On the Weyl problem in Minkowski space.

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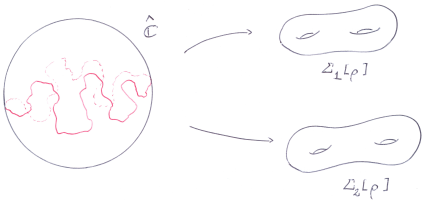
18/06/2021

Background I - Quasi-Fuchsian representations

Σ is a compact surface with fundamental group Π.

$\mathrm{PSO}_0(3,1)$ is the conformal group of $\hat{\mathbb{C}}$.

$\rho : \Pi \to \mathrm{PSO}_0(3,1)$ is **quasi-Fuchsian** whenever it is discrete, injective and preserves a unique Jordan curve Γ.



Background II - Ahlfors-Bers double parametrisation

$\hat{\mathbb{C}} \setminus \Gamma/\rho$ is two marked Riemann surfaces $\Sigma_1[\rho]$ and $\Sigma_2[\rho]$ each with fundamental group Π.

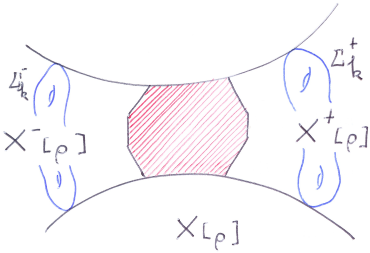
By Ahlfors-Bers, (Σ_1, Σ_2) defines a *bijection* from the space of quasi-Fuchsian representations into $\mathrm{Teich}[\Sigma] \times \mathrm{Teich}[\Sigma]$.

We consider Ahlfors-Bers result as a global parametrisation of a space of representations.

Background III - Quasi-Fuchsian manifolds

$\mathrm{PSO}_0(3,1)$ is also $\mathrm{Isom}_0(\mathbb{H}^3)$.

$X[\rho] := \mathbb{H}^3/\rho$ is a **quasi-Fuchsian manifold**.



$X[\rho]$ is homeomorphic to $\Sigma \times \mathbb{R}$.

It has two opposing ends $X^{\pm}[\rho]$.

Background IV - Labourie

By Labourie, for all $k \in]-1,0[$, there exists a unique, embedded surface $\Sigma_k^{\pm}[\rho]$ in $X^{\pm}[\rho]$ of constant intrinsic curvature equal to k and onto which $X^{\pm}[\rho]$ contracts.

By Labourie, for all $k \in]-1,0[$, (Σ_k^-, Σ_k^+) defines a *bijection* from the space of quasi-Fuchsian representations into $\mathrm{Teich}[\Sigma] \times \mathrm{Teich}[\Sigma]$.

This result has applications to hyperbolic geometry.

We consider it as a global parametrisation of a space of representations.

Background V - Labourie-Schlenker

By Labourie, for every pair (h_+, h_-) of marked hyperbolic metrics, there exists a unique quasi-Fuchsian ρ such that $(\Sigma, h_\pm) = \Sigma_k^\pm[\rho]$.

This extends naturally to prescription of quasi-Fuchsian representations by pairs of negatively-curved metrics.

By Labourie-Schlenker, for every pair (h_+, h_-) of marked negatively-curved metrics with curvature in $]-1, 0[$, there exists a unique quasi-Fuchsian ρ such that $(\Sigma, h_\pm)$ embeds isometrically into $X^\pm[\rho]$.

Background VI - Labourie-Schlenker

This result also has applications to hyperbolic geometry.

We consider it again as result about the prescription of quasi-Fuchsian representations.

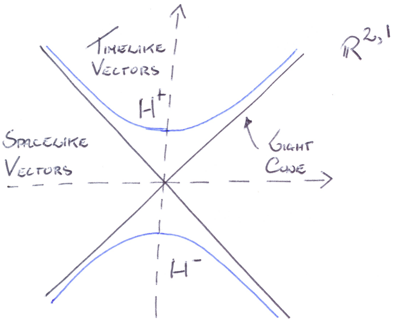
Problems of parametrisation and prescription have been extended to other types of data and other groups.

We will be concerned with prescription of representations in the affine isometry group of Minkowski space.

Minkowski space

Minkowski space is $\mathbb{R}^{2,1}$ with the **Minkowski metric**

$$g^{2,1} := dx^1 dx^1 + dx^2 dx^2 - dx^3 dx^3.$$



The unit pseudosphere and hyperbolic space

The **unit pseudosphere** is

$$H := \left\{ x \mid \|x\|^2 = -1 \right\}.$$

Its upper sheet

$$H^+ := \left\{ x \mid \|x\|^2 = -1, x_3 > 0 \right\}$$

models 2-dimensional hyperbolic space $\mathbb{H}^2$.

The isometry group

The *linear* isometry group $\mathbb{R}^{2,1}$ is $\mathrm{O}(2,1)$.

It has 4 connected components determined by spatial and temporal orientation.

Its identity component is $\mathrm{SO}_0(2,1)$.

This component preserves H^+ and identifies with $\mathrm{Isom}_0(\mathbb{H}^2)$.

The *affine* isometry group has identity component $\mathrm{SO}_0(2,1) \ltimes \mathbb{R}^{2,1}$.

Its group law is

$$(M, x) \cdot (N, y) := (MN, My + x).$$

Fuchsian representations and affine deformations

Σ is a marked, compact surface with fundamental group Π.

$\rho : \Pi \to \mathrm{SO}_0(2,1)$ is **Fuchsian** whenever it is discrete and injective.

The space $\mathrm{Teich}[\Sigma]$ is conjugacy classes of ρ is **Teichmüller space**.

$\sigma := (\rho, \tau) : \Pi \to \mathrm{SO}_0(2,1) \ltimes \mathbb{R}^{2,1}$ is an **affine deformation** whenever ρ is Fuchsian.

The space of conjugacy classes of σ is $\mathrm{TTeich}[\Sigma]$.

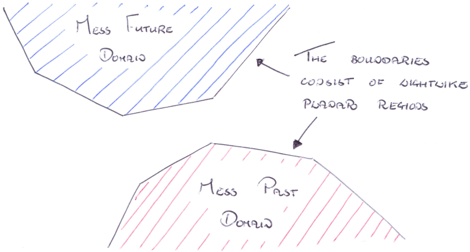
Maximal invariant convex sets

$K \subseteq \mathbb{R}^{2,1}$ is **future (past) complete** whenever every future (past) oriented causal geodesic ray can be continued indefinitely.

Mess shows that there exists a unique σ-invariant, future (past) complete, closed, convex, proper subset of $\mathbb{R}^{2,1}$ maximal amongst subsets with these properties.

Mess future and past domains

$K^{\pm}(\sigma)$ is **Mess future (past) domain.**



Denoting $\sigma^* := (\rho, -\tau)$, there is a trivial **involution**

$$K^{\pm}(\sigma^*) = -K^{\pm}(\sigma).$$

GHMC Minkowski spacetimes

σ acts properly discontinuously on $K^{\pm}(\sigma)^{o}$.

The quotient $X^{\pm}(\sigma) := K^{\pm}(\sigma)^{o}/\sigma$ is a future (past) complete GHMC Minkowski spacetime.

All future (past) complete GHMC Minkowski spacetimes arise in this manner (surjectivity).

Two affine deformations yield isometric spacetimes if and only if their are conjugate (injectivity).

The trivial involution defines an involution of GHMC Minkowski spacetimes whose fixed point set is Fuchsian spacetimes.

Equivariant isometric embeddings

(Σ, g) is a compact riemannian surface with curvature $-\kappa < 0$.

$e : \tilde{\Sigma} \to \mathbb{R}^{2,1}$ is a σ-equivariant, future-oriented, isometric embedding.

σ is called the **holonomy** of e.

Two such embeddings are **equivalent** if they differ by an element of $\mathrm{SO}_0(2,1) \ltimes \mathbb{R}^{2,1}$.

Equivalent embeddings have equivalent holonomies.

Equivalent classes of embeddings thus define points of $\mathrm{TTeich}[\Sigma]$.

$e(\tilde{\Sigma})$ is contained in $K^+(\sigma)$ and descends to an embedding of Σ in $X^+(\sigma)$.

The fundamental theorem of surface theory

The **shape operator** of e is a section A of $\mathrm{End}(T\Sigma)$.

A **Codazzi tensor** is a symmetric section B of $\mathrm{End}(T\Sigma)$ such that $d^{\nabla}B = 0$.

A **Labourie tensor** is a Codazzi tensor satisfying $\mathrm{Det}(A) = \kappa$.

A is a Labourie tensor.

Equivalent embeddings have the same Labourie tensor.

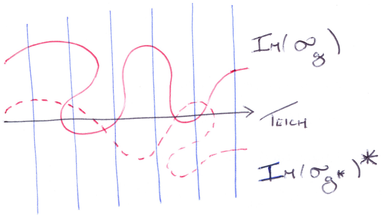
Conversely, by the fundamental theorem of surface theory, every Labourie tensor integrates to a unique equivalence class of equivariant isometric immersions.

Via holonomy, every Labourie tensor defines a unique point $\sigma(g, A)$ of $\mathrm{TTeich}[\Sigma]$.

A half-dimensional submanifold

The space of Labourie tensors is $(6g-6)$-dimensional.

For any given g, $A \mapsto \sigma(g, A)$ traces out a $(6g-6)$-dimensional subset of $\mathrm{TTeich}[\Sigma]$.



Intersection theory

Theorem

Let g and g^* be marked, negatively curved metrics over Σ. There exist unique Labourie tensors A and A^* of g and g^* respectively such that

$$\sigma(g, A) = \sigma(g^*, A^*)^*$$

Theorem

There exists a unique future-complete GHMC Minkowski spacetime X such that (Σ, g) and (Σ, g^*) embed isometrically as Cauchy surfaces in X and X^* respectively. Furthermore, the embeddings are also unique.

Marked hyperbolic metrics

$h(g, A) := g(A\cdot, A\cdot)$ is a marked hyperbolic metric.

$B := A^{-1}$ is a Codazzi tensor of $h(g, A)$.

$A \mapsto h(g, A)$ sends the space of Labourie tensors into $\mathrm{Teich}[\Sigma]$.

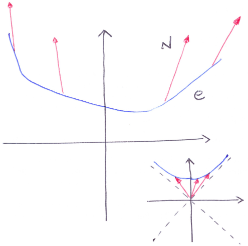
By Trapani-Valli, this function is a diffeomorphism.

Given any marked, compact hyperbolic surface R there exists a unique diffeomorphism $\Phi : R \to \Sigma$ and positive-definite Codazzi tensor B such that $\Phi^* g = h(B\cdot, B\cdot)$.

In particular $A = \Phi_* B^{-1}$.

Identifying ρ and h

As a point of $\mathrm{Teich}[\Sigma]$, $h(g, A) = \rho(g, A)$.



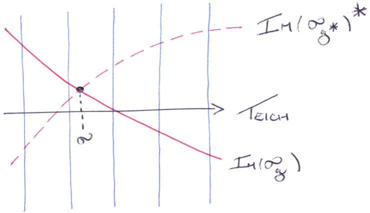
- N IS A ρ-EQUIVARIANT IMMERSION IN $\mathbb{H}^2 = H^+$
- $h := g(A\cdot, A\cdot) = \mathrm{III}$ IS THE THIRD FUNDAMENTAL FORM OF e.

By Trapani-Valli, the first component of σ is a diffeomorphism.

Reformulating the main result

$\mathrm{Im}(\sigma(g,\cdot))$ is a section τ_g of $\mathrm{TTeich}[\Sigma]$:

$$\rho \mapsto h \mapsto (g, \Phi, B) \mapsto (g, A) \mapsto \sigma(g, A) = (\rho(g, A), \tau(g, A)).$$



Theorem

Let g and g^* be marked, negatively curved metrics over Σ. There exists a unique $\rho \in \mathrm{Teich}[\Sigma]$ such that $\tau_g(\rho) + \tau_{g^*}(\rho) = 0$.

Energy functionals

For each g, the **Trapani-Valli energy** is

$$\mathrm{E}_g[\rho] := \mathrm{E}_g[h] := \int_R \mathrm{Tr}(B)\mathrm{dArea}.$$

E_g is proper and strictly convex.

τ_g is minus the Weil-Petersson gradient of E_g.

Since $\mathrm{Teich}[\Sigma]$ is geodesically convex, there exists a unique ρ such that

$$\tau_g + \tau_{g^*} = -(\nabla \mathrm{E}_g + \nabla \mathrm{E}_{g^*}) = 0.$$

The gradient of the energy functional - Step I

Define

$$B_{g,0} := B - (f\mathrm{Id} - \mathrm{Hess}(f)),$$

where f is the unique ρ-periodic function solving

$$(\Delta - 2)f = \mathrm{Tr}(B).$$

$B_{g,0}$ is trace free and for every trace free M,

$$\int_R \mathrm{Tr}(B_g M)\mathrm{dArea} = \int_R \mathrm{Tr}(B_{g,0} M)\mathrm{dArea}.$$

The gradient of the energy functional - Step II

First-order variations of h are given by trace-free Codazzi fields M.

By Bonsante-Mondello-Schlenker, for all M,

$$\mathrm{DE}_g \cdot M = -\int_R \mathrm{Tr}(B_g M)\mathrm{dArea} = -\int_R \mathrm{Tr}(B_{g,0} M)\mathrm{dArea}.$$

Consequently,

$$\nabla \mathrm{E}_g = -B_{g,0}.$$

The gradient of the energy functional - Step III

A trace-free Codazzi field M defines a first-order variation of ρ.

Since $\mathbb{H}^2 = H^+ \subseteq \mathbb{R}^{2,1}$, M lifts to a ρ-equivariant section $\tilde{M}$ of $T\mathbb{H}^2 \otimes \mathbb{R}^{2,1}$.

The Codazzi equation $d^\nabla M = 0$ means that $\tilde{M}$ is exact.

After choosing a base point x_0, up to a trivial cocycle, the first order variation of ρ is

$$\mathrm{ad}[M](\gamma) := \int_\gamma \tilde{M}$$

By Bonsante-Seppi, this function is a linear isomorphism between trace-free Codazzi fields and cocycles of ρ.

The gradient of the energy functional - Step IV

After choosing a base point x_0, the equivariant immersion of $g = h(B_g\cdot, B_g\cdot)$ is

$$e_g(x) := \int_{x_0}^{x} \tilde{B}_g.$$

Its cocycle is thus equivalent to

$$\tau_g(\gamma) = \int_\gamma \tilde{B}_g = \int_\gamma \tilde{B}_{g,0} = \mathrm{ad}[B_{g,0}] = -\mathrm{ad}[\nabla E_g].$$

Danke!