ker}(D^R)$, then $f = 0$.

Prologue - Glueing IV

Step I: For all $L, \epsilon > 0$ and for sufficiently large R ,

$$\|f|_{[-L,L]}\|_{H^1} \leq \epsilon.$$

Indeed, define

$$\eta^R(t) = \xi(4(x-1)/R)\xi(4(1-x)/R).$$

Then, denoting $A := A_1$,

$$\begin{aligned}(\partial_t - A)(\eta^R f) &= D_{02}^R(\eta^R f) + \text{Err}^R(\eta^R f) \\ &= [D_{02}^R, \eta^R]f + \eta^R D_{02}^R f + \text{Err}^R(\eta^R f) \\ &= (\partial_t \eta^R)f + \text{Err}^R(\eta^R f).\end{aligned}$$

This is small for large R , and the assertion follows by invertibility of $\partial_t - A$.

Prologue - Glueing V

Step II: For all $\epsilon > 0$ and for sufficiently large R ,

$$\|\xi f\|_{H^1} \leq \epsilon.$$

Indeed

$$\begin{aligned} D_{01}(\xi f) &= D_{02}^R(\xi f) + \text{Err}^R(\xi f) \\ &= [D_{02}^R, \xi]f + \xi D_{02}^R f + \text{Err}^R(\xi f) \\ &= (\partial_t \xi)f + \text{Err}^R(\xi f). \end{aligned}$$

This is small for large R , and the assertion follows since $\xi f \in \text{Ker}(D_{01})^\perp$.

Mean Curvature Flow

Let M be a surface. Let $i_t : M \rightarrow \mathbb{R}^3$ be a smooth family of complete, smooth immersions of M into \mathbb{R}^3 .

Let $N_t : M \rightarrow S^2$ be the unit normal vector field over i_t .

Let $H_t : M \rightarrow \mathbb{R}$ be the mean curvature of M .

We say that i_t is a **mean curvature flow** whenever it satisfies

$$\langle \partial_t i_t, N_t \rangle + H_t = 0.$$

This is a degenerate-parabolic partial differential equation.

Mean Curvature Solitons

Let A_t be a group of isometries of \mathbb{R}^3 .

Let $i : M \rightarrow \mathbb{R}^3$ be a complete, smooth immersion.

We say that i is a **soliton** for the mean curvature flow whenever the family $i_t := A_t \circ i$ is a mean curvature flow.

When A_t is a group of translations (usually in the z -direction), we say that i is a **translating soliton**.

Translating solitons are solutions of

$$\langle e_z, N_t \rangle + H_t = 0,$$

where e_z here denotes the unit vector in the z direction.

This is a degenerate-elliptic partial differential equation.

Invariant Examples I - Translation Invariance

Simple examples are constructed by supposing invariance with respect to a continuous group of isometries of \mathbb{R}^3 .

This reduces the dimension of the problem from 2 to 1.

First suppose horizontal translation invariance.

This reduces the problem to mean curvature solitons in \mathbb{R}^2 . Up to translation, there are two examples.

(1) The vertical line in \mathbb{R}^2 . This yields the vertical plane in \mathbb{R}^3 .

(2) The graph of the function

$$g :] - \pi/2, \pi/2[\rightarrow \mathbb{R}, \quad t \mapsto \log(\sec(t)).$$

This is known as the **Grim Reaper Curve**. It yields the **Grim plane** in \mathbb{R}^3 .

Invariant Examples II - Rotation Invariance I

Now suppose rotation invariance about the z-axis.

There are two types of solutions: **simply connected** solutions and **doubly connected** solutions.

The simply connected solution, S_0 , is called the **Grim paraboloid**.

S_0 is the graph of a function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$.

For large x ,

$$g(x) \sim C + \frac{1}{2}\|x\|^2 - \log(\|x\|) + \dots$$

Up to vertical translation, S_0 is unique.

Invariant Examples - Rotation Invariance II

The doubly connected solution, S_r , is called the **Grim catenoid**.

S_r is the union of the graphs of two functions

$$g_{u,r}, g_{l,r} : A(r, \infty) \rightarrow \mathbb{R}.$$

For large x ,

$$g_{u,r}(x), g_{l,r}(x) \sim C + \frac{1}{2}\|x\|^2 - \log(\|x\|) + \dots$$

Up to vertical translation, this family is 1-dimensional, parametrised by the inner-radius, r .

As r tends to 0, S_r converges to 2 copies of S_0 .

Invariant Examples - Discrete Translation Invariance

In 2013, Nguyen constructed solutions that are invariant under the action of a discrete group of translations.

She calls these solutions **translating tridents**.

They are obtained by desingularising the union of a **Grim plane** and a **vertical plane** along a **singly-periodic Sherk surface**.

In particular, like the singly-periodic Sherk surface, they have infinite genus.

Invariant Examples - Discrete Rotation Invariance

In 2014, Martín, Savas-Halilaj and Smoczyk ask for the existence of solutions of non-trivial finite topology.

We achieve this by singularising the union of a **Grim catenoid** and a **Grim paraboloid** about the **Costa-Hoffman-Meeks surface**, C_g , of genus g .



Revision - Costa-Hoffman-Meeks Surfaces

C_g is a complete, embedded minimal surface of genus g with 3 horizontal ends.

The upper and lower ends are catenoidal, that is, they are asymptotic to the graphs of

$$f_{\pm}(x) = \pm A \pm B \log(\|x\|) + \dots$$

The middle end is planar.

For all $k \in \{0, \dots, g\}$, C_g is invariant by reflection in the plane spanned by the vectors

$$\cos(k\pi/(g+1))e_x + \sin(k\pi/(g+1))e_y \text{ and } e_z.$$

We refer to the isometry group spanned by these reflections as the group of **horizontal symmetries** of C_g .

Solitons of Non-Trivial, Finite Genus

Theorem A

Let g be a positive integer and fix $\eta > 0$. For all sufficiently large Δ , and for all $\epsilon, R > 0$ such that

$$\epsilon R^{4+\eta} < \frac{1}{\Delta}, \quad \epsilon R^{5-\eta} > \Delta,$$

there exists a complete, embedded, translating soliton, $M_{g,\epsilon,R}$, with genus g and 3 ends. Furthermore

- (1) $M_{g,\epsilon,R}$ is preserved by the horizontal symmetries of C_g ;
- (2) $M_{g,\epsilon,R} \setminus B_{\epsilon R}(0)$ consists of 3 disjoint Grim ends each of which converges towards the Grim paraboloid as Δ tends to infinity; and
- (3) upon rescaling by $1/\epsilon$, $M_{g,\epsilon,R} \cap B_{\epsilon R}(0)$ converges towards C_g as Δ tends to infinity.

Transition Regions and Cut-Off Functions

For $R > 0$, define

$$B(R) := \{(x, y, z) \mid \|x, y\| \leq R\}.$$

$$A(R, 2R) := \{(x, y, z) \mid R \leq \|x, y\| \leq 2R\}.$$

Define $\chi : [0, \infty[\rightarrow [0, 1]$ such that

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq 1, \\ 0 & \text{if } t \geq 2. \end{cases}$$

For $R > 0$, define $\chi_R : \mathbb{R}^3 \rightarrow [0, 1]$ by

$$\chi_R(x, y, z) = \chi\left(\frac{\sqrt{x^2 + y^2}}{R}\right).$$

We call χ_R the **cut-off function** over the **transition region** $A(R, 2R)$.

Lies, Damned Lies, and Mathematics

We pretend that C_g is the graph of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

(It is outside a compact set.)

We pretend that J_C is invertible, with Green's operator G_C .

(It's almost invertible.)

We pretend that G is the graph of a function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$.

(It is outside a compact set.)

We pretend that J_G is invertible, with Green's operator G_G .

(It is.)

Building the Approximate Soliton

Given $\epsilon > 0$, we define the rescaled Grim end

$$g_\epsilon(x, y) := g(\epsilon x, \epsilon y).$$

Given $R > 0$, we define the approximate soliton

$$h(x, y) := h_{\epsilon, R}(x, y) = \chi_R(x, y)f(x, y) + (1 - \chi_R)(x, y)g_\epsilon(x, y).$$

Let H be the graph of h .

H solves the translating soliton equation over $A(2R, \infty)$.

The error over $B(2R)$ is bounded by ϵ .

We consider H as an **approximate soliton**.

Building the Green's operator

We construct the translating soliton by perturbing H .

This is a simple application of Schauder's fixed point theorem.

However, we require a Green's operator for J_H .

Furthermore, the Green's operator is not unique: we need to choose the right one.

We construct a suitable Green's operator out of G_C and G_G .

We do this via a “ping-pong” argument.

Semi-Norms

For $\phi \in C_0^\infty(B(0, 2R))$, we define

$$\|\phi\|_{k,C}$$

to be the C^k -norm of ϕ with respect to the intrinsic metric of C .

For $\psi \in C_0^\infty(A(R, \infty))$, we define

$$\|\phi\|_{k,G}$$

to be the C^k -norm of ϕ with respect to the intrinsic metric of G .

Care should be taken, as G has been **rescaled** by a factor of $1/\epsilon!$

The “Ping-Pong” Argument - Part I

For ϕ supported in $B(0, 2R)$, we define

$$A\phi := J_H \chi_{R^4} G_C \phi - \phi.$$

$A\phi$ measures the obstruction to $\chi_{R^4} G_C \phi$ being the J_H inverse of ϕ .

We rewrite A as

$$A\phi = \chi_{R^4} (J_H - J_C) G_C \phi - [J_G, \chi_{R^4}] G_C \phi.$$

This is supported in $A(R, \infty)$. Furthermore

$$\|A\phi\|_{0,G} \leq \frac{1}{R^{6+\delta}} \|\phi\|_{0,C}.$$

The “Ping-Pong” Argument - Part II

For ψ supported in $A(R, \infty)$, we define

$$B\phi := J_H(1 - \chi_{R/2})G_G\psi - \psi.$$

$B\phi$ measures the obstruction to $(1 - \chi_{R/2})G_G\psi$ being the J_H inverse of ψ .

We rewrite B as

$$B\phi = \chi_{R/4}(J_H - J_C)G_C\phi + [J_G, \chi_{R/4}]G_C\phi.$$

This is supported in $B(2R)$. Furthermore

$$\|B\psi\|_{0,C} \leq \frac{R^2}{\epsilon R} \|\psi\|_{0,C}.$$

The “Ping-Pong” Argument - Part III

We define

$$Q_C := \sum_{m=0}^{\infty} (BA)^m, \quad Q_G := \sum_{m=0}^{\infty} (AB)^m$$

If ϕ is supported in $B(0, 2R)$, we define

$$G_{H,C}\phi := \chi_{R^4} G_C Q_C \phi - (1 - \chi_{R/4}) G_G A Q_C \phi.$$

This yields

$$\begin{aligned} J_H G_{H,C}\phi &= J_H \chi_{R^4} G_C Q_C \phi - J_H (1 - \chi_{R/4}) G_G A Q_C \phi, \\ &= A Q_C \phi + Q_C \phi - B A Q_C \phi - A Q_C \phi, \\ &= \phi. \end{aligned}$$

We define $G_{H,G}$ in a similar manner.

Information Loss I - Differentiation Operators

Let D denote the differentiation operator of \mathbb{R}^2 and define

$$D_C := RD, \quad D_G := \frac{1}{\epsilon}D$$

D_C is the natural differentiation operator over the Costa-Hoffman-Meeks surface.

D_G is the natural differentiation operator over the rescaled Grimm end.

In particular

$$\|\phi\|_{2,C} = \|\phi\|_{L^\infty} + \|D_C\phi\|_{L^\infty} + \|D_C^2\phi\|_{L^\infty}$$

$$\|\phi\|_{2,G} = \|\phi\|_{L^\infty} + \|D_G\phi\|_{L^\infty} + \|D_G^2\phi\|_{L^\infty}$$

Information Loss II - Changing Norms

For ϕ supported in $A(R/4, 2R)$,

$$\begin{aligned}\|\phi\|_{L^\infty} &= \|\phi\|_{L^\infty}, \\ \|D_G\phi\|_{L^\infty} &= (\epsilon R)^{-1} \|D_C\phi\|_{L^\infty}, \\ \|D_G^2\phi\|_{L^\infty} &= (\epsilon R)^{-2} \|D_C^2\phi\|_{L^\infty}.\end{aligned}$$

In particular

$$\|\phi\|_{2,G} \leq (\epsilon R)^{-2} \|\phi\|_{2,C}.$$

This constitutes a loss of information concerning the lower order derivatives of ϕ !

Information Recovery - The Hybrid Norm

To recover this lost information, we actually use the norm

$$\|\phi\|_{k,G} := \|\phi\|_{C^k(G)} + \frac{1}{(\epsilon R)} \|\phi\|_{H^k(G)},$$

where $\|\cdot\|_{C^k(G)}$ is the rescaled C^k norm over G and $H^k(G)$ is the rescaled Sobolev norm over G .

In particular,

$$\|\phi\|_{C^1(G)} \leq \|\phi\|_{H^2(G)} \leq (\epsilon R) \|\phi\|_{2,G}.$$

Well... not quite...

Thankyou!

